

CS 5785 – Applied Machine Learning – Lec. 15

Prof. Serge Belongie, Cornell Tech
Scribe: Mor Cohen, Kelly Wang, Vinson Wang

Oct. 26, 2018

1 Separating Hyperplanes - SVM

Support Vector machine, SVMs for short, is one of the most common Machine Learning techniques. Recall that Logistic Regression and LDA are both methods that make some assumptions about the class densities and they just differ in the way the coefficients are estimated. At the time, we briefly discussed an alternative way to look at the problem in terms of separating line/plane/hyperplane, see Figure 1. We also discussed generative vs discriminative methods, see Figure 2.

In some cases where the classes are separable, like in Figure 1, there are infinite number of possible solutions to the problem, i.e., lines that separate between the classes. We are interested in finding the best solution which is the most generalizable solution.

In Figure 1 the orange line represents the least squares fit which misclassifies one of the data points for response $Y \in \{1, -1\}$ on X :

$$x : \hat{\beta}_0 + \hat{\beta}_1 x_1 + \hat{\beta}_2 x_2 = 0$$

Classifiers that compute a linear combination of input features and return the sign are sometimes called *perceptrons*. Frank Rosenblatt, in 1958, was the first to think of the idea of linear combinations of the input to represent the output, see Figure 3.

The Perceptron Learning algorithm tries to find a separating hyperplane by minimizing the distance of misclassified points to the decision boundary. This is a stepping stone to more advanced methods that deal with data that's not linearly separable.

First we need a quick review of some vector algebra. Figure 4 shows an *affine set* L (marked by the green line) defined by the equation: $f(x) = \beta_0 + \beta^\top x = 0$ for $x \in \mathbb{R}^2$. Think of $f(x)$ as a tilted plane intersecting the (x_1, x_2) plane; this intersection is a line: on one side the values of $f(x)$ are positive, on the other negative, this leads to the notion of *signed distance*.

Properties:

1. For any 2 points¹ x_1 and x_2 lying in L , $\beta^\top(x_1 - x_2) = 0$, and hence $\beta^* = \frac{\beta}{\|\beta\|}$ is the unit vector normal to the surface L . Note that L is the

¹Note that HTF switches convention here: x_1 and x_2 referred to individual feature variables in the affine set example above, and now we're back to x_i referring to the i th feature vector.

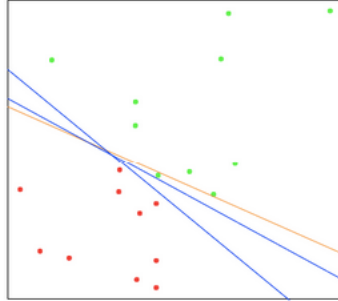



FIGURE 4.14. A toy example with two classes separable by a hyperplane. The orange line is the least squares solution, which misclassifies one of the training points. Also shown are two blue separating hyperplanes found by the perceptron learning algorithm with different random starts.

Figure 1: [HTF]

Model	Classif/reg	Gen/Discr	Param/Non	Section
Discriminant analysis	Classif	Gen	Param	Sec. 4.2.2, 4.2.4
Naive Bayes classifier	Classif	Gen	Param	Sec. 3.5, 3.5.1.2
Tree-augmented Naive Bayes classifier	Classif	Gen	Param	Sec. 10.2.1
Linear regression	Regr	Discrim	Param	Sec. 1.4.5, 7.3, 7.6,
Logistic regression	Classif	Discrim	Param	Sec. 1.4.6, 8.3.4, 8.4.3, 21.8.1.1
Sparse linear/ logistic regression	Both	Discrim	Param	Ch. 13
Mixture of experts	Both	Discrim	Param	Sec. 11.2.4
Multilayer perceptron (MLP)/ Neural network	Both	Discrim	Param	Ch. 16
Conditional random field (CRF)	Classif	Discrim	Param	Sec. 19.6
K nearest neighbor classifier	Classif	Gen	Non	Sec. 1.4.2, 14.7.3
(Infinite) Mixture Discriminant analysis	Classif	Gen	Non	Sec. 14.7.3
Classification and regression trees (CART)	Both	Discrim	Non	Sec. 16.2
Boosted model	Both	Discrim	Non	Sec. 16.4
Sparse kernelized lin/logreg (SKLR)	Both	Discrim	Non	Sec. 14.3.2
Relevance vector machine (RVM)	Both	Discrim	Non	Sec. 14.3.2
Support vector machine (SVM)	Both	Discrim	Non	Sec. 14.5
Gaussian processes (GP)	Both	Discrim	Non	Ch. 15
Smoothing splines	Regr	Discrim	Non	Section 15.4.6

