MA-240 Review

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Contents

Chapter 1: Introduction to Differential Equations

1.1: Definitions and Terminology

Differential Equation An equation containing the derivatives of one or more dependent variables with respect to one or more independent variables.

Classification by Order The order of the highest derivative in the equation.

Classification by Type Whether the differential equation has partial derivatives in it. If it does, it's called a Partial Differential Equation, otherwise it's called an Ordinary Differential Equation.

Classification By Linearity A differential equation is linear if it can be put in the form

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1} y}{dx^{n-1}} + \cdots + a_0 y = g(x)$$

where $a_i(x)$ and $g(x)$ depend at most on the independent variable.

Solutions Suppose ϕ is a function defined on some interval. If substituting ϕ into the ODE reduces the equation to an identity, then ϕ solves the ODE on the interval. The graph of ϕ is called a Solution Curve.

Particular Solution In general, an n^{th} order Ordinary Differential Equation will be solved by an n-parameter family of solutions. A solution of a differential equation that is free of arbitrary parameters is said to be particular.

Singular Solution A particular solution which does not belong to an n-parameter family of solutions.

General Solution If the n-parameter family contains all the solutions, then the n-parameter family is said to be the general solution.

1.2: Initial-Value Problems

An Initial-Value Problem is an n^{th} order Ordinary Differential Equation paired with conditions for the solution and it's first $(n-1)$ derivatives. Below is an example of an Initial-Value Problem.

$$y'' + 3y' - y = \sin x \qquad y(\pi) = 1 \qquad y'(\pi) = 3$$

Picard's Theorem discusses the existence of a unique solution. Suppose R is a region in the xy-plane containing (x_0, y_0). If $f(x,y)$ and $\frac{\partial f}{\partial y}$ are continuous on R, there exists some interval I which belongs to x_0 such that $\frac{dy}{dx} = f(x,y)$ has one and only one solution passing through x_0.

1.3: Differential Equations as Mathematical Models

Population Dynamics The rate at which population grows at a certain time is proportional to the total population of the country at that time: $\frac{dP}{dt} = kP$

Radioactive Decay The rate at which the nuclei of a substance decays is proportional to the amount of substance remaining at a given time: $\frac{dA}{dt} = kA$

Newton's Law of Cooling The rate at which the temperature of a body changes is proportional to the difference between the temperature of the body and the temperature of the surrounding medium: $\frac{dT}{dt} = k(T - T_m)$

Mixtures The mixing of two salt solutions of differing concentrations results in a first-order differential equation for the amount of salt contained in a mixture: $\frac{dA}{dt} = R_{in} - R_{out}$

Chapter 2: First-Order Differential Equations

2.1: Solution Curves Without a Solution

It's not always easy to solve a differential equation, we may just want to know what they look like. This is where Direction Fields come in. They provide a visual representation of solutions. Evaluate some $y' = f(x, y)$ on a dense grid.

An Autonomous first-order differential equation is a differential equation which is only a function of y: $y' = f(x, y) = f(y)$. The points at which $f(x, y) = 0$ are called Critical Points. A constant solution of an Autonomous differential equation is called an Equilibrium Solution. A Phase Portrait depicts the behavior of the differential equation on certain intervals. When two arrowheads of a phase portrait point towards eachother, it is said to be an attractor, which is stable. When two arrowheads of a phase portrait point away from eachother, it is said to be a repeller.

2.2: Separable Equations

A first-order equation of the following form is said to be separable.

$$\frac{dy}{dx} = g(x)h(y)$$

This equation can be solved by integration.

$$\int p(y)y'dx = \int g(x)dx \quad \rightarrow \quad \int p(y)dy = \int g(x)dx$$

2.3: Linear Equations

Recall that a differential equation is linear if it can be put in the following form:

$$a_n(x)\frac{d^n y}{dx^n} + a_{n-1}(x)\frac{d^{n-1}y}{dx^{n-1}} + \cdots + a_0 y = g(x) \text{ where } y' = P(x)y$$

We look for a function μ by multiplying by μ on both sides of y'. We will get a derivative on one side.

$$\text{Let } \mu(x) = e^{\int P(x)dx} \quad \rightarrow \quad \frac{d}{dx}\left[\mu(x)y\right] = f(x)e^{\int P(x)dx}$$

For a linear differential equation, if $f(x) = 0$ we call the differential equation homogeneous, $f(x)$ is sometimes called a forcing function.

