Let C be a nonsingular projective curve in $\mathbb{P}^2_k$, where k is an algebraically closed field. For each line L in $\mathbb{P}^2$, consider the finite set of points on both L and C, denoted $L \cap C$. If C is a curve of degree d, then $L \cap C$ will have exactly d points (counted with the respective multiplicities). We then write the formal sum $L \cap C = \sum n_i P_i$, where $P_i \in C$ are the points and n_i the multiplicities. This formal sum is a divisor on C. As L varies, we obtain a family of divisors on C, parametrized by the set of all lines in $\mathbb{P}^2$.

Using the duality between lines and points in $\mathbb{P}^2$ given by

$$[a : b : c] \leftrightarrow \mathfrak{Z}(ax + by + cz)$$

the set of all lines in $\mathbb{P}^2$ is the dual projective space $(\mathbb{P}^2)^*$. In Proj$k[x, y, z]$ this duality is given by

$$\mathfrak{V}(x - a, y - b, z - c) \leftrightarrow \mathfrak{V}(ax + by + cz)$$

where $a \neq 0$, $b \neq 0$, or $c \neq 0$. Call this set of divisors a linear system of divisors on C. Note that the embedding of C in $\mathbb{P}^2$ can be recovered just from knowing this linear system: if P is a point of C, we consider the set of divisors in the linear system which contain P. They correspond to the lines $L \in (\mathbb{P}^2)^*$ passing through P, and this set determines P uniquely as a point of $\mathbb{P}^2$.

Example 1. Consider the nonsingular curve $C = \mathfrak{Z}(x^2 + y^2 - z^2) \in \mathbb{P}^2_k$, and the point $P = [4 : 3 : 5]$ on C. Then consider the divisor $P + [-1 : 0 : 1]$. This divisor corresponds to the line $\mathfrak{Z}\,(x - 3y + z)$ (the line that connects P and $[-1 : 0 : 1]$). The dual of this line is $[1 : -3 : 1]$.
Now consider the divisor $2P$. This divisor corresponds to the line $\mathfrak{Z}\,(4x + 3y - 5z)$. The dual of this line is $[4 : 3 : -5]$.
Now consider the divisor $P+[a : b : c]$ where $[a : b : c] \in C\setminus\{P\}$. Then the line corresponding to this divisor is $\mathfrak{Z}((5b-3c)x+(4c-5a)y+(3a-4b)z)$ with dual point $[5b-3c : 4c-5a : 3a-4b]$. Then let us explore the set

$$Z = \{[4 : 3 : -5], [5b - 3c : 4c - 5a : 3a - 4b] : a^2 + b^2 - c^2 = 0, [a : b : c] \neq [4 : 3 : 5]\}.$$

Then $Z = \mathfrak{Z}(4x + 3y + 5z)$, which is the dual of P.

Example 2. Consider the stellar scheme $X = \text{Proj}\mathbb{C}[x, y, z, w]/(yw - x^2)$ and the subscheme $Y = \text{Proj}\mathbb{C}[x, y, z, w]/(yw - x^2, z)$. Note that Y is an integral scheme since $\Gamma(Y) = \mathbb{C}[x, y, z, w]/(yw - x^2, z)$ is an integral domain. Additionally, Y is of codimension 1 in X. Thus Y is a prime divisor of X.

Then Y has generic point $\eta = \sqrt{0\Gamma(Y)} = 0\Gamma(Y)$, since $\Gamma(Y)$ is integral domain and hence both reduced and lucky. Then consider the quotient map $q : \Gamma(X) \to \Gamma(Y)$ and observe $q^{-1}(\eta) = (z) \in X$.

Now see that $\mathcal{O}_{(z),X} \cong \Gamma(X)_{(z)}$ is local, noetherian, has maximal ideal (z) which is principal, and is an integral domain. Thus $\mathcal{O}_{(z),X}$ is a discrete valuation ring.

Now every element of $K(X)$ (the field of fractions on X) can be assigned an integer under the discrete valuation v_z or v_Y as

$$v_z(\alpha) = -\min\left\{n \in \mathbb{Z} : z^n\alpha \in \mathcal{O}^{\times}_{(z),X}\right\}.$$

We see for example $v_z(1/z) = -1$ and $v_z(z^3 + wz^2) = 2$ and $v_z(xy) = 0$ and $v_z(z + x) = 0$ and $v_z(z + yz) = 1$ and $v_z(xy/z^3) = -3$ and $v_z((wy + x)/(z^6 - wyz^4)) = -4$.

Lemma 6.1. Let X be a stellar scheme and fix $f \in K(X)^{\times}$. Then $v_Y(f) = 0$ for all except finitely many prime divisors Y.

Proof. Let $U_f = \mathrm{Spec}A$ be an open affine subset of X on which f is regular. Then $Z = X \setminus U_f$ is a proper closed subset of X. Since X is noetherian, Z can contain at most finitely many prime divisors of X (since $U_f \neq \emptyset$ and thus $\dim Z < \dim X$ and prime divisors are irreducible closed subsets of dimension $= \dim X - 1$. If $\mathrm{codim}Z \geq 2$ then Z can contain no prime divisors. If $\mathrm{codim}Z = 1$ then any prime divisor Y of X contained in Z is an irreducible component of Y. Since X is noetherian there are only finitely many irreducible components of Z, so Z can contain only finitely many prime divisors of X). These are the prime divisors Y for which $v_Y(f) < 0$ (if U_f is the complement of the pole set of f).

