Problem 2.4. *Funny Curves.* Let $\operatorname{char} k = 3$ and let $X = \mathfrak{V}(x^3y + y^3z + z^3x)$. Show that X is nonsingular and that every point is an inflection point. Furthermore, show that the dual curve X^* is isomorphic to X, however the natural map $X \to X^*$ is purely inseparable.

Proof. Observe $X_x = 3x^2y + z^3 = z^3$ (since $\operatorname{char} k = 3$), and $X_y = x^3$ and $X_z = y^3$. Then X is singular at the point $P = (a : b : c)$ satisfying $a^3 = b^3 = c^3 = 0$. Since no such P exists in $\mathbb{P}^2_k$, the curve X must be nonsingular.

Now we will show that every point of X is an inflection point. Let $P = (0 : 0 : 1) \in X$. Then consider the affine localization of X, $\mathfrak{D}(z) \cap X = \mathfrak{V}(x^3y + y^3 + x)$. Then $3x^2y\mathsf{d}x + x^3\mathsf{d}y + 3y^2\mathsf{d}y + \mathsf{d}x = x^3\mathsf{d}y + \mathsf{d}x = 0$. So at point $P = (0 : 0 : 1)$, we have $0\mathsf{d}y + \mathsf{d}x = 0$ and so the line tangent to P at X is $T_P(X) = \mathfrak{V}(x)$. Then let us calculate $i(X \cap T_P(X); P)$. It is the length of the $k[x, y, z]_{(x,y)}$-module $(k[x, y, z]/(x^3y + y^3z + z^3x, x))_{(y)} = k[y, z]/(y^3z)_{(y)}$, which is 3 (since z is a unit). This $i(X \cap T_P(X); P) = 3$ and so P is an inflection point.

Now let $Q = (a : b : 1) \in (\mathfrak{D}(z) \cap X) \setminus \{P\}$. Then $T_Q(X) = \mathfrak{V}(a^3y + x)$ (with the calculation from above). Then $i(X \cap T_Q(X); Q)$ is the length of the $k[x, y, z]_{(x-az,y-bz)}$-module $(k[x, y, z]/(x^3y + y^3z + z^3x))_{(x-az,y-bz)} = (k[y, z]/(-a^9y^4 + y^3z - a^3yz^3))_{(y-bz)} \cong (k[y, z]/(-a^9(y+bz)^4 + (y+bz)^3z - a^3(y+bz)z^3))_{(y)}$, which is 4. Thus $i(X \cap T_Q(X); Q) = 4$, so Q is an inflection point of X.

Now let $R = (0 : 1 : 0) \in X$. Then $\mathfrak{D}(y) \cap X = \mathfrak{V}(x^3 + z + z^3x)$ and so $\mathsf{d}z + z^3\mathsf{d}x = 0$. Thus $T_R(X) = \mathfrak{V}(z)$. Then $i(X \cap T_R(X); R)$ is the length of the $k[x, y, z]_{(x,z)}$-module $(k[x, y, z]/(x^3y + y^3z + z^3x, z))_{(x)} = (k[x, y]/(x^3y))_{(x)}$, which is 3. Thus $i(X \cap T_R(X); R) = 3$, so R is an inflection point of X.

A very similar argument shows that $i(X \cap T_{(1:0:0)}(X); (1 : 0 : 0)) = 3$. Thus every point of X is an inflection point.

The dualizing map $\delta : X \to X^*$ is given by the map of rings $\delta^\sharp : k[x^*, y^*, z^*] \to S_X$ given by $\delta^\sharp(x^*) = X_x = z^3$, $\delta^\sharp(y^*) = X_y = x^3$ and $\delta^\sharp(z^*) = X_z = y^3$. We wish to show that $\ker \delta^\sharp = ((x^*)^3(y^*) + (y^*)^3(z^*) + (z^*)^3(x^*))$.

First note that $\delta^\sharp((x^*)^3(y^*) + (y^*)^3(z^*) + (z^*)^3(x^*)) = (z^3)^3(x^3) + (x^3)^3(y^3) + (y^3)^3(z^3) = (z^3x + x^3y + y^3z)^3 = 0^3 = 0$. Therefore $((x^*)^3(y^*) + (y^*)^3(z^*) + (z^*)^3(x^*)) \subseteq \ker \delta^\sharp$.

This map is purely inseparable because for every $f \in S_X^*$, the polynomial $\alpha^3 - \delta^\sharp(f) \in S_X[\alpha]$ has a root. $\square$

Problem 2.5. Let $f : X \to Y$ be a degree n map and let $g(X) \geq 2$.

