The space of complete conics is the closure of the space

$$\{(C, C^*) \subseteq \mathbb{P}^5 \times \mathbb{P}^{5*} : C \text{ is a smooth conic and } C^* \text{ is the dual of } C\}.$$

Note that we can consider a conic as a symmetric 3×3 matrix $A = \begin{pmatrix} a & b/2 & c/2 \\ b/2 & d & e/2 \\ c/2 & e/2 & f \end{pmatrix}$ and derive the polynomial for C from the product $\begin{pmatrix} x & y & z \end{pmatrix} A \begin{pmatrix} x \\ y \\ z \end{pmatrix}$. This is the same as $\langle A(\vec{v}), \vec{v} \rangle = 0$. When C is smooth, its dual can thus be calculated simply by taking the inverse A^{-1} or equivalently by the adjoint $\mathrm{adj}(A) \sim A^{-1}$ since $\mathrm{adj}(A) = \det(A)A^{-1}$.

Let us denote the variety of non-smooth conics in $\mathbb{P}^5$ by X and the subvariety of double lines S. Note X is a degree-3 variety of dimension 4, and S has dimension 2, because S is the image of the embedding

$$f : \mathbb{P}^2 \to \mathbb{P}^5$$

defined by $f(a, b, c) = (a^2, 2ab, 2ac, b^2, 2bc, c^2)$. Note that X is the image of the 2-to-1 map $g : \mathbb{P}^2 \times \mathbb{P}^2 \to \mathbb{P}^5$ given by $g((a, b, c), (d, e, f)) = (ad, ae + bd, af + cd, be, bf + ce, cf)$. This map is 2-to-1 because $g((a, b, c), (d, e, f)) = g((d, e, f), (a, b, c))$. The map g ramifies on the surface S.

In the space of complete conics, a singular conic C_0 does not always uniquely determine its pair. Rather, when C_0 is a double line, the direction of a line containing C_0 determines its dual. Let C_t be a pencil of conics containing C_0 at $t = 0$. Its dual will hence be defined at $C_0^* := \lim_{t\to 0} C_t^* \neq (C_0)^*$, since $(C_0)^*$ is simply a double point. (When C_0 is the union of two distinct lines $\ell_1 + \ell_2$, the limit $\lim_{t\to 0} C_t^* = 2\overline{\ell_1 \ell_2}$ always, so C_0 has a unique dual.)

We thus have four kinds of complete conics (C, C^*):

1. C is smooth and C^* is its dual.

2. C is the union of two distinct lines, and C^* is the double line of the dual of the unique singularity.

3. C is a double line and C^* is a double line. (When C_t is linear, C^* will be the double line given by the dual of the unique intersection of C_t with C. This means that the pencil has a unique base point of multiplicity 4.)

4. C is a double line and C^* is the union of two distinct lines. (When C_t is linear C^* will be the union of the two dual lines given by the two base points of multiplicity 2 of the pencil.)

Note C_t may be quadratic and its tangent line within the intersection with S may be contained in X. We will provide some examples.

Example 1. Let C_0 be of type two in a pencil with four distinct base points. By the four point theorem, we can say that $C_0 = V(xy)$ and $C_1 = V(z(x + y + z))$. The four base points are $(1, 0, 0)$, $(0, 1, 0)$, $(-1, 0, 1)$ and $(0, -1, 1)$. Then we can construct a pencil of conics $uxy + v(xz + yz + z^2)$ where $(u, v) \in \mathbb{P}^1$. This pencil is described by the matrix $A = \begin{pmatrix} 0 & u/2 & v/2 \\ u/2 & 0 & v/2 \\ v/2 & v/2 & v \end{pmatrix}$, whose inverse is equivalent to $\begin{pmatrix} -v^2 & v^2 - 2uv & uv \\ v^2 - 2uv & -v^2 & uv \\ uv & uv & -u^2 \end{pmatrix}$, plugging in $v = 0$ we get the conic z^2, which indeed is the dual of the point $(0, 0, 1)$. By plugging in $u = 0$, we get the conic $-x^2 + 2xy - y^2 = -(x - y)^2$, which is the dual of the point $(1, 1, 0)$, which is the singularity of $z(x + y + z)$.

Example 2. Now let C_0 be of type 2 in a linear pencil with one base point of multiplicity 2 and two ordinary base points. We can take $C_0 = V(xy)$ and $C_1 = V(z(x + y))$. The two ordinary base points of the pencil are $(1, 0, 0)$ and $(0, 1, 0)$ and the base point of multiplicity 2 is $(0, 0, 1)$. Then C_0^* is z^2 again, and C_1^* is $(x - y)^2$, which is dual of $(1, -1, 0)$, which is the intersection of z and $x + y$.

Example 3. Now let C_0 be of type 2 in a linear pencil with two base points of multiplicity 2. Take $C_0 = V(xy)$ and $C_1 = V(z^2)$. Then C_0^* is z^2 and C_1^* is xy.

