Some desultory comments on ideals of spaces of reducible forms
September 12, 2014

For simplicity, assume the ground field is the complex numbers. We start by finding the equation for the variety of reducible conics, and making other comments for this case, as a warmup for looking at varieties of reducible cubics.

Reducible ternary quadratic forms, necessarily of type (1,1):

This has dimension 4 in a P^5, hence is a hypersurface. There are at least three ways to get the equation.

(1) Use the fact that it is just the locus of singular conics. If F is a singular quadratic form, then $F_x = F_y = F_z = 0$ defines the locus. This can be written as $A(x, y, z)^t$, where A is the coefficient matrix of the gradient of F (i.e., of the partials of F). The locus of singular conics is given by the vanishing of the determinant of A. Note that this is a cubic, which is what we would expect since a general line in P^5 defines a pencil of conics with exactly three singular members. (The pencil is given by the conics through 4 general points of P^2; there are three singular conics, viz. the three pairs of lines through the 4 points.)

Let $F = ax^2 + bxy + cy^2 + dxz + eyz + fz^2 = a * x^2 + b * x * y + c * y^2 + d * x * z + e * y * z + f * z^2$. In M2, we get the 3 by 1 matrix of partials of F by $jacobian(ideal(F))$. To get the 3 by 3 matrix of double partials, take $jacobian(transposejacobian(ideal(F)))$. To get the Hessian, take $detjacobian(transposejacobian(ideal(F)))$. This gives $-2 * c * d^2 + 2 * b * d * e - 2 * a * e^2 - 2 * b^2 * f + 8 * a * c * f$, or $-2cd^2 + 2bde - 2ae^2 - 2b^2 f + 8acf$.

This uses the special fact that the partials are linear. If we ignore that, then we can do as follows, using a saturation step in place of the matrix factorization:

```
R=QQ[a..f]
S=frac(R)[x,y,z]
F=a*x^2+b*x*y+c*y^2+d*x*z+e*y*z+f*z^2
J=jacobian(ideal(F)) = matrix {{2*a*x+b*y+d*z}, {b*x+2*c*y+e*z}, {d*x+e*y+2*f*z}}
R=QQ[a..f,x,y,z]
I=ideal(2*a*x+b*y+d*z, b*x+2*c*y+e*z, d*x+e*y+2*f*z)
M=ideal(x,y,z)
J=saturate(I,M)
J=ideal(d*x+e*y+2*f*z,b*x+2*c*y+e*z,2*a*x+b*y+d*z,c*d^2-b*d*e+a*e^2+b^2*f-4*a*c*f)
```

Thus the locus of singular conics is defined by $c * d^2 - b * d * e + a * e^2 + b^2 * f - 4 * a * c * f$, as before.

(2) Consider the pencil of lines through the point $p = (0, 0, 1)$; i.e., $ny = mx$, $z = 1$. Assume F is an irreducible quadratic form which does not vanish at p. The pencil of lines has two distinct members which meet the zero locus of F in a double point. If F is reducible or contains p, there is only one such member of the pencil (and it occurs with multiplicity 2). Thus the condition that there is only one such member of the pencil also is a condition for F to either be reducible or to contain the point $p = (0, 0, 1)$. This condition can be found by restricting $F = ax^2 + bxy + cy^2 + dxz + eyz + fz^2$ to the pencil (i.e., let $y = mx/n$); this gives $ax^2 + bmx^2/n + cm^2x^2/n^2 + dxz + emxz/n + fz^2$, or $an^2x^2 + bmnx^2 + cm^2x^2 + dn^2xz + emnxz + fn^2z^2$, which is $(an^2 + bmn + cm^2)x^2 + (dn^2 + emn)xz + fn^2z^2$. The values of m and n such that $(an^2 + bmn + cm^2)x^2 + (dn^2 + emn)xz + fn^2z^2$ has a double root in x and z is given by the discriminant $D = (dn^2 + emn)^2 - 4(an^2 + bmn + cm^2)fn^2 = (e^2 - 4cf)m^2n^2 + (2de - 4bf)mn^3 + (d^2 - 4af)n^4$. Factoring out n^2 gives $(e^2 - 4cf)m^2 + (2de - 4bf)mn + (d^2 - 4af)n^2$. When this has a double root, then only one member of the pencil of lines meets F in a double point and hence F is reducible (or contains p). I.e., the condition is

$0 = (2de-4bf)^2-4(e^2-4cf)(d^2-4af) = 16cd^2f-16bdef+16ae^2f+16b^2f^2-64acf^2$. The condition that F contains $(0,0,1)$ is $f=0$, so factoring this out gives $16cd^2-16bde+16ae^2+16b^2f-64acf$ as the equation of the reducible conics. This is just -8 times what we found in (1).

