Exercise 7.3. Let $Y \subseteq \mathbb{P}^2$ be a curve. For each nonsingular point $P \in Y$, show there is a unique line $T_P(Y)$ whose intersection multiplicity with Y at P is > 1.

Proof. Note that Y has coordinate ring $k[x,y,z]/(f)$ for some $f \in k[x,y,z]$ of degree n. We claim that

$$D = \frac{\partial f}{\partial x}(P)x + \frac{\partial f}{\partial y}(P)y + \frac{\partial f}{\partial z}(P)z = 0$$

is the line tangent to Y at P. We will show that $k[x,y,z]/(f,D)_P$ has length > 1 as a $k[x,y,z]_P$-module.

Since the group of automorphisms $\mathrm{Aut}\mathbb{P}^2$ acts transitively on $\mathbb{P}^2$, it is sufficient to prove this in the case $P = (0,0,1)$ by changing coordinates. Since $(0,0,1) \in Y$, we know that f has no z^n-term. Since Y is nonsingular at $(0,0,1)$, we know that $\frac{\partial f}{\partial x}(0,0,1) \neq 0$ or $\frac{\partial f}{\partial y}(0,0,1) \neq 0$ or $\frac{\partial f}{\partial z}(0,0,1) \neq 0$. We claim that $\frac{\partial f}{\partial z}(0,0,1) = 0$. This follows from there being no z^n-term in f, and so $\frac{\partial f}{\partial z}$ has no z^{n-1}-term. Therefore either $\frac{\partial f}{\partial x}(0,0,1) \neq 0$ or $\frac{\partial f}{\partial y}(0,0,1) \neq 0$. Thus f has terms that are linear in x or y. Thus we can write $f = axz^{n-1} + byz^{n-1} + g$ where $g \in (x,y)^2$ and $a,b \in k$ with $a \neq 0$ or $b \neq 0$.

Note in this case that $\frac{\partial f}{\partial x}(0,0,1) = a$ and $\frac{\partial f}{\partial y}(0,0,1) = b$. Therefore in our case we have $D = ax + by$ with $a \neq 0$ or $b \neq 0$. Then we will show that $k[x,y,z]/(f, ax+by)_{(x,y)}$ has length > 1 as a $k[x,y,z]_{(x,y)}$-module.

Observe that $k[x,y,z]/((ax+by+g)/z^{n-1}, ax+by)_{(x,y)} \cong k[t,z]/(h)_{(t)}$ where $t = -ax/b$ or $t = -by/a$ depending whether a or b is nonzero, and $h \in (t)^2$. This has length > 1 because it definitely has at least 2 submodules, generated by 1 and generated by t. Therefore $ax+by$ is the line tangent to Y at $P = (0,0,1)$.

To show that $T_P(Y)$ is unique, let us calculate the length of $k[x,y,z]/(f, ix+jy+kz)_{(x,y)}$ as a $k[x,y,z]_{(x,y)}$-module, where $ix+jy+kz \neq ax+by$. If $k \neq 0$, then our line $ix+jy+kz$ doesn't even go through our point $(0,0,1)$ and we would be killing a unit gives us the 0-module. Thus let $k=0$. Then $ix+jy \neq ax+by$. Thus $k[x,y,z]/(f, ix+jy)_{(x,y)} \cong k[t,z]/(\ell)_{(t)}$ with $\ell \in (t) \setminus (t^2)$. Thus $\ell/t \notin (t)$ is a unit and also $t(\ell/t) = \ell = 0$ and so $t = 0$. Thus $k[t,z]/(\ell)_{(t)}$ has length $= 1$. Thus $ix+jy$ is not tangent to Y at $P = (0,0,1)$. Therefore $T_P(Y)$ is unique. □

I would also like to put the Hessian problem from *Geometry of Schemes* by Eisenbud and Harris here.

Exercise IV-1. Let K be an algebraically closed field of characteristic zero, $C \subseteq \mathbb{P}^2_K$ a plane curve and $p \in C$ a nonsingular point of C. Show that the projective tangent line $T_p(C)$ has contact of order at least 3 with C at p if and only if $H(p) = 0$.