(a) If $P \in X$ is a ramification point, and $e_P = r$, show that $f^{-1}f(P)$ consists of exactly n/r points, each having index r. Let $P_1, \ldots, P_s$ be a maximal set of ramification points of X lying over distinct points of Y, and let $e_{P_i} = r_i$. Then show that Hurwitz's Theorem implies that

$$(2g - 2)/n = 2g(Y) - 2 + \sum_{i=1}^{s}(1 - 1/r_i).$$

(b) Since $g \geq 2$, let left hand side of the equation is > 0. Show that if $g(Y) \geq 0$, $s \geq 0$, and $r_i \geq 2$ for $1 \leq i \leq s$ are integers such that

$$2g(Y) - 2 + \sum_{i=1}^{s}(1 - 1/r_i) > 0,$$

then the minimum value of this expression is $1/42$. Conclude that $n \leq 84(g-1)$.

Proof.

(a) Let $P, Q \in f^{-1}f(P)$. Then we will show that $e_P = e_Q$. Let t be a uniformizing parameter of $\mathcal{O}_{Y,f(P)}$, let u be a uniformizing parameter of $\mathcal{O}_{X,P}$ and let w be a uniformizing parameter of $\mathcal{O}_{X,Q}$. Then there is an $a \in \mathcal{O}_{X,P}^\times$ and a $b \in \mathcal{O}_{X,Q}^\times$ such that $f^\sharp(t) = au^{e_P}$ and $f^\sharp(t) = bw^{e_Q}$. Then $au^{e_P} = bw^{e_Q}$ in $K(X)$. Then $e_P = v_P(au^{e_P}) = v_P(bw^{e_Q})$ and $e_Q = v_Q(bw^{e_Q}) = v_Q(au^{e_P})$. Thus $(au^{e_P})/(bw^{e_Q})$ is a unit in both $\mathcal{O}_{X,P}$ and $\mathcal{O}_{X,Q}$. Thus $v_P(au^{e_P}) = v_Q(bw^{e_Q})$ and so $e_P = e_Q$.

Now we know that for all $Q, P \in f^{-1}f(P)$, that $e_P = e_Q$. Now consider the divisor $f(P)$ and its image $f^*f(P) = \sum_{R \to f(P)} e_R \cdot R$. Since $\deg f(P) = 1$, we know $\deg f^*f(P) = n$ (II, 6.9). Furthermore, we know that e_R is constant by the above proof, so $f^*f(P) = e_R \sum_{R\to f(P)} R$ and so $e_R \deg\left(\sum_{R\to f(P)} R\right) = n$, so there are n/e_R many points in $f^{-1}f(P)$, each having ramification index e_R.

Now let f have s many branch points and let $P_1, \ldots, P_s$ be a maximal set of ramification points over distinct branch points in Y. Hurwitz's Theorem guarantees $2g-2 = n(2g(Y)-2) + \deg R$, where R is the ramification divisor of f.

We wish to show that $n\sum_{i=1}^s (1-1/r_i) = \deg R$. Since $\operatorname{char} k = 0$, f has only tame ramification points and so $\deg R = \sum_{P \in X}(e_P - 1) = \sum_{P \text{ a ramification point}} (e_P - 1) = sn - n/r_1 - \cdots - n/r_s = n(s - 1/r_1 - \cdots - 1/r_s) = n\sum_{i=1}^s (1-1/r_i)$. This is because there are n/r_i many ramification points for the i^{th} branch point, each having ramification index r_i, and there are s many branch points.

Therefore $\deg R = n\sum_{i=1}^s(1-1/r_i)$ and so Hurwitz's Theorem implies that $2g-2 = n(2g(Y)-2) + n\sum_{i=1}^s(1-1/r_i)$ and so $(2g-2)/n = 2g(Y) - 2 + \sum_{i=1}^s(1-1/r_i)$.

(b) Now we have the equality $(2g-2)/n = 2g(Y) - 2 + \sum_{i=1}^s(1-1/r_i)$, and since the left side of this equality is > 0, so is the right side. Let $g(Y) \geq 0$, $s \geq 0$ and $r_i \geq 2$ for all $1 \leq i \leq s$.

Call the right side of the equation T.