(3) Define a homomorphism $H : k[a,b,\dots,f] \to k[A,\dots,F]$ by sending $a,\dots,f$ to the coefficients of $(Ax+By+Cz)(Dx+Ey+Fz) = (A*x+B*y+C*z)*(D*x+E*y+F*z) = A*D*x^2+B*D*x*y+A*E*x*y+B*E*y^2+C*D*x*z+A*F*x*z+C*E*y*z+B*F*y*z+C*F*z^2 = ADx^2+(BD+AE)xy+BEy^2+(CD+AF)xz+(CE+BF)yz+CFz^2 = A*D*x^2+(B*D+A*E)*x*y+B*E*y^2+(C*D+A*F)*x*z+(C*E+B*F)*y*z+C*F*z^2$. Then the reducible forms are the locus defined by the kernel of H; using Macaulay2 we find that it is generated by $c*d^2-b*d*e+a*e^2+b^2*f-4*a*c*f$.

In M2:

```
R=QQ[a..f]
S=QQ[A..F]
H=map(S,R,{A*D,B*D+A*E,B*E,C*D+A*F,C*E+B*F,C*F})
kernel(H) = ideal(c*d^2-b*d*e+a*e^2+b^2*f-4*a*c*f)
```

Note: The map $P^2 \to P^5$ given by $L \mapsto L^2$ is the 2-uple morphism (up to a diagonal change of basis involving binomial coefficients); i.e., its image is the Veronese V. Its image has degree 4 (since the intersection of two conics has 4 points).

Now we look at the spaces of secant lines. This is well known for the Veronese V: the points in P^5 in the secant space $Sec_2(V)$ are the closure of the space of conics which lie in pencils defined by $\langle L_1^2, L_2^2\rangle$, where L_1 and L_2 are linearly independent linear forms. But every conic in the pencil $\langle L_1^2, L_2^2\rangle$ is reducible, so the secant variety is contained in the locus of the space of singular conics, hence has dimension 4, so is deficient (i.e., has dimension less than $2*2+1$).

The secant variety $Sec_2(S)$ for the locus S of singular conics is all of P^5, since every irreducible conic is in a pencil defined by 2 (or more) singular conics: take any irreducible conic, C. Pick any four points on C. The set of all conics through those four points is a pencil, and this pencil contains three reducible conics, namely the three pairs of lines through the four points. This shows that not only is every irreducible conic in the secant variety, but every irreducible conic is on a 3-secant line. This is not remarkable since the locus S of singular conics is a hypersurface of degree 3, every line is a 3 secant line unless the line is tangent to S. It is also true that every irreducible conic is on a secant line containing exactly two reducible conics. Let C be an irreducible conic. Pick any three points $p_1, p_2, p_3 \in C$. Let L_1 be tangent to C at p_1 and let L_2 be the line through p_2 and p_3. Let L_3 be the line through p_1 and p_2 and let L_4 be the line through p_1 and p_3. Then C is in the pencil defined by L_1L_2 and L_3L_4, and L_1L_2 and L_3L_4 are the only two reducible conics in the pencil (note that the pencil is again defined by 4 points, these being p_1, p_2, p_3 and the fourth point being the point of L_1 infinitely near to p_1, which is how we know that C is in the pencil, since it contains all 4 points). In fact, the secant line defined by L_1L_2 and L_3L_4 is tangent to S at L_3L_4, as we now explain.

First we find the singular locus of S and the tangent space to smooth points of S. Up to change of coordinates, there are only two points in S: x^2 and xy. Since S is defined by $cd^2-bde+ae^2+b^2f-4acf$ and x^2 is represented by the coordinates $(1,0,0,0,0,0)$, the equation for S in coordinates centered at the point $(1,0,0,0,0,0)$ is $cd^2-bde+e^2+b^2f-4cf$. There are no linear terms so we see that x^2 gives a singular point of S, hence S is singular along all squares, i.e., along V. To see that the singular locus of S is exactly V it's enough to check that xy gives a smooth point of S. Now xy is represented by the point $(0,1,0,0,0,0)$, so the equation becomes $cd^2-de+ae^2+f-4acf=0$, hence the tangent space at xy is $f=0$; i.e., it consists of all conics of the form $ax^2+bxy+cy^2+dxz+eyz$. It is easy to check that this is this span is independent of the choice of the coordinate z; i.e., the smooth points of S are those representing conics of the form L_1L_2 where L_1 and L_2 are linearly independent linear forms, and the tangent hyperplane to

S at L_1L_2 are the span of the conics of the form $L_1^2, L_1L_2, L_2^2, L_1L_3, L_2L_3$, where L_3 is any choice of a third line such that L_1, L_2, L_3 are linearly independent. Another way to say this is that the tangent hyperplane to S at a smooth point L_1L_2 of S is the span of all conics that vanish at the singular point of L_1L_2.

