The third rule of sheaves states that whenever V_i forms an open cover of U, and $s|_{V_i} = 0$ for some $s \in \mathcal{F}(U)$ for all i, then $s = 0$. We can illustrate this rule with an example of $V_1, V_2, V_3 \subseteq U$.

$$\begin{tikzcd}
& \mathcal{F}(V_1) & \\
& \mathcal{F}(U) \arrow[u] \arrow[dl] \arrow[dr] & \\
\mathcal{F}(V_2) & & \mathcal{F}(V_3)
\end{tikzcd}$$

$$\begin{tikzcd}
& \color{blue}{0} & \\
& \color{blue}{s} = \color{red}{0} \arrow[u, mapsto, blue] \arrow[dl, mapsto, blue] \arrow[dr, mapsto, blue] & \\
\color{blue}{0} & & \color{blue}{0}
\end{tikzcd}$$

So whenever we have the blue, we get the red. This can be told in a story that whenever there is nowhere you can send a section so that it will not die, then it must already be dead! Another way of telling this is that there is always a way to "save" a living (i.e. nonzero) section through restriction maps.

This has implications for restricting to stalks as well. Say there is a section $s \in \mathcal{F}(U)$ such that for each restriction map $\rho_x : \mathcal{F}(U) \to \mathcal{F}_x$, we have $\rho_x(s) = 0$. We can assume $\mathcal{F}(U)$ is in the direct limit chain for the construction of $\mathcal{F}_x$, and so this means that there is an open neighborhood $x \in V_x \subseteq U$ such that $s|_{V_x} = 0$ for each x. This $\{V_x\}_{x \in U}$ forms the open cover required to enact Rule 3 and conclude that $s = 0$.

The fourth rule of sheaves is that whenever V_i is an open cover of U, and there are $s_i \in \mathcal{F}(V_i)$ such that for $s_i|_{V_i \cap V_j} = s_j|_{V_i \cap V_j}$ for each i, j, then there is an $s \in \mathcal{F}(U)$ such that $s|_{V_i} = s_i$ for each i. We can illustrate this with the example $V_1, V_2, V_3 \subseteq U$.

$$\begin{tikzcd}
& \mathcal{F}(V_1) \arrow[dl] \arrow[dr] & \\
\mathcal{F}(V_1 \cap V_2) & & \mathcal{F}(V_1 \cap V_3) \\
& \mathcal{F}(U) \arrow[uu] \arrow[dl] \arrow[dr] & \\
\mathcal{F}(V_2) \arrow[uu] \arrow[dr] & & \mathcal{F}(V_3) \arrow[uu] \arrow[dl] \\
& \mathcal{F}(V_2 \cap V_3) &
\end{tikzcd}$$

$$\begin{tikzcd}
& \color{blue}{s_1} \arrow[dl, mapsto, blue] \arrow[dr, mapsto, blue] & \\
\color{blue}{a} & & \color{blue}{b} \\
& \color{red}{s} \arrow[uu, mapsto, red] \arrow[dl, mapsto, red] \arrow[dr, mapsto, red] & \\
\color{blue}{s_2} \arrow[uu, mapsto, blue] \arrow[dr, mapsto, blue] & & \color{blue}{s_3} \arrow[uu, mapsto, blue] \arrow[dl, mapsto, blue] \\
& \color{blue}{c} &
\end{tikzcd}$$

Whenever we have the stuff in blue, we get the stuff in red. This can be told in the story that whenever you have a bunch of sections and any pair of sections will join each other upon meeting, then that means they were all descended from a common ancestor! Or were cut from the same cloth. Or chipped from the same stone. Whatever your metaphor of choice is. Note that Rule 3 makes this ancestor unique.

Now assume $\mathcal{F}$ is a pre-sheaf, and we can define the pre-sheaf $G(\mathcal{F})$ by

$$G(\mathcal{F})(U) = \prod_{x \in U} \mathcal{F}_x.$$

Note there is a canonical map $\gamma : \mathcal{F} \to G(\mathcal{F})$ by defining $\gamma(U)(s) = (s|_x)_{x\in U}$ for each $U \subseteq X$ and each $s \in \mathcal{F}(U)$.

