

Groupoids of Configurations of Lines

Jake Kettinger

Colorado State University

Speaking at Tulane University's Algebra and Combinatorics Seminar

12 November 2025

Definition

A finite set Z in \mathbb{P}_k^n is **geproci** if the projection \overline{Z} of Z from a general point P to a hyperplane $H = \mathbb{P}_k^{n-1}$ is a complete intersection in H .

Geproci stands for **g**eneral **p**rojection is a **c**omplete **i**ntersection. The only nontrivial examples known are for $n = 3$. In this case a hyperplane is a plane. A reduced set of points in a plane is a complete intersection if it is the transverse intersection of two algebraic curves. For $\#Z = ab$ ($a \leq b$), Z is (a, b) -geproci if \overline{Z} is the intersection of a degree a curve and a degree b curve.

Background: k -Spreads in \mathbb{P}_F^n

Definition

Given a $k \geq 0$ and field F , a **k -spread** of \mathbb{P}_F^n is a set of mutually-skew k -planes that partition \mathbb{P}_F^n .

Spreads are known to exist for $F =$ any finite field if and only if $k + 1$ divides $n + 1$, also and for $k + 1 = (n + 1)/2$ for $F = \mathbb{R}$.

Spreads are instrumental for the proof that $\mathbb{P}_{\mathbb{F}_q}^3$ is geproci under $\mathbb{P}_{\mathbb{F}_q}^3$. In this case, a spread is a partition of $\mathbb{P}_{\mathbb{F}_q}^3$ into lines.

The Hopf Fibration over \mathbb{R}

The Hopf fibration $H : S^3 \rightarrow S^2$ can yield a spread over $\mathbb{P}_{\mathbb{R}}^3$.

$$\begin{array}{ccc} S^3 & \xrightarrow{H} & S^2 \\ \downarrow A & & \downarrow \cong \\ \mathbb{P}_{\mathbb{R}}^3 & \xrightarrow{F} & \mathbb{P}_{\mathbb{C}}^1 \end{array}$$

Let $L_{a,b}$ denote the line joining the points $(1, 0, a, b)$ and $(0, 1, -b, a)$, and let L_{∞} denote the line joining $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$. Then $\mathcal{S} = \{L_{a,b} : a, b \in \mathbb{R}\} \cup \{L_{\infty}\}$ is the spread yielded by the Hopf fibration. You can also use a similar method in positive characteristic to define a spread known as the **Hopf spread**.

According to Gorla, it is possible to construct a spread of k -dimensional hyperplanes inside $\mathbb{P}_{\mathbb{F}_q}^n$ if and only if $k + 1$ divides $n + 1$.

Let $p(x) \in \mathbb{F}_q[x]$ be irreducible, monic, and degree $k + 1$, and let P be its companion matrix. Then one can construct a spread of spaces of the form

$$\underbrace{\text{rowsp} \left(0 \quad \cdots \quad 0 \quad \boxed{I_{k+1}} \quad A_1 \quad \cdots \quad A_j \right)}_{\frac{n+1}{k+1}} : A_i \in \mathbb{F}_q[P].$$

When $k = 1$ and $n = 3$, this construction is identical to the Hopf spread! But this also provides new examples of spreads in higher dimensions.

Non-Hopf Spreads

Definition

A **regulus** is a set of at least three mutually-skew lines \mathcal{R} such that there is a quadric surface Q where $\bigcup_{R \in \mathcal{R}} R \subseteq Q$, [like this](#).

Every regulus \mathcal{R} admits an **opposite regulus** \mathcal{R}^* .

The Hopf spread contains reguli: for example $\mathcal{R} = \{L_{a,0} : a \in k\} \cup \{L_\infty\}$ is a regulus.

Given a spread \mathcal{S} containing a regulus \mathcal{R} , the set of lines $(\mathcal{S} \setminus \mathcal{R}) \cup \mathcal{R}^*$ is also a spread.

It was once conjectured that every spread could be constructed by starting with the Hopf spread and replacing a succession of reguli with their opposites. This is now known to be false.

Maximal Partial Spreads

Note that a spread over $\mathbb{P}_{\mathbb{F}_q}^3$ comprises $q^2 + 1$ mutually-skew lines.

