1 Planes in $\mathbb{P}^4$

It is interesting to me to consider configurations of planes in $\mathbb{P}^4$. Let us start by considering the Chow ring $A = A(\mathfrak{Gr}(3,5))$.

First note that $A \cong \mathbb{Z}^{10}$, we have

$$\begin{tikzcd}
 & \sigma_{0,0,0} \arrow[d, dash] & \\
 & \sigma_{1,0,0} \arrow[dl, dash] \arrow[dr, dash] & \\
\sigma_{1,1,0} \arrow[d, dash] \arrow[drr, dash] & & \sigma_{2,0,0} \arrow[d, dash] \\
\sigma_{1,1,1} \arrow[d, dash] & & \sigma_{2,1,0} \arrow[dll, dash] \arrow[d, dash] \\
\sigma_{2,1,1} \arrow[dr, dash] & & \sigma_{2,2,0} \arrow[dl, dash] \\
 & \sigma_{2,2,1} \arrow[d, dash] & \\
 & \sigma_{2,2,2} &
\end{tikzcd}$$

and the following geometric interpretations of each Schubert class:

$\Sigma_{0,0,0}$	All planes
$\Sigma_{1,0,0}$	Planes touching a given line ℓ
$\Sigma_{1,1,0}$	Planes whose intersection with a given plane π is at least some line
$\Sigma_{2,0,0}$	Planes containing a given point p
$\Sigma_{1,1,1}$	Planes in a given threeperplane τ
$\Sigma_{2,1,0}$	Planes intersecting a given plane π at at least a line and contain a given point $p \in \pi$
$\Sigma_{2,1,1}$	Planes contained in a given threeperplane τ and contain a given point $p \in \tau$
$\Sigma_{2,2,0}$	Planes containing a given line ℓ
$\Sigma_{2,2,1}$	Planes contained in a given threeperplane τ and contain a given line $\ell \subseteq \tau$
$\Sigma_{2,2,2}$	One plane

We also have the following multiplication table:

$\times$	σ_0	σ_1	$\sigma_{1,1}$	σ_2	$\sigma_{1,1,1}$	$\sigma_{2,1}$	$\sigma_{2,1,1}$	$\sigma_{2,2}$	$\sigma_{2,2,1}$	$\sigma_{2,2,2}$
σ_0	σ_0	σ_1	$\sigma_{1,1}$	σ_2	$\sigma_{1,1,1}$	$\sigma_{2,1}$	$\sigma_{2,1,1}$	$\sigma_{2,2}$	$\sigma_{2,2,1}$	$\sigma_{2,2,2}$
σ_1		$\sigma_{1,1}+\sigma_2$	$\sigma_{1,1,1}+\sigma_{2,1}$	$\sigma_{2,1}$	$\sigma_{2,1,1}$	$\sigma_{2,1,1}+\sigma_{2,2}$	$\sigma_{2,2,1}$	$\sigma_{2,2,1}$	$\sigma_{2,2,2}$	0
$\sigma_{1,1}$			$\sigma_{2,1,1}+\sigma_{2,2}$	$\sigma_{2,1,1}$	$\sigma_{2,2,1}$	$\sigma_{2,2,1}$	$\sigma_{2,2,2}$	0	0	0
σ_2				$\sigma_{2,2}$	0	$\sigma_{2,2,1}$	0	$\sigma_{2,2,2}$	0	0
$\sigma_{1,1,1}$					$\sigma_{2,2,2}$	0	0	0	0	0
$\sigma_{2,1}$						$\sigma_{2,2,2}$	0	0	0	0
$\sigma_{2,1,1}$							0	0	0	0
$\sigma_{2,2}$								0	0	0
$\sigma_{2,2,1}$									0	0
$\sigma_{2,2,2}$										0

