MA-240-A Midterm Exam Corrections

Jacob Sigman

3/31/22

Contents

Question 1

1. **False** - Use the Wronskian to determine if a S is linearly independent. If the Wronskian is 0, then S is linearly dependent. Since $f = 0 \in S$, this makes the Wronskian 0, since the whole column of the Wronskian is 0, therefore S is linearly dependent.
2. **False** - The leading term for a critically damped oscillator is te^{-t}, but for an overdamped oscillator the leading term is e^{-t+b} which will cause the oscillator to move in the other direction, making it's approach to 0 slower.
3. **False** - Solutions are not defined as points, but as curves.
4. **False**
5. **True**
6. **True** - Since the derivative of the function is 0 at that point, while it may touch the line, it'll never cross the line.
7. **False** - Some ODEs have solutions that can only be represented visually, so not every ODE would possess an exact solution.
8. **True** - A linear homogeneous ODE possesses at least one solution, that solution being 0 (the trivial solution).

Question 2

Define a bounded region in the xy-plane containing the point (x_0, y_0). If $f(x, y)$ and $\frac{\partial f}{\partial y}$ are continuous in the region, then there exists an interval I and a unique function $y(x) \in I$ that is a solution for this problem.

Question 3

Let L be the differential operator defined as

$$L = \sum_{n=0}^{k} a_n(x) D^{(n)}$$

This means that:

$$L(y) = g_1(x),\, g_2(x),\, \cdots,\, g_n(x)$$

if and only if the following is defined as the particular solution:

$$y_p = \sum_{j=1}^{k} y_{p_j}(x)$$

By the superposition principle, the result is:

$$L(y_p) = L\left(\sum_{j=1}^{k} y_{p_j}(x)\right) = \sum_{j=1}^{k} L\left(y_{p_j}(x)\right) = \sum_{j=1}^{k} g_j(x)$$

Question 4

Part 1

$$3(1+t^2)y' = 2ty(y^3-1)$$

$$3(1+t^2)\frac{dy}{dt} = 2t(y^4-y)$$

$$\frac{3}{y^4-y}dy = \frac{2t}{1+t^2}dt$$

Perform partial fraction decomposition on $\frac{3}{y^4-y}$.

$$\frac{3}{y^4-y} = \frac{3}{y(y^3-1)} = \frac{3}{y(y-1)(y^2+y+1)} = -\frac{3}{y} + \frac{1}{y-1} + \frac{Ay+B}{y^2+y+1}$$

$$-\frac{3}{y} + \frac{1}{y-1} + \frac{Ay+B}{y^2+y+1} = 3$$

$$-3(y-1)(y^2+y+1) + y(y^2+y+1) + (Ay+B)(y)(y-1) = 3$$

$$-3y^3 + 3 + y^3 + y^2 + y + Ay^3 - Ay^2 + By^2 - By = 3$$

$$Ay^3 - 3y^3 + y^3 = 0 \qquad A = 2$$

$$y - By = 0 \qquad B = 1$$

$$\frac{3}{y^4-y} = -\frac{3}{y} + \frac{1}{y-1} + \frac{2y+1}{y^2+y+1}$$

Substitute partial fraction decomposition into seperable equations.

$$\int \left[-\frac{3}{y} + \frac{1}{y-1} + \frac{2y+1}{y^2+y+1}\right] dy = \int \frac{2t}{1+t^2}dt$$

$$\int \left[-\frac{3}{y} + \frac{1}{y-1} + \frac{2y+1}{y^2+y+1}\right] dy = -3\ln|y| + \ln|y-1| + \ln|y^2+y+1|$$

$$\int \frac{2t}{1+t^2}dt = \ln|1+t^2| + c$$

$$-3\ln|y| + \ln|y-1| + \ln|y^2+y+1| = \ln|1+t^2| + c$$

$$\boxed{y = \frac{1}{\sqrt[3]{-e^c t^2 - e^c + 1}} \quad t \in \mathbb{R}}$$

Part 2

$$y^{(5)} - 9y^{(4)} - y^{(3)} - 8y'' - 90y' = -\sin(3x)$$

$$\text{Auxillary Equation: } m^5 - 9m^4 - m^3 - 81m^2 - 90m = 0$$

$$m(m-1)(m+10)(m-3i)(m+3i) = 0 \qquad m = 0,\, 1,\, -10,\, \pm\sqrt{3}i$$

$$y_c = c_1 + c_2e^x + c_3e^{-10x} + c_4\sin(3x) + c_5\cos(3x)$$

$$\text{Guess: } y_p = Ax\sin(3x) + Bx\cos(3x)$$

$$y_p' = -3Bx\sin(3x) + A\sin(3x) + 3Ax\cos(3x) + B\cos(3x) = (A - 3Bx)\sin(3x) + (3Ax + B)\cos(3x)$$

