TCSS 343 - Assignment 3

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1 UNDERSTAND

In this problem use the Master Theorem to find and prove tight bounds for these recurrences (6 points each).

To solve this problem I used the master theorem taken from CLRS. I used a bit of a stronger statement than what was stated in the theorem but they are equivalent mathematically due to the nature of limits. I will include the theorem for the reader's benefit. To check my work I decided to include what I obtained using the Akra-Bazzi method.

Master Theorem

The Master Theorem applies to recurrences of the following form:

$$T(n) = aT(n/b) + f(n)$$

where $a \geq 1$ and $b > 1$ are constants and $f(n)$ is an asymptotically positive function.

There are 3 cases:

1. If $f(n) = O(n^{\log_b a - \epsilon})$ for some constant $\epsilon > 0$, then $T(n) = \Theta(n^{\log_b a})$.
2. If $f(n) = \Theta(n^{\log_b a} \log^k n)$ with[1] $k \geq 0$, then $T(n) = \Theta(n^{\log_b a} \log^{k+1} n)$.
3. If $f(n) = \Omega(n^{\log_b a + \epsilon})$ with $\epsilon > 0$, and $f(n)$ satisfies the regularity condition, then $T(n) = \Theta(f(n))$. Regularity condition: $af(n/b) \leq cf(n)$ for some constant $c < 1$ and all sufficiently large n.

1.

$$T(n) = \begin{cases} c & \text{if } n < 8 \\ 16T(\frac{n}{8}) + n \log n & \text{if } n \geq 8 \end{cases}$$

By Master Method I obtain case 1:

$$\begin{aligned} \lim_{n\to\infty} \frac{n \log n}{n^{\log_8 16 + \varepsilon}} &= \lim_{n\to\infty} \frac{n \log n}{n^{\frac{4}{3}+\frac{2}{3}}} \\ &= \lim_{n\to\infty} (\frac{n}{n})(\frac{\log n}{n}) \to 0 \\ T(n) &\in \Theta(n^{\frac{4}{3}}) \end{aligned}$$

By Akra-Bazzi I obtain:

$$\begin{aligned} 16(\frac{1}{8})^p &= 1 \\ p &= \frac{4}{3} \end{aligned}$$

$$\begin{aligned} T(n) &\in \Theta(n^p(1 + \int_1^n \frac{u \log u}{u^{p+1}} du)) \\ &\in \Theta(-9n + 10n^{\frac{4}{3}} - 3n \log n) \\ &\in \Theta(n^{\frac{4}{3}}) \end{aligned}$$

2.

$$T(n) = \begin{cases} c & \text{if } n < 4 \\ 8T(\frac{n}{4}) + n \log n & \text{if } n \geq 4 \end{cases}$$

By Master Method I obtain case 2:

$$\begin{aligned} \lim_{n \to \infty} \frac{n^{\frac{1}{3}}}{n^{\log_8 2 + \varepsilon}} &= \lim_{n \to \infty} \frac{n^{\frac{1}{3}}}{n^{\frac{1}{3}}} \\ &= \lim_{n \to \infty} 1 \to 1 \\ &\in \Theta(n^{\frac{1}{3}} \log n) \end{aligned}$$

By Akra-Bazzi I obtain:

$$\begin{aligned} 2(\frac{1}{8})^p &= 1 \\ p &= \frac{1}{3} \end{aligned}$$

$$\begin{aligned} T(n) &\in \Theta(n^p(1 + \int_1^n \frac{u^{\frac{1}{3}}}{u^{p+1}} du)) \\ &\in \Theta(n^{\frac{1}{3}} + n^{\frac{1}{3}} \log n) \\ &\in \Theta(n^{\frac{1}{3}} \log n) \end{aligned}$$

3.

$$T(n) = \begin{cases} c & \text{if } n < 2 \\ 3T(\frac{n}{2}) + 9^n & \text{if } n \geq 2 \end{cases}$$

By Master Method I obtain case 3:

$$\begin{aligned}\lim_{n\to\infty}\frac{9^n}{n^{\log_3 2+\varepsilon}} &= \lim_{n\to\infty}\frac{9^n}{n^{\frac{\log 2}{\log 3}-(\frac{\log 2}{\log 3}+1)}}\\ &= \lim_{n\to\infty} n9^n \to \infty\\ &\in \Theta(9^n)\end{aligned}$$

This recurrence is not well suited for Akra-Bazzi.

4.

$$T(n) = \begin{cases} c & \text{if } n \le 1 \\ 3T(\frac{3n}{5}) + n^2 & \text{if } n > 1 \end{cases}$$

By Master Method I obtain case 1:

$$\begin{aligned}\lim_{n\to\infty}\frac{n^2}{n^{\log_{\frac{5}{3}} 3+\varepsilon}} &= \lim_{n\to\infty}\frac{n^2}{n^{\log_{\frac{5}{3}} 3+(3-\log_{\frac{5}{3}} 3)}}\\ &= \lim_{n\to\infty}\frac{n^2}{n^3} \to 0\\ \log_{\frac{5}{3}} 3 &= 2.15066...\\ &\in \Theta(n^{2.15066...})\end{aligned}$$

By Akra-Bazzi I obtain:

$$\begin{aligned}3(\frac{3}{5})^p &= 1\\ p &= \log_{\frac{5}{3}} 3\end{aligned}$$

$$\begin{aligned}T(n) &\in \Theta(n^p(1+\int_1^n \frac{u^2}{u^{p+1}}du))\\ &\in \Theta(7.63746n^{2.15066...} - 6.63746n^2)\\ &\in \Theta(n^{2.15066...})\end{aligned}$$