According to my (possibly erroneous) calculations, in $\mathbb{P}^4_{\mathbb{F}_2}$ we have the following sizes of Schubert cycles:

$$\begin{aligned}
\#\Sigma_{0,0,0} &= 155\\
\#\Sigma_{1,0,0} &= 91\\
\#\Sigma_{1,1,0} &= 43\\
\#\Sigma_{2,0,0} &= 35\\
\#\Sigma_{1,1,1} &= 15\\
\#\Sigma_{2,1,0} &= 19\\
\#\Sigma_{2,1,1} &= 7\\
\#\Sigma_{2,2,0} &= 7\\
\#\Sigma_{2,2,1} &= 3\\
\#\Sigma_{2,2,2} &= 1
\end{aligned}$$

Note $\#\Sigma_0 = \mathfrak{Gr}(3,5) = 155 = \frac{\frac{31*30*28}{6}}{\frac{7*6*4}{6}}$.

Question: Can we cover $\mathbb{P}^4_{\mathbb{F}_2}$ with planes that pairwise intersect at points?

Question: Can we come up with a configuration $(31_?, ?_7)$ covering $\mathbb{P}^4_{\mathbb{F}_2}$? Possibly a $(31_7, 31_7)$.

Unfortunately, I think that, given a point $p \in \mathbb{P}^4_{\mathbb{F}_2}$, a maximal set of planes π_i that contain p and intersect each other only at p (that is, $\pi_i \cap \pi_j = \{p\}$) has size 5. Think of it like this: take a threeperplane τ that does not contain p. Then every plane containing p must intersect τ at a line. So we want to find a maximal set of lines in $\tau \cong \mathbb{P}^3_{\mathbb{F}_2}$ that are mutually-skew. That is a spread in $\mathbb{P}^3_{\mathbb{F}_2}$, which is well-known to consist of 5 lines.

Therefore in that $(31_7, 31_7)$ configuration, we must attain planes that meet at lines: every set of seven planes containing a point p must have planes whose intersection is a line.

We could thus make a "spread" of 5 planes in $\mathbb{P}^4_{\mathbb{F}_2}$ that all intersect at the same point.