In particular, a pencil $\langle L_1L_2, L_3L_4\rangle$ is tangent to S at a smooth point L_1L_2 if and only if $p \in L_3L_4$ where p is the point where L_1 and L_2 cross (here we are not distinguishing a line from the form that defines it).

Ternary cubic forms:

Here there are various strata:
L^3 where L is a linear form: dimension 2, degree 9 (this is a Veronese, the image of the 3-uple morphism, up to a diagonal change of basis)
L^2M where L and M are linear: dimension 4
$LM(aL + bM)$ where L and M are linear: dimension 5
LMN where L, M and N are linear: dimension 6, closure of degree 15 (see below)
LQ where L is linear, Q is a conic and L is tangent to Q: dimension 6
LQ where L is linear and Q is a conic: dimension 7, closure of degree 21 (see below)
C where C is a cuspidal cubic: dimension 7, closure of degree 24 (see below)
D where D is a nodal cubic: dimension 8, closure of degree 12 (since a general pencil of cubics, by an Euler characteristic calculation, has 12 singular members)
F where F is any cubic: dimension 9.

M2 commands to compute locus of singular cubics:

```
R=QQ[a..j]
S=frac(R)[x,y,z]
F=a*x^3+b*x^2*y+c*x*y^2+d*y^3+e*x^2*z+f*x*y*z+g*y^2*z+h*x*z^2+i*y*z^2+j*z^3
J=jacobian(ideal(F))
I=ideal(J_0_0, J_0_1,J_0_2)
Unfortunately, we want this ideal in R[x,y,z], not S.
So use the fact that
J_0_0=3*a*x^2+2*b*x*y+c*y^2+2*e*x*z+f*y*z+h*z^2
J_0_1=b*x^2+2*c*x*y+3*d*y^2+f*x*z+2*g*y*z+i*z^2
J_0_2=e*x^2+f*x*y+g*y^2+2*h*x*z+2*i*y*z+3*j*z^2
but start over:
R=QQ[a..j,x,y,z]
I=ideal(3*a*x^2+2*b*x*y+c*y^2+2*e*x*z+f*y*z+h*z^2,
b*x^2+2*c*x*y+3*d*y^2+f*x*z+2*g*y*z+i*z^2,
e*x^2+f*x*y+g*y^2+2*h*x*z+2*i*y*z+3*j*z^2)
M=ideal(x,y,z)
J=saturate(I,M)
Here is the betti table for J:

            0  1  2   3  4  5 6
o7 = total: 1 22 77 114 88 36 6
         0: 1  .  .   .  .  . .
         1: .  .  .   .  .  . .
         2: .  3  .   .  .  . .
         3: .  .  .   .  .  . .
         4: .  3 11   6  .  . .
```

```
 5: .  .  .   .  .  . .
 6: . 15 63 105 87 36 6
 7: .  .  .   .  .  . .
 8: .  .  .   .  .  . .
 9: .  .  .   .  .  . .
10: .  .  .   .  .  . .
11: .  1  3   3  1  . .
```

The only generator of J which doesn't involve x, y, z is the last one, of degree 12, hence this is the equation of the locus S of singular cubics. It is included as commented out text at this point in the source file.

Alternatively, the ring of invariants (with respect to projective changes of coordinates) for ternary cubics is known. The nodal cubics form a single 8 dimensional orbit, so the closure of the locus of nodal cubics is the zero locus of some invariant, which must be of degree 12.

Note that every singular cubic is in the closure of the space of nodal cubics: for example, a cuspidal cubic $zy^2 - x^3$ is the limit of $z(y^2 - tx^2) - x^3$ as $t \to 0$. This is possible since the orbit of $y^2 - x^3$ is 7 dimensional with a 1 dimensional stabilizer (one orbit can be in the closure of another if it has smaller dimension). Every other singular cubic is reducible, so (up to projective change of coordinates) is either $y(y^2 - xz)$, $x(y^2 - xz)$, xyz, $xy(x - y)$, x^2y or x^3. But $y(y^2 - xz)$ is the limit as $t \to 0$ of $y(y^2 - xz) + tz^3$, xyz is the limit as $t \to 0$ of $y(-ty^2 + xz) - tz^3$, $xy(x - y)$ is the limit as $t \to 0$ of $xy(x - y) + tz(x^2 + y^2)$, x^2y is the limit as $t \to 0$ of $xy(x - y) + tz(x^2 + y^2)$, and x^3 is the limit as $t \to 0$ of $x^3 - tz(y^2 - x^2)$. Thus the closure of the space of nodal cubics contains every singular cubic. Also, any nonsingular cubic has an 8 dimensional orbit so no nonsingular cubic is in the closure of the nodal cubics. I.e., the closure of the orbit of the nodal cubics is precisely the space of all singular cubics.

