

Groupoids of Configurations of Lines

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20 September 2024

Definition

A finite set Z in \mathbb{P}_k^n is **geproci** if the projection \overline{Z} of Z from a general point P to a hyperplane $H = \mathbb{P}_k^{n-1}$ is a complete intersection in H .

Geproci stands for **g**eneral **p**rojection is a **c**omplete **i**ntersection. The only nontrivial examples known are for $n = 3$. In this case a hyperplane is a plane. A reduced set of points in a plane is a complete intersection if it is the transverse intersection of two algebraic curves, [like this](#).

For $\#Z = ab$ ($a \leq b$), Z is (a, b) -geproci if \overline{Z} is the intersection of a degree a curve and a degree b curve.

Background: Spreads in \mathbb{P}_k^{2t+1}

Definition

Given a $t \geq 0$ and field k , a **spread** of \mathbb{P}_k^{2t+1} is a set of mutually-skew t -planes that partition \mathbb{P}_k^{2t+1} .

Spreads are known to exist for any $t \geq 0$ for $k = \text{any finite field}$, and for $t = 0, 1$ for $k = \mathbb{R}$.

Spreads are instrumental for the proof that $\mathbb{P}_{\mathbb{F}_q}^3$ is geaproci under $\mathbb{P}_{\mathbb{F}_q}^3$. In this case, a spread is a partition of $\mathbb{P}_{\mathbb{F}_q}^3$ into lines.

The Hopf Fibration over \mathbb{R}

The Hopf fibration $H : S^3 \rightarrow S^2$ can yield a spread over $\mathbb{P}_{\mathbb{R}}^3$.

$$\begin{array}{ccc} S^3 & \xrightarrow{H} & S^2 \\ \downarrow A & & \downarrow \cong \\ \mathbb{P}_{\mathbb{R}}^3 & \xrightarrow{F} & \mathbb{P}_{\mathbb{C}}^1 \end{array}$$

Let $L_{a,b}$ denote the line joining the points $(1, 0, a, b)$ and $(0, 1, -b, a)$, and let L_{∞} denote the line joining $(0, 0, 1, 0)$ and $(0, 0, 0, 1)$. Then $\mathcal{S} = \{L_{a,b} : a, b \in \mathbb{R}\} \cup \{L_{\infty}\}$ is the spread yielded by the Hopf fibration.

The Hopf Fibration over \mathbb{R} , continued

Note that $L_{a,b}$ and $L_{c,d}$ are indeed skew for $(a,b) \neq (c,d)$. We can see this because

$$\begin{vmatrix} 1 & 0 & a & b \\ 0 & 1 & -b & a \\ 1 & 0 & c & d \\ 0 & 1 & -d & c \end{vmatrix} = (a-c)^2 + (b-d)^2,$$

which can only be 0 if $(a,b) = (c,d) \in \mathbb{R}^2$.

Furthermore, the point $(a,b,c,d) \in \mathbb{P}_{\mathbb{R}}^3$ is on the line $L_{\frac{ac+bd}{a^2+b^2}, \frac{ad-bc}{a^2+b^2}}$ if $(a,b) \neq (0,0)$, and on L_{∞} otherwise.

So this is indeed a spread over $\mathbb{P}_{\mathbb{R}}^3$!

Spreads over \mathbb{F}_q

Since \mathbb{F}_q is not algebraically closed, we can mimic the construction of the Hopf spread!

- First let q be odd. Then there is some $\theta \in \mathbb{F}_q$ such that $x^2 - \theta \in \mathbb{F}_q[x]$ is irreducible. Defining $L_{a,b} = \overline{(1, 0, a, b), (0, 1, \theta b, a)}$ for $(a, b) \in \mathbb{F}_q$ and $L_\infty = \overline{(0, 0, 1, 0), (0, 0, 0, 1)}$ yields a spread over $\mathbb{P}_{\mathbb{F}_q}^3$.
- Now let q be even. Then there is some $\psi \in \mathbb{F}_q$ such that $x^2 + x + \psi \in \mathbb{F}_q[x]$ is irreducible. Defining $L_{a,b} = \overline{(1, 0, a, b), (0, 1, \psi b, a + b)}$ and $L_\infty = \overline{(0, 0, 1, 0), (0, 0, 0, 1)}$ yields a spread over $\mathbb{P}_{\mathbb{F}_q}^3$.

