Through observation in Macaulay2, we appear to get this table for unexpected cubic hypersurfaces through all the points through $\mathbb{P}^n_{\mathbb{Z}/2\mathbb{Z}} \subseteq \mathbb{P}^n_k$ and having a generic double/triple point $P$:

| $Z$ | $\dim I(Z+2P)_3$ | $\dim I(Z+3P)_3$ |
|---|---|---|
| $\mathbb{P}^2_{\mathbb{Z}/2\mathbb{Z}}$ | 1 | 0 |
| $\mathbb{P}^3_{\mathbb{Z}/2\mathbb{Z}}$ | 3 | 1 |
| $\mathbb{P}^4_{\mathbb{Z}/2\mathbb{Z}}$ | 6 | 3 |
| $\mathbb{P}^5_{\mathbb{Z}/2\mathbb{Z}}$ | 10 | 6 |
| $\mathbb{P}^6_{\mathbb{Z}/2\mathbb{Z}}$ | 15 | 10 |

The pattern appears to be $\binom{n}{2}$ and $\binom{n-1}{2}$.

The pattern for cubics with a double point at $P = (a, b, c, \dots)$ appears to be

2. $\mathbb{P}^2$: $a^2y^2z + a^2yz^2 + b^2x^2z + b^2vz^2 + c^2x^2y + c^2xy^2$
3. $\mathbb{P}^3$:
   (a) $b^2(z^2w + zw^2) + c^2(y^2w + yw^2) + d^2(y^2z + yz^2)$
   (b) $a^2(z^2w + zw^w) + c^2(x^2w + xw^2) + d^2(x^2z + xz^2)$
   (c) $a^2(y^2w + yw^2) + b^2(x^2w + xw^2) + d^2(x^2y + xy^2)$
4. $\mathbb{P}^4$:
   (a) $c^2(w^2u + wu^2) + d^2(z^2u + zu^2) + e(z^2w + zw^2)$
   (b) $b^2(w^2u + wu^2) + d^2(y^2u + yu^2) + e(y^2w + yw^2)$
   (c) $b^2(z^2u + zu^2) + c^2(y^2u + yu^2) + e(y^2z + yz^2)$
   (d) $a^2(w^2u + wu^2) + d^2(x^2u + xu^2) + e(x^2w + xw^2)$
   (e) $a^2(z^2u + zu^2) + c^2(x^2u + xu^2) + e(x^2z + xz^2)$
   (f) $a^2(y^2u + yu^2) + b^2(x^2u + xu^2) + e(x^2y + xy^2)$

and so on like that. The unique cubic cone in $\mathbb{P}^3$ that contains $Z$ and has a triple point at $P = (a, b, c, d)$ is

$$\begin{gathered}(c^2d + cd^2)(x^2y + xy^2) + (b^2d + bd^2)(x^2z + xz^2)\\ (b^2c + bc^2)(x^2w + xw^2) + (a^2d + ad^2)(y^2z + yz^2)\\ (a^2c + ac^2)(y^2w + yw^2) + (a^2b + ab^2)(z^2w + zw^2)\end{gathered}$$

so the pattern in higher dimensions should continue similarly.

In general, pick a point $(a_0, \dots, a_n) \in \mathbb{P}^n$, with variables $x_0, \dots, x_n$. To build the double-point cubic through $Z_n$, pick three distinct $i, j, \ell \in \{0, \dots, n\}$. The cubics are of the form

$$a_i^2(x_j^2x_\ell + x_jx_\ell^2) + a_j^2(x_i^2x_\ell + x_ix_\ell^2) + a_\ell^2(x_i^2x_j + x_ix_j^2).$$

This means there are $\binom{n+1}{3}$ such cubics. But fixing $i = 0$ reduces this number to $\binom{n}{2}$.