Definition

A **partial spread** of $\mathbb{P}_{\mathbb{F}_q}^3$ with **deficiency** d is a set of $q^2 + 1 - d$ mutually-skew lines. A **maximal partial spread** is a partial spread of positive deficiency that is not contained in any larger partial spread.

Maximal partial spreads are also instrumental to the study of geproci sets; in fact, given a maximal partial spread \mathcal{M} , the set $\mathbb{P}_{\mathbb{F}_q}^3 \setminus (\bigcup_{L \in \mathcal{M}} L)$ is geproci.

Projecting a Line to a Line via a Line

Definition

Given three lines $L_1, L_2, L_3 \subseteq \mathbb{P}_k^3$ where $L_1 \cap L_2 = L_2 \cap L_3 = \emptyset$, we can define the function $\pi(L_1, L_2, L_3) : L_1 \rightarrow L_3$ as follows: take $p \in L_1$. Then there is a unique line T such that $p \in T$, $T \cap L_2 \neq \emptyset$, and $T \cap L_3 \neq \emptyset$. Then define $\pi(L_1, L_2, L_3)(p) = T \cap L_3$. This is the **projection** of L_1 to L_3 **via** L_2 .

[Here is a demonstration.](#)

Definition

A **groupoid** is a category \mathcal{G} where every morphism is invertible.

- For any object $G \in \mathcal{G}$, $\text{Hom}_{\mathcal{G}}(G, G) = \text{Aut}_{\mathcal{G}}(G)$ is a group. $\text{Aut}_{\mathcal{G}}(G)$ is a “group of the groupoid.”
 - Whenever $\text{Hom}_{\mathcal{G}}(G_1, G_2) \neq \emptyset$, then $\text{Aut}_{\mathcal{G}}(G_1) \cong \text{Aut}_{\mathcal{G}}(G_2)$.
- So when $\text{Hom}_{\mathcal{G}}(G_1, G_2) \neq \emptyset$ for all $G_1, G_2 \in \mathcal{G}$, \mathcal{G} induces only one group of the groupoid, up to isomorphism. Then it makes sense to say “the” group of the groupoid, $\text{Aut}_{\mathcal{G}}$.

Theorem

Let \mathcal{L} be a set of lines in \mathbb{P}_F^3 . Define Π to be the composition-closure of the set of functions $\{\pi(L_i, L_j, L_k) : L_i, L_j, L_k \in \mathcal{L}, L_i \cap L_j = L_j \cap L_k = \emptyset\}$. Then (\mathcal{L}, Π) is a groupoid.

In this case, any group of the groupoid is a subgroup of $\text{Aut}(\mathbb{P}_F^1) \cong \text{PGL}(2, F)$.

What can we say about this groupoid and its corresponding group(s)? In characteristic 0, when is it finite versus infinite? When does $\text{Aut}_{(\mathcal{L}, \Pi)}(L)$ have finite orbits, or finitely many orbits? What is the relationship (if \exists) between $\text{Aut}_{(\mathcal{L}, \Pi)} \leq \text{Aut}(\mathbb{P}^1)$ and $\text{Aut}(\mathcal{L}) \leq \text{Aut}(\mathbb{P}^3)$?

Ganger's Results

In her 2024 thesis, Ganger used the technique of **transversals** to prove the following theorem:

Theorem (Ganger Corollary 2.5)

The group of the groupoid for the Hopf spread induced by the degree-2 field extension $\mathbb{F}_{q^2}/\mathbb{F}_q$ over a finite field is isomorphic to the quotient $\mathbb{F}_{q^2}^/\mathbb{F}_q^* \cong C_{q+1}$.*

Definition

Given a set of lines \mathcal{L} in \mathbb{P}_F^3 , a **transversal** is a line T in \mathbb{P}_F^3 such that $T \cap \bar{L} \neq \emptyset$ for all $L \in \mathcal{L}$.

The Hopf spread has exactly two transversals T_1, T_2 for any finite field. The intersection of transversal with a line $L \in \mathcal{S}$ is a fixed point of $\text{Aut}_{(\mathcal{S}, \Pi)}(L)$!