Figure 2: [Murphy 8.1]



Frank Rosenblatt
1928–1969

Rosenblatt's perceptron played an important role in the history of machine learning. Initially, Rosenblatt simulated the perceptron on an IBM 704 computer at Cornell in 1957, but by the early 1960s he had built special-purpose hardware that provided a direct, parallel implementation of perceptron learning. Many of his ideas were encapsulated in "Principles of Neurodynamics: Perceptrons and the Theory of Brain Mechanisms" published in 1962. Rosenblatt's work was criticized by Marvin Minsky, whose objections were published in the book "Perceptrons", co-authored with

Seymour Papert. This book was widely misinterpreted at the time as showing that neural networks were fatally flawed and could only learn solutions for linearly separable problems. In fact, it only proved such limitations in the case of single-layer networks such as the perceptron and merely conjectured (incorrectly) that they applied to more general network models. Unfortunately, however, this book contributed to the substantial decline in research funding for neural computing, a situation that was not reversed until the mid-1980s. Today, there are many hundreds, if not thousands, of applications of neural networks in widespread use, with examples in areas such as handwriting recognition and information retrieval being used routinely by millions of people.

Figure 3: [Bishop]

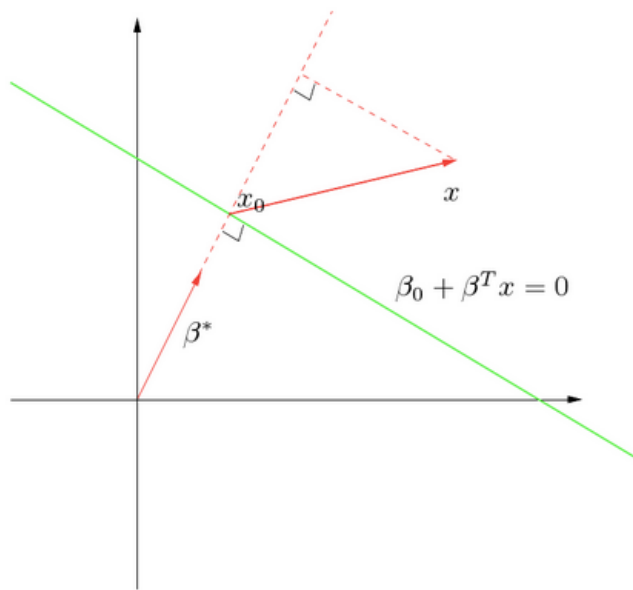


FIGURE 4.15. *The linear algebra of a hyperplane (affine set).*

Figure 4: [HTF]

set of tips of vectors from the origin that have a dot product of $-\beta_0$ with β . Notice that $x_1 - x_2$ is a vector parallel to L and therefore has a dot product of 0 with β .

2. For any point x_0 in L we have that $\beta^\top x_0 = -\beta_0$.
3. The signed distance between any point x to L is given by:

$$\beta^{*\top}(x - x_0) = \frac{1}{\|\beta\|}(\beta^\top x + \beta_0) = \frac{1}{\|f'(x)\|}f(x)$$

Hence $f(x)$ is proportional to the signed distance from x to the affine set L defined by $f(x) = 0$. This is important because our observations, such as data points and training examples, are vectors in this space and we would like to characterize their relationship to L .

1.1 Perceptron Learning

We will focus on Rosenblatt's perceptron learning algorithm, which finds a separating hyperplane by minimizing the distance of misclassified points to the decision boundary.

