Problem V.4.4. Let Θ be an irreducible cubic curve. Let L be a line meeting Θ at points P, Q, and R. Let L' be a line meeting Θ at points P', Q', and R'. Let L_P, L_Q, and L_R be the lines adjoining P and P', Q and Q', and R and R', respectively. Denote the third points where L_P, L_Q, and L_R meet Θ as P'', Q'', and R'', respectively. Show P'', Q'', and R'' are collinear.

Solution. Let Θ^* be the cubic $L_P + L_Q + L_R$. Then

$$\Theta.\Theta^* = P + Q + R + P' + Q' + R' + P'' + Q'' + R'',$$

which is a divisor of nine points. Thus by Theorem 4.5, we know that any cubic containing any 8 of those points must also contain the ninth.

Now consider the line L'' adjoining P'' and Q''. We will show that $R'' \in L''$. Consider the cubic $\Theta^{**} = L + L' + L''$. This is a cubic containing P, Q, R, P', Q', R', P'', and Q'', so by Theorem 4.5, it must contain R''. We know that $R'' \notin L$, since then R' would also be on L and so R' would be one of P or Q. Similarly, R'' cannot be on L'. Thus R'' must be on L''. Thus R'' is collinear with P'' and Q''. □

Part (b): Let C be an elliptic curve with inflection point I. For $P, Q \in C$, define $P + Q$ as the intersection with C of the line adjoining I to the point on the intersection with C and the line adjoining P and Q. Prove that for all $P, Q, R \in C$, $(P + Q) + R = P + (Q + R)$.

Proof. Let $\pi_Q : C \to C$ be projection from Q, and let $\pi_I : C \to C$ be projection from I. Then $P + Q = \pi_I(\pi_Q(P))$. Consider the two cubics

$$\begin{aligned}\Theta_1 &= PQ + \pi_Q(R)I + R\pi_I(\pi_Q(P))\\ \Theta_2 &= RQ + \pi_Q(P)I + P\pi_I(\pi_Q(R)).\end{aligned}$$

Then

$$\Theta_1.\Theta_2 = Q + \pi_Q(P) + P + \pi_Q(R) + I + \pi_I(\pi_Q(R)) + R + \pi_I(\pi_Q(P)) + S$$

where S is the intersection of the lines $R\pi_I(\pi_Q(P))$ and $P\pi_I(\pi_Q(R))$. By Theorem 4.5, $S \in C$ since the other 8 points are on C. Thus $S = \pi_R(\pi_I(\pi_Q(P))) = \pi_P(\pi_I(\pi_Q(R)))$. Therefore we get

$$(P + Q) + R = \pi_I(\pi_R(\pi_I(\pi_Q(P)))) = \pi_I(S) = \pi_I(\pi_P(\pi_I(\pi_Q(R)))) = P + (Q + R).$$

Thus the group action is associative. □

Problem V.4.5. Let A, B, C, A', B', and C' be distinct points on a conic Γ. Let

$$\begin{aligned}P &= AB'.A'B\\ Q &= AC'.A'C\\ R &= BC'.B'C.\end{aligned}$$

Show that P, Q, and R are collinear.

Solution. First let us construct two conics

$$\Theta = AB' + A'C + BC'$$
$$\Theta' = A'B + AC' + B'C.$$

Then

$$\begin{aligned}\Theta.\Theta' &= AB'.A'B + AB'.AC' + AB'.B'C + A'C.A'B \\ &\quad + A'C.AC' + A'C.B'C + BC'.A'B + BC'.AC' + BC'.B'C \\ &= P + A + B' + A' + Q + C + B + C' + R.\end{aligned}$$

Thus the nine points are on both cubics, and so by Theorem 4.5 any cubic containing any 8 of these points must contain the ninth.

