For a curve C on a surface X, $h^0(X;C) + C.K_X = g(C) = h^0(C;K_C) = h^0(C;(C+K_X)|_C) = h^0(X|_C;(C+K_X)|_C)$.

Let $X = \mathbb{P}^1 \times \mathbb{P}^1$ and let C be a smooth $(2,3)$ curve. Then $K_X = (-2,-2)$ and $C.K_X = -4-6 = -10$. (Note $(a,b).(c,d) = ad+bc$ in $\mathbb{P}^1 \times \mathbb{P}^1$). And $h^0(X;C) = (2+1)(3+1) = 12$. Thus $g(C) = 12-10 = 2$. Thus C is an example of a smooth genus-2 curve, which does not exist in $\mathbb{P}^2$.

In $\mathbb{P}^2$, smooth curves of degree d have a genus $\binom{d+2}{2} - 3d = \frac{d^2+3d+2}{2} - \frac{6d}{2} = \frac{d^2-3d+2}{2} = \binom{d-1}{2}$. As d increments up from 1, we get geni of 0, 0, 1, 3, 6, 10, 15, 21, 28, 36,. . . .

Let D be a smooth curve in $X = \mathbb{P}^1 \times \mathbb{P}^1$ of degree (a,b). Then $h^0(X;D) = (a+1)(b+1)$ and $C.K_X = -2a-2b$. Then $g(C) = ab+a+b+1-2a-2b = ab-a-b+1 = (a-1)(b-1)$. Thus we can get a smooth curve of any genus we want by taking $a = 2$ and b to be one greater than the desired genus.

$\mathbb{P}^1 \times \mathbb{P}^1$ embeds into $\mathbb{P}^3$ as a quadric surface. What can be said of the degree of an (a,b) curve under this embedding? A $(1,0)$ curve is a line on the surface, and so is a degree one curve in $\mathbb{P}^3$. Perhaps $a+b$? The emdedding goes $((u,s),(t,v)) \longmapsto (ut,uv,st,sv)$. So we may get the map $k[x,y,z,w] \longrightarrow S(\mathbb{P}^1 \times \mathbb{P}^1)$ by $x \longmapsto u+t$, $y \longmapsto u+v$, $z \longmapsto s+t$, and $w \longmapsto s+v$. On the $w \neq 0$ affine neighborhood of $V(xw-yz)$, we get $(a,b,c,1) \longmapsto ((b,1),(c,1))$. The image is the intersection of the $s \neq 0$ and $v \neq 0$ affine neighborhoods. So we can say u/s maps to y/w, and t/v maps to z/w. So $f(u/s,1), g(t/v,1)$ maps to $f(y/w,1)+g(z/w,1)$. For example, let $f(u,s) = u^2+us+s^2$ be homogeneous and let $g(t,v) = t^3+t^2v+tv^2+v^3$ be homogeneous. Then $(f(u/s,1), g(t/v,1))$ maps to $(y/w)^2+(y/w)+1+(z/w)^3+(z/w)^2+(z/w)+1$. We can homogenize with w to get $y^2w+yw^2+w^3+z^3+z^2w+zw^2+w^3$. This is a cubic surface, and the surface $V(xw-yz)$ is quadric, so do we just get a sextic curve?