All other prime divisors must meet U_f. Thus we must show that there are only finitely many prime divisors Y of U_f for which $v_Y(f) \neq 0$. Since f is regular on U_f, we have $v_Y(f) \geq 0$ in any case. And $v_Y(f) > 0$ if and only if Y is contained in the closed subset of U defined by the ideal Af in A. Since $f \neq 0$, this is a proper closed subset, hence contains only finitely many closed irreducible subsets of codimension one of U. □

Example 3. Let $X = \mathrm{Spec}\mathbb{C}[x, y, z]/(x^2 + y^2 - z^2 - 1)$ and let $f = \frac{x+y}{z-1} + (x^2 + y^2 - z^2 - 1) \in K(X)^{\times}$. Then $U_f = X \setminus \mathfrak{V}_X(z - 1)$ is an open set on which f is regular. Note that $v_{\mathfrak{V}_X(z-1)}(f) = -1$ and $v_{\mathfrak{V}_X(x+y)}(f) = 1$. On every other prime divisor Y of X, $v_Y(f) = 0$ and so

$$(f) = \mathfrak{V}_X(x + y) - \mathfrak{V}_X(z - 1).$$

Moreover, if we have $g = \frac{(x+y)^3(2x-z)^6}{(z-1)^2(x+5)^7}$ then

$$(g) = 3\mathfrak{V}_X(x + y) + 6\mathfrak{V}_X(2x - z) - 2\mathfrak{V}_X(z - 1) - 7\mathfrak{V}_X(x + 5).$$

Furthermore, we have the concept of adding principal divisors. Observe

$$\begin{aligned} &(f) + (g) \\ &= \mathfrak{V}_X(x + y) - \mathfrak{V}_X(z - 1) + 3\mathfrak{V}_X(x + y) + 6\mathfrak{V}_X(2x - z) - 2\mathfrak{V}_X(z - 1) - 7\mathfrak{V}_X(x + 5) \\ &= 4\mathfrak{V}_X(x + y) + 6\mathfrak{V}_X(2x - z) - 3\mathfrak{V}(z - 1) - 7\mathfrak{V}_X(x + 5) = \left(\frac{(x + y)^4(2x - z)^6}{(z - 1)^3(x + 5)^7}\right) \\ &= (fg). \end{aligned}$$

Note: When X is affine and $\Gamma(X, \mathcal{O}_X)$ is a UFD (as in Example 3), every formal sum of prime divisors of X can be realized as the principal divisor of some $f \in K(X)^{\times}$. In this case, $\mathrm{Cl}(X) = 0$.

Proposition 6.2. Let A be a noetherian ring. Then A is a UFD if and only if Spec$A$ is normal and ClSpec$A = 0$.

Proposition 6.3A. Let A be an integrally closed noetherian domain. Then

$$A = \bigcap_{\text{ht}\mathfrak{p}=1} A_{\mathfrak{p}}$$

where the intersection is taken over all prime ideals of height 1. Remember height corresponds with codimension. This is like saying the ring of regular functions is the set of functions that are regular on every hypersurface. (Similar to the containment shown in Theorem I.3.2.)

Example 6.3.1. If $X = \mathbb{A}_k^n$ over some field k, then Cl$X = 0$ (since $X = \text{Spec}k[x_1, \ldots, x_n]$ and $k[x_1, \ldots, x_n]$ is a UFD). We also get X is normal.

Example 6.3.2. Let A be a Dedekind domain. Then Cl(Spec$A$) is the ideal class group of A (from algebraic number theory). Generalizing the fact that Cl(Spec$A$) $= 0$ if and only if A is a UFD. The ideal class group of an algebraic number field K is the quotient group J_K/P_K where J_K is the group of fractional ideals of the ring of integers of K and P_K is its subgroup of principal ideals. It measures the extent to which unique factorization fails.

Note: In a projective scheme, the degree of the scheme matters a lot more than in affine space. For example, $X = \text{Spec}\mathbb{C}[x, y]/(y-x^2)$ has $\text{Cl}(X) = 0$ becuase $\mathbb{C}[x, y]/(y-x^2) \cong \mathbb{C}[x]$ is a UFD. However, $Y = \text{Proj}\mathbb{C}[x, y, z]/(yz - x^2)$ is a different story because single points do not make up principal divisors in this projective scheme. For example $P = (0, 0) \sim 0$ in ClX since the line $x = 0$ goes through P and no other point on X, but $Q = [0 : 0 : 1] \not\sim 0$ in ClY since any line that goes through Q on Y must go through some other point on Y, or go through Q with multiplicity 2. So $2Q \sim 0$ and $Q + R \sim 0$ for any point R on Y, but $Q \not\sim 0$ in ClY. In fact, this tells us that $R \sim -R \sim S$ for any two points R, S on Y. Thus Cl$Y \cong \mathbb{Z}/2\mathbb{Z}$. This is actually $\text{Cl}(\mathbb{P}^2 \setminus Y) \cong \mathbb{Z}/2\mathbb{Z}$, since our divisors are lines through $\mathbb{P}^2$. We will see from 6.4 and 6.5 that actually Cl$Y \cong \mathbb{Z}$. This makes more sense because $Y \cong \mathbb{P}^2$ and so Cl$Y \cong \text{Cl}\mathbb{P}^2 \cong \mathbb{Z}$. But I believe we get $\text{Cl}^\circ Y \cong \mathbb{Z}/2\mathbb{Z}$, since $\text{Cl}^\circ Y = \ker(\text{Cl}Y \twoheadrightarrow \mathbb{Z})$

