Definition 1.1. A **Steiner triple system** is an ordered pair (S, T) where S is a finite set of points or symbols, and T is a set of 3-element subsets of S called triples, such that each pair of distinct elements of S occurs together in exactly one triple of T.

One example is the Fano plane. An equivalent way of thinking of a Steiner triple system is as a partitioning of the edges of the complete graph $K_{|S|}$ into triangles.

Theorem 1.1.3. A Steiner triple system of order v exists if and only if $v \equiv 1$ or $3 \mod 6$.

Proof. If (S, T) is a Steiner triple system of order v, the triple $\{a, b, c\}$ contains exactly three two-element subsets: $\{a, b\}$, $\{a, c\}$, and $\{b, c\}$. Note that S itself contains $\binom{v}{2}$-two element subsets. Since every pair much exist within one triple in T, we must have

$$3|T| = \binom{v}{2} = \frac{v(v-1)}{2}.$$

And so

$$|T| = \frac{v(v-1)}{6}.$$

For any $x \in S$, define the set $T(x) = \{t \setminus \{x\} : x \in t \in T\}$. Then $T(x)$ partitions $S \setminus \{x\}$ into two-element subsets. This is because every $s \in S \setminus \{x\}$ is in exactly one triple alongside x, so if $y \in S$ is such that $\{x, y, s\} \in T$, then $\{y, s\}$ is the only pair in $T(x)$ containing s.

Thus $v - 1$ is even, so v must be odd. Therefore $v \equiv 1$ or 3 or $5 \mod 6$. However, $\frac{v(v-1)}{6}$ is never an integer for $v \equiv 5 \mod 6$. This is because $5(5-1) = 20 \equiv 2 \mod 6 \not\equiv 0 \mod 6$. So we are left with $v \equiv 1$ or $3 \mod 6$. □

It remains to show there exists a Steiner system for every such number. We will see this with various construction methods.

Exercise 1.1.4. Let S be a finite set of size v and let T be a set of triples of S satisfying

1. each pair of distinct elements of S belongs to *at least* one triple in T, and
2. $|T| \leq v(v-1)/6$.

Show that (S, T) is a Steiner triple system.

Proof. Since $|T| \leq v(v-1)/6$, we know that $3|T| \leq \binom{v}{2}$. For each pair $\{x, y\} \in \mathcal{P}(S, 2)$, denote by $N_{\{x,y\}}$ the number of triples $t \in T$ that satisfy $\{x, y\} \subseteq t$. Let N be the sum total of the number of times each pair is present in a triple of T. Because of property (a), $N \geq \binom{v}{2}$. But because each triple always contains exactly three pairs, we have $N = 3|T|$. Therefore $N = 3|T| = \binom{v}{2}$.

Suppose there existed $z \neq w \in S$ such that $\{x, y, z\} = \{x, y, w\}$. Then $N_{\{x,y\}} > 1$. But then $N > \binom{v}{2}$, which contradicts our earlier result. So we must have $N_{\{x,y\}} = 1$, and so we have a Steiner triple system.

□

Definition 1.2. A **latin square** of order n is an $n \times n$ array, each cell of which contains exactly one of the numbers $\{1, \ldots, n\}$ and each column and row of which contains exactly one of each of the numbers $\{1, \ldots, n\}$. A **quasigroup** of order n is a pair $(Q, \circ)$ where Q is a set of n elements and $\cdot \circ \cdot : Q \times Q \to Q$ is a binary operation such that for every pair of elements $a, b \in Q$ the equations $a \circ x = b$ and $y \circ a = b$ have unique solutions.

A latin square and a quasigroup are essentially the same thing. The former is merely the multiplication table of the latter.

