Statistics

LLN

$$\lim_{n\to\infty}\left|\frac{1}{n}\sum_{i=1}^{n} x_i - \mathbb{E}\left[x\right]\right| = 0$$

Unbiasedness

$$\text{Bias}_\theta = \mathbb{E}\left[\hat{\theta}\right] - \theta = \mathbb{E}\left[\hat{\theta} - \theta\right] = 0$$

Consistency

$$\lim_{n\to\infty} \mathbf{P}\left(\left|\widehat{X} - \frac{1}{n}\sum_{i=1}^{n} X_i\right| \le \epsilon\right) = 1$$

Properties of Gaussians

$$X \sim \mathcal{N}(\mu,\, \sigma^2) \implies Y = aX + b \sim \mathcal{N}(a\mu + b,\, a^2\sigma^2)$$

$$X \sim \mathcal{N}(\mu_x,\, \sigma_x^2), \quad Y \sim \mathcal{N}(\mu_y,\, \sigma_y^2), \quad \implies \quad Z = X + Y \sim \mathcal{N}(\mu_x + \mu_y,\, \sigma_x^2 + \sigma_y^2)$$

Information Theory

Information

$$\mathbf{I}\left(S\right) = \log_2 \frac{1}{\mathbf{P}\left(S\right)}$$

Information Gain

$$\mathbf{I}\left(Y,\, X_i\right) = \mathbf{H}\left[Y\right] - \mathbf{H}\left[Y \,|\, X_i\right]$$

Entropy

$$\mathbf{H}\left[S\right] = \sum_{s\in\{S\}} \mathbf{P}\left(S{=}s\right)\cdot\mathbf{I}\left(S{=}s\right) = \sum_{s\in\{S\}} \mathbf{P}\left(S{=}s\right)\log_2 \frac{1}{\mathbf{P}\left(S{=}s\right)}$$

$$= \mathbb{E}\left[\log_2 1/\mathbf{P}\left(S{=}s\right)\right] = -\mathbb{E}\left[\log_2 \mathbf{P}\left(S{=}s\right)\right]$$

$$= -\sum_{s\in\{S\}} \mathbf{P}\left(S{=}s\right)\log_2 \mathbf{P}\left(S{=}s\right)$$

Learning Theory

PAC Learning

$$\mathbf{P}\left(\left|\hat{\theta} - \theta^*\right| \le \epsilon\right) \ge 1 - \delta \qquad\qquad \mathbf{P}\left(\left|\hat{\theta} - \theta^*\right| > \epsilon\right) < \delta$$

Decision Theory

Risk

$$\text{Risk}(f) = R(f) := \mathbb{E}_{(X,Y)}\left[\ell(Y,\, f(X))\right]$$

Bayes Risk

$$R(f^*) \le R(f),\ \forall\, f \in \mathcal{F}$$

Bayes Optimal Rule

$$f^*(P) = \arg\min_f \mathbb{E}_{(X,Y)\sim P}\left[\ell(Y,\, f(X))\right]$$

Emperical Risk Minimization

$$\hat{f}_n = \arg\min_f \frac{1}{n}\sum_{i=1}^{n}\left[\ell(Y_i,\, f(x_i))\right] \xrightarrow[n\to\infty]{\text{LNN}} \arg\min_f \mathbb{E}_{(X,\,Y)}\left[\ell(Y,\, f(X))\right]$$

Risk

True Risk

$$R(f) := \mathbb{E}_{(X,Y)}\left[\ell(f(X),\, Y)\right]$$

Emperical Risk

$$\widehat{R}_{\mathcal{D}}(f) := \frac{1}{|\mathcal{D}|}\sum_{i\in\mathcal{D}}\left[\ell(f(X_i),\, Y_i)\right]$$

Excess Risk

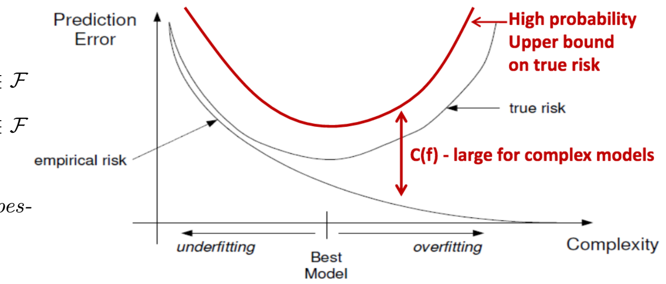
$$\mathbb{E}\left[R(\hat{f}_n)\right] - R(f^*)$$

Structural Risk

$$\left|R(f) - \widehat{R}_n(f)\right| \le C(f) \quad \forall\, f\in\mathcal{F}$$

$$R(f) \le \widehat{R}_n(f) + C(f) \quad \forall\, f\in\mathcal{F}$$

Use $\widehat{R}_n(\hat{f}_n) + C(\hat{f}_n)$ as a *pessimistic* estimate of true risk.