Proposition 6.4. Let $X = \mathbb{P}_k^n$ for some field k. For any divisor $D = \sum n_i Y_i$, define the degree of D by $\deg D = \sum n_i \deg Y_i$, where $\deg Y_i$ is the degree of the hypersurface Y_i (degree is $(\dim Y_i)!$ times the leading coefficient of the Hilbert polynomial of the ring $\Gamma(Y_i, \mathcal{O}_{X|Y_i})$). Let H be the hypersurface $x_0 = 0$. Then:

(a) if D is any divisor of degree d, then $D \sim dH$;

(b) for any $f \in K^\times$, $\deg f = 0$;

(c) the degree function gives an isomorphism $\deg : \text{Cl}X \to \mathbb{Z}$.

Proof. Let $S = k[x_0, \dots, x_n]$ be the homogeneous coordinate ring of X. If g is a homogeneous element of degree d, we can factor it into irreducible polynomials $g = g_1^{n_1} \cdots g_r^{n_r}$. Then g_i defines a hypersurface Y_i of degree $d_i = \deg g_i$, and we can define the divisor of g to be $(g) = \sum n_i Y_i$. Then $\deg g = d$. Now a rational function f on X is a quotient g/h of homogeneous polynomials of the same degree (in order to be well-defined in compatibility with the scaling rule of points in projective space). Then $(f) = (g) - (h)$, so we see that $\deg(f) = 0$, proving (b).

If D is any divisor of degree d, we can write it as a difference $D_1 - D_2$ of effective divisors of degrees d_1, d_2 with $d_1 - d_2 = d$. Let $D_1 = (g_1)$ and $D_2 = (g_2)$. This is possible because an irreducible hypersurface in $\mathbb{P}^n$ corresponds to a homogeneous prime ideal of height 1 in S, which is principal. Taking power products we can get any effective divisor as (g) for some homogeneous g. Now $D - dH = (f)$ where $f = g_1/x_0^d g_2$ is a rational function on X. This proves (a). Statement (c) follows from (a), (b), and the fact that $\deg H = 1$. $\square$

Proposition 6.5. Let X be stellar and let Z be a proper closed subset of X and let $U = X \setminus Z$. Then:

(a) there is a surjective homomorphism $\mathrm{Cl}X \to \mathrm{Cl}U$ defined by

$$D = \sum n_i Y_i \mapsto \sum n_i (Y_i \cap U),$$

where we ignore those $Y_i \cap U$ which are empty;

(b) if $\mathrm{codim}(Z, X) \geq 2$, then $\mathrm{Cl}X \to \mathrm{Cl}U$ is an isomorphism;

(c) if Z is an irreducible subset of codimension 1, then there is an exact sequence

$$\mathbb{Z} \to \mathrm{Cl}X \to \mathrm{Cl}U \to 0$$

where the first map is defined by $1 \mapsto 1 \cdot Z$.

Proof.

(a) If Y is a prime divisor on X, then $Y \cap U$ is either empty or a prime divisor on U. If $f \in K^\times$, and $(f) = \sum n_i Y_i$, then considering f as a rational function on U, we have $(f)_U = \sum n_i (Y_i \cap U)$, so indeed we have a homomorphism $\mathrm{Cl}X \to \mathrm{Cl}U$. It is surjective because every prime divisor of U is the restriction of its closure in X.

(b) The groups $\mathrm{Div}X$ and $\mathrm{Cl}X$ depend only on subsets of codimension 1, so removing a closed subset Z of codimension ≥ 2 doesn't change anything.

(c) The kernel of $\mathrm{Cl}X \to \mathrm{Cl}U$ consists of divisors whose support is contained in Z. If Z is irreducible, the kernel is just the subgroup of $\mathrm{Cl}X$ generated by $1 \cdot Z$.

$\square$

Example 6.5.1. Let Y be an irreducible curve of degree d in $\mathbb{P}^2_k$. Then $\mathrm{Cl}(\mathbb{P}^2 \setminus Y) = \mathbb{Z}/d\mathbb{Z}$, immediately from (6.4) and (6.5) (mainly 6.4a and 6.5c).

