AMATH 583 - HW 1

Jesse Ayers

1.) a.) $\int_0^{\pi/2} \ln \sin x \, dx$

-> let $x = \pi/2 - x$

$I = \int_0^{\pi/2} \ln(\sin(\pi/2 - x))dx = \int \ln \cos x \, dx$

$2I = \int_0^{\pi/2} \ln \sin x + \ln \cos x \, dx = \int_0^{\pi/2} \ln(\sin x \cos x) \, dx$

-> massage to make a trig identity -> $\int_0^{\pi/2} \ln \frac{2 \sin x \cos x}{2} dx$

$2 \sin x \cos x = \sin 2x$ -> $2I = \int_0^{\pi/2} \ln \sin 2x \, dx - \int_0^{\pi/2} \ln 2 \, dx$

so let's u-sub the first part

$u = 2x$
$du = 2\,dx$ -> $\frac{1}{2}\int_0^{\pi} \ln(\sin u) \, du$

This is the original function:

$= \frac{1}{2} \cdot \int_0^{\pi} \ln \sin x \, dx$ -> $\frac{1}{2} \cdot x \cdot 2 \int_0^{\pi/2} \ln(\sin x) dx$

$2I = I - \int_0^{\pi/2} \ln(2) \, dx$ -> $I = -\ln(2) \, [x]_0^{\pi/2}$

$$\boxed{= -\frac{\pi}{2} \ln(2)}$$

1b.) $\int_0^{\pi} \frac{x\sin x}{1+\cos^2(x)}\,dx$

let $x = \pi - x$ $\rightarrow$ $I = \int_0^{\pi} \frac{(\pi-x)\sin(\pi-x)}{1+\cos^2(\pi-x)}\,dx$

$= \int_0^{\pi} \frac{\pi - x\sin x}{1+\cos^2 x}\,dx = \pi\int_0^{\pi} \frac{\sin x}{1+\cos^2 x} - I$

$= \pi/2 \int_0^{\pi} \frac{\sin x}{1+\cos^2 x}$ $\rightarrow$ u sub $\rightarrow$ $u = \cos x$
$du = -\sin x\,dx$

$= -\pi/2 \int_0^{\pi} \frac{u}{1+u^2}\,du$

$= -\pi/2 \cdot -\pi/2 = \boxed{\pi^2/4}$

1.c.) $\int_0^{\pi/2} \frac{1}{1+(\tan(x))^{\sqrt{2}}}\,dx$

like before, let's have $u = \pi/2 - x$ ; sub to use $\int f(x) = \int f(a-x)$

$I = \int_0^{\pi/2} \frac{1}{1+(\tan(\pi/2 - x))^{\sqrt{2}}}\,du$ $\rightarrow$ Identity $\tan(\pi/2 - x) = \cot x$

$I = \int_0^{\pi/2} \frac{1}{1+\cot u^{\sqrt{2}}}\,du \cdot \frac{(\tan u)^{\sqrt{2}}}{(\tan u)^{\sqrt{2}}} = \int_0^{\pi/2} \frac{(\tan u)^{\sqrt{2}}}{(\tan u)^{\sqrt{2}}+1}\,du$

$2I = \int_0^{\pi/2} \frac{1}{1+(\tan(x))^{\sqrt{2}}}\,dx + \int_0^{\pi/2} \frac{(\tan x)^{\sqrt{2}}}{1+(\tan x)^{\sqrt{2}}}\,dx = \int_0^{\pi/2} \frac{1+(\tan x)^{\sqrt{2}}}{1+(\tan x)^{\sqrt{2}}}\,dx$

$2I = [x]_0^{\pi/2} = \pi/2$ $\rightarrow$ $I = \boxed{\pi/4}$

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2.) a.) Find $a, b \in \mathbb{R} \ni (1+i\sqrt{3})^{11} = a+ib$

So, we can use Binomial theorem to expand this:

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k$$

In our case $x=1$, so kind of discard all that. We also know that Pascals triangle gives us the coefficients i.e. the $\binom{n}{k}$ part, so I looked that up to get this expansion:

$$\binom{11}{0} 1^{11} (i\sqrt{3})^0 + \binom{11}{1} 1^{10} (i\sqrt{3}) \ldots$$

$$= 1 + 11i\sqrt{3} + 55(-3) + 165(-3i\sqrt{3}) + 330(9) + 462(9i\sqrt{3})$$

$$+ 462(-27) + 330(-27i\sqrt{3}) + 165(81) + 55(81i\sqrt{3})$$

$$+ 11(-243) + (-243i\sqrt{3})$$

$\rightarrow$ simplify like terms $\rightarrow$ $= 1024 - 1024i\sqrt{3}$

so, $\boxed{a = 1024, \; b = -1024\sqrt{3}}$

b). Find values of $(1+i\sqrt{3})^{1/5}$

First, let's convert the complex # to polar form

$r = \sqrt{1^2 + \sqrt{3}^2} = \sqrt{4} = 2$

$\theta = \tan^{-1}(\sqrt{3}/1) = \pi/3$

complex polar $\rightarrow$ $z = r\cos\theta + ir\sin\theta$ $\rightarrow$ $z = 2(\cos\pi/3 + i\sin\pi/3)$

we can then use demoivre's formula to get the 5 roots.

$z^{1/5} = 2^{1/5}(\cos\pi/3 + i\sin\pi/3)^{1/5} = 2^{1/5}(\cos\pi/15 + i\sin\pi/15)$

The roots are then $\boxed{2^{1/5}\left(\cos\frac{\pi + 6k\pi}{15} + i\sin\frac{\pi + 6k\pi}{15}\right)}$

$\boxed{\text{for } k = 0, 1, 2, 3, 4.}$

Not plugging in since I don't think it's that insightful

c.) Solve $w \in \mathbb{C}$ given $w^{4/3} + 2i = 0$

$w^{4/3} + 2i = 0 \rightarrow (w^{4/3})^{3/4} = (-2i)^{3/4}$

$w = (-2i)^{3/4} \rightarrow (6i)^{1/4} \rightarrow$

$$\boxed{w = \pm(6i)^{1/4}}$$

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3.) write the following as a ratio of integers

a.) $1 + 10^{-2} + 10^{-4} + 10^{-6} + \ldots$

$= 1.\overline{01}$ $\rightarrow$ can rewrite this as a geometric series as $1 + \frac{1}{100} + \frac{1}{10000} \ldots$

The sum of a geomtric series is $\frac{a}{1-r}$

our case $a = 1$, $r = \frac{1}{100}$ ... $\frac{1}{1 - \frac{1}{100}} = \frac{1}{99/100} = \boxed{\frac{100}{99}}$

b.) $376.\overline{376}$

Just like before, $376 + \frac{376}{1000} + \frac{376}{1000000} \ldots$

$= 376 \underbrace{\left[1 + \frac{1}{1000} + \frac{1}{1000000} \ldots\right]}_{\text{Geometric series}}$ $\Rightarrow a = 1$, $r = \frac{1}{1000}$

$376 \cdot \left[\frac{1}{1 - \frac{1}{1000}}\right] = \frac{1000}{999} \cdot 376 = \boxed{\frac{376{,}000}{999}}$

c.) $0.999\overline{9}$

$= 0.999 + 0.000\overline{9} = \frac{999}{1000} + \frac{9}{10{,}000} + \frac{9}{100000} \ldots$

$= \frac{999}{1000} + \left[\frac{9}{10{,}000}\left[1 + \frac{1}{10} + \frac{1}{100} \ldots\right]\right]$

$a = \frac{9}{10{,}000}$ $r = \frac{1}{10}$ $\rightarrow$ $\frac{9/10{,}000}{1 - 1/10}$

$= \frac{9}{10000} \cdot \frac{10}{9} = \frac{90}{90000}$

$\frac{999}{1000} + \frac{90}{90000} = \boxed{1/1 = 1}$

4.) Estimate as ratio of integers using Secant approx.

a) $(1.1)^{1/3}$

rewrite as function $f(X) = X^3 - 1.1$

Initial guesses $X_0 = 1$, $X_1 = 1.1$

Secant method: $X_n = X_{n-1} - f(X_{n-1}) \dfrac{X_{n-1} - X_{n-2}}{f(X_{n-1}) - f(X_{n-2})}$

$X_2 = 1.1 - f(1.1) \dfrac{1.1 - 1}{f(1.1) - f(1)} = 1.1 - 0.231 \left( \dfrac{0.1}{0.231 - 0.1} \right) = \cancel{1.033669}$ 1.030211