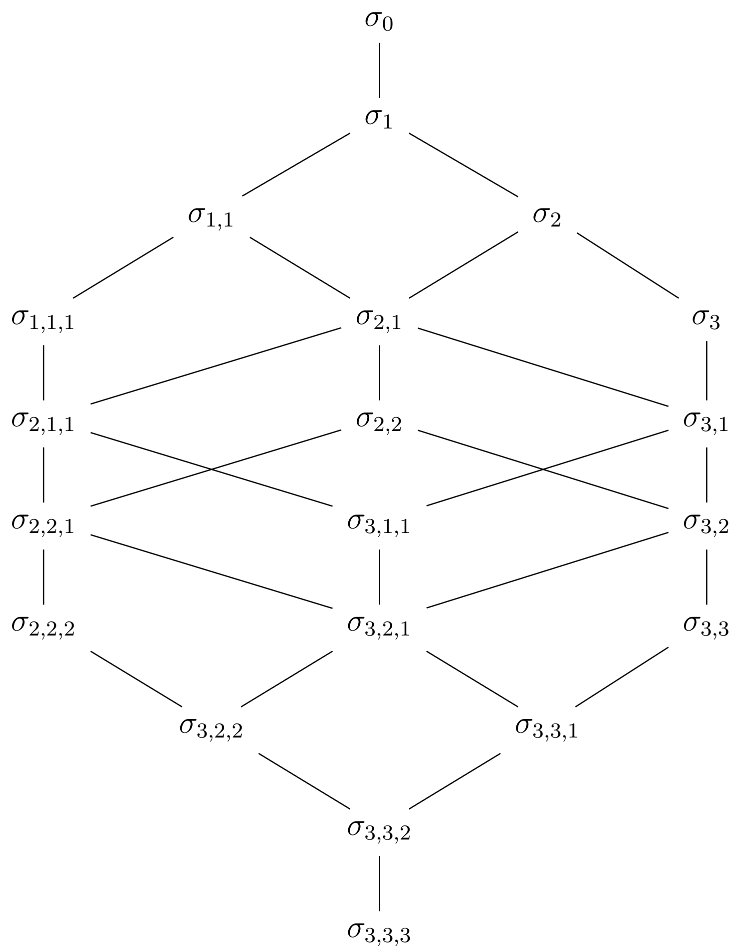
New stuff as of 7 March 2025

We can also find how many Schubert cycles there are of a given type. For example $\#\{\Sigma_0\} = 1$ since there is just one Σ_0 (all of $\mathfrak{Gr}(3,5)$). Similarly, $\#\{\Sigma_{2,2,2}\} = 155$ since each $\Sigma_{2,2,2}$ is a point in $\mathfrak{Gr}(3,5)$, which has 155 points. Enumerating these is really just a matter of enumerating the flags containing a combinations of a given point, line, plane, or threeperplane.

$$
\begin{aligned}
&\#\{\Sigma_{0,0,0}\} = 1\\
&\#\{\Sigma_{1,0,0}\} = 155 = \#\mathfrak{Gr}(2,5)\\
&\#\{\Sigma_{1,1,0}\} = 155 = \#\mathfrak{Gr}(3,5)\\
&\#\{\Sigma_{2,0,0}\} = 31 = \#\mathbb{P}^4_2\\
&\#\{\Sigma_{1,1,1}\} = 31 = \#\mathfrak{Gr}(4,5)\\
&\#\{\Sigma_{2,1,0}\} = 1085 = \#\mathfrak{Gr}(3,5) * \#\mathbb{P}^2_2\\
&\#\{\Sigma_{2,1,1}\} = 465 = \#\mathfrak{Gr}(4,5) * \#\mathbb{P}^3_2\\
&\#\{\Sigma_{2,2,0}\} = 155 = \#\mathfrak{Gr}(2,5)\\
&\#\{\Sigma_{2,2,1}\} = 1085 = \#\mathfrak{Gr}(4,5) * \#\mathfrak{Gr}(2,4)\\
&\#\{\Sigma_{2,2,2}\} = 155 = \#\mathfrak{Gr}(3,5)
\end{aligned}
$$

2 Planes in $\mathbb{P}^5$

Now let's look at planes in $\mathbb{P}^5$. We have the Chow ring $A = A(\mathfrak{Gr}(3,6))$. Note that $\dim \mathfrak{Gr}(3,6) = 3(6-3) = 9$, and $\mathrm{rank} A(\mathfrak{Gr}(3,6)) = \binom{6}{3} = 20$.



And the following geometric interpretations of each Schubert class:

$\Sigma_{0,0,0}$	All planes
$\Sigma_{1,0,0}$	Planes touching a given plane π
$\Sigma_{1,1,0}$	Planes whose intersection with a given threeperplane τ is at least some line
$\Sigma_{2,0,0}$	Planes touching a given line ℓ
$\Sigma_{1,1,1}$	Planes contained in a given fourperplane φ
$\Sigma_{2,1,0}$	Planes intersecting a given threeperplane τ at at least a line and touch a given line $\ell \in \tau$
$\Sigma_{3,0,0}$	Planes containing in a given point p
$\Sigma_{2,1,1}$	Planes contained in a fourperplane φ and that touch a given line $\ell \subseteq \varphi$
$\Sigma_{2,2,0}$	Planes whose intersection with a given plane π is at least a line
$\Sigma_{3,1,0}$	Planes whose intersection with a given threeperplane τ is at least a line and who contain a given point $p \in \tau$
$\Sigma_{2,2,1}$	Planes contained in a given fourperplane φ and whose intersection with a given plane $\pi \subseteq \varphi$ is at least a line
$\Sigma_{3,1,1}$	Planes contained in a given fourperplane φ and contain a given point $p \in \varphi$
$\Sigma_{3,2,0}$	Planes whose intersection with a given plane π is at least some line $\ell \subseteq \pi$ and who contain a given point $p \in \pi$
$\Sigma_{2,2,2}$	Planes contained in a given threeperplane τ
$\Sigma_{3,2,1}$	Planes contained in a given fourperplane φ and whose intersection with a given plane $\pi \subseteq \varphi$ is at least some line $\ell \subseteq \pi$ and who contain a given point $p \in \pi$
$\Sigma_{3,3,0}$	Planes that contain a given line ℓ
$\Sigma_{3,2,2}$	Planes contained in a given threeperplane τ and whose intersection with a given plane $\pi \subseteq \tau$ is at least some line $\ell \subseteq \pi$ and who contain a given point $p \in \pi$
$\Sigma_{3,3,1}$	Planes contained in a given fourperplane φ and contain a given line $\ell \subseteq \varphi$
$\Sigma_{3,3,2}$	Planes contained in a given threeperplane τ and contain a given line $\ell \subseteq \tau$
$\Sigma_{3,3,3}$	One plane