Example 6.5.2. Let $A = k[x,y,z]/(xy - z^2)$ and let X be the affine cone Spec$A$. We will show Cl$X = \mathbb{Z}/2\mathbb{Z}$, and that it is generated by a ruling of the cone, say

$$Y = \mathrm{Spec} k[x,y,z]/(xy - z^2, y, z) \cong \mathrm{Spec} k[x].$$

First note that Y is a prime divisor (integral, codimension 1), so by 6.5 we have an exact sequence

$$\mathbb{Z} \to \mathrm{Cl}X \to \mathrm{Cl}(X \setminus Y) \to 0,$$

where the first map sends $1 \mapsto 1 \cdot Y$. Now Y can be cut out set-theoretically by the function y. In fact, the divisor of y is $(y) = 2 \cdot Y$. This is because $y = 0$ implies $z^2 = 0$ ~~and z generates the maximal ideal of the local ring at the generic point of Y. So $y \in (z)^2$ in $\mathcal{O}_{(z),X}$ ((z) is the generic point of Y)~~ and thus $v_Y(y) = 2$, so $(y) = 2Y$. I think the generic point of Y is (y,z), so the localization is $\Gamma(X)_{(y,z)}$, and $y \in (y,z)^2$ since $xy = z^2$ and so $y = x^{-1}z^2 \in (y,z)^2$ (we get x^{-1} from the localization) and thus $v_Y(y) = 2$. Also note (y,z) is prime in A whereas my other candidates for the generic point ((z) and $(y,z^2) = (y)$) are not.

Hence $X \setminus Y = \mathfrak{D}(y) \cong \mathrm{Spec}A_y$. Primes that don't contain y. I think we don't mention z because $\mathfrak{V}(x,z)$ is another divisor of X distinct from Y, so we can still have primes that contain z. Now $A_y = k[x,y,y^{-1},z]/(xy - z^2)$. In this ring, $x = y^{-1}z^2$, so we can eliminate x, and find $A_y \cong k[y,y^{-1},z]$. This is a UFD, so by 6.2, we have $\mathrm{Cl}(X \setminus Y) = 0$.

Thus we see that ClX is generated by Y, and that $2Y = 0$. It remains to show that Y itself is not a principal divisor. Since A is integrally closed (Ex. 6.4), it is equivalent to show that the prime ideal of Y, namely $\mathfrak{p} = (y,z)$, is not principal. Let $\mathfrak{m} = (x,y,z)$, and note that $\mathfrak{m}/\mathfrak{m}^2$ is a 3-dimensional vector space over k generated by $\overline{x}, \overline{y}, \overline{z}$, the images of x,y,z. Now $\mathfrak{p} \subseteq \mathfrak{m}$, and the image of $\mathfrak{p}$ in $\mathfrak{m}/\mathfrak{m}^2$ contains $\overline{y}$ and $\overline{z}$. Hence $\mathfrak{p}$ cannot be a principal ideal.

Proposition 6.6. Let X be stellar. Then $X \times \mathbb{A}^1 (= X \times_{\mathrm{Spec}\mathbb{Z}} \mathrm{Spec}\mathbb{Z}[t])$

$$(\text{since } \Gamma(X, \mathcal{O}_X) \otimes_{\mathbb{Z}} \mathbb{Z}[t] \cong \Gamma(X, \mathcal{O}_X)[t])$$

and so $X \times \mathbb{A}^1 = X \times_{\mathrm{Spec}\mathbb{Z}} \mathrm{Spec}\mathbb{Z}[t] = \mathrm{Spec}(\Gamma(X,\mathcal{O}_X) \otimes_{\mathbb{Z}} \mathbb{Z}[t]) = \mathrm{Spec}\Gamma(X,\mathcal{O}_X)[t]$ is also stellar, and $\mathrm{Cl}X \cong \mathrm{Cl}(X \times \mathbb{A}^1)$.

Proof. "Clearly" $X \times \mathbb{A}^1$ is noetherian, integral and separated. To see that it is regular in codimension one, we note that there are two kinds of points of codimension one on $X \times \mathbb{A}^1$. The first type is a point x whose image in X is a point y of codimension 1. In this case x is the generic point of $\pi^{-1}(y)$, where $\pi : X \times \mathbb{A}^1 \to X$ is the projection. Its local ring is $\mathcal{O}_x \cong \mathcal{O}_y[t]_{\mathfrak{m}_y}$ which is a DVR since $\mathcal{O}_y$ is. The corresponding prime divisor $\mathrm{cl}_{X \times \mathbb{A}^1}(\{x\})$ is just $\pi^{-1}(\mathrm{cl}_X(\{y\}))$.

The second type is a point $x \in X \times \mathbb{A}^1$ of codemension 1, whose image in X is the generic point of X. In this case $\mathcal{O}_x$ is a localization of $K[t]$ at some maximal ideal, where K is the function field of X. It is a DVR since $K[t]$ is a PID. Thus $X \times \mathbb{A}^1$ is stellar.

We define a map $\mathrm{Cl}X \to \mathrm{Cl}(X \times \mathbb{A}^1)$ by $D = \sum n_i Y_i \mapsto \pi^* D = \sum n_i \pi^{-1}(Y_i)$. If $f \in K^\times$, then $\pi^*((f))$ is the divisor of f considered as an element of $K(t)$, the function field of $X \times \mathbb{A}^1$. Thus we have a homomorphism $\pi^* : \mathrm{Cl}X \to \mathrm{Cl}(X \times \mathbb{A}^1)$.

To show that π^* is injective, suppose $D \in \mathrm{Div} X$, and $\pi^* D = (f)$ for some $f \in K(t)$. Since $\pi^* D$ involves only prime divisors of type 1, f must be in K. For otherwise we could write $f = g/h$, with $g, h \in K[t]$ relatively prime. If g, h are not both in K, then (f) will involve some prime divisor of type 2 on $X \times \mathbb{A}^1$. Now if $f \in K$, then $D = (f)$ so π^* is injective.