$$y_p'' = -3(3Ax+B)\sin(3x) - 3B\sin(3x) + 3(A-3Bx)\cos(3x) + 3A\cos(3x) = (-9Ax-6B)\sin(3x) + (6A-9Bx)\cos(3x)$$

$$y_p^{(3)} = -3(6A - 9Bx)\sin(3x) - 9A\sin(3x) + 3(-9Ax - 6B)\cos(3x) - 9B\cos(3x)$$

$$y_p^{(3)} = (-27A + 27Bx)\sin(3x) + (-27Ax - 27B)\cos(3x)$$

$$y_p^{(4)} = (81Ax + 81B)\sin(3x) + 27B\sin(3x) + (81Bx - 81A)\cos(3x) - 21A\cos(3x)$$

$$y_p^{(4)} = (81Ax + 108B)\sin(3x) + (81Bx - 108A)\cos(3x)$$

$$y_p^{(5)} = -3(81Bx - 108A)\sin(3x) + 81A\sin(3x) + 3(81Ax + 108B)\cos(3x) + 81B\cos(3x)$$

$$y_p^{(5)} = (405A - 243Bx)\sin(3x) + (243Ax + 405B)\cos(3x)$$

$$405A + 27A - 90A + 48B - 972B = -1 \qquad 342A - 924B = -1$$

$$405B + 27B - 90B - 48A + 972A = 0 \qquad 342B + 924A = 0$$

$$A = -\frac{342}{924}B \qquad B = \frac{77}{80895}$$

$$A = -\frac{342}{924}B \qquad A = -\frac{26334}{74744208}$$

$$\boxed{y = c_1 + c_2e^x + c_3e^{-10x} + c_4\sin(3x) + c_5\cos(3x) - \frac{26334x}{74744208}\sin(3x) + \frac{77x}{80895}\cos(3x) \quad x \in \mathbb{R}}$$

Part 3

$$y' = \frac{y}{e^{-y}\sin(2y) - (1+y)x}$$

$$\frac{dy}{dx} = \frac{y}{e^{-y}\sin(2y) - (1+y)x}$$

$$\frac{dx}{dy} = \frac{e^{-y}\sin(2y) - (1+y)x}{y} = \frac{-y-1}{y}x + \frac{e^{-y}}{y}\sin(2y)$$

$$\frac{dx}{dy} + \frac{y+1}{y}x = \frac{e^{-y}}{y}\sin(2y)$$

$$\mu = e^{\int \frac{y+1}{y}dy} = ye^y$$

$$\frac{d}{dy}\left[xye^y\right] = \frac{e^{-y}}{y}\sin(2y)\,ye^y = \sin(2y)$$

$$xye^y = -\frac{1}{2}\cos(2y) + c$$

$$\boxed{x = -\frac{1}{2ye^y}\cos(2y) + \frac{c}{ye^y} \quad y \in (0, \infty)}$$

Part 4

$$y' = \frac{1 + \ln(x) + \frac{y}{x}}{1 - \ln(x)}$$

$$\frac{dy}{dx}(1 - \ln(x)) = 1 + \ln(x) + \frac{y}{x}$$

$$(1 - \ln(x))dy = \left(1 + \ln(x) + \frac{y}{x}\right) dx$$

$$\left(1 + \ln(x) + \frac{y}{x}\right) dx - (1 - \ln(x))dy = 0$$

$$\text{Let } \mathbf{M} = 1 + \ln(x) + \frac{y}{x} \text{ and } \mathbf{N} = 1 - \ln(x)$$

$$\mathbf{M}_y = \mathbf{N}_x = -\frac{1}{x}$$

$$\int \mathbf{N}dy = y - y\ln|x| \qquad \int \mathbf{M}dx = -x - x\ln|x| + x - y\ln|x| = -x\ln|x| - y\ln|x|$$

