Definition. A topological space X is *irreducible* if for any two closed subsets $V_1, V_2 \subseteq X$ such that $V_1 \cup V_2 = X$, then $V_1 = X$ or $V_2 = X$.

Proposition 1. Let X be a topological space. Then the following are equivalent:

(1) X is irreducible.

(2) Every nonempty open subset of X is dense.

(3) Every pair of nonempty open subsets of X intersect nontrivially.

Proof. (1) $\Rightarrow$ (2) : Suppose there is a nonempty open subset $U \subseteq X$ such that $\overline{U} \neq X$. Then $\overline{U}$ is a proper closed subset of X. Also $X \setminus U$ is a proper closed subset of X, since U is nonempty. Furthermore, $X = \overline{U} \cup X \setminus U$, so X is not irreducible.

(2) $\Rightarrow$ (3) : Suppose there are two nonempty open subsets $U_1, U_2 \subseteq X$ such that $U_1 \cap U_2 = \emptyset$. So $U_1 \subseteq X \setminus U_2$, which is closed. Thus $\overline{U_1} \subseteq X \setminus U_2 \neq X$. Therefore U_1 is not dense.

(3) $\Rightarrow$ (1) : Suppose every pair of open subsets of X intersect nontrivially. Then let $X = V_1 \cap V_2$ be the union of two closed subsets. Then $X \setminus V_1$ and $X \setminus V_2$ are open subsets, but $(X \setminus V_1) \cap (X \setminus V_2) = \emptyset$. Therefore $X \setminus V_1 = \emptyset$ (so $X = V_1$) or vice versa. Therefore X is irreducible. $\square$

Proposition 1 will prove to be very handy. We also have the following two equivalences:

Proposition 2. Let X be a topological space. Then X is irreducible if and only if every nonempty open subset of X is connected.

Proof. ($\Rightarrow$) Suppose $U \subseteq X$ is not connected. Then there are $W_1, W_2 \subseteq U$ open in U such that $W_1 \cup W_2 = U$ and $W_1 = U \setminus W_2$. Then W_1, W_2 are also closed in U. Then $W_1 = U \cap Y_1$ and $W_2 = U \cap Y_2$ for some closed sets Y_1, Y_2 closed in X. Furthermore, $X \setminus U$ is closed. Then $Y_1 \cup Y_2 \cup X \setminus U = X$, so X is not irreducible (since none of these are X).

($\Leftarrow$) Suppose X is not irreducible. Then $X = C_1 \cup C_2$ (each not equal to X). Let $C_1 + C_2$ denote the symmetric difference of C_1 and C_2. Then $C_1 + C_2 = X \setminus C_1 \cup X \setminus C_2$ is open. Then $X \setminus C_1 \subseteq C_1 + C_2$ is closed in $C_1 + C_2$ since $X \setminus C_1 = C_2 \cap (C_1 + C_2)$. Then symmetrically, $X \setminus C_1$ is open in $C_1 + C_2$. Then $X \setminus C_1$ is proper and closed and open in $C_1 + C_2$, so $C_1 + C_2$ is not connected. $\square$

Proposition 3. Let X be a topological space. Then X is irreducible if and only if every open subset of X is irreducible.

Proof. ($\Rightarrow$) Let $U \subseteq X$ not be irreducible. Then $U = C_1 \cup C_2$. Then there are closed subsets $D_1, D_2 \subseteq X$ such that $C_1 = U \cap D_1$ and $C_2 = U \cap D_2$. Then $X = D_1 \cup D_2 \cup X \setminus U$, so X is not irreducible.

($\Leftarrow$) X is open in itself, so X is irreducible. $\square$

Definition. A ring A is *lucky* if its nilradical $\sqrt{0A}$ is prime. (This is equivalent to $\sqrt{0A}$ being the only minimal prime of A.)

Note that a (commutative) ring A is an integral domain if and only if A is both lucky and reduced.

Definition. A *generic point* (also called a *lucky point* (by me)) of a topological space X is a point $\omega \in X$ such that $\overline{\{\omega\}} = X$.

Note that a topological space X need not have a unique generic point in general. For example in $\mathbb{R}$ under the trivial topology, every point is a generic point.

Lemma 1. Let X be any topological space with a generic point $\omega \in X$. Then X is irreducible.

Proof. Since ω is a generic point, then $\overline{\{\omega\}} = X$ and so the smallest closed set that contains ω is X. Then let U_1, U_2 be nontrivial open sets of X. Then $V_1 = X \setminus U_1$ and $V_2 = X \setminus U_2$ are proper closed subsets of X. Therefore $\omega \notin V_1 \cup V_2$, so $\omega \in U_1 \cap U_2$. Thus all nonempty open sets intersect nontrivially (as all nonempty open sets contain ω), so X is irreducible. $\square$

Note that the converse does not necessarily hold for all topological spaces. Counterexamples include $\mathbb{R}$ with the finite-complement topology (for a compact example) or $\mathbb{R}$ with the infinite-ray-to-the-right topology (for a non-compact example).

Proposition 4. Let $f : X \to Y$ be a continuous map of topological spaces and let Z be an irreducible subset of X. Then $f(Z)$ is an irreducible subset of Y. Furthermore, if ω is a generic point of Z, then $f(\omega)$ is a generic point of $f(Z)$.

Proof. Let U_1 and U_2 be nonempty open subsets of $f(Z)$. Then $V_1 = f^{-1}(U_1) \cap Z$ and $V_2 = f^{-1}(U_2) \cap Z$ are nonempty open subsets of $Z \subseteq f^{-1}(f(Z))$. Since Z is irreducible, $V_1 \cap V_2 \neq \emptyset$ and so $U_1 \cap U_2 \neq \emptyset$. Since U_1 and U_2 are arbitrary nonempty open subsets of $f(Z)$, we know that $f(Z)$ must be irreducible.

Now let Z be irreducible with generic point ω. Then $Z = \mathrm{cl}_Z(\{\omega\})$. We want to show that $f(Z) = \mathrm{cl}_{f(Z)}(\{f(\omega)\})$. Again, let U be a nonempty open subset of $f(Z)$. Then $V = f^{-1}(U) \cap Z$ is a nonempty open subset of Z. Therefore $\omega \in V$. Then $f(\omega) \in U$. and so $f(\omega) \notin f(Z) \setminus U$. Thus $f(\omega)$ is in every open subset of $f(Z)$ (except $\emptyset$) and so $f(\omega)$ is excluded from every closed subset of $f(Z)$ other than $f(Z)$ itself. Thus $f(Z)$ is the smallest closed subset of $f(Z)$ containing $f(\omega)$ and so $f(Z) = \mathrm{cl}_{f(Z)}(\{f(\omega)\})$. Thus $f(\omega)$ is a generic point of $f(Z)$. $\square$

Proposition 5. Let A be a ring. Then the following are equivalent.

(1) A is lucky.

(2) $\sqrt{0A} \in \mathrm{Spec}A$ is the unique generic point of $\mathrm{Spec}A$ under the Zariski topology.

(3) $\mathrm{Spec}A$ is irreducible under the Zariski topology.

Proof. (1) $\Rightarrow$ (2) : Suppose A is lucky. Then $\sqrt{0A}$ is the only minimal prime of A. So $\sqrt{0A} \in \operatorname{Spec} A$ and for all $\mathfrak{p} \in \operatorname{Spec} A$, $\sqrt{0A} \subseteq \mathfrak{p}$. Thus $\mathfrak{V}\left(\sqrt{0A}\right) = \operatorname{Spec} A$. Now consider $\left\{\sqrt{0A}\right\}$. Let $\sqrt{0A} \in \mathfrak{V}(\mathfrak{a})$ for some radical ideal $\mathfrak{a} \subseteq A$. Then $\mathfrak{a} \subseteq \sqrt{0A}$ and so all primes contain $\mathfrak{a}$. Thus $\mathfrak{V}(\mathfrak{a}) = \operatorname{Spec} A$. Therefore $\sqrt{0A}$ is a generic point.

Now to show uniqueness. Let $\mathfrak{q} \in \operatorname{Spec} A$ be a generic point of $\operatorname{Spec} A$. Then every prime ideal contains $\mathfrak{q}$. Thus $\mathfrak{q} \subseteq \bigcap_{\mathfrak{p} \in \operatorname{Spec} A} \mathfrak{p} = \sqrt{0A}$. Since $\mathfrak{q}$ is prime and $\sqrt{0A}$ is the unique minimal prime, we have $\mathfrak{q} = \sqrt{0A}$.

(2) $\Rightarrow$ (3) : This is Lemma 1.

(3) $\Rightarrow$ (1) : Suppose A is not lucky, so $\sqrt{0A}$ is not prime. Then there is more than one minimal prime of A. Let $\{\mathfrak{p}_\alpha\}_{\alpha \in J}$ be the set of all minimal primes of A (of size greater than 1). Then $\mathfrak{V}(\mathfrak{p}_\alpha) \neq \operatorname{Spec} A$, since $\mathfrak{p}_\beta \notin \mathfrak{V}(\mathfrak{p}_\alpha)$ for all $\beta \neq \alpha \in J$. Furthermore, $\bigcup_{\alpha \in J} \mathfrak{V}(\mathfrak{p}_\alpha) = \mathfrak{V}\left(\bigcap_{\alpha \in J} \mathfrak{p}_\alpha\right) = \mathfrak{V}\left(\sqrt{0A}\right) = \operatorname{Spec} A$. Thus $\operatorname{Spec} A$ can be realized as the union of closed subsets and is equal to none of those subsets. Thus $\operatorname{Spec} A$ is not irreducible.

$\square$

Problem 1. (Hartshorne 2.2.9) Let X be a scheme. Then every irreducible closed subset $Z \subseteq X$ contains a unique generic point.

Proof. First we know that if X is an affine scheme, then every irreducible closed subset Z is $\operatorname{Spec} A_Z$ for some lucky ring A_Z. Then by Proposition 5, Z has a unique generic point.

Now let X be an arbitrary scheme. Then for every point $P \in X$, there is a neighborhood U containing P such that U is an affine scheme (with the restricted sheaf). Then X can be covered by affine schemes as open subsets. So let Z be an irreducible closed subset of X. Then for each $P \in Z$, there is a neighborhood $U_P \subseteq Z$ of P such that U_P is an affine scheme. Since Z is irreducible, U_P is irreducible (Proposition 3). Thus U_P has a unique generic point ω_P. Thus $\operatorname{cl}_{U_P}(\{\omega_P\}) = U_P$. Note that Z can be covered by such open sets, and such open sets are dense in Z (Z is irreducible).

