Definition 1. Let $C = V(f) \subseteq \mathbb{P}^2$ be a curve of degree d and let $Q = (Q_0, Q_1, Q_2) \in \mathbb{P}^2$ be any point. Then the **polar** (or **first polar**) of C with respect to Q is

$$\mathfrak{P}_Q(C) = V(Q_0 \cdot f_x + Q_1 \cdot f_y + Q_2 \cdot f_z).$$

It is the unique degree $d-1$ curve whose intersection with C is exactly the points of C whose tangent lines contains Q. Furthermore, the **polar conic** of C with respect to Q, denoted $\mathfrak{P}_Q^{(2)}(C)$, is the result of iterating taking the polar curve with respect to Q until one obtains a degree 2 curve.

Lemma 1. Given a curve $C \subseteq \mathbb{P}^2$ of degree d and any $Q \in \mathbb{P}^2$ and any $A \subseteq \mathrm{Aut}(\mathbb{P}^2)$, one has

$$A(\mathfrak{P}_Q(C)) = \mathfrak{P}_{AQ}(AC).$$

Proof. Note again that $\mathfrak{P}_Q(C)$ is the unique degree $d-1$ curve whose intersection with C is exactly the points of C whose tangent lines contain Q. In other words,

$$\mathfrak{P}_Q(C).C = P_1 + P_2 + \cdots + P_{d(d-1)}$$

where

$$\{Q\} = \bigcap_{i=1}^{d(d-1)} T_{P_i}(C).$$

Note that in some cases, like if $Q \in C$, we can have $P_i = P_j$ for $i \neq j$.

Since authomorphisms preserve incidence and linearity, we have

$$A(\mathfrak{P}_Q(C)).AC = AP_1 + AP_2 + \cdots + AP_{d(d-1)}$$

where

$$\{AQ\} = \bigcap_{i=1}^{d(d-1)} T_{AP_i}(AC).$$

Thus $A(\mathfrak{P}_Q(C))$ is a degree $d-1$ curve whose intersection with AC is exactly the points whose tangent lines contain AQ. Since $\mathfrak{P}_{AQ}(AC)$ is the unique curve with such properties, we must have $A(\mathfrak{P}_Q(C)) = \mathfrak{P}_{AQ}(AC)$. $\square$

Lemma 2. Given a curve $C \subseteq \mathbb{P}^2$ and an automorphism $A \subseteq \mathrm{Aut}(\mathbb{P}^2)$, we have $\mathfrak{H}AC = A\mathfrak{H}C$.

Proof. Note that $\mathfrak{H}C$ satisfies

$$\mathfrak{H}C = \{Q \in \mathbb{P}^2 : \mathfrak{P}_Q^{(2)}(C) \text{ is reducible}\}.$$

By the previous lemma, $A(\mathfrak{P}_Q(C)) = \mathfrak{P}_{AQ}(AC)$, so automorphisms commute with finding the polar curve. Thus we have $A(\mathfrak{P}_Q^{(2)}(C)) = \mathfrak{P}_{AQ}^{(2)}(AC)$. So if $Q \in \mathfrak{H}C$, then $\mathfrak{P}_Q^{(2)}(C)$ is reducible and so $\mathfrak{P}_{AQ}^{(2)}(AC) = A(\mathfrak{P}_Q^{(2)}(C))$ is reducible. Thus $AQ \in \mathfrak{H}AC$, so $A\mathfrak{H}C \subseteq \mathfrak{H}AC$. A similar argument works for the reverse containment. Thus we have equality. $\square$