Consider the map of $P^8 \to P^9$ given by $M \mapsto MC$ where M is a 3 by 3 matrix (a coordinate change matrix when M is nonsingular) acting on cubics and C is the nodal cubic $x^3 - (y^2 - x^2)z$. This map defines a homomorphism whose kernel is generated by the equation of the locus of singular cubics. If we take M to be the matrix whose entries are the letters A through I, then $x^3 - (y^2 - x^2)z$ maps to $(A*x + B*y + C*z)^3 - ((D*x + E*y + F*z)^2 - (A*x + B*y + C*z)^2)*(G*x + H*y + I*z)$. We can use M2 to compute the equation of S:

```
S=QQ[A..I,x,y,z]
(A*x+B*y+C*z)^3-((D*x+E*y+F*z)^2-(A*x+B*y+C*z)^2)*(G*x+H*y+I*z)
this simplifies to
(A^3+A^2*G-D^2*G)*x^3+
(3*A^2*B+2*A*B*G-2*D*E*G+A^2*H-D^2*H)*x^2*y+
(3*A*B^2+B^2*G-E^2*G+2*A*B*H-2*D*E*H)*x*y^2+
(B^3+B^2*H-E^2*H)*y^3+
(3*A^2*C+2*A*C*G-2*D*F*G+A^2*I-D^2*I)*x^2*z+
(6*A*B*C+2*B*C*G-2*E*F*G+2*A*C*H-2*D*F*H+2*A*B*I-2*D*E*I)*x*y*z+
(3*B^2*C+2*B*C*H-2*E*F*H+B^2*I-E^2*I)*y^2*z+
(3*A*C^2+C^2*G-F^2*G+2*A*C*I-2*D*F*I)*x*z^2+
(3*B*C^2+C^2*H-F^2*H+2*B*C*I-2*E*F*I)*y*z^2+
(C^3+C^2*I-F^2*I)*z^3.

M2:

R=QQ[a..j]
S=QQ[A..I]
HO=map(S,R,{A^3+A^2*G-D^2*G,3*A^2*B+2*A*B*G-2*D*E*G+A^2*H-D^2*H,
```

```
3*A*B^2+B^2*G-E^2*G+2*A*B*H-2*D*E*H,B^3+B^2*H-E^2*H,
3*A^2*C+2*A*C*G-2*D*F*G+A^2*I-D^2*I,
6*A*B*C+2*B*C*G-2*E*F*G+2*A*C*H-2*D*F*H+2*A*B*I-2*D*E*I,
3*B^2*C+2*B*C*H-2*E*F*H+B^2*I-E^2*I,3*A*C^2+C^2*G-F^2*G+2*A*C*I-2*D*F*I,
3*B*C^2+C^2*H-F^2*H+2*B*C*I-2*E*F*I,C^3+C^2*I-F^2*I})
kernel(HO) = ideal(???) [This computation seems pretty slow; it hasn't finished yet.]
```

Alternatively, we can instead use the known ring of invariants (see B. Sturmfels, Algorithms in invariant theory, or url: http://math.stanford.edu/ notzeb/aronhold.html): given the general cubic $a*x^3+b*y^3+c*z^3+3*d*x^2*y+3*e*y^2*z+3*f*z^2*x+3*g*x*y^2+3*h*y*z^2+3*i*z*x^2+6*j*x*y*z$, the ring of invariants is $\mathbf{Q}[S,T]$, where

$$S = a*g*e*c - a*g*h^2 - a*j*b*c + a*j*e*h + a*f*b*h - a*f*e^2 - d^2*e*c + d^2*h^2 + d*i*b*c - d*i*e*h + d*g*j*c - d*g*f*h - 2*d*j^2*h + 3*d*j*f*e - d*f^2*b - i^2*b*h + i^2*e^2 - i*g^2*c + 3*i*g*j*h - i*g*f*e - 2*i*j^2*e + i*j*f*b + g^2*f^2 - 2*g*j^2*f + j^4$$