Representing $\text{Aut}_{\mathcal{G}} \leq \text{PGL}(2, F)$

Let $U = \overline{u_0 u_1}$, $V = \overline{v_0 v_1}$, $W = \overline{w_0 w_1}$ be lines in \mathbb{P}^3 . Then there is a unique matrix $A \in \text{PGL}(2, F)$ such that the following diagram commutes.

$$\begin{array}{ccc} \mathbb{P}^1 & \xrightarrow{A} & \mathbb{P}^1 \\ \downarrow \overline{u_0 u_1} & & \downarrow \overline{w_0 w_1} \\ U & \xrightarrow{\pi(U, V, W)} & W \end{array}$$

A can be taken to be $\begin{pmatrix} -u_0 \wedge v_0 \wedge v_1 \wedge w_1 & -u_1 \wedge v_0 \wedge v_1 \wedge w_1 \\ u_0 \wedge v_0 \wedge v_1 \wedge w_0 & u_1 \wedge v_0 \wedge v_1 \wedge w_0 \end{pmatrix}$.

Example: The D_4 configuration

Theorem

Let \mathcal{L} be the 16 lines of the $(12_4, 16_3)$ configuration D_4 and let Π be the composition-closure of the projection functions. Then the group of the groupoid $\mathcal{G} = (\mathcal{L}, \Pi)$ is $\text{Aut}_{\mathcal{G}} \cong S_3$.

Argument boils down to:

- $\text{Hom}_{\mathcal{G}}(L, L') \neq \emptyset$ for $L, L' \in \mathcal{L}$, so $\text{Aut}_{\mathcal{G}}$ is well-defined.
- Let $q \in L$ be a quadruple point and $\pi \in \text{Hom}_{\mathcal{G}}(L, L')$. Then $\pi(q)$ is a quadruple point. So $\text{Aut}_{\mathcal{G}} \leq S_3$.
- We have found automorphisms in $\text{Aut}_{\mathcal{G}}(L)$ of orders 2 and 3, so $\text{Aut}_{\mathcal{G}} \cong S_3$.

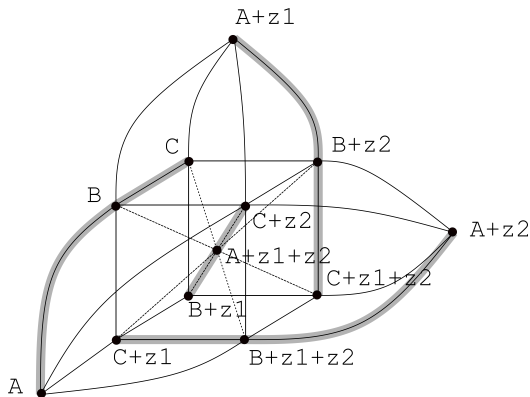
In fact, one can find a subset of six lines whose group of the groupoid is S_3 !

A Helpful Labeling

Let $\{A, B, C\}$ be a set of three letters, and consider the group $(\mathbb{Z}/2\mathbb{Z})^2 = \langle z_1, z_2 \rangle$. Then one can label the vertices of the D_4 configuration with the elements of $\{A, B, C\} \oplus (\mathbb{Z}/2\mathbb{Z})^2$ in such a way that there is a bijective correspondence between the lines of the D_4 and triples of the form $\{A + g, B + g', C + g'' : g + g' + g'' = 0\} \subseteq \{A, B, C\} \oplus (\mathbb{Z}/2\mathbb{Z})^2$.

This makes the theorem on the previous slide easier to prove because you can divide the vertices into “types” A , B , and C .

A Helpful Labeling



- $\{A, B, C\}$
- $\{A + z_1, B + z_2, C + z_1 + z_2\}$
- $\{A + z_2, B + z_1 + z_2, C + z_1\}$
- $\{A + z_1 + z_2, B + z_1, C + z_2\}$

Other Finite Groupoids

Using a similar analysis, we have found finite groupoids with automorphism groups isomorphic to A_4 and S_4 within the Penrose and Klein configurations.