If a response $y_i = 1$ is misclassified, then $x^\top \beta + \beta_0 < 0$, and the opposite for a misclassified response with $y_i = -1$

In Figure 5 the perceptron algorithm is shown step by step, from the top left to the bottom right. We start with a random guess for β and tweak the guess based on a randomly chosen misclassified point. In each step we add the

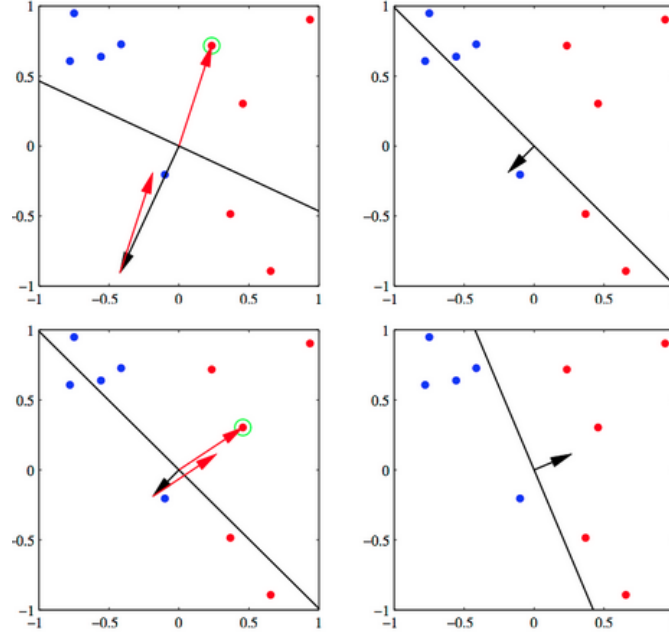


Figure 5: From top left to bottom right. The black line represents the vector defined by β , a randomly generated starting point. The green marked red dot is a chosen misclassified point.[Bishop 4.7]

signed distance between the origin and the red dot circled in green by placing the red vector at the tip of the black vector. We continue until there are no misclassifications. If the two classes are not linearly separable the algorithm would never stop, if the classes are linearly separable then the resulting boundary depends on the random guess we started with.

The goal is to minimize:

$$D(\beta, \beta_0) = - \sum_{i \in \mathcal{M}} y_i (x_i^\top \beta + \beta_0)$$

Where \mathcal{M} is the set of misclassified points. $D(\cdot, \cdot)$ is nonnegative and proportional to the distance of the misclassified points to L (defined by $\beta^\top x + \beta_0$). If we fix \mathcal{M} the gradient is given by:

$$\frac{\partial D(\beta, \beta_0)}{\partial \beta} = - \sum_{i \in \mathcal{M}} y_i x_i$$

$$\frac{\partial D(\beta, \beta_0)}{\partial \beta_0} = - \sum_{i \in \mathcal{M}} y_i$$

Rosenblatt's algorithm uses *Stochastic Gradient Descent* to minimize this piecewise linear criterion.

1.2 Stochastic Gradient Descent (SGD)

With SGD, instead of computing the sum of the gradient contributions of each observation followed by a step in the negative gradient direction, we take a step after each observation is visited. The missclassified observations are visited in some order (for example: random), and we update the parameters as follows:

$$\begin{pmatrix} \beta \\ \beta_0 \end{pmatrix} \leftarrow \begin{pmatrix} \beta \\ \beta_0 \end{pmatrix} + \rho \begin{pmatrix} y_i x_i \\ y_i \end{pmatrix}$$

ρ is called the *learning rate*. Let $\rho = 1$ w.l.o.g. The errors “kick around” the hyperplane until it stabilizes, as seen in Figure 5. If the classes are linearly separable one can show that the algorithm converges to a separating hyperplane in a finite number of steps. There are some problems that arise:

- For separable data, we have many solutions, and the one found depends on our initialization of the algorithm.
- The finite number of steps can be very large. The smaller the gap, the more time to find it.
- For non-separable data, the algorithm won’t converge and cycles – possibly quite long and, as such, hard to detect – can develop.

One way to address the second problem (the number of steps) is to “lift” the data out of the original space via some sort of basis-function transformations, and seek a hyperplane in that space. This may risk overfitting. Let’s consider the XOR problem:

$$\begin{array}{cc} \times & \circ \\ \circ & \times \end{array}$$

This is clearly not linearly separable with any line. So how can we linearly separate these two classes? By moving the problem into a higher dimension, we can magically make it linearly separable. The way to choose the added dimension is by a *kernel*.

So now that we can “force” data to be linearly separable we need a way to find the best separator.

1.3 Optimal Separating Hyperplanes

In 1996, Vapnik said that we should add the constraint of maximizing the distance to the closest point from either class in order to find the optimal hyperplane. By *maximizing the margin* between the two classes on the training data, we get better classification performance on the test data, see Figure 6.