Example 4. Now let C_0 be of type 2 in a linear pencil with one base point of multiplicity 3 and one ordinary base point. We can take $C_0 = V(xy)$ and $C_1 = V(x^2 - yz)$. This has a base point of multiplicity 3 at $(0, 0, 1)$ and an ordinary base point at $(0, 1, 0)$. Then C_0^* is once again z^2.

Note that if C_0 is type 2, there is no non-trivial linear pencil of conics containing C_0 that has one base point of multiplicity 4: every such pencil would be a line contained in X.

Example 5. Let C_0 be a double line in a linear pencil with two base points of multiplicity 2. Then C_0^* will be a union of two lines. Take $C_0 = V(y^2)$ and $C_1 = V(xz)$, then the pencil comprises conics of the form $uy^2 + vxz$ for $(u, v) \in \mathbb{P}^1$. This pencil has two base points $(1, 0, 0)$ and $(0, 0, 1)$. This is described by the matrix $A = \begin{pmatrix} 0 & 0 & v/2 \\ 0 & u & 0 \\ v/2 & 0 & 0 \end{pmatrix}$. Then $A^{-1} \sim \begin{pmatrix} 0 & 0 & 2u \\ 0 & v & 0 \\ 2u & 0 & 0 \end{pmatrix}$ so when $v = 0$ we get the dual of C_0 is xz, which is the the union of the duals of the two base points $(1, 0, 0)$ and $(0, 0, 1)$.

Example 6. Let C_0 be a double line in a linear pencil with one base point of multiplicity 4. Then C_0^* will be a double line. Take $C_0 = V(y^2)$ and $C_1 = V(x^2 - yz)$. Then the pencil comprises conics of the form $uy^2 + v(x^2 - yz)$, and has one base point of multiplicity 4 at $(0, 0, 1)$. The matrix describing this pencil is $A = \begin{pmatrix} v & 0 & 0 \\ 0 & u & -v/2 \\ 0 & -v/2 & 0 \end{pmatrix}$. Then $A^{-1} \sim \begin{pmatrix} v & 0 & 0 \\ 0 & 0 & -2v \\ 0 & -2v & -4u \end{pmatrix}$ so when $v = 0$ we get the dual of C_0 is z^2, this is the dual of the base point $(0, 0, 1)$.

Example 7. Now let C_0 be the double line $V(y^2)$ in the quadratic pencil of conics given by $4u^2x^2 + v^2y^2 + 4uvyz$. That is, C_0 is the fiber $u = 0$. This pencil has a base point at $(0, 0, 1)$ of multiplicity 4 and every fiber is tangent to the lines $x = z$ and $x = -z$. Then the matrix associated with this pencil is $A = \begin{pmatrix} 4u^2 & 0 & 0 \\ 0 & v^2 & 2uv \\ 0 & 2uv & 0 \end{pmatrix}$. Then $A^{-1} \sim \begin{pmatrix} v & 0 & 0 \\ 0 & 0 & 2u \\ 0 & 2u & -v \end{pmatrix}$ and so we get the linear pencil $vx^2 + 4uyz - vz^2$. When $u = 0$ we get the dual of C_0 is $x^2 - z^2 = (x+z)(x-z)$, which is the product of the duals of the points $(1, 0, 1)$ and $(-1, 0, 1)$, which are the limits of the points of tangency to the lines $V(x + z)$ and $V(x - z)$ in the pencil as the fibers approach C_0.

Note that this example that this complete conic is an example of type 4, even though the pencil has one base point of multiplicity 4 like in type 3. This is because the pencil is quadratic and the quadratic curve in $\mathbb{P}^5$ has a tangent line at its intersection with S inside X.

Observe: the quadratic curve cut out by this pencil is given by $(4u^2, 0, 0, v^2, 4uv, 0)$, parametrized by $(u, v) \in \mathbb{P}^1$. This curve sits in the plane $V(\lambda_1, \lambda_2, \lambda_5)$, so we can ignore the zeroes to simply write $(4u^2, v^2, 4uv)$. The fiber at $u = 0$ is the point $(0, 1, 0)$. This curve is tangent to the line $x = 0$ at $(0, 1, 0)$, which corresponds to the x^2 axis in $\mathbb{P}^5$. Therefore the tangency of $4u^2x^2 + v^2y^2 + 4uvyz$ at $y^2 \in S$ is the line L along the x^2 axis. Other points on L are conics of the form $uy^2 + vx^2$, which is a union of lines (but only a square when $uv = 0$). So $L \subseteq X$.

Another way to think of complete conics: let V be a three-dimensional vector space and take $\mathbb{P}^5 = \mathbb{P}(\mathrm{Sym}^2 V^*)$ and $\mathbb{P}^{5*} = \mathbb{P}(\mathrm{Sym}^2 V)$. Let e_1, e_2, and e_3 generate V: then $\mathrm{Sym}^2 V$ is generated by e_1^2, e_1e_2, e_1e_3, e_2^2, e_2e_3, and e_3^2, so $\mathbb{P}(\mathrm{Sym}^2 V)$ is indeed five dimensional projective space.

