MA-240-A Midterm Exam Corrections

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Contents

Question 1

1. **False** - Use the Wronskian to determine if a S is linearly independent. If the Wronskian is 0, then S is linearly dependent. Since $f = 0 \in S$, this makes the Wronskian 0, since the whole column of the Wronskian is 0, therefore S is linearly dependent.
2. **False** - The leading term for a critically damped oscillator is te^{-t}, but for an overdamped oscillator the leading term is e^{-t+b} which will cause the oscillator to move in the other direction, making it's approach to 0 slower.
3. **False** - Solutions are not defined as points, but as curves.
4. **False**
5. **True**
6. **True** - Since the derivative of the function is 0 at that point, while it may touch the line, it'll never cross the line.
7. **False** - Some ODEs have solutions that can only be represented visually, so not every ODE would possess an exact solution.
8. **True** - A linear homogeneous ODE possesses at least one solution, that solution being 0 (the trivial solution).

Question 2

Define a bounded region in the xy-plane containing the point (x_0, y_0). If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous in the region, then there exists an interval I and a unique function $y(x) \in I$ that is a solution for this problem.

Question 3

Let L be the differential operator defined as

$$L = \sum_{n=0}^{k} a_n(x) D^{(n)}$$

This means that:

$$L(y) = g_1(x),\, g_2(x),\, \cdots,\, g_n(x)$$

if and only if the following is defined as the particular solution:

$$y_p = \sum_{j=1}^{k} y_{p_j}(x)$$

By the superposition principle, the result is:

$$L(y_p) = L\left(\sum_{j=1}^{k} y_{p_j}(x)\right) = \sum_{j=1}^{k} L\left(y_{p_j}(x)\right) = \sum_{j=1}^{k} g_j(x)$$

Question 4

Part 1

$$3(1+t^2)y' = 2ty(y^3-1)$$

$$3(1+t^2)\frac{dy}{dt} = 2t(y^4-y)$$

$$\frac{3}{y^4-y}dy = \frac{2t}{1+t^2}dt$$

Perform partial fraction decomposition on $\frac{3}{y^4-y}$.

$$\frac{3}{y^4-y} = \frac{3}{y(y^3-1)} = \frac{3}{y(y-1)(y^2+y+1)} = -\frac{3}{y} + \frac{1}{y-1} + \frac{Ay+B}{y^2+y+1}$$

$$-\frac{3}{y} + \frac{1}{y-1} + \frac{Ay+B}{y^2+y+1} = 3$$

$$-3(y-1)(y^2+y+1) + y(y^2+y+1) + (Ay+B)(y)(y-1) = 3$$

$$-3y^3 + 3 + y^3 + y^2 + y + Ay^3 - Ay^2 + By^2 - By = 3$$

$$Ay^3 - 3y^3 + y^3 = 0 \qquad A = 2$$

$$y - By = 0 \qquad B = 1$$

$$\frac{3}{y^4-y} = -\frac{3}{y} + \frac{1}{y-1} + \frac{2y+1}{y^2+y+1}$$

Substitute partial fraction decomposition into seperable equations.

$$\int \left[-\frac{3}{y} + \frac{1}{y-1} + \frac{2y+1}{y^2+y+1}\right] dy = \int \frac{2t}{1+t^2}dt$$

$$\int \left[-\frac{3}{y} + \frac{1}{y-1} + \frac{2y+1}{y^2+y+1}\right] dy = -3\ln|y| + \ln|y-1| + \ln|y^2+y+1|$$

$$\int \frac{2t}{1+t^2}dt = \ln|1+t^2| + c$$

$$-3\ln|y| + \ln|y-1| + \ln|y^2+y+1| = \ln|1+t^2| + c$$

$$\boxed{y = \frac{1}{\sqrt[3]{-e^c t^2 - e^c + 1}} \quad t \in \mathbb{R}}$$

Part 2

$$y^{(5)} - 9y^{(4)} - y^{(3)} - 81y'' - 90y' = -\sin(3x)$$

$$\text{Auxillary Equation: } m^5 - 9m^4 - m^3 - 81m^2 - 90m = 0$$

$$m(m-1)(m+10)(m-3i)(m+3i) = 0 \qquad m = 0,\, 1,\, -10,\, \pm\sqrt{3}i$$

$$y_c = c_1 + c_2e^x + c_3e^{-10x} + c_4\sin(3x) + c_5\cos(3x)$$

$$\text{Guess: } y_p = Ax\sin(3x) + Bx\cos(3x)$$

$$y_p' = -3Bx\sin(3x) + A\sin(3x) + 3Ax\cos(3x) + B\cos(3x) = (A - 3Bx)\sin(3x) + (3Ax + B)\cos(3x)$$