Let $P \in Z$ be fixed, then we claim that ω_P is a generic point for all of Z. That is, $\operatorname{cl}_Z(\{\omega_P\}) = Z$. We know that $\operatorname{cl}_{U_P}(\{\omega_P\}) = U_P$ (so the smallest closed set in U_P that contains ω_P is U_P) and $\operatorname{cl}_Z(U_P) = Z$ (so the smallest closed set in Z that contains U_P is Z), then the smallest closed set in Z that contains ω_P must be Z. Thus ω_P is a generic point of Z.

Now to prove uniqueness. Since ω_P is a generic point of Z, we know (by Lemma 1) that ω_P is in every open subset of Z (including those which are themselves affine schemes). Thus ω_P is a generic point in every affine subscheme, which from Proposition 5 have a *unique* generic point. Thus ω_P is the unique generic point of every affine subscheme of Z. Thus ω_P is the unique generic point of Z. $\square$

Proposition 6. Let X be a topological space. Then X is noetherian if and only if every subspace of X is compact.

Proof. ($\Rightarrow$) Let $Y \subseteq X$ be an arbitrary subspace. Then let $\mathscr{U} = \{U_\alpha\}_{\alpha \in J}$ be an open cover of Y. Consider the collection $\mathscr{A} = \{\bigcup \mathscr{F} : \mathscr{F} \text{ is a finite subset of } \mathscr{U}\}$. First we will show that $\mathscr{A}$ has a maximal element.

Let $\{\alpha_1, \alpha_2, \dots\}$ be a countable subset of J. Then

$$U_{\alpha_1} \subseteq U_{\alpha_1} \cup U_{\alpha_2} \subseteq U_{\alpha_1} \cup U_{\alpha_2} \cup U_{\alpha_3} \subseteq \cdots$$

is an ascending chain of open subsets of Y. We know that for each $U_\alpha \in \mathscr{U}$, there is a V_α open in X such that $U_\alpha = X \cap V_\alpha$. Since X is noetherian, the chain

$$V_{\alpha_1} \subseteq V_{\alpha_1} \cup V_{\alpha_2} \subseteq V_{\alpha_1} \cup V_{\alpha_2} \cup V_{\alpha_3} \subseteq \cdots$$

stabilizes in X. Thus the chain stabilizes in Y.

Thus an arbitary chain in the poset $\langle \mathscr{A}, \subseteq \rangle$ has a maximal element. By Zorn's Lemma, $\mathscr{A}$ then has a maximal element $U = U_{\alpha_1} \cup \cdots \cup U_{\alpha_n}$ where $n \in \mathbb{N}$. We claim $U = Y$. If $U \neq Y$ then there is an $y \in Y \setminus U$. Since $\mathscr{U}$ covers Y, there is an $\alpha \in J$ such that $y \in U_\alpha$. Thus $U \subsetneq U \cup U_\alpha \in \mathscr{A}$. This contradicts the maximality of U. Thus $U = Y$ and so $Y \in \mathscr{A}$. Thus Y is compact.

($\Leftarrow$) Let

$$C_1 \supseteq C_2 \supseteq C_3 \supseteq \cdots$$

be a descending chain of closed subsets of X. Then define $U_i = X \setminus C_i$ for all $i \geq 1$, and see that

$$U_1 \subseteq U_2 \subseteq U_3 \subseteq \cdots$$

is an ascending chain of open subsets of X. Observe that $\{U_i\}_{i\geq 1}$ is an open cover of the subspace $U := \bigcup_{i\geq 1} U_i$. Since every subspace of X is compact, the open cover $\{U_i\}_{i\geq 1}$ must have a finite subcover. Since $U_i \subseteq U_j$ for all $i < j$, we know that there must exist an $N \in \mathbb{N}$ such that $U_i = U_N$ for all $i \geq N$, and $U = U_N$. Thus we know that $C_i = C_N$ for all $i \geq N$, and so the descending chain of closed subsets stabilizes. Thus X is noetherian. □

Definition. A topological space X is a *Zariski space* if it is noetherian and every (nonempty) closed irreducible subset has a unique generic point.

Lemma 2. Let X be a topological space and P be a point in X. Then $\mathrm{cl}_X(\{P\})$ is irreducible. (This can also work as an alternative proof of Lemma 1.)

Proof. Let $\mathrm{cl}_X(\{P\}) = C_1 \cup C_2$. Then one of C_1 or C_2 is a closed set containing P, and thus is equal to $\mathrm{cl}_X(\{P\})$. □

Problem 2. (Hartshorne 2.3.17)

(a) Show that if X is a noetherian scheme, then $\mathrm{sp}(X)$ is a Zariski space.

(b) Show that any minimal nonempty closed subset of a Zariski space consists of one point. (These are called *closed points*.)

(c) Show that any Zariski space X is T_0: Given any two distinct points in X, there is an open set containing one but not the other.

(d) If X is an irreducible Zariski space, then its generic point is contained in every nonempty open subset. (Lemma 1!)

(e) If x_0, x_1 are points of a topological space X, we say x_1 *specializes* x_0 (written $x_1 \rightsquigarrow x_0$) if $x_0 \in \mathrm{cl}_X(\{x_1\})$. We also say x_0 is a *specialization* of x_1, or x_1 is a *generization* of x_0. Now let X be a Zariski space partially ordered by $x_1 > x_0$ if $x_1 \rightsquigarrow x_0$. Show that the minimal points of the poset X are the closed points, and the maximal points are the generic points of the irreducible components of X. Show that a closed set contains every specialization of any of its points. (We say closed sets are *stable under specialization* and open sets are *stable under generization.*)

(f) Let t be the Zariski functor. If X is a noetherian topological space, show $t(X)$ is a Zariski space. Furthermore X itself is a Zariski space if and only if the map $\alpha : X \to t(X)$ is a homeomorphism.

Proof.

(a) $\mathrm{sp}(X)$ is a noetherian topological space because X is locally noetherian and so it is covered by noetherian open sets $\mathrm{Spec}A_i$ (each A_i is a noetherian ring, so $\mathrm{Spec}A_i$ is a noetherian space) and quasi-compact, so $\mathrm{sp}(X)$ is covered by finitely many noetherian subspaces.

Since being noetherian is equivalent with every subspace being compact, if we take an arbitrary subspace Y of $\mathrm{sp}(X)$, we see that $Y \cap \mathrm{Spec}A_i \subseteq \mathrm{Spec}A_i$ is compact for each of the finitely many i. Thus an arbitrary open cover of Y can be intersected with each of the $\mathrm{Spec}A_i$ to create an open cover of $Y \cap \mathrm{Spec}A_i$. Each of these intersections of the open cover is equal to some finite subcover of $Y \cap \mathrm{Spec}A_i$ for each of the finitely many i. Thus the open cover of Y can be replaced by the finite union of each of these finite subcovers. Thus Y is compact for an arbitrary $Y \subseteq \mathrm{sp}(X)$. Thus $\mathrm{sp}(X)$ is noetherian.

Furthermore, by Problem 2.9, every nonempty irreducible closed subset of any scheme has a unique generic point. Thus X is Zariski.

(b) Let X be a Zariski space. Let C be a minimal nonempty closed set of X. That is, the only closed set of X properly contained in C is $\emptyset$. Then C is a nonempty irreducible closed subset of X, and so C contains a unique generic point ω_C. However, *every* point $P \in C$ is a generic point of C, since $\emptyset \neq \mathrm{cl}_X(\{P\}) \subseteq C$, with equality coming from the minimality of C. Thus C contains just the unique point ω_C.

(c) Let $P \neq Q \in X$. Then by Lemma 2, $\mathrm{cl}_X(\{P\})$ is a nonempty irreducible closed subset of X (its unique generic point is P). The same goes for $\mathrm{cl}_X(\{Q\})$. We claim that either $P \notin \mathrm{cl}_X(\{Q\})$ or $Q \notin \mathrm{cl}_X(\{P\})$.

Suppose $P \in \mathrm{cl}_X(\{Q\})$. Then every closed set that contains Q also contains P. That is, every open set excluding Q also excludes P. Then $\mathrm{cl}_X(\{P\}) \subseteq \mathrm{cl}_X(\{Q\})$. If equality holds, then $\mathrm{cl}_X(\{Q\})$ has both P and Q as generic points. Since X is Zariski, generic points must be unique, but $P \neq Q$. Thus equality cannot hold. Therefore we have $\mathrm{cl}_X(\{P\}) \subsetneq \mathrm{cl}_X(\{Q\})$. Thus $Q \notin \mathrm{cl}_X(\{P\})$. Thus $Q \in X \setminus \mathrm{cl}_X(\{P\})$ is open and excludes P. Thus X is T_0.

(d) This is proved in the proof of Lemma 1 (but it's not the statement of Lemma 1).

(e) Let $m \in X$ be a minimal point under this partial ordering. That is, m specializes nothing other than m, or there are no points other than m in $\mathrm{cl}_X(\{m\})$. Thus $\mathrm{cl}_X(\{m\}) = \{m\}$ and so m is a closed point of X.

Now let $M \in X$ be a maximal point under this partial ordering. That is, nothing other than M specializes M, or $M \in \mathrm{cl}_X(\{P\})$ only if $P = M$. We will interpret Hartshorne to mean show M is the unique generic point for some *maximal* irreducible closed set of X, as showing $\mathrm{cl}_X(\{M\})$ is irreducible is Lemma 2.

Let C be an irreducible closed subset such that $\mathrm{cl}_X(\{M\}) \subseteq C$. Since X is Zariski, C has a unique generic point ω. Then $C = \mathrm{cl}_X(\{\omega\})$. Then $M \in \mathrm{cl}_X(\{\omega\})$, but by maximality of M, this implies $M = \omega$, and so $\mathrm{cl}_X(\{M\}) = C$. Thus $\mathrm{cl}_X(\{M\})$ is a maximal irreducible closed subset of X.

Let C be a closed set and let $c \in C$. Let $z \in \mathrm{cl}_X(\{c\}) \subseteq C$, so $z \in C$. Thus C is closed under specialization.