and

$$T = a^2*b^2*c^2-3*a^2*e^2*h^2-6*a^2*b*e*h*c+4*a^2*b*h^3+4*a^2*e^3*c-6*a*d*g*b*c^2+18*a*d*g*e*h*c-12*a*d*g*h^3+12*a*d*j*b*h*c-24*a*d*j*e^2*c+12*a*d*j*e*h^2-12*a*d*f*b*h^2+6*a*d*f*b*e*c+6*a*d*f*e^2*h+6*a*i*g*b*h*c-12*a*i*g*e^2*c+6*a*i*g*e*h^2+12*a*i*j*b*e*c+12*a*i*j*e^2*h-6*a*i*f*b^2*c+18*a*i*f*b*e*h-24*a*g^2*j*h*c-24*a*i*j*b*h^2-12*a*i*f*e^3+4*a*g^3*c^2-12*a*g^2*f*e*c+24*a*g^2*f*h^2+36*a*g*j^2*e*c+12*a*g*j^2*h^2+12*a*g*j*f*b*c-60*a*g*j*f*e*h-12*a*g*f^2*b*h+24*a*g*f^2*e^2-20*a*j^3*b*c-12*a*j^3*e*h+36*a*j^2*f*b*h+12*a*j^2*f*e^2-24*a*j*f^2*b*e+4*a*f^3*b^2+4*d^3*b*c^2-12*d^3*e*h*c+8*d^3*h^3+24*d^2*i*e^2*c-12*d^2*i*e*h^2+12*d^2*g*j*h*c+6*d^2*g*f*e*c-24*d^2*j^2*h^2-12*d^2*i*b*h*c-3*d^2*g^2*c^2-24*g^2*j^2*f^2+24*g*j^4*f-12*d^2*g*f*h^2+12*d^2*j^2*e*c-24*d^2*j*f*b*c-27*d^2*f^2*e^2+36*d^2*j*f*e*h+24*d^2*f^2*b*h+24*d*i^2*b*h^2-12*d*i^2*b*e*c-12*d*i^2*e^2*h+6*d*i*g^2*h*c-60*d*i*g*j*e*c+36*d*i*g*j*h^2+18*d*i*g*f*b*c-6*d*i*g*f*e*h+36*d*i*j^2*b*c-12*d*i*j^2*e*h-60*d*i*j*f*b*h+36*d*i*j*f*e^2+6*d*i*f^2*b*e+12*d*g^2*j*f*c-12*d*g*j^3*c-12*d*g*j^2*f*h+36*d*g*j*f^2*e-12*d*g*f^3*b+24*d*j^4*h+12*d*j^2*f^2*b+4*i^3*b^2*c+24*i^2*g^2*e*c-27*i^2*g^2*h^2-36*d*j^3*f*e-12*i^3*b*e*h+8*i^3*e^3-24*i^2*g*j*b*c+36*i^2*g*j*e*h+6*i^2*g*f*b*h+12*i^2*j^2*b*h-3*i^2*f^2*b^2-12*d*g^2*f^2*h-12*i^2*g*f*e^2-24*i^2*j^2*e^2+12*i^2*j*f*b*e-12*i*g^3*f*c+12*i*g^2*j^2*c+36*i*g^2*j*f*h-12*i*g^2*f^2*e-36*i*g*j^3*h-12*i*g*j^2*f*e+12*i*g*j*f^2*b+24*i*j^4*e-12*i*j^3*f*b+8*g^3*f^3-8*j^6.$$

An example of a nodal cubic is $x^3+3(y^2-x^2)z$, so $a=1, e=1, i=-1$ and the other coefficients are 0. Plugging in shows $S=1$ and $T=-8$ for a nodal cubic, so a degree 12 form vanishing for $x^3+3(y^2-x^2)z$ must be of the form mS^3-nT^2 such that $m-64n=0$, hence we can take $64S^3-T^2$, or $(4S)^3-T^2$. (This is related to the j-invariant for smooth cubics: $j = 1728*(4S)^3/((4S)^3-T^2)$.)

In short, the locus of singular cubics is given by $(4S)^3-T^2$. Its expansion is included at this point of the source file as commented out text. Note that this equation for the locus of singular cubics is based on using the general cubic written as $a*x^3+b*y^3+c*z^3+3*d*x^2*y+3*e*y^2*z+3*f*z^2*x+3*g*x*y^2+3*h*y*z^2+3*i*z*x^2+6*j*x*y*z$, whereas above we used $a*x^3+b*x^2*y+c*x*y^2+d*y^3+e*x^2*z+f*x*y*z+g*y^2*z+h*x*z^2+i*y*z^2+j*z^3$, so the order and scaling of the monomials is different. By performing the appropriate change of basis to the equation we found before we should get the equation we found here, and this is indeed the case. After the change of basis, the equation before is just 3^9 times the equation we found here (checked with Macaulay2).