The error function and complimentary error function are defined as follows.

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}dt \qquad \text{erfc}(x) = \frac{2}{\sqrt{\pi}}\int_x^\infty e^{-t^2}dt$$

2.4: Exact Equations

Consider a differential equation of the following form:

$$\mathbf{M}(x, y)dx + \mathbf{N}(x, y)dy = 0 \quad \rightarrow \quad \mathbf{M}(x, y) + \mathbf{N}(x, y)y' = 0$$

If $z = f(x, y)$ is a 2-variable function with continuous partials in some region R, then on R, then on R, the total differential of z is said to be:

$$dz = f_x dx + f_y dy$$

of particular interest to us is the case where $f(x, y) = c$ so $f_x dx + f_y dy = 0$. $f(x, y) = c$ is a family of functions defined by the parameter c.

An expression $\mathbf{M}dx + \mathbf{N}dy$ is an exact differential form if there exists a function on a region R such that $\mathbf{M} = f_x$ and $\mathbf{N} = f_y$ are continuous. A differential equation is exact if:

$$\frac{\partial \mathbf{M}}{\partial y} = \frac{\partial \mathbf{N}}{\partial x}$$

If $\mathbf{M}_y \neq \mathbf{N}_x$, then multiply both sides by a factor μ such that $(\mu\mathbf{M})_y = (\mu\mathbf{N})_x$.

$$\mu_x + \left(\frac{\mathbf{N}_x - \mathbf{M}_y}{\mathbf{N}}\right)\mu = 0 \quad \text{or} \quad \mu_y + \left(\frac{\mathbf{M}_y - \mathbf{N}_x}{\mathbf{M}}\right)\mu = 0$$

2.5: Solutions by Substitutions

A function f is homogeneous if $f(tx, ty) = t^\alpha f(x, y)$ for all t, x, y of degree α. A differential equation $\mathbf{M}dx + \mathbf{N}dy = 0$ is homogeneous if $\mathbf{M}$ and $\mathbf{N}$ are homogeneous of the same degree. Substitutions for y and x are made as follows:

$$y = ux \quad \text{or} \quad x = vy$$

Bernoulli's Equation is as follows.

$$y' + P(x)y = f(x)y^n \qquad \text{Substitution: } u = y^{1-n}$$

Chapter 3: Modeling with First-Order Differential Equations

3.1: Linear Models

Growth and Decay	Newton's Law of Cooling	Mixtures
$\frac{dx}{dt} = kx$	$\frac{dT}{dt} = k(T - T_m)$	$\frac{dA}{dt} = R_{in} - R_{out}$

Series Circuits

$$L\frac{di}{dt} + Ri = E(t) \qquad R\frac{dq}{dt} + \frac{1}{C}q = E(t)$$

Chapter 4: Higher-Order Differential Equations

4.1: Preliminary Theory - Linear Equations

An n^{th}-order Initial-Value Problem is defined as

$$\sum_{i=0}^{n} a_i(x)y^{(i)}(x) = g(x) \text{ such that } y(x_0) = y_0, y'(x_0) = y_1, \cdots, y^{(n-1)}(x_0) = y_{(n-1)}$$

Let a_i be continuous on I. $a_n(x) \neq 0$ for all $x \in I$, then there is a unique solution to the n^{th}-order Initial-Value Problem on I. A boundary-value problem is defined as

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = g(x) \text{ such that } y(c) = y_0 \text{ and } y(b) = y_1$$

If $g(x) = 0$ in an n^{th}-order Initial-Value Problem, the differential equation is homogeneous. Otherwise, it's not homogeneous. The differential operator is defined as follows:

$$Df = f'$$

If $\{y_i\}_{i=1}^{k}$ are solutions to the homogeneous n^{th}-order Initial-Value Problem, then

$$\sum_{i=1}^{k} c_i y_i$$

is also a solution to the homogeneous n^{th}-order Initial-Value Problem. This is called the superposition principle.