Let’s revisit $D(\beta, \beta_0)$ with this in mind. Consider the optimization criterion:

$$\max_{\beta, \beta_0, \|\beta\|=1} M$$

Subject to

$$y_i(x_i^\top \beta + \beta_0) \geq M$$

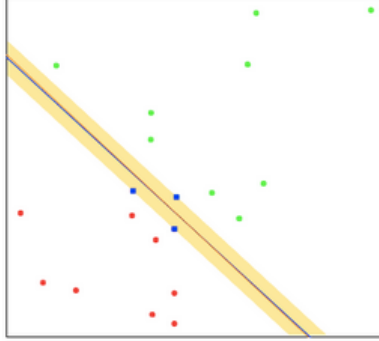


FIGURE 4.16. The same data as in Figure 4.14. The shaded region delineates the maximum margin separating the two classes. There are three support points indicated, which lie on the boundary of the margin, and the optimal separating hyperplane (blue line) bisects the slab. Included in the figure is the boundary found using logistic regression (red line), which is very close to the optimal separating hyperplane (see Section 12.3.3).

Figure 6: The yellow area represents the margin on each side of the boundary. [HTF]

For $i = 1, 2, \dots, N$. This ensures that all points are a signed distance M from the decision boundary defined by (β, β_0) , and we seek the largest such M . We can absorb the $\|\beta\| = 1$ constraint by writing the conditions as:

$$\frac{1}{\|\beta\|} y_i (x_i^\top \beta + \beta_0) \geq M$$

By doing that we essentially redefine β_0 . Equivalently, we could state the constraint as:

$$y_i (x_i^\top \beta + \beta_0) \geq M \|\beta\|$$

Since for any β and β_0 satisfying this inequality, any positively scaled multiple also satisfies it. We can arbitrarily set $\|\beta\| = \frac{1}{M}$ and replace our maximizing problem with the following minimizing problem:

$$\min_{\beta, \beta_0} = \frac{1}{2} \|\beta\|^2$$

Subject to

$$y_i (x_i^\top \beta + \beta_0) \geq 1$$

For $i = 1, 2, \dots, N$.

These constraints define an empty “slab” or margin around the linear decision boundary of thickness $\frac{1}{\|\beta\|}$. This is a convex optimization problem. It is a quadratic criterion with linear inequality constraints. The Lagrange (primal)



Figure 4.8 Illustration of the Mark 1 perceptron hardware. The photograph on the left shows how the inputs were obtained using a simple camera system in which an input scene, in this case a printed character, was illuminated by powerful lights, and an image focussed onto a 20×20 array of cadmium sulphide photocells, giving a primitive 400 pixel image. The perceptron also had a patch board, shown in the middle photograph, which allowed different configurations of input features to be tried. Often these were wired up at random to demonstrate the ability of the perceptron to learn without the need for precise wiring, in contrast to a modern digital computer. The photograph on the right shows one of the racks of adaptive weights. Each weight was implemented using a rotary variable resistor, also called a potentiometer, driven by an electric motor thereby allowing the value of the weight to be adjusted automatically by the learning algorithm.

Figure 7: [Bishop]

function, to be minimized with regard to β and β_0 is:

$$L_p = \frac{1}{2} \|\beta\|^2 - \sum_{i=1}^N \alpha_i [y_i (x_i^\top \beta + \beta_0) - 1]$$

This basically takes the constraint and turns it into weighted penalties.

2 Deriving the SVM

To solve this problem, set the derivative with respect to β_1 , β_0 equal to zero.

$$\beta = \sum_{i=1}^N \alpha_i y_i x_i \quad (1)$$

$$0 = \sum_{i=1}^N \alpha_i y_i \quad (2)$$

Recall that $\|\beta\|^2 = \beta^\top \beta$. So $\frac{1}{2} \frac{d}{d\beta} \|\beta\|^2 = \beta$ thinking about doing a derivative with vectors. This is telling us that the coefficients are a linear combination of all observations weighted by α and the label. If you plug β back into the expression for L_p you get the “Wolfe dual”:

$$L_D = \sum_{i=1}^N \alpha_i - \frac{1}{2} \sum_{i=1}^N \sum_{k=1}^N \alpha_i x_k y_i y_k x_i^\top x_k \quad \text{subject to } \alpha_i \geq 0 \quad (3)$$

The $x_i^\top x_k$ in the above equation is an inner product that can be swapped in for a kernel instead, which is a generalization of a dot product.