The Bose Construction ($v \equiv 3 \mod 6$): Let $v = 6n + 3$ and let Q be an idempotent quasigroup of order $2n + 1$. Let $S = Q \times \{a, b, c\}$ and define T to contain the following two types:

1. For $1 \le i \le 2n + 1$, $\{(i, a), (i, b), (i, c)\} \in T$.
2. For $1 \le i < j \le 2n + 1$ $\{(i, a), (j, a), (i \circ j, b)\}, \{(i, b), (j, b), (i \circ j, c)\}, \{(i, c), (j, c), (i \circ j, a)\} \in T$.

Then (S, T) is a Steiner triple system. This can be proven by applying Exercise 1.1.4.

Proof. First we will show that $|T| \le v(v - 1)/6$. Recall now that $v = 6n + 3$. The triples of Type 1 can be counted as $2n+1$. The triples of Type 2 can be counted as $3(2n+1)(2n)/2 = 3(2n^2 + n) = 6n^2 + 3n$. Therefore $|T| = 6n^2 + 5n + 1$. Note here that $v(v - 1)/6 = (6n + 3)(6n + 2)/6 = (36n^2 + 30n + 6)/6 = 6n^2 + 5n + 1$. So in fact $|T| = v(v - 1)/6$.

Next we need to demonstrate that every pair is in at least one triple of T. Let $\{(i, p), (j, q)\}$ be a pair, where $i, j \in Q$ and $p, q \in \{a, b, c\}$. First suppose $i = j$. Then $\{(i, p), (j, q)\} \subseteq \{(i, a), (i, b), (i, c)\}$, the Type 1 triple.

Now suppose $p = q$. Then $\{(i, p), (j, q)\} \subseteq \{(i, p), (j, p), (i \circ j, p + 1)\}$, the Type 2 triple.

Finally, suppose $i \ne j$ and $p \ne q$. Then either $p = q + 1$ or $q = p + 1$. First suppose $q = p+1$. Then we want to show there is a $k \in Q$ such that $\{(i, p), (k, p), (j, q = p+1)\} \in T$. The element k must satisfy $i \circ k = j$. Because Q is a quasigroup, this equation has a unique solution k, so we are done. NOTE: it is impossible that $k = i$ because Q is idempotent.

The same thing goes for the case $p = q + 1$. Therefore every pair is in at least one triple in T. By Exercise 1.1.4, (S, T) is thus a Steiner triple system. □

Example JK1. Let us construct a Steiner triple system of nine elements. First let us take the quasigroup

$Q =$

$\circ$	1	2	3
1	1	3	2
2	3	2	1
3	2	1	3

Then the Bose Steiner triple system $(Q \times \{a,b,c\}, T)$ contains $|T| = 12$ triples.

$$\begin{gathered}\{(1,a),(1,b),(1,c)\},\{(2,a),(2,b),(2,c)\},\{(3,a),(3,b),(3,c)\},\\ \{(1,a),(2,a),(3,b)\},\{(1,b),(2,b),(3,c)\},\{(1,c),(2,c),(3,a)\},\\ \{(1,a),(3,a),(2,b)\},\{(1,b),(3,b),(2,c)\},\{(1,c),(3,c),(2,a)\},\\ \{(2,a),(3,a),(1,b)\},\{(2,b),(3,b),(1,c)\},\{(2,c),(3,c),(1,a)\}.\end{gathered}$$

This can be expressed in Grünbaum's configuration notation as a $(9_4, 12_3)$, because there are 9 points, 4 "lines" per point, 12 "lines" and 3 points per "line." This is the same kind of configuration as that formed by the nine flex points of an elliptic curve.

So the Bose method can form a $([6n+3]_{[3n+1]}, [6n^2+5n+1]_3)$-configuration. So we can make a $(15_7, 35_3)$-configuration, a $(21_{10}, 70_3)$-configuration, a $(27_{13}, 117_3)$-configuration, and a $(33_{16}, 176_3)$-configuration etc. An $(81_{40}, 1080_3)$-configuration etc.