A set of functions $\{f_i\}_{i=1}^{n}$ is linearly independent if

$$\sum_{i=1}^{n} c_i f_i(x) = 0$$

This implies $c_i = 0$ for all i otherwise it's linearly dependent. This means that there exists a set of constants $\{c_i\}$ that are not all zero such that

$$\sum_{i=1}^{n} c_i f_i(x) = 0$$

.

Let F be a set of functions. If $0 \in F$, then F is linearly dependent. $F = \{f_1, f_2\}$ is linearly dependent if and only if $f_1 = kf_2$. The Wronskian of a set of functions $\{f_i\}_{i=1}^{n}$ where f_i are differentiable up to $n-1$ degrees is:

$$W(f_1, \cdots, f_n) = \begin{vmatrix} f_1 & f_2 & \cdots & f_n \\ f_1' & f_2' & \cdots & f_n' \\ \vdots & \vdots & \ddots & \vdots \\ f_1^{(n-1)} & f_2^{(n-1)} & \cdots & f_n^{(n-1)} \end{vmatrix}$$

A set of n solutions to the homogeneous n^{th}-order Initial-Value Problem is linearly independent if and only if the Wronskian is nonzero everywhere. If $\{y_i\}_{i=1}^{n}$ are solutions to the homogeneous n^{th}-order Initial-Value Problem, then $W(y_1, \cdots, y_n)$ is always or never zero on I. A set of linearly independent solutions to the homogeneous n^{th}-order Initial-Value Problem $\{y_i\}_{i=1}^{n}$ is called a fundamental set. There is always a fundamental set for the homogeneous n^{th}-order Initial-Value Problem.

If $\{y_i\}_{i=1}^n$ are solutions to the homogeneous n^{th}-order Initial-Value Problem and y_p is a solution to the non-homogeneous n^{th}-order Initial-Value Problem then

$$\sum_{i=1}^{n} c_i y_i + y_p$$

is a solution to the non-homogeneous n^{th}-order Initial-Value Problem. If $\{y_i\}_{i=1}^n$ are a fundamental set for the homogeneous n^{th}-order Initial-Value Problem and y_p solves the non-homogeneous n^{th}-order Initial-Value Problem, then the general solution of the non-homogeneous n^{th}-order Initial-Value Problem is

$$y = \sum_{i=1}^{n} c_i y_i + y_p$$

To solve a non-homogeneous n^{th}-order Initial-Value Problem, solve for the homogeneous case first, find the particular solution, then find the general solution. Say that y_{p_j} solves

$$\sum_{i=0}^{n} a_i(x) y_i^{(i)} = g_j(x) \text{ for } j = 1, \cdots, m$$

then

$$\sum_{j=1}^{m} y_{p_j} \text{ solves } \sum_{i=0}^{n} a_i(x) y_i^{(i)} = \sum_{j=1}^{m} g_j(x)$$

4.2: Reduction of Order

Say y_1 is a known solution of

$$y'' + P(x)y' + Q(x)y = f(x)$$

If y_2 is another solution and $\{y_1, y_2\}$ is a fundamental set, then we've found our solution. Use the following to find the second solution.

$$y_2(x) = u(x) y_1(x)$$

Below is the formula the textbook provides for reduction of order.

$$y_2(x) = y_1(x) \int \frac{e^{-\int P(x)dx}}{y_1(x)^2} dx$$

4.3: Homogeneous Linear Equations with Constant Coefficients

The equation that is trying to be solved is

$$ay'' + by' + cy = 0$$

The characteristic equation of the above equation can be written as follows

$$am^2 + bm + c = 0$$

There are three cases that can be analyzed based on the discriminant $(b^2 - 4ac)$ of the above characteristic equation. The first case is when the discriminant is greater than zero, and the characteristic equation has two distinct real roots m_1 and m_2.

$$y = c_1 e^{m_1 x} + c_2 e^{m_2 x}$$

The second case is when the discriminant is equal to zero, and the characteristic equation has a repeated root m.

$$y = c_1 e^{mx} + c_2 x e^{mx}$$

The third case is when the discriminant is less than zero, and the characteristic equation has complex conjugate roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$.

$$y = e^{\alpha x}(c_1 \cos(\beta x)) + c_2 \sin(\beta x))$$

4.4: Undetermined Coefficients - Superposition Approach

The Method of Undetermined Coefficients is one way of determining a particular solution. In solving differential equations, find the complementary solution, then find the particular solution, then put them both together using the superposition principle.

