This talk is based on the findings of Dutta, Halbeisen, and Hungerbühler in their 2023 paper *Properties of Hesse derivatives of cubic curves*, and on the speaker's own research. We also look at *Hesse Pencils and 3-Torsion Structures* by Anema, Top, and Tuijp.

Definition 1. Let k be a field and let $f \in k[x, y, z]$ be a polynomial. The *hessian* $H(f)$ of f is determinant of the matrix

$$\begin{pmatrix} f_{xx} & f_{xy} & f_{xz} \\ f_{xy} & f_{yy} & f_{yz} \\ f_{xz} & f_{yz} & f_{zz} \end{pmatrix}.$$

Note that the degree of $H(f)$ is $3(\deg f - 2)$. Thus when $\deg f = 3$, we have $\deg H(f) = 3$.

For the next definition we will use the Bengali letter হ, pronounced like "haw."

Definition 2. Let $C = V(f)$ be a curve in $\mathbb{P}^2_k$. Then the *Hesse derivative* হC of C is $V(h(f))$.

An important property of the Hesse derivative of C is that

$$C \cap \text{হ}C = \{P : P \text{ is a flex point of } C\}.$$

I first got interested in this topic from reading ATT, who talk about the *Hesse pencil*: the elliptic fibration spanned by an elliptic curve E and its Hesse derivative হE.

Proposition 2, ATT. Let $\mathcal{P}$ be the pencil spanned by E and হE (the so-called *Hesse pencil*). Then হ$^n E \in \mathcal{P}$ for any $n \in \mathbb{N}$.

Proof. This follows from Corollary 2.2 of ATT. □

This implies that হ is an action of the Hesse pencil of E. So if you keep iterating the Hesse derivative over and over again, you will keep landing in the Hesse pencil, but the j-invariant may be different!

The j-invariant of an elliptic curve is a complex number associated to the isomorphism class of that curve. Important j-invariants are 1728 and 0.

In much of their paper, the authors DHH consider cubic curves of the form

$$\Gamma_c = V(x^3 + y^3 + z^3 + cxyz)$$

and explore the discrete dynamical system on c that comes from iterating the Hesse derivative.

Lemma 3. (Lemma 8 of DHH.) Let $c_0 \neq 0$. Then the Hesse derivative of Γ_{c_0} is হ$\Gamma_{c_0} = \Gamma_{c_1}$ where

$$c_1 = -\frac{108 + c_0^3}{3c_0^2}.$$

The Hesse derivative of Γ_0 is হ$\Gamma_0 = \Gamma_\infty$, and the Hesse derivative of Γ_∞ is হ$\Gamma_\infty = \Gamma_\infty$.

DHH then talk about a broader family of rational functions

$$h(x) = -\frac{a+x^3}{3x^2}$$

and analyze their fixed points and critical points under iteration.

Lemma 4. (Lemma 9 of DHH.) The rational function $h(x) = -\frac{a+x^3}{3x^2}$ has a unique real fixed point

$$\varphi = -\sqrt[3]{\frac{a}{4}}$$

and the unique critical point

$$\kappa = \sqrt[3]{2a}, \qquad h(\kappa) = -\sqrt[3]{\frac{a}{4}} = \varphi.$$

Proposition 5. (Proposition 10 of DHH.) Let $a \neq 0$. If we define

$$h^n = \underbrace{h \circ \cdots \circ h}_{n \text{ times}}$$

and if κ_n is a critial point of h^n then $h^n(\kappa_n) = \varphi$. Conversely, if $h^n(x) = \varphi$, then $\frac{\mathsf{d}}{\mathsf{d}x}h^n(x) = 0$ or $x = \varphi$.

Proposition 6. (Proposition 11 of DHH.) Let χ_n be the number of critical points of h^n. Then

$$\chi_{2r+1} = 2 \times 3^r - 1 \text{ and } \chi_{2r} = 3^r - 1$$

for all $r \geq 0$.