Spreads constructed using this method will be known as “Hopf spreads.”

Non-Hopf Spreads

Definition

A **regulus** is a set of mutually-skew lines \mathcal{R} such that there is a quadric surface Q where $\bigcup_{R \in \mathcal{R}} R = Q$, [like this](#).

Every regulus \mathcal{R} admits an **opposite regulus** \mathcal{R}^* .

The Hopf spread contains reguli: for example $\mathcal{R} = \{L_{a,0}\} \cup \{L_\infty\}$ is a regulus.

Given a spread \mathcal{S} containing a regulus \mathcal{R} , the set of lines $(\mathcal{S} \setminus \mathcal{R}) \cup \mathcal{R}^*$ is also a spread.

It was once conjectured that every spread could be constructed by starting with the Hopf spread and replacing a succession of reguli with their opposites. This is now known to be false. In fact, there are spreads that contain no reguli whatsoever!

Maximal Partial Spreads

Note that a spread over $\mathbb{P}_{\mathbb{F}_q}^3$ comprises $q^2 + 1$ mutually-skew lines.

Definition

A **partial spread** of $\mathbb{P}_{\mathbb{F}_q}^3$ with **deficiency** d is a set of $q^2 + 1 - d$ mutually-skew lines. A **maximal partial spread** is a partial spread of positive deficiency that is not contained in any larger partial spread.

Maximal partial spreads are also instrumental to the study of geproci sets; in fact, given a maximal partial spread \mathcal{M} , the set $\mathbb{P}_{\mathbb{F}_q}^3 \setminus (\bigcup_{L \in \mathcal{M}} L)$ is geproci.

Projecting a Line to a Line via... a Line

Definition

Given three lines $L_1, L_2, L_3 \subseteq \mathbb{P}_k^3$ where $L_1 \cap L_2 = L_2 \cap L_3 = \emptyset$, we can define the function $\pi(L_1, L_2, L_3) : L_1 \rightarrow L_3$ as follows: take $p \in L_1$. Then there is a unique line T such that $p \in T$, $T \cap L_2 \neq \emptyset$, and $T \cap L_3 \neq \emptyset$. Then define $\pi(L_1, L_2, L_3)(p) = T \cap L_3$. This is the **projection** of L_1 to L_3 **via** L_2 .

[Here is a demonstration.](#)

Definition

A **groupoid** is a category \mathcal{G} where every morphism is invertible.

- For any object $G \in \mathcal{G}$, $\text{Hom}_{\mathcal{G}}(G, G) = \text{Aut}_{\mathcal{G}}(G)$ is a group. $\text{Aut}_{\mathcal{G}}(G)$ is a “group of the groupoid.”
 - Whenever $\text{Hom}_{\mathcal{G}}(G_1, G_2) \neq \emptyset$, then $\text{Aut}_{\mathcal{G}}(G_1) \cong \text{Aut}_{\mathcal{G}}(G_2)$.
- So when $\text{Hom}_{\mathcal{G}}(G_1, G_2) \neq \emptyset$ for all $G_1, G_2 \in \mathcal{G}$, \mathcal{G} induces only one group of the groupoid, up to isomorphism. Then it makes sense to say “the” group of the groupoid, $\text{Aut}_{\mathcal{G}}$.

Theorem

Let \mathcal{L} be a set of lines in \mathbb{P}_k^3 . Define Π to be the composition-closure of the set of functions $\{\pi(L_1, L_2, L_3) : L_1, L_2, L_3 \in \mathcal{L}, L_1 \cap L_2 = L_2 \cap L_3 = \emptyset\}$. Then (\mathcal{L}, Π) is a groupoid.

In this case, any group of the groupoid is a subgroup of $\text{Aut}(\mathbb{P}_k^1) \cong \text{PGL}(2, k)$.

What can we say about this groupoid and its corresponding group(s)? In characteristic 0, when is it finite versus infinite? When does $\text{Aut}_{(\mathcal{L}, \Pi)}(L)$ have finite orbits, or finitely many orbits?