Guess a solution based on the non-homogeneous component of the differential equation and determine the coefficients of the guessed particular solution. Below are some examples of some guesses.

$g(x)$	Form of y_p
1	A
$5x+7$	$Ax+B$
$3x^2-2$	Ax^2+Bx+C
$\sin(4x)$	$A\cos(4x)+B\sin(4x)$
e^{5x}	Ae^{5x}
$(9x-2)e^{5x}$	$(Ax+B)e^{5x}$
$(x^2)e^{5x}$	$(Ax^2+Bx+C)e^{5x}$
$e^{3x}\sin(4x)$	$Ae^{3x}\cos(4x)+Be^{3x}\sin(4x)$
$xe^{3x}\sin(4x)$	$(Ax+B)e^{3x}\cos(4x)+(Cx+D)e^{3x}\sin(4x)$

4.6: Variation of Parameters

Variation of Parameters is another approach to determining a particular solution. The formula below is used:

$$y_p = y_1 \int \frac{y_1 f(x)}{W(y_1,y_2)}dx + y_2 \int \frac{y_2 f(x)}{W(y_1,y_2)}dx$$

A more general form:

$$y_p = \sum_{i=1}^{n} u_i y_i \quad \text{where} \quad u_i' = \frac{\Delta_i}{W(y_1,\cdots,y_n)}$$

$$\text{Where } \Delta_i = W(y_1,\cdots,y_2) \text{ where the } i^{\text{th}} \text{ column is } \begin{bmatrix} 0 \\ \vdots \\ f(x) \end{bmatrix}$$

4.7: Cauchy-Euler Equation

The Cauchy-Euler Equation is a linear differential equation of the following form:

$$\sum_{i=0}^{n} a_i x^i y^{(i)}(x) = g(x)$$

The characteristic equation for a Cauchy-Euler Equation is

$$am(m-1)+bm+c=0$$

There are three cases that can be analyzed based on the discriminant ($b^2 - 4ac$) of the above characteristic equation. The first case is when the discriminant is greater than zero, and the characteristic equation has two distinct real roots m_1 and m_2.

$$y = c_1 x^{m_1} + c_2 x^{m_2}$$

The second case is when the discriminant is equal to zero, and the characteristic equation has a repeated root m.

$$y = c_1 x^m + c_2 x^m \ln x$$

The third case is when the discriminant is less than zero, and the characteristic equation has complex conjugate roots $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$.

$$y = x^\alpha (c_1 \cos(\beta \ln x)) + c_2 \sin(\beta \ln x))$$

Chapter 5: Modeling with Higher-Order Differential Equations

5.1: Linear Models: Initial-Value Problems

Linear models can describe simple harmonic motion. The equation for the oscillator below is said to be undamped.

$$\frac{d^2x}{dt^2} + \omega^2 x = 0 \text{ where } \omega^2 = \frac{k}{m}$$

Sometimes, damping forces act on an oscillator. The equation for a damped oscillator with damping constant β is

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = 0 \text{ where } 2\lambda = \frac{\beta}{m} \text{ and } \omega^2 = \frac{k}{m}$$

Sometimes, driving forces act on an oscillator. The equation for a driven oscillator with driving force $f(t)$ is

$$\frac{d^2x}{dt^2} + 2\lambda\frac{dx}{dt} + \omega^2 x = F(t) \text{ where } 2\lambda = \frac{\beta}{m},\ \omega^2 = \frac{k}{m}, \text{ and } F(t) = \frac{f(t)}{m}$$

Chapter 6: Series Series Solutions of Linear Equations

6.1: Review of Power Series

A power series centered at a is a series in the form:

$$\sum_{n=0}^{\infty} c_n (x-a)^n$$

Convergence A power series is convergent if it's sequence of partial sums converges. The limit of the partial sums is: $\lim_{N\to\infty} \sum_{n=0}^{N} c_n (x-a)^n$ If this limit doesn't exist, the power series is divergent.