(f) Recall that given a topological space X, then $t(X)$ is the set of all nonempty irreducible closed subsets of X, with the topology given by the closed sets are of the form $t(Y)$, where Y is a closed subset of X.

Let X be a noetherian space. Then we will show $t(X)$ is noetherian. Let

$$t(Y_0) \supseteq t(Y_1) \supseteq t(Y_2) \supseteq \cdots$$

be a descending chain of closed sets of $t(X)$. Then $t(Y_i)$ is the set of nonempty irreducible closed subsets of Y_i for all i. Since $t(Y_{i+1}) \subseteq t(Y_i)$, that means that every irreducible closed subset of Y_{i+1} is also an irreducible closed subset of Y_i. So let $y \in Y_{i+1}$. Then $\mathrm{cl}_X(\{y\})$ is an irreducible closed subset of Y_{i+1} (Lemma 2 and the fact Y_{i+1} is closed), and thus is an irreducible closed subset of Y_i. Thus $y \in Y_i$, so $Y_{i+1} \subseteq Y_i$. Therefore we have the descending chain of closed sets in X

$$Y_0 \supseteq Y_1 \supseteq Y_2 \supseteq \cdots$$

which stabilizes since X is noetherian. Thus the original descending chain of closed sets of $t(X)$ stabilizes, so X is noetherian.

Now to show $t(X)$ is Zariski. Let $t(Y) \subseteq t(X)$ be a nonempty irreducible closed subset. So $t(Y)$ is the set of all nonempty irreducible closed subsets of Y, and it *itself* is irreducible as a nonempty closed subset of $t(X)$. This means that if $t(Y) = t(Y_1) \cup t(Y_2)$, then $t(Y) = t(Y_1)$ or $t(Y) = t(Y_2)$.

We will show that $t(Y) = t(Y_1) \cup t(Y_2)$ if and only if $Y = Y_1 \cup Y_2$. For the forwards direction, let $y \in Y$. Then $\mathrm{cl}_X(\{y\}) \in t(Y)$, as shown earlier, so $\mathrm{cl}_X(\{y\}) \in t(Y_1)$ or $t(Y_2)$. Thus $y \in Y_1 \cup Y_2$. Now let $z \in Y_1 \cup Y_2$. Then $\mathrm{cl}_X(\{z\}) \subseteq Y_1$ or $\mathrm{cl}_X(\{z\}) \subseteq Y_2$, so $\mathrm{cl}_X(\{z\}) \in t(Y)$. Thus $z \in Y$.

For the backwards direction, let $C \in t(Y)$. Then $C \subseteq Y$, so $C \subseteq Y_1 \cap Y_2$. Then $C = (Y_1 \cap C) \cap (Y_2 \cap C)$, but since C is irreducible then $C = Y_1 \cap C$ or $C = Y_2 \cap C$. Without loss of generality, let $C = Y_1 \cap C$, so then $C \subseteq Y_1$. Then $C \in t(Y_1) \cup t(Y_2)$. Now let $D \in t(Y_1) \cup t(Y_2)$. Then D is a nonempty irreducible closed subset of Y_1 or

Y_2. Without loss of generality, let $D \in t(Y_1)$. Then since $t(Y_1) \subseteq t(Y)$, we know from earlier that $D \in t(Y)$. Thus $t(Y) = t(Y_1) \cup t(Y_2)$.

Now suppose $Y = Y_1 \cup Y_2$. Then $t(Y) = t(Y_1) \cup t(Y_2)$. Since $t(Y)$ is irreducible, then $t(Y) = t(Y_1)$ or $t(Y) = t(Y_2)$. Then $Y = Y_1$ or $Y = Y_2$. Therefore Y is irreducible. Thus $Y \in t(Y)$. We claim that Y is the unique generic point of $t(Y)$.

First we will show that $\mathrm{cl}_{t(Y)}(\{Y\}) = t(Y)$. Note that $\mathrm{cl}_{t(Y)}(\{Y\}) \subseteq t(Y)$ since $t(Y)$ is closed. Now suppose $Z \in t(Y)$. Then Z is a nonempty irreducible closed subset of Y. Then any closed set in $t(X)$ that contains Y must also contain $Z \subseteq Y$. Therefore $Z \in \mathrm{cl}_{t(Y)}(\{Y\})$, so $\mathrm{cl}_{t(Y)}(\{Y\}) = t(Y)$.

Thus Y is a generic point of $t(Y)$. Now we must show uniqueness. Let $Z \in t(Y)$ be such that $t(Y) = \mathrm{cl}_{t(Y)}(\{Z\}) = t(Z)$. Then $Y = Z$. Therefore $t(X)$ is a Zariski space.

($\Rightarrow$) Now suppose X is Zariski. Then we will show $\alpha : X \to t(X)$ is a homeomorphism. Recall that $\alpha(x)$ is defined to be $\mathrm{cl}_X(\{x\})$. Since X is Zariski, we know that every nonempty irreducible closed subset of X (i.e. every element of $t(X)$) has a unique generic point. That is, for all $Y \in t(X)$, there is a unique $y \in Y$ such that $Y = \mathrm{cl}_X(\{y\})$.

Thus let us define $\beta : t(X) \to X$ by $\beta(\mathrm{cl}_X(\{x\})) = x$. We will show that α and β are continuous, and $\beta = \alpha^{-1}$.

First we will show that α is continuous. Let $C \subseteq t(X)$ be closed, so then $C = t(Y)$ for some closed subset Y of X. Then $\alpha^{-1}(t(Y)) = \{x \in X : \mathrm{cl}_X(\{x\}) \subseteq Y\} =: Q$. We will show $Q = Y$. Let $x \in Y$, then $\mathrm{cl}_X(\{x\}) \subseteq Y$ since Y is closed. Thus $x \in Q$. Now let $x \in Q$. Then $\mathrm{cl}_X(\{x\}) \subseteq Y$, so $x \in Y$. Thus $Q = Y$. Therefore $\alpha^{-1}(t(Y))$ is closed, so α is continuous.

Now let us first note that β is well-defined since every element Y of $t(X)$ can be realized as the closure of a unique generic point $\omega_Y \in Y$. To show β is continuous, let us take a closed set $C \subseteq X$ and its preimage $\beta^{-1}(C)$. We will show that $\beta^{-1}(C) = t(C)$. First let $D \in t(C)$. Then D is a nonempty irreducible closed subset of C, so $D = \mathrm{cl}_X(\{d\})$ for some $d \in D$. Then $\beta(D) = \beta(\mathrm{cl}_X(\{d\}) = d \in D \subseteq C$. Thus $t(C) \subseteq \beta^{-1}(C)$.

Now let $D \in \beta^{-1}(C)$. Then D is an irreducible closed subset of X, so $D = \mathrm{cl}_X(\{d\})$ for some $d \in D$. Then since $\beta(D) = d \in C$, we have $D \subseteq C$. So D is a nonempty irreducible closed subset of C, and thus $D \in t(C)$. Thus $\beta^{-1}(C) = t(C)$ is closed. Thus β is continuous.

Now to show that $\beta = \alpha^{-1}$. Let $\mathrm{cl}_X(\{y\}) \in t(X)$. Then $\alpha(\beta(\mathrm{cl}_X(\{y\}))) = \alpha(y) = \mathrm{cl}_X(\{y\})$. Thus $\alpha \circ \beta = \mathrm{id}_{t(X)}$ Now let $x \in X$. Then $\beta(\alpha(x)) = \beta(\mathrm{cl}_X(\{x\})) = x$. Thus $\beta \circ \alpha = \mathrm{id}_X$. Thus $\beta = \alpha^{-1}$ and so α is a homeomorphism.

($\Leftarrow$) Now suppose $\alpha : X \to t(X)$ is a homeomorphism. Then $X \cong t(X)$ and we've already shown that $t(X)$ is a Zariski space. Thus X is a Zariski space.

□

Lemma 3. Let A be a ring with ideal $\mathfrak{a} \subseteq A$. Then $\mathfrak{V}(\mathfrak{a})$ is an irreducible subset of Spec$A$ if and only if $\sqrt{\mathfrak{a}} \in \mathrm{Spec}A$. In this case, $\sqrt{\mathfrak{a}}$ is the unique generic point of $\mathfrak{V}(\mathfrak{a})$.

Proof. (⇒) Suppose $\sqrt{\mathfrak{a}} \notin \mathrm{Spec}A$. Then there is more than one minimal prime containing $\mathfrak{a}$, so $\mathfrak{V}(\mathfrak{a}) \cap \mathrm{minSpec}A$ contains at least two distinct primes $\mathfrak{p}_1$ and $\mathfrak{p}_2$. Then

$$\mathfrak{V}(\mathfrak{a}) = \mathfrak{V}\left(\bigcap_{\mathfrak{a}\subsetneq\mathfrak{p}\in\mathrm{minSpec}A} \mathfrak{p}\right) = \bigcup_{\mathfrak{a}\subsetneq\mathfrak{p}\in\mathrm{minSpec}A} \mathfrak{V}(\mathfrak{p}).$$

But $\mathfrak{V}(\mathfrak{a}) \neq \mathfrak{V}(\mathfrak{p})$ for any $\mathfrak{p} \in \mathrm{minSpec}A$, since for example $\mathfrak{p}_1 \in \mathfrak{V}(\mathfrak{a}) \setminus \mathfrak{V}(\mathfrak{p})$ for all $\mathfrak{p} \neq \mathfrak{p}_1$ and $\mathfrak{p}_2 \in \mathfrak{V}(\mathfrak{a}) \setminus \mathfrak{V}(\mathfrak{p})$ for all $\mathfrak{p} \neq \mathfrak{p}_2$ (and $\mathfrak{p}_1 \neq \mathfrak{p}_2$). Thus $\mathfrak{V}(\mathfrak{a})$ is not irreducible.