Proposition 7. (Proposition 12 of DHH) Let Φ_n be the number of fixed points of h^n. Then

$$\Phi_{2r+1} = 1 \text{ and } \Phi_{2r} = 2\chi_{2r} - 1 = 2 \times 3^r - 3$$

for all $r \geq 0$.

Consider the cubic curve in Weierstrass form:

$$C = y^2z - x^3 - axz^2 - bz^3$$

where a and b are some elements of a field k.

Applying the Hessian to C gives us the curve

$$H(C)(x,y,z) = 24xy^2 - 8a^2z^3 + 24ax^2z + 72bxz^2 = 8(3xy^2 - a^2z^3 + 3ax^2z + 9bxz^2).$$

Applying the change of coordinates by swapping x and z yields

$$H(C)(z,y,x)/8 = 3y^2z - a^2x^3 + 3axz^2 + 9bx^2z.$$

Then replacing z with a^2z and x with a^2x+3bz yields

$$H(C)(a^2z, y, a^2x+3bz)/8 = 3a^2y^2z - a^8x^3 + (3a^7+27a^4b^2)xz^2 + (9a^5b+54a^2b^3)z^3.$$

This is almost in Weierstrass form. We now need the first two coefficients to be 1 and -1, respectively.

Let $\alpha y^2z + \beta x^3 + \gamma xz^2 + \delta z^3 = 3a^2y^2z - a^8x^3 + (3a^7+27a^4b^2)xz^2 + (9a^5b+54a^2b^3)z^3$. Let us transform x to $-\sqrt[3]{\alpha}x$ and transform y to $\sqrt{\beta}y$. Then we get

$$H(C)\left(a^2z, \sqrt{\beta}y, -a^2\sqrt[3]{\alpha}x+3bz\right)/8 = \alpha\beta y^2z - \alpha\beta x^3 - \sqrt[3]{\alpha}\gamma xz^2 + \delta z^3.$$

So finally we have

$$H(C)\left(a^2z, \sqrt{\beta}y, -a^2\sqrt[3]{\alpha}x+3bz\right)/(8\alpha\beta) = y^2z - x^3 - \frac{\sqrt[3]{\alpha}\gamma}{\alpha\beta}xz^2 - \frac{-\delta}{\alpha\beta}z^3$$

in Weierstrass form.

Now note that the j-invariant of C is $j(C) = 1728 \cdot \dfrac{4a^3}{4a^3+27b^2}$.

Since the curves defined by the polynomials $H(C)(x,y,z)$ and $\frac{H(C)\left(a^2z, \sqrt{\beta}y, -a^2\sqrt[3]{\alpha}x+3bz\right)}{8\alpha\beta}$ are isomorphic, they have the same j-invariant.

Mathematica tells us

$$\begin{aligned} &H(C)\left(a^2z, \sqrt{\beta}y, -a^2\sqrt[3]{\alpha}x+3bz\right)/(8\alpha\beta) \\ &= -xz^2\left(-\frac{9\sqrt[3]{3}\sqrt[3]{a^2}b^2}{a^6} - \frac{\sqrt[3]{3}\sqrt[3]{a^2}}{a^3}\right) - z^3\left(\frac{18b^3}{a^8}+\frac{3b}{a^5}\right) - x^3 + y^2z. \end{aligned}$$

Using Mathematica, the j-invariant of this is

$$j(H(C)) = 1728 \cdot \frac{4(a^3+9b^2)^3}{a^6(4a^3+27b^2)} = j(C)\cdot\frac{(a^3+9b^2)^3}{a^9}.$$

We can also use Macaulay2 code to compute this.

```
i1:k=frac(QQ[a,b]);

i2:R=k[x,y,z];

i3:f=3*x*y^2-a^2*z^3+3*a*x^2*z+9*b*x*z^2;
```

`f` here is the hessian of $y^2z - x^3 - axz^2 - bz^3$, divided by 8.

```
i4:g=sub(f,x=>z,y=>y,z=>x);

i5:h=sub(g,x=>a^2*x+3*b*z,y=>y,z=>a^2*z);
```

