Quiz 11 Solutions: Markov Chains

Consider a Markov chain with states 1, 2, 3, and state transition matrix

$$\mathbf{P} = \begin{bmatrix} 0.25 & 0.25 & 0.5 \\ 0 & 0.4 & 0.6 \\ 0 & 0.8 & 0.2 \end{bmatrix}.$$

Problem 11.1 Given that the system starts at time 0 in the state $X_0 = 3$, what is the probability that $X_1 = 2$?

Solution:

The answer is $P_{32} = 0.8$.

Problem 11.2 Given that the system starts at time 0 in state $X_0 = 1$, what is the probability that $X_1 = 3$ and $X_2 = 2$?

Solution:

This requires we go from state 1 to state 3, and then to state 2. This is $P_{13}P_{32} = 0.5 \cdot 0.8 = 0.4$.

Problem 11.3 Which states are transient? choose from:

(a) Only state 1.

(b) Only state 2.

(c) Only state 3.

(d) Both states 1 and 2.

Solution:

The answer is (a), as only state 1 is transient.

Problem 11.4 What is the limiting probability as time increases to infinity of being in state 2? Enter your answer up to two decimal places.

Solution:

Since state 1 is transient, it has no probability in the limit, so $\pi_1 = 0$. Probability balance between states 2 and 3 requires:

$$0.6\pi_2 = 0.8\pi_3.$$

Alternatively, since you may not have read about probability balance, the Markov chain steady state equations are

$$\mathbf{P}^T \underline{\pi} = \underline{\pi}$$

Using only the middle equation yields:

$$0.25\pi_1 + 0.4\pi_2 + 0.8\pi_3 = \pi_2.$$

Substituting for $\pi_1 = 0$, we get the same balance equation $0.6\pi_2 = 0.8\pi_3$.

Together with the normalization equation $\pi_1 + \pi_2 + \pi_3 = \pi_2 + \pi_3 = 1$, this yields $\pi_2 = \frac{4}{7}, \pi_3 = \frac{3}{7}$. In decimals, $\pi_2 \approx 0.57$.

Problem 11.5 What is the limiting probability as time increases to infinity of being in state 1? Enter your answer up to two decimal places.

Solution:

It is a transient state. The answer is 0.