We get the solution by maximizing L_D in the *positive orthant*, i.e., $\alpha_i \geq 0$, a simple convex optimization problem for which standard solvers can be used.

The solution must also satisfy the Karush-Kuhn-Tucker (KKT) conditions which consist of Equations (1), (2), (3), and below (4).

$$\alpha_i [y_i (x_i^\top \beta + \beta_0) - 1] = 0 \quad \forall i \quad (4)$$

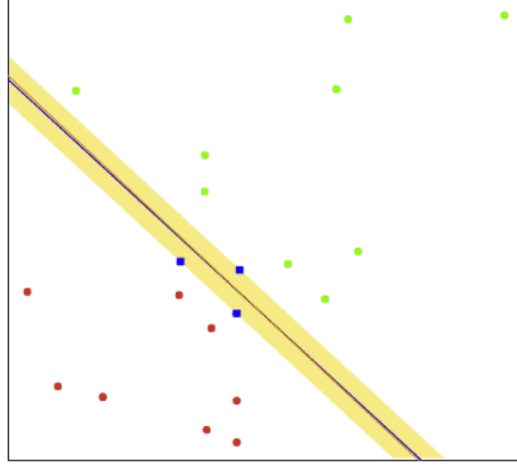


FIGURE 4.16. *The same data as in Figure 4.14. The shaded region delineates the maximum margin separating the two classes. There are three support points indicated, which lie on the boundary of the margin, and the optimal separating hyperplane (blue line) bisects the slab. Included in the figure is the boundary found using logistic regression (red line), which is very close to the optimal separating hyperplane (see Section 12.3.3).*

Figure 8: Figure 4.16

This tells us that:

1. If $\alpha_i > 0$ then $y_i(x_i^\top \beta + \beta_0) = 1$, i.e. X_i is on the boundary of the slab, as seen in Fig. 8.
2. If $y_i(x_i^\top \beta + \beta_0) > 1$, x_i is not on the boundary of the slab, and $\alpha_i = 0$.

From Equation (1) we see that β is defined as a linear combination of the support points α_i .

We get β_0 by solving for Equation (4) for any of the support points. Finally, the optimal separating hyperplane is given by:

$$\begin{aligned}\hat{f}(x) &= x^\top \beta + \beta_0 \\ \hat{G}(x) &= \text{sign } \hat{f}(x)\end{aligned}$$

Every point in the data set has the potential to have a non-zero α but the data points that contribute to the separating hyperplane are the ones on the margin, on the periphery of the slab. This is extremely efficient since most of

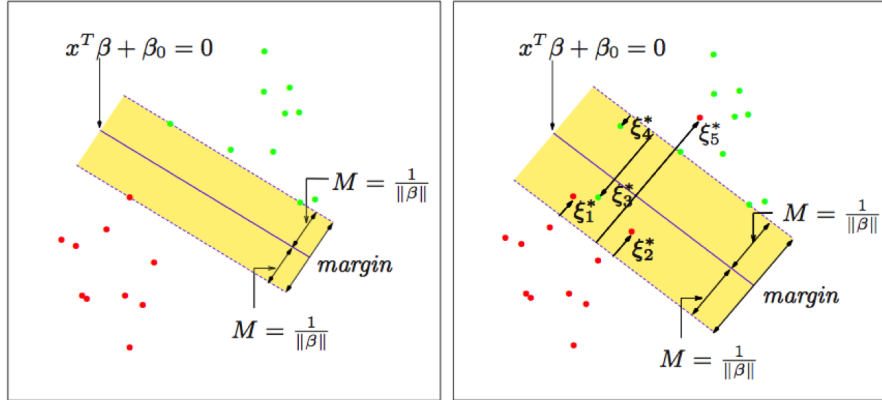


FIGURE 12.1. Support vector classifiers. The left panel shows the separable case. The decision boundary is the solid line, while broken lines bound the shaded maximal margin of width $2M = 2/\|\beta\|$. The right panel shows the nonseparable (overlap) case. The points labeled ξ_j^* are on the wrong side of their margin by an amount $\xi_j^* = M\xi_j$; points on the correct side have $\xi_j^* = 0$. The margin is maximized subject to a total budget $\sum \xi_i \leq \text{constant}$. Hence $\sum \xi_j^*$ is the total distance of points on the wrong side of their margin.

