Example: Each smooth cubic in $\mathbb{P}^3$ contains exactly 27 distinct lines.

Sketch: Given a smooth cubic surface $X \subseteq \mathbb{P}^3$ determined by the vanishing of a cubic form F in four variables, we wish to determine the degree of the locus in $\mathrm{PGr}(1,3)$ of lines contained in X. We linearize the problem using the observation that, if we fix a particular line $L \subseteq \mathbb{P}^3$, then the condition that L lies on X can be expressed as four linear conditions on the coefficients of F: to see this, note that the restriction map from the 20-dimensional vector space of cubic forms in $\mathbb{P}^3$ to the four-dimensional vector space $V_L = H^0(\mathcal{O}_L(3))$ of cubic forms on a line $L \cong \mathbb{P}^1 \subseteq \mathbb{P}^3$ (four dimensions are x^3, x^2y, xy^2, and y^3) is a linear surjection, and the condition for the inclusion $L \subseteq X$ is that F maps to 0 in V_L.

As the line L varies over $\mathrm{PGr}(1,3)$, the four-dimensional spaces V_L of cubic forms on the varying lines L fit together to form a vector bundle $\mathcal{V}$ of rank 4 on $\mathrm{PGr}(1,3)$. A cubic form F on $\mathbb{P}^3$, through its restriction to each V_L, defines an algebraic global section σ_F of this vector bundle. Thus the locus of lines contained in the cubic surface X is the zero locus of the section σ_F. Assuming for the moment that this zero locus is zero-dimensional, we call its class in $A(\mathrm{PGr}(1,3))$ the *fourth Chern class* of $\mathcal{V}$, denoted $c_4(\mathcal{V})$.

There are powerful tools for computing Chern classes of vector bundles, especially when those bundles can be built up from simpler bundles by linear-algebraic constructions. The spaces $H^0(\mathcal{O}(1))$ fit together to form the dual $\mathcal{S}^*$ of the tautological bundle of rank 2 on $\mathrm{PGr}(1,3)$, and the bundle $\mathcal{V}$ is the symmetric cube $\mathrm{Sym}^3\mathcal{S}^*$ of $\mathcal{S}^*$, which allows us to express the Chern classes of $\mathcal{V}$ in terms of those of $\mathcal{S}^*$. We will see that $\deg c_4(\mathcal{V}) = 27$. (You can also prove the statement without Chern classes like in Hartshorne: X is the blowup of $\mathbb{P}^2$ at six general points.)

Let $\mathcal{E}$ be a vector bundle on a variety X of dimension n. We will introduce Chern classes $c_i(\mathcal{E}) \in A_{n-i}(X)$, extending the definition of $c_1(\mathcal{L})$ for a line bundle. As with our treatment of the intersection product, we will give an appealingly intuitive characterization rather than a proof of existence.

Recall that we defined the first Chern class $c_1(\mathcal{L})$ of a line bundle $\mathcal{L}$ on a variety X to be

$$c_1(\mathcal{L}) = [\mathrm{Div}(\tau)] \in A_{n-1}(X)$$

for any rational section τ of $\mathcal{L}$. We define $c_i(\mathcal{L}) = 0$ for all $i \geq 2$. In this section we will characterize Chern classes $c_i(\mathcal{E})$ for any vector bundle $\mathcal{E}$ and any integer $i \geq 0$.

We first sketch the situation in the case of a bundle $\mathcal{E}$ generated by its global sections (this circumstance is in fact the case in most of our applications, and in particular in the example of the 27 lines). Let $r = \mathrm{rank}\mathcal{E}$.

In the case $r = 1$ already treated, the class $c_1(\mathcal{L})$ may be regarded as a measure of nontriviality: if $c_1(\mathcal{L}) = 0$, then $\mathcal{L}$ has a nowhere-vanishing section, whence $\mathcal{L} \cong \mathcal{O}_X$. We extend this idea of measuring nontriviality using the idea of the "degeneracy locus" of a collection of sections – roughly, this is the locus where the sections become linearly dependent in the fibers of $\mathcal{E}$ (is this like the bundle of plane conics over $\mathbb{P}^{3*}$??? Where double lines exist on multiple planes at once?). To make the meaning precise, we use multilinear algebra.

