Quiz 9 Solutions: Estimation and Sums of Random Variables

Problem 9.1 If $\hat{x}_{\text{MMSE}}(y) = ay + b$, then $\hat{x}_{\text{LLSE}}(y) = ay + b$.

Solution:

True. The MMSE estimator is the optimal estimator while the LLSE estimator is the optimal linear estimator. If the MMSE estimator is $ay + b$, then we know that the optimal estimator is linear and we also know its coefficients a and b. Since these are the best possible coefficients for a linear estimator, they will be the coefficients for the LLSE estimator as well.

Problem 9.2 If $\rho_{X,Y} = 0$, then $\mathsf{MSE}_{\text{LLSE}} = \mathbb{E}[(\hat{x}_{\text{LLSE}}(Y) - X)^2] = \mathsf{Var}[X]$.

Solution:

True. For the LLSE estimator, we have that

$$\mathsf{MSE}_{\text{LLSE}} = \mathbb{E}[(\hat{x}_{\text{LLSE}}(Y) - X)^2] = (1 - \rho_{X,Y}^2)\mathsf{Var}[X] \ .$$

Problem 9.3 Let $X_1, \dots, X_n$ be i.i.d. random variables with mean $\mathbb{E}[X_i] = \mu$. Then, the value of sample mean $M_n = \dfrac{1}{n}\displaystyle\sum_{i=1}^{n} X_i$ always gets closer to the true mean as we include additional samples: for all $k < n$, $|M_n - \mu| < |M_k - \mu|$.

Solution:

False. While the variance decreases $\mathsf{Var}[M_k] = \frac{\mathsf{Var}[X]}{k} > \frac{\mathsf{Var}[X]}{n} = \mathsf{Var}[M_n]$, this does not guarantee that any particular realization satisfies $M_k < M_n$.

Problem 9.4 X and Y are jointly Gaussian with $\mathbb{E}[X] = 1$, $\mathbb{E}[Y] = 2$, $\mathsf{Var}[X] = 1$, $\mathsf{Var}[Y] = 4$, $\rho_{X,Y} = -\frac{1}{2}$. In this scenario, the MMSE estimator is of the form $\hat{x}_{\text{MMSE}}(y) = ay + \frac{3}{2}$. Determine a and enter the answer, up to three decimal places.

Solution:

Since X and Y are jointly Gaussian,

$$\begin{aligned}
\hat{x}_{\text{MMSE}}(y) &= \mathbb{E}[X] + \rho_{X,Y}\frac{\sigma_X}{\sigma_Y}(y - \mathbb{E}[Y]) \\
&= 1 - \frac{1}{2} \cdot \frac{1}{2}(y - 2) \\
&= -\frac{y}{4} + \frac{3}{2}
\end{aligned}$$

Problem 9.5 X and Y are jointly Gaussian with $\mathbb{E}[X] = 1$, $\mathbb{E}[Y] = 2$, $\mathsf{Var}[X] = 1$, $\mathsf{Var}[Y] = 4$, $\rho_{X,Y} = -\frac{1}{2}$. Determine the mean-squared error (MSE) of the MMSE estimator and enter the answer, up to three decimal places.

Solution:

Since X and Y are jointly Gaussian,

$$\begin{aligned}\text{MSE}_{\text{MMSE}} &= \mathsf{Var}[X](1-\rho_{X,Y}^2) \\ &= 1\cdot\Big(1-\Big(-\frac{1}{2}\Big)^2\Big) \\ &= \frac{3}{4}\end{aligned}$$

Problem 9.6 $X_1, \ldots, X_{20}$ are i.i.d. Uniform$(0,2)$ random variables. Let $M_{20} = \frac{1}{20}\sum_{i=1}^{20} X_i$ be the sample mean. Determine the variance of M_{20} and enter the answer, up to three decimal places.

Solution:

We know that, for i.i.d. random variables, $\mathsf{Var}[M_n] = \dfrac{\mathsf{Var}[X]}{n}$.

$$\mathsf{Var}[M_{20}] = \frac{\mathsf{Var}[X]}{20} = \frac{\frac{(2-0)^2}{12}}{20} = \frac{\frac{1}{3}}{20} = \frac{1}{60} \approx 0.017$$

Problem 9.7 $X_1, \ldots, X_{20}$ are i.i.d. Uniform$(0,2)$ random variables. Let $M_{20} = \frac{1}{20}\sum_{i=1}^{20} X_i$ be the sample mean. Using the Central Limit Theorem approximation, estimate the probability that M_{20} deviates from its mean $\mu = 1$ by more than 0.1, $\mathbb{P}\big[|M_{20}-\mu| > 0.1]$ and enter the answer, up to three decimal places.

Solution:

We know that, using the Central Limit Theorem approximation, $\mathbb{P}\big[|M_n - \mu| > \epsilon] \approx 2Q\Big(\dfrac{\epsilon\sqrt{n}}{\sigma}\Big)$ where $\sigma^2 = \mathsf{Var}[X]$ and $\mu = \mathbb{E}[X]$. Here, we know that $\frac{(2-0)^2}{12} = \frac{1}{3}$.

$$\mathbb{P}\big[|M_{20}-\mu| > 0.1] \approx 2Q\left(\frac{0.1\sqrt{20}}{\sqrt{1/3}}\right) = 2Q(0.7746) \approx 0.439$$