To show π^* is surjective, show that any prime divisor of type 2 on $X \times \mathbb{A}^1$ is linearly equivalent to a linear combination of prime divisors of type 1. So let $Z \subseteq X \times \mathbb{A}^1$ be a prime divisor of type 2. Localizing at the generic point of X, we get a prime divisor in $\mathrm{Spec} K[t]$, which corresponds to a prime ideal $\mathfrak{p} \subseteq K[t]$. This is principal, so let f be a generator. Then $f \in K(t)$, and the divisor of f consists of Z plus perhaps something purely of type 1. It cannot have any other prime divisors of type 2. Thus Z is linearly equivalent to a divisor purely of type 1. This completes the proof. □

Corollary 6.10. A principal divisor on a complete nonsingular curve X has degree zero. Consequently the degree function induces a surjective homomorphism $\deg : \mathrm{Cl} X \to \mathbb{Z}$.

Proof. Let $f \in K(X)^\times$. If $f \in k$, then $(f) = 0$ so there is nothing to prove. If $f \notin k$, then the inclusion of fields $k(f) \subseteq K(X)$ induces a finite morphism $\varphi : X \to \mathbb{P}^1$. It is a morphism by (I,6.12) and it is finite by (6.8). Now $(f) = \varphi^*(\{0\} - \{\infty\})$. Since $\{0\} - \{\infty\}$ is a divisor of degree 0 on X (since $\deg(\{0\} - \{\infty\}) = \deg\{0\} - \deg\{\infty\} = 1 - 1 = 0$), we conclude that (f) has degree 0 on X.

This is tricky to see. As an example, let $X = \mathrm{Proj} k[x, y, z]/(yz - x^2)$, then $K(X) = k[x, y, z]/(yz - x^2)_{((0))}$. Let us choose $f = \frac{x-z}{y} \in K(X)$. Note that f is zero at $[1 : 1 : 1]$ and $[0 : 1 : 0]$ and has a pole at $[0 : 0 : 1]$ (with multiplicity 2). Also note that f induces a morphism $\varphi_f : X \to \mathbb{P}^1$ defined by $\varphi_f([a : b : c]) = [a - c : b]$. This further induces a group homomorphism $\varphi_f^* : \mathrm{Cl}\mathbb{P}^1 \to \mathrm{Cl} X$. Then

$$\begin{aligned} \varphi_f^*(\{[0:1]\} - \{[1:0]\}) &= \varphi_f^{-1}(\{[0:1]\}) - \varphi_f^{-1}(\{[1:0]\}) \\ &= \{[1:1:1], [0:1:0]\} - \{[0:0:1]\} = (f). \end{aligned}$$

Is it $[1 : 1 : 1] + [0 : 1 : 0] - 2[0 : 0 : 1]$? Either way, since $0 - \infty$ is a divisor of degree zero on $\mathbb{P}^1$, we know from (6.9) that $\varphi_f^*(0 - \infty) = (f)$ has degree 0 on X.

Thus the degree of a divisor on X depends only on its linear equivalence class, and we obtain a homomorphism $\mathrm{Cl} X \to \mathbb{Z}$ as stated. It is surjective because the degree of a single point is 1. □

Example 6.10.1. A complete nonsingular curve X is rational if and only if there exist two distinct points $P, Q \in X$ with $P \sim Q$. Recall that *rational* means birational to $\mathbb{P}^1$. If X is rational, then it is infact isomorphic to $\mathbb{P}^1$ by (6.7). And on $\mathbb{P}^1$ we have already seen that any two points are linearly equivalent (6.4). Conversely, suppose X has two points $P \neq Q$ with $P \sim Q$. Then there is a rational function $f \in K(X)$ with $(f) = P - Q$. Consider the morphism $\varphi_f : X \to \mathbb{P}^1$ determined by f as in the proof of (6.10). We have $\varphi_f^*(\{0\}) = P$ so φ_f must be a morphism of degree 1. In other words, φ_f is birational, so X is rational.

Remark 6.10.3. This example of the cubic curve (as in the group law of a projective elliptic curve E, which is $\mathrm{Cl}^\circ(E) = \ker(\mathrm{Cl}E \twoheadrightarrow \mathbb{Z})$) illustrates the general fact that the divisor class group of a variety a has a discrete component (in this case $\mathbb{Z}$) and a continuous component (in this case $\mathrm{Cl}^\circ X$) which itself has the structure of an algebraic variety.

More specifically, if X is any complete nonsingular curve, then the group $\mathrm{Cl}^\circ X$ is isomorphic to the group of closed points of an abelian variety called the *Jacobian variety* of X. An *abelian variety* is a complete group variety over k. The dimension of the Jacobian variety is the *genus* of the curve. Thus the whole divisor class group of X is an extension of $\mathbb{Z}$ by the group of closed points of the Jacobian variety of X.

If X is a nonsingular projective variety of dimension ≥ 2, then one can define a subgroup $\mathrm{Cl}^\circ X$ of $\mathrm{Cl}X$, namely the subgroup of divisor classes *algebraically equivalent* to zero, such that $\mathrm{Cl}X/\mathrm{Cl}^\circ X$ is a finitely generated abelian group, called the *Néron-Severi group* of X, and $\mathrm{Cl}^\circ X$ is isomorphic to the group of closed points of an abelian variety called the *Picard variety* of X.

