Let $\mathbb{F}_q$ be a finite field of size q and characteristic p, where p is an odd prime. Let $r \in \mathbb{F}_q$ be such that the polynomial $x^2 - r \in \mathbb{F}_q[x]$ is irreducible; that is, r has no square root in $\mathbb{F}_q$. Denote by $L_r(a,b)$ the line in $\mathbb{P}^3_{\mathbb{F}_q}$ connecting the points $(1,0,a,b)$ and $(0,1,rb,a)$. Denote by $L(\infty)$ the line connecting the points $(0,0,1,0)$ and $(0,0,0,1)$.

Proposition 1. The set of lines $S_r = \{L_r(a,b), L(\infty) : a,b \in \mathbb{F}_q\}$ is a spread in $\mathbb{P}^3_{\mathbb{F}_q}$. We will call this the r-spread of $\mathbb{P}^3_{\mathbb{F}_q}$.

Proof. First note that for all $(a,b) \in \mathbb{F}_q^2$, $L_r(a,b) \cap L(\infty) = \varnothing$. Now it is enough to show that for distinct pairs $(a,b),(c,d) \in \mathbb{F}_q^2$, $L_r(a,b) \cap L_r(c,d) = \varnothing$.

Note that the lines $L_r(a,b)$ and $L_r(c,d)$ are skew if and only if the two vector subspaces of $\mathbb{F}_q^4$ $\operatorname{span}\{(1,0,a,b),(0,1,rb,a)\}$ and $\operatorname{span}\{(1,0,c,d),(0,1,rd,c)\}$ intersect only at the origin. This is true if and only if the set of vectors $\{(1,0,a,b),(0,1,rb,a),(1,0,c,d),(0,1,rd,c)\}$ is linearly independent.

We can check this by confirming that

$$\det \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & rb & a \\ 1 & 0 & c & d \\ 0 & 1 & rd & c \end{pmatrix} \neq 0.$$

In fact, we get $\det \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & rb & a \\ 1 & 0 & c & d \\ 0 & 1 & rd & c \end{pmatrix} = (a-c)^2 - (b-d)^2 r$. For contradiction, let us assume $(a-c)^2 - (b-d)^2 r = 0$. In the case that $b = d$, then we get $(a-c)^2 = 0$, and so $a = c$, which means $(a,b) = (c,d)$, which contradicts the assumption that (a,b) and (c,d) are distinct.

If $b \neq d$, then $b-d$ is a unit. Then $(a-c)^2 - (b-d)^2 r = 0$ if and only if $\left(\frac{a-c}{b-d}\right)^2 - r = 0$. This contradicts the assumption that r does not have any square roots.

Therefore $\det \begin{pmatrix} 1 & 0 & a & b \\ 0 & 1 & rb & a \\ 1 & 0 & c & d \\ 0 & 1 & rd & c \end{pmatrix} \neq 0$ for all $(a,b) \neq (c,d)$, and so the lines $L_r(a,b)$ and $L_r(c,d)$ are skew in $\mathbb{P}^3_{\mathbb{F}_q}$. Since $\#S_r = q^2+1$ and each line contains $q+1$ unique points, S_r must be a spread of $\mathbb{P}^3_{\mathbb{F}_q}$ □

Remark 2. In the case $\operatorname{char} \mathbb{F}_q = 2$, we want to choose $r \in \mathbb{F}_q$ to be such that the polynomial x^2+x+r is irreducible in $\mathbb{F}_q[x]$. Then define $L_r(a,b)$ to be the line in $\mathbb{P}^3_{\mathbb{F}_q}$ connecting the points $(1,0,a,b)$ and $(0,1,br,a+b)$. Then a similar argument shows that $S_r = \{L_r(a,b), L(\infty) : a,b \in \mathbb{F}_q\}$ is a spread.

This construction gives us a unique spread for each quadratic non-residue of $\mathbb{F}_q$ (q is again odd). This constructs $\frac{q-1}{2}$ spreads for $\mathbb{P}^3_{\mathbb{F}_q}$.

Now we will see that there are $q+1$ lines of S_r that are invariant based on r. Let Q denote the set of elements of $\mathbb{F}_q$ that do not have square roots: for all $r,s \in Q$, the set of lines $I = \{L_r(a,0), L(\infty) : a \in \mathbb{F}_q\}$ is contained in the spread S_s. Indeed, $L_r(a,0) = L_s(a,0)$ for all $r, s \in Q$. For this reason we can simply call these lines $L(a)$.

In fact, I forms a $(q+1) \times (q+1)$ grid. For $a \in \mathbb{F}_q$, let us define the line $\Gamma(a)$ to be the line connecting the points $(0,0,1,a)$ and $(1,a,0,0)$ and $\Gamma(\infty)$ to be the line through the points $(0,0,0,1)$ and $(0,1,0,0)$. Then $J = \{\Gamma(a), \Gamma(\infty) : a \in \mathbb{F}_q\}$ is a set of $q+1$ mutually-skew lines in $\mathbb{P}^3_{\mathbb{F}_q}$. Furthermore, each line in I intersects each line in J exactly once. Thus for each $\Gamma \in J$, all of the $q+1$ points of Γ are contained in one of the $q+1$ lines of I. Therefore none of the lines of J meet any of the lines in $S_r \setminus I$ for any $r \in Q$. Thus

$$T_r := (S_r \setminus I) \cup J$$

is a spread of $\mathbb{P}^3_{\mathbb{F}_q}$. Thus each quadratic non-residue r of $\mathbb{F}_q$ gives us two unique spreads: S_r and T_r for a total of $q-1$ unique spreads. **Is there a way to construct a maximal partial spread out of mixing up lines from these $q-1$ spreads?**