Interval of Convergence The set of all real numbers x for which the series converges. The center of this interval is the center a of the series.

Radius of Convergence The radius R of the interval of convergence. If the series only converges at it's center, $R = 0$, otherwise, a power series will converge for $|x - a| < R$ and will diverge for $|x - a| > R$.

Absolute Convergence Within the interval of convergence, the power series converges absolutely.

One can test the convergence of a power series using the ratio test. Use the following limit:

$$\lim_{n\to\infty}\left|\frac{c_{n+1}(x-a)^{n+1}}{c_n(x-a)^n}\right| = L$$

If $L < 1$, the series converges absolutely, if $L > 1$ the series diverges, and if $L = 1$ the test is inconclusive.

The derivatives of power series are defined as follows.

$$y' = \sum_{n=1}^{\infty} c_n n x^{n-1} \qquad y'' = \sum_{n=2}^{\infty} c_n n(n-1)x^{n-2}$$

If a power series is 0 for all numbers x in some open interval, then $c_n = 0$ for all n. A function is analytic at a if it has a power series representation at a. Power series can also be combined through addition, multiplication, and division. You would do these as you would with polynomials. Below are some Maclaurin Series representations of common functions.

$\boldsymbol{f(x)}$	**Maclaurin Series Representation**
e^x	$\sum_{n=0}^{\infty} \frac{1}{n!}x^n$
$cos x$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!}x^{2n}$
$\sin x$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!}x^{2n+1}$
$\arctan x$	$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}x^{2n+1}$
$\cosh x$	$\sum_{n=0}^{\infty} \frac{1}{(2n)!}x^{2n}$
$\sinh x$	$\sum_{n=0}^{\infty} \frac{1}{(2n+1)!}x^{2n+1}$
$\ln(1+x)$	$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}x^n$
$\frac{1}{1-x}$	$\sum_{n=0}^{\infty} x^n$

6.2: Solutions about Ordinary Points

A point $x = x_0$ is said to be an ordinary point of a differential equation if both $P(x)$ and $Q(x)$ are analytic at x_0 in the differential equation below. A point that is not ordinary is said to be a singular point.

$$y'' + P(x)y' + Q(x)y = 0$$

If x_0 is an ordinary point of the above equation, then we can always find two linearly independent solutions of the above equation on an interval containing x_0 of the form

$$y = \sum_{n=0}^{\infty} c_n(x-x_0)^n \text{ on } |x-x_0| < R$$

Where the radius of convergence is the distance from x_0 to the closest singular point. The general process is to combine all the series in a differential equation then to solve the recurrence relation once it's obtained.

6.3: Solutions about Singular Points

There are two types of singular points. A singular point is said to be a regular singular point if $(x-x_0)P(x)$ and $(x-x_0)^2Q(x)$ are both analytic at x_0. Otherwise, the point is said to be an irregular singular point.

Frobenius's Theorem states that if $x = x_0$ is a regular singular point of the standard form of a differential equation then there exists at least one solution of the form

$$y = (x-x_0)^r \sum_{n=0}^{\infty} c_n (x-x_0)^n$$

Where r is some constant. After substituting the power series solution into the differential equation and simplifying, the indicial equation is a quadratic equation in r that results from equating the total coefficient of the lowest power of x to zero. The values of r, or the indicial roots can be obtained and plugged into the recurrence relation.

Chapter 7: The Laplace Transform

7.1: Definition of the Laplace Transform

Let k be a continuous function of two real variables, and g be a continuous function of one real variable. The integral transform with kernel k is defined as follows.

$$I_k(g) = \int_a^b k(s,t)g(t)dt = f(s)$$

Let f be a function defined for $t \geq 0$. The Laplace transform of f is

$$\mathscr{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t)dt$$

The Laplace transform is a linear transform. A function f is said to be of exponential order if there exist constants c, $M > 0$, and $T > 0$ such that $|f(t)| \leq Me^{ct}$ for all $t > T$. If f is piecewise continuous on $[0, \infty)$ and of exponential order, then $\lim_{s\to\infty} = 0$. Below are some common Laplace transforms.

