Jacob Miller - STAT641 - Homework #3

① show $var(\hat{y}) = \sigma^2 H$

$\hat{y} = Hy$

$var(\hat{y}) = var(Hy)$ $\longrightarrow$ $var(cx) = c^2 var(x)$

$var(\hat{y}) = H^2 var(y)$ $\longrightarrow$ H idempotent, $H^2 = H$

$\boxed{var(\hat{y}) = \sigma^2 H}$

② prove $[H]$ & $[I-H]$ are symmetric & idempotent

$(I-H)(I-H) = I^2 - IH - IH + H^2$
$= I - H - H + H$ $\longrightarrow$ assuming $H^2 = H$, which I prove below
$= I - H$ $\therefore$ idempotent

$(I-H)^T = I^T - H^T$ $\longrightarrow$ assuming $H^T = H$, which I prove below
$= I - H$ $\therefore$ symmetric

$H^T = (X(X^TX)^{-1}X^T)^T$
$= X[(X^TX)^{-1}]^T X^T$ $\longrightarrow$ $[(X^TX)^{-1}]^T = (X^TX)^{-1}$
$= X(X^TX)^{-1}X^T$ $\therefore$ symmetric

$HH = (X(X^TX)^{-1}X^T)(X(X^TX)^{-1}X^T)$
$= X(X^TX)^{-1} \cancel{[X^TX][X^TX]^{-1}} X^T$
$= X(X^TX)^{-1}X^T$ $\therefore$ idempotent

③ show residuals can be expressed as $e = (I - H)y$

$$e = y - X\hat{\beta}$$
$$= y - \hat{y}$$
$$= y - Hy$$
$$\underline{e = (I - H)y}$$

derive $E(e)$ & $Var(e)$

$var(e) = \cancel{var(I-y)I}\ var[(I-H)y]$ → $var(c \cdot x) = c^2 \cdot var(x)$

$= (I-H)^2 var(y)$ → $(I-H)$ idempotent, proved earlier

$$\underline{var(e) = (I-H)\sigma^2}$$

$$e = (I-H)y$$
$$= (I-H)(X\beta + \varepsilon)$$
$$= (I-H)X\beta + (I-H)\varepsilon$$
$$= \cancel{IX\beta - HX\beta} + (I-H)\varepsilon$$
$$= (I-H)\varepsilon$$

↓

$E(e|X) = E((I-H)\varepsilon|X)$

$= (I-H)\,E(\varepsilon|X)$ → we know expected value of error = $\emptyset$

$= (I-H) \cdot 0$

$$\underline{E(e|X) = 0}$$