Let $\varphi : V \to V^*$ and $\psi : V^* \to V$ be symmetric (and therefore determine conics in $\mathbb{P}^2$). The variety Y of complete conics comes from pairs (φ, ψ) such that $\varphi \circ \psi$ has its diagonal entries equal to each other (two equations) and its off-diagonal entries all equal to zero (six equations) (although really that should be three equations if we are taking φ and ψ to be symmetric??). So $\varphi \circ \psi$ is a scalar multiple of the identity matrix. We get the following types of complete conics (φ, ψ):

1. If φ has rank 3, then ψ is its inverse.

2. If φ has rank 2, then the products $\varphi \circ \psi$ and $\psi \circ \varphi$ must both be 0. It follows that ψ is the unique (up to scalars) symmetric map $V^* \to V$ whose kernel is the image of φ and whose image is the kernel of φ.

3. If φ has rank 1, then ψ may have rank 1 or 2; in the latter case, it may be any symmetric map $V^* \to V$ whose image is the kernel of φ and whose kernel is the image of φ.

4. If $\mathrm{rank}\ \varphi = \mathrm{rank}\ \psi = 1$, then they simply have to satisfy the condition that $\mathrm{img}\ \varphi \subseteq \ker \psi$ and $\mathrm{img}\ \psi \subseteq \ker \varphi$.

We can describe duals of quadrics in general dimension as follows.

Proposition 8.1. Let $Q \subseteq \mathbb{P}(V) = \mathbb{P}^n$ be the quadric corresponding to the symmectric map $\varphi : V \to V^*$, and let $v \in V$ be a nonzero vector such that $\langle \varphi(v), v\rangle = 0$, so that $v \in Q$. The tangent hyperplane to Q at v is

$$T_vQ = \{w \in \mathbb{P}(V) : \langle \varphi(v), w\rangle = 0\}$$

and the dual of Q is thus

$$Q^* = \{\varphi(v) \in \mathbb{P}(V^*) : v \in Q \text{ and } \varphi(v) \neq 0\}.$$

In particular, if Q is nonsingular (that is, if the rank of $\varphi$is $n + 1$), then Q^* is the image $\varphi(Q)$ of Q under the induced map $\varphi : \mathbb{P}V \to \mathbb{P}V^*$ and Q^* is the quadric corresponding to the cofactor map φ^c (what I've been calling the adjoint).

On the other hand, if the rank of Q is n and Q^c is the quadric corresponding to the cofactor map φ^c, then Q^c is the unique double hyperplane containing Q^*; that is, the support of Q^c is the hyperplane corresponding to the annihilator of the singular point of Q.

Proof. For any $w \in V$, the line $\overline{vw} \subseteq \mathbb{P}V$ spanned by v and w is tangent to Q at v if and only if

$$\langle \varphi(v + \varepsilon w), v + \varepsilon w\rangle = 0 \mod \varepsilon^2.$$

Expanding this out, we get

$$\langle \varphi(v), v\rangle + \varepsilon\langle \varphi(v), w\rangle + \varepsilon\langle \varphi(w), v\rangle + \varepsilon^2\langle \varphi(w), w\rangle$$

and remembering that $\langle \varphi(v), v\rangle = 0$ and $\varepsilon^2 = 0$, we get

$$\langle \varphi(w), v\rangle + \langle \varphi(v), w\rangle = 0,$$

and by symmetry of φ and the assumption that we are not in characteristic 2, this is the case if and only if

$$\langle \varphi(v), w\rangle = 0,$$

proving that first statement and identifying the dual variety as $Q^* = \varphi(Q)$. (Remember symmetry means $\langle \varphi(v), w\rangle = \langle v, \varphi(w)\rangle$.) Also remember that the dual of a vector subspace is its orthogonal complement. Considering the cone CT of T_vQ in V, the orthogonal complement of CT is the linear subspace generated by $\varphi(v)$, since $T_vQ = \{w \in \mathbb{P}(V) : \langle \varphi(v), w\rangle = 0\}$. Thus the image of the tangent hyperplane T_vQ under the dualization map is the point $\varphi(v)$. Doing this for all the points, we get the equality $Q^* = \varphi(Q)$.

Suppose the rank of Q is n or $n + 1$. Let φ^c be the matrix of cofactors of φ, so that $\varphi^c \circ \varphi = \det \varphi \circ I$ (this is what I've been calling the adjoint). Since rank Q = rank $\varphi \geq n$, the map φ^c is nonzero. The quadric Q^c is by definition the set of all $w \in V^*$ such that $\langle w, \varphi^c(w)\rangle = 0$. If $v \in Q$ then

$$\langle \varphi(v), \varphi^c\varphi(v)\rangle = \det(\varphi)\langle \varphi(v), v\rangle = 0,$$

so $\varphi(Q)$ is contained in Q^c.

