Proposition 1. Let $\mathbb{F}_q$ be a finite field of size q. Let $\mathrm{Gr}_{\mathbb{F}_q}(k,n)$ be the space of $(k-1)$-dimensional projective hyperplanes in $\mathbb{P}^{n-1}_{\mathbb{F}_q}$. Then $\#\mathrm{Gr}_{\mathbb{F}_q}(k,n) = \displaystyle\prod_{i=1}^{k} \frac{q^{n+1-i}-1}{q^i-1}$.

Proof. First consider $\mathrm{Gr}(1,n)$. This is the set of points in $\mathbb{P}^{n-1}_{\mathbb{F}_q}$. The number of points can be calculated as $\dfrac{q^{n+1-1}-1}{q^1-1} = \dfrac{q^n-1}{q-1}$. Thus the proposition holds for the case $k=1$.

Now consider $\mathrm{Gr}(k,n)$. We will show $\#\mathrm{Gr}(k,n) = \#\mathrm{Gr}(k-1,n)\dfrac{q^{n+1-k}-1}{q^k-1}$. We want to form a $(k-1)$-dimensional hyperplane by grabbing a $(k-2)$-dimensional hyperplane and a point not on the plane we grabbed, and then correct for any overcounting. There are $\#\mathrm{Gr}(k-1,n)$ many $(k-2)$-dimensional hyperplanes. Given any hyperplane, there are $\dfrac{q^n-1}{q-1} - \dfrac{q^{k-1}-1}{q-1} = \dfrac{q^n-q^{k-1}}{q-1} = q^{k-1}\dfrac{q^{n+1-k}-1}{q-1}$ points not on that hyperplane.

Multiplying $q^{k-1}\dfrac{q^{n+1-k}-1}{q-1}$ and $\#\mathrm{Gr}(k-1,n)$ gives us the number of $(k-1)$-dimensional hyperplanes there are with some factor of overcounting. To correct for this, we need to divide by a factor given by how many $(k-2)$-dimensional hyperplanes there are in $\mathbb{P}^{k-1}_{\mathbb{F}_q}$, multiplied by how many points in $\mathbb{P}^{k-1}_{\mathbb{F}_q}$ not contained on the given hyperplane.

By duality, there are as many $(k-2)$-dimensional hyperplanes in $\mathbb{P}^{k-1}_{\mathbb{F}_q}$ as there are points in $\mathbb{P}^{k-1}_{\mathbb{F}_q}$. Indeed, $\mathrm{Gr}(k-1,k) \cong \mathrm{Gr}(1,k)$. Thus there are $\dfrac{q^k-1}{q-1}$ many $(k-2)$-dimensional hyperplanes in $\mathbb{P}^{k-1}_{\mathbb{F}_q}$. Now there are $\dfrac{q^k-1}{q-1} - \dfrac{q^{k-1}-1}{q-1} = \dfrac{q^k-q^{k-1}}{q-1} = q^{k-1}$ points in $\mathbb{P}^{k-1}_{\mathbb{F}_q}$ not contained in a given $(k-2)$-dimensional hyperplane. Thus the factor of overcounting is $q^{k-1}\dfrac{q^k-1}{q-1}$.

Diving gives us

$$\#\mathrm{Gr}(k,n) = \frac{\#\mathrm{Gr}(k-1,n)q^{k-1}\dfrac{q^{n+1-k}-1}{q-1}}{q^{k-1}\dfrac{q^k-1}{q-1}} = \#\mathrm{Gr}(k-1,n)\frac{q^{n+1-k}-1}{q^k-1}.$$

Because we know the explicit formula for $\#\mathrm{Gr}(1,n) = \dfrac{q^n-1}{q-1}$, we finally get $\#\mathrm{Gr}_{\mathbb{F}_q}(k,n) = \displaystyle\prod_{i=1}^{k} \frac{q^{n+1-i}-1}{q^i-1}$, and so the proposition holds. ☐

Example 1. There are $\dfrac{(2^6-1)(2^5-1)(2^4-1)}{(2-1)(2^2-1)(2^3-1)} = 1395$ planes in $\mathbb{P}^5_{\mathbb{Z}/2\mathbb{Z}}$.