`h` here is $H(C)(a^2z, y, a^2x+3bz)/8$. Now we must define the coefficients α, β, γ, and δ, as above.

```
i6:A=3*a^2;
```

```
i7:B=-a^8;

i8:G=(3*a^7+27*a^4*b^2);

i9:D=(9*a^5*b+54*a^2*b^3);
```

Next we will define $c = \sqrt[3]{\alpha}$ and $s = \sqrt{\beta}$.

```
i10:S=R[c,s]/ideal(c^3-A,s^2-B);

i11:i=sub(h,x=>-c*x,y=>s*y,z=>z);

i12:l=i*(1/A)*(1/B)
```

So now `l` is $H(C)\left(a^2z, \sqrt{\beta}y, -a^2\sqrt[3]{\alpha}x + 3bz\right)/(8\alpha\beta) = y^2z - x^3 - \frac{\sqrt[3]{\alpha}\gamma}{\alpha\beta}xz^2 - \frac{-\delta}{\alpha\beta}z^3$.

Now let us define $V = \frac{\sqrt[3]{\alpha}\gamma}{\alpha\beta}$ and $U = \frac{-\delta}{\alpha\beta}$.

```
i13:V=c*G*(1/A)*(1/B)

i14:U=-D*(1/A)*(1/B)
```

By definition, the j-invariant must be $1728 * (4V^3)/(4V^3 + 27U^2)$. We can test the following equalities.

```
i15:4*V^3==-12*(a^3+9*b^2)^3*(1/a^16)
o15:true

i16:(4*V^3+27*U^2)==-3*(4*a^3+27*b^2)*(1/a^10)
o16:true
```

So we can put the j-invariant in terms of a and b as

$$1728 * -12 * \frac{(a^3+9b^2)^3}{a^{16}} * \frac{1}{-3} * \frac{a^{10}}{4a^3+27b^2} = 1728 \cdot \frac{4(a^3+9b^2)^3}{a^6(4a^3+27b^2)} = j(C) \cdot \frac{(a^3+9b^2)^3}{a^9}.$$

Proposition 8. Consider the elliptic curve E with j-invariant j. Then the j-invariant of हE is $\dfrac{(6912-j)^2}{27j^3}$.

Proof. Now let us find $j(H(C))$ in terms of $j(C)$. Note that $j(H(C)) = j(C) \cdot \dfrac{(a^3+9b^2)^3}{a^9} = j(C) \cdot \left(\dfrac{a^3+9b^2}{a^3}\right)^3$. Note that $\dfrac{a^3+9b^2}{a^3} = \dfrac{4a^3+36b^2}{4a^3} = \dfrac{4a^3+27b^2}{4a^3} + \dfrac{9b^2}{4a^3} = \dfrac{1728}{j(C)} + \dfrac{9b^2}{4a^3}$.

Now note that $\dfrac{9b^2}{4a^3} - \dfrac{1728}{j(C)} = \dfrac{9b^2}{4a^3} - \dfrac{4a^3+27b^2}{4a^3} = \dfrac{-4a^3-18b^2}{4a^3} = -1 - 2\left(\dfrac{9b^2}{4a^3}\right)$. Solving for $\dfrac{9b^2}{4a^3}$, we get $\dfrac{9b^2}{4a^3} = \dfrac{1728j(C)^{-1}-1}{3}$.

Therefore $j(H(C)) = j(C) \cdot \left(\dfrac{1728}{j(C)} + \dfrac{1728j(C)^{-1}-1}{3}\right)^3$. Simplifying this yields

$$j(H(C)) = \frac{(4 \cdot 1728 - j(C))^3}{27j(C)^2}.$$

$\square$

Therefore we can understand $j(H(C))$ using the function $H : k \setminus \{0\} \to k$ defined by $H(j) = \dfrac{(6912-j)^3}{27j^2}$. Note that $H(j) = 1728$ whenever $(6912-j)^3 = 46656j^2$. This gives us the polynomial $j^3 + 25920j^2 + 3 \cdot 6912^2 j - 6912^3 = (j-1728)(j+13824)^2$. This gives us two roots: one at $j = 1728$ and one at $j = -13824 = -8 \cdot 1728$.