$$y_p'' = -3(3Ax+B)\sin(3x) - 3B\sin(3x) + 3(A-3Bx)\cos(3x) + 3A\cos(3x) = (-9Ax-6B)\sin(3x) + (6A-9Bx)\cos(3x)$$

$$y_p^{(3)} = -3(6A - 9Bx)\sin(3x) - 9A\sin(3x) + 3(-9Ax - 6B)\cos(3x) - 9B\cos(3x)$$

$$y_p^{(3)} = (-27A + 27Bx)\sin(3x) + (-27Ax - 27B)\cos(3x)$$

$$y_p^{(4)} = (81Ax + 81B)\sin(3x) + 27B\sin(3x) + (81Bx - 81A)\cos(3x) - 21A\cos(3x)$$

$$y_p^{(4)} = (81Ax + 108B)\sin(3x) + (81Bx - 108A)\cos(3x)$$

$$y_p^{(5)} = -3(81Bx - 108A)\sin(3x) + 81A\sin(3x) + 3(81Ax + 108B)\cos(3x) + 81B\cos(3x)$$

$$y_p^{(5)} = (405A - 243Bx)\sin(3x) + (243Ax + 405B)\cos(3x)$$

$$405A + 27A - 90A + 486B - 972B = -1 \qquad 342A - 486B = -1$$

$$405B + 27B - 90B - 486A + 972A = 0 \qquad 342B + 486A = 0$$

$$A = -\frac{342}{486}B = -\frac{19}{27}B \qquad B = \frac{3}{2180}$$

$$A = -\frac{342}{486}B \qquad A = -\frac{57}{58860}$$

$$\boxed{y = c_1 + c_2e^x + c_3e^{-10x} + c_4\sin(3x) + c_5\cos(3x) - \frac{57x}{58860}\sin(3x) + \frac{3x}{2180}\cos(3x) \quad x \in \mathbb{R}}$$

Part 3

$$y' = \frac{y}{e^{-y}\sin(2y) - (1+y)x}$$

$$\frac{dy}{dx} = \frac{y}{e^{-y}\sin(2y) - (1+y)x}$$

$$\frac{dx}{dy} = \frac{e^{-y}\sin(2y) - (1+y)x}{y} = \frac{-y-1}{y}x + \frac{e^{-y}}{y}\sin(2y)$$

$$\frac{dx}{dy} + \frac{y+1}{y}x = \frac{e^{-y}}{y}\sin(2y)$$

$$\mu = e^{\int \frac{y+1}{y}dy} = ye^y$$

$$\frac{d}{dy}\left[xye^y\right] = \frac{e^{-y}}{y}\sin(2y)\,ye^y = \sin(2y)$$

$$xye^y = -\frac{1}{2}\cos(2y) + c$$

$$\boxed{x = -\frac{1}{2ye^y}\cos(2y) + \frac{c}{ye^y} \quad y \in (0, \infty)}$$

Part 4

$$y' = \frac{1 + \ln(x) + \frac{y}{x}}{1 - \ln(x)}$$

$$\frac{dy}{dx}(1 - \ln(x)) = 1 + \ln(x) + \frac{y}{x}$$

$$(1 - \ln(x))dy = \left(1 + \ln(x) + \frac{y}{x}\right) dx$$

$$\left(1 + \ln(x) + \frac{y}{x}\right) dx - (1 - \ln(x))dy = 0$$

$$\text{Let } \mathbf{M} = 1 + \ln(x) + \frac{y}{x} \text{ and } \mathbf{N} = 1 - \ln(x)$$

$$\mathbf{M}_y = \mathbf{N}_x = -\frac{1}{x}$$

$$\int \mathbf{N} dy = y - y \ln|x| \qquad \int \mathbf{M} dx = -x - x\ln|x| + x - y\ln|x| = -x\ln|x| - y\ln|x|$$

$$y - y\ln|x| - x\ln|x| = c$$

$$y(1 - \ln|x|) - x\ln|x| = c$$

$$\boxed{y = \frac{c + x\ln|x|}{1 - \ln|x|} \quad x \in (e, \infty)}$$

Part 5

$$y' = 5 - 3y + \frac{1}{2}e^{-3x}$$

$$y' + 3y = 5 + \frac{1}{2}e^{-3x}$$

$$\mu = e^{3x} \qquad \frac{d}{dx}\left[e^{3x}y\right] = 5e^{3x} + \frac{1}{2}$$

$$e^{3x}y = \frac{5}{3}e^{3x} + \frac{x}{2} + c$$

$$\boxed{y = \frac{5}{3} + \frac{x}{2}e^{-3x} + ce^{-3x} \quad x \in \mathbb{R}}$$