Figure 9: Figure 12.1

the points are zero and don't contribute to the decision. You get β and the offset β_0 and then threshold it to get the final decision.

Let's compare this with LDA and logistic regression, assuming a 2-class problem that's separable. For LDA, we assume the classes are distributed as bivariate Gaussians that have the same means but different covariances. But it's probably not the case that your data is like this. For logistic, you also get a decision line or hyperplane using not the points themselves but with log-odds as a linear combination. Logistic is pretty similar qualitatively to SVM because we get driven to two extreme values. In Newton-Raphson, the coefficients are most important close to the boundary. All points become support vectors or not, so $\alpha = 0$ or $\alpha \neq 0$.

3 Non-separable Hyperplanes

What do you do when the data is not separable (not some slab away from a hyperplane)?

We won't do derivations for this non-separable case but we will talk about this, as seen in 9. When you look at the non-separable case, you think that have a margin that is a perfectly separating margin and have some points on the wrong side of the boundary. You attach to every single data point a *slack variable*, ξ_i . The slack variable has the ability to move into the correct side like a rubber band, if you're willing to pay the price.

However, we may overfit if the slack variables are left unconstrained. The amount of stretch is specified by the variable c , a regularization constant or

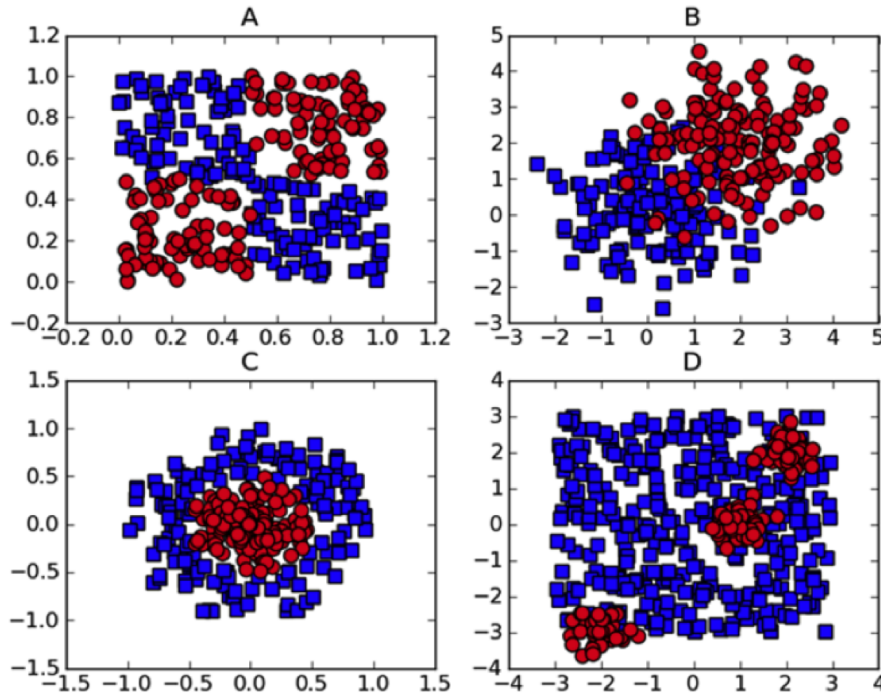


Figure 6.1 Four examples of datasets that aren't linearly separable

Figure 10: Figure 6.1

cost, which we last saw with decision trees. We want a compromise allowing some data points to have slack. The regularization is like having a budget for the slack variables. If $c = 0$ then you're not constraining anything. If c is really big, you're being very inflexible assuming that nothing is allowed to violate the constraint, on the left, which is very inflexible. You can use cross-validation to find c . Fig. 10 shows an example.

Support-vector machines let you visualize the prototypes that sit $\pm\epsilon$ around the boundary. For images, When a new face comes in, it's quite efficient with SVM. For kNN you would have to compute the distances to every single face. In SVM, you bring in the new face and do an inner product with the support vectors and if you look at sex classifications for faces in Fig. 11 you see that these two faces are just ϵ more or less feminine or masculine. You can see these exemplars.