Proof. Note that the $H(x, y, z)$ function here is the *Hessian:*

$$H(x,y,z) = \det \begin{pmatrix} \dfrac{\partial^2 C}{\partial x^2} & \dfrac{\partial^2 C}{\partial xy} & \dfrac{\partial^2 C}{\partial xz} \\ \\ \dfrac{\partial^2 C}{\partial yx} & \dfrac{\partial^2 C}{\partial y^2} & \dfrac{\partial^2 C}{\partial yz} \\ \\ \dfrac{\partial^2 C}{\partial zx} & \dfrac{\partial^2 C}{\partial zy} & \dfrac{\partial^2 C}{\partial z^2} \end{pmatrix}.$$

Again let us assume that $p = (0, 0, 1)$. Then similarly as in Hartshorne Ex. 7.3,

$$C = axz^{n-1} + byz^{n-1} + cx^2z^{n-2} + dxyz^{n-2} + ey^2z^{n-2} + g$$

where $g \in (x, y)^3$ and $n = \deg C$. Then we can write

$$H(0,0,1) = \det \begin{pmatrix} 2c & d & (n-1)a \\ d & 2e & (n-1)b \\ (n-1)a & (n-1)b & 0 \end{pmatrix} = 2(n-1)^2(abd - b^2c - a^2e).$$

Now again consider the $k[x, y, z]_{(x,y)}$-module $k[x, y, z]/(C, ax+by)_{(x,y)}$. We've seen earlier that the fact $ax+by = 0$ annihilates the $(x, y) \setminus (x, y)^2$-terms of C. Now we wish to show the $(x, y)^2 \setminus (x, y)^3$-terms of C are annihilated if and only if $H(0, 0, 1) = 0$. More precisely, we wish to show– given $ax+by = 0$– that $cx^2+dxy+ey^2 = 0$ if and only if $ea^2 - dab + cb^2 = 0$.

($\Rightarrow$) Assume that $ax + by = 0$ and $cx^2 + dxy + ey^2 = 0$. Then

$$y^2(ea^2 - dab + cb^2) = a^2(cx^2 + dxy + ey^2) = 0.$$

Since $y^2 \neq 0$ and C is an irreducible curve, hence $k[x, y, z]/(C, ax + by)_{(x,y)}$ is an integral domain, we have $ea^2 - dab + cb^2 = 0$. Hence if p is a flex point, then the Hessian at p is zero.

($\Leftarrow$) Assume $ax + by = 0$ and $ea^2 - dab + cb^2 = 0$. Assume further that $b \neq 0$. Then $y = -ax/b$. Then

$$cx^2 + dxy + ey^2 = cx^2 - dax^2/b + ea^2x^2/b^2 = (x^2/b^2)(cb^2 - dab + ea^2) = 0.$$

Since $x^2/b^2 \neq 0$ and C is an irreducible curve, hence $k[x, y, z]/(C, ax + by)_{(x,y)}$ is an integral domain, we have $cx^2 + dxy + ey^2 = 0$. The case is similar for $a \neq 0$. Hence if the Hessian is zero, p is a flex point. □

Note that since $H(x,y,z)$ is a degree-$3(n-2)$ polynomial, Bézout's Theorem says that C can have at most $3n(n-2) = 3n^2 - 6n$ inflection points.

Now we wish to show that $P \in \mathrm{He}(X)$ if and only if there is a $Q \in \mathbb{P}^2$ such that P is singular on $\mathcal{P}_Q(X)$, the first polar curve of X with respect to Q.

First let $X = V(F)$. Then $\mathcal{P}_Q(X) = \nabla F \cdot (Q_0, Q_1, Q_2) = Q_0F_x + Q_1F_y + Q_2F_z$. For P to be a singularity on this curve, we must have $\nabla(\nabla F \cdot Q)(P) = 0$. Then

$$\begin{aligned} Q_0F_{xx}(P) + Q_1F_{xy}(P) + Q_2F_{xz}(P) &= 0 \\ Q_0F_{xy}(P) + Q_1F_{yy}(P) + Q_2F_{yz}(P) &= 0 \\ Q_0F_{xz}(P) + Q_1F_{yz}(P) + Q_2F_{zz}(P) &= 0 \end{aligned}$$

Therefore

$$\begin{pmatrix} F_{xx}(P) & F_{xy}(P) & F_{xz}(P) \\ F_{xy}(P) & F_{yy}(P) & F_{yz}(P) \\ F_{xz}(P) & F_{yz}(P) & F_{zz}(P) \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

with $(Q_0, Q_1, Q_2) \neq 0$. Thus the matrix $\mathrm{He}(X)(P)$ is not injective and so $\det \mathrm{He}(X)(P) = 0$. Thus P is on the Hessian curve.