$X_3 = 1.0321$ $X_4 = 1.0322$, 1.0322

$$\boxed{= \frac{5161}{5000}}$$

b) $\sqrt{8.5}$ → $f(x) = x^2 - 8.5$

Initial guesses $X_0 = 2.5$, $X_1 = 3$

Secant method: $X_n = X_{n-1} - f(X_{n-1}) \dfrac{(X_{n-1} - X_{n-2})}{f(X_{n-1}) - f(X_{n-2})}$

$X_2 = 3 - f(3) \dfrac{3 - 2.5}{f(3) - f(2.5)} = \cancel{3.01754}$ 2.909

$X_3 = 2.9153$, $X_4 = 2.9154$, $X_5 = 2.9154$

$$\boxed{= \frac{14577}{5000}}$$

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5. given $(x+y+z)^7$, find the coefficients

a.) $x^2 y^2 z^3$

trinomial expansion : $(a+b+c)^n = \sum_{\substack{i,j,k \\ i+j+k=n}}^{n} \binom{n}{i,j,k} a^i b^j c^k$

so $x^2 y^2 z^3$ coefficient $= \binom{7}{2,2,3} x^2 y^2 z^3$

$$\binom{7}{2,2,3} = \frac{7!}{2!\,2!\,3!} = \frac{7\cdot6\cdot5\cdot4\cdot\cancel{3!}}{2\cdot2\cdot\cancel{3!}} = \frac{7\cdot6\cdot5\cdot\cancel{4}}{\cancel{4}} = 7\cdot6\cdot5 = \boxed{210}$$

b.) $x^3 z^4$

Same as above, trinomial expansion.

$$\binom{7}{3,0,4} = \frac{7!}{3!\,0!\,4!} = \frac{7\cdot6\cdot5\cdot\cancel{4!}}{3!\,\cancel{4!}} = \frac{7\cdot\cancel{6}\cdot5}{\cancel{6}} = \boxed{35}$$

6. Given $(x+2y-3z+2w+5)^{16}$

a.) $x^2 y^3 z^2 w^5$

From multinomial theorem $= \binom{n}{k_1, \dots k_m}$

the coefficient is $\underbrace{\frac{16!}{2!\,3!\,2!\,5!\,4!}}_{\text{coefficient}} \cdot \underbrace{2^3 \cdot (-3)^2 \cdot 2^5 \cdot 5^4}_{\text{From the function}}$

$= \boxed{435891456000000}$

7. Two #'s $a, b \in \mathbb{Z}$ are relatively prime when $\gcd(a,b) = 1$

a) for any $n \in \mathbb{Z}^+$ prove $8n+3$ and $5n+2$ are relatively prime.

Using euclid's algorithm

$\gcd(8n+3, 5n+2) \rightarrow 8n+3 = 1(5n+2) + (3n+1)$

$\rightarrow 5n+2 = 1 \cdot (3n+1) + (2n+1) \rightarrow \gcd(3n+1, 2n+1)$

$3n+1 = 1 \cdot (2n+1) + n \rightarrow \gcd(2n+1, n)$

$2n+1 = 2(n) + 1$

~~$n = 1 \cdot 1 + 0$~~

$\gcd(n, 1) = 1$

Thus $8n+3$ & $5n+2$ are relatively prime!

b.) find $\gcd(250, 111)$

$\gcd(250, 111) \rightarrow 250 = 2(111) + 28$

Euclid's Algo

$\gcd(111, 28) \rightarrow 111 = 3(28) + 27$

$\gcd(28, 27) \rightarrow 28 = 1(27) + 1$

$\gcd(27, 1) \rightarrow 27 = 27(1) + 0$

$\boxed{\gcd(250, 111) = 1}$

As a linear combination:

Sum the gcd's backwards: $\boxed{1 = 4(250) - 9(111)}$

AMATH 583 HW 1 Jesse Ames

8. Write the prime factorization of 980220

980220
- 2
- 490110
  - 2
  - 245055
    - 3
    - 81685
      - 5
      - 16337
        - 17
        - 961
          - 31
          - 31

so $\boxed{2^2 \cdot 3 \cdot 5 \cdot 17 \cdot 31^2}$