If rank $\varphi = n + 1$, so that φ is an isomorphism, then $Q^* = \varphi(Q)$ is again a quadric hypersurface and we must have $Q^* = Q^c$. If rank $\varphi = n$, then since $\varphi^c\varphi = 0$ the rank of φ^c is 1, and the associated quadric is a double hyperplane. On the other hand, Q is the cone over a nonsingular quadric in $\mathbb{P}^{n-1}$, and Q^* is the dual of that quadric inside a hyperplane (corresponding to the vertex of Q) in $\mathbb{P}^{n*}$. Thus Q^* spans the plane contained in Q^c. □

Corollary 8.2. If Q and Q' are smooth quadrics, then Q and Q' have the same tangent hyperplane $\ell = 0$ at some point of intersection $v \in Q \cap Q'$ if and only if Q^* and Q'^* have the common tangent hyperplane $v^* = 0$ at the point of intersection $\ell^* \in Q^* \cap Q'^*$.

To solve the five conic problem, we will need to address these four issues:

- We have to show that in passing from the "naive" compactification $\mathbb{P}^5$ of the space U of smooth conics to the more sensitive compactification Y, we have in fact eliminated the problem of extraneous intersection; in other words, we have to show that for five general conics C_i the corresponding divisors $Z_{C_i} \subseteq Y$ intersect only in points $(C, C') \in Y$ with C and $C' = C^*$ smooth.

- We have to show that the five divisors Z_{C_i} are transverse at each point where they intersect.

- We have to determine the Chow ring of the space Y, or at least the structore of a subring of $A(Y)$ containing the class ζ of the hypersurfaces Z_{C_i} we wish to intersect.

- We have to identify the class ζ in this ring and find the degree of the fifth power $\zeta^5 \in A^5(X)$.

We will start by showing that no complete conic (C, C') of type 2 lies in the intersection of the divisors associated to five general conics. The first thing we need to do is to describe the points (C, C') of type 2 lying in Z_D for a smooth conic D. This is straightforward: if $C = L \cup M$ is a conic of rank 2 which is a limit of smooth conic tangent to D, then C must also have a point of intersection multiplicity 2 or more with D. (Note that by symmetry a similar description holds for the points of type 3: the complete conic $(2L, p^* + q^*)$ will lie on Z_D only if L is tangent to D, or p or q lie on D.)

Now suppose that (C, C') is a complete conic of type 2 lying in the intersection of the divisors $Z_i = Z_{C_i}$ associated to five general conics C_i. Write $C = L \cup M$ and set $p = L \cap M$. We note that since the C_i are general, no three are concurrent; thus p can lie on at most two of the conics C_i. We will proceed by considering three cases in turn:

- p lies on line of the conics C_i. This is the most immediate case: Since the conics C_i^* are also general, it is like wise the case that no three of them are concurrent. In other words, no line in the plane is tangent to more than two of the C_i and correspondingly $(L \cup M, p) \in Z_{C_i}$ for at most four of the C_i.

- p lies on two of the conics C_i, say C_1 and C_2. Since C_3, C_4, and C_5 are general with respect to C_1 and C_2, none of the finitely many lines tangent to two of them passes through a point of $C_1 \cap C_2$; thus L and M can each be tangent to at most one of the conics C_3, C_4, and C_5, and again we see that $(L \cup M, 2p^*) \in Z_{C_i}$ for at most four of the conics C_i.

- p lies on exactly one of the conics, say C_1. Now since C_1 is general with respect to the other four, it will not contain any of the finitely many points of pairwise intersection of lines tangent to two of them. Thus L and M cannot each be tangent to two of the other four conics and once more we see that $(L \cup M, 2p^*) \in Z_{C_i}$ for at most four of the C_i.

Thus no conic of type 2 can lie in the intersection of the Z_{C_i}; by symmetry, no complete conic of type 3 can either.

It remains to verify that no complete conic (C, C') of type 4 can lie in the intersection $\bigcap Z_i$ and again we have to start by characterizing the intersection of a cycle $Z = Z_D$ with the locus of complete conics of type 4.

To do this, write an arbitrary complete conic of type 4 as $(2M, 2q^*)$ with $q \in M$. If $(2M, 2q^*) \in Z_D$, then there is a one parameter family $(C_t, C_t') \in Z_D$ with C_t smooth, $C_t' = C_t^*$ for $t \neq 0$ and $(C_0, C_0') = (2M, 2q^*)$; let $p_t \in C_t \cap D$ be the point of tangency of C_t with D, and set $p = \lim_{t\to 0} p_t \in M$. The tangent line $T_{p_t}C_t = T_{p_t}D$ to C_t at p_t will have as limit the tangent line L to D at p, so $L^* \in q^*$. Thus both p and q are in both L and M. If $p = q$ then in particular $q \in D$. On the other hand, if $p \neq q$, then we must have $M = \overline{pq} = L$, so $M^* \in D^*$. We conclude therefore, that a complete conic $(2M, 2q^*)$ of type 4 can lie in Z_D only if either $q \in D$ or $M^* \in D^*$.

