MA-223 Review

Jacob Sigman

Contents

Chapter 15: Multiple Integrals

15.1: Double and Iterated Integrals over Rectangles

Double Integrals

Subdivide a region R into n rectangular pieces. The rectangular pieces have an area of

$$\Delta A = \Delta x \Delta y$$

Then, a sample point (x_k, y_k) is picked. To obtain the Reimann sum, you multiply the area by the function evaluated at the sample point.

$$S_n = \sum_{k=1}^{n} f(x_k, y_k)\Delta A_k$$

Using this sum, a double integral can be taken over a rectangle using the form below.

$$\iint_R f(x,y)\,dA \quad \text{or} \quad \iint_R f(x,y)\,dx\,dy$$

Fubini's Theorem

Fubini's Theorem states that the double integration over a region will evaluate to the same numerical value regardless of the order of integration.

$$\iint_R f(x,y)\,dA = \int_c^d \int_a^b f(x,y)\,dx\,dy \; = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

15.2: Double Integrals over General Regions

Double Integrals over Bounded, Nonrectangular Regions

The double integral can be taken over any region R with area A using the formula from the previous section.

$$\iint_R f(x,y)\,dA$$

Volumes

Firstly, determine the area of the solid. This would be done by taking a single integral between two curves $y = g_1(x)$ and $y = g_2(x)$

$$A(x) = \int_{g_1(x)}^{g_2(x)} f(x,y)dy$$

Then, integrate the area from $x = a$ to $x = b$ to get a double integral for volume.

$$V = \int_a^b A(x)\,dx = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Fubini's Theorem also applies to Volume, and therefore the order of integration doesn't matter, and the region can be defined in any way.

15.3: Area by Double Integration

Areas of Bounded Regions in the Plane

If we take $f(x, y) = 1$ in the definition of the double integral over a region R, the Reimann sum reduces to

$$S_n = \sum_{k=1}^{n} \Delta A_k$$

Hence, the area of a closed, bounded plane region R is

$$A = \iint\limits_R dA$$

Average Value

The average value of f over R is

$$\text{Average Value} = \frac{1}{\text{Area of } R} \iint\limits_R f \, dA$$

15.4: Double Integrals in Polar Form

Area in Polar Coordinates

There's no proof here for how it came to be, but when taking an area in polar coordinates, the integral looks as follows.

$$\iint\limits_R r \, dr \, d\theta$$

Changing Cartesian Integrals into Polar Integrals

Recall that $x = r \sin\theta$ and $y = r \cos\theta$. The transformation for a Cartesian integral is as follows.

$$\iint\limits_R f(x, y) \, dx \, dy = \iint\limits_G f(r\cos\theta, \, r\sin\theta) \, r \, dr \, d\theta$$

15.5: Triple Integrals in Rectangular Coordinates

Triple Integrals

Similar to double integrals, a triple integral for volume can be written for a sample point (x_k, y_k, z_k) as follows.

$$S_n = \sum_{k=1}^{n} f(x_k, y_k, z_k) \Delta V_k$$

Volume of a Region in Space

The volume of a closed, bounded region D in space is

$$V = \iiint\limits_D dV$$

Average Value of a Function in Space

The average value of a function F over a region D is

$$\text{Average Value} = \frac{1}{\text{Volume of } D} \iiint\limits_D F\, dV$$

15.6: Moments and Centers of Mass

Masses and First Moments

If $\delta(x, y, z)$ is the density of an object occupying a region D in space, the integral of δ over D gives the mass of the object, which is as follows.

$$\iiint\limits_D \delta(x, y, z)\, dV$$

The first moment of a solid region D about a coordinate plane is the triple integral over D of the distance from a point (x, y, z) in D to the plane multiplied by the density of the solid at that point. The moment about the coordinate planes are as follows.

