DSP Weekly Summary

I. Week 1

A. Euler's Forward Rectangular Method

Find the difference equation for:

$$D(s) = \frac{U(s)}{E(s)} = K_0 \frac{s+a}{s+b}$$

$$\rightarrow sU(s) + bU(s) = K_0(sE(s) + aE(s))$$

The corresponding differential equation is:

$$\dot{u} + bu[k] = K_0(\dot{e} + ae[k]$$

The forward rectangular method gives:

$$\frac{u[k+1]-u[k]}{T} + bu[k] = K_0(\frac{e[k+1]-e[k]}{T} + ae[k]$$

$$\rightarrow u[k+1] = (1+bT)u[k] + (K_0aT + K_0)e[k] + K_0e[k+1]$$

B. System Stabilisation Refresher

To stabilise a system plant, $G(s)$, with a lead compensator of the form:

$$D(s) = \frac{s+a}{s+b}$$

the closed-loop equation will be:

$$G_{cl}(s) = \frac{D(s)G(s)}{1+D(s)G(s)}$$

So the characteristic equation we need to solve is:

$$1 + D(s)G(s) = 0$$

The zeroes of this formula must all be in the left-half plane to ensure it is stable. Alternatively, there is the PID controller which takes the form:

$$PID(s) = K_p + \frac{K_i}{s} + K_d s$$

II. Week 2

A. Z-Transform

$$\mathcal{Z}\{f[k]\}(z) = \sum_{k=-\infty}^{\infty} f[k]z^{-k}$$

$$\mathcal{Z}\{f[k+n]\} = z^n\mathcal{Z}\{f[k]\} - z^n f[0] - z^{n-1}f[1] - z^{n-2}f[2] ... \{n > 0\}$$

To link between the z and Laplace transforms, we take the Laplace transform of $f_s(t)$:

$$\mathcal{L}\{f_s(t)\}(s) = \sum_{k=0}^{\infty} f[kT](e^{Ts})^{-k}$$

where $z = e^{Ts}$ (note: by this same token, $s = \frac{1}{t}ln(z)$). For the z transform to have a stable result, the poles must be located INSIDE the unit circle:

$$s = a + jb \rightarrow z = e^{sT} = e^a e^{jb} \rightarrow e^a < 1$$

If the system transfer function is stable, we can then use the Tustin transformation to be sure that the z-poles are also stable.

B. Trapezoidal Rule

The trapezoidal rule maps between the s and z domains:

$$s \rightarrow \frac{2}{T}\frac{z-1}{z+1} \quad z \rightarrow \frac{2+sT}{2-sT}$$

C. Tustin (Bilinear) Transformation

This rule maps between the $j\omega$ and z domains:

$$\frac{z-1}{z+1} \rightarrow jtan(\frac{j\omega T}{2})$$

D. Final Value Theorem

If the poles of $(z-1)F(z)$ are inside the unit circle, then:

$$\lim_{k\to\infty} f[k] = \lim_{z\to 1} \frac{z-1}{z}F(z)$$

E. Comb Function

The comb function is used for sampling:

$$comb(t) = \sum_{k=-\infty}^{\infty} e^{j2\pi kt}$$

F. Finite Geometric Series Refresher

$$\sum_{n=0}^{N-1} r^n = \frac{1-r^N}{1-r}$$

Example use with the z transform:

$$|a| < 1 \quad k \in \mathbb{Z}$$

$$f[k] = \begin{cases} a^k, & k \geq 0 \\ 0, & else \end{cases}$$

Using the definition of the z transform, we get:

$$\rightarrow \lim_{K\to\infty} \sum_{k=0}^{K-1} a^k z^{-k} = \lim_{K\to\infty} \sum_{k=0}^{K-1} (\frac{a}{z})^k$$

$$= \lim_{K\to\infty} \frac{1-(\frac{a}{z})^K}{1-(\frac{a}{z})} = \frac{1-0}{1-(\frac{a}{z})}$$

$$= \frac{1}{1-(\frac{a}{z})} = \frac{z}{z-a}$$

This derivation is particularly important to the tutorial questions for this week.