Given this, we see that the first conidtion $(q \in C_i)$ can be satisfied for at most two of the C_i, and the latter $(M^* \in C_i^*)$ likewise for at most two; thus no complete conic $(2M, 2q^*)$ of type 4 can lie in Z_{C_i} for all i.

Now on to transversality. In order to prove that the cycles $Z_{C_i} \subseteq Y$ intersect transversely when the conics $C_1, \ldots, C_5$ are general, we need a description of the tangent spaces to the Z_{C_i} at points of $\bigcap Z_i$. We have just shown that such points are represented by smooth conics, and the open subscheme parametrizing smooth conics is the same whether we are working in $\mathbb{P}^5$ or in Y, so we may express the answer in terms of the geometry of $\mathbb{P}^5$.

Lemma 8.5. Let $D \subseteq \mathbb{P}^2$ be a smooth conic curve and $Z_D^\circ \subseteq \mathbb{P}^5$ the variety of smooth plane conics C tangent to D.

(a) If C has a point p of simple tangency with D and is otherwise transverse to D, then Z_D° is smooth at $[C]$.

(b) In this case, the projective tangent plane $T_{[C]}Z_D^\circ$ to Z_D° at $[C]$ is the hyperplane $H_p \subseteq \mathbb{P}^5$ of conics containing p.

Proof. First, identify D with $\mathbb{P}^1$ and consider the restriction map

$$H^0(\mathcal{O}_{\mathbb{P}^2}(2)) \to H^0(\mathcal{O}_D(2)) = H^0(\mathcal{O}_{\mathbb{P}^1}(4)).$$

This map is surjective, with kernel the one-dimensional subspace spanned by the section representing D itself. Interms of projective spaces, the restriction induces a rational map

$$\pi_D : \mathbb{P}^5 = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^2}(2)) \to \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(4)) = \mathbb{P}^4$$

(this is just the linear projection of $\mathbb{P}^5$ from the point $D \in \mathbb{P}^5$ to $\mathbb{P}^4$). The closure Z_D° in $\mathbb{P}^5$ is thus the cone with vertex $D \in \mathbb{P}^5$ over the hypersurface $\mathcal{D} \subseteq \mathbb{P}^4$ of singular divisors in the linear system $|\mathcal{O}_{\mathbb{P}^1}(4)|$ (the singular divisors correspond to points of tangency); Lemma 8.5 will follow directly from the next result.

Proposition 8.6. Let $\mathbb{P}^d = \mathbb{P}H^0(\mathcal{O}_{\mathbb{P}^1}(d))$ be the space of polynomials of degree d on $\mathbb{P}^1$ and $\mathcal{D} \subseteq \mathbb{P}^d$ the discriminant hypersurface; that is, the locus of polynomials with a repeated root. If $F \in \mathcal{D}$ is a point corresponding to a polynomial with exactly one double root p and $d-2$ simple roots, then $\mathcal{D}$ is smooth at F with tangent space the space of polynomials vanishing at p.

Proof. Let us introduce the incidence correspondence

$$\Psi = \{(F,p) \in \mathbb{P}^d \times \mathbb{P}^1 : \mathrm{ord}_p(F) \geq 2\},$$

and write down its equations in local coordinates $(a,x) \in \mathbb{P}^d \times \mathbb{P}^1$. Ψ is the zero locus of the polynomials

$$R(a,x) - a_d x^d + a_{d-1}x^{d-1} + \cdots + a_1 x + a_0$$

and

$$S(a,x) = da_d x^{d-1} + (d-1)a_{d-1}x^{d-2} + \cdots + 2a_2 x + a_1.$$

Evaluated at a general point (a,x) where $a_1 = a_0 = x = 0$, all the partial derivatives of R and S vanish except

$$\begin{pmatrix} \frac{\partial R}{\partial a_1} & \frac{\partial R}{\partial a_0} & \frac{\partial R}{\partial x} \\ \frac{\partial S}{\partial a_1} & \frac{\partial S}{\partial a_0} & \frac{\partial S}{\partial x} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 2a_2 \end{pmatrix}.$$

The fact that the first 2×2 minor is nonzero assures us that Ψ is smooth at the point, and the fact that $a_2 \neq 0$ and the characteristic is not 2 assures us that the differential $\mathsf{d}\pi : T_{(a,0)}\Psi \to T_a\mathbb{P}^d$ of the projection $\mathcal{D} \to \mathbb{P}^d$ is injective, with the image the plane $a_0 = 0$. (Remember $a_0 = 0$ is the minimum condition for a vanishing at $x = 0$.) Finally, the fact that π is one-to-one at such a point tells us the image $\mathcal{D} = \pi(\Psi)$ is smooth at the image point. □

So Ψ is like a $\mathcal{D}$-fibration? $\Psi|_{x=\texttt{anything}}$ is a $\mathcal{D}$. In other words, the fibers of the map $\Psi \to \mathbb{P}^1$ given by $(a,x) = x$ are all isomorphic to $\mathcal{D}$. I think π is supposed to be $\Psi \to \mathbb{P}^d$ given by $(a,x) \mapsto a$. Look up differential geometry notes?