$$M_{yz} = \iiint\limits_D x\, \delta(x, y, z)\, dV \qquad M_{xz} = \iiint\limits_D y\, \delta(x, y, z)\, dV \qquad M_{xy} = \iiint\limits_D z\, \delta(x, y, z)\, dV$$

The center of mass can be found from the first moments. The coordinates for the center of mass are as follows.

$$\bar{x} = \frac{M_{yz}}{M} \qquad \bar{y} = \frac{M_{xz}}{M} \qquad \bar{z} = \frac{M_{xy}}{M}$$

These formulas can also be applied in two dimensions.

$$M = \iint\limits_R \delta(x, y)\, dA$$

$$M_y = \iint\limits_R x\, \delta(x, y)\, dA \qquad M_x = \iint\limits_R y\, \delta(x, y)\, dA$$

$$\bar{x} = \frac{M_y}{M} \qquad \bar{y} = \frac{M_x}{M}$$

Moments of Inertia

If r is the distance from the point (x, y, z), then the moment of inertia about the line L of the mass with linear density δ is

$$I_L = \iiint\limits_D r^2 \delta\, dV$$

The line L can be any coordinate axis to give the moment of inertia about that axis. The moments of inertia about each coordinate axis are as follows.

$$I_x = \iiint\limits_D (y^2 + z^2)\, \delta(x, y, z)\, dV \quad I_y = \iiint\limits_D (x^2 + z^2)\, \delta(x, y, z)\, dV \quad I_z = \iiint\limits_D (x^2 + y^2)\, \delta(x, y, z)\, dV$$

These formulas can also be applied in two dimensions.

$$I_x = \iint\limits_R y^2\, \delta(x, y)\, dA \qquad I_y = \iint\limits_R x^2\, \delta(x, y)\, dA$$

$$I_L = \iint\limits_R r^2\,\delta(x,y)\,dA$$

The polar moment of inertia, or the moment of inertia about the origin, is as follows.

$$I_0 = \iint\limits_R (x^2+y^2)\,\delta(x,y)\,dA = I_x + I_y$$

15.7: Triple Integrals in Cylindrical and Spherical Coordinates

Integration in Cylindrical Coordinates

In cylindrical coordinates, r and θ are polar coordinates on the xy-plane and z is the rectangular vertical coordinate. Conversions between rectangular and cylindrical coordinates can be done as follows.

$$x = r\cos\theta \qquad y = r\sin\theta \qquad z = z$$

$$r^2 = x^2 + y^2 \qquad \tan\theta = \frac{y}{x}$$

The triple integral in cylindrical coordinates of a function f over a region D is as follows.

$$\iiint\limits_D f(r,\theta,z)\,dz\,r\,dr\,d\theta$$

Spherical Coordinates and Integration

Spherical coordinates locate points in space with two angles (θ and ϕ) and a distance (ρ). ρ is the distance from the point in space to the origin. ϕ is the angle the line formed between the point and the origin makes with the positive z-axis. θ is the angle from cylindrical coordinates. Spherical coordinates can be related to rectangular and cylindrical coordinates as follows.

$$r = \rho\sin\phi \qquad x = r\cos\theta = \rho\sin\phi\cos\theta$$

$$z = \rho\sin\phi \qquad y = r\sin\theta = \rho\sin\phi\sin\theta$$

$$\rho = \sqrt{x^2+y^2+x^2} = \sqrt{r^2+z^2}$$

The triple integral in spherical coordinates of a function f over a region D is as follows.

$$\iiint\limits_D f(\rho,\phi,\theta)\,\rho^2\sin\phi\,d\rho\,d\phi\,d\theta$$

15.8: Substitutions in Multiple Integrals

Substitutions in Double Integrals

Suppose that region G in the uv-plane is transformed one-to-one into the region R in the xy-plane by equations of the form $x = g(u,v)$ and $y = h(u,v)$. The transformed integral can be written as follows.