Now pick four distinct indices $i, j, \ell, m \in \{0, \ldots, n\}$. The cubic cones through $Z_n$ are of the form

$$\begin{aligned}&(a_i^2 a_j + a_i a_j^2)(x_\ell^2 x_m + x_\ell x_m^2) + (a_i^2 a_\ell + a_i a_\ell^2)(x_j^2 x_m + x_j x_m^2)\\ +&(a_i^2 a_m + a_i a_m^2)(x_j^2 x_\ell + x_j x_\ell^2) + (a_j^2 a_\ell + a_j a_\ell^2)(x_i^2 x_m + x_i x_m^2)\\ +&(a_j^2 a_m + a_j a_m^2)(x_i^2 x_\ell + x_i x_\ell^2) + (a_\ell^2 a_m + a_\ell a_m^2)(x_i^2 x_j + x_i x_j^2).\end{aligned}$$

Thus there are $\binom{n+1}{4}$ such cubics. Fixing $i = 0$ reduces this to $\binom{n}{3}$ and fixing $j = 1$ reduces this to $\binom{n-1}{2}$.

**Proposition 1.** The cubics of the form

$$a_0^2(x_j^2 x_\ell + x_j x_\ell^2) + a_j^2(x_0^2 x_\ell + x_0 x_\ell^2) + a_\ell^2(x_0^2 x_j + x_0 x_j^2)$$

generate the space of all cubics that contain every point of $\mathbb{P}^n_{\mathbb{Z}/2\mathbb{Z}}$ and have a double point at $(a_0, \ldots, a_n)$.

*Proof.* First note that $p_j^2 p_\ell + p_j p_\ell^2 = 0$ for all $(p_0, \ldots, p_n) \in \mathbb{P}^n_{\mathbb{Z}/2\mathbb{Z}}$, and so the hypersurface contains $\mathbb{P}^2_{\mathbb{Z}/2\mathbb{Z}}$.

Now note that

$$\begin{aligned}&a_0^2((x_j + a_j)^2(x_\ell + a_\ell) + (x_j + a_j)(x_\ell + a_\ell)^2)\\ &+a_j^2((x_0 + a_0)^2(x_\ell + a_\ell) + (x_0 + a_0)(x_\ell + a_\ell)^2)\\ &+a_\ell^2((x_0 + a_0)^2(x_j + a_j) + (x_0 + a_0)(x_j + a_j)^2)\\ &= a_0^2(x_j^2 x_\ell + a_\ell x_j^2 + a_j^2 x_\ell + a_j^2 a_\ell + x_j x_\ell^2 + a_j x_\ell^2 + a_\ell^2 x_j + a_j a_\ell^2)\\ &+a_j^2(x_0^2 x_\ell + a_\ell x_0^2 + a_0^2 x_\ell + a_0^2 a_\ell + x_0 x_\ell^2 + a_0 x_\ell^2 + a_\ell^2 x_0 + a_0 a_\ell^2)\\ &+a_\ell^2(x_0^2 x_j + a_j x_0^2 + a_0^2 x_j + a_0^2 a_j + x_0 x_j^2 + a_0 x_j^2 + a_j^2 x_0 + a_0 a_j^2)\\ &= S + a_0^2 a_j^2 x_\ell + a_0^2 a_\ell^2 x_j + a_0^2 a_j^2 x_\ell + a_j^2 a_\ell^2 x_0 + a_0^2 a_\ell x_j + a_j^2 a_\ell x_0\\ &+a_0^2 a_j^2 a_\ell + a_0^2 a_j a_\ell^2 + a_0^2 a_j^2 a_\ell + a_0 a_j^2 a_\ell^2 + a_0^2 a_j a_\ell^2 + a_0 a_j^2 a_\ell^2\\ &= S\end{aligned}$$

where $S \in (x_0, x_j, x_\ell)^2$. Therefore $S$ has a double point at $(a_0, \ldots, a_n)$.

Now what's left to prove is

$$\{a_0^2(x_j^2 x_\ell + x_j x_\ell^2) + a_j^2(x_0^2 x_\ell + x_0 x_\ell^2) + a_\ell^2(x_0^2 x_j + x_0 x_j^2) : 0 \neq j, \ell; j \neq \ell\}$$

generates the cubic hypersurfaces that contain $\mathbb{P}^n_{\mathbb{Z}/2\mathbb{Z}}$ and have a double point at $(a_0, \ldots, a_n)$. $\square$