$(1, 0, 0, 0)$	$(0, 1, 0, 0)$	$(0, 0, 1, 0)$	$(0, 0, 0, 1)$
$(0, 1, -1, 1)$	$(1, 0, -1, -1)$	$(1, -1, 0, 1)$	$(1, 1, 1, 0)$
$(0, 1, -t, t^2)$	$(1, 0, -t, -t^2)$	$(1, -t, 0, t^2)$	$(1, t, t^2, 0)$
$(0, 1, -t^2, t)$	$(1, 0, -t^2, -t)$	$(1, -t^2, 0, t)$	$(1, t^2, t, 0)$
$(0, 1, -t, 1)$	$(1, 0, -1, -t)$	$(1, -t^2, 0, t^2)$	$(1, t, 1, 0)$
$(0, 1, -t^2, t^2)$	$(1, 0, -t, -1)$	$(1, -1, 0, t)$	$(1, t^2, t^2, 0)$
$(0, 1, -1, t)$	$(1, 0, -t^2, -t^2)$	$(1, -t, 0, 1)$	$(1, 1, t, 0)$
$(0, 1, -t^2, 1)$	$(1, 0, -1, -t^2)$	$(1, -t, 0, t)$	$(1, t^2, 1, 0)$
$(0, 1, -1, t^2)$	$(1, 0, -t, -t)$	$(1, -t^2, 0, 1)$	$(1, 1, t^2, 0)$
$(0, 1, -t, t)$	$(1, 0, -t^2, -1)$	$(1, -1, 0, t^2)$	$(1, t, t, 0)$

$(0, 0, 1, 1)$	$(0, 0, 1, i)$	$(0, 0, 1, -1)$	$(0, 0, 1, -i)$
$(0, 1, 0, 1)$	$(0, 1, 0, i)$	$(0, 1, 0, -1)$	$(0, 1, 0, -i)$
$(0, 1, 1, 0)$	$(0, 1, i, 0)$	$(0, 1, -1, 0)$	$(0, 1, -i, 0)$
$(1, 0, 0, 1)$	$(1, 0, 0, i)$	$(1, 0, 0, -1)$	$(1, 0, 0, -i)$
$(1, 0, 1, 0)$	$(1, 0, i, 0)$	$(1, 0, -1, 0)$	$(1, 0, -i, 0)$
$(1, 1, 0, 0)$	$(1, i, 0, 0)$	$(1, -1, 0, 0)$	$(1, -i, 0, 0)$
$(1, 0, 0, 0)$	$(0, 1, 0, 0)$	$(0, 0, 1, 0)$	$(0, 0, 0, 1)$
$(1, 1, 1, 1)$	$(1, 1, 1, -1)$	$(1, 1, -1, 1)$	$(1, 1, -1, -1)$
$(1, -1, 1, 1)$	$(1, -1, 1, -1)$	$(1, -1, -1, 1)$	$(1, -1, -1, -1)$
$(1, 1, i, i)$	$(1, 1, i, -i)$	$(1, 1, -i, i)$	$(1, 1, -i, -i)$
$(1, -1, i, i)$	$(1, -1, i, -i)$	$(1, -1, -i, i)$	$(1, -1, -i, -i)$
$(1, i, 1, i)$	$(1, i, 1, -i)$	$(1, -i, 1, i)$	$(1, -i, 1, -i)$
$(1, i, -1, i)$	$(1, i, -1, -i)$	$(1, -i, -1, i)$	$(1, -i, -1, -i)$
$(1, i, i, 1)$	$(1, i, -i, 1)$	$(1, -i, i, 1)$	$(1, -i, -i, 1)$
$(1, i, i, -1)$	$(1, i, -i, -1)$	$(1, -i, i, -1)$	$(1, -i, -i, -1)$

A preprint now available on the [arXiv!](#)

Further Questions

- Configurations of hyperplanes?
- Groupoids of maximal partial spreads in finite geometry?
- Generalize geproci to configurations in the Grassmannian?
- Configurations of lines in $\mathbb{P}_{\mathbb{C}}^3$ with $\text{Aut}_{(\mathcal{L}, \Pi)} \cong A_5$?

Thank you!

Thanks for your attention! ¹

¹(Pssst, I am on the job market!)