We wish to show that $G(\mathcal{F})$ is a flasque sheaf and that $\mathcal{F} \to G(\mathcal{F})$ is a monomorphism if $\mathcal{F}$ is a sheaf.

Proof. First we will show that $G(\mathcal{F})$ is a sheaf. Let $s \in G(\mathcal{F})(U)$ and let $\{V_i\}$ be an open cover of U such that $s|_{V_i} = 0$ for each i. Note that each of the restriction maps $\rho_i : G(\mathcal{F})(U) \to G(\mathcal{F})(V_i)$ is simply a projection map $\prod_{x\in U} \mathcal{F}_x \to \prod_{x\in V_i \subseteq U} \mathcal{F}_x$. Since $\rho_i(s) = 0$ then $s|_x = 0$ for each $x \in V_i$. Since this is true for all the V_i, which cover U, we then know that $s|_x = 0$ for all $x \in U$. Thus s is an element of a product with all of its coordinates equal to 0, and so $s = 0$. Thus $G(\mathcal{F})$ satisfies the third rule of sheaves.

Now let $\{V_i\}$ be an open cover of U and let $s_i \in G(\mathcal{F})(V_i)$ satisfy that $s_i|_{V_i\cap V_j} = s_j|_{V_i\cap V_j}$ for all i, j. Then we can construct an element $s \in G(\mathcal{F})(U)$ such that $s|_{V_i} = s_i$ for all i, by gluing together the coordinates of each of the s_i. We can do this because we know $s_i|_x = s_j|_x$ for all $x \in V_i \cap V_j$. We can construct s by starting with $x \in U$ and choosing any V_i containing x and defining $s|_x = s_i|_x$. This is well-defined because of the precedent. Thus $G(\mathcal{F})$ is a sheaf.

Now we will show that $G(\mathcal{F})$ is flasque. Let $\rho : G(\mathcal{F})(U) \to G(\mathcal{F})(V)$ be a restriction map. We can see that ρ is surjective because it is a simple projection map.

Now we will show that the morphism $\gamma : \mathcal{F} \to G(\mathcal{F})$ is a monomorphism if $\mathcal{F}$ is a sheaf. Let us consider $\gamma(U) : \mathcal{F}(U) \to G(\mathcal{F})(U)$ and let us look at the kernel $\ker \gamma(U)$. Consider $s \in \ker \gamma(U)$. Then $\gamma(U)(s) = (s|_x)_{x\in U} = 0$ and so $s|_x = 0$ for each $x \in U$. Thus we know that there is an open cover $\{V_x\}_{x\in U}$ of U such that $s|_{V_x} = 0$ for each V_x. Since $\mathcal{F}$ is a sheaf and thus satisfies Rule 3, we know that $s = 0$ and thus $\ker \gamma(U) = 0$. Thus γ is a monomorphism of sheaves. □

Note that it is not necessarily true that $\mathcal{F}_x \to G(\mathcal{F})_x$ is the identity or that $G(\mathcal{F})_x = \mathcal{F}_x$ even. My hypothesis is that $G(\mathcal{F})_x \cong \prod_{y\in X} \mathcal{F}_y / \{(s(y))_{y\in X}$: there is a neighborhood $V \ni x$ such that $s(y) = 0$ for all $y \in V\}$. See Sheaves on Spaces Example 11.5. For example if $s(x) = r(x)$ but $s(y) \neq r(y)$ for all other $y \in X$, then $(s(y))_{y\in X} \not\sim (r(y))_{y\in X} \in G(\mathcal{F})_x$.

Also note there is no unique way to reconstruct a sheaf just from knowing all the stalks. It is true that a morphism of sheaves is an epi/mono/iso if and only if the induced maps on stalks are epi's/mono's/iso's, but isomorphisms on all the stalks does not necessarily translate to isomorphims of sheaves.