- If $g(Y) \geq 2$, then $T \geq 2$ and $n \leq g-1$.
- If $g(Y) = 1$, then $s \geq 1$ (since if $s = 0$ then T would be 0, which is not allowed) and $T \geq 0 + 1 - 1/2 = 1/2$ so $n \leq 4(g-1)$.
- If $g(Y) = 0$, then $s \geq 3$ and
  - if $s \geq 5$ then $T \geq -2 + s(1-1/2) \geq 1/2$, so that $n \leq 4(g-1)$.
  - if $s = 4$ then $T \geq -2 + 4 - 1/2 - 1/2 - 1/2 - 1/3 = 1/6$, so $n \leq 12(g-1)$
  - if $s = 3$, then we may assume $2 \leq r_1 \leq r_2 \leq r_3$.
    - If $r_1 \geq 3$ then $T \geq -2 + 3 - 1/3 - 1/3 - 1/4 = 1/12$ so $n \leq 24(g-1)$.
    - If $r_1 = 2$ then
      - if $r_2 \geq 4$ then $T \geq -2 + 3 - 1/2 - 1/4 - 1/5 = 1/20$ so $n \leq 40(g-1)$
      - if $r_2 = 3$ then $T \geq -2 + 3 - 1/2 - 1/3 - 1/7 = 1/42$ so $n \leq 84(g-1)$.

In conclusion, $n \leq 84(g-1)$. Note these numbers were obtained from the fact that the resulting number must be positive, and a smaller integer would result in a nonpositive sum.

□

Problem 2.2. *Classification of Curves of genus 2.* Fix an algebraically closed field k of characteristic $\neq 2$.

1. If X is a curve of genus 2 over k, the canonical linear system $|K|$ determines a finite morphism $f : X \to \mathbb{P}^1$ of degree 2. Show that it is ramified at exactly 6 points, with ramification index of 2 at each one. Note that f is uniquely determined, up to automorphism of $\mathbb{P}^1$, so X determines an (unordered) set of 6 points of $\mathbb{P}^1$, up to automorphism of $\mathbb{P}^1$.

2. Conversely, given six distinct elements $\alpha_1, \ldots, \alpha_6 \in k$, let K be the extension of $k(x)$ determined by the equation $z^2 = (x-\alpha_1)\cdots(x-\alpha_6)$. Let $f : X \to \mathbb{P}^1$ be the corresponding morphism of curves. Show that $g(X) = 2$, the map f is the same as the one determined by the canonical linear system, and f is ramified over the six points $x = \alpha_i$ of $\mathbb{P}^1$ and nowhere else. II.Ex.6.4: Let k be a field of characteristic $\neq 2$ and let f be a square-free polynomial in $k[x_1, \ldots, x_n]$. Let $A = k[x_1, \ldots, x_n, z]/(z^2 - f)$. Show that A is an integrally closed ring.

3. Using I.Ex.6.6, show that if P_1, P_2, P_3 are three distinct points in $\mathbb{P}^1$, then there exists a unique $\varphi \in \mathrm{Aut}\mathbb{P}^1$ such that $\varphi(P_1) = 0$, $\varphi(P_2) = 1$ and $\varphi(P_3) = \infty$. Thus we may assume X is ramified over $0, 1, \infty, \beta_1, \beta_2, \beta_3$ where $\beta_1, \beta_2, \beta_3$ are three distinct elements of $k \neq 0, 1$.

4. Let Σ_6 be the symmetric group on 6 letters. Define an action of Σ_6 on sets of three distinct elements $\beta_1, \beta_2, \beta_3 \in k \neq 0, 1$ as follows: reorder the set $0, 1, \infty, \beta_1, \beta_2, \beta_3$ according to a given elements in Σ_6, then renormalize as in 3 so that the first three become $0, 1, \infty$ again. Then the last three are the new $\beta_1', \beta_2', \beta_3'$.

5. Summing up, conclude that there is a one-to-one correspondence between the set of isomorphism classes of curves of genus 2 over k, and triples of distinct elements $\beta_1, \beta_2, \beta_3 \in k \neq 0, 1$ modulo the action of Σ_6 described in 4. In particular, there are many non-isomorphic curves of genus 2. We say that curves of genus 2 depend on three parameters, since they correspond to the points of an open subset of $\mathbb{A}^3_k$ modulo a finite group.

Proof.

1. By Hurwitz's Theorem, $2 \cdot 2 - 2 = 2 \cdot (2 \cdot 0 - 2) + \deg R$, so $\deg R = 6$. Since for each $P \in X$, $1 \leq e_P \leq 2$, we know that $e_P = 2$ for each ramification point of f. Then $e_P - 1 = 1$ so $R = P_1 + \cdots + P_6$.

2. We know that $A = k[x,z]/(z^2 - h)$ is an integrally closed ring. We know that the field of fractions of A is $K = k(x)[z]/(z^2 - h)$, which is a Galois extension of $k(x)$ with Galois group $z \mapsto -z$.

   We can consider the abstract nonsingular curve with function field equal to K, $Y := C_K$ (I.6). The inclusion $k \hookrightarrow K$ gives the map $f : Y \to \mathbb{P}^2$.

$\square$