TCSS 343 - Assignment 3

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In this problem use the Master Theorem to find and prove tight bounds for these recurrences (6 points each).

To solve this problem I used the master theorem taken from CLRS. I used a bit of a stronger statement than what was stated in the theorem but they are equivalent mathematically due to the nature of limits. I will include the theorem for the reader's benefit. To check my work I decided to include what I obtained using the Akra-Bazzi method.

Master Theorem

The Master Theorem applies to recurrences of the following form:

$$T(n) = aT(n/b) + f(n)$$

where $a \geq 1$ and $b > 1$ are constants and $f(n)$ is an asymptotically positive function.

There are 3 cases:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a} \log^k n)$ with[1] $k \geq 0$, then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ with $\epsilon > 0$, and $f(n)$ satisfies the regularity condition, then $T(n) = \Theta(f(n))$. Regularity condition: $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n.

1.

$$T(n) = \begin{cases} c & \text{if } n < 8 \\ 16T(n/8) + n \log n & \text{if } n \geq 8 \end{cases}$$

$$\begin{aligned}
\lim_{n\to\infty} \lim_{\varepsilon\to 0} \frac{n \log n}{n^{\log_8 16 + \varepsilon}} &= \lim_{n\to\infty} \lim_{\varepsilon\to 0} \frac{n \log n}{n^{\frac{4}{3}+\varepsilon}} \\
&= \lim_{n\to\infty} \lim_{\varepsilon\to 0} (\frac{n}{n})(\frac{\log n}{n^{\frac{1}{3}} n^{\varepsilon}}) \\
&= \lim_{n\to\infty} (1)(\frac{\log n}{\sqrt[3]{n}(1)}) \\
&= \lim_{n\to\infty} \frac{\log n}{\sqrt[3]{n}} \to 0
\end{aligned}$$

From Knuth's concrete mathematics I know that any polynomial, even fractional powers, grows asytotically faster than a logarithm so this limit goes to 0. We have that problem one is the first case:

$$T(n) \in \Theta(n^{\frac{4}{3}})$$

By Akra-Bazzi I obtain:

$$\begin{aligned} 16(1/8)^p &= 1 \\ p &= \frac{4}{3} \end{aligned}$$

$$\begin{aligned} T(n) &= \Theta(n^p(1 + \int_1^n \frac{u \log u}{u^{p+1}} du)) \\ &= \Theta(-9n + 10n^{\frac{4}{3}} - 3n \log n) \\ &= \Theta(n^{\frac{4}{3}}) \end{aligned}$$