(⇐) Let $\sqrt{\mathfrak{a}} \in \mathrm{Spec}A$. We will show $\mathrm{cl}_{\mathrm{Spec}A}\left(\{\sqrt{\mathfrak{a}}\}\right) = \mathfrak{V}(\mathfrak{a})$. Note $\mathrm{cl}_{\mathrm{Spec}A}\left(\{\sqrt{\mathfrak{a}}\}\right) \subseteq \mathfrak{V}(\mathfrak{a})$ since $\sqrt{\mathfrak{a}} \in \mathfrak{V}(\mathfrak{a})$ is closed and $\mathrm{cl}_{\mathrm{Spec}A}\left(\{\sqrt{\mathfrak{a}}\}\right)$ is the smallest closed set that contains $\sqrt{\mathfrak{a}}$. Note that $\mathrm{cl}_{\mathrm{Spec}A}\left(\{\sqrt{\mathfrak{a}}\}\right) = \mathfrak{V}(\mathfrak{b})$ for some ideal $\mathfrak{b}$ of A. Then $\sqrt{\mathfrak{a}} \in \mathfrak{V}(\mathfrak{b})$ and so $\mathfrak{b} \subseteq \sqrt{\mathfrak{a}}$ and so every prime that contains $\sqrt{\mathfrak{a}}$ also contains $\mathfrak{b}$. But we also know that every prime ideal that contains $\mathfrak{b}$ is in $\mathrm{cl}_{\mathrm{Spec}A}\left(\{\sqrt{\mathfrak{a}}\}\right) \subseteq \mathfrak{V}(\mathfrak{a})$. Thus every prime ideal that contains $\mathfrak{b}$ also contains $\sqrt{\mathfrak{a}}$. Thus $\sqrt{\mathfrak{a}} = \sqrt{\mathfrak{b}}$ and so $\mathfrak{V}(\mathfrak{a}) = \mathfrak{V}(\mathfrak{b}) = \mathrm{cl}_{\mathrm{Spec}A}\left(\sqrt{\mathfrak{a}}\right)$. Thus $\sqrt{\mathfrak{a}}$ is a generic point for $\mathfrak{V}(\mathfrak{a})$, and so $\mathfrak{V}(\mathfrak{a})$ is irreducible by Lemma 1. □

Proposition 7. Let A be a noetherian ring. Then $\mathrm{Spec}A$ is a Zariski space.

Proof. First we will show $\mathrm{Spec}A$ is noetherian. Let

$$\mathfrak{V}(\mathfrak{a}_0) \supseteq \mathfrak{V}(\mathfrak{a}_1) \supseteq \mathfrak{V}(\mathfrak{a}_2) \supseteq \cdots$$

be an arbitrary descending chain of closed subsets of $\mathrm{Spec}A$. Then we have the ascending chain of ideals in A

$$\mathfrak{a}_0 \subseteq \mathfrak{a}_1 \subseteq \mathfrak{a}_2 \subseteq \cdots$$

which stabilizes after finitely many links. Thus the descending chain of closed subsets of $\mathrm{Spec}A$ stabilizes after finitely many links. Thus $\mathrm{Spec}A$ is a noetherian space.

Now let $\mathfrak{V}(\mathfrak{a})$ be an irreducible subset of $\mathrm{Spec}A$. We know from Lemma 3 that $\sqrt{\mathfrak{a}} \in \mathrm{Spec}A$ and that $\sqrt{\mathfrak{a}}$ is a generic point of $\mathfrak{V}(\mathfrak{a})$.

Now to show uniqueness. Suppose $\mathfrak{V}(\mathfrak{a}) = \mathrm{cl}_{\mathrm{Spec}A}(\{\mathfrak{b}\})$ for some $\mathfrak{b} \in \mathrm{Spec}A$. Then as we've already shown, $\mathfrak{a} = \mathfrak{b}$. Thus $\mathfrak{a}$ is the unique generic point for $\mathfrak{V}(\mathfrak{a})$ and so $\mathrm{Spec}A$ is Zariski. □

Definition. A scheme is *connected* if its topological space is connected. A scheme is *irreducible* if its topological space is irreducible.

Definition. A scheme is *reduced* if for every open set U, the ring $\mathcal{O}_X(U)$ has no (nonzero) nilpotent elements (equivalently if the all the stalks have no nonzero nilpotent elements).

Definition. A scheme is *integral* if for every open set $U \subseteq X$, the ring $\mathcal{O}_X(U)$ is an integral domain. (Note a scheme is integral if and only if it is reduced and irreducible.)

Definition. A scheme X is *locally noetherian* if it can be covered by open affine subsets $\mathrm{Spec}A_i$ where each A_i is a noetherian ring. X is *noetherian* if it is locally noetherian and quasi-compact (eqv. if it can be covered by finitely many such $\mathrm{Spec}A_i$).

Definition. A morphism $f : X \to Y$ of schemes is *locally of finite type* if there is a covering of Y by open affine subsets $V_i = \mathrm{Spec}B_i$ such that for each i, $f^{-1}(V_i)$ can be covered by open affine subsets $U_{ij} = \mathrm{Spec}A_{ij}$, where each A_{ij} is a finitely-generated B_i-algebra. The morphism f is *of finite type* if in addition each $f^{-1}(V_i)$ can be covered by finitely many of the U_{ij}

Definition. A morphism $f : X \to Y$ is a *finite* morphism if there is a covering of Y by open affine subsets $V_i = \mathrm{Spec}B_i$ such that for each i, $f^{-1}(V_i)$ is affine, equal to $\mathrm{Spec}A_i$, where A_i is a B_i-algebra which is a finitely generated B_i-module.

Definition. A morphism of schemes $f : X \to Y$ with Y irreducible is called *generically finite* if $f^{-1}(\omega)$ is a finite set, where ω is the unique generic point of Y.

Definition. A morphism of schemes $f : X \to Y$ is *dominant* if $f(X)$ is dense in Y.

Definition. If X is an integral scheme with (unique) generic point ω, the local ring $\mathcal{O}_\omega$ is a field called the *function field* of X (also denoted $K(X)$).

Definition. Let S be a graded ring. Then the *irrelevant ideal* S_+ is the ideal generated by all homogeneous elements of degree at least one. Any prime homogeneous ideal that is not S_+ is called a *relevant* prime ideal.

Definition. Let S be a graded ring. Then the *projectrum* $\mathrm{Proj}S$ of S is the set of all relevant prime ideals of S.

Note that $X = \mathbb{P}^n_k = \mathrm{Proj}k[x_0, \ldots, x_n]$ is a scheme for all fields k and all $n \in \mathbb{N}$. Let $a = [a_0 : \cdots : a_n] \in X$. Then there is some $a_i \neq 0$ by necessity. The open set given by $X \setminus \mathfrak{V}(x_i) = \mathfrak{D}_+(x_i) \cong \mathrm{Spec}k[x_0, \ldots, x_n]_{(x_i)}$ (with sheaf $\mathcal{O}_{X|\mathfrak{D}_+(x_i)}$) contains a and is homeomorphic to the affine scheme $\mathrm{Spec}k[x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n]$.

Definition. Let S be a scheme and let X, Y be schemes over S (schemes with morphisms to S). The *fibred product* of X and Y over S, denoted $X \times_S Y$ is a scheme, together with morphsims $p_1 : X \times_S Y \to X$ and $p_2 : X \times_S Y \to Y$, which make a commutative diagram with the given morphisms $X \to S$and $Y \to S$, such that any scheme Z over S, and given morphisms $f : Z \to X$ and $g : Z \to Y$ which make a commutative diagram with the given morphisms $X \to S$ and $Y \to S$, then there exists a unique morphism $\theta : Z \to X \times_S Y$ such that $f = p_1 \circ \theta$ and $g = p_2 \circ \theta$. The morphisms p_1 and p_2 are called the *projection morphisms* of the fibred product onto its factors.

$$
\begin{tikzcd}
Z \arrow[dr, dashed, "\exists!"] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& X \times_S Y \arrow[r] \arrow[d] & Y \arrow[d] \\
& X \arrow[r] & S
\end{tikzcd}
$$

The *product* of X and Y, denoted $X \times Y$, is $X \times_{\mathrm{Spec}\mathbb{Z}} Y$ ($\mathrm{Spec}\mathbb{Z}$ is the terminal object in the category of schemes).

Definition. An *open subscheme* of a scheme X is a scheme U whose topological space is an open subset of $\mathrm{sp}(X)$ and whose structure sheaf is isomorphic to the restriction $\mathcal{O}_{X|U}$ of the structure sheaf of X. An *open immersion* is a morphism $f : X \to Y$ which induces an isomorphism between X and an open subscheme of Y.

Definition. A *closed immersion* is a morphism of schemes $f : Y \to X$ which induces a homeomorphism between $\mathrm{sp}(Y)$ and a closed subspace of $\mathrm{sp}(X)$ and furthermore the induced map $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$ of sheaves on X is surjective. A *closed subscheme* of a scheme X is an equivalence class of closed immersions, where we say $f : Y \to X$ is equivalent to $f' : Y' \to X$ if there is an isomorphism $i : Y' \to Y$ where $f' = f \circ i$.

Definition. Let $f : X \to Y$ be a morphism of schemes. The *diagonal map* is a unique map $\Delta : X \to X \times_Y X$ whose composition with both the projection maps $p_1, p_2 : X \times_Y X \to X$ is the identity map $X \to X$. The morphism f is *separated* if the diagonal map Δ is a closed immersion. In this case X is *separated* over Y. A scheme is *separated* if it is separated over Spec$\mathbb{Z}$.

Proposition 8. If A is a ring, then Spec$A \cong$ SpecA_{red}.

Proof. First consider the canonical quotient map $q : A \to A/\sqrt{0A}$. Then we will define $\tilde{q} : \text{Spec}A \to \text{Spec}(A/\sqrt{0A})$ by $\tilde{q}(\mathfrak{p}) = q(\mathfrak{p})$. First we will show that $\tilde{q}$ is well-defined.

Let $\mathfrak{p} \in \text{Spec}A$. Then we will show $q(\mathfrak{p})$ is prime in $A/\sqrt{0A}$. Let $ab + \sqrt{0A} \in \mathfrak{p} + \sqrt{0A}$. Then there is a nilpotent $n \in \sqrt{0A}$ such that $ab - n \in \mathfrak{p}$. Note there is an $N > 0$ such that $n^N = 0$. Then $(ab-n)(ab+n) = a^2b^2 - n^2 \in \mathfrak{p}$. Also $(a^2b^2 - n^2)(a^2b^2 + n^2) = a^4b^4 - n^4 \in \mathfrak{p}$. Continuing on we will eventually reach $a^{2^m}b^{2^m} - n^{2^m} \in \mathfrak{p}$ where $2^m \geq N$. Then $n^{2^m} = 0$ and so $a^{2^m}b^{2^m} \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime we have either $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. We will remember this **trick** for later.