Part 6

$$x^3 y^{(3)} - 6y = 0$$

$$\text{Auxillary Equation: } 1(m-2)(m-1)m - 6 = 0$$

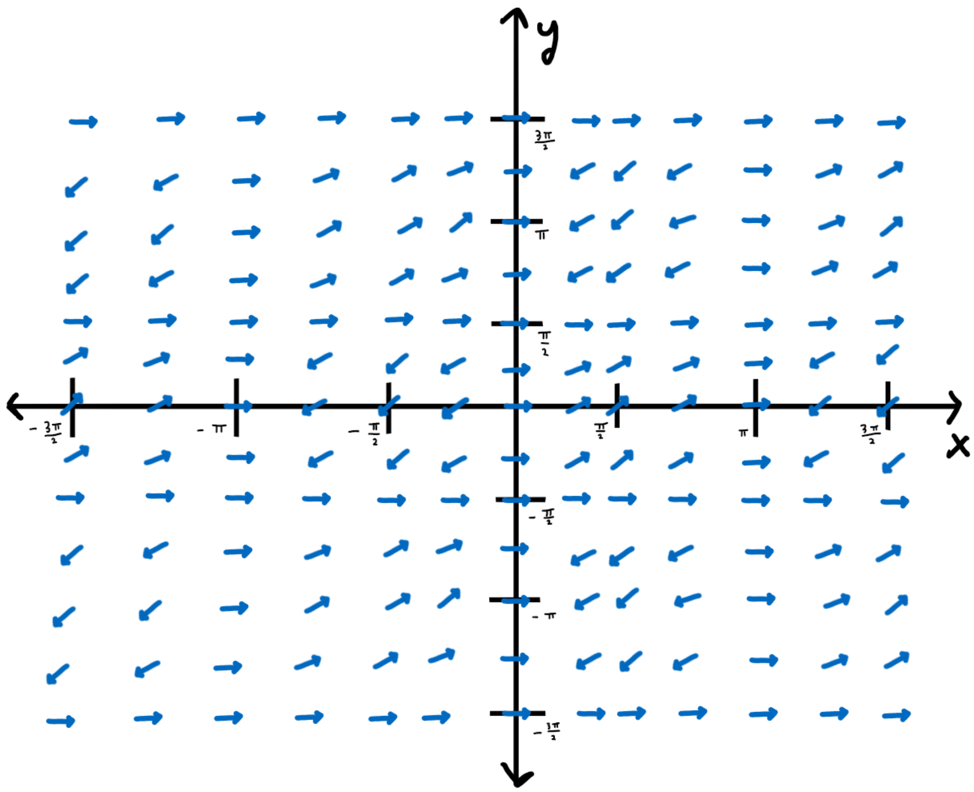
$$m^3 - 3m^2 + 2m - 6 = 0$$

$$(m-3)(m+2i)(m-2i) = 0 \qquad m = 3, \pm\sqrt{2}i$$

$$\boxed{y_c = c_1x^3 + c_2\cos\left(\sqrt{2}\ln|x|\right) + c_3\sin\left(\sqrt{2}\ln|x|\right) \quad x \in (0, \infty)}$$

Question 5

$$y' = \sin(x)\cos(y)$$



Question 6

$$m = 1 \text{ kg} \qquad F = kx \qquad k = 10$$

$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 10x = F(t)$$

$$\text{Auxillary Equation: } m^2 + 8m + 10 = 0 \qquad m = \frac{-8 \pm \sqrt{64 - 40}}{2} = -4 \pm \sqrt{6}$$

$$x_c = c_1 e^{\left(-4+\sqrt{6}\right)t} + c_2 e^{\left(-4-\sqrt{6}\right)t}$$

All terms of the complementary solution are transient.

Part (a)

$$\text{Guess: } x_p = A\sin(4t) + B\cos(4t)$$

$$x_p' = 4A\cos(4t) - 4B\sin(4t)$$

$$x_p'' = -16A\sin(4t) - 16B\cos(4t)$$

$$-16A - 32B + 10A = F_0$$

$$-16B + 32A + 10B = 0$$

$$A = \frac{6}{32}B \qquad -16\frac{6}{32}B - 32B - \frac{6}{32}B * 10 = F_0 \qquad B = -\frac{8F_0}{265} \qquad A = -\frac{3F_0}{530}$$

$$\boxed{x = c_1 e^{\left(-4+\sqrt{6}\right)t} + c_2 e^{\left(-4-\sqrt{6}\right)t} - \frac{3F_0}{530}\sin(4t) - \frac{8F_0}{265}\cos(4t) \quad t \in \mathbb{R}}$$

Only the complementary solution is transient.

