Let

$$p(x,y,z) = x^3a(3,0,0) + x^2ya(2,1,0) + x^2za(2,0,1) + xy^2a(1,2,0) \\ + xyza(1,1,1) + xz^2a(1,0,2) + y^3a(0,3,0) + y^2za(0,2,1) + yz^2a(0,1,2)$$

and let

$$q(x,y,z) = x^3b(3,0,0) + x^2yb(2,1,0) + x^2zb(2,0,1) + xy^2b(1,2,0) \\ + xyzb(1,1,1) + xz^2b(1,0,2) + y^3b(0,3,0) + y^2zb(0,2,1) + yz^2b(0,1,2).$$

These are two cubics that go through $(0,0,1) \in \mathbb{P}^2$. Then a pencil of cubics can be given by the parametrization $jp(x,y,z) + kq(x,y,z)$ for $(j,k) \in \mathbb{P}^1$. The tangent line of this (j,k) linear combination at the point $(0,0,1)$ is $(j * p_x[0,0,1] + k * q_x[0,0,1]) * x + (j * p_y[0,0,1] + k * q_y[0,0,1]) * y + (j * p_z[0,0,1] + k * q_z[0,0,1]) * z$.

Note that $jp_x(0,0,1) + kq_x(0,0,1) = ja(1,0,2) + kb(1,0,2)$ and $jp_y(0,0,1) + kq_y(0,0,1) = ja(0,1,2) + kb(0,1,2)$. Also, $jp_z(0,0,1) + kq_z(0,0,1) = 0$. So this line gives us the equation $y = -x\dfrac{ja(1,0,2)+kb(1,0,2)}{ja(0,1,2)+kb(0,1,2)}$. Plugging this is for the equation for the curve gives us $jp\left(x, -\dfrac{x(ja(1,0,2)+kb(1,0,2))}{ja(0,1,2)+kb(0,1,2)}, 1\right) + kq\left(x, -\dfrac{x(ja(1,0,2)+kb(1,0,2))}{ja(0,1,2)+kb(0,1,2)}, 1\right)$. This is a polynomial in the variable x, and finding the roots of this polynomial tells (other than the multiplicity-two $x = 0$) tells us the reflection point of the Bertini involution centered at $(0,0,1)$.

The root the quotient of

$$\begin{aligned} A = -((ja(0,1,2) + kb(0,1,2))(-j^2ka(1,0,2)a(1,1,1)b(0,1,2) \\ +2j^2ka(0,1,2)a(2,0,1)b(0,1,2) + j^2ka(1,0,2)^2b(0,2,1) + 2j^2ka(0,2,1)a(1,0,2)b(1,0,2) \\ -j^2ka(0,1,2)a(1,1,1)b(1,0,2) - j^2ka(0,1,2)a(1,0,2)b(1,1,1) + j^2ka(0,1,2)^2b(2,0,1) \\ +jk^2a(2,0,1)b(0,1,2)^2 + jk^2a(0,2,1)b(1,0,2)^2 \\ -jk^2a(1,1,1)b(0,1,2)b(1,0,2) + 2jk^2a(1,0,2)b(0,2,1)b(1,0,2) \\ -jk^2a(1,0,2)b(0,1,2)b(1,1,1) - jk^2a(0,1,2)b(1,0,2)b(1,1,1) + 2jk^2a(0,1,2)b(0,1,2)b(2,0,1) \\ +j^3a(0,2,1)a(1,0,2)^2 - j^3a(0,1,2)a(1,0,2)a(1,1,1) \\ +j^3a(0,1,2)^2a(2,0,1) + k^3b(0,2,1)b(1,0,2)^2 \\ -k^3b(0,1,2)b(1,0,2)b(1,1,1) + k^3b(0,1,2)^2b(2,0,1) \end{aligned}$$