$$\iint\limits_R f(x,y)\,dx\,dy = \iint\limits_G f(g(u,v),h(u,v))\,|J(u,v)|\,du\,dv$$

$J(u,v)$ is the Jacobian and is defined as

$$J(u,v) = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \frac{\partial x}{\partial u}\frac{\partial y}{\partial v} - \frac{\partial y}{\partial u}\frac{\partial x}{\partial v}$$

Chapter 16: Integration in Vector Fields

16.1: Line Integrals

If f is defined on a curve C given parametrically by $\mathbf{r}(t) = g(t)\mathbf{i} + h(t)\mathbf{j} + k(t)\mathbf{k}$ and $a \leq t \leq b$, the line integral of C can be taken as

$$\int\limits_C f(x, y, z)\, ds = \int_a^b f(g(t), h(t), k(t))\, |\mathbf{v}(t)|\, dt$$

The mass and moment formulas for anything lying along a smooth cuve C in space are as follows. Note that dm is equivalent to $\delta\, ds$.

$$M = \int\limits_C \delta\, ds$$

$$M_{yz} = \int\limits_C x\, \delta\, ds \qquad M_{xz} = \int\limits_C y\, \delta\, ds \qquad M_{xy} = \int\limits_C z\, \delta\, ds$$

$$\bar{x} = \frac{M_{yz}}{M} \qquad \bar{y} = \frac{M_{xz}}{M} \qquad \bar{z} = \frac{M_{xy}}{M}$$

$$I_x = \int\limits_C (y^2 + z^2)\, \delta\, ds \qquad I_y = \int\limits_C (x^2 + z^2)\, \delta\, ds \qquad I_z = \int\limits_C (x^2 + y^2)\, \delta\, ds$$

$$I_L = \int\limits_C r^2 \delta\, ds$$

16.2: Vector Fields and Line Integrals - Work, Circulation, and Flux

Vector Fields

A vector field is a function that assigns a vector to each point in its domain. The field is continuous if it's component functions are continuous. The field is differentiable if it's component functions are differentiable. A three-dimensional vector field looks as follows.

$$\mathbf{F}(x, y, z) = M(x, y, z)\mathbf{i} + N(x, y, z)\mathbf{j} + P(x, y, z)\mathbf{k}$$

Gradient Fields

The gradient field of a differentiable function $f(x, y, z)$ is defined as

$$\nabla f = \frac{\partial f}{\partial x}\mathbf{i} + \frac{\partial f}{\partial y}\mathbf{j} + \frac{\partial f}{\partial z}\mathbf{k}$$

Line Integrals of Vector Fields

If $\mathbf{F}$ is a vector field with continuous components defined along a smooth curve C parameterized by $\mathbf{r}(t)$ and $a \leq t \leq b$, then the line integral of $\mathbf{F}$ along C is

$$\int\limits_C \mathbf{F} \cdot \mathbf{T}\, ds = \int\limits_C \left[\mathbf{F} \cdot \frac{d\mathbf{r}}{ds}\right] ds = \int\limits_C \mathbf{F} \cdot d\mathbf{r} = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt$$

Line integrals can also be written in differential form, which is as follows.

$$\int\limits_C M(x, y, z)\, dx + \int\limits_C N(x, y, z)\, dy + \int\limits_C P(x, y, z)\, dz = \int\limits_C M\, dx + N\, dy + P\, dz$$

Work Done by a Force over a Curve in Space

If $\mathbf{F}$ is a continuous force field along a smooth curve C parameterized by $\mathbf{r}(t)$ and $a \leq t \leq b$, then the work done along C is

$$W = \int_C \mathbf{F} \cdot \mathbf{T}\, ds = \int_a^b \mathbf{F}(\mathbf{r}(t)) \cdot \frac{d\mathbf{r}}{dt}\, dt$$

Flow Integrals and Circulation for Velocity Fields

If $\mathbf{r}(t)$ parameterizes a smooth curve C in the domain of a continuous velocity field $\mathbf{F}$, the flow along the curve is

$$\text{Flow} = \int_C \mathbf{F} \cdot \mathbf{T}\, ds$$

This is called a flow integral, if the curve starts and ends at the same point, the flow is called the circulation around the curve.

