Let L be a line in $\mathbb{P}^3_k$. We want to consider the blowup $\mathrm{Bl}_L(\mathbb{P}^3)$. We have

$$\mathrm{Bl}_L(\mathbb{P}^3) = \{(P, \Pi) \in \mathbb{P}^3 \times \mathbb{P}^{3*} : P \in \Pi, L \subseteq \Pi\}$$

with map $\pi_L : \mathrm{Bl}_L \to \mathbb{P}^3$ given by $\pi_L(P, \Pi) = P$. Then

$$E_L = \pi_L^{-1}(L) = \{(P, \Pi) : P \in L \subseteq \Pi\} \cong \mathbb{P}^1 \times \mathbb{P}^1.$$

Now consider $\mathbb{P}^d \subseteq \mathbb{P}^n$. We have

$$\mathrm{Bl}_{\mathbb{P}^d}(\mathbb{P}^n) = \{(P, \mathbb{P}^{d+1}) \in \mathbb{P}^n \times \mathrm{PGr}(d+1, n) : P \in \mathbb{P}^{d+1}, \mathbb{P}^d \subseteq \mathbb{P}^{d+1}\}.$$

We can see that this doesn't do anything when $d = n - 1$. For example, when $d = 1$ and $n = 2$,

$$\mathrm{Bl}_{\mathbb{P}^1}(\mathbb{P}^2) = \{(P, \mathbb{P}^2) : P \in \mathbb{P}^2, \mathbb{P}^1 \subseteq \mathbb{P}^2\} = \mathbb{P}^2$$

because there is only one $\mathbb{P}^2$ containing $\mathbb{P}^1$ in $\mathbb{P}^2$.

In the case $Z = \mathbb{P}^d \subseteq \mathbb{P}^n$,

$$E_Z = \{(P, \mathbb{P}^{d+1}) : P \in Z \subseteq \mathbb{P}^{d+1}\} = Z \times \{\mathbb{P}^{d+1} \subseteq \mathbb{P}^n : Z \subseteq \mathbb{P}^{d+1}\}.$$

If you have a $d+1$-dimensional subspace $\widetilde{Z}$ in k^{n+1}, what is the space of $d+2$-dimensional spaces containing $\widetilde{Z}$? It is $\mathbb{P}(k^{n+1}/\widetilde{Z}) \cong \mathbb{P}_k^{n-d-1}$. In general, blowing up a d-dimensional variety Z in $\mathbb{P}^n$ yields the exceptional divisor

$$E_Z = Z \times \mathbb{P}^{n-d-1}.$$

Question: is this also true for nonlinear Z's?

1. Note when $d = 1$ and $n = 3$, we get the match $n - d - 1 = 1$.
2. When $d = 0$ and $n = 2$, we get $n - d - 1 = 1$, which matches blowing up $\mathbb{P}^2$ at a point: the exceptional locus is $\{\mathtt{pt}\} \times \mathbb{P}^1 = \mathbb{P}^1$.
3. When $d = 1$ and $n = 4$, we get $n - d - 1 = 2$. So the exceptional locus is $\mathbb{P}^1 \times \mathbb{P}^2$.
4. When $d = 2$ and $n = 5$, we get $n - d - 1 = 2$, so the exceptional locus is $\mathbb{P}^2 \times \mathbb{P}^2$.

Afterword: We have $\dim \mathrm{PGr}(d+1, n) = (d+2)(n-d-1)$. (Remember: $\dim \mathbb{P}(\bigwedge^d k^n) = \binom{n}{d} - 1$, but the Grassmannian is a *subvariety* of this space! Specifically, the subvariety given by *simple wedges*.) And furthermore $\dim \mathrm{PGr}(d, n) = (d+1)(n-d)$.

Example 1. The Chow ring $A(\mathbb{P}^2 \times \mathbb{P}^2)$ is equal to $A(\mathbb{P}^2) \otimes A(\mathbb{P}^2)$. We have:

- $A^0(\mathbb{P}^2 \times \mathbb{P}^2)$ is generated by $\mathbb{P}^2 \otimes \mathbb{P}^2$.
- $A^1(\mathbb{P}^2 \times \mathbb{P}^2)$ is generated by $\ell_1 \otimes \mathbb{P}^2$ and $\mathbb{P}^2 \otimes \ell_2$ (where ℓ_i generates $A^1(\mathbb{P}^2)_i$).
- A^2 is generated by $p_1 \otimes \mathbb{P}^2$, $\ell_1 \otimes \ell_2$, and $\mathbb{P}^2 \otimes p_2$.