More resources available in Lang, Mumford.

Definition: Let X be a scheme (not necessarily stellar anymore). For each open affine subset $U = \mathrm{Spec}A$, let S be the set of elements of A which are not zero divisors, and let $K(U)$ be the localization of A by the multiplicative set S. We call $K(U)$ the *total quotient ring* of A. For each open set U, let $S(U)$ denote the elements of $\Gamma(U, \mathcal{O}_X)$ which are not zero divisors in each local ring $\mathcal{O}_x$ for $x \in U$. Then the rings $S(U)^{-1}\Gamma(U, \mathcal{O}_X)$ form a presheaf, whose associated sheaf of rings $\mathcal{K}$ we call the *sheaf of total quotients* of $\mathcal{O}$. On an arbitrary scheme, $\mathcal{K}$ replaces the concept of a function field for an integral scheme. We denote with $\mathcal{K}^\times$ the sheaf (of multiplicative groups) of invertible elements in the sheaf of rings $\mathcal{K}$. So $\mathcal{K}^\times(U) = \mathcal{K}(U)^\times$. Similarly $\mathcal{O}^\times$ is the sheaf of invertible elements in $\mathcal{O}$.
Definition: A *Cartier divisor* on a scheme X is a global section of the sheaf $\mathcal{K}^\times/\mathcal{O}^\times$. Thinking of the properties of quotient sheaves, we see that a Cartier divisor on X can be described by giving an open cover $\{U_i\}$ if X, and for each i an element $f_i \in \Gamma(U_i, \mathcal{K}^\times)$, such that for each i, j, $f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^\times)$. A Cartier divisor is *principal* if it is in the image of the natural map $\Gamma(X, \mathcal{K}^\times) \to \Gamma(X, \mathcal{K}^\times/\mathcal{O}^\times)$. Two Cartier divisors are *linearly equivalent* if their difference is principal. (Although the group operation $\mathcal{K}^\times/\mathcal{O}^\times$ is multiplication, we will use the language of additive groups when speaking of Cartier divisors, so as to preserve the analogy with Weil divisors).

Example: Let $X = \mathbb{A}^2_k$. Then $\mathcal{K}^\times(X) = k(x,y)^\times = k(x,y) \setminus 0$, and $\mathcal{O}^\times(X) = k[x,y]^\times = k^\times = k \setminus 0$. Note that $\Gamma(X, \mathcal{K}^\times/\mathcal{O}^\times)$ is not $k(x,y)^\times/k^\times$ because we need to sheafify. We have $\Gamma(X, \mathcal{K}^\times/\mathcal{O}^\times) = \{(U_i, f_i) : f_i \in \Gamma(U_i, \mathcal{K}^\times), f_i/f_j \in \Gamma(U_i \cap U_j, \mathcal{O}^\times)\}$. Note that since X is irreducible (infact it is integral), we have that $\Gamma(U, \mathcal{K}^\times) = \Gamma(X, \mathcal{K}^\times)$ for all open $U \subseteq X$. Then $f_i \in \Gamma(U_i, \mathcal{K}^\times)$ if and only if $f_i \in \Gamma(X, \mathcal{K}^\times) = k(x,y)^\times$. Then the divisor (U_i, f_i) is principal, since it is in the image of the map $\Gamma(X, \mathcal{K}^\times) \to \Gamma(X, \mathcal{K}^\times/\mathcal{O}^\times)$ $(f_i \mapsto (U_i, f_i))$.

Proposition 6.11. Let X be an integral separated noetherian scheme, all of whose local rings are unique factorization domains (in which case we say X is *locally factorial*). Then the group $\mathrm{Div}X$ of Weil divisors on X is isomorphic to the group of Cartier divisors $\Gamma(X, \mathcal{K}^\times/\mathcal{O}^\times)$, and furthermore, the principal Weil divisors correspond to the principal Cartier divisors under this isomorphism.

For a Cartier divisor $\{(U_i, f_i)\}$, where $f_i \in \Gamma(U_i, \mathcal{K}^\times)$ and $\{U_i\}$ is an open cover of X. For each prime divisor Y, take the coefficient $v_Y(f_i)$, where i is any index for which $Y \cap U_i \neq \emptyset$. If j is another such index, then f_i/f_j is invertible on $U_i \cap U_j$ and so $v_Y(f_i/f_j) = 0$ and thus $v_Y(f_i) = v_Y(f_j)$. Thus we obtain a well-defined Weil divisor $\sum_Y v_Y(f_i)Y$ on X. This is finite since X is noetherian.

Example: Let $X = \mathbb{P}^1_k = k[x_0, x_1]$. Then $H = \{[1 : 0]\}$. Then H corresponds to the pair of pairs $(\mathfrak{D}(x_0), x_1/x_0)$ and $(\mathfrak{D}(x_1), 1)$. Note that $x_1/x_0 \in \mathcal{O}(\mathfrak{D}(x_0) \cap \mathfrak{D}(x_1)) = \mathcal{O}(\mathfrak{D}(x_0x_1)) = \mathcal{O}(\text{Spec}k[x_0, x_1]_{x_0x_1}) = k[x_0, x_1]_{x_0x_1}$ since $x_1^2, 1/(x_0x_1) \in k[x_0, x_1]_{x_0x_1}$ and $x_1/x_0 = (x_1)^2/(x_0x_1)$. For any prime divisor Y of $\mathfrak{D}(x_0)$, we have $v_Y(x_1/x_0) = 0$ for all Y but $Y = H$, in which case $v_H(x_1/x_0) = 1$. Note that $v_{\{[0:1]\}}(x_1/x_0) = -1$ but we don't count it since $\{[0 : 1]\} \cap \mathfrak{D}(x_0) = \emptyset$. Now note that $v_Y(1) = 0$ for all prime divisors Y (also note in this case we only count prime divisors Y such that $Y \cap \mathfrak{D}(x_1) \neq \emptyset$). Thus we get back our original prime divisor $v_H(x_0/x_1) \cdot H = H$.