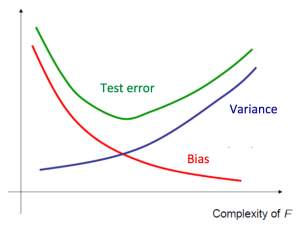
Risk Estimation

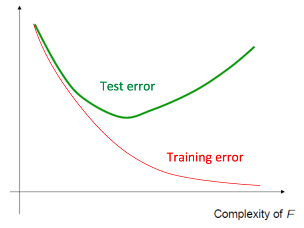
Bias–Variance Tradeoff

$$\text{Bias} = \mathbb{E}\left[f(X)\right] - f^*(X)$$

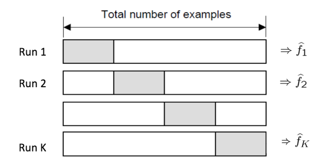
$$\text{Variance} = \mathbb{E}\left[\left(f(X) - \mathbb{E}\left[f(X)\right]\right)^2\right]$$

$$\text{Bias}^2 + \text{Variance} = \mathbb{E}\left[\left(f(X) - f^*(X)\right)^2\right]$$

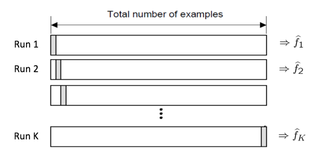




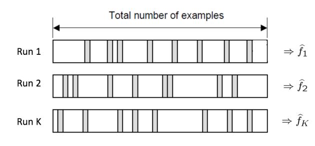
K–Fold CV



LOO CV



Random Subsampling



Hold–out Method

1. split into two sets
2. use $\mathcal{D}_T$ to train a predictor $\hat{f}_{\mathcal{D}_T}$
3. use $\mathcal{D}_V$ to evaluate the predictor, $\widehat{R}_{\mathcal{D}_V}(\hat{f}_{\mathcal{D}_T})$

$$\mathcal{D} = \left\{(X_i,\, Y_i)\right\}_{i=1}^{n}$$

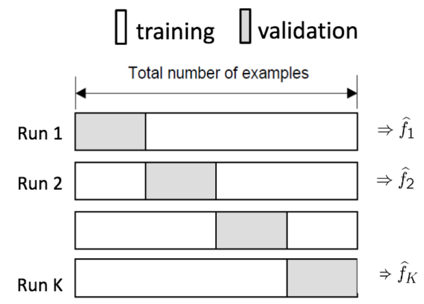
$$\downarrow$$

$$\underbrace{\mathcal{D}_T = \left\{(X_i,\, Y_i)\right\}_{i=1}^{m}}_{\text{training set}} \qquad \underbrace{\mathcal{D}_V = \left\{(X_i,\, Y_i)\right\}_{i=m+1}^{n}}_{\text{holdout set}}$$

Estimating True Risk

Estimate the error of a predictor on n data points.
If K is large (close to n), bias of error estimate is small since each training set has close to n data points.
However, variance of error estiamtes is high since each validation set has fewer data points and $\widehat{R}_{V_k}$ might deviate a lot from the mean.

$$\text{Error estimate} = \frac{1}{K}\sum_{k=1}^{K}\widehat{R}_{V_k}(\hat{f}_{T_k})$$



Risk Minimization

Emperical Risk Minimization

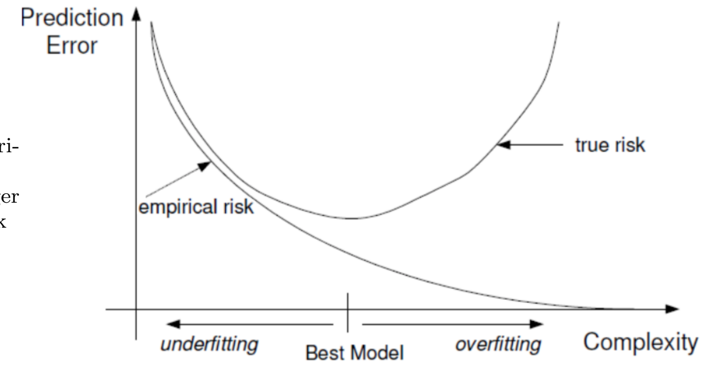
$$\widehat{f}_n = \arg\min_f \frac{1}{n}\sum_{i=1}^{n}\left[\ell\left(Y_i,\, f(x_i)\right)\right]$$

Structural Risk Minimization

$$\hat{f}_n = \arg\min_{f\in\mathcal{F}}\left[\widehat{R}_n(f) + \lambda C(f)\right]$$