$$y - y\ln|x| - x\ln|x| = c$$

$$y(1 - \ln|x|) - x\ln|x| = c$$

$$\boxed{y = \frac{c\,x\ln|x|}{1 - \ln|x|} \quad x \in (e, \infty)}$$

Part 5

$$y' = 5 - 3y + \frac{1}{2}e^{-3x}$$

$$y' + 3y = 5 + \frac{1}{2}e^{-3x}$$

$$\mu = e^{3x} \qquad \frac{d}{dx}\left[e^{3x}y\right] = 5e^{3x} + \frac{1}{2}$$

$$e^{3x}y = \frac{5}{3}e^{3x} + \frac{x}{2} + c$$

$$\boxed{y = \frac{5}{3} + \frac{x}{2}e^{-3x} + ce^{-3x} \quad x \in \mathbb{R}}$$

Part 6

$$x^3y^{(3)} - 6y = 0$$

$$\text{Auxillary Equation: } 1(m-2)(m-1)m - 6 = 0$$

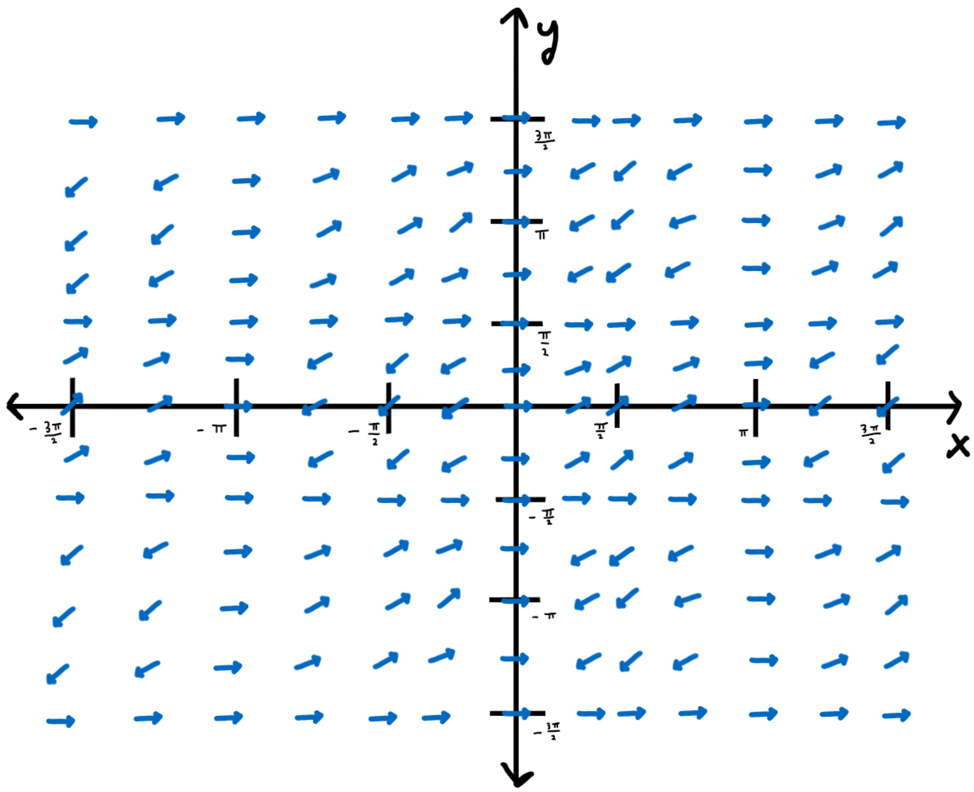
$$m^3 - 3m^2 + 2m - 6 = 0$$

$$(m-3)(m+2i)(m-2i) = 0 \qquad m = 3,\ \pm\sqrt{2}i$$

$$\boxed{y_c = c_1x^3 + c_2\cos\left(\sqrt{2}\ln|x|\right) + c_3\sin\left(\sqrt{2}\ln|x|\right) \quad x \in (0, \infty)}$$

Question 5

$$y' = \sin(x)\cos(y)$$



Question 6

$$m = 1 \text{ kg} \qquad F = kx \qquad k = 10$$

$$\frac{d^2x}{dt^2} + 8\frac{dx}{dt} + 10x = F(t)$$

$$\text{Auxillary Equation: } m^2 + 8m + 10 = 0 \qquad m = \frac{-8 \pm \sqrt{64 - 40}}{2} = -4 \pm \sqrt{6}$$

$$x_c = c_1 e^{\left(-4+\sqrt{6}\right)t} + c_2 e^{\left(-4-\sqrt{6}\right)t}$$

All terms of the complementary solution are transient.

