We will consider the set of nine (not distinct) points in $\mathbb{P}^3_{\mathbb{Z}/2\mathbb{Z}}$:

$$Z = \{(1,0,0,0)\times 2, (0,1,0,0)\times 2, (0,0,1,0)\times 2, (0,0,0,1)\times 2, (1,1,1,1)\}$$

by choosing as our infinitely-near points for $(1,0,0,0)$, $(0,1,0,0)$, $(0,0,1,0)$, and $(0,0,0,1)$ as the point that corresponds to the (respective) direction of the point $(1,1,1,1)$.

Consider the 20 cubic monomials:

$$\begin{matrix} x^3 & x^2y & x^2z & x^2w & xy^2 \\ xyz & xyw & xz^2 & xzw & xw^2 \\ y^3 & y^2z & y^2w & yz^2 & yzw \\ yw^2 & z^3 & z^2w & zw^2 & w^3 \end{matrix}.$$

In order to contain the four coordinate vertices, any cubic containing Z must have the x^3, y^3, z^3, and w^3 coefficient as 0.

We also need the tangent plane of the cubic at $(0,0,0,1)$ to contain the point $(1,1,1,1)$. If we localize at $w=1$, we get the linear forms $xw^2 \mapsto x$, $yw^2 \mapsto y$, and $zw^2 \mapsto z$. For the tangent plane to contain $(1,1,1,1)$, we need two of the x, y, and z monomials as terms of the cubic polynomial.

Thus, for example the three cubic polynomials xw^2+yw^2, xw^2+zw^2, and yw^2+zw^2 satisfy this property. Notice that xw^2+yw^2 contains $(0,0,0,1)$, $(0,0,1,0)$, $(0,1,0,0)$, $(1,0,0,0)$, and $(1,1,1,1)$, and also the tangent plane at $(0,0,0,1)$ is $x+y$, which contains $(1,1,1,1)$. That tangent planes at the other three coordinate vertices are not defined.

Also notice that xw^2+zw^2 has a tangent plane of $x+z$ at $(0,0,0,1)$. It is okay that it is not the same plane as in the xw^2+yw^2 case: it just needs to also contain $(1,1,1,1)$.

Also note that any sum of two monomials with three different variables contains Z. For example, consider $xyz+xyw$. This contains $(1,1,1,1)$ and the four coordinate vertices, and the tangent plane is not defined at any of the four coordinate vertices.

So we can take the ideal

$$\begin{aligned} I = (&x^2y+x^2z, x^2y+x^2w, x^2z+x^2w, xy^2+y^2z, xy^2+y^2w, y^2z+y^2w, \\ &xz^2+yz^2, xz^2+z^2w, yz^2+z^2w, xw^2+yw^2, xw^2+zw^2, yw^2+zw^2, \\ &xyz+xyw, xyz+xzw, xyz+yzw, xyw+xzw, xyw+yzw, xzw+yzw) \end{aligned}$$

which Macaulay2 tells us can be generated by the 11 polynomials

$$\begin{gathered} (yw^2+zw^2, xw^2+zw^2, xzw+yzw, xyw+yzw, yz^2+z^2w, \\ xz^2+z^2w, y^2z+y^2w, xyz+yzw, x^2z+x^2w, xy^2+y^2w, x^2y+x^2w) = I \end{gathered}$$

The ideal I contains all cubic polynomials that contain $(1,1,1,1)$, the four coordinate vertices, and whose tangent planes at the four coordinate vertices (when they are defined) contain $(1,1,1,1)$. Thus we should have $I = I(Z)_3$.

Let us define the point $(a,b,c) \in \mathbb{A}^3_k$, where $\operatorname{char} k = 2$, and its corresponding ideal $P = (x-aw, y-bw, z-cw)$. According to Macaulay2, the ideal $J = \texttt{intersect}(I, P^3)$ has two generators of degree 3.