NOTE: If \mathcal{L} contains lines L_1, L_2, L_3 where $L_1 \cap L_2 \neq \emptyset$, then neither $\pi(L_1, L_2, L_3)$ nor $\pi(L_3, L_2, L_1)$ are defined. So when can we characterize whether $\text{Hom}_{(\mathcal{L}, \Pi)}(L_1, L_3) = \emptyset$? If $\mathcal{L} = \mathcal{R} \cup \mathcal{R}^*$, then $\text{Hom}(R, R') = \emptyset$ for all $R \in \mathcal{R}$ and $R' \in \mathcal{R}^*$.

Ganger's Results

In her 2024 thesis, Ganger used the technique of **transversals** to prove the following theorem:

Theorem (Ganger Corollary 2.5)

The group of the groupoid for the Hopf spread induced by the degree-2 field extension $\mathbb{F}_{q^2}/\mathbb{F}_q$ over a finite field is isomorphic to the quotient $\mathbb{F}_{q^2}^/\mathbb{F}_q^* \cong C_{q+1}$.*

Definition

Given a set of lines \mathcal{L} in \mathbb{P}_k^3 , a **transversal** is a line T in \mathbb{P}_k^3 such that $T \cap \bar{L} \neq \emptyset$ for all $L \in \mathcal{L}$.

The Hopf spread has exactly two transversals T_1, T_2 for any finite field. The intersection of transversal with a line $L \in \mathcal{S}$ is a fixed point of $\text{Aut}_{(\mathcal{S}, \Pi)}(L)$!

Representing $\text{Aut}_{\mathcal{G}} \leq \text{PGL}(2, k)$

Let $U = \overline{u_0 u_1}$, $V = \overline{v_0 v_1}$, $W = \overline{w_0 w_1}$ be lines in \mathbb{P}^3 . Then any point on U can be written as $au_0 + bu_1$ for $(a, b) \in \mathbb{P}^1$ and any point on W can be written as $cw_0 + dw_1$ for $(c, d) \in \mathbb{P}^1$. Then if

$$cw_0 + dw_1 = \pi(U, V, W)(au_0 + bu_1),$$

we must have $\overline{(au_0 + bu_1)(cw_0 + dw_1)} \cap V \neq \emptyset$.

Therefore the wedge product

$$(au_0 + bu_1) \wedge v_0 \wedge v_1 \wedge (cw_0 + dw_1) = 0.$$

We can write this as a matrix formula

$$\begin{pmatrix} a & b \end{pmatrix} \begin{pmatrix} u_0 \wedge v_0 \wedge v_1 \wedge w_0 & u_0 \wedge v_0 \wedge v_1 \wedge w_1 \\ u_1 \wedge v_0 \wedge v_1 \wedge w_0 & u_1 \wedge v_0 \wedge v_1 \wedge w_1 \end{pmatrix} \begin{pmatrix} c \\ d \end{pmatrix} = 0,$$

or rewrite this equivalently:

$$\begin{pmatrix} -u_0 \wedge v_0 \wedge v_1 \wedge w_1 & -u_1 \wedge v_0 \wedge v_1 \wedge w_1 \\ u_0 \wedge v_0 \wedge v_1 \wedge w_0 & u_1 \wedge v_0 \wedge v_1 \wedge w_0 \end{pmatrix} \begin{pmatrix} a \\ b \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}.$$

Representing $\text{Aut}_{\mathcal{G}} \leq \text{PGL}(2, k)$

The 2×2 matrix

$$\begin{pmatrix} -u_0 \wedge v_0 \wedge v_1 \wedge w_1 & -u_1 \wedge v_0 \wedge v_1 \wedge w_1 \\ u_0 \wedge v_0 \wedge v_1 \wedge w_0 & u_1 \wedge v_0 \wedge v_1 \wedge w_0 \end{pmatrix}$$