Thus $a + \sqrt{0A} \in q(\mathfrak{p})$ or $b + \sqrt{0A} \in q(\mathfrak{p})$. Thus $q(\mathfrak{p}) \in \text{Spec}A_{\text{red}}$ and so $\tilde{q}$ is well-defined.

Now we shall show that $\tilde{q}$ is continuous. Let $\mathfrak{a} \subseteq A_{\text{red}}$ be an ideal. Then $\mathfrak{V}(\mathfrak{a}) \subseteq \text{Spec}A_{\text{red}}$ is closed. We shall show that $\tilde{q}^{-1}(\mathfrak{V}(\mathfrak{a}))$ is closed. Namely, that $\tilde{q}^{-1}(\mathfrak{V}(\mathfrak{a})) = \mathfrak{V}(q^{-1}(\mathfrak{a}))$.

Let $\mathfrak{p} \in \tilde{q}^{-1}(\mathfrak{V}(\mathfrak{a}))$. Then $\tilde{q}(\mathfrak{p}) = q(\mathfrak{p}) \in \mathfrak{V}(\mathfrak{a})$. Thus $\mathfrak{a} \subseteq q(\mathfrak{p})$. Then we will show that $q^{-1}(\mathfrak{a}) \subseteq \mathfrak{p}$. Let $x \in q^{-1}(\mathfrak{a})$. Then $q(x) \in \mathfrak{a} \subseteq q(\mathfrak{p})$, so $x + \sqrt{0A} \in \mathfrak{p} + \sqrt{0A}$. Then there exists an $n \in \sqrt{0A}$ such that $x - n \in \mathfrak{p}$. By the same **trick** from earlier, we then have $x \in \mathfrak{p}$. Therefore $q^{-1}(\mathfrak{a}) \subseteq \mathfrak{p}$. Then $\mathfrak{p} \in \mathfrak{V}(q^{-1}(\mathfrak{a}))$. So $\tilde{q}^{-1}(\mathfrak{V}(\mathfrak{a})) \subseteq \mathfrak{V}(q^{-1}(\mathfrak{a}))$.

Now let $\mathfrak{p} \in \mathfrak{V}(q^{-1}(\mathfrak{a}))$. Then $q^{-1}(\mathfrak{a}) \subseteq \mathfrak{p}$, meaning if $q(y) \in \mathfrak{a}$, then $y \in \mathfrak{p}$. Now let $x \in \mathfrak{a}$. Then since q is surjective, there is a $y \in A$ such that $q(y) = x$. Therefore $y \in \mathfrak{p}$, and so $x \in q(\mathfrak{p})$. Therefore $\mathfrak{a} \subseteq q(\mathfrak{p})$. Thus $q(\mathfrak{p}) = \tilde{q}(\mathfrak{p}) \in \mathfrak{V}(\mathfrak{a})$. Thus $\mathfrak{p} \in \tilde{q}^{-1}(\mathfrak{V}(\mathfrak{a}))$. Therefore $\mathfrak{V}(q^{-1}(\mathfrak{a})) = \tilde{q}^{-1}(\mathfrak{V}(\mathfrak{a}))$. So $\tilde{q}$ is continuous.

Now let us define $h : \text{Spec}A_{\text{red}} \to \text{Spec}A$ as $h(\mathfrak{q}) = q^{-1}(\mathfrak{q})$. We know that h is well-defined because the preimage of a prime ideal is always a prime ideal. We shall show that h is continuous.

Let $\mathfrak{V}(\mathfrak{b}) \subseteq \text{Spec}A$ be closed. We will show that $h^{-1}(\mathfrak{V}(\mathfrak{b})) = \mathfrak{V}(q(\mathfrak{b}))$. Let $\mathfrak{q} \in h^{-1}(\mathfrak{V}(\mathfrak{b}))$. Then $h(\mathfrak{q}) = q^{-1}(\mathfrak{q}) \in \mathfrak{V}(\mathfrak{b})$. So $\mathfrak{b} \subseteq q^{-1}(\mathfrak{q})$, meaning if $y \in \mathfrak{b}$ then $q(y) \in \mathfrak{q}$. Let $x \in q(\mathfrak{b})$. Then there is a $y \in \mathfrak{b}$ such that $q(y) = x$. But we know that $q(y) \in \mathfrak{q}$. Thus $x \in \mathfrak{q}$. Therefore $q(\mathfrak{b}) \subseteq \mathfrak{q}$. Therefore $\mathfrak{q} \in \mathfrak{V}(q(\mathfrak{b}))$. So $h^{-1}(\mathfrak{V}(\mathfrak{b})) \subseteq \mathfrak{V}(q(\mathfrak{b}))$.

Now let $\mathfrak{q} \in \mathfrak{V}(q(\mathfrak{b}))$. Then $q(\mathfrak{b}) \subseteq \mathfrak{q}$. Then for all $x \in \mathfrak{b}$, $q(x) \in \mathfrak{q}$, and so $x \in q^{-1}(\mathfrak{q})$. Thus $\mathfrak{b} \subseteq q^{-1}(\mathfrak{q}) = h(\mathfrak{q})$. So $h(\mathfrak{q}) \in \mathfrak{V}(\mathfrak{b})$, so $\mathfrak{q} \in h^{-1}(\mathfrak{V}(\mathfrak{b}))$. Therefore $h^{-1}(\mathfrak{V}(\mathfrak{b})) = \mathfrak{V}(q(\mathfrak{b}))$ is closed, so h is continuous.

Now we will show $h = \tilde{q}^{-1}$. Let $\mathfrak{p} \in \text{Spec}A$. Then we will show $h(\tilde{q}(\mathfrak{p})) = q^{-1}(q(\mathfrak{p})) = \mathfrak{p}$. First let $x \in q^{-1}(q(\mathfrak{p}))$. Then $q(x) \in q(\mathfrak{p})$, so there is an $n \in \sqrt{0A}$ such that $x - n \in \mathfrak{p}$. By the **trick**, we know that $x \in \mathfrak{p}$. Thus $q^{-1}(q(\mathfrak{p})) \subseteq \mathfrak{p}$. The other inclusion is true in general, so $h(\tilde{q}(\mathfrak{p})) = \mathfrak{p}$. So $h \circ \tilde{q}$ is the identity on Spec$A$

Since q is surjective, we have the equality $q(q^{-1}(\mathfrak{p})) = \mathfrak{p}$. Thus $\tilde{q} \circ h$ is the identity on SpecA_{red}. So $h = \tilde{q}^{-1}$. Thus $\tilde{q}$ is a homeomorphism. □

Note: A very similar proof shows that $\mathfrak{V}(\mathfrak{a}) \cong \mathrm{Spec}A/\mathfrak{a}$ for all ideals $\mathfrak{a} \subseteq A$. Thus every closed subscheme of an affine scheme is affine.

Let A be a ring and let $X = \mathrm{Spec}A$ be an affine scheme. Then $\mathcal{O}_X$ is the sheaf corresponding to X which takes open sets $U \subseteq X$ and sends them to $\mathcal{O}_X(U)$ (also called $\Gamma(U, \mathcal{O}_X)$) consisting of functions (called *sections* on U) $s : U \to \coprod_{\mathfrak{p}\in U} A_{\mathfrak{p}}$ that satisfy the following criteria:

(1) $s(\mathfrak{p}) \in A_{\mathfrak{p}}$ for all $\mathfrak{p} \in U$.

(2) s is locally a quotient of elements of A: that is, for each $\mathfrak{p} \in U$ there is a neighborhood $V_{\mathfrak{p}} \subseteq U$ of $\mathfrak{p}$, and elements $a, f \in A$ such that for each $\mathfrak{q} \in V$, we have $f \notin \mathfrak{q}$ and $s(\mathfrak{q}) = a/f \in A_{\mathfrak{q}}$.

Proposition 9. Let A be a lucky ring, then $X = \mathrm{Spec}A$ be an irreducible affine scheme, and let $U \subseteq X$ be open. Then for all $s \in \Gamma(U, \mathcal{O}_X)$, we have s is *globally* a quotient on A. That is, there are $a, f \in A$ such that $f \notin \mathfrak{p}$ for all $\mathfrak{p} \in U$ and $s(\mathfrak{p}) = a/f$ for all $\mathfrak{p} \in U$.

Proof. Let $\mathfrak{p}, \mathfrak{q} \in U$ and let $s \in \Gamma(U, \mathcal{O}_X)$. Then since s is locally a quotient on A, there are $V_{\mathfrak{p}} \subseteq U$ and $V_{\mathfrak{q}} \subseteq U$ and $a_{\mathfrak{p}}, a_{\mathfrak{q}}, f_{\mathfrak{p}}, f_{\mathfrak{q}} \in A$ such that $f_{\mathfrak{p}} \notin V_{\mathfrak{p}}$ and $f_{\mathfrak{q}} \notin V_{\mathfrak{q}}$ and $s(\mathfrak{p}') = a_{\mathfrak{p}}/f_{\mathfrak{p}}$ for all $\mathfrak{p}' \in V_{\mathfrak{p}}$ and $s(\mathfrak{q}') = a_{\mathfrak{q}}/f_{\mathfrak{q}}$ for all $\mathfrak{q}' \in V_{\mathfrak{q}}$. Then since X is irreducible, all nonempty open subsets intersect nontrivially. Thus there is an $\mathfrak{a} \in V_{\mathfrak{p}} \cap V_{\mathfrak{q}}$ such that $s(\mathfrak{a}) = a_{\mathfrak{p}}/f_{\mathfrak{p}} = a_{\mathfrak{q}}/f_{\mathfrak{q}}$. Thus $f_{\mathfrak{p}} \notin \mathfrak{q}$ and $f_{\mathfrak{q}} \notin \mathfrak{p}$ and $s(\mathfrak{p}') = a_{\mathfrak{q}}/f_{\mathfrak{q}}$ for all $\mathfrak{p}' \in V_{\mathfrak{p}}$ and $s(\mathfrak{q}') = a_{\mathfrak{p}}/f_{\mathfrak{p}}$ for all $\mathfrak{q}' \in V_{\mathfrak{q}}$. Thus $s(\mathfrak{p}) = a_{\mathfrak{q}}/f_{\mathfrak{q}}$ and $s(\mathfrak{q}) = a_{\mathfrak{p}}/f_{\mathfrak{p}}$. Thus the same choice of a and f can be used to represent $s(\mathfrak{q})$ and $s(\mathfrak{p})$. Since $\mathfrak{p}$ and $\mathfrak{q}$ were arbitrary, we have that s is globally a quotient on A. □

Example 1. Let $A = \mathbb{C}[x, y]/(y - x^2)$ and let $X = \mathrm{Spec}A$. Consider the open set $U = X \setminus \mathfrak{V}(x + 1)$. Note that $\mathfrak{V}(x + 1) = \{(x + 1)\}$ and thus $U = \{(x - z), 0 : z \in \mathbb{C} \setminus \{-1\}\}$. An example of a section $s \in \Gamma(U, \mathcal{O}_X)$ is that which sends all elements of U to $\dfrac{1}{x+1}$, since $x + 1 \in A_{(x-z)}$ for $z \in \mathbb{C} \setminus \{-1\}$ and $x + 1 \in A_0$.

