Claim: Let A be a square matrix and $\vec{v}, \vec{u}$ vectors such that

$$\vec{v}^\top A\vec{u} = 0.$$

Then given a non-fixing quarter rotation Q, we have

$$\operatorname{adj}(A)Q\vec{v} = \vec{u}.$$

We want to show that $\vec{v}^\top A\operatorname{adj}(A)Q\vec{v} = 0$. Note that $A\operatorname{adj}(A) = \det(A)I_n$. So if $\det(A) = 0$, we are done. If $\det(A) \neq 0$, we can divide both sides by the determinant and show that $\vec{v}^\top Q\vec{v} = 0$.

Let $\{\vec{b}_1, \ldots, \vec{b}_n\}$ be an orthonormal basis for k^n. Then Q satisfies $\vec{b}_i^\top Q\vec{b}_i = 0$ for all $1 \leq i \leq n$.

Write $\vec{v} = \sum_{i=1}^n s_i\vec{b}_i$. Then

$$\vec{v}^\top Q\vec{v} = \sum_{i=1}^n \sum_{j=1}^n s_i\vec{b}_i^\top Q s_j\vec{b}_j.$$

We already know that $s_i\vec{b}_i^\top Q s_i\vec{b}_i = 0$ for all $1 \leq i \leq n$. We wish to show that for each $s_i\vec{b}_i\top Q s_j\vec{b}_j$, there is an $s_k\vec{b}_k^\top Q s_\ell\vec{b}_\ell$ that cancels it out... actually I think we need Q to be even and to have the same number of 1's and -1's. For example

$$Q = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}.$$