Now define $C^0(\mathcal{F}) = \mathcal{F}$, $C^1(\mathcal{F}) = \text{coker}\,[\mathcal{F} \to G(\mathcal{F})]$, and $C^{n+1} = C^1C^n(\mathcal{F})$. Then we construct cohomology groups of $(X, \mathcal{F})$:

$$\begin{aligned} H^0(X,\mathcal{F}) &= \Gamma(X,\mathcal{F}) \\ H^1(X,\mathcal{F}) &= \text{coker}\,[\Gamma(X,G(\mathcal{F})) \to \Gamma(X,C^1(\mathcal{F}))] \\ H^{n+1}(X,\mathcal{F}) &= H^1(X,C^n(\mathcal{F})). \end{aligned}$$

Impressions: Working with global sections is nice, but the recursion is trash. Also there is an exercise in Arapura that says that if $\mathcal{F}$ is flasque, then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$.

Note that

$$\prod_{P \in U} \mathcal{F}_P \cong \left\{ s : U \to \coprod_{P \in U} \mathcal{F}_P : s(P) \in \mathcal{F}_P \right\}$$

by associating $(a_P)_{P \in U}$ with the function $P \mapsto a_P$ for each $P \in U$. This will be useful presently.

Given a pre-sheaf $\mathcal{F}$, we construct the sheafification of $\mathcal{F}$, denoted $\mathcal{F}^+$, which sends U to the group of maps $s : U \to \coprod_{P \in U} \mathcal{F}_P$ satisfying

(1) for each $P \in U$, $s(P) \in \mathcal{F}_P$, and

(2) for each $P \in U$, there is a neighborhood $V \subseteq U$ of P and a $t \in \mathcal{F}(V)$ such that for each $Q \in V$, we have $s(Q) = t_Q (= t|_Q)$.

We can immediately see that this can be realized as a subset of $\prod_{P \in U} \mathcal{F}_P$. Thus a section of $\mathcal{F}^+(U)$ can be realized as an element of $\prod_{P \in U} \mathcal{F}_P$ where each of the coordinate germs locally descend from the same sections of $\mathcal{F}(V)$ where $V \subseteq U$. Germs in a sequence must locally descend from a common ancestor, or are "cut from the same cloth," "chipped from the same stone" whathaveyou. Note that $\mathcal{F}^+_P = \mathcal{F}_P$ for all $P \in X$. Again, showing that there is not a unique way to reconstruct a pre-sheaf from knowing all the stalks, as $\mathcal{F}^+ \neq \mathcal{F}$ in general.

If U is irreducible then all nonempty open subsets of U intersect nontrivially. In this case we can get rid of the "local" qualifier in the statement. For each $P \in U$, there is a $t \in \mathcal{F}(U)$ such that for each $Q \in U$, we have $s(Q) = t_Q$. Thus we can associate the section $s \in \mathcal{F}^+(U)$ with the section $t \in \mathcal{F}(U)$. Thus for U irreducible, we have $\mathcal{F}^+(U) \cong \mathcal{F}(U)$. This is a way of showing that any pre-sheaf defined on an irreducible topological space is automatically a sheaf!

Conjecture: Let $\{V_i\}$ be an open *partition* of U which are also closed irreducible subspaces of U. Then $\mathcal{F}^+(U) \cong \prod_i \mathcal{F}(V_i)$. We can define $\prod_i \mathcal{F}(V_i) \to \mathcal{F}^+(U)$ by $(s_i)_i \mapsto ((s_i|_P)_{P\in V_i})_i$. Now let $(t_P)_{P\in U} \in \mathcal{F}^+(U)$. Since each of the V_i are irreducible, we know that for each $P \in V_i$, we have $t_P = r_P$ for some $r \in \mathcal{F}(V_i)$. Then let $P_i \in V_i$ be any point and denote the section of $\mathcal{F}(V_i)$ associated to t_{P_i} as r_i. Then we can define a map $\mathcal{F}^+(U) \to \prod_i \mathcal{F}(V_i)$ via $(t_P)_{P\in U} \mapsto (r_i)_i$. These are inverse maps.

Examples of such a U are parallel lines in affine space, or skew lines in dimension > 2 space.