$\boldsymbol{f(t)}$	$\boldsymbol{\mathscr{L}\{s\}}$
1	$\frac{1}{s}$
t^n	$\frac{n!}{s^{n+1}}$
e^{at}	$\frac{1}{s-a}$
$\sin kt$	$\frac{k}{s^2+k^2}$
$\cos kt$	$\frac{s}{s^2+k^2}$
$\sinh kt$	$\frac{k}{s^2-k^2}$
$\cosh kt$	$\frac{s}{s^2-k^2}$

7.2: Inverse Transforms and Transforms of Derivatives

The Laplace transform can also be applied in reverse. Partial fractions play a key role in determining inverse Laplace transforms so that each term can be factored into distinct linear factors. Below are some common inverse Laplace transforms.

$\mathscr{L}\{s\}$	$f(t)$
$\frac{1}{s}$	1
$\frac{n!}{s^{n+1}}$	t^n
$\frac{1}{s-a}$	e^{at}
$\frac{k}{s^2+k^2}$	$\sin kt$
$\frac{s}{s^2+k^2}$	$\cos kt$
$\frac{k}{s^2-k^2}$	$\sinh kt$
$\frac{s}{s^2-k^2}$	$\cosh kt$

The transform of a derivative can be determined as well. If $f, f', \ldots, f^{(n-1)}$ are continuous on $[0, \infty)$ and are of exponential order, and if $f^{(n)}(t)$ is piecewise continuous on $[0, \infty)$, then

$$\mathscr{L}\{f^{(n)}(t)\} = s^n F(s) - s^{n-1} f(0) - s^{n-2} f'(0) - \cdots - f^{(n-1)}(0)$$

7.3: Operational Properties I

It is possible to compute the Laplace transform of an exponential multiple of f by shifting the Laplace transform. If $\mathscr{L}\{f(t)\} = F(s)$ and a is any real number, then the first shifting theorem states that

$$\mathscr{L}\{e^{at} f(t)\} = F(s-a)$$

The unit step function is defined as follows

$$\mathscr{U}(t-a) = \mathscr{U}_a(t) = \begin{cases} 0, & 0 \le t < a \\ 1, & t \ge a \end{cases}$$

Let $F(s) = \mathscr{L}\{f(t)\}$. The properties of the unit step function with regard to Laplace transforms are as follows

$$\mathscr{L}\{\mathscr{U}_a(t)\} = \frac{e^{-sa}}{s} \qquad \mathscr{L}\{f(t-a)\,\mathscr{U}(t-a)\} = e^{-sa} F(s)$$

7.4: Operational Properties II

The Laplace transform of the product of a function $f(t)$ with t raised to a numerical exponent. This gives the derivative of a transform. If $F(s) = \mathscr{L}\{f(t)\}$ then

$$\mathscr{L}\{t^n f(t)\} = (-1)^n \frac{d^n}{ds^n} F(s)$$

If functions f and g are piecewise continuous on the interval $[0, \infty)$, then the convolution of f and g is defined by

$$f * g = \int_0^t f(\tau)\, g(t-\tau) d\tau$$

The convolution theorem states that if $f(t)$ and $g(t)$ are piecewise continuous on $[0, \infty)$ and of exponential order, then

$$\mathscr{L}\{f * g\} = \mathscr{L}\{f(t)\}\,\mathscr{L}\{g(t)\} = F(s)\,G(s)$$

A Volterra integral equation for $f(t)$ is as follows

$$f(t) = g(t) + \int_0^t f(\tau)\,h(t-\tau)d\tau$$

A function is periodic with period T if $f(t+T) = f(t)$ for all t. If $f(t)$ is piecewise continuous on $[0, \infty)$, of exponential order, and periodic with period T, then

$$\mathscr{L}\{f(t)\} = \frac{1}{1-e^{-sT}} \int_0^T e^{-st} f(t)dt$$

7.5: The Dirac Delta Function

A function that is a unit impulse is defined as follows

$$\delta_a(t-t_0) = \begin{cases} 0, & 0 \le t < t_0 - a \\ \frac{1}{2a}, & t_0 - a \le t < t_0 + a \\ 0, & t \ge t_0 + a \end{cases}$$