Example JK2. Let us construct a Steiner triple system of fifteen elements. First let us take the quasigroup

$$Q = \begin{array}{|c|c|c|c|c|c}
\circ & 1 & 2 & 3 & 4 & 5 \\ \hline
1 & 1 & 5 & 4 & 3 & 2 \\ \hline
2 & 5 & 4 & 3 & 2 & 1 \\ \hline
3 & 4 & 3 & 2 & 1 & 5 \\ \hline
4 & 3 & 2 & 1 & 5 & 4 \\ \hline
5 & 2 & 1 & 5 & 4 & 3
\end{array}$$

Note that this quasigroup is commutative but not idempotent! Does it fail to produce a Steiner triple system? Then the Bose construction yields $(Q \times \{a,b,c\}, T)$ contains $|T| = 35$ triples. The five Type 1 triples:

$$\begin{gathered}\{(1,a),(1,b),(1,c)\},\\ \{(2,a),(2,b),(2,c)\},\\ \{(3,a),(3,b),(3,c)\},\\ \{(4,a),(4,b),(4,c)\},\\ \{(5,a),(5,b),(5,c)\},\end{gathered}$$

and the thirty Type 2 triples:

$$\begin{gathered}
\{(1,a),(2,a),(5,b)\},\{(1,b),(2,b),(5,c)\},\{(1,c),(2,c),(5,a)\},\\
\{(1,a),(3,a),(4,b)\},\{(1,b),(3,b),(4,c)\},\{(1,c),(3,c),(4,a)\},\\
\{(1,a),(4,a),(3,b)\},\{(1,b),(4,b),(3,c)\},\{(1,c),(4,c),(3,a)\},\\
\{(1,a),(5,a),(2,b)\},\{(1,b),(5,b),(2,c)\},\{(1,c),(5,c),(2,a)\},\\
\{(2,a),(3,a),(3,b)\},\{(2,b),(3,b),(3,c)\},\{(2,c),(3,c),(3,a)\},\\
\{(2,a),(4,a),(2,b)\},\{(2,b),(4,b),(2,c)\},\{(2,c),(4,c),(2,a)\},\\
\{(2,a),(5,a),(1,b)\},\{(2,b),(5,b),(1,c)\},\{(2,c),(5,c),(1,a)\},\\
\{(3,a),(4,a),(1,b)\},\{(3,b),(4,b),(1,c)\},\{(3,c),(4,c),(1,a)\},\\
\{(3,a),(5,a),(5,b)\},\{(3,b),(5,b),(5,c)\},\{(3,c),(5,c),(5,a)\},\\
\{(4,a),(5,a),(4,b)\},\{(4,b),(5,b),(4,c)\},\{(4,c),(5,c),(4,a)\}.
\end{gathered}$$

This fails to be a Steiner triple system because for example the pair $\{(5,a),(5,b)\}$ appears in the triple $\{(5,a),(5,b),(5,c)\}$ of Type 1 and the triple $\{(3,a),(5,a),(5,b)\}$ of Type 2. This ultimately results from the non-idempotent nature of $3 \circ 5 = 5$.

Definition 1.3.1. A quasigroup of even order $2n$ is **half-idempotent** if $i \circ i = i$ for all $i \leq n/2$ and $i \circ i = i - n/2$ for all $i > n/2$.

Note that for every odd n, you can make a commutative idempotent quasigroup out of $\mathbb{Z}/n\mathbb{Z}$ by $i \circ j = \frac{i+j}{2}$, since $2 \in (\mathbb{Z}/n\mathbb{Z})^\times$. For even $2n$, you can partition $\mathbb{Z}/2n\mathbb{Z}$ into $A = \{0,\dots,n-1\}$ and $B = \{n,\dots,2n-1=-1\}$. Then for the equation $y = 2x$, denote by A_y the solution x in the set A, and denote by B_y the solution x in the set B. Then

$$i \circ j = \begin{cases} A_{i+j} & i+j \text{ is even} \\ B_{i+j-1} & i+j \text{ is odd} \end{cases}$$

is a half-idempotent commutative quasigroup.