Note that $s(0) = s(x - z)$ for all $z \in \mathbb{C} \setminus \{-1\}$ since 0 is in every nonempty open set of X (and thus U). Since X is irreducible, we know that all open subsets of X intersect nontrivially, and thus s must be globally a quotient. This means that there are $a, f \in A$ such that $f \notin \mathfrak{p}$ and $s(\mathfrak{p}) = a/f \in A_{\mathfrak{p}} \subseteq \coprod_{\mathfrak{p}\in U} A_{\mathfrak{p}}$ for all $\mathfrak{p} \in U$.

Proposition 10. Let A be a ring. Then the open sets of the form $\mathfrak{D}(f)$ where $f \in A$, form a basis for $\mathrm{Spec}A$ and $\mathfrak{D}(f) \cong \mathrm{Spec}A_f$

Proof. First note that $\mathfrak{D}(f)$ is open since $\mathfrak{D}(f) = \mathrm{Spec}A \setminus \mathfrak{V}((f))$. Furthermore, the $\mathfrak{D}(f)$ cover $\mathrm{Spec}A$ since $\mathfrak{D}(1) = \mathrm{Spec}A$.

Now let $f, g \in A$ and consider $\mathfrak{D}(f) \cap \mathfrak{D}(g)$. Then $\mathfrak{D}(fg) = \mathfrak{D}(f) \cap \mathfrak{D}(g)$ since if $\mathfrak{p} \in \mathfrak{D}(fg)$, then $fg \notin \mathfrak{p}$, and so $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$, and thus $\mathfrak{p} \in \mathfrak{D}(f) \cap \mathfrak{D}(g)$. Furthermore, if $\mathfrak{p} \in \mathfrak{D}(f) \cap \mathfrak{D}(g)$, then $f \notin \mathfrak{p}$ and $g \notin \mathfrak{p}$. Since $\mathfrak{p}$ is prime, this implies $fg \notin \mathfrak{p}$. Thus for all $\mathfrak{p} \in \mathfrak{D}(f) \cap \mathfrak{D}(g)$, we have $\mathfrak{p} \in \mathfrak{D}(fg) \subseteq \mathfrak{D}(f) \cap \mathfrak{D}(g)$, and so $\{\mathfrak{D}(f)\}_{f\in A}$ forms a basis on $\mathrm{Spec}A$.

Now let us define a map $\varphi : \mathfrak{D}(f) \to \mathrm{Spec}A_f$ by $\varphi(\mathfrak{p}) = \mathfrak{p}_f$. Let $\frac{a}{f^n}\frac{b}{f^m} \in \mathfrak{p}_f$. Then $f^{n+m}\frac{a}{f^n}\frac{b}{f^m} = ab \in \mathfrak{p}_f = \mathfrak{p}A_f$. Then since $ab \in A$, we have $ab \in \mathfrak{p}A_f \cap A = \mathfrak{p}$. Since $\mathfrak{p}$ is prime, then $a \in \mathfrak{p}$ or $b \in \mathfrak{p}$. Therefore $\frac{a}{f^n} \in \mathfrak{p}_f$ or $\frac{a}{f^m} \in \mathfrak{p}_f$, respectively. Thus $\mathfrak{p}_f$ is prime in A_f. Note also that $\mathfrak{p}_f \neq A_f$ since $1 \notin \mathfrak{p}_f$ since $f \notin \mathfrak{p}$. Thus φ is well-defined.

Now let $\mathfrak{a}$ be an ideal of A_f. Then $\mathfrak{V}(\mathfrak{a}) \subseteq \mathrm{Spec}A_f$ is closed. We will show that $\varphi^{-1}(\mathfrak{V}_{A_f}(\mathfrak{a})) = \mathfrak{V}_A(\mathfrak{a} \cap A) \cap \mathfrak{D}(f)$. First let $\mathfrak{p} \in \varphi^{-1}(\mathfrak{V}_{A_f}(\mathfrak{a}))$. Then $\varphi(\mathfrak{p}) \in \mathfrak{V}_{A_f}(\mathfrak{a})$ and so $\mathfrak{a} \subseteq \varphi(\mathfrak{p}) = \mathfrak{p}A_f$. Then $\mathfrak{a} \cap A \subseteq \mathfrak{p}A_f \cap A = \mathfrak{p}$ and so $\mathfrak{p} \in \mathfrak{V}_A(\mathfrak{a} \cap A)$. Furthermore, $\mathfrak{p} \in \mathfrak{D}(f)$ by assumption of being in a preimage under φ.

Now let $\mathfrak{p} \in \mathfrak{V}_A(\mathfrak{a} \cap A) \cap \mathfrak{D}(f)$. Then $f \notin \mathfrak{p}$ and $\mathfrak{a} \cap A \subseteq \mathfrak{p}$. Then $(\mathfrak{a} \cap A)A_f = \mathfrak{a} \subseteq \mathfrak{p}A_f = \varphi(\mathfrak{p})$, so $\mathfrak{p} \in \varphi^{-1}(\mathfrak{V}_{A_f}(\mathfrak{a}))$. Thus φ is continuous.

Now to define a map $\mu : \mathrm{Spec}A_f \to \mathfrak{D}(f)$ by $\mu(\mathfrak{q}) = \mathfrak{q} \cap A$. First let $ab \in \mathfrak{q} \cap A$. Then $ab \in \mathfrak{q}$ and since $\mathfrak{q}$ is prime, $a \in \mathfrak{q}$ or $\mathfrak{b} \in \mathfrak{q}$, so $a \in \mathfrak{q} \cap A$ or $b \in \mathfrak{q} \cap A$ respectively. So μ is well-defined.

Now we will show that μ is continuous. Let $V = \mathfrak{V}_A(\mathfrak{a}) \cap \mathfrak{D}(f)$ be closed in $\mathfrak{D}(f)$. Then we will show $\mu^{-1}(V) = \mathfrak{V}_{A_f}(\mathfrak{a}A_f)$. First let $\mathfrak{q} \in \mu^{-1}(V)$. Then $\mathfrak{a} \subseteq \mu(\mathfrak{q}) = \mathfrak{q} \cap A$ and $f \notin \mu(\mathfrak{q})$. Then $\mathfrak{a}A_f \subseteq (\mathfrak{q} \cap A)A_f = \mathfrak{q}$, so $\mathfrak{q} \in \mathfrak{V}_{A_f}(\mathfrak{a}A_f)$.

Now let $\mathfrak{q} \in \mathfrak{V}_{A_f}(\mathfrak{a}A_f)$. Then $\mathfrak{a}A_f \subseteq \mathfrak{q}$ and also $f \notin \mathfrak{q} \cap A = \mu(\mathfrak{q})$ since if $f \in \mathfrak{q} \cap A$ then $1 \in \mathfrak{q}$ and thus $\mathfrak{q}$ is not prime. Thus $(\mathfrak{a}A_f) \cap A = \mathfrak{a} \subseteq \mathfrak{q} \cap A = \mu(\mathfrak{q})$. Thus $\mu(\mathfrak{q}) \in V$ and so $\mathfrak{q} \in \mu^{-1}(V)$. Thus μ is continuous.

Now note that $\varphi(\mu(\mathfrak{q})) = (\mathfrak{q} \cap A)A_f = \mathfrak{q}$ and $\mu(\varphi(\mathfrak{p})) = (\mathfrak{p}A_f) \cap A = \mathfrak{p}$. Thus φ is a homeomorphism. $\square$

So if $\varphi : A \to B$ is a ring homomorpism then $\mathrm{Spec}\varphi(A) \cong \mathfrak{V}(\ker\varphi) \subseteq \mathrm{Spec}A$. Additionally, if $\psi : \mathcal{F} \to \mathcal{G}$ is a surjective morphism of sheaves over X, then $\psi(X) : \Gamma(X, \mathcal{F}) \to \Gamma(X, \mathcal{G})$ is a surjective ring homomorphism, so $\mathfrak{V}(\ker\psi(X)) \cong \mathrm{Spec}\Gamma(X, \mathcal{G}) \subseteq \mathrm{Spec}\Gamma(X, \mathcal{F})$.

Proposition 11. Let X be a scheme. Then affine schemes make up a basis for X.

Proof. We already know from the definition of a scheme that affine schemes form an open cover of X. Now let $U = \mathrm{Spec}A$ and $V = \mathrm{Spec}B$ be two nondisjoint open affine subschemes of X. Then let $\mathfrak{p} \in W = \mathrm{Spec}A \cap \mathrm{Spec}B \neq$ is an open subspace of $\mathrm{Spec}A$. Thus W is a union of open sets of the form $\mathfrak{D}(f)$, where $f \in A$, since $\{\mathfrak{D}(f)\}_{f \in A}$ forms a basis on $\mathrm{Spec}A$. Then there is an $f \in A$ such that $\mathfrak{p} \in \mathfrak{D}(f) \subseteq W = \mathrm{Spec}A \cap \mathrm{Spec}B$ and $\mathfrak{D}(f) \cong \mathrm{Spec}A_f$. Thus affine subschemes form a basis on X. Furthermore, every open subscheme of an any scheme contains an affine scheme. $\square$

Lemma 4. Let S be a graded ring and let $\mathfrak{s}$ be an ideal of S. Then $\mathfrak{s}$ is a homogeneous ideal if and only if for all $s \in \mathfrak{s}$, we have $s_i \in \mathfrak{s}$ where s_i is the i^{th} homogeneous component of s, for all i.