Note that $H(H(j))$ factors as

$$H(H(j)) = \frac{\left(j^3 + 165888j^2 + 143327232j - 330225942528\right)^3}{729(j-6912)^6 j^2}.$$

Thus $H(H(j)) = j$ when

$$\left(j^3 + 165888j^2 + 143327232j - 330225942528\right)^3 = 729(j-6912)^6 j^3.$$

This gives us a degree-9 polynomial with roots

$$\begin{aligned}
j &= 1728 \\
j &= \frac{3456}{7}\left(-1 - 3i\sqrt{3}\right) \\
j &= \frac{3456}{7}\left(-1 + 3i\sqrt{3}\right) \\
j &= 3456\left(5 - 3\sqrt{3}\right) \\
j &= 3456\left(3\sqrt{3} + 5\right) \\
j &= -5184i\sqrt{3} - \frac{1}{2}\sqrt{-\frac{4514807808}{13} - \frac{1}{13}644972544i\sqrt{3}} + 1728 \\
j &= -5184i\sqrt{3} + \frac{1}{2}\sqrt{-\frac{4514807808}{13} - \frac{1}{13}644972544i\sqrt{3}} + 1728 \\
j &= 5184i\sqrt{3} - \frac{1}{2}\sqrt{-\frac{4514807808}{13} + \frac{644972544i\sqrt{3}}{13}} + 1728 \\
j &= 5184i\sqrt{3} + \frac{1}{2}\sqrt{-\frac{4514807808}{13} + \frac{644972544i\sqrt{3}}{13}} + 1728.
\end{aligned}$$

So when the j-invariant of a cubic curve C is one of these nine numbers, then $H(H(C)) \cong C$. Note that for six of these values, we will have $H(C) \ncong C$. For example, H transposes the two values $3456\left(5 - 4\sqrt{3}\right)$ and $3456\left(3\sqrt{3} + 5\right)$.

An interesting question is whether there is a nonsingular curve C such that $H(H(C)) = C$, and not just isomorphic. Furthermore, is there a nonsingular curve C such that $H(H(C)) = C$ but $H(C) \ncong C$? If such a curve exists, its j-invariant must be one of the last six numbers in the above list.

We can classify cubic curves C as **stable** if $H(C) = C$, **semi-stable** if there is an $n > 0$ such that $H^n(C) = H^{n+1}(C)$, **periodic** if there is an $n > 0$ such that $H^n(C) = C$,

semi-periodic if there are $m > n > 0$ such that $H^m(C) = H^n(C)$, and **aperiodic** if $H^m(C) \neq H^n(C)$ for all $n \neq m$.

Furthermore, we can classify cubic curves C as j**-stable** if $H(C) \cong C$, j**-semi-stable** if there is an $n > 0$ such that $H^n(C) \cong H^{n+1}(C)$, j**-periodic** if there is an $n > 0$ such that $H^n(C) \cong C$, j**-semi-periodic** if there are $m > n > 0$ such that $H^m(C) \cong H^n(C)$, and j**-aperiodic** if $H^m(C) \ncong H^n(C)$ for all $n \neq m$.

The only stable curves are reducible curves. Examples of semi-stable curves are those with j-invariant of 0 or 6912.

Examples of j-stable curves are those with j-invariant 1728. Examples of j-semi-stable curves are those with j-invariant -13824. Examples of j-periodic curves are those with j-invariant $3456\left(5 - 3\sqrt{3}\right)$. Which curves among these are actually periodic? Examples of j-semi-periodic curves are those with j-invariant $3456\left(39\sqrt{3} \pm 3i\sqrt{582\sqrt{3} - 1008} - 67\right)$, which both map to $3456\left(5 - 3\sqrt{3}\right)$ under H. Which of these curves are actually semi-periodic?