Part (a)

$$\text{Guess: } x_p = A\sin(4t) + B\cos(4t)$$

$$x_p' = 4A\cos(4t) - 4B\sin(4t)$$

$$x_p'' = -16A\sin(4t) - 16B\cos(4t)$$

$$-16A - 32B + 10A = F_0$$

$$-16B + 32A + 10B = 0$$

$$A = \frac{6}{32}B \qquad -16\frac{6}{32}B - 32B - \frac{6}{32}B * 10 = F_0 \qquad B = -\frac{8F_0}{265} \qquad A = -\frac{3F_0}{530}$$

$$\boxed{x = c_1 e^{\left(-4+\sqrt{6}\right)t} + c_2 e^{\left(-4-\sqrt{6}\right)t} - \frac{3F_0}{530}\sin(4t) - \frac{8F_0}{265}\cos(4t) \quad t \in \mathbb{R}}$$

Only the complementary solution is transient.

Part (b)

$$\text{Guess: } x_p = Ae^{-4t}\sin(4t) + Be^{-4t}\cos(4t)$$

$$x_p' = 4Ae^{-4t}\cos(4t) - 4Ae^{-4t}\sin(4t) - 4Be^{-4t}\sin(4t) - 4Be^{-4t}\cos(4t)$$

$$x_p'' = \frac{d}{dt}\left[4Ae^{-4t}\cos(4t) - 4Ae^{-4t}\sin(4t) - 4Be^{-4t}\sin(4t) - 4Be^{-4t}\cos(4t)\right]$$

$$x_p'' = \frac{d}{dt}\left[-4e^{-4t}((B+A)\sin(4t) + (B-A)\cos(4t))\right]$$

$$x_p'' = 16e^{-4t}((B+A)\sin(4t) + (B-A)\cos(4t)) - 4e^{-4t}(4(B+A)\cos(4t) - 4(B-A)\sin(4t))$$

$$x_p'' = 32Be^{-4t}\sin(4t) - 32Ae^{-4t}\cos(4t)$$

$$32B - 32A - 32B + 10A = F_0$$

$$-32A + 32A - 32B + 10B = 0$$

$$A = -\frac{F_0}{22} \qquad B = 0$$

$$\boxed{x = c_1 e^{\left(-4+\sqrt{6}\right)t} + c_2 e^{\left(-4-\sqrt{6}\right)t} - \frac{F_0}{22}e^{-4t}\sin(4t) \quad t \in \mathbb{R}}$$

Both the complementary and particular solutions are transient.

Part (c)

$$\text{Guess: } x_p = Ae^{-4t}\sin\left(\sqrt{10}t\right) + Be^{-4t}\cos\left(\sqrt{10}t\right)$$

$$x_p' = \sqrt{10}Ae^{-4t}\cos\left(\sqrt{10}t\right) - 4Ae^{-4t}\sin\left(\sqrt{10}t\right) - \sqrt{10}Be^{-4t}\sin\left(\sqrt{10}t\right) - 4Be^{-4t}\cos\left(\sqrt{10}t\right)$$

$$x_p'' = \frac{d}{dt}\left[\sqrt{10}Ae^{-4t}\cos\left(\sqrt{10}t\right) - 4Ae^{-4t}\sin\left(\sqrt{10}t\right) - \sqrt{10}Be^{-4t}\sin\left(\sqrt{10}t\right) - 4Be^{-4t}\cos\left(\sqrt{10}t\right)\right]$$

$$x_p'' = \frac{d}{dt}\left[-e^{-4t}\left(\left(\sqrt{10}B + 4A\right)\sin\left(\sqrt{10}t\right) + \left(4B - \sqrt{10}A\right)\cos\left(\sqrt{10}t\right)\right)\right]$$