Flux Across a Simple Plane Curve

A curve is simple if it does not cross itself. When a curve starts and ends at the same point, it is closed. The flux along a simple, closed curve, is the line integral of the scalar component of the velocity field in the direction of the curve's outward-pointing normal vector, which can be written as

$$\text{Flux of } \mathbf{F} \text{ across } C = \int_C \mathbf{F} \cdot \mathbf{n}\, ds$$

Flux can also be calculated across a smooth, closed plane curve. This is done by letting $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$. This makes the flux integral

$$\text{Flux of } \mathbf{F} \text{ across } C = \int_C \mathbf{F} \cdot \mathbf{n}\, ds = \oint_C M\, dy - N\, dx$$

16.3: Path Independence, Conservative Fields, and Potential Functions

Path Independence

Let $\mathbf{F}$ be a vector field defined on an open region D in space. If the line integral along any two paths are the same, the line integrals are path independent and $\mathbf{F}$ is conservative on D.

If $\mathbf{F}$ is a vector field defined on D and $\mathbf{F} = \nabla f$ for some scalar function f on D, then f is called a potential function for $\mathbf{F}$.

Line Integrals in Conservative Fields

Let C be a smooth curve parameterized by $\mathbf{r}(t)$ joining the points A and B in space. If f is a differentiable function with a continuous gradient $\mathbf{F} = \nabla f$ on a domain D containing C, then

$$\int_C \mathbf{F} \cdot d\mathbf{r} = f(B) - f(A)$$

If $\mathbf{F} = M\mathbf{i} + N\mathbf{j} + P\mathbf{k}$ is a vector field whose components are continuous throughout an open connected region D in space, then $\mathbf{F}$ is conservative if and only if $\mathbf{F}$ is a gradient field ∇f for a differentiable function f.

Exact Differential Forms

A differential form is exact if for some scalar function f on D is as follows.

$$M\,dx + N\,dy + P\,dz = \frac{\partial f}{\partial x}\,dx + \frac{\partial f}{\partial y}\,dy + \frac{\partial f}{\partial z}\,dz = df$$

A differential form is exact on a connected and simply connected domain if and only if

$$\frac{\partial P}{\partial y} = \frac{\partial N}{\partial z} \qquad \frac{\partial M}{\partial z} = \frac{\partial P}{\partial x} \qquad \frac{\partial N}{\partial x} = \frac{\partial M}{\partial y}$$

16.4: Green's Theorem in the Plane

Divergence

The divergence (flux density) of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is

$$\text{div }\mathbf{F} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}$$

Spin Around an Axis

The circulation density, or the k-component of the curl, of a vector field $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ at the point (x, y) is the scalar expression

$$(\text{curl }\mathbf{F}) \cdot \mathbf{k} = \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$$

Two Forms for Green's Theorem

Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the outward flux of $\mathbf{F}$ across C equals the double integral of div $\mathbf{F}$ over the region R enclosed C.

$$\oint_C \mathbf{F} \cdot \mathbf{n}\,ds = \oint_C M\,dy - N\,dx = \iint_R \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right) dx\,dy$$

Let C be a piecewise smooth, simple closed curve enclosing a region R in the plane. Let $\mathbf{F} = M\mathbf{i} + N\mathbf{j}$ be a vector field with M and N having continuous first partial derivatives in an open region containing R. Then the counterclockwise circulation of $\mathbf{F}$ around C equals the double integral of (curl $\mathbf{F}$) $\cdot$ $\mathbf{k}$ over R.

$$\oint_C \mathbf{F} \cdot \mathbf{T}\,ds = \oint_C M\,dx + N\,dy = \iint_R \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$