Part (b)

$$\text{Guess: } x_p = Ae^{-4t}\sin(4t) + Be^{-4t}\cos(4t)$$

$$x_p' = 4Ae^{-4t}\cos(4t) - 4Ae^{-4t}\sin(4t) - 4Be^{-4t}\sin(4t) - 4Be^{-4t}\cos(4t)$$

$$x_p'' = \frac{d}{dt}\left[4Ae^{-4t}\cos(4t) - 4Ae^{-4t}\sin(4t) - 4Be^{-4t}\sin(4t) - 4Be^{-4t}\cos(4t)\right]$$

$$x_p'' = \frac{d}{dt}\left[-4e^{-4t}((B+A)\sin(4t) + (B-A)\cos(4t))\right]$$

$$x_p'' = 16e^{-4t}((B+A)\sin(4t) + (B-A)\cos(4t)) - 4e^{-4t}(4(B+A)\cos(4t) - 4(B-A)\sin(4t))$$

$$x_p'' = 32Be^{-4t}\sin(4t) - 32Ae^{-4t}\cos(4t)$$

$$32B - 32A - 32B + 10A = F_0$$

$$-32A + 32A - 32B + 10B = 0$$

$$A = -\frac{F_0}{22} \qquad B = 0$$

$$\boxed{x = c_1 e^{\left(-4+\sqrt{6}\right)t} + c_2 e^{\left(-4-\sqrt{6}\right)t} - \frac{F_0}{22}e^{-4t}\sin(4t) \quad t \in \mathbb{R}}$$

Both the complementary and particular solutions are transient.

Part (c)

$$\text{Guess: } x_p = Ae^{-4t}\sin\left(\sqrt{10}t\right) + Be^{-4t}\cos\left(\sqrt{10}t\right)$$

$$x_p' = \sqrt{10}Ae^{-4t}\cos\left(\sqrt{10}t\right) - 4Ae^{-4t}\sin\left(\sqrt{10}t\right) - \sqrt{10}Be^{-4t}\sin\left(\sqrt{10}t\right) - 4Be^{-4t}\cos\left(\sqrt{10}t\right)$$

$$x_p'' = \frac{d}{dt}\left[\sqrt{10}Ae^{-4t}\cos\left(\sqrt{10}t\right) - 4Ae^{-4t}\sin\left(\sqrt{10}t\right) - \sqrt{10}Be^{-4t}\sin\left(\sqrt{10}t\right) - 4Be^{-4t}\cos\left(\sqrt{10}t\right)\right]$$

$$x_p'' = \frac{d}{dt}\left[-e^{-4t}\left(\left(\sqrt{10}B + 4A\right)\sin\left(\sqrt{10}t\right) + \left(4B - \sqrt{10}A\right)\cos\left(\sqrt{10}t\right)\right)\right]$$

$$\begin{aligned} x_p'' = 4e^{-4t}&\left(\left(\sqrt{10}B + 4A\right)\sin\left(\sqrt{10}t\right) + \left(4B - \sqrt{10}A\right)\cos\left(\sqrt{10}t\right)\right) \\ &- e^{-4t}\left(\sqrt{10}\left(\sqrt{10}B + 4A\right)\cos\left(\sqrt{10}t\right) - \sqrt{10}\left(4B - \sqrt{10}A\right)\sin\left(\sqrt{10}t\right)\right) \end{aligned}$$

$$x_p'' = \left(8\sqrt{10}B + 6A\right)e^{-4t}\sin\left(\sqrt{10}t\right) + \left(6B - 8\sqrt{10}A\right)e^{-4t}\cos\left(\sqrt{10}t\right)$$

$$8\sqrt{10}B + 6A - 32A - 8\sqrt{10}B + 10A = F_0$$

$$6B - 8\sqrt{10}A + 8\sqrt{10}A - 32B + 10B = 0$$

$$A = -\frac{F_0}{16} \qquad B = 0$$

$$\boxed{x = c_1 e^{\left(-4+\sqrt{6}\right)t} + c_2 e^{\left(-4-\sqrt{6}\right)t} - \frac{F_0}{16}e^{-4t}\sin\left(\sqrt{10}t\right) \quad t \in \mathbb{R}}$$

Both the complementary and particular solutions are transient.