The Skolem Construction ($v \equiv 1 \mod 6$): Let Q be a half-idempotent commutative quasigroup of order $2n$, where $Q = \{1,\dots,2n\}$. Define $S = \{\infty\} \cup (Q \times \{a,b,c\})$. Define T as following three types:

1. For $1 \leq i \leq n$, $\{(i,a),(i,b),(i,c)\} \in T$.
2. For $1 \leq i \leq n$, $\{\infty,(n+i,a),(i,b)\},\{\infty,(n+i,b),(i,c)\},\{\infty,(n+i,c),(i,a)\} \in T$.
3. For $1 \leq i < j \leq 2n$, $\{(i,a),(j,a),(i \circ j,b)\},\{(i,b),(j,b),(i \circ j,c)\},\{(i,c),(j,c),(i \circ j,a)\} \in T$.

Then (S,T) is a Steiner triple system.

Proof. Let us again turn to Exercise 1.1.4. Counting up the triples of T, we get n triples of Type 1, $3n$ triples of Type 2, and $3 * 2n(2n-1)/2 = 6n^2 - 3n$ triples of Type 3. Adding these up, we get $|T| = 6n^2 + n$. Note that with $v = 6n+1$, we have $v(v-1)/6 = (6n+1)(6n)/6 = 6n^2 + n$. And so $|T| = v(v-1)/6$.

Next, we must show that every pair is present in at least one triple of T.

- First let us consider pairs of the form $\{(i,p),(j,q)\}$ where $i,j \in Q$ and $p,q \in \{a,b,c\}$. If $i = j \leq n$, then $\{(i,p),(j,q)\}$ is in a triple of Type 1.
- If $i = j > n$, then write $i = j = k+n$ for some $k \in Q$. So we wish to find a triple of T containing $\{(k+n,p),(k+n,q)\}$. Suppose that $q = p+1$. Then we wish to find an $\ell \in Q$ such that $\{(\ell,p),(k+n,p),(k+n,q=p+1)\} \in T$. This is a Type 3 triple. This is true if $\ell \circ (k+n) = k+n$. Because Q is a quasigroup, ℓ exists and is unique. So the pair $\{(i,p),(j,q)\}$ is in at least one triple. The same goes for if $p = q+1$. (NOTE: importantly, $\ell \neq k+n$ because in a half-idempotent quasigroup $(k+n)\circ(k+n) = k \neq k+n$.)
- Now suppose $p = q$. Then $\{(i,p),(j,p)\}$ is in the Type 3 triple $\{(i,p),(j,p),(i\circ j,p{+}1)\}$.
- Now suppose $i \neq j$ and $p \neq q$. Suppose $j = n+i$ and $p = q+1$. then $\{(i,p),(j,q)\}$ is in the Type 2 triple $\{\infty,(j,q),(i,p)\}$.
- Now suppose $j = n+i$ and $q = p+1$. Then $\{(i,p),(j,q)\}$ is in the Type 3 triple $\{(i,p),(x,p),(j,q)\}$ where x solves $i \circ x = j$. Note it is impossible that $x = i$ because $i \circ i = i$ since $i \leq n$ and Q is half-idempotent.
- Now suppose $i \neq j$ do not satisfy $|i-j| = n$ and $p \neq q$. Suppose $q = p+1$. Then we want to find an $x \in Q$ such that $\{(i,p),(x,p),(j,q)\}$ is a Type 3 triple. Then $i \circ x = j$. It is impossible that $x = i$, because $i-(i\circ i) = \begin{cases} 0 & i \leq n \\ n & i > n \end{cases}$. The former case contradicts $i \neq j$ and the latter case contradicts $|i-j| \neq n$. Thus x truly provides us with a Type 3 triple.
- Finally, consider the pair $\{\infty,(i,p)\}$. If $i \leq n$, then $\{\infty,(i,p),(n+i,p-1)\}$ is a Type 2 triple and if $i > n$ then $\{\infty,(i,p),(i-n,p+1)\}$ is a Type 2 triple.