This time the multiplication table is too large to fit into the document. But one can test the products by using the SchurRings package with the following commands:

```
i1:  loadPackage "SchurRings"

i2:  S=schurRing(QQ,s,3)
```

And then multiply the desired s_{i_1,j_1,k_1}'s and s_{i_2,j_2,k_2}'s together. Just remember to interpret any s whose subscript contains a number greater than 3 as a 0 because those are invalid Schubert classes in this particular Grassmannian of $\mathfrak{Gr}(3,6)$.

And the following count of the sizes of each Schubert cycles over $\mathbb{F}_2$.

$$
\begin{aligned}
\#\Sigma_{0,0,0} &= 1395\\
\#\Sigma_{1,0,0} &= 883\\
\#\Sigma_{1,1,0} &= 435\\
\#\Sigma_{2,0,0} &= 435\\
\#\Sigma_{1,1,1} &= 155\\
\#\Sigma_{2,1,0} &= 243\\
\#\Sigma_{3,0,0} &= 155\\
\#\Sigma_{2,1,1} &= 91\\
\#\Sigma_{2,2,0} &= 99\\
\#\Sigma_{3,1,0} &= 91\\
\#\Sigma_{2,2,1} &= 43\\
\#\Sigma_{3,1,1} &= 35\\
\#\Sigma_{3,2,0} &= 43\\
\#\Sigma_{2,2,2} &= 15\\
\#\Sigma_{3,2,1} &= 19\\
\#\Sigma_{3,3,0} &= 15\\
\#\Sigma_{3,2,2} &= 7\\
\#\Sigma_{3,3,1} &= 7\\
\#\Sigma_{3,3,2} &= 3\\
\#\Sigma_{3,3,3} &= 1
\end{aligned}
$$

Note: $\#\Sigma_{1,1,0}$ can be calculated either by taking the 35 lines in τ times the 48 points in $\mathbb{P}^5 \setminus \tau$, dividing by the four points of the plane not in τ, and then adding the 15 planes in τ, or by dualizing: the set of planes whose join with a given line ℓ is at most some threeperplane; this is simply the planes that touch ℓ, so the dual of $\Sigma_{1,1,0}$ is $\Sigma_{2,0,0}$.

For $\#\Sigma_{2,1,0}$ we can take the 19 lines in τ touching ℓ, times the 48 points in $\mathbb{P}^5 \setminus \tau$, dividing by the four points of the plane not in τ, and then adding the 15 planes in τ (each plane in τ necessarily touches ℓ).

For $\#\Sigma_{2,2,0}$ we can take the 14 planes containing $\ell \subseteq \pi$ that are **not** equal to π itself, multiplying by the 7 lines of π, and then adding the plane π back in.

$883 = (155 - 15 - 15 - 15 + 2) * 7 + 99.$