and

$$
\begin{aligned}
B = {} & a(0,1,2)a(1,0,2)^2a(1,2,0)j^4 - a(0,1,2)^2a(1,0,2)a(2,1,0)j^4 + a(0,1,2)^3a(3,0,0)j^4 \\
& + ka(1,0,2)^2a(1,2,0)b(0,1,2)j^3 - 2ka(0,1,2)a(1,0,2)a(2,1,0)b(0,1,2)j^3 \\
& + 3ka(0,1,2)^2a(3,0,0)b(0,1,2)j^3 - ka(1,0,2)^3b(0,3,0)j^3 - 3ka(0,3,0)a(1,0,2)^2b(1,0,2)j^3 \\
& + 2ka(0,1,2)a(1,0,2)a(1,2,0)b(1,0,2)j^3 - ka(0,1,2)^2a(2,1,0)b(1,0,2)j^3 \\
& + ka(0,1,2)a(1,0,2)^2b(1,2,0)j^3 - ka(0,1,2)^2a(1,0,2)b(2,1,0)j^3 \\
& + ka(0,1,2)^3b(3,0,0)j^3 - k^2a(1,0,2)a(2,1,0)b(0,1,2)^2j^2 \\
& + 3k^2a(0,1,2)a(3,0,0)b(0,1,2)^2j^2 - 3k^2a(0,3,0)a(1,0,2)b(1,0,2)^2j^2 \\
& + k^2a(0,1,2)a(1,2,0)b(1,0,2)^2j^2 + 2k^2a(1,0,2)a(1,2,0)b(0,1,2)b(1,0,2)j^2 \\
& - 2k^2a(0,1,2)a(2,1,0)b(0,1,2)b(1,0,2)j^2 - 3k^2a(1,0,2)^2b(0,3,0)b(1,0,2)j^2 \\
& + k^2a(1,0,2)^2b(0,1,2)b(1,2,0)j^2 \\
& + 2k^2a(0,1,2)a(1,0,2)b(1,0,2)b(1,2,0)j^2 - 2k^2a(0,1,2)a(1,0,2)b(0,1,2)b(2,1,0)j^2 \\
& - k^2a(0,1,2)^2b(1,0,2)b(2,1,0)j^2 + 3k^2a(0,1,2)^2b(0,1,2)b(3,0,0)j^2 \\
& + k^3a(3,0,0)b(0,1,2)^3j - k^3a(0,3,0)b(1,0,2)^3j + k^3a(1,2,0)b(0,1,2)b(1,0,2)^2j \\
& - 3k^3a(1,0,2)b(0,3,0)b(1,0,2)^2j - k^3a(2,1,0)b(0,1,2)^2b(1,0,2)j \\
& + k^3a(0,1,2)b(1,0,2)^2b(1,2,0)j + 2k^3a(1,0,2)b(0,1,2)b(1,0,2)b(1,2,0)j \\
& - k^3a(1,0,2)b(0,1,2)^2b(2,1,0)j - 2k^3a(0,1,2)b(0,1,2)b(1,0,2)b(2,1,0)j \\
& + 3k^3a(0,1,2)b(0,1,2)^2b(3,0,0)j - j^4a(0,3,0)a(1,0,2)^3 \\
& - k^4b(0,3,0)b(1,0,2)^3 + k^4b(0,1,2)b(1,0,2)^2b(1,2,0) \\
& - k^4b(0,1,2)^2b(1,0,2)b(2,1,0) + k^4b(0,1,2)^3b(3,0,0)
\end{aligned}
$$

Thus the root is $x = A/B$, and so $y = -\dfrac{A(ja(1,0,2)+kb(1,0,2))}{B(ja(0,1,2)+kb(0,1,2))}$. Thus the reflection point is at $(A(ja(0,1,2)+kb(0,1,2)), -A(ja(1,0,2)+kb(1,0,2)), B(ja(0,1,2)+kb(0,1,2)))$ and is parametrized by (j,k). These three parametrization functions written explicitly are:

$$
\begin{aligned}
x = {} & -(ja(0,1,2)+kb(0,1,2))^2(-j^2ka(1,0,2)a(1,1,1)b(0,1,2) + 2j^2ka(0,1,2)a(2,0,1)b(0,1,2) \\
& + j^2ka(1,0,2)^2b(0,2,1) + 2j^2ka(0,2,1)a(1,0,2)b(1,0,2) \\
& - j^2ka(0,1,2)a(1,1,1)b(1,0,2) - j^2ka(0,1,2)a(1,0,2)b(1,1,1) \\
& + j^2ka(0,1,2)^2b(2,0,1) + jk^2a(2,0,1)b(0,1,2)^2 + jk^2a(0,2,1)b(1,0,2)^2 \\
& - jk^2a(1,1,1)b(0,1,2)b(1,0,2) + 2jk^2a(1,0,2)b(0,2,1)b(1,0,2) \\
& - jk^2a(1,0,2)b(0,1,2)b(1,1,1) - jk^2a(0,1,2)b(1,0,2)b(1,1,1) \\
& + 2jk^2a(0,1,2)b(0,1,2)b(2,0,1) + j^3a(0,2,1)a(1,0,2)^2 \\
& - j^3a(0,1,2)a(1,0,2)a(1,1,1) + j^3a(0,1,2)^2a(2,0,1) \\
& + k^3b(0,2,1)b(1,0,2)^2 - k^3b(0,1,2)b(1,0,2)b(1,1,1) \\
& + k^3b(0,1,2)^2b(2,0,1))
\end{aligned}
$$