Both $S=0$ and $T=0$ for $3x^3+3zy^2$, so all invariants vanish on the cuspidal cubics. If C is any cubic for which S and T both vanish, then $(4S)^3-T^2=0$ so C is singular. But C can't be nodal since neither S nor T vanishes for a nodal cubic. In fact $S=T=0$ defines the locus of all cuspidal and all simply connected reducible conics. To see this we check each such cubic. The numbers following each cubic below are the coefficients a through i that define the specific cubic with respect to $a{*}x^3{+}b{*}y^3{+}c{*}z^3{+}3{*}d{*}x^2{*}y{+}3{*}e{*}y^2{*}z{+}3{*}f{*}z^2{*}x{+}3{*}g{*}x{*}y^2{+}3{*}h{*}y{*}z^2{+}3{*}i{*}z{*}x^2{+}6{*}j{*}x{*}y{*}z$.

cuspidal $3x^3 - 3y^2z$:

```
a b c d  e f g h i j
3 0 0 0 -1 0 0 0 0 0
```

$(S, T) = (0, 0)$

irreducible conic and general line $6y^3 - 6xyz$:

```
a b c d e f g h i  j
0 6 0 0 0 0 0 0 0 -1
```

$(S, T) = (1, -8)$

irreducible conic and tangent line $3xy^2 - 3x^2z$:

```
a b c d e f g h  i j
0 0 0 0 0 0 1 0 -1 0
```

$(S, T) = (0, 0)$

three general lines $6xyz$:

```
a b c d e f g h i j
0 0 0 0 0 0 0 0 0 1
```

$(S, T) = (1, -8)$

three concurrent lines $3xy^2 - 3x^2y$:

```
a b c  d e f g h i j
0 0 0 -1 0 0 1 0 0 0
```

$(S, T) = (0, 0)$

a line squared and a general line $3x^2y$:

```
a b c d e f g h i j
0 0 0 1 0 0 0 0 0 0
```

$(S, T) = (0, 0)$

a line cubed x^3:

```
a b c d e f g h i j
1 0 0 0 0 0 0 0 0 0
```

$(S, T) = (0, 0)$

An awk program to evaluate S and T (it replaces a through i in the echo statement with the desired values) is included at this point as commented out text in the source file. One can also do this in M2 this way: $S[a, b, c, d, e, f, g, h, i, j]$ will evaluate S at the values inserted for a through i.

Thus the locus $S = T = 0$ consists of exactly the closure of the cuspidal cubics, so (S, T) is a complete intersection with irreducible zero locus, hence primary.

I tried to use M2 to check if ideal(S,T) is radical; it ran all night under M2 and is not done yet. However, if in M2 we set $B = jacobian(ideal(S))$ and $C = jacobian(ideal(T))$, we can check the values of the gradients of S and T at a cuspidal cubic $3y^2z - 3x^3$ (hence at $(a, b, c, d, e, f, g, h, i, j) = (-3, 0, 0, 0, 1, 0, 0, 0, 0, 0)$). Since the gradient is a matrix, to see the values we must run through the entries:

```
for i from 0 to 9 do print (B_0_i[-3, 0, 0, 0, 1, 0, 0, 0, 0, 0])
output: [0, 0, 0, 0, 0, 3, 0, 0, 0, 0]
for i from 0 to 9 do print (C_0_i[-3, 0, 0, 0, 1, 0, 0, 0, 0, 0])
output: [0, 0, 36, 0, 0, 0, 0, 0, 0, 0]
```

Doing so we find that the gradients are linearly independent, hence S and T meet transversely at a general point of their intersection, hence (S, T) is a prime ideal, so radical, hence the space of cuspidal cubics has degree $\deg(S)\deg(T) = 24$.

Now we look at the space of ternary cubic forms of type $(1,2)$. $(ax+by+cz)(dx^2+exy+fy^2+gxz+hyz+iz^2) = (a*x+b*y+c*z)*(d*x^2+e*x*y+f*y^2+g*x*z+h*y*z+i*z^2) = a*d*x^3+(b*d+a*e)*x^2*y+(b*e+a*f)*x*y^2+b*f*y^3+(c*d+a*g)*x^2*z+(c*e+b*g+a*h)*x*y*z+(c*f+b*h)*y^2*z+(c*g+a*i)*x*z^2+(c*h+b*i)*y*z^2+c*i*z^3 = adx^3+(bd+ae)x^2y+(be+af)xy^2+bfy^3+(cd+ag)x^2z+(ce+bg+ah)xyz+(cf+bh)y^2z+(cg+ai)xz^2+(ch+bi)yz^2+ciz^3$.

With respect to the general cubic being written $F = a*x^3+b*x^2*y+c*x*y^2+d*y^3+e*x^2*z+f*x*y*z+g*y^2*z+h*x*z^2+i*y*z^2+j*z^3$ we get a homomorphism whose kernel is the space of forms of the given type. The kernel computation was pretty fast.

```
M2:

R=QQ[A..J]
S=QQ[a..i]
HO=map(S,R,{a*d,b*d+a*e,b*e+a*f,b*f,c*d+a*g,c*e+b*g+a*h,c*f+b*h,c*g+a*i,c*h+b*i,c*i})
II=kernel(HO) =
degree(II) = 21
```

The actual list of generators is long; it's included at this point in the source file as commented out text.