**Proposition 2.** The cubics of the form

$$\begin{aligned}
&(a_0^2a_1+a_0a_1^2)(x_\ell^2x_m+x_\ell x_m^2)+(a_0^2a_\ell+a_0a_\ell^2)(x_1^2x_m+x_1x_m^2)\\
+&(a_0^2a_m+a_0a_m^2)(x_1^2x_\ell+x_1x_\ell^2)+(a_1^2a_\ell+a_1a_\ell^2)(x_0^2x_m+x_0x_m^2)\\
+&(a_1^2a_m+a_1a_m^2)(x_0^2x_\ell+x_0x_\ell^2)+(a_\ell^2a_m+a_\ell a_m^2)(x_0^2x_1+x_0x_1^2).
\end{aligned}$$

generate the space of all cubics that contain every point of $\mathbb{P}^n_{\mathbb{Z}/2\mathbb{Z}}$ and have a triple point at $(a_0,\ldots,a_n)$.

*Proof.* First note that $p_j^2p_\ell+p_jp_\ell^2=0$ for all $(p_0,\ldots,p_n)\in\mathbb{P}^n_{\mathbb{Z}/2\mathbb{Z}}$, and so the hypersurface contains $\mathbb{P}^2_{\mathbb{Z}/2\mathbb{Z}}$.

Now note that

$$\begin{aligned}
F=&(a_0^2a_1+a_0a_1^2)((x_\ell+a_\ell)^2(x_m+a_m)+(x_\ell+a_\ell)(x_m+a_m)^2)\\
&+(a_0^2a_\ell+a_0a_\ell^2)((x_1+a_1)^2(x_m+a_m)+(x_1+a_1)(x_m+a_m)^2)\\
&+(a_0^2a_m+a_0a_m^2)((x_1+a_1)^2(x_\ell+a_\ell)+(x_1+a_1)(x_\ell+a_\ell)^2)\\
&+(a_1^2a_\ell+a_1a_\ell^2)((x_0+a_0)^2(x_m+a_m)+(x_0+a_0)(x_m+a_m)^2)\\
&+(a_1^2a_m+a_1a_m^2)((x_0+a_0)^2(x_\ell+a_\ell)+(x_0+a_0)(x_\ell+a_\ell)^2)\\
&+(a_\ell^2a_m+a_\ell a_m^2)((x_0+a_0)^2(x_1+a_1)+(x_0+a_0)(x_1+a_1)^2)
\end{aligned}$$

$$\begin{aligned}
=&(a_0^2a_1+a_0a_1^2)(x_\ell^2x_m+a_mx_\ell^2+a_\ell^2x_m+a_\ell^2a_m+x_\ell x_m^2+a_\ell x_m^2+a_m^2x_\ell+a_\ell a_m^2)\\
&+(a_0^2a_\ell+a_0a_\ell^2)(x_1^2x_m+a_mx_1^2+a_1^2x_m+a_1^2a_m+x_1x_m^2+a_1x_m^2+a_m^2x_1+a_1a_m^2)\\
&+(a_0^2a_m+a_0a_m^2)(x_\ell^2x_1+a_1x_\ell^2+a_\ell^2x_1+a_\ell^2a_1+x_\ell x_1^2+a_\ell x_1^2+a_1^2x_\ell+a_\ell a_1^2)\\
&+(a_\ell^2a_1+a_\ell a_1^2)(x_0^2x_m+a_mx_0^2+a_0^2x_m+a_0^2a_m+x_0x_m^2+a_0x_m^2+a_m^2x_0+a_0a_m^2)\\
&+(a_m^2a_1+a_ma_1^2)(x_\ell^2x_0+a_0x_\ell^2+a_\ell^2x_0+a_\ell^2a_0+x_\ell x_0^2+a_\ell x_0^2+a_0^2x_\ell+a_\ell a_0^2)\\
&+(a_\ell^2a_m+a_\ell a_m^2)(x_0^2x_1+a_1x_0^2+a_0^2x_1+a_0^2a_1+x_0x_1^2+a_0x_1^2+a_1^2x_0+a_0a_1^2)
\end{aligned}$$