Those are all the possible pairs!!! They are all in some kind of triple in T, so by Exercise 1.1.4, we are done!!! $\square$

So this is a method of constructing a $([6n+1]_{[3n]},[6n^2+n]_3)$-configuration, I suppose. We can make a $(7_3,7_3)$, a $(13_6,26_3)$, $(19_9,57_3)$, a $(25_{12},100_3)$ etc.

Example JK3. Consider the half-idempotent commutative quasigroup Q represented by the matrix $\begin{pmatrix} 1 & 4 & 2 & 3 \\ 4 & 2 & 3 & 1 \\ 2 & 3 & 1 & 4 \\ 3 & 1 & 4 & 2 \end{pmatrix}$. Then we can construct the $(13_6,26_3)$ Steiner triple system on $S = \{\infty\} \cup (Q \times \{a,b,c\})$ has two triples of Type 1:

$$\{(1,a),(1,b),(1,c)\},\{(2,a),(2,b),(2,c)\},$$

six triples of Type 2:

$$\{\infty,(3,a),(1,b)\},\{\infty,(3,b),(1,c)\},\{\infty,(3,c),(1,a)\},$$
$$\{\infty,(4,a),(2,b)\},\{\infty,(4,b),(2,c)\},\{\infty,(4,c),(2,a)\},$$

and eighteen triples of Type 3:

$$\begin{gathered}
\{(1,a),(2,a),(4,b)\},\{(1,b),(2,b),(4,c)\},\{(1,c),(2,c),(4,a)\},\\
\{(1,a),(3,a),(2,b)\},\{(1,b),(3,b),(2,c)\},\{(1,c),(3,c),(2,a)\},\\
\{(1,a),(4,a),(3,b)\},\{(1,b),(4,b),(3,c)\},\{(1,c),(4,c),(3,a)\},\\
\{(2,a),(3,a),(3,b)\},\{(2,b),(3,b),(3,c)\},\{(2,c),(3,c),(3,a)\},\\
\{(2,a),(4,a),(1,b)\},\{(2,b),(4,b),(1,c)\},\{(2,c),(4,c),(1,a)\},\\
\{(3,a),(4,a),(4,b)\},\{(3,b),(4,b),(4,c)\},\{(3,c),(4,c),(4,a)\},
\end{gathered}$$

Pairwise Balanced Designs There is no Steiner triple system of order $6n+5$, but we can generalize the concept as a **pairwise balanced design** to approximate.

Definition 1.4.1. A **pairwise balanced design** is a set of elements S together with set of blocks $B \subseteq 2^S$ such that every pair of elements of S is in exactly one block in B. (That is, it is like a Steiner triple system without the requirement that all the blocks are size 3, or indeed all the same size at all!)

Example 1.4.1. $S = \{1, 2, \ldots, 11\}$ and B contains the following 16 blocks:

$$\begin{aligned}
\{1,2,3,4,5\}\\
\{1,6,7\}\\
\{1,8,9\}\\
\{1,10,11\}\\
\{2,6,9\}\\
\{2,7,11\}\\
\{2,8,10\}\\
\{3,6,11\}\\
\{3,7,8\}\\
\{3,9,10\}\\
\{4,6,10\}\\
\{4,7,9\}\\
\{4,8,11\}\\
\{5,6,8\}\\
\{5,7,10\}\\
\{5,9,11\}
\end{aligned}$$

Construction not recorded. **Quasigroups with holes and Steiner triple systems:**

Let $Q = \{1, 2, \ldots, 2n\}$ and let $H = \{\{1, 2\}, \{3, 4\}, \ldots, \{2n-1, 2n\}\}$. In what follows, the two-element subsets $\{2i-1, 2i\}$ are called **holes**. A quasigroup Q with holes H is a quasigroup $(Q, \circ)$ of order $2n$ in which for each $h \in H$, $(h, \circ)$ is a subquasigroup of $(Q, \circ)$.