Here is the betti table:

```
i6 : betti res II

            0  1  2  3 4
o6 = total: 1 35 70 45 9
         0: 1  .  .  . .
         1: .  .  .  . .
         2: .  .  .  . .
         3: .  .  .  . .
         4: .  .  .  . .
         5: .  .  .  . .
         6: .  .  .  . .
         7: . 35 70 45 8
         8: .  .  .  . 1
```

Since the space of $(1,2)$ forms have projective dimension 7, we would expect that the secant space would be all of P^9, and this is in fact the case, since every smooth cubic is in a pencil that contains four reducible members of type $(1,2)$. The construction works like this. Let C be a smooth plane cubic. It has a flex point p. Take four lines through p, none of which are the flex line. These four lines meet C at 8 points other than p, two per line. Care must be taken so that no three of these points, taken from 3 different lines from among the four lines, are collinear (this is an open condition so easy to assure). Using the group law on C one can now show that each line comes with an irreducible conic through the 6 of the 8 points not on the line, and the pencil consisting of C and any choice of one of the lines and its associated conic contains the 3 other cubics comprising any of the lines and its associated conic. It's possible that a line and its associated conic may be tangent; a more delicate argument is needed if one wants to avoid this possibility. (For any smooth cubic one can pick 9 points such that there is a pencil of cubics through the 9 points, 6 members of which are line-conic pairs where the conics are irreducible and each conic meets its associated line in two distinct points.)

Thus each smooth conic is on at least a 4-secant for the space of $(1,2)$ cubics, hence the secant variety is all of P^9.

Now we look at the space of ternary cubic forms of type $(1,1,1)$. $(ax+by+cz)(dx+ey+fz)(gx+hy+iz) = (a*x+b*y+c*z)*(d*x+e*y+f*z)*(g*x+h*y+i*z) =$

$a*d*g*x^3+b*d*g*x^2*y+a*e*g*x^2*y+a*d*h*x^2*y+b*e*g*x*y^2+b*d*h*x*y^2+a*e*h*x*y^2+b*e*h*y^3+c*d*g*x^2*z+a*f*g*x^2*z+a*d*i*x^2*z+c*e*g*x*y*z+b*f*g*x*y*z+c*d*h*x*y*z+a*f*h*x*y*z+b*d*i*x*y*z+a*e*i*x*y*z+c*e*h*y^2*z+b*f*h*y^2*z+b*e*i*y^2*z+c*f*g*x*z^2+c*d*i*x*z^2+a*f*i*x*z^2+c*f*h*y*z^2+c*e*i*y*z^2+b*f*i*y*z^2+c*f*i*z^3 = a*d*g*x^3+(b*d*g+a*e*g+a*d*h)*x^2*y+(b*e*g+b*d*h+a*e*h)*x*y^2+b*e*h*y^3+(c*d*g+a*f*g+a*d*i)*x^2*z+(c*e*g+b*f*g+c*d*h+a*f*h+b*d*i+a*e*i)*x*y*z+(c*e*h+b*f*h+b*e*i)*y^2*z+(c*f*g+c*d*i+a*f*i)*x*z^2+(c*f*h+c*e*i+b*f*i)*y*z^2+c*f*i*z^3 = adgx^3+(bdg+aeg+adh)x^2y+(beg+bdh+aeh)xy^2+behy^3+(cdg+afg+adi)x^2z+(ceg+bfg+cdh+afh+bdi+aei)xyz+(ceh+bfh+bei)y^2z+(cfg+cdi+afi)xz^2+(cfh+cei+bfi)yz^2+cfiz^3.$

With respect to the general cubic being written $F = a*x^3+b*x^2*y+c*x*y^2+d*y^3+e*x^2*z+f*x*y*z+g*y^2*z+h*x*z^2+i*y*z^2+j*z^3$ we get a homomorphism whose kernel is the space of forms of the given type. The kernel computation again was pretty fast.

```
M2:

R=QQ[A..J]
S=QQ[a..i]
HO=map(S,R,{a*d*g,b*d*g+a*e*g+a*d*h,
b*e*g+b*d*h+a*e*h,b*e*h,c*d*g+a*f*g+a*d*i,
c*e*g+b*f*g+c*d*h+a*f*h+b*d*i+a*e*i,c*e*h+b*f*h+b*e*i,
c*f*g+c*d*i+a*f*i,c*f*h+c*e*i+b*f*i,c*f*i})
II=kernel(HO) = [gens included as commented out text in source]
degree(II) = 15

betti res II
               0  1   2   3   4   5   6  7  8 9
o10 = total: 1 35 119 211 252 210 120 45 10 1
          0: 1  .   .   .   .   .   .  .  . .
          1: .  .   .   .   .   .   .  .  . .
          2: .  .   .   .   .   .   .  .  . .
          3: . 35 119 210 252 210 120 45 10 1
          4: .  .   .   1   .   .   .  .  . .
```