5.

$$T(n) = \begin{cases} c & \text{if } n \le 1 \\ 3T(\frac{3n}{5}) + n^{2.5} & \text{if } n > 1 \end{cases}$$

By Master Method I obtain case 3:

$$\begin{aligned}\lim_{n\to\infty}\frac{n^{2.5}}{n^{\log_{\frac{5}{3}} 3+\varepsilon}} &= \lim_{n\to\infty}\frac{n^{2.5}}{n^{\log_{\frac{5}{3}} 3+0.1}}\\ &= \lim_{n\to\infty}\frac{n^2.5}{n^{2.25}} \to \infty\\ &= \lim_{n\to\infty} n^{0.25...} \to \infty\\ &\in \Theta(n^{2.5})\end{aligned}$$

2 EXPLORE

For the following problems stated as pseudo-code, let $A[\ell \ldots r]$ denote the sublist of the integer list A from the ℓ-th to the r-th element inclusive, let Cubic($A[1 \ldots n]$) denote an algorithm that runs in time $\Theta(n^3)$, and let Swift($A[1 \ldots n]$) denote an algorithm that runs in time $\Theta(n \log(\log n))$.

```
Three(A[1...n])
      If n ≤ 1 Then Return // nothing to do
      Cubic(A[1...n])
      Three(A[1...⌊n/2⌋])
      Three(A[⌊n/4⌋+1...⌊3n/4⌋])
      Three(A[⌊n/2⌋+1...n])
End Three.
```

(6 points) 1. State a recurrence that gives the complexity $T(n)$ for algorithm Three.

For this problem I decided to analyze the cost of running each line individually.

```
Three(A[1...n]) O(1)
      If n ≤ 1 Then Return // nothing to do O(1)
      Cubic(A[1...n]) O(n^3)
      Three(A[1...⌊n/2⌋]) T(n/2)
      Three(A[⌊n/4⌋+1...⌊3n/4⌋]) T(n/2)
      Three(A[⌊n/2⌋+1...n]) T(n/2)
End Three.  O(1)
```

Which gives me the following recurrece relation:

$$T(n) = \begin{cases} c & \text{if } n \leq 1 \\ 3T(\frac{n}{2}) + n^3 & \text{if } n > 1 \end{cases}$$

(6 points) 2. Find the tight complexity of algorithm Three.

By Master Theorem I obtain case 3:

$$\begin{aligned} \lim_{n\to\infty} \frac{n^3}{n^{\log 3+\varepsilon}} &= \lim_{n\to\infty} \frac{n^3}{n^{\log 3+(2-\log 3)}} \\ &= \lim_{n\to\infty} \frac{n^3}{n^2} \\ &= \lim_{n\to\infty} n \to \infty \\ T(n) &\in \Theta(n^3) \end{aligned}$$

By Akra-Bazzi I obtain:

$$\begin{aligned} 3(\frac{1}{2})^p &= 1 \\ p &= \log_2 3 \end{aligned}$$

$$
\begin{aligned}
T(n) &\in \Theta(n^p(1+\int_1^n \frac{u^3}{u^{p+1}}du)) \\
&\in \Theta(0.293305n^{1.58496} + 0.706695n^3) \\
&\in \Theta(n^3)
\end{aligned}
$$

```
One(A[1...n])
      If n ≤ 1 Then Return // nothing to do
      Swift(A[1...n])
      One(A[1...⌊n/2⌋])
End One.
```

(6 points) 3. State a recurrence that gives the complexity $T(n)$ for algorithm `One`.

```
One(A[1...n])O(1)
      If n ≤ 1 Then Return // nothing to do O(1)
      Swift(A[1...n])O(n log log n)
      One(A[1...⌊n/2⌋]) T(n/2)
End One.
```

Which gives me the following recurrece relation:

$$
T(n) = \begin{cases} c & \text{if } n \leq 1 \\ T(\frac{n}{2}) + n\log\log n & \text{if } n > 1 \end{cases}
$$

(12 points) 4. Find the tight complexity of algorithm `One`.

$$
\begin{aligned}
\lim_{n\to\infty} \frac{n\log\log n}{n^{\log 1+\varepsilon}} &= \lim_{n\to\infty} \frac{n\log\log n}{n^{0+\gamma}} \\
\gamma &= 0.577216 \\
&= \lim_{n\to\infty} n^{0.422784} \log\log n \to \infty \\
T(n) &\in \Theta(n\log\log n)
\end{aligned}
$$

By Akra-Bazzi I obtain:

$$
\begin{aligned}
(\frac{1}{2})^p &= 1 \\
p &= 0
\end{aligned}
$$

As for computing this integral, it is non-elementary. It involes computing the $Li(u)$ which I've looked up and it defintely grows slower than any of the functions we normally use.

It's worth noting that we don't need to work about that, Akra-Bazzi tells us that if $p < k$ then we get case 3 of the master theorem.

$$\begin{aligned}
T(n) &\in \Theta(n^p(1 + \int_1^n \frac{u \log\log u}{u^{p+1}} du)) \\
&\in \Theta(1 + \gamma - i\pi + n \log\log n - Li(n)) \\
&\in \Theta(n \log\log n)
\end{aligned}$$