Getting back to the statement of Lemma 8.5, if $C \subseteq \mathbb{P}^5$ is a conic with a point p of simply tangency with D and is otherise transverse to D, then by Proposition 8.6 $\mathcal{D}$ is smooth at the image point in $\mathbb{P}^4$, with tangent space the space of polynomials vanishing at p. Since Z_D is the cone over $\mathcal{D}$ it follows that Z_D is smooth at C; the tangent space statement follows as well. □

In order to apply Lemma 8.5 we need to establish some more facts about a conic tangent to five general conics.

Lemma 8.7. Let $C_1, \ldots C_5 \subseteq \mathbb{P}^2$ be general conics and $C \subseteq \mathbb{P}^2$ any smooth conic tangent to all five. Each conic C_i is simply tangent to C at a point p_i and is otherwise transverse to C, and the points $p_i \in C$ are distinct.

Proof. Let U be the set of smooth conics, and consider incidence correspondences

$$\Phi = \{(C_1, \ldots, C_5; C) \in U^5 \times U : \text{each } C_i \text{ is tangent to } C\}$$
$$\subseteq \Phi' = \{(C_1, \ldots, C_5 : C) \in (\mathbb{P}^5)^5 \times U : \text{each } C_i \text{ is tangent to } C\}.$$

The set Φ is an open subset of Φ'. Since U is irreducible of dimension 5 and the projection map $\Phi' \to U$ on the last factor has irreducible fibers $(Z_C)^5$ of dimension 20, we see that Φ'– and thus also Φ– is irreducible of dimension 25. There are certainly points in Φ where the conditions of the lemma are satisfied: simply choose a conic C and five general conics C_i tangent to it. Thus the set of $(C_1, \ldots, C_5; C) \in \Phi$ where the conditions of the lemma are not satisfied is a proper closed subset, and as such it can have dimension at most 24, and cannot dominate U^5 under the projection to the first factor. This proves the lemma. □

To complete the argument for transversality, let $[C] \in \bigcap Z_i$ be a point corresponding to the conic $C \subseteq \mathbb{P}^2$. By Lemma 8.7 the points p_i of tangency of C with the C_i are distinct points on C. Since C is the unique conic through these five points, the intersectino of the tangent spaces to Z_i at $[C]$

$$\bigcap T_{[C]} Z_i = \bigcap H_{p_i} = \{[C]\}$$

is zero-dimensional, proving transversality.

Now onto the Chow ring of Y. Having confirmed that the intersection $\bigcap Z_i$ indeed behaves well, let us turn now to computing the intersection number. We start by describing the relevant subgroup of the Chow group $A(Y)$.

First, let $\alpha, \beta \in A^1(Y)$ be the pullbacks to $Y \subseteq \mathbb{P}^5 \times \mathbb{P}^{5*}$ of the hyperplane classes on $\mathbb{P}^5$ and $\mathbb{P}^{5*}$. These are respectively represented by the divisors

$$A_p = \{(C, C^*) : p \in C\}$$

(for any point $p \in \mathbb{P}^2$) and

$$B_L = \{(C, C^*) : L \in C^*\}$$

(for any point $L \in \mathbb{P}^{2*}$).

Also, let $\gamma, \varphi \in A^4(Y)$ be the classes of the curves Γ and Φ that are the pullbacks to Y of general lines in $\mathbb{P}^5$ and $\mathbb{P}^{5*}$. These are, respectively, the classes of the loci of complete conics (C, C^*) such that C contains four general points in the plane, and such that C^* contains four points $L_i \in \mathbb{P}^{2*}$ (that is, C is tangent to four lines in $\mathbb{P}^2$).

Lemma 8.8. The group $A^1(X)$ of divisor classes on X has rank 2, and is generated over the rationals by α and β. The intersection number of these classes with γ and φ are given by the table.

$\times$	α	β
γ	1	2
φ	2	1

Proof. We first show that the rank of $A^1(Y)$ is at most 2. The open subset $U \subseteq Y$ of pairs (C, C^*) with C and C^* smooth is isomorphic to the complement of a hypersurface in $\mathbb{P}^5$, and hence has torsion Picard group: Any line bundle on U extends to a line bundle on $\mathbb{P}^5$, a power of which is represented by a divisor supported on the complement $\mathbb{P}^5 \setminus U$. Thus, if L is any line bundle on Y, a power of L is trivial on U and so is represented by a divisor supported on the complement $Y \setminus U$. But the complement in X has just two irreducible components: the closures D_2 and D_3 of the loci of complete conics of type 2 and 3. Any divisor class on X is thus a rational linear combination of the classes D_2 and D_3, from which we see that the rank of the Picard group of Y is at most 2.