Typically it's convenient to work with another type of unit impulse called the Dirac delta function, which can be characterized by two properties:

$$\delta(t-t_0) = \begin{cases} \infty, & t = t_0 \\ 0, & t \ne t_0 \end{cases} \qquad \int_0^\infty \delta(t-t_0)\,dt = 1$$

The Laplace transform of the Dirac delta function is

$$\mathscr{L}\{\delta(t-t_0)\} = e^{-st_0}$$

Chapter 11: Fourier Series

11.1: Orthogonal Functions

An inner product is a function $(\mathbf{u}, \mathbf{v})$ such that

$$(\mathbf{u}, \mathbf{v}) = (\mathbf{v}, \mathbf{u}) \qquad (\mathbf{u}, \mathbf{u}) = 0 \text{ if } \mathbf{u} = 0 \text{ and } (\mathbf{u}, \mathbf{u}) > 0 \text{ if } u \ne 0$$
$$(k\mathbf{u}, \mathbf{v}) = k(\mathbf{u}, \mathbf{v}) \qquad (\mathbf{u}+\mathbf{v}, \mathbf{w}) = (\mathbf{u}, \mathbf{w}) + (\mathbf{v}, \mathbf{w})$$

It's defined as the number

$$(f_1, f_2) = \int_a^b f_1(x)\,f_2(x)\,dx$$

The two functions f_1 and f_2 are orthogonal if their inner product is 0. A set of real-valued functions $\{\phi_i\}_{i=1}^n$ is said to be orthogonal if

$$(\phi_m, \phi_n) = \int_a^b \phi_m(x)\,\phi_n(x)\,dx = 0$$

Additionally, if $(\phi_m, \phi_m) = 1$, the set is said to be orthonormal. The value $\sqrt{(\phi_m, \phi_m)}$ denoted by $||\phi_m||$ is called the norm of ϕ_m induced by the inner product. Any orthogonal set of nonzero functions can be made into an orthonormal set by normalizing each function, dividing each function by its norm. It's also possible to represent funcitons in the form of a series. Given a function, when we can find $\{\phi_n\}_{n=0}^{\infty}$ and $\{c_n\}_{n=0}^{\infty}$ such that

$$f(x) = \sum_{n=0}^{\infty} c_n \phi_n(x)$$

If we can find $\{\phi_n\}_{n=0}^{\infty}$, then

$$c_n = \frac{(f, \phi_n)}{||\phi_n||^2} \text{ and } f(x) = \sum_{n=0}^{\infty} \frac{(f, \phi_n)}{||\phi_n||^2} \phi_n(x)$$

A set of real-valued functions is said to be orthogonal with respect to a weight function on an interval if

$$\int_a^b w(x)\, \phi_m(x)\, \phi_n(x)\, dx = 0$$

11.2: Fourier Series

The Fourier series of a function f on an interval $(-p, p)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{p}\right) + b_n \sin\left(\frac{n\pi x}{p}\right) \right]$$

$$a_0 = \frac{1}{p} \int_{-p}^{p} f(x) dx \qquad a_n = \frac{1}{p} \int_{-p}^{p} f(x) \cos\left(\frac{n\pi x}{p}\right) dx \qquad b_n = \frac{1}{p} \int_{-p}^{p} f(x) \sin\left(\frac{n\pi x}{p}\right) dx$$

Let f and f' be piecewise continuous on $[-p, p]$, then for all x on the interval for which f is continuous, the Fourier series of f converges to $f(x)$. At points of discontinuity

$$f_s(x) = \frac{f(x^+) + f(x^-)}{2}$$

Where $f(x^+)$ and $f(x^-)$ are the right-hand and left-hand limits of f at x, respectively.