The group of Cartier Divisors modulo the principal Cartier divisors of a scheme X is called CaClX and the kernel of the map CaCl$X \to \mathbb{Z}$ is CaCl$^\circ X$.

An *invertible sheaf* on a ringed space X is defined to be a locally free $\mathcal{O}_X$-module of rank 1. We will see that invertible sheaves on a scheme are closely related to divisor classes modulo linear equivalence.

Proposition 6.12. If $\mathcal{L}$ and $\mathcal{M}$ are invertible sheaves on a ringed space X, so is $\mathcal{L} \otimes \mathcal{M}$. Recall that $\mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{M}$ is a sheaf that sends an open set U of X to $\mathcal{L}(U) \otimes_{\Gamma(X,\mathcal{O}_X)} \mathcal{M}(U)$.If $\mathcal{L}$ is any invertible sheaf on X, then there exists an invertible sheaf $\mathcal{L}^{-1}$ on X such that $\mathcal{L} \otimes \mathcal{L}^{-1} \cong \mathcal{O}_X$.

Proof. The first statement is clear, since $\mathcal{L}$ and $\mathcal{M}$ are both locally free of rank 1, and $\mathcal{O}_X \otimes \mathcal{O}_X \cong \mathcal{O}_X$. For the second statement, let be any invertible sheaf, and take $\mathcal{L}^{-1}$ to be the dual sheaf $\mathcal{L}^\vee = \mathcal{H}om(\mathcal{L}, \mathcal{O}_X)$. Then $\mathcal{L}^\vee \otimes_{\mathcal{O}_X} \mathcal{L} \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{L}) = \mathcal{O}_X$ by (Ex 5.1). □

Definition: For any ringed space X, we define the *Picard group* of X, Pic$X$, to be the group of isomorphism classes of invertible sheaves on X, under the operation $\otimes$. The proposition shows this is infact a group.

Definition: Let D be a Cartier divisor on a scheme X, represented by $\{(U_i, f_i)\}$. We define the subsheaf $\mathcal{L}(D)$ of the sheaf of total quotient rings $\mathcal{K}$ by taking $\mathcal{L}(D)$ to be the sub-$\mathcal{O}_X$-module of $\mathcal{K}$ generated by f_i^{-1} on U_i. This is well-defined, since f_i/f_j is invertible on $U_i \cap U_j$, so f_i^{-1} and f_j^{-1} generate the same $\mathcal{O}_X$-module. We call $\mathcal{L}(D)$ the *sheaf associated* to D.

Proposition 6.13. Let X be a scheme. Then:

(a) for any Cartier divisor D, $\mathcal{L}(D)$ is an invertible sheaf on X. The map $D \mapsto \mathcal{L}(D)$ gives a 1-1 correspondence between Cartier divisors on X and invertible subsheaves of $\mathcal{K}$;

(b) $\mathcal{L}(D_1 - D_2) \cong \mathcal{L}(D_1) \otimes \mathcal{L}(D_2)^{-1}$;

(c) $D_1 \sim D_2$ if and only if $\mathcal{L}(D_1) \cong \mathcal{L}(D_2)$ as abstract invertible sheaves (i.e., disregarding the embedding in $\mathcal{K}$).

Corollary 6.14. On any scheme X, the map $D \mapsto \mathcal{L}(D)$ gives an injective homomorphism of the group $\mathrm{CaCl}X$ of Cartier divisors modulo linear equivalence to $\mathrm{Pic}X$.

Definition: Let S be a graded ring, and let $X = \mathrm{Proj}S$. For any $n \in \mathbb{Z}$, we define the sheaf $\mathcal{O}_X(n)$ to be $S(n)^{\sim}$. We call $\mathcal{O}_X(1)$ the *twisting sheaf* of Serre. For any sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$, we denote by $\mathcal{F}(n)$ the *twisted sheaf* $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{O}_X(n)$. Furthermore $\mathcal{O}(n)$ is an invertible sheaf for all $n \in \mathbb{Z}$.

Definition: A *quasi-coherent* sheaf on a ringed space $(X, \mathcal{O}_X)$ is a sheaf of $\mathcal{O}_X$-modules $\mathcal{F}$ such that for each point in X there is an open set U of X there is an exact sequence

$$\mathcal{O}_X^{\oplus I}|_U \to \mathcal{O}_X^{\oplus J}|_U \to \mathcal{F} \to 0$$

for some (possibly infinite) indexing sets I and J.