A linear SVM probably isn't powerful enough to calculate this. We need a mechanism to bring in more powerful notions of similarity between patterns.

4 Kernels

What is a kernel? A kernel is a function that satisfies the following conditions:

$$\sum_{i,j} k(x_i, x_j) c_i c_j \geq 0$$

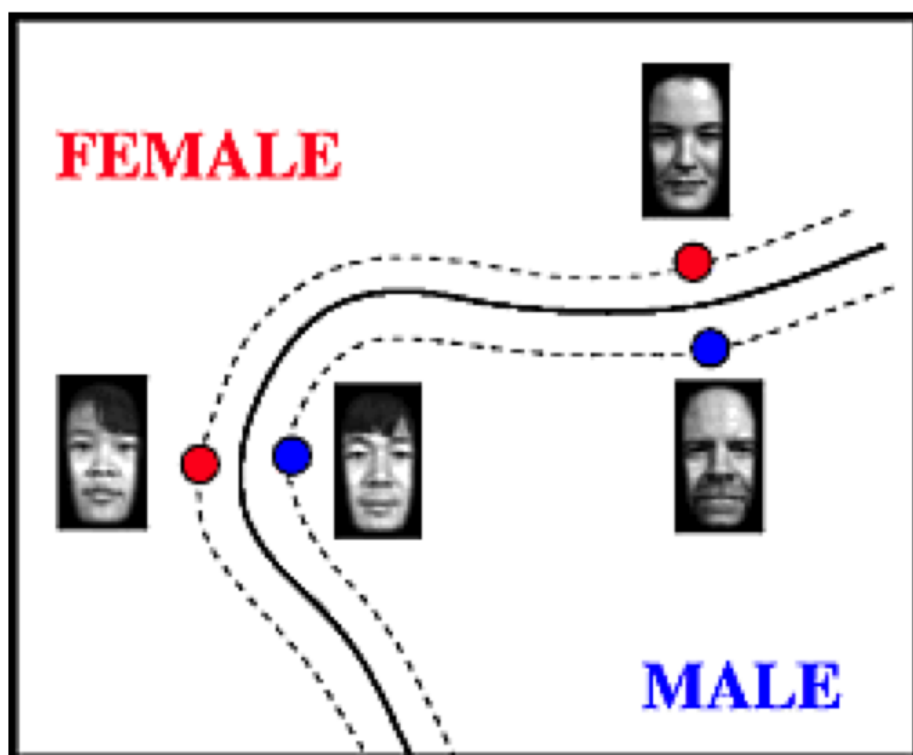


Figure 11: Male Female

for some set of feature vectors $\{x_i\}$ and any vector c .

We say the kernel is “Mercer” or that it is positive semidefinite. If this were just:

$$\sum_{i,j} A_{ij} c_i c_j \geq 0$$

or equivalently

$$c^\top A c \geq 0 \text{ because positive semidefinite } A = Q Q^\top$$

If a matrix is positive semidefinite, we can write it as a product of a matrix with its transpose. Also each entry in a positive semidefinite matrix can be expressed by the pairwise inner products of a set of vectors.

$$k(x, y) = \phi(x) \cdot \phi(y)$$

The kernel function, $k(\cdot, \cdot)$, is operating on all data points and pops out numbers for all possible pairs. If the resulting matrix satisfies the Mercer condition, that means the function k represents an inner product between two vectors. We’re generalizing the dot product so if it satisfies this condition, there exists some features vectors that give you the number by computing k .

This is called the “kernel trick”. So if you learn a method built around inner products, you look at the algorithm and everywhere it has a dot product, you pull in a kernel instead. That means you do the algorithm on a higher-dimensional version of the data instead of the data itself.

Whatever $k(x, y)$ is doing, you’re lifting the vector into a higher dimensional feature space by using $\phi(\cdot)$, and computing the inner product between $\phi(x)$ and $\phi(y)$.

Let’s say $\phi(x)$ represents an empirical feature mapping:

$$\begin{aligned} \phi : (x_1, x_2)^\top &\rightarrow (x_1, x_2, x_1^2 - x_2^2)^\top \\ x \in \mathbb{R}^2, \phi(x) &\in \mathbb{R}^3 \end{aligned}$$

Here we went from the native data that is 2-dimensional to mapped 3-dimensional data, supplemented by the sum of squares. ϕ is implicit. You just choose the kernel.