Since passing through a point is one linear condition on a quadric, we have $\deg(\alpha\gamma) = 1$ and dually $\deg(\beta\varphi) = 1$. Similarly, since a gneral pencil of conics will cut out on a line $L \subseteq \mathbb{P}^2$ a pencil of degree 2, which will have two branch points, we see that $\deg(\alpha\varphi) = 2$ (meaning: we can use $C_{(u,v)}$ in the pencil to map a pair of points in $L \cong \mathbb{P}^1$ to $(u, v) \in \mathbb{P}^1$ corresponding to $C_{(u,v)} \cap L$ giving a degree-2 map $\mathbb{P}^1 \to \mathbb{P}^1$ and using Hurwitz we get two branch points). Again by duality $\deg(\beta\gamma) = 2$. Since the matrix of intersections between α, β and γ, φ is nonsingular, we conclude that α and β generate $\operatorname{Pic}(Y) \otimes \mathbb{Q}$. $\square$

In fact, α and β generate $A^1(Y)$ over $\mathbb{Z}$ as well, as we could see from the description of Y as a blow up of $\mathbb{P}^5$.

It follows from Lemma 8.8 that we can write $\zeta = p\alpha + q\beta \in A^1(X) \otimes \mathbb{Q}$ for some $p, q \in \mathbb{Q}$. to compute p and q, we use the fact that, restricted to the open set $U \subseteq Y$, the divisor Z is a sextic hypersurface; it follows that $\deg(\zeta\gamma) = p + 2q = 6$ (γ is a line), and since ζ is a symmetric in α and β as well $\deg(\zeta\varphi) = q + 2p = 6$ as well. Thus

$$\zeta = 2\alpha + 2\beta \in A^1(Y) \otimes \mathbb{Q}.$$

From this we see that $\deg(\zeta^5) = 32\deg(\alpha+\beta)^5$, and it suffices to evaluate the degree of the class $\alpha^{5-i}\beta^i \in A^5(X)$ for $i = 0, \ldots, 5$. By symmetry, it is enough to do this for $i = 0$, 1 and 2.

To do this, observe first that the projection of $X \subseteq \mathbb{P}^5 \times \mathbb{P}^{5*}$ onto the first factor is an isomorphism on the set U_1 of pairs (C, C') such that rank $C \geq 2$ (the map $U \to \mathbb{P}^{5*}$ sending a smooth conic C to its dual in fact extends to a regular map on U_1 sending a conic $C = L \cup M$ of rank 2 to the double line $2p^* \in \mathbb{P}^5$, where $p = L \cap M$). Since all conics passing through three given general points have rank ≥ 2, the intersections needed will occur only in U_1. since the degree of a zero-dimensional intersection is equal to the degree of the intersection scheme, this implies that we can make the computations on $\mathbb{P}^5$ instead of Y. For this we use Bèzout's theorem:

- $i = 0$: Passing through a point is a linear condition of quadrics. There is a unique quadric passing through five general points, and the intersection of five hyperplanes in $\mathbb{P}^5$ has degree 1, so $\deg(\alpha^5) = 1$.

- $i = 1$: The quadrics tangent to a given line form a quadric hypersurface in $\mathbb{P}^5$ (that is, a general pencil of conics in $\mathbb{P}^2$ will have to fibers tangent to a given line L, but Hurwitz). Since not all conics in the one-dimensional linear space of conics through four general points will be tangent to a general line, $\deg(\alpha^4\beta) = 2$.

- $i = 2$: Similarly, we see that the conics passing through three given general points and tangent to a general line form a conic curve $U_1 \subseteq \mathbb{P}^5$. (That is, β is the class of the same quadric hypersurface from previous and α^3 is a plane in $\mathbb{P}^5$, so their intersection is a plane conic in $\mathbb{P}^5$.) Not all these conics are tangent to another given general line. (For example, after fixing coordinates we may this of circles as the conics passing through the points $(\pm i, 1, 0)$. Certainly there are circles through a given point and tangent to a given line that are not tangent to another given line.) It follows that $\deg(\alpha^3\beta^2)$ is the degree of the zero-dimensional intersection of a plane with two quadrics, that is, 4. Meaning for L, M lines in $\mathbb{P}^2$, $\alpha^3\beta_L$ is a plane conic and $\alpha^3\beta_M$ is a conic in the same plane α^3, and so they intersect at 4 points.

Thus

$$\begin{aligned}\deg((\alpha+\beta)^5) = \binom{5}{0} + \binom{5}{1}\cdot 2 + \binom{5}{2}\cdot 4 + \binom{5}{3}\cdot 4 + \binom{5}{4}\cdot 2 + \binom{5}{5} \\ = 1 + 10 + 40 + 40 + 10 + 1 = 102\end{aligned}$$

and correspondingly

$$\zeta^5 = 2^5 \cdot 102 = 3264.$$

This proves that there are 3264 plane conics tangent to five general plane conics.

Section 13.3.5 gives an alternative proof of the five conics problem using excess intersection formula.

In 9.7 they do the plane conics in $\mathbb{P}^3$ meeting 8 lines. Kind of like how fibrations generalize products, do bundles generalize fibrations? I think bundles are the domains of fibrations actually...