Now consider the line L adjoining P and Q, and consider the cubic $\Gamma + L$. We know that $\Gamma + L$ contains A, B, C, A', B', C', P, and Q, and so it must contain R as well. We know that R cannot be on Γ since R is on the line BC', which already intersects Γ at the two points B and C' and lines only intersect conics at two points; so for R to be on Γ would mean that R is either B or C' and similarly, R would have to simultaneously be one of either B' or C. This is impossible, since the points were given to be distinct. Thus R cannot be on Γ and so R must be on L. Thus R is collinear with P and Q. □

Problem V.4.6. Given 13 points $P_1, \dots, P_{13}$ in the plane, show there are three additional determined points P_{14}, P_{15}, P_{16} such that all quartic curves through $P_1, \dots, P_{13}$ necessarily also pass through P_{14}, P_{15}, P_{16}.

Proof. Let $\mathfrak{d}$ be the linear system of all quartic curves on $\mathbb{P}^2$. First let us begin by calculating the dimension of $\mathfrak{d}$. There are $\binom{4+3-1}{3-1} = 15$ degree-4 monomials, so $\dim \mathfrak{d} = 15 - 1 = 14$. Each time we assign a base point, the dimension decreases by 1, so after assigning 13 base points $P_1, \dots, P_{13}$, we have a new linear system $\mathfrak{d}'$ such that $\dim \mathfrak{d}' = 1$. We need to know how many unassigned base points $\mathfrak{d}'$ has.

Let $Q, Q' \in \mathfrak{d}'$. By Bézout's Theorem, Q and Q' must meet at $4 \times 4 = 16$ points. We know 13 of them are $P_1, \dots, P_{13}$. Call the other 3 of them P_{14}, P_{15}, P_{16}. Since $\dim \mathfrak{d}' = 1$, any other curve $Q'' \in \mathfrak{d}'$ is a linear combination of Q and Q', so it must also pass through P_{14}, P_{15}, P_{16}. Thus P_{14}, P_{15}, and P_{16} are the unassigned base points of $\mathfrak{d}'$. □

In general, it appears a linear system of n-degree curves will pick up $n^2 - \binom{n+2}{2} + 2$ unassigned base points after assigning the first $\binom{n+2}{2} - 2$ base points. Then it takes the final $n^2 + 1^{\text{th}}$ point to uniquely determine the curve, but it's really the $\binom{n+2}{2} - 1^{\text{th}}$ freely-chosen point. This could be wrong and it actually might get more complicated as degree increases in a way I don't see right now, but this is how it looks.

The set of monomials with three variables of degree d is $\begin{pmatrix} 2+d \\ 2 \end{pmatrix}$. In general, it requires $\begin{pmatrix} 2+d \\ 2 \end{pmatrix} - 1$ general points to uniquely determine a curve of degree d.

Denote these points $P_\ell = (a_\ell, b_\ell, c_\ell)$ for $2 \leq \ell \leq \binom{2+d}{2}$. We can find the equation of this curve using a $\binom{2+d}{2} \times \binom{2+d}{2}$ matrix. The top row consists of all the monomials of the form $x^i y^j z^k$ where $0 \leq i, j, k \leq d$, and $i + j + k = d$. The ℓ^{th} row will consist of monomials of the form $a_\ell^i b_\ell^j c_\ell^k$ where i, j, and k are equal to the exponents of the x, y, and z in the corresponding column.

Call this set of points $\mathsf{P} = \left\{ P_\ell : 2 \leq \ell \leq \binom{2+d}{2} \right\}$. Then denote the aforementioned matrix M_{P}. Then $\det(M_{\mathsf{P}})$ is a homogeneous polynomial of degree d. We claim that $\det(M_{\mathsf{P}})(P_\ell) = 0$ for all $P_\ell \in \mathsf{P}$.

Note that $\det(M_{\mathsf{P}})(P_\ell)$ is equal to the determinant of the matrix M_P with $x = a_\ell$, $y = b_\ell$, and $z = c_\ell$. Since this row repeats later on in the matrix in row ℓ, the determinant must be 0. Thus $\det(M_{\mathsf{P}})(P_\ell) = 0$ for all $P_\ell \in \mathsf{P}$.