If you imagine a clump with an annulus, the data may not be perfectly linearly separable but by adding a dimension becomes more separable, perhaps with a plane.

4.1 Examples of Kernels

1. RBF (radial basis function):

$$k(x, y) = e^{-\|x-y\|^2/\alpha}$$

2. For histogram data h_i, g_i , use χ^2 kernel:

$$\chi^2 = \frac{1}{2} \sum_{i=1}^K \frac{(h_i - g_i)^2}{h_i + g_i}$$

$$k(h, g) = e^{-\chi^2(h, g)/\alpha}$$

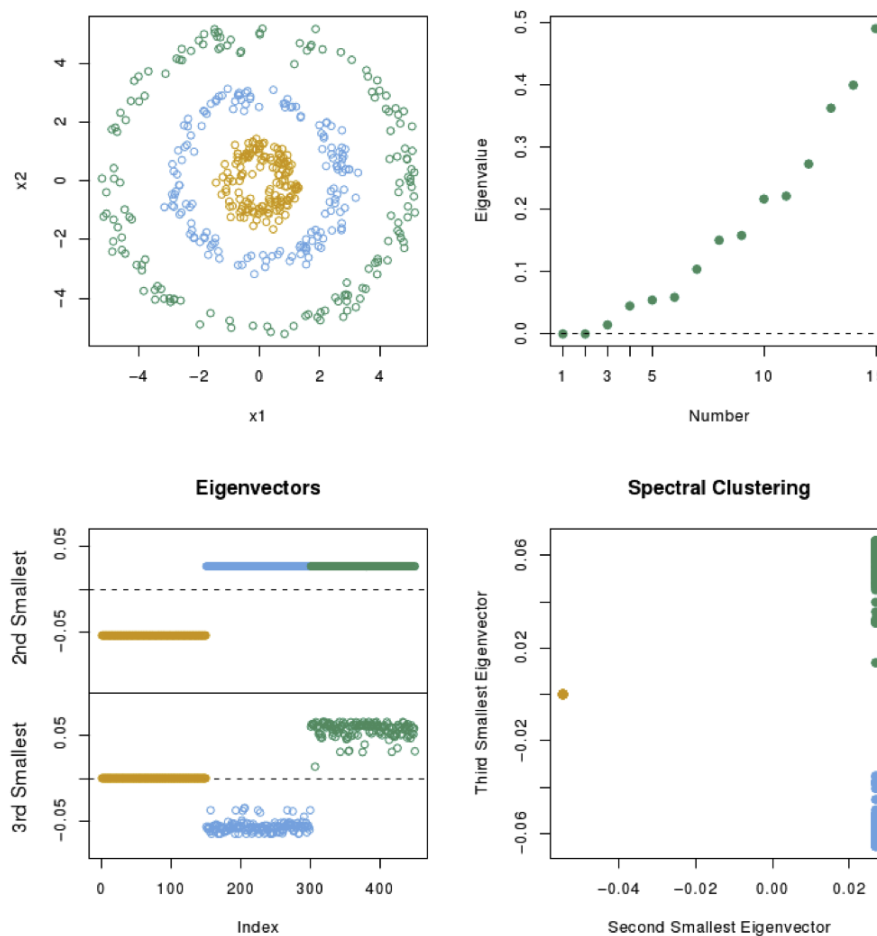


Figure 12: Fig. 14.29

3. Trivial Kernel (dot product)

$$k(x, y) = x^T y$$

There is some $\phi(x)$ that would give you a new vector that if you do the inner product on would give you the same as the kernel. However, some of these feature mappings would be infinitely long.

The main place you see these things is with SVMs with kernels, kPCA, and spectral clustering. We can see some of this in Fig. 12.

If you did K-Means with the top left you probably wouldn't get anything useful. With spectral clustering, you still use K-Means but first evaluate a Gaussian between all pairs of points, creating a kernel matrix using one of the example kernels above. You get a Gaussian-weighted distance matrix and find its leading eigenvectors and keep the first k. If you plot the eigenvectors and keep the first k, you do K-Means and then get this nice separation between the clusters. The first eigenvector separates from the second and the second separates the third. You still use k-means after producing the kernel matrix

and its eigenvectors. You do clustering on the leading eigenvectors.