Proof. ($\Rightarrow$) Let $\mathfrak{s}$ be a homogeneous ideal of S. Then $\mathfrak{s}$ can be generated by homogeneous elements of S. Let $s = s_0 + s_1 + \cdots + s_n \in \mathfrak{s}$ where s_i is a homogeneous element of S for all $1 \leq i \leq n$. Then since $s \in \mathfrak{s}$, we know that there exist homogeneous elements $h_0, \ldots, h_m \in \mathfrak{s}$ and elements $r_0, \ldots, r_m \in S$ such that $s = r_0h_0 + r_1h_1 + \cdots + r_mh_m$. Without loss of generality, let $r_i \in S_0$ for all $1 \leq i \leq m$. If $r_i \notin S_0$, then we know that either r_i is homogeneous, in which case r_ih_i is itself an homogeneous element of $\mathfrak{s}$ that we could have taken instead, or

r_ih_i is not homogeneous but could be broken down into homogeneous components, each of which are in $\mathfrak{s}$ because they are each products of $h_i \in \mathfrak{s}$. In fact, we can assume without loss of generality that $r_i = 1$ for all $1 \leq i \leq m$. Thus $s = h_0 + \cdots + h_m = s_0 + \cdots + s_n$. Thus $m = n$ and $h_i = s_i$ for all $1 \leq i \leq n$ and so $s_i \in \mathfrak{s}$ for all $1 \leq i \leq n$.

($\Leftarrow$) Now assume that for all $s \in \mathfrak{s}$, each homogeneous component of s is in $\mathfrak{s}$. Then $\mathfrak{s}$ is generated by $\{r : r \text{ is a homogeneous component of } s, s \in \mathfrak{s}\}$ and so $\mathfrak{s}$ is a homogeneous ideal. $\square$

Corollary 1. The arbitrary intersection of homogeneous ideals is homogeneous.

Proof. Let $\mathfrak{s} = \bigcap_{\alpha \in J} \mathfrak{s}_\alpha$ be an intersection of arbitrarily many homogeneous ideals of S. Then let $s \in \mathfrak{s}$. Then $s \in \mathfrak{s}_\alpha$ for each α and so each homogeneous component s_i of s is in s_α for each α. Then $s_i \in \mathfrak{s}$ and so $\mathfrak{s}$ is homogeneous. $\square$

Lemma 5. Let S be a graded ring and let $\mathfrak{s}$ be a homogeneous ideal of S. Then $\sqrt{\mathfrak{s}}$ is a homogeneous ideal.

Proof. Let $s \in \sqrt{\mathfrak{s}}$, so $s^n \in \mathfrak{s}$ for some $n \in \mathbb{N}$. Let $s = s_0 + s_1 + \cdots + s_k$ where $s_i \in S_i$ and $k \in \mathbb{N}$. Then $s^n = u_0 + u_1 + \cdots + u_m$ for some $u_i \in S_i$ and $m \in \mathbb{N}$. Then we know that $s_k^n = u_m$, since they are each the highest-degree homogeneous component of s^n. Then $u_m \in \mathfrak{s}$ since $\mathfrak{s}$ is homogeneous and so $s_k^n \in \mathfrak{s}$ and thus $s_k \in \sqrt{\mathfrak{s}}$. Then $s - s_k \in \sqrt{\mathfrak{s}}$. This process repeats (ending in finite time) to show that all homogeneous components of s lie in $\sqrt{\mathfrak{s}}$. Thus $\sqrt{\mathfrak{s}}$ is homogeneous. $\square$

Corollary 2. For any graded ring S, the nilradical $\sqrt{0S}$ is homogeneous.

Proof. Since $0S$ is a homogeneous ideal of S (generated by 0), take $\mathfrak{s} = 0S$ and apply Lemma 5. $\square$

Problem 3. (Hartshorne 2.2.14) Let S be a graded ring.

(a) Show $\text{Proj}S = \emptyset$ if and only if every element of S_+ is nilpotent.

(b) Let $\varphi : S \to T$ be a graded homomorphism of graded rings (preserving degrees). Let $U = \{\mathfrak{p} \in \text{Proj}T : \varphi(S_+) \not\subseteq \mathfrak{p}\}$. Show that U is an open subset of $\text{Proj}T$ and that U determines a natural morphism $f : U \to \text{Proj}S$.

(c) The morphism f can be an isomorphism even when φ is not. For example, suppose $\varphi_d : S_d \to T_d$ is an isomorphism for all $d \geq d_0$, where d_0 is some integer. Then show $U = \text{Proj}T$ and the morphism $f : \text{Proj}T \to \text{Proj}S$ is an isomorphism.

(d) Let V be a projective variety with homogeneous coordinate ring S. Show that $t(V) \cong \text{Proj}S$.

Proof.

(a) ($\Rightarrow$) Let hSpec$S$ denote the set of all homogeneous primes of S. Suppose Proj$S = \emptyset$. Then there are no homogeneous primes of Spec$S$ not containing the irrelevant ideal. In other words, every homogeneous prime of S contains S_+, so $S_+ \subseteq \bigcap_{\mathfrak{p} \in \text{hSpec}S} \mathfrak{p}$. Now we wish to show that every prime of S contains S_+, or that $S_+ \subseteq \sqrt{0S}$.

Then since $\sqrt{0S}$ is homogeneous, we know that an element $s \in S$ is nilpotent if and only if each of its homogeneous components are nilpotent.

Then since hSpec$S \subseteq$ Spec$S$, we have that

$$\sqrt{0S} = \bigcap_{\mathfrak{p} \in \text{Spec}S} \mathfrak{p} \subseteq \bigcap_{\mathfrak{p} \in \text{hSpec}S} \mathfrak{p}.$$

Now let $s \in \bigcap_{\mathfrak{p} \in \text{hSpec}S} \mathfrak{p}$. Then every homogeneous component of s is in every homogeneous prime.

(b) Let us consider $V = \text{Proj}T \setminus U = \{\mathfrak{p} \in \text{Proj}T : \varphi(S_+) \subseteq \mathfrak{p}\}$. Then $V = \mathfrak{V}(\varphi(S_+))$ and so is closed, so U is open.

Now let us define a function $f : U \to \text{Proj}S$ by $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. We know that $\varphi^{-1}(\mathfrak{p})$ is a homogeneous prime because $\mathfrak{p}$ is a homogeneous prime. Furthermore, $S_+ \not\subseteq \varphi^{-1}(\mathfrak{p})$ by the definition of U. Thus $f\varphi^{-1}(\mathfrak{p}) \in \text{Proj}S$ and so f is well-defined.

Now f is continuous because contraction preserves containment relations.

(c) Let's have the example $S = \mathbb{C}[x,z]/(z^2)$ and $T = \mathbb{C}[y]$. Then let's have $\varphi : S \to T$ be defined so that $\varphi_0 : \mathbb{C} \to \mathbb{C}$ is $\text{id}_{\mathbb{C}}$, and φ_1 sends x and z to y. Thus φ_1 is not an isomorphism, but φ_d is for all $d \geq 2$.

Now $U = \{\mathfrak{p} \in \text{Proj}T : \varphi((x,z)) \not\subseteq \mathfrak{p}\}$. Note that $\varphi((x,z)) = (y) = T_+$. Then $U = \{\mathfrak{p} \in \text{Proj}T : T_+ \not\subseteq \mathfrak{p}\} = \text{Proj}T$.

Then $f : \text{Proj}T \to \text{Proj}S$ is defined by $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$.

$\square$

Problem 4. (Hartshorne 2.2.18) Let $\varphi : A \to B$ be a ring homomorphism and $f : Y = \text{Spec}B \to X = \text{Spec}A$ be the induced morphism of affine schemes.

(a) Let $a \in A$. Show that a is nilpotent if and only if $\mathfrak{D}(a)$ is empty.

(b) Show that φ is injective if and only if the induced map on sheaves $f^\sharp : \mathcal{O}_X \to f_*\mathcal{O}_Y$ is injective. Show furthermore that f is dominant in that case ($f(Y)$ contains every minimal prime of A?).

(c) Show that if φ is surjective, then $f^\sharp$ is surjective and f is a homeomorphism of Y onto a closed subset of X.

(d) Prove the converse of (c).

Proof.

(a) Let $a \in \sqrt{0A}$. Then a is in every prime ideal of A. Since $\mathfrak{D}(a)$ is defined to be the set of prime ideals that do not contain a, it follows that $\mathfrak{D}(a)$ must be empty.

Now let $\mathfrak{D}(a) = \emptyset$. Then there are no primes of A which do not contain a. In other words, every prime ideal of A contains a, and so $a \subseteq \bigcap_{\mathfrak{p}\in \mathrm{Spec}A} \mathfrak{p} = \sqrt{0A}$.

(b) ($\Rightarrow$) Let $U \subseteq X$ be open. We must show that $f^\sharp(U) : \mathcal{O}_X(U) \to \mathcal{O}_Y(f^{-1}(U))$ is injective. Since φ is injective, we know $\varphi_\mathfrak{q} : A_{\varphi^{-1}(\mathfrak{q})} \to B_\mathfrak{q}$ is injective for all $\mathfrak{q} \in \mathrm{Spec}B$. Note that $f^\sharp$ satisfies the condition that for any point $\mathfrak{q} \in Y$, and any neighborhood U of $\mathfrak{p} = f(\mathfrak{q})$ in X, a section $s \in \mathcal{O}_X(U)$ vanishes at $\mathfrak{p}$ if and only if the section $f^\sharp(U)(s) \in f_*\mathcal{O}_Y = \mathcal{O}_Y(f^{-1}(U))$ vanishes at $\mathfrak{q}$. (Not necessarily $f^\sharp(s) = s \circ f$, see Eisenbud, Harris pg. 28-30).

Now suppose $f^\sharp(U)(s) = 0$. Thus $f^\sharp(U)(s)$ vanishes at $\mathfrak{q}$ for all $\mathfrak{q} \in f^{-1}(U)$. Therefore s vanishes at $f(\mathfrak{q})$, by the definition of $f^\sharp$. So for all $\mathfrak{q} \in f^{-1}(U)$, we have $s(f(\mathfrak{q})) = 0$. If φ is an **integral injection**, then f is surjective. Thus for all $\mathfrak{p} \in U$, there is a $\mathfrak{q} \in f^{-1}(U)$ such that $f(\mathfrak{q}) = \mathfrak{p}$. We know that $s(f(\mathfrak{q})) = 0$ and so $s(\mathfrak{p}) = 0$, and so s is zero. Thus $f^\sharp(U)$ is injective.