Anyway here is my incorrect attempt at enumerating the plane conics in $\mathbb{P}^3$ that intersect 8 general lines.

First thing I did was take our space of plane conics in $\mathbb{P}^3$ to be $X = \mathbb{P}^{3*} \times \mathbb{P}^5$. This was the mistake: its actually a $\mathbb{P}^5$-fibration over $\mathbb{P}^{3*}$, not a literal product.

Then I determined the Chow ring $A(X) = \mathbb{Z}[\alpha, \beta]/(\alpha^4, \beta^6)$ where α is the pullback of a plane in $\mathbb{P}^3$ and β is the pullback of a hyperplane in $\mathbb{P}^5$. I wanted to determine $\zeta = p\alpha + q\beta \in A^1(X)$ the class of all plane conics through a given line L and find $\deg(\zeta^8)$. To this effect, I wanted to find the degree of ζ in each component: $\mathbb{P}^3$ and $\mathbb{P}^5$. That is, what is $\deg(\zeta\alpha^2\beta^5)$ and $\deg(\zeta\alpha^3\beta^4)$? Note $\alpha^2\beta^5$ is a line in the $\mathbb{P}^{3*}$ (so it represents a pencil of planes with a "fixed" conic) and $\alpha^3\beta^4$ is a line in the $\mathbb{P}^5$ (so it represents a pencil of conics in a fixed plane).

I calculated that $\deg(\zeta\alpha^3\beta^4) = 1$ because a general pencil of conics will have one fibre containing a general point (the general point where is the intersection of the general line L with the fixed plane).

I then calculated that $\deg(\zeta\alpha^2\beta^4) = 2$ because the intersection of L with a general pencil of planes will trace out a line in a plane containing the fixed conic C, which will intersect C at two points.

From this we see that $\zeta = 2\alpha + \beta$. (THIS PART WAS RIGHT!!) Now we can calculate

$$\begin{aligned} \zeta^8 &= (2\alpha+\beta)^8 \\ &= 256\alpha^8 + \binom{8}{1}128\alpha^7\beta + \binom{8}{2}64\alpha^6\beta^2 + \binom{8}{3}32\alpha^5\beta^3 \\ &+ \binom{8}{4}16\alpha^2\beta^6 + \binom{8}{5}8\alpha^3\beta^5 + \binom{8}{6}4\alpha^2\beta^6 + \binom{8}{7}2\alpha\beta^7 + \beta^8 \\ &= \binom{8}{5}8\alpha^3\beta^5 = 448\alpha^3\beta^5 \end{aligned}$$

and so $\deg(\zeta^8) = 448$. This is wrong because X is wrong and so I'm using the wrong Chow ring A.

Apparently the correct thing to do involves taking $G = PGr(2,3) = \mathbb{P}^{3*}$ and looking that the "tautological bundle" $S = \mathcal{O}_G(-1)$ and then $X = \mathrm{Sym}^2(S^*)(\overset{?}{=} \mathrm{Sym}^2(\mathcal{O}_G(1)))$ is the bundle over G whose fiber over each point is the family of conics in the plane associated to point.

The space of complete conics is also isomorphic to the space $U = \mathbb{P}(\mathrm{Sym}^2\mathcal{O}_{\mathbb{P}^2}(1))$ blown up along the space I of nonreduced conics (i.e. squares of lines). Let E be the exceptional divisor of this blowup W. Then the space of conics that are tangent to a given conic $C \subseteq \mathbb{P}^2$ has the class $C_T = 6H - 2E \in A(W)$. This is because C_T is a degree-6 hypersurface of U (and therefore also W) by Riemann-Hurwitz, and C_T intersects I with a multiplicity of 2. This can be seen as follows: let L^2 be a double line (and so automatically intersects C with multiplicty 2 twice), and consider a general pencil of conics $\mathcal{P}$ containing L^2. Then $\mathcal{P}$ corresponds to a line in U, which must intersect C_T at six points as established by Riemann-Hurwitz, by the induced map $C \to \mathbb{P}^1$ given by $c \mapsto [t : u]$ where $c \in \mathcal{P}(t : u) \cap C$. This map has six ramification points, but two of them are the intersection of L^2 with C, and so there are only four more ramification points of the map $C \to \mathbb{P}^1$. Thus the line induced by $\mathcal{P}$ intersects C_T at four points not equal to $L^2 \in I$. Thus C_T contains I with multiplicity 2.

So now we need to calculate $(6H-2E)^5$. We can use the properties of $L_T = 2H-E$, which is the class of conics tangent to a given line (this can also be verified by Riemann-Hurwitz). Then

$$\begin{aligned} H^5 &= 1 \\ H^4 L_T &= 2 \\ H^3 L_T^2 &= 4 \\ H^2 L_T^3 &= 4 \\ H L_T^4 &= 2 \\ L_T^5 &= 1 \end{aligned}$$

and so

$$(6H - 2E)^5 = (2H + 2L_T)^5 = 3264.$$

This was originally known as Steiner's Conic Problem.