Specifically, the two generators are as follows:

$$J_0 = xy^2 + \frac{a+1}{c+1}y^2z + \frac{ab^3+b^3c+ab^2+b^2c}{ac^3+bc^3+ac^2+bc^2}xz^2 + \frac{a^2b^2+ab^2c+ab^2+b^2c}{ac^3+bc^3+ac^2+bc^2}yz^2 + \frac{a+c}{c+1}y^2w + \frac{ab^2+b^2c}{c^3+c^2}z^2w + \frac{ab^3+ab^2c+b^3+b^2c}{ac+bc+a+b}xw^2 + \frac{a^2b^2+ab^2c+ab^2+b^2c}{ac+bc+a+b}yw^2 + \frac{ab^2+b^2}{c+1}zw^2$$

and

$$J_1 = x^2y + \frac{b+1}{c+1}x^2z + \frac{a^2b^2+a^2bc+a^2c+a^2c}{ac^3+bc^3+ac^2+bc^2}xz^2 + \frac{a^3b+a^3c+a^2b+a^2c}{ac^3+bc^3+ac^2+bc^2}yz^2 + \frac{b+c}{c+1}x^2w + \frac{a^2b+a^2c}{c^3+c^2}z^2w + \frac{a^2b^2+a^2bc+a^2b+a^2c}{ac+bc+a+b}xw^2 + \frac{a^3b+a^2bc+a^3+a^2c}{ac+bc+a+b}yw^2 + \frac{a^2b+a^2}{c+1}zw^2.$$

Therefore, Z is $(3,3)$-geproci.

Now consider the 6 points $Y = \{(1,0,0,0)\times 2, (0,1,0,0)\times 2, (0,0,1,0)\times 2\}$, where the infinitely near point for each is in the direction of $(0,0,0,1)$. We will show that this is $(2,3)$-geproci.

First we will look at a configuration of points in $\mathbb{P}^2$:

$$Y' = \{(1,0,0)\times 2, (0,1,0)\times 2, (0,0,1)\times 2\}$$

where the infinitely-near point for each is in the direction of $(1,1,1)$. We will show that this set of 6 points is a complete intersection of a conic and a cubic, and then show that a general projection of Y onto any plane is isomorphic to Y'. Note that Y' is contained in the conic $A = xy+xz+yz$ and the cubic $B = (x+y)(x+z)(y+z) = 0$. Also note that A is not a component of B, since A is an irreducible conic and B is the union of three lines. Therefore Y' is a complete intersection of a conic and a cubic.

Now let us return to $Y \subseteq \mathbb{P}^3$. Let us project Y from a general point $P \in \mathbb{P}^3$ onto a general plane $\Pi \subseteq \mathbb{P}^3$. Since the lines corresponding to each infinitely-near point meet at $(0,0,0,1)$, and since projection from a point preserves lines (and therefore the intersection of lines), the images of the three infinitely-near points under the projection $\pi_{P,\Pi}$ will also correspond to three concurrent lines. In other words, Y will map to the set

$$\pi_{P,\Pi}(Y) = \{\pi_{P,\Pi}(1,0,0,0)\times 2, \pi_{P,\Pi}(0,1,0,0)\times 2, \pi_{P,\Pi}(0,0,1,0)\times 2\}$$

where each infinitely-near point is in the direction of $\pi_{P,\Pi}(0,0,0,1)$. For a general point P, the images of the three ordinary points in Y and the point $\pi_{P,\Pi}(0,0,0,1)$ will not be collinear. Therefore we can map Π to $\mathbb{P}^2$ and use an automorphism of the plane to map $\pi_{P,\Pi}(1,0,0,0)$

to $(1,0,0)$, $\pi_{P,\Pi}(0,1,0,0)$ to $(0,1,0)$, $\pi_{P,\Pi}(0,0,1,0)$ to $(0,0,1)$, and $\pi_{P,\Pi}(0,0,0,1)$ to $(1,1,1)$. Then we are in the same situation as Y', which is a complete intersection of a conic and a cubic.

Note that Y is a half-grid, since the cubic containing Y is a union of three lines, but the conic is irreducible.