($\Leftarrow$) Now suppose that $f^\sharp(U)$ is injective. Then let $a \in A$ be such that $\varphi(a) = 0$. Since $f^\sharp(U)$ is injective, that means that if there is any $\mathfrak{q} \in f^{-1}(U)$ for which $f^\sharp(U)(s)(\mathfrak{q}) \neq 0$, then $s \neq 0$. In this case let $U = \mathfrak{D}(a)$ (note that if $a \in \sqrt{0A}$ then $\mathfrak{D}(a) = \emptyset$ and so $\mathcal{O}_X(\mathfrak{D}(a)) = 0 = f_*\mathcal{O}_Y(\mathfrak{D}(a))$, and so $f^\sharp(\mathfrak{D}(a))$ is vacuously injective).

If $a \notin \sqrt{0A}$ then $\mathfrak{D}(a) \neq \emptyset$ and so there is at least one prime $\mathfrak{p} \in \mathfrak{D}(a)$ not containing a. Then $f^{-1}(\mathfrak{D}(a)) = \{\mathfrak{q} \in \mathrm{Spec}B : a \notin f(\mathfrak{q})\} = \{\mathfrak{q} \in \mathrm{Spec}B : a \notin \varphi^{-1}(\mathfrak{q})\} = \emptyset$ since $\varphi(a) = 0 \in \mathfrak{q}$. Thus $f_*\mathcal{O}_Y(\mathfrak{D}(a)) = 0$. Since we're given that $f^\sharp$ is injective, we know that $\mathcal{O}_X(\mathfrak{D}(a)) = 0$ as well. Thus $\mathfrak{D}(a) = \emptyset$ and so $a \in \sqrt{0A}$. (wlog A is reduced?)

Now we will show that f is dominant if φ is injective. One way to accomplish this is to show that $f(Y)$ contains every minimal prime of A. Let $\mathfrak{p} \in \mathrm{minSpec}A$. Then we want to show that there is a prime $\mathfrak{q} \in \mathrm{Spec}B$ such that $\mathfrak{p} = \varphi^{-1}(\mathfrak{q})$. Note that $\mathfrak{p}$ is the generic point for a maximal irreducible component of X, by Lemma 3. That is, $X = \bigcup_{\mathfrak{p}\in \mathrm{minSpec}A} \mathfrak{V}(\mathfrak{p})$.

Let $\mathfrak{q} \in \mathrm{minSpec}B$. Then $\mathfrak{V}(\mathfrak{q})$ is a maximal irreducible closed subset of $\mathrm{Spec}B$, with generic point $\mathfrak{q}$. Then $f(\mathfrak{V}(\mathfrak{q}))$ is an irreducible subset of $\mathrm{Spec}A$, and $f(\mathfrak{V}(\mathfrak{q})) = \mathrm{cl}_{f(\mathfrak{V}(\mathfrak{q}))}(\{f(\mathfrak{q})\})$ by Proposition 4.

Let $\mathfrak{p} \in \mathrm{minSpec}A$ and consider the ideal $\varphi(\mathfrak{p})B$. First we wish to show $\mathfrak{p} = \varphi^{-1}(\varphi(\mathfrak{p})B)$. Let $a \in \mathfrak{p}$. Then $\varphi(a) \in \varphi(\mathfrak{p})B$ and so $a \in \varphi^{-1}(\varphi(\mathfrak{p})B)$. Then $\mathfrak{p} \subseteq \varphi^{-1}(\varphi(\mathfrak{p})B)$.

□

Problem 5. (Hartshorne 2.3.13) *Properties of Morphisms of Finite Type*

(a) A closed immersion is a morphism of finite type.

(b) A quasi-compact immersion is a morphism of finite type.

(c) A composition of two morphisms of finite type is of finite type.

(d) Morphisms of finite type are stable under base extension.

(e) If X and Y are schemes of finite type over S, then $X \times_S Y$ is a scheme of finite type over S.

(f) If $X \xrightarrow{f} Y \xrightarrow{g} Z$ are two morphisms, and f is quasi-compact, and $g \circ f$ is of finite type, then f is of finite type.

(g) If $f : X \to Y$ is a morphism of finite type and Y is noetherian, then X is noetherian.

Proof.

(a) Let $U \subseteq Y$ be open affine. Then $f|_{f^{-1}(U)} : f^{-1}(U) \to U$ is a closed immersion, since it is a homeomorphism onto the the closed subset $f(X) \cap U$ of U and because $f|^{\sharp}_{f^{-1}(U)} : \mathcal{O}_{Y|U} \to f_*\mathcal{O}_{X|f^{-1}(U)}$ is surjective. We will show this latter assertion by showing the map is surjective on the stalks. We know that $f^\sharp : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is surjective and is therefore surjective on each local morphism of stalks $f^\sharp_p : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is surjective for $p \in X$. Since for all $p \in f^{-1}(U)$, we have $\mathcal{O}_{Y,f(p)} = \mathcal{O}_{Y|U,f(p)}$ and $\mathcal{O}_{X,p} = \mathcal{O}_{X|f^{-1}(U),p}$, then surjectivity of $f|^{\sharp}_{f^{-1}(U),p}$ is inherited from that of $f^\sharp_p$.

□

Proposition 12. Let $(f, f^\sharp) : (\mathrm{Spec}A, \mathcal{O}_{\mathrm{Spec}A}) \to (\mathrm{Spec}B, \mathcal{O}_{\mathrm{Spec}B})$ be a morphism of schemes. Then $(f, f^\sharp)$ is induced by a B-algebra homomorphism $\varphi : B \to A$. (See Hartshorne 2.3c)

Proof. Recall $f^\sharp$ goes from $\mathcal{O}_{\mathrm{Spec}B}$ to $f_*\mathcal{O}_{\mathrm{Spec}A}$. By taking global sections, we see that $f^\sharp$ induces a ring homomorphism $\varphi : \Gamma(\mathrm{Spec}B, \mathcal{O}_{\mathrm{Spec}B}) \to \Gamma(\mathrm{Spec}A, \mathcal{O}_{\mathrm{Spec}A})$. By Hartshorne 2.2c, these rings are isomorphic to B and A, respectively. Now we will show that φ induces f.

For any $\mathfrak{p} \in \mathrm{Spec}A$, we have the induced local homomorphism on the stalks, $\mathcal{O}_{\mathrm{Spec}B,f(\mathfrak{p})} \to \mathcal{O}_{\mathrm{Spec}A,\mathfrak{p}}$, or $B_{f(\mathfrak{p})} \to A_{\mathfrak{p}}$, which must be compatible with φ on the global sections and localization homomorphisms. In other words, we have the commutative diagram

$$\begin{tikzcd} B \arrow[r, "\varphi"] \arrow[d] & A \arrow[d] \\ B_{f(\mathfrak{p})} \arrow[r, "f^\sharp_{\mathfrak{p}}"] & A_{\mathfrak{p}} \end{tikzcd}.$$

Since $f^\sharp$ is a local homomorphism, it follows that $f(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$, which shows f coincides with the spectral map $\mathrm{Spec}A \to \mathrm{Spec}B$ induced by φ. Then $f^\sharp$ also comes from φ, so the morphism of schemes $(f, f^\sharp)$ is indeed induced by φ. □

Note that $\mathfrak{D}(f) \subseteq \mathfrak{D}(g)$ if and only if $f \in \sqrt{(g)}$. So there is an $a \in A$ and $n \in \mathbb{N}$ such that $f^n = ag$. Thus the induced restriction map under the $\mathcal{O}_{\mathrm{Spec}A}$ sheaf is

$$\rho : A\left[\frac{1}{g}\right] = A\left[\frac{a}{f^n}\right] \to A\left[\frac{1}{f}\right]$$

given by inclusion.

Thus, $\mathfrak{D}(f) = \mathfrak{D}(g)$ if and only if $\sqrt{(f)} = \sqrt{(g)}$.

Now let B be a Boolean ring and let $f \in B$. We shall show that $\mathfrak{D}(f)$ is open and closed in Spec$B$. By construction, $\mathfrak{D}(f)$ is open. Now we claim that $\mathfrak{D}(f) = \mathfrak{V}(f+1)$.

First let $\mathfrak{p} \in \mathfrak{D}(f)$. Then $f \notin \mathfrak{p}$. We know that $f(f+1) = f^2 + f = f + f = 0 \in \mathfrak{p}$. Since $\mathfrak{p}$ is prime and $f \notin \mathfrak{p}$, we can conclude $f + 1 \in \mathfrak{p}$. Thus $\mathfrak{p} \in \mathfrak{V}(f+1)$. Now let $\mathfrak{p} \in \mathfrak{V}(f+1)$. Then $f + 1 \in \mathfrak{p}$ and so $f \notin \mathfrak{p}$ since $1 \notin \mathfrak{p}$. Thus $\mathfrak{D}(f) = \mathfrak{V}(f+1)$.

Now let $f, g \in B$. Then

$$\mathfrak{D}(f) \cup \mathfrak{D}(g) = \mathfrak{V}(f+1) \cup \mathfrak{V}(g+1) = \mathfrak{V}((f+1)(g+1)) = \mathfrak{V}(fg+f+g+1) = \mathfrak{D}(fg+f+g).$$

Note that in a Bézout Domain A and $a, a' \in A$, that $\mathfrak{D}(a) \cup \mathfrak{D}(a') = \mathfrak{D}(\gcd(a, a'))$. In fact, the Boolean ring B is a Bézout Domain with $\gcd(f, g) = fg + f + g =: h$.

We can see that $h|f$ as $fh = fg + f + fg = f$ and $h|g$ as $gh = fg + fg + g = g$. Now let $b \in B$ be such that $b|f$ and $b|g$. We must show that $b|h$. Let $bx = f$ and $by = g$. Then $bxg = fg$ and so

$$bxg + bx + by = b(xg + x + y) = h.$$

Thus $b|h$ and so $h = \gcd(f, g)$.