Fulton 5.41 Let C be a smooth cubic curve with flex $O \in C$, and let $P_1, \ldots, P_{3m} \in C$ with $m \in \mathbb{N}$. Show that $P_1 \oplus P_2 \oplus \cdots \oplus P_{3m} = O$ if and only if there is a degree-m curve F such that $F.C = P_1 + \cdots + P_{3m}$

Proof. First let us see the case $m = 1$. Suppose P_1, P_2, P_3 are collinear. Then $P_1 \oplus P_2$ is attained by forming the line connecting P_1 and P_2, which meets C again at P_3, then by forming the line connecting P_3 and O, which meets the curve again at the point we will call $-P_3$. The line connecting P_3 with $-P_3$ meets the curve again at O, so $-P_3 \oplus P_3 = P_1 \oplus P_2 \oplus P_3 = O$.

Now assume that $P_1 \oplus P_2 \oplus P_3 = O$. Then $P_1 \oplus P_2 = -P_3$, so the line connecting $-P_3$ and O meets the line connecting P_1 and P_2 at P_3. Therefore P_1, P_2, P_3 are collinear.

Now let $m = 2$. First assume that $P_1, \ldots, P_6$ are coconical. Then define $Q = -(P_1 \oplus P_2)$, $R = -(P_3 \oplus P_4)$, and $S = -(Q \oplus R)$, so $S = P_1 \oplus P_2 \oplus P_3 \oplus P_4$. That is, P_1, P_2, and Q are collinear, P_3, P_4, and R are collinear, and Q, R, and S are collinear. We can form a cubic curve as the union of the conic through $P_1, \ldots, P_6$ and the line through S, Q, and R. Since these points are also on C, any cubic that passes through 8 of the nine points will also pass through the ninth by Cayley-Bacharach. We can form another cubic with the lines P_1P_2Q, P_3P_4R, and SP_5. This cubic contains eight of the nine points, and so must also contain P_6. The point P_6 cannot be on the first two lines, since lines must intersect C at a maximum of three points, so P_6 must be on the line connecting S and P_5. Thus S, P_5, and P_6 are collinear and so $P_5 \oplus P_6 \oplus S = O$. But recall that $S = P_1 \oplus P_2 \oplus P_3 \oplus P_4$, so $P_1 \oplus \cdots \oplus P_6 = O$.

Now assume $P_1 \oplus \cdots \oplus P_6 = O$. Then $S \oplus P_5 \oplus P_6 = O$, where S is as defined previously. Then S, P_5, and P_6 are collinear. We can form a cubic by the lines P_1P_2Q, P_3P_4R, and P_5P_6S, so any cubic through eight of the nine must contain the ninth as well. So the conic through the points P_1, P_2, P_3, P_4, and Q must contain R as well by Cayley-Bacharach. Recall also that S, Q, and R are collinear, and so $P_1, \ldots, P_6$ must be coconical as well.

Now for the inductive step. Suppose that for some $m \geq 1$, $P_1 \oplus \cdots \oplus P_{3m} = O$ if and only if there is a curve F of degree m such that $F.C = P_1 + \cdots + P_{3m}$. Then let $P_1, \ldots, P_{3(m+1)} \in C$.

First suppose $P_1 \oplus \cdots \oplus P_{3(m+1)} = O$. Define $Q = -(P_1 \oplus P_2)$, $R = -(P_3 \oplus P_4)$, and $S = -(Q \oplus R)$. Then $S \oplus P_5 \oplus \cdots P_{3(m+1)} = O$. By the inductive hypothesis, there is a degree m curve F' such that $F'.C = S + P_5 + \cdots + P_{3(m+1)}$. We can then form a degree $m+2$ curve F'' by taking the union of F' with the lines P_1P_2Q and P_3P_4R. Note that Q, R, and S form a line, so by Noether's Theorem there is another degree $m+2$ curve formed as the union of the line QRS and some degree $m+1$ curve through $P_1, \ldots, P_{3(m+1)}$. This curve F is the one we were looking for.