The bundle $\mathcal{E}$ is trivial if and only if it has r everywhere-independent global sections $\tau_0, \ldots, \tau_{r-1}$ (does that mean it's a product?); in this case, any set of r general sections will do. Thus a first measure of nontriviality is the locus where r general sections $\tau_0, \ldots, \tau_{r-1}$ are dependent (I think that would be the space of double lines in $\mathbb{P}^3$ in the plane conics example). If we write $\tau : \mathcal{O}_X^r \to \mathcal{E}$ of the map sending the i^{th} basis vector to τ_i, then this is the locus where τ fails to be a surjection, or equivalently, where the determinant of τ is zero. We can interpret this as the vanishing of a special section of an exterior power of $\mathcal{E}$: it is the zero scheme of the section

$$\tau_0 \wedge \cdots \wedge \tau_{r-1} \in \bigwedge^r \mathcal{E}.$$

Since rank$\mathcal{E} = r$, the bundle $\bigwedge^r \mathcal{E}$ has rank 1 and the class of the zero locus is by definition $c_1(\bigwedge^r \mathcal{E})$; this is a class in $A_{\dim X-1}(X)$ depending only on the isomorphism class of $\mathcal{E}$. We call it the *first Chern class of* $\mathcal{E}$, written $c_1(\mathcal{E})$.

More generally, we can consider for any i the scheme where $r-i$ general sections of $\mathcal{E}$ fail to be independent, defined by the vanishing of

$$\tau_0 \wedge \cdots \wedge \tau_{r-i} \in \bigwedge^{r-i+1} \mathcal{E}.$$

This is called the *degeneracy locus* of the sections $\tau_0, \ldots, \tau_{r-i}$. Since the degeneracy loci are central to our understanding and applications of Chern classes, we should first say what we expect them to look like.

To see how this should go, consider first the "degeneracy locus of one section." A section τ of $\mathcal{E}$ is locally given by r functions $f_1, \ldots, f_r$, so that by the principal ideal theorem the codimension of each component of $V(\tau)$ is at most r. Moreover, if $\mathcal{E}$ is generated by global sections and τ is a general section, then the function f_{i+1} will not vanish identically on any component of the locus where $f_1, \ldots, f_i$ vanish, and it follows that every component of $V(\tau)$ has codimension exactly r. Under our standing assumption of characteristic 0, a version of Bertini's theorem tells us that $V(\tau)$ is reduced as well. (This may fail in positive characteristic, for example in the case of a line bundle whose complete linear system defines an inseparable morphism.) It turns out that this is typical.

Lemma 5.2. Suppose $\mathcal{E}$ is a vector bundle of rank r on a variety X, and let i be an integer with $1 \leq i \leq r$. Let $\tau_0, \ldots, \tau_{r-i}$ be global sections of $\mathcal{E}$, and let $D = V(\tau_0 \wedge \cdots \wedge \tau_{r-i})$ be the degeneracy locus where they are dependent.

(a) No component of D has codimension $> i$.

(b) If the τ_i are general elements of a vector space $W \subseteq H^0(\mathcal{E})$ of global sections generating $\mathcal{E}$, then D is generically reduced and has codimension i in X.

Proof. (a) This is Macaulay's "generalized unmixedness theorem." He proved it for the case of polynomial rings, and the general case was proved by Eagon and Northcott.

(b) Let W be an m-dimensional vector space of global sections of $\mathcal{E}$ that generate $\mathcal{E}$, and let $\varphi : X \to \mathfrak{Gr}(m-r, W)$ be the associated morphism sending $p \in X$ to the kernel of the evaluation map $W \to \mathcal{E}_p$. If $U \subseteq W$ is a subspace of dimension $r-i+1$ spanned by $\tau_0, \ldots, \tau_{r-i}$ then the locus $V(\tau_0 \wedge \cdots \wedge \tau_{r-i}) \subseteq X$ is the preimage $\varphi^{-1}(\Sigma)$ of the Schubert cycle

$$\Sigma_i(U) = \{\Lambda \in \mathfrak{Gr}(m-r, W) : \Lambda \cap U \neq \varnothing\}$$

of $(m-r)$-planes in W meeting U nontrivially. By Kleiman's theorem if $U \subseteq W$ is general this locus is generically reduced of codimension i. $\square$