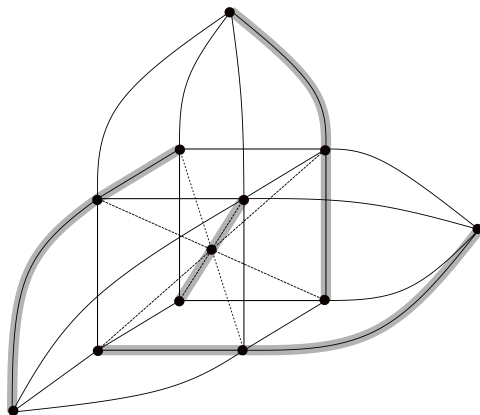
represents $\pi(U, V, W)$, given parametrizations

$$U = \overline{u_0 u_1}, V = \overline{v_0 v_1}, W = \overline{w_0 w_1}.$$

Note $r \wedge s \wedge t \wedge u = \det \begin{pmatrix} r & s & t & u \end{pmatrix}$.

This allows us to use computational methods to experiment on the group of the groupoid!

Example: The D_4 configuration



The D_4 configuration is a (3,4)-geproci half-grid. It is a $(12_4, 16_3)$ -configuration. What is the group of the groupoid of the 16 lines?

Example: The D_4 configuration

Theorem

Let \mathcal{L} be the 16 lines of the D_4 configuration and let Π be the composition-closure of the projection functions. Then the group of the groupoid $\mathcal{G} = (\mathcal{L}, \Pi)$ is $\text{Aut}_{\mathcal{G}} \cong \mathbb{Z}/3\mathbb{Z}$.

Argument boils down to:

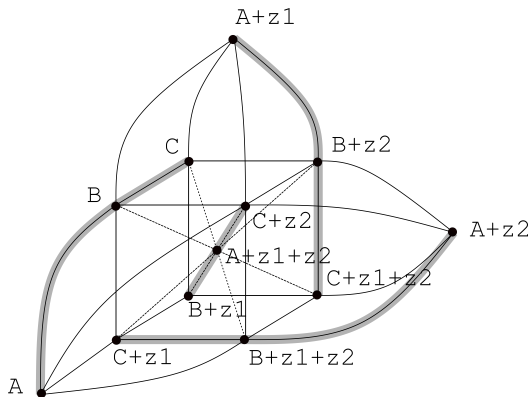
- $\text{Hom}_{\mathcal{G}}(L, L') \neq \emptyset$ for $L, L' \in \mathcal{L}$, so $\text{Aut}_{\mathcal{G}}$ is well-defined.
- Let $q \in L$ be a quadruple point and $\pi \in \text{Hom}_{\mathcal{G}}(L, L')$. Then $\pi(q)$ is a quadruple point. So $\text{Aut}_{\mathcal{G}} \leq S_3$.
- Let $\phi \in \text{Aut}_{\mathcal{G}}(L)$ fix the quadruple point q . Then ϕ fixes the other two quadruple points of L , so $\phi = \text{Id}_L$ and $\text{Aut}_{\mathcal{G}} \leq \mathbb{Z}/3\mathbb{Z}$.
- $\text{Aut}_{\mathcal{G}}$ is nontrivial, so $\text{Aut}_{\mathcal{G}} \cong \mathbb{Z}/3\mathbb{Z}$.

A Helpful Labeling

Let $\{A, B, C\}$ be a set of three letters, and consider the group $(\mathbb{Z}/2\mathbb{Z})^2 = \langle z_1, z_2 \rangle$. Then one can label the vertices of the D_4 configuration with the elements of $\{A, B, C\} \oplus (\mathbb{Z}/2\mathbb{Z})^2$ in such a way that there is a bijective correspondence between the lines of the D_4 and triples of the form $\{A + g, B + g', C + g'' : g + g' + g'' = 0\} \subseteq \{A, B, C\} \oplus (\mathbb{Z}/2\mathbb{Z})^2$.

This makes the theorem on the previous slide easier to prove because you can divide the vertices into “types” A , B , and C .

A Helpful Labeling



- $\{A, B, C\}$
- $\{A + z_1, B + z_2, C + z_1 + z_2\}$
- $\{A + z_2, B + z_1 + z_2, C + z_1\}$
- $\{A + z_1 + z_2, B + z_1, C + z_2\}$