Question 7

Part (a)

$$y' = \frac{y}{x}$$
$$\frac{1}{y}dy = \frac{1}{x}dx$$
$$\int \frac{1}{y}dy = \int \frac{1}{x}dx$$
$$\ln|y| = \ln|x| + c$$
$$\boxed{y = c\,x \quad x \in \mathbb{R}}$$

Part (b)

$$-ydx + xdy = 0$$
$$-\frac{y}{x^2}dx + \frac{1}{x}dy = 0$$
$$\text{Let } \mathbf{M} = -\frac{y}{x^2} \text{ and } \mathbf{N} = \frac{1}{x}$$
$$\mathbf{M}_y = \mathbf{N}_x = -\frac{1}{x^2}$$
$$\int \mathbf{N}dy = \frac{y}{x} \qquad \int \mathbf{M}dx = \frac{y}{x}$$
$$\frac{y}{x} = c$$
$$\boxed{y = c\,x \quad x \in (0, \infty)}$$

Part (c)

$$-ydx + xdy = 0$$
$$-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = 0$$
$$\text{Let } \mathbf{M} = -\frac{y}{x^2+y^2} \text{ and } \mathbf{N} = \frac{x}{x^2+y^2}$$
$$\mathbf{M}_y = \mathbf{N}_x = -\frac{x^2-y^2}{(x^2+y^2)^2}$$
$$\int \mathbf{N}dy = \arctan\left(\frac{y}{x}\right) + g(y)$$
$$\frac{\partial}{\partial x}\left[\arctan\left(\frac{y}{x}\right) + g(y)\right] = -\frac{y}{x^2+y^2} + g'(y)$$
$$g'(y) = 0 \qquad g(y) = c$$
$$\arctan\left(\frac{y}{x}\right) = c$$

An approximation for arctan can be used as follows:

$$\frac{y}{x} = c \qquad c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
$$\boxed{y = c\,x \quad x \in \mathbb{R}}$$

Part (d)

The answers to (a), (b), and (c) are the same. Since it's the same differential equation, different methods of solving will lead to the same general solution. Initial/Boundary conditions would lead to different answers. Despite them being the same, there are some domain restrictions with each solution since the methods used involve approximations and may not be the best way of solving a differential equation. However, they give the same solution nonetheless, despite the domain restrictions on the three solutions, even though they were not solved using the same method.

Question 8

Part (a)

Let some function f_2 be a constant multiple k of some function f_1. The Wronskian of the two functions is as follows:

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} f_1 & kf_1 \\ f_1' & kf_1' \end{vmatrix} = k\, f_1 f_1' - k f_1 f_1' = 0$$

Since the Wronskian of the two functions is 0, the functions are linearly dependent.

Part (b)

$$\frac{d}{dx}\left(|x|^3\right) = \frac{d}{dx}\left(|x|\, x^2\right) = 2x\,|x| + \frac{x^3}{|x|} = \frac{3x^3}{|x|} = 3x\,|x|$$

$$W\left(x^3, |x|^3\right) = \begin{vmatrix} x^3 & |x|^3 \\ 3x^2 & 3x\,|x| \end{vmatrix} = 3x^2|x|^3 - 3x^4|x| = \boxed{0}$$

Part (c)

$$\sum_{i=0}^{n} a_i(x) y^{(i)}(x) = g(x) \text{ such that } y(x_0) = y_0, y'(x_0) = y_1, \cdots, y^{(n-1)}(x_0) = y_{(n-1)}$$

The uniqueness theorem for Linear ODEs states that if $a_n(x), a_{n-1}(x), \cdots, a_0(x)$ and $g(x)$ are continuous on an interval, and if $x = x_0$ is any point on this interval, then a unique solution exists on the interval. Firstly, both x^3 and $|x|^3$ are continuous. Additionally, both differential equations can be arranged so they are linear.

$$xy' = 3y \quad \rightarrow \quad xy' - 3y = 0 \quad \rightarrow \quad a_1 = x, \quad a_0 = -3, \quad g(x) = 0$$

$$xy'' = 2y' \quad \rightarrow \quad xy'' - 2y' = 0 \quad \rightarrow \quad a_2 = x, \quad a_1 = -2, \quad a_0 = 0, \quad g(x) = 0$$

For both differential equations, $a_n(x), a_{n-1}(x), \cdots, a_0(x)$ and $g(x)$ are continuous.

For $x \geq 0$: $x^3 = |x|^3 = x^3$ This does not violate the uniqueness theorem as the functions are the same and linearly dependent.

For $x < 0$: $x^3 = |x|^3 = -x^3$ This does not violate the uniqueness theorem as the functions are linearly dependent and constant multiples of eachother.