- A^3 is generated by $p_1 \otimes \ell_2$ and $\ell_1 \otimes p_2$.
- A^4 is generated by $p_1 \otimes p_2$.

We can simplify these generators by rewriting them as follows:

- $\mathbb{P}^2 \otimes \mathbb{P}^2 = 1$
- $\ell_1 \otimes \mathbb{P}^2 = \ell_1$
- $\mathbb{P}^2 \otimes \ell_2 = \ell_2$
- $p_1 \otimes \mathbb{P}^2 = \ell_1^2$
- $\ell_1 \otimes \ell_2 = \ell_1\ell_2$
- $\mathbb{P}^2 \otimes p_2 = \ell_2^2$
- $p_1 \otimes \ell_2 = \ell_1^2\ell_2$
- $\ell_1 \otimes p_2 = \ell_1\ell_2^2$
- $p_2 \otimes p_2 = \ell_1^2\ell_2^2$

Then we see $A = \mathbb{Z}[\ell_1, \ell_2]/(\ell_1^3, \ell_2^3)$. Blowing up $\mathbb{P}^5$ at a plane $\Pi \cong \mathbb{P}^2$ yields an exceptional divisor $E_\Pi \cong \mathbb{P}^2 \times \mathbb{P}^2$. As we can see, $A(\mathbb{P}^5) = \mathbb{Z}[h]/(h^6)$ and $A(E_\Pi) = \mathbb{Z}[\ell_1, \ell_2]/(\ell_1^3, \ell_2^3)$.

Let $\zeta \in A^1(E)$ denote the first Chern class of $\mathcal{O}_{\mathbb{P}\mathcal{N}}(1)$. What is ζ?? I don't know what that is.

Consider the diagram

$$\begin{tikzcd} \mathrm{Bl}_\Pi(\mathbb{P}^5) \arrow[r, "\pi"] & \mathbb{P}^5 \\ E_\Pi \arrow[u, hook, "j"] \arrow[r, "\pi|_{E_\Pi}"] & \Pi \arrow[u, hook, "i"'] \end{tikzcd} .$$

We can construct the Chow ring $A(\mathrm{Bl}_\Pi(\mathbb{P}^5))$ as being generated by $\pi^*(A(\mathbb{P}^5))$ and $j_*(A(E_\Pi))$ under the following rules:

- $\pi^*\alpha \cdot \pi^*\beta = \pi^*(\alpha\beta)$
- $\pi^*\alpha \cdot j_*\gamma = j_*(\gamma \cdot \pi|_{E_\Pi}^* i^*\alpha)$
- $j_*\gamma \cdot j_*\delta = -j_*(\gamma\delta\zeta)$

So we can identify the generators of each A^i as follows:

- A^0: 1
- A^1: π^*h, j_*1

- A^2: $\pi^* h^2$, $j_* \ell_1$, $j_* \ell_2$
- A^3: $\pi^* h^3$, $j_* \ell_1^2$, $j_* \ell_1 \ell_2$, $j_* \ell_2$
- A^4: $\pi^* h^4$, $j_* \ell_1^2 \ell_2$, $j_* \ell_1 \ell_2^2$
- A^5: p (every point is rationally equivalent to every other I think?)

Note that for an affine scheme Y and subscheme $A \subseteq Y$ with ideal $(f_1, \ldots, f_k)$, the blowup $\mathrm{Bl}_A Y \to Y$ of Y along A is the closure in $Y \times \mathbb{P}^{k-1}$ of the graph of the map $Y \setminus A \to \mathbb{P}^{k-1}$ given by $[f_1, \ldots, f_k]$. We can globalize this: let Y be any scheme and $A \subseteq Y$ a subscheme. If $\mathcal{L}$ is a line bundle on Y and $\sigma_1, \ldots, \sigma_k \in H^0(\mathcal{L})$ sections generating the subsheaf $\mathcal{I}_{A/Y} \otimes \mathcal{L}$, then the closure of the graph of the map $Y \setminus A \to \mathbb{P}^{k-1}$ given by $[f_1, \ldots, f_k]$ is the blowup $\mathrm{Bl}_A Y \to Y$ along A.