$$\begin{aligned}
y = (ja(0,1,2)+kb(0,1,2))(ja(1,0,2)+kb(1,0,2))(&-j^2ka(1,0,2)a(1,1,1)b(0,1,2) \\
&+2j^2ka(0,1,2)a(2,0,1)b(0,1,2)+j^2ka(1,0,2)^2b(0,2,1) \\
&+2j^2ka(0,2,1)a(1,0,2)b(1,0,2)-j^2ka(0,1,2)a(1,1,1)b(1,0,2) \\
&-j^2ka(0,1,2)a(1,0,2)b(1,1,1)+j^2ka(0,1,2)^2b(2,0,1)+jk^2a(2,0,1)b(0,1,2)^2 \\
&+jk^2a(0,2,1)b(1,0,2)^2-jk^2a(1,1,1)b(0,1,2)b(1,0,2) \\
&+2jk^2a(1,0,2)b(0,2,1)b(1,0,2)-jk^2a(1,0,2)b(0,1,2)b(1,1,1) \\
&-jk^2a(0,1,2)b(1,0,2)b(1,1,1)+2jk^2a(0,1,2)b(0,1,2)b(2,0,1) \\
&+j^3a(0,2,1)a(1,0,2)^2-j^3a(0,1,2)a(1,0,2)a(1,1,1) \\
&+j^3a(0,1,2)^2a(2,0,1)+k^3b(0,2,1)b(1,0,2)^2-k^3b(0,1,2)b(1,0,2)b(1,1,1) \\
&+k^3b(0,1,2)^2b(2,0,1))
\end{aligned}$$

and

$$\begin{aligned}
z = (ja(0,1,2)+kb(0,1,2))(&a(0,1,2)a(1,0,2)^2a(1,2,0)j^4 \\
&-a(0,1,2)^2a(1,0,2)a(2,1,0)j^4+a(0,1,2)^3a(3,0,0)j^4 \\
&+ka(1,0,2)^2a(1,2,0)b(0,1,2)j^3-2ka(0,1,2)a(1,0,2)a(2,1,0)b(0,1,2)j^3 \\
&+3ka(0,1,2)^2a(3,0,0)b(0,1,2)j^3-ka(1,0,2)^3b(0,3,0)j^3 \\
&-3ka(0,3,0)a(1,0,2)^2b(1,0,2)j^3+2ka(0,1,2)a(1,0,2)a(1,2,0)b(1,0,2)j^3 \\
&-ka(0,1,2)^2a(2,1,0)b(1,0,2)j^3+ka(0,1,2)a(1,0,2)^2b(1,2,0)j^3 \\
&-ka(0,1,2)^2a(1,0,2)b(2,1,0)j^3+ka(0,1,2)^3b(3,0,0)j^3 \\
&-k^2a(1,0,2)a(2,1,0)b(0,1,2)^2j^2+3k^2a(0,1,2)a(3,0,0)b(0,1,2)^2j^2 \\
&-3k^2a(0,3,0)a(1,0,2)b(1,0,2)^2j^2+k^2a(0,1,2)a(1,2,0)b(1,0,2)^2j^2 \\
&+2k^2a(1,0,2)a(1,2,0)b(0,1,2)b(1,0,2)j^2 \\
&-2k^2a(0,1,2)a(2,1,0)b(0,1,2)b(1,0,2)j^2-3k^2a(1,0,2)^2b(0,3,0)b(1,0,2)j^2 \\
&+k^2a(1,0,2)^2b(0,1,2)b(1,2,0)j^2+2k^2a(0,1,2)a(1,0,2)b(1,0,2)b(1,2,0)j^2 \\
&-2k^2a(0,1,2)a(1,0,2)b(0,1,2)b(2,1,0)j^2 \\
&-k^2a(0,1,2)^2b(1,0,2)b(2,1,0)j^2+3k^2a(0,1,2)^2b(0,1,2)b(3,0,0)j^2 \\
&+k^3a(3,0,0)b(0,1,2)^3j-k^3a(0,3,0)b(1,0,2)^3j \\
&+k^3a(1,2,0)b(0,1,2)b(1,0,2)^2j-3k^3a(1,0,2)b(0,3,0)b(1,0,2)^2j \\
&-k^3a(2,1,0)b(0,1,2)^2b(1,0,2)j+k^3a(0,1,2)b(1,0,2)^2b(1,2,0)j \\
&+2k^3a(1,0,2)b(0,1,2)b(1,0,2)b(1,2,0)j-k^3a(1,0,2)b(0,1,2)^2b(2,1,0)j \\
&-2k^3a(0,1,2)b(0,1,2)b(1,0,2)b(2,1,0)j+3k^3a(0,1,2)b(0,1,2)^2b(3,0,0)j \\
&-j^4a(0,3,0)a(1,0,2)^3-k^4b(0,3,0)b(1,0,2)^3+k^4b(0,1,2)b(1,0,2)^2b(1,2,0) \\
&-k^4b(0,1,2)^2b(1,0,2)b(2,1,0) \\
&+k^4b(0,1,2)^3b(3,0,0))
\end{aligned}$$

The inversion locus with respect to $(0,0,1)$ is a degree - 4 curve parametrized by $(A*(ja[0,1,2]+kb[0,1,2]), -A*(ja[1,0,2]+kb[1,0,2]), B*(ja[0,1,2]+kb[0,1,2]))$ for $(j,k) \in \mathbb{P}^1$. It seems like a quintic curve at first but notice that each coordinate in the parametrization has a factor of $(ja[0,1,2]+kb[0,1,2])$. Factoring this out yields reveals the curve as quartic

. Also note that since the reflection locus can be parametrized by $\mathbb{P}^1$, it is a rational curve (Lüroth).

An *additional fact* is that the reflection locus passes through $(0, 0, 1)$ three times and goes through each other base point of the pencil once.