First we will show that the only Halphen pencils with Bertini involutions are those of index 1 or 2.

Let E be a smooth elliptic curve with nine base points $P_1, \ldots, P_9 \in E$. That is, that $P_1 \oplus \cdots \oplus P_9 = 0 \mod O$ where O is some flex point of E. Note that a point $Q \in E$ is fixed under the Bertini involution centered at P_9 if and only if $Q \in E[2]_{P_9}$. If $Q = P_9$, then we get an Halphen pencil of index 1 by blowing up $P_1, \ldots, P_8, Q$. If $Q \in E[2]_{P_9} \setminus \{P_9\}$, then we must show that we get an Halphen pencil of index 2 by blowing up $P_1, \ldots, P_8, Q$.

Note that $2Q = 0 \mod P_9$. Therefore, the tangent line $T_Q(E)$ meets E again exactly where the tangent line $T_{P_9}(E)$ meets E again. We call this point P_9^2. Thus when we re-mod the curve at O, we get the equality $2Q = 2P_9 = -P_9^2 \mod O$.

Therefore $2P_1 \oplus \cdots \oplus 2P_8 \oplus 2Q = 2P_1 \oplus \cdots \oplus 2P_8 \oplus 2P_9 = 0 \mod O$. So blowing up $P_1, \ldots, P_8, Q$ yields an Halphen pencil of index 2. Thus the only Halphen pencils with Bertini involutions are those of index 1 and 2, because the center of the Bertini involution must be fixed.

Now we will show that if $\beta_{P_9}(R) = S$ (that is, $R \oplus S = 0 \mod P_9$), then the unique sextic curve S that has double points at $P_1, \ldots, P_8, Q$ (where again $Q \in E[2]_{P_9} \setminus \{P_9\}$) and also contains R must additionally contain S.

Since $R \oplus S = 0 \mod P_9$, we know that R, S, and P_9^2 are collinear, so $R \oplus S \oplus P_9^2 = 0 \mod O$. We also know that $2Q = 2P_9 = -P_9^2 \mod O$. We want to show that $2P_1 \oplus \cdots \oplus 2P_8 \oplus R \oplus S = 0 \mod O$.

Note that

$$2P_1 \oplus \cdots \oplus 2P_8 \oplus R \oplus S = 2P_1 \oplus \cdots \oplus 2P_8 \oplus -P_9^2 = 2P_1 \oplus \cdots \oplus 2P_9 = 0 \mod O.$$

Therefore there is a sextic curve S that meets E with multiplicity 2 at $P_1, \ldots, P_8$ and with multiplicity 1 at R and S.