G. Some Common Z-Transforms from the Table

$$\frac{1}{s} \rightarrow \frac{z}{z-1}$$

$$\frac{1}{s^2} \rightarrow T\frac{z}{(z-1)^2}$$

$$\frac{2}{s^3} \rightarrow T^2\frac{z(z+1)}{(z-1)^3}$$

$$\frac{1}{s+a} \rightarrow \frac{z}{z-e^{-aT}}$$

$$\frac{1}{(s+a)^2} \rightarrow T\frac{ze^{-aT}}{(z-e^{-aT})^2}$$

$$\frac{1}{(s+a)(s+b)} \rightarrow \frac{z(e^{-aT}-e^{-bT})}{(b-a)(z-e^{-aT})(z-e^{-bT})}$$

III. Week 3

A. Sampled Continuous Functions

The sampled expression for the continuous function, $r(t)$, is $r_s(t)$. The Laplace transform of this sampled function is given by:

$$R_s(s) = \sum_{k=-\infty}^{\infty} r(kT)e^{-kTs}$$

Using the z-transform definition of $z = e^{Ts}$ (and $s = \frac{1}{t}ln(z)$), we get the z-transform of this function to be:

$$R(z) = \sum_{k=-\infty}^{\infty} r(kT)z^{-k}$$

The key point here is that $R_s(s) \leftrightarrow R(z)$. You get the z-transform from the sampled signal.

B. Zero Order Hold Operation

The zero order hold function is effectively a block in a block diagram that we can use to hold the sampled function at a value for a sampled period T. The transfer function is given by:

$$ZOH(s) = \frac{1-e^{-Ts}}{s}$$

C. Open and Closed Loop Discrete Systems

The transfer function of an open loop discrete system is:

$$G(z) = (1-z^{-1})\mathcal{Z}\{\frac{G(s)}{s}\}$$

Adding a controller and closing the loop represents the system as per its continuous counterpart:

$$G_{cl}(z) = \frac{C(z)G(z)}{1+C(z)G(z)}$$

D. Pitfalls Regarding Sampling of Signals

If the input to a system, $R(s)$, is sampled and then passed through a function block, $G(s)$, then the output is:

$$Y(s) = R_s(s)G(s)$$

and the sampled output is:

$$Y_s(s) = R_s(s)G_s(s)$$

When converted to the z domain, the transform would be represented as:

$$Y(z) = R(z)G(z)$$

However, if the unsampled input to a system, $R(s)$, is passed through a function block, $G(s)$, then the output is:

$$Y(s) = R(s)G(s)$$

and the sampled output is:

$$Y_s(s) = [R(s)G(s)]_s$$

When converted to the z domain, the transform would be represented as:

$$Y(z) = RG(z)$$

E. Loop Gain

The loop gain of a system is defined as any function blocks that are present in the system if the loop is open. A feedback controller ($H(s)$, for example) would not be part of this function. Loop gain is defined as:

$$L(z) = C(z)G(z)$$

More generally, the loop gain is defined as:

$$L(z) = \frac{N(z)}{(z-1)^n D(z)} \quad forn \geq 0$$

where $N(z)$ and $D(z)$ are the numerator and denominator polynomials with no roots at unity. $L(z)$ has n poles at unity. It is therefore called a type n system.

F. Steady State Error

Applying the final value theorem above, the steady state error function of a system is defined as:

$$e(\infty) = \lim_{z\to 1} \frac{(z-1)R(z)}{z(1+L(z))}$$

G. Position, Velocity and Acceleration Error Constants

$$K_p = \lim_{z\to 1} L(z)$$

$$K_v = \lim_{z\to 1} \frac{(z-1)L(z)}{T}$$

$$K_p = \lim_{z\to 1} \frac{(z-1)^2L(z)}{T^2}$$

There is a table in Lecture 3c that shows the steady state error values in response to different inputs, which shows where these constants come in. To remove any steady state error of a system, add poles at $z = 1$ (note: the closed loop system should still remain stable).