$$\begin{pmatrix} \mathbf{1} & \mathbf{2} & 5 & 6 & 7 & 8 & 3 & 4 \\ \mathbf{2} & \mathbf{1} & 8 & 7 & 3 & 4 & 6 & 5 \\ 5 & 8 & \mathbf{3} & \mathbf{4} & 2 & 7 & 1 & 6 \\ 6 & 7 & \mathbf{4} & \mathbf{3} & 8 & 1 & 5 & 2 \\ 7 & 3 & 2 & 8 & \mathbf{5} & \mathbf{6} & 4 & 1 \\ 8 & 4 & 7 & 1 & \mathbf{6} & \mathbf{5} & 2 & 3 \\ 3 & 6 & 1 & 5 & 4 & 2 & \mathbf{7} & \mathbf{8} \\ 4 & 5 & 6 & 2 & 1 & 3 & \mathbf{8} & \mathbf{7} \end{pmatrix}$$

Exercise 1.5.10. Let $(\{1, 2\}, \circ_1)$ be any quasigroup of order 2 (there are two of them), and let $(Q, \circ_2)$ be an idempotent quasigroup of order $2n+1$ (for example, $a \circ b = \frac{a+b}{2} \mod 2n+1$, which always exists since 2 is a unit mod $2n+1$).

Let $S = \{1, 2\} \times Q$. Define a binary operation on S by $(a, b) \otimes (c, d) = (a \circ_1 c, b \circ_2 d)$. Then $(S, \otimes)$ is a commutative quasigroup of order $4n+2$ with holes $H = \{\{(1, i), (2, i)\} : i \in \mathbb{Q}\}$.

Proof. The fact that S is a commutative quasigroup is immediate as both $\{1, 2\}$ and Q are commutative quasigroups. Then note that $(1, i) \otimes (1, i) = (1 \circ_1 1, i)$, $(1, i) \otimes (2, i) = (1 \circ_1 2, i)$, $(2, i) \otimes (1, i) = (2 \circ_1, i) = (1 \circ_1 2, i)$, and $(2, i) \otimes (2, i) = (2 \circ_1 2, i)$, which forms a subquasigroup since $(\{1, 2\}, \circ_1)$ is a quasigroup.

Maybe I want to show that if Q is a quasigroup and with holes H of size 2 and $h, h' \in H$ with $h \cap h' \neq \varnothing$, then $h = h'$. Then $(h, \circ)$ is a quasigroup and $(h', \circ)$ is a quasigroup. Then is $(h \cap h', \circ)$ a quasigroup? Not necessarily. Well, let $a \in h \cap h'$, so $h = \{a, b\}$ and $h' = \{a, b'\}$. Then

$$\begin{pmatrix} \circ & a & b \\ a & a \circ a & a \circ b \\ b & b \circ a & b \circ b \end{pmatrix}$$

is a quasigroup and

$$\begin{pmatrix} \circ & a & b' \\ a & a \circ a & a \circ b' \\ b' & b' \circ a & b' \circ b' \end{pmatrix}$$

is a quasigroup. In order for both of these to be true, $a \circ a = a$. Thus $a \circ b = b \circ a = b$ and $a \circ b' = b' \circ a = b'$ and $b \circ b = b' \circ b' = a$. So we have

$$\begin{pmatrix} \circ & a & b \\ a & a & b \\ b & b & a \end{pmatrix}$$

and

$$\begin{pmatrix} \circ & a & b' \\ a & a & b' \\ b' & b' & a \end{pmatrix}.$$

Then

$$\begin{pmatrix} \circ & a & b & b' \\ a & a & b & b' \\ b & b & a & * \\ b' & b' & * & a \end{pmatrix}$$

Actually my claim is not true! For example, in

$$\begin{pmatrix} \circ & 1 & 2 & 3 & 4 \\ 1 & 1 & 2 & 3 & 4 \\ 2 & 2 & 1 & 4 & 3 \\ 3 & 3 & 4 & 1 & 2 \\ 4 & 4 & 3 & 2 & 1 \end{pmatrix},$$

the number 1 forms a subquasigroup with 2, 3, and 4, so the holes are $\{1,2\}$, $\{1,3\}$, and $\{1,4\}$.