Every periodic j is the solution to some $H^n(j) = j$. There are only finitely many solutions for a given $n \in \mathbb{N}$ and so there are only countably many isomorphism classes for periodic curves. Thus there are uncountably infinitely many aperiodic curves (and even isomorphism classes of aperiodic curves!).

Is there some kind of distribution? For a given j, $H^n(j)$ will be within some interval $x\%$ of the time? That's a dynamics question.

Question 9. Let $f : \mathcal{P} \to \mathbb{P}^1$ be an elliptic fibration. What is the action that इ induces on $\mathbb{P}^1 \to \mathbb{P}^1$, factoring through f?

Question 10. Let χ_n be the number of real critical points of H^n. What is the sequence χ_n?

Question 11. Let Φ_n be the number of real fixed points of H^n. What is the sequence Φ_n?

Question 12. Fix a number j and a prime p. Does the sequence $H^n(j)$ converge in the p-adic metric?

Question 13. I predict that there are uncountably many isomorphism classes of aperiodic curves. What is an example of an aperiodic curve?

Conjecture 14, Peterson. Let $j \in \mathbb{R}$ be general and consider the sequence $\{H^n(j)\}_{n\in\mathbb{N}}$. Define

$$\begin{aligned} L_n &= \#\{H^i(j) : i \leq n, H^i(j) < 0\} \\ M_n &= \#\{H^i(j) : i \leq n, 0 < H^i(j) < 6912\} \\ R_n &= \#\{H^i(j) : i \leq n, 6912 < H^i(j)\} \end{aligned}$$

. Then

$$\lim_{n\to\infty} \frac{L_n}{n} = \lim_{n\to\infty} \frac{M_n}{n} = \lim_{n\to\infty} \frac{R_n}{n} = \frac{1}{3}.$$

We can refine this even further: let

$$\begin{aligned}
L'_n &= \#\{H^i(j) : i \le n, H^i(j) < -13824\}\\
L''_n &= \#\{H^i(j) : i \le n, -13824 < H^i(j) < 0\}\\
M'_n &= \#\{H^i(j) : i \le n, 0 < H^i(j) < 1728\}\\
M''_n &= \#\{H^i(j) : i \le n, 1728 < H^i(j) < 6912\}\\
R'_n &= \#\left\{H^i(j) : i \le n, 6912 < H^i(j) < 6912\left(19 + 15\sqrt[3]{2} + 12\sqrt[3]{4}\right)\right\}\\
R''_n &= \#\left\{H^i(j) : i \le n, 6912\left(19 + 15\sqrt[3]{2} + 12\sqrt[3]{4}\right) < H^i(j)\right\}
\end{aligned}$$

. Then

$$\lim_{n\to\infty}\frac{L'_n}{n} = \lim_{n\to\infty}\frac{L''_n}{n} = \lim_{n\to\infty}\frac{M'_n}{n} = \lim_{n\to\infty}\frac{M''_n}{n} = \lim_{n\to\infty}\frac{R'_n}{n} = \lim_{n\to\infty}\frac{R''_n}{n} = \frac{1}{6}.$$

As a map $H : \overline{\mathbb{C}} \to \overline{\mathbb{C}}$ on the Riemann sphere, H has four fixed points:

$$1728,\ \frac{3456}{7}(-1+3i\sqrt{3}),\ \frac{3456}{7}(-1-3i\sqrt{3}),\ \text{and } \infty.$$

The four fixed points have λ multipliers of

$$-3,\ -\frac{3}{2} - i\frac{\sqrt{3}}{2},\ -\frac{3}{2} + i\frac{\sqrt{3}}{2},\ \text{and } -27$$

respectively. All of the multipliers are have absolute values greater than 1, so all four fixed points are repelling.

The Julia set appears to be an off-center ellipse of x-radius ≈ 36.376047 and y-radius ≈ 36.478225, shown below.