$$\begin{aligned} x_p'' = 4e^{-4t}&\left(\left(\sqrt{10}B + 4A\right)\sin\left(\sqrt{10}t\right) + \left(4B - \sqrt{10}A\right)\cos\left(\sqrt{10}t\right)\right) \\ &- e^{-4t}\left(\sqrt{10}\left(\sqrt{10}B + 4A\right)\cos\left(\sqrt{10}t\right) - \sqrt{10}\left(4B - \sqrt{10}A\right)\sin\left(\sqrt{10}t\right)\right) \end{aligned}$$

$$x_p'' = \left(8\sqrt{10}B + 6A\right)e^{-4t}\sin\left(\sqrt{10}t\right) + \left(6B - 8\sqrt{10}A\right)e^{-4t}\cos\left(\sqrt{10}t\right)$$

$$8\sqrt{10}B + 6A - 32A - 8\sqrt{10}B + 10A = F_0$$

$$6B - 8\sqrt{10}A + 8\sqrt{10}A - 32B = 0$$

$$A = -\frac{F_0}{16} \qquad B = 0$$

$$\boxed{x = c_1 e^{\left(-4+\sqrt{6}\right)t} + c_2 e^{\left(-4-\sqrt{6}\right)t} - \frac{F_0}{16}e^{-4t}\sin\left(\sqrt{10}t\right) \quad t \in \mathbb{R}}$$

Both the complementary and particular solutions are transient.

Question 7

Part (a)

$$y' = \frac{y}{x}$$
$$\frac{1}{y}dy = \frac{1}{x}dx$$
$$\int \frac{1}{y}dy = \int \frac{1}{x}dx$$
$$\ln|y| = \ln|x| + c$$
$$\boxed{y = c\,x \quad x \in \mathbb{R}}$$

Part (b)

$$-ydx + xdy = 0$$
$$-\frac{y}{x^2}dx + \frac{1}{x}dy = 0$$
$$\text{Let } \mathbf{M} = -\frac{y}{x^2} \text{ and } \mathbf{N} = \frac{1}{x}$$
$$\mathbf{M}_y = \mathbf{N}_x = -\frac{1}{x^2}$$
$$\int \mathbf{N}dy = \frac{y}{x} \qquad \int \mathbf{M}dx = \frac{y}{x}$$
$$\frac{y}{x} = c$$
$$\boxed{y = c\,x \quad x \in (0, \infty)}$$

Part (c)

$$-ydx + xdy = 0$$
$$-\frac{y}{x^2+y^2}dx + \frac{x}{x^2+y^2}dy = 0$$
$$\text{Let } \mathbf{M} = -\frac{y}{x^2+y^2} \text{ and } \mathbf{N} = \frac{x}{x^2+y^2}$$
$$\mathbf{M}_y = \mathbf{N}_x = -\frac{x^2-y^2}{(x^2+y^2)^2}$$
$$\int \mathbf{N}dy = \arctan\left(\frac{y}{x}\right) + g(y)$$
$$\frac{\partial}{\partial x}\left[\arctan\left(\frac{y}{x}\right) + g(y)\right] = -\frac{y}{x^2+y^2} + g'(y)$$
$$g'(y) = 0 \qquad g(y) = c$$
$$\arctan\left(\frac{y}{x}\right) = c$$

An approximation for arctan can be used as follows:

$$\frac{y}{x} = c \qquad c \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$$
$$\boxed{y = c\,x \quad x \in \mathbb{R}}$$

Part (d)

The answers to (a), (b), and (c) are the same.

Question 8

Part (a)

Let some function f_2 be a constant multiple k of some function f_1. The Wronskian of the two functions is as follows:

$$W(f_1, f_2) = \begin{vmatrix} f_1 & f_2 \\ f_1' & f_2' \end{vmatrix} = \begin{vmatrix} f_1 & kf_1 \\ f_1' & kf_1' \end{vmatrix} = k\, f_1 f_1' - k f_1 f_1' = 0$$

Since the Wronskian of the two functions is 0, the functions are linearly dependent.

Part (b)

$$\frac{d}{dx}\left(|x|^3\right) = \frac{d}{dx}\left(|x|\, x^2\right) = 2x\,|x| + \frac{x^3}{|x|} = \frac{3x^3}{|x|} = 3x\,|x|$$

$$W\left(x^3, |x|^3\right) = \begin{vmatrix} x^3 & |x|^3 \\ 3x^2 & 3x\,|x| \end{vmatrix} = 3x^2|x|^3 - 3x^4|x| = \boxed{0}$$

Part (c)