11.3: Fourier Cosine and Sine Series

An even function is a function such that

$$f(-x) = f(x)$$

An odd function is a function such that

$$f(-x) = -f(x)$$

The Fourier series of an even function f defined on the interval $(-p, p)$ is a cosine series

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{p}\right) \right]$$

$$a_0 = \frac{2}{p} \int_0^p f(x) dx \qquad a_n = \frac{2}{p} \int_0^p f(x) \cos\left(\frac{n\pi x}{p}\right) dx$$

The Fourier series of an odd function f defined on the interval $(-p, p)$ is a sine series

$$f(x) = \sum_{n=1}^{\infty} \left[b_n \sin\left(\frac{n\pi x}{p}\right) \right]$$

$$b_n = \frac{2}{p} \int_0^p f(x) \sin\left(\frac{n\pi x}{p}\right) dx$$

Chapter 12: Boundary-Value Problems in Rectangular Coordinates

12.1: Seperable Partial Differential Equations

The general form of a linear second-order partial differential equation is given by

$$A\frac{\partial^2 u}{\partial x^2} + B\frac{\partial^2 u}{\partial x \partial y} + C\frac{\partial^2 u}{\partial y^2} + D\frac{\partial u}{\partial x} + E\frac{\partial u}{\partial y} + Fu = G$$

If $G = 0$ the equation is said to be homogeneous, otherwise it's said to be non-homogeneous. Separation of variables is the primary technique used in partial differential equations. Consider the following guess

$$u(x, y) = X(x)\,Y(y) \qquad \frac{\partial u}{\partial x} = X'Y \qquad \frac{\partial u}{\partial y} = XY' \qquad \frac{\partial^2 u}{\partial x^2} = X''Y \qquad \frac{\partial^2 u}{\partial x^2} = XY''$$

Additionally, it's practice to equate the differential equation to a separation constant $-\lambda$. There are three cases that must be analyzed, the first case being if $\lambda = 0$, the second case being if $\lambda = -\alpha^2$, and the third case being if $\lambda = \alpha^2$. The superposition principle also applies.

There are three ways of classifying partial differential equations. A partial differential equation is hyperbolic if $B^2 - 4AC > 0$, parabolic if $B^2 - 4AC = 0$, and elliptic if $B^2 - 4AC < 0$.

12.2: Classical PDEs and Boundary-Value Problems

Heat Equation	**Wave Equation**	**Laplace's Equation**
$k\frac{\partial^2 u}{\partial x^2} = \frac{\partial u}{\partial t}$	$a^2\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 u}{\partial t^2}$	$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$

There are three types of boundary conditions as follows

$$u(L, t) = u_0 \qquad u_x(L, t) = 0 \qquad u_x(L, t) = -h(u(L, t) - u_m)$$

12.3: Heat Equation

This section of the textbook solves the heat equation. The solution is as follows

$$u(x, t) = \frac{2}{L}\sum_{n=1}^{\infty}\left[\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx\right]e^{-k\left(\frac{n^2\pi^2}{L^2}\right)t}\sin\left(\frac{n\pi x}{L}\right)$$

12.4: Wave Equation

This section of the textbook solves the wave equation. The solution is as follows

$$u(x, t) = \sum_{n=1}^{\infty}\left[A_n\cos\left(\frac{n\pi a}{L}t\right) + B_n\sin\left(\frac{n\pi a}{L}t\right)\right]\sin\left(\frac{n\pi x}{L}\right)$$

$$A_n = \frac{2}{L}\int_0^L f(x)\sin\left(\frac{n\pi x}{L}\right)dx \qquad B_n = \frac{2}{n\pi a}\int_0^L g(x)\sin\left(\frac{n\pi x}{L}\right)dx$$

12.5: Laplace's Equation

This section of the textbook solves Laplace's equation. The solution is as follows

$$u(x,y) = A_0 y + \sum_{n=1}^{\infty} A_n \sinh\left(\frac{n\pi y}{a}\right)\cos\left(\frac{n\pi x}{a}\right)$$

$$A_0 = \frac{1}{ab}\int_0^a f(x)dx \qquad A_n = \frac{2}{a\sinh\left(\frac{n\pi b}{a}\right)}\int_0^a f(x)\cos\left(\frac{n\pi x}{a}\right)dx$$