This is a variation of the traditional twelve coin puzzle, where one coin has a different weight than eleven other coins that all have the same weight (but you do not know whether the odd coin is heavier or lighter than the others), and you must use a balance scale three times to determine which coin is the odd one out and whether it is heavier or lighter. In this variation, you do not need to determine whether the odd coin is heavier or lighter than the others (although there is a solution that will give you that information for all but one coin), **but you must have a predetermined strategy of how you will place the coins, without relying in information learned by using the scales**. The full statement of the puzzle is given below.

Puzzle : Imagine you have 12 coins marked A through L. All but one of the coins have the same weight, and you do not know which one has a different weight nor whether it is lighter or heavier than the others. You may use a balance scale only three times to determine which coin is the odd one out. You may place any arrangement of coins you desire on the LEFT side of the balance scale, the RIGHT side of the balance scale, or OFF the balance scale.

Come up with a **fixed** arrangement of coins for each of the three rounds that is guaranteed to inform you of which coin is the odd one out. That is, your strategy for placing coins on Round 2 cannot be informed by the information you received from Round 1. Your strategy must be predetermined.

See page 2 for a hint to get started!

See page 3 for a solution!

Hint. You can start by taking each coin and assigning it three values from $\{-1, 0, 1\}$ depending on whether that coin spends the first, second, and third rounds on the left side of the balance scale, off the balance scale, or the right side of the balance scale. For example assigning coin A the triple $(-1, 0, 1)$ means that coin A spends the first round on the left side of the balance scale, the second round off the balance scale, and the third round on the right side of the balance scale. Or assigning coin B the triple $(0, 1, -1)$ means that coin B spends the first round off the balance scale, the second round of the right side of the balance scale, and the third round on the left side of the balance scale. Try using the geometry of these points in $(\mathbb{Z}/3\mathbb{Z})^3$!

There is a solution where each round has four coins on the left side, four coins on the right side, and four coins off the balance scale. $\square$

Solution. There are multiple solutions to this puzzle, which you can read more about here. We will focus on the solution where each round has four coins on the left side, four coins on the right side, and four coins off the balance scale.

You can start by taking each coin and assigning it three values from $\{-1, 0, 1\}$ depending on whether that coin spends the first, second, and third rounds on the left side of the balance scale, off the balance scale, or the right side of the balance scale. This gives us an arrangement of 12 points in $\mathbb{F}_3^3$. But are they all distinct points?

Consider if coins A and B were assigned the same point $(x, y, z) \in \mathbb{F}_3^3$. Then it would be impossible to distinguish whether A or B is the odd one out, since they are always together in all three rounds. So the 12 points in $\mathbb{F}_3^3$ must be distinct. Let us call this set of 12 points X. What other conditions must X follow?

No two distinct points in X may be negatives of each other. That is, if $p \in X$, then $-p \notin X$. Consider this: $A \mapsto (x, y, z)$ and $B \mapsto (-x, -y, -z)$. Then for every round either A and B are on opposing sides of the balance scale or they are both off the balance scale. With this strategy, it would be impossible to distinguish between the scenarios of A being heavier than the other coins, or B being lighter than the other coins (and vice versa). Since our strategy must be able to give us the answer regardless of which coin is heavier or lighter, we cannot accept this strategy.

Now we have enough information about X to construct a winning strategy. We want X to be a set of 12 distinct points in $\mathbb{F}_3^3$ where

- If $p \in X$ then $-p \notin X$,
- Each plane of the form $x = -1, 0, 1$, $y = -1, 0, 1$ and $z = -1, 0, 1$ contains exactly four points of X. That is, each round has exactly four coins on the left side of the balance, four on the right side of the balance, and four off the balance. For example, the plane $x = -1$ intersects X at the four coins that spend round 1 on the left side of the balance, $y = 0$, intersects X at the four coins that spend round 2 off the balance, and $z = 1$ intersects X at the four points that spend round 3 on the right side of the balance.

Any arrangement of 12 points that fulfills the two properties listed above yields a viable strategy. For example we can use the points $(-1, -1, -1)$ $(-1, 0, 1)$, $(-1, 1, 0)$, $(-1, 1, 1)$, $(0, -1, -1)$, $(0, -1, 0)$, $(0, 0, -1)$, $(0, 0, 0)$, $(1, , -1, 1)$, $(1, 0, 1)$, $(1, 1, 0)$, $(1, 1, -1)$.

Then we can assign each coin to a point by the following table:

A	B	C	D	E	F	G	H	I	J	K	L
-1	-1	-1	-1	0	0	0	0	1	1	1	1
-1	0	1	1	-1	-1	0	0	-1	0	1	1
-1	1	0	1	-1	0	-1	0	1	1	-1	0

Then the three rounds look like

	Left side	Right side		Off
Round 1	ABCD	IJKL		EFGH
Round 2	AEFI	CDKL		BGHJ
Round 3	AEGK	BDIJ		CFHL

Then we can make an interpretation key for any way the balance might tip in rounds 1, 2, and 3 to determine which coin is the odd one out and whether it is heavier or lighter than the others.

Let's denote every time the scale tips left with a -1, every time the scale tips right with a 1, and every time the scale is balanced with a 0. Then again each scenario is a point $(x, y, z) \in \mathbb{F}_3^3$. Let's explore each scenario in a case-by-case basis.

- $(-1,-1,-1)$: If the scale tips left all three times, then A must be heavier than the others.
- $(-1,-1,0)$: If the scale tips left twice and then is balanced on round 3, then coin L must be lighter than the others.
- $(-1,-1,1)$: Coin K is light.
- $(-1,0,-1)$: Coin J is light.
- $(-1,0,0)$: This scenario cannot happen. If the scale is balanced in rounds 2 and 3, then H must be the odd one out because H is the only coin that is not on the balance in both rounds 2 and 3. But that is contradicted by the evidence of round 1, so it is impossible for this to happen.
- $(-1,0,1)$: Coin B is heavy.
- $(-1,1,-1)$: Coin I is light.
- $(-1,1,0)$: Coin C is heavy.
- $(-1,1,1)$: Coin D is heavy.
- $(0,-1,-1)$: Coin E is heavy.
- $(0,-1,0)$: Coin F is heavy.
- $(0,-1,1)$: This is another impossible scenario. If the scale does not tip in round 1 then one of E, F, G, or H is the odd one out. But if it's E, then the scales would tip the same way in rounds 2 and 3. If it's F, then the scales would not tip in round 3. If it's G, then the scales would not tip in round 2, and if it's H, then the scales would not tip in rounds 2 and 3.
- $(0,0,-1)$: G is heavy.
- $(0,0,0)$: H is the odd one out, but we will not be able to determine if it's heavier or lighter than its comrades.

Every other scenario is simply the reverse of a scenario listed above; for example $(1,-1,1)$ is the opposite of $(-1,1,-1)$, so we would conclude that coin I is heavy instead of light. □

As you can see, this predetermined strategy is guaranteed to reveal which coin is the odd one out. Furthermore, we will be able to determine whether the odd coin is heavier or lighter than its friends, *unless the odd coin is H*, which is the only coin never placed on the scale.