A *coherent sheaf* on $(X, \mathcal{O}_X)$ is a quasi-coherent sheaf $\mathcal{F}$ satisfying the following two properties:

1. $\mathcal{F}$ is of finite type over $\mathcal{O}_X$: for every point in X there is an open set U of X and an $n \in \mathbb{N}$ such that $\mathcal{O}_X^n|_U \to \mathcal{F}|_U$ is surjective.

2. For any $U \subseteq X$, any $n \in \mathbb{N}$ and morphism $\varphi : \mathcal{O}_X^n|_U \to \mathcal{F}_U$ of $\mathcal{O}_X$-modules, $\ker \varphi$ is of finite type.

Theorem 7.1. Let A be a ring, and let X be a scheme over A.

(a) If $\varphi : X \to \mathbb{P}^n_A$ is an A-morphism, then $\varphi^*(\mathcal{O}(1))$ is an invertible sheaf on X, which is generated by the global sections $s_i = \varphi^*(x_i)$, $i = 0, 1, \ldots, n$.

(b) Conversely, if $\mathcal{L}$ is an invertible sheaf on X, and if $s_0, \ldots, s_n \in \Gamma(X, \mathcal{L})$ are global sections which generate $\mathcal{L}$, then there exists a unique A-morphism $\varphi : X \to \mathbb{P}^n_A$ such that $\mathcal{L} \cong \varphi^*(\mathcal{O}(1))$ and $s_i = \varphi^*(x_i)$ under this isomorphism.

Definition: An invertible sheaf $\mathcal{L}$ on a noetherian scheme X is said to be *ample* if for every sheaf $\mathcal{F}$ on X, there is an integer $n_0 > 0$ (depending on $\mathcal{F}$) such that for every $n \geq n_0$, the sheaf $\mathcal{F} \otimes \mathcal{L}^n$ is generated by its global sections. (Here $\mathcal{L}^n = \mathcal{L}^{\otimes n}$ denotes the n-fold tensor power of $\mathcal{L}$ with itself.)

A sheaf on X is *very ample relative to Y* (where X is a scheme over Y) if there is an immersion $i : X \to \mathbb{P}^n_Y$ for some n such that $\mathcal{L} \cong i^*\mathcal{O}(1)$.

Definition: Let A be a ring, B be an A-algebra, and M be a B-module. An A-derivation of B into M is a map $d : B \to M$ such that (1) d is additive, (2) $d(bb') = bdb' + b'db$, and (3) $da = 0$ for all $a \in A$.

Definition: The *module of relative differential forms* of B over A is a B-module $\Omega_{B/A}$, together with an A-derivation $d : B \to \Omega_{B/A}$ which satisfies the following universal property: for any B-module M, and for any A-derivation $d' : B \to M$, there exists a unique B-module homomorphism $f : \Omega_{B/A} \to M$ such that $d' = f \circ d$.

Definition: A *sheaf of ideals* $\mathcal{I}$ of a scheme X is an $\mathcal{O}_X$-module that is a subsheaf of $\mathcal{O}_X$. That is $\mathcal{I}(U)$ is an ideal of $\mathcal{O}_X(U)$ for all open sets U of X.

Let $f : X \to Y$ be a morphism of schemes. We consider the diagonal morphism $\Delta : X \to X \times_Y X$. It follows from the proof of (4.2) that Δ gives an isomorphism of X onto its image $\Delta(X)$ which is a *locally closed* subscheme of $X \times_Y X$, i.e., a closed subscheme of an open subset W of $X \times_Y X$.

Definition: Let $\mathcal{I}$ be the sheaf of ideals of $\Delta(X)$ in W. Then we define the *sheaf of relative differentials* of X over Y to be the sheaf $\Omega_{X/Y} = \Delta^*(\mathcal{I}/\mathcal{I}^2)$ on X.

Let A be a ring and let M be an A-module. Let $T^n(M) = \bigotimes_{n\geq 1} M$. For $n = 1$, have $T^0(M) = A$. Then $T(M) = \bigoplus_{n\geq 0} T^n(M)$ is a (noncommutative) A-algebra, called the *tensor algebra* of M. We define the *symmetric algebra* $S(M) = \bigoplus_{n\geq 0} S^n(M)$ of M to be the quotient of $T(M)$ by the two-sided ideal generated by all expressions $x \otimes y - y \otimes x$, for all $x, y \in M$. Then $S(M)$ is a commutative A-algebra. We call $S^n(M)$ the n^{th} symmetric product of M. In other words, $S(M)$ is the "largest" commutative A-algebra that $T(M)$ will surject onto.

We define the *exterior algebra* $\bigwedge M = \bigoplus_{n\geq 0} \bigwedge^n M$ of M to be the quotient of $T(M)$ by the two sided ideal generated by all expressions $x \otimes x$ for $x \in M$. Note that this ideal contains all expressions of the form $x \otimes y + y \otimes x$, so that $\bigwedge M$ is a *skew commutative* graded A-algebra. This means that if $u \in \bigwedge^r M$ and $v \in \bigwedge^s M$, then $u \wedge v = (-1)^{rs} v \wedge u$ (here we denote by $\wedge$ the multiplication in this algebra; so the image of $x \otimes y$ in $\bigwedge^2 M$ is denoted by $x \wedge y$). The n^{th} component $\bigwedge^n M$ is called the n^{th} *exterior power* of M.

Tensor algebras, *symmetric algebras*, and *exterior algebras* also apply to sheaves of modules as well.