Since the space of $(1,1,1)$ forms has projective dimension 6, we would still expect that the secant space would be all of P^9, and this is the case, since every smooth cubic is in a pencil that contains exactly four reducible members of type $(1,1,1)$ (and no other singular members). Moreover, each reducible member of type $(1,1,1)$ consists of three non-concurrent lines. The construction works like this. Let C be a smooth cubic. Consider the pencil defined by C and its Hessian. The Hessian vanishes at the flex points of C, and there are 12 lines through pairs of flex points. Each line goes through a third flex, and each line meets two other of the 12 lines away from the 9 flex points. Thus the 12 lines form fours triples, where each line in a triple meets the other two lines in the triple away from the 9 flex points. These four triples are in the pencil. An Euler characteristic calculation shows that the lines in each triple are nonconcurrent, and that there are no other singular members of the pencil. Thus each smooth cubic lies on a 4-secant line for the space of cubics which consist of three nonconcurrent lines.

Ternary quartic forms of type $(1,1,1,1)$:

$(a*x+b*y+c*z)*(d*x+e*y+f*z)*(g*x+h*y+i*z)*(j*x+k*y+l*z) = a*d*g*j*x^4+b*d*g*j*x^3*y+a*e*g*j*x^3*y+a*d*h*j*x^3*y+a*d*g*k*x^3*y+b*e*g*j*x^2*y^2+b*d*h*j*x^2*y^2+a*e*h*j*x^2*y^2+b*d*g*k*x^2*y^2+a*e*g*k*x^2*y^2+a*d*h*k*x^2*y^2+b*e*h*j*x*y^3+b*e*g*k*x*y^3+b*d*h*k*x*y^3+a*e*h*k*x*y^3+b*e*h*k*y^4+c*d*g*j*x^3*z+a*f*g*j*x^3*z+a*d*i*j*x^3*z+a*d*g*l*x^3*z+c*e*g*j*x^2*y*z+b*f*g*j*x^2*y*z+c*d*h*j*x^2*y*z+a*f*h*j*x^2*y*z+b*d*i*j*x^2*y*z+a*e*i*j*x^2*y*z+c*d*g*k*x^2*y*z+a*f*g*k*x^2*y*z+a*d*i*k*x^2*y*z+b*$

$d*g*l*x^2*y*z+a*e*g*l*x^2*y*z+a*d*h*l*x^2*y*z+c*e*h*j*x*y^2*z+b*f*h*j*x*y^2*z+b*e*i*j*x*y^2*z+c*e*g*k*x*y^2*z+b*f*g*k*x*y^2*z+c*d*h*k*x*y^2*z+a*f*h*k*x*y^2*z+b*d*i*k*x*y^2*z+a*e*i*k*x*y^2*z+b*e*g*l*x*y^2*z+b*d*h*l*x*y^2*z+a*e*h*l*x*y^2*z+c*e*h*k*y^3*z+b*f*h*k*y^3*z+b*e*i*k*y^3*z+b*e*h*l*y^3*z+c*f*g*j*x^2*z^2+c*d*i*j*x^2*z^2+a*f*i*j*x^2*z^2+c*d*g*l*x^2*z^2+a*f*g*l*x^2*z^2+a*d*i*l*x^2*z^2+c*f*h*j*x*y*z^2+c*e*i*j*x*y*z^2+b*f*i*j*x*y*z^2+c*f*g*k*x*y*z^2+c*d*i*k*x*y*z^2+a*f*i*k*x*y*z^2+c*e*g*l*x*y*z^2+b*f*g*l*x*y*z^2+c*d*h*l*x*y*z^2+a*f*h*l*x*y*z^2+b*d*i*l*x*y*z^2+a*e*i*l*x*y*z^2+c*f*h*k*y^2*z^2+c*e*i*k*y^2*z^2+b*f*i*k*y^2*z^2+c*e*h*l*y^2*z^2+b*f*h*l*y^2*z^2+b*e*i*l*y^2*z^2+c*f*i*j*x*z^3+c*f*g*l*x*z^3+c*d*i*l*x*z^3+a*f*i*l*x*z^3+c*f*i*k*y*z^3+c*f*h*l*y*z^3+c*e*i*l*y*z^3+b*f*i*l*y*z^3+c*f*i*l*z^4$

$R = QQ[A..O]$

$S = QQ[a..l]$

$HO = map(S, R, a*d*g*j, b*d*g*j + a*e*g*j + a*d*h*j + a*d*g*k, b*e*g*j + b*d*h*$

$II = kernel(HO)$