$$\begin{aligned}
=T&+(a_0^2a_1+a_0a_1^2)(x_\ell^2a_m+a_\ell x_m^2)+(a_0^2a_\ell+a_0a_\ell^2)(x_1^2a_m+a_1x_m^2)\\
&+(a_0^2a_m+a_0a_m^2)(x_1^2a_\ell+a_1x_\ell^2)+(a_1^2a_\ell+a_1a_\ell^2)(x_0^2a_m+a_0x_m^2)\\
&+(a_1^2a_m+a_1a_m^2)(x_0^2a_\ell+a_0x_\ell^2)+(a_\ell^2a_m+a_\ell a_m^2)(x_0^2a_1+a_0x_1^2)
\end{aligned}$$

$$\begin{aligned}
&+(a_0^2a_1+a_0a_1^2)(a_\ell^2x_m+x_\ell a_m^2)+(a_0^2a_\ell+a_0a_\ell^2)(a_1^2x_m+x_1a_m^2)\\
&+(a_0^2a_m+a_0a_m^2)(a_1^2x_\ell+x_1a_\ell^2)+(a_1^2a_\ell+a_1a_\ell^2)(a_0^2x_m+x_0a_m^2)\\
&+(a_1^2a_m+a_1a_m^2)(a_0^2x_\ell+x_0a_\ell^2)+(a_\ell^2a_m+a_\ell a_m^2)(a_0^2x_1+x_0a_1^2)
\end{aligned}$$

$$\begin{aligned}
&+(a_0^2a_1+a_0a_1^2)(a_\ell^2a_m+a_\ell a_m^2)+(a_0^2a_\ell+a_0a_\ell^2)(a_1^2a_m+a_1a_m^2)\\
&+(a_0^2a_m+a_0a_m^2)(a_1^2a_\ell+a_1a_\ell^2)+(a_1^2a_\ell+a_1a_\ell^2)(a_0^2a_m+a_0a_m^2)\\
&+(a_1^2a_m+a_1a_m^2)(a_0^2a_\ell+a_0a_\ell^2)+(a_\ell^2a_m+a_\ell a_m^2)(a_0^2a_1+a_0a_1^2)
\end{aligned}$$

Where $T \in (x_0, x_1, x_\ell, x_m)^3$. Observe that the $x_0$-term of this polynomial is

$$(a_1^2 a_\ell a_m^2 + a_1 a_\ell^2 a_m + a_1^2 a_\ell^2 a_m + a_1 a_\ell^2 a_m^2 + a_1^2 a_\ell^2 a_m + a_1^2 a_\ell a_m^2)x_0 = 0.$$

By symmetry, the linear forms of $F$ are all 0. Similarly, observe that the $x_0^2$-term of $F$ is

$$(a_1^2 a_\ell a_m + a_1 a_\ell^2 a_m + a_1^2 a_\ell a_m + a_1 a_\ell a_m^2 + a_1 a_\ell^2 a_m + a_1 a_\ell a_m^2)x_0^2 = 0.$$

By symmetry, all the quadratic forms of $F$ are 0. Therefore $F$ has a triple point at $(a_0, \ldots, a_n)$.

Now what's left to show is

$$\begin{aligned}\{(a_0^2 a_1 + a_0 a_1^2)(x_\ell^2 x_m + x_\ell x_m^2) + (a_0^2 a_\ell + a_0 a_\ell^2)(x_1^2 x_m + x_1 x_m^2) \\ +(a_0^2 a_m + a_0 a_m^2)(x_1^2 x_\ell + x_1 x_\ell^2) + (a_1^2 a_\ell + a_1 a_\ell^2)(x_0^2 x_m + x_0 x_m^2) \\ +(a_1^2 a_m + a_1 a_m^2)(x_0^2 x_\ell + x_0 x_\ell^2) + (a_\ell^2 a_m + a_\ell a_m^2)(x_0^2 x_1 + x_0 x_1^2) : 0 \neq \ell, m \neq 1; \ell \neq m\}\end{aligned}$$

generates the cubic hypersurfaces that contain $\mathbb{P}^n_{\mathbb{Z}/2\mathbb{Z}}$ and have a triple point at $(a_0, \ldots, a_n)$. $\square$

There are $\binom{d+1}{2}$ generators for cubic surfaces containing $Z$ in $\mathbb{P}^3_k$. They are $x_i^2 x_j + x_i x_j^2$ for $i \neq j$. Prove that adding a double point imposes $d$ additional condition and adding a triple point imposes $2d-1$ conditions?