It could be the definition of holes requires a pairwise partition of the quasigroup.

Whatever... QED or something. □

BASICALLY Exercise 1.5.10 is all about taking an idempotent commutative $2n+1$ quasigroup and basically duplicating every entry as its own individual 2×2 quasigroup, like so

$$Q = \begin{pmatrix} \circ & A & B & C \\ A & A & C & B \\ B & C & B & A \\ C & B & A & C \end{pmatrix}, A = \begin{pmatrix} \circ & 1 & 2 \\ 1 & 1 & 2 \\ 2 & 2 & 1 \end{pmatrix}, B = \begin{pmatrix} \circ & 3 & 4 \\ 3 & 3 & 4 \\ 4 & 4 & 3 \end{pmatrix}, C = \begin{pmatrix} \circ & 5 & 6 \\ 5 & 5 & 6 \\ 6 & 6 & 5 \end{pmatrix}$$

and so

$$Q = \begin{pmatrix} \circ & 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & \mathbf{1} & \mathbf{2} & 5 & 6 & 3 & 4 \\ 2 & \mathbf{2} & \mathbf{1} & 6 & 5 & 4 & 3 \\ 3 & 5 & 6 & \mathbf{3} & \mathbf{4} & 1 & 2 \\ 4 & 6 & 5 & \mathbf{4} & \mathbf{3} & 2 & 1 \\ 5 & 3 & 4 & 1 & 2 & \mathbf{5} & \mathbf{6} \\ 6 & 4 & 3 & 2 & 1 & \mathbf{6} & \mathbf{5} \end{pmatrix}.$$

That's why it only works if $\#Q = 2 \mod 4$. If $\#Q = 0 \mod 4$ I suppose we can start with a half-idempotent commutative quasigroup and doing the same sorta thing?

Let's try it:

$$Q = \begin{pmatrix} A & D & B & C \\ D & B & C & A \\ B & C & A & D \\ C & A & D & B \end{pmatrix}$$

is a half-idempotent commutative quasigroup. Then let $A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix}$ and I can already tell this isn't going to work...

Let $(\{1, \ldots, 2n\}, \circ)$ be a commutative quasigroup with holes H. Then

1. $(\{\infty\} \cup (\{1, 2, \ldots, 2n\} \times \{a, b, c\}), B)$ is an STS$(6n + 1)$ where B is:

   (a) for $1 \leq i \leq n$, let B_i contain the triples in an STS(7) on the symbols $\{\infty\} \cup (\{2i - 1, 2i\} \times \{a, b, c\})$ and $B_i \subseteq B$, and

   (b) for $1 \leq i \neq j \leq 2n$, $\{i, j\} \notin H$ place the triples $\{(i, a), (j, a), (i \circ j, b)\}$, $\{(i, b), (j, b), (i \circ j, c)\}$, and $\{(i, c), (j, c), (i \circ j, a)\}$ in B.

2. $(\{\infty_1, \infty_2, \infty_3\} \cup (\{1, 2, \ldots, 2n\} \times \{a, b, c\}), B')$ is an STS$(6n + 3)$ where B' replaces (a) in 1. with:

   (a) for $1 \leq i \leq n$ let B' contain the triples in an STS(9) on the symbols $\{\infty_1, \infty_2, \infty_3\} \cup (\{2i - 1, 2i\} \times \{a, b, c\})$ in which $\{\infty_1, \infty_2, \infty_3\}$ is a triple.