Overfitting

discrepency between emperical risk and true risk,
so emperical risk is no longer a good indicator of true risk



Regularization

Complexity Regularization

$$\hat{f}_n = \arg\min_{f\in\mathcal{F}}\left\{\widehat{R}_n(f) + \lambda C(f)\right\}$$

$C(f)$: bound on deviation from true risk
λ : regularization penalty

Information Criteria

AIC (Akiake IC) $\quad C(f) = \#\,parameters$
Allows # parameters to be infinite as # training data n becomes large

BIC (Bayesian IC) $\quad C(f) = \#\,parameters * \log n$
Penalizes complex models more heavily – limits complexity of models as # training data n becomes large

Model Selection

- define a finite set of model classes
- estimate true risk for each model class
- select model class with lowest estimated true risk

Model classes $\{\mathcal{F}_\lambda\}$ of increasing complexity $\mathcal{F}_1 < \mathcal{F}_2 < \ldots$

$$\min_\lambda \min_{f\in\mathcal{F}_\lambda} J(f,\, \lambda)$$

- given λ estimate $\hat{f}_\lambda$ using *emperical* / *structural* / *complexity regularized* risk minimization
- select λ for which $\hat{f}_\lambda$ has minimum true risk estimated using *cross–validation* / *hold–out* / *information criteria*

Generalization Error

Terms

estimated predictor $:= \hat{f}_n$ | optimal predictor $:= f^*$

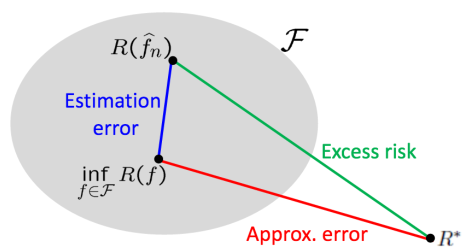
risk of estimated predictor $:= R(\hat{f}_n)$ | risk of optimal predictor $:= R(f^*)$

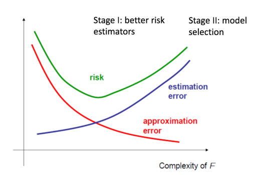
expected risk $:= \mathbb{E}\left[R(\hat{f}_n)\right]$ | excess risk $:= \mathbb{E}\left[R(\hat{f}_n)\right] - R(f^*)$

True Risk Decomposition

$$\mathbb{E}\left[R(\hat{f}_n)\right] - R^* = \underbrace{\left(\mathbb{E}\left[R(\hat{f}_n)\right] - \inf_{f\in\mathcal{F}} R(f)\right)}_{\text{estimation error}} + \underbrace{\left(\inf_{f\in\mathcal{F}} R(f) - R^*\right)}_{\text{approximation error}}$$

Estimation Error : due to randomness of training data
Approximation Error : due to restriction of model class





Regression

Linear Regression

Ridge Regression

$$\hat{\theta}_{\text{MAP}} \operatorname*{arg\,min}_{\theta} \sum_{i=1}^{n} (Y_i - X_i\theta)^2 + \lambda\|\theta\|_2^2$$

Lasso Regression

$$\hat{\theta}_{\text{MAP}} \operatorname*{arg\,min}_{\theta} \sum_{i=1}^{n} (Y_i - X_i\theta)^2 + \lambda\|\theta\|_1$$

Polynomial Regression

Classification

Logistic Regression

Naive Bayes

Boosting

Decision Trees

$$\begin{aligned}
\operatorname*{arg\,max}_{X_i} \left[\mathbf{H}\left[Y\right] - \mathbf{H}\left[Y \,\middle|\, X_i\right] \right] &= \operatorname*{arg\,min}_{X_i} \mathbf{H}\left[Y \,\middle|\, X_i\right] \\
&= \operatorname*{arg\,min}_{X_i} \sum_{x \in \{X_i\}} \left[\mathbf{P}\left(X_i{=}x\right) \mathbf{H}\left[Y \,\middle|\, X_i{=}x\right] \right] \\
&= \operatorname*{arg\,min}_{X_i} - \sum_{x \in \{X_i\}} \left[\mathbf{P}\left(X_i{=}x\right) \sum_{y \in \{Y\}} \left[\mathbf{P}\left(Y{=} \,\middle|\, X_i{=}x\right) \log_2 \mathbf{P}\left(Y{=}y \,\middle|\, X_i{=}x\right) \right] \right]
\end{aligned}$$

Support Vector Machines

Deep Learning

Common Activation Functions

Backpropagation

Gradient Descent

Perceptron

MLP

CNN

RNN

LSTM

Clustering

KNN

Kernel Regression

Kernel Trick