Now suppose that there is a degree $m+1$ curve F through the points $P_1, \ldots, P_{3(m+1)}$. Then consider the points $S, P_5, \ldots, P_{3(m+1)}$. There is a degree $m+2$ curve formed by taking the union of F with the line through the points Q, R, and S. Since P_1, P_2, and Q are collinear, and P_3, P_4, and R are collinear, there is another degree $m+2$ curve attained with the union of the two lines P_1P_2Q and P_3P_4R with some curve F' of degree m through the remaining points $S, P_5, \ldots, P_{3(m+1)}$, by Noether's Theorem. Then by the inductive hypothesis, $S \oplus P_5 \oplus \cdots \oplus P_{3(m+1)} = O$, but $S = P_1 \oplus P_2 \oplus P_3 \oplus P_4$, and so $P_1 \oplus \cdots \oplus P_{3(m+1)} = O$. $\square$

Note this still works if $P_i = P_j$ for $i \neq j$. The curve F will likely not be irreducible.

Let us explore what happens if we choose O to be a general point on the curve, and not necessarily a flex point. Note that the tangent line $T_O(C)$ meets C at a distinct other point that we will denote O^2. First we will show that three points are collinear if and only if their sum is O^2.

First let $P_1, P_2, P_3 \in C$ be collinear. Then to get $P_1 \oplus P_2$, we first connect P_1 to P_2 with a line and get P_3. Then connect O with P_3 to get $P_1 \oplus P_2$. Then to get $P_1 \oplus P_2 \oplus P_3$, we connect $P_1 \oplus P_2$ with P_3, which we already know meets C again at O. We then connect O with O and get O^2. Thus $P_1 \oplus P_2 \oplus P_3 = O^2$.

Now assume $P_1 \oplus P_2 \oplus P_3 = O^2$. Then $(P_1 - O) + (P_2 - O) + (P_3 - O) \sim O^2 - O$. Thus $P_1 + P_2 + P_3 - O^2 - 2O \sim 0$ in $\mathrm{Pic}^0 C$. We know that $\mathcal{O}_C(2O + O^2) \cong \mathcal{O}_C(1)$, i.e., $2O + O^2$ represents the class of a line through C. Therefore we must also have $\mathcal{O}_C(P_1 + P_2 + P_3) \cong \mathcal{O}_C(1)$, and so P_1, P_2, and P_3 are collinear.

We claim that for $3m$ points $P_1, \ldots, P_{3m} \in C$, we have a degree-$m$ curve F such that $F.C = P_1 + \cdots + P_{3m}$ if and only if $P_1 \oplus \cdots \oplus P_{3m} = mO^2$. When O is flex, we happen to have $O^2 = O$.

Suppose $P_1 + \cdots + P_{3m} - 3mO \sim m(O^2 - O)$. Then $P_1 + \cdots + P_{3m} - mO^2 - 2mO \sim 0 \in \mathrm{Pic}^0 C$. We know that $\mathcal{O}(2mO + mO^2) \cong \mathcal{O}_C(m)$ and so $\mathcal{O}_C(P_1 + \cdots + P_{3m}) \cong \mathcal{O}_C(m)$ as well. Thus there is a degree-m curve F such that $F.C = P_1 + \cdots + P_{3m}$. The reverse direction is the same idea. So if there is a degree-m curve F such that $F.C = P_1 + \cdots P_{3m}$, then $P_1 \oplus \cdots \oplus P_{3m} = mO^2$.

Now let us show that, given a choice of flex point $O \in C$ for the base point, nine points $P_1, \ldots, P_9$ will add up to a point of order n if and only if there is a degree $3n$ curve which intersects C with multiplicity n at each point P_i.

First suppose $P_1 \oplus \cdots \oplus P_9 = \nu$ has order n. Then $nP_1 \oplus \cdots \oplus nP_9 = O$, so we have a sum of $9n$ points which add to O, so we know that there exists a curve F of degree $9n/3 = 3n$ that intersects C at each of the $9n$ points. Since the $9n$ points are just n copies of the original 9 points, we know that F must intersect C n times at each P_i.

Now suppose there is a curve F of degree $3n$ that intersects C n times at each P_i. It follows directly from Fulton 5.41 that $nP_1 \oplus \cdots \oplus nP_9 = O$, and so (if F is irreducible), the nine points add to an element of order n.

Note that if O is not flex, then the nine points $P_1, \ldots, P_9$ will add to an n^{th} root of $3nO^2$, like for example $3O^2$, but there may be more examples? I believe adding to $3O^2$ corresponds to F not being irreducible, and adding to some other root will correspond to F being irreducible.