3. $(\{\infty_1, \infty_2, \infty_3, \infty_4, \infty_5\} \cup (\{1, 2, \ldots, 2n\} \times \{a, b, c\}), B'')$ is a PBD$(6n + 5)$ with one block of size 5 and the rest of size 3 where B'' replaces (a) of 1. with

   (a) for $1 \leq i \leq n$ B_i'' contains the blocks in a PBD(11) on the symbols $\{\infty_1, \ldots, \infty_5\} \cup (\{2i - 1, 2i\} \times \{a, b, c\})$ in which $\{\infty_1, \ldots, \infty_5\}$ is a block.

Let's use Exercise 1.1.4 to prove that constructions 1 and 2 are indeed Steiner triple systems.

Proof. First let us start with construction 1. First we want to show that each pair is in at least one triple. First take $\{\infty, (2i, a)\}$ as the pair. Then this pair is in the triple in the Steiner triple system STS(7) on the symbols $\{\infty\} \cup (\{2i - 1, 2i\} \times \{a, b, c\})$. Same for $2i - 1$ and b and c.

Now let us consider the pair $\{(i, a), (j, a)\}$. First, if $\{i, j\} \in H$ then $\infty \cup (\{i, j\} \times \{a, b, c\})$ forms an STS(7) and so $\{(i, a), (j, a)\}$ is in one of those triples. Now if $\{i, j\}$ is not a hole, then this pair is in the triple $\{(i, a), (j, a), (i \circ j, b)\}$. (Same for b and c.) Now let us consider the pair $\{(i, a), (j, b)\}$ where $\{i, j\}$ is not a hole. Then there is a $k \in Q$ such that $i \circ k = j$ and so $\{(i, a), (k, a), (j, b)\}$ is a triple containing the pair. (Same for a, b, c whatever.) Thus every pair is in at least one triple presto!

Now we must show that the number of triples is at most $(6n+1)(6n)/6 = 6n^2 + n$. There are $7n$ triples of type (a) and $3(\frac{2n(2n-2)}{2}) = 3(n)(2n - 2) = 6n^2 - 6n$ triples of type (b). NOTE: it is not $3\binom{2n}{2}$ because the type (b) triples preclude $\{i, j\} \in H$, so it is $3(\frac{2n(2n-2)}{2})$ instead of $3(\frac{2n(2n-1)}{2})$.

Adding these together, we have $7n + 6n^2 - 6n = 6n^2 + n$ as desired. Thus Exercise 1.1.4 works to form a Steiner triple system here.

Now consider the type 2. system. First, every pair is in at least one triple. Let us start with the pair $\{\infty, (i, a)\}$. Then $\{\infty\} \cup (\{i, j\} \times \{a, b, c\})$ with $\{i, j\} \in H$ is an STS(9) and so the pair $\{\infty, (i, a)\}$ is in one of the triples there. Same for b and c. Then consider $\{(i, a), (j, a)\}$. If $\{i, j\} \in H$, then we have the same STS(9) as before. If $\{i, j\} \notin H$, then the pair $\{(i, a), (j, a)\}$ is in a triple $\{(i, a), (j, a), (i \circ j, b)\}$ of type (b) like in construction 1. Same for $\{(i, a), (j, b)\}$.

Now the fun part. We want to show there are at most $(6n+3)(6n+2)/6 = (36n^2 + 30n+6)/6 = 6n^2+5n+1$ triples. There are 12 triples in each STS(9), giving ostensibly $12n$ triples of type (a'). But $\{\infty_1, \infty_2, \infty_3\}$ is a triple shared by each STS(9). So there are really $11n+1$ triples of type (a'). There are also $3\frac{(2n)(2n-2)}{2} = 6n^2-6n$ triples of type (b). So adding these together, we have $11n+1+6n^2-6n = 6n^2+5n+1$ triples. So Exercise 1.1.4 works. $\square$