Given $H(X)(P)(Q) = 0$, we know that $P \in \mathcal{P}_Q(X)$ because

$$\begin{pmatrix} F_{xx}(P) & F_{xy}(P) & F_{xz}(P) \\ F_{xy}(P) & F_{yy}(P) & F_{yz}(P) \\ F_{xz}(P) & F_{yz}(P) & F_{zz}(P) \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

implies

$$\begin{pmatrix} P_0 & P_1 & P_2 \end{pmatrix} \begin{pmatrix} F_{xx}(P) & F_{xy}(P) & F_{xz}(P) \\ F_{xy}(P) & F_{yy}(P) & F_{yz}(P) \\ F_{xz}(P) & F_{yz}(P) & F_{zz}(P) \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \end{pmatrix} = 0$$

and so

$$\begin{aligned} &Q_0(P_0F_{xx}(P) + P_1F_{xy}(P) + P_2F_{xz}(P)) \\ &+Q_1(P_0F_{xy}(P) + P_1F_{yy}(P) + P_2F_{yz}(P)) \\ &+Q_2(P_0F_{xz}(P) + P_1F_{yz}(P) + P_2F_{zz}(P)) = 0. \end{aligned}$$

Let d be the degree of F. Then this implies

$$(d-1)Q_0F_x(P) + (d-1)Q_1F_y(P) + (d-1)Q_2F_z(P) = 0$$

by Euler's Lemma, and so

$$Q_0F_x(P) + Q_1F_y(P) + Q_2F_z(P) = 0.$$

Thus $P \in \mathcal{P}_Q(X)$.

Now we want to show that if $P \in X$, that $T_P(X) = T_P(\mathcal{P}_P(X))$.

Note that $T_P(X)$ is given by the polynomial

$$xF_x(P) + yF_y(P) + zF_z(P)$$

and that $\mathcal{P}_P(X)$ is given by the polynomial

$$P_0F_x + P_1F_y + P_2F_z.$$

Therefore $T_P(\mathcal{P}_P(X))$ is given by the polynomial

$$\begin{aligned} &x(P_0F_{xx}(P) + P_1F_{xy}(P) + P_2F_{xz}(P)) \\ +&y(P_0F_{xy}(P) + P_1F_{yy}(P) + P_2F_{yz}(P)) \\ +&z(P_0F_{xz}(P) + P_1F_{yz}(P) + P_2F_{zz}(P)). \end{aligned}$$

By Euler's Lemma, $P_0F_{xx}(P) + P_1F_{xy}(P) + P_2F_{xz}(P) = (d-1)F_x(P)$ where d is the degree of F. Thus $T_P(X) = T_P(\mathcal{P}_P(X))$.

Now we want to show that $P \in \mathrm{He}(X)$ if and only if the polar quadric $\mathcal{P}_{P^{d-2}}(X)$ is singular. According to Lopez, the polar quadric of a curve at P is given by the polynomial

$$\begin{pmatrix} x & y & z \end{pmatrix} \begin{pmatrix} F_{xx}(P) & F_{xy}(P) & F_{xz}(P) \\ F_{xy}(P) & F_{yy}(P) & F_{yz}(P) \\ F_{xz}(P) & F_{yz}(P) & F_{zz}(P) \end{pmatrix} \begin{pmatrix} x \\ y \\ z \end{pmatrix}.$$

If this is true, then the result follows because we already saw that $P \in \mathrm{He}(X)$ if and only if the matrix $H(X)(P)$ is not injective. So there is a $Q \in \mathbb{P}^2$ such that

$$\begin{pmatrix} F_{xx}(P) & F_{xy}(P) & F_{xz}(P) \\ F_{xy}(P) & F_{yy}(P) & F_{yz}(P) \\ F_{xz}(P) & F_{yz}(P) & F_{zz}(P) \end{pmatrix} \begin{pmatrix} Q_0 \\ Q_1 \\ Q_2 \end{pmatrix} = 0.$$

Therefore

$$\begin{aligned} Q_0F_{xx}(P) + Q_1F_{xy}(P) + Q_2F_{xz}(P) &= 0 \\ Q_0F_{xy}(P) + Q_1F_{yy}(P) + Q_2F_{yz}(P) &= 0 \\ Q_0F_{xz}(P) + Q_1F_{yz}(P) + Q_2F_{zz}(P) &= 0. \end{aligned}$$

Note that these three equations are exactly half the three partial derivatives of $\mathcal{P}_P(X)$ evaluated at Q. Thus $\mathcal{P}_P(X)$ has a singularity at Q. We know that $Q \in \mathcal{P}_P(X)$ because Q solves the product of the three matrices from the definition of $\mathcal{P}_P(X)$.

Also Mathematica says that if C and D are cubics that share a flex point P, then $uC+vD$ will also be flex at P for all $[u : v] \in \mathbb{P}^1$.