Consider for example the field $\mathbb{F}_4 = \mathbb{F}_2[\alpha]$ where $\alpha^2 = \alpha + 1$. Then the polynomial $x^3 + \alpha \in \mathbb{F}_4[x]$ is irreducible. Then let $\mathbb{F}_{64} = \mathbb{F}_4[\beta]$ where $\beta^3 = \alpha$.

Then we can consider the plane $[A, B]$ with $A, B \in \mathbb{F}_{64}$. We can write $A = a_0 + a_1\beta + a_2\beta^2$ and $B = b_0 + b_1\beta + b_2\beta^2$ for $a_i, b_i \in \mathbb{F}_4$.

The plane $[A, B]$ is the set $\{(x, Ax + Bx^4) : x \in \mathbb{F}_{64}\}$. That is, we can choose $x = 1, \beta, \beta^2$ to identify three points contained in the plane. For $x = 1$, we get $(1, A + B) = (1, 0, 0, a_0 + b_0, a_1 + b_1, a_2 + b_2)$. For $x = \beta$, we get

$$Ax + Bx^4 = A\beta + B\beta\alpha = a_0\beta + a_1\beta^2 + a_2\alpha + b_0\beta\alpha + b_1\beta^2\alpha + b_2\alpha + b_2.$$

So $(\beta, A\beta + B\beta^4)$ corresponds to the point $(0, 1, 0, a_2\alpha + b_2\alpha + b_2, a_0 + b_0\alpha, a_1 + b_1\alpha)$.

Finally, for $x = \beta^2$, we have $A\beta^2 + B\beta^8 = A\beta^2 + B\beta^2(\alpha + 1)$.

$$A\beta^2 + B\beta^2(\alpha + 1) = a_0\beta^2 + a_1\alpha + a_2\beta\alpha + b_0\beta^2(\alpha + 1) + b_1 + b_2\beta$$

and so $(\beta^2, A\beta^2 + B\beta^8)$ corresponds to the point $(0, 0, 1, a_1\alpha + b_1, a_2\alpha + b_2, a_0 + b_0\alpha + b_0)$. So

$$\begin{aligned}[A, B] = \operatorname{join}\{&(1, 0, 0, a_0 + b_0, a_1 + b_1, a_2 + b_2),\\ &(0, 1, 0, a_2\alpha + b_2\alpha + b_2, a_0 + b_0\alpha, a_1 + b_1\alpha),\\ &(0, 0, 1, a_1\alpha + b_1, a_2\alpha + b_2, a_0 + b_0\alpha + b_0)\}.\end{aligned}$$

In particular, the plane

$$\begin{aligned}[A, 0] = \operatorname{join}\{&(1, 0, 0, a_0, a_1, a_2),\\ &(0, 1, 0, a_2\alpha, a_0, a_1),\\ &(0, 0, 1, a_1\alpha, a_2\alpha, a_0)\}.\end{aligned}$$

Also define the plane $\Pi_\infty = \operatorname{join}\{(0, 0, 0, 1, 0, 0), (0, 0, 0, 0, 1, 0), (0, 0, 0, 0, 0, 1)\}$.

I claim that $\mathcal{S} = \{[A, 0] : A \in \mathbb{F}_{64}\} \cup \{\Pi_\infty\}$ is a spread of $\mathbb{P}^5_{\mathbb{F}_4}$. First note that $\#\mathbb{P}^5_{\mathbb{F}_4} = \dfrac{4^6 - 1}{4 - 1} = 1365$. And $\#\mathbb{P}^2_{\mathbb{F}_4} = 21$. And $1365/21 = \#\mathcal{S}$.

$$\begin{vmatrix} 1 & 0 & 0 & a_0 & a_1 & a_2 \\ 0 & 1 & 0 & \alpha a_2 & a_0 & a_1 \\ 0 & 0 & 1 & \alpha a_1 & \alpha a_2 & a_0 \\ 1 & 0 & 0 & b_0 & b_1 & b_2 \\ 0 & 1 & 0 & \alpha b_2 & b_0 & b_1 \\ 0 & 0 & 1 & \alpha b_1 & \alpha b_2 & b_0 \end{vmatrix} = (a_0 + b_0)^3 + \alpha(a_1 + b_1)^3 + \alpha^2(a_2 + b_2)^3 + \alpha(a_0 + b_0)(a_1 + b_1)(a_2 + b_2),$$

which we can rewrite as

$$X^3 + \alpha Y^3 + \alpha^2 Z^3 + \alpha XYZ \in \mathbb{F}_4[X, Y, Z].$$

We want to show the only root in $\mathbb{F}_4$ is $(X, Y, Z) = (0, 0, 0)$. Suppose $X \neq 0$. Then $X^3 = 1$. So we have $1 + \alpha Y^3 + \alpha^2 Z^3 + \alpha XYZ$.

For the case $\mathbb{F}_2$, we can construct a spread via the field extension $\mathbb{F}_8 \cong \mathbb{F}_2/(x^3+x+1)$ and defining the planes

$$\begin{aligned}[A,B] = \mathrm{join}\{&(1,0,0,a_0+b_0,a_1+b_1,a_2+b_2),\\ &(0,1,0,a_2+b_1,a_0+a_2+b_1+b_2,a_1+b_0+b_2),\\ &(0,0,1,a_1+b_1+b_2,a_1+a_2+b_0+b_1,a_0+a_2+b_0+b_1+b_2)\}\end{aligned}$$

for $A,B \in \mathbb{F}_8$ and the spread comprising

$$\begin{aligned}[A,0] = \mathrm{join}\{&(1,0,0,a_0,a_1,a_2),\\ &(0,1,0,a_2,a_0+a_2,a_1),\\ &(0,0,1,a_1,a_1+a_2,a_0+a_2)\}.\end{aligned}$$

Do $[0,0]$, $[1,0]$, and Π_∞ form a "regulus" in $\mathbb{P}^5_{\mathbb{F}_2}$? We have $[0,0]=V(x_3,x_4,x_5)$, $[1,0]=V(x_0+x_3,x_1+x_4,x_2+x_5)$, and $\Pi_\infty = V(x_0,x_1,x_2)$. Is there some quadric 4-dimensional hypersurface containing all three planes?

The family

$$I = (x_0x_4+x_1x_3, x_0x_5+x_2x_3, x_1x_5+x_2x_4)$$

contains all three planes. According to Macaulay2, I is codimension 2 and degree 3, so it carves out a cubic threefold containing the planes.

I should further study the plane-bearing properties of cubic threefolds in $\mathbb{P}^5_k$, for example

$$V(x_0x_4-x_1x_3, x_0x_5-x_2x_3, x_1x_5-x_2x_4).$$

I believe this is isomorphic to the Segre embedding

$$\mathbb{P}^2\times\mathbb{P}^1 \hookrightarrow \mathbb{P}^5$$

where

$$(a_0,a_1,a_2),(b_0,b_1)\mapsto(a_0b_0,a_0b_1,a_1b_0,a_1b_1,a_2b_0,a_2b_1).$$

In this case, we have

$$V(x_0x_3-x_1x_2, x_0x_5-x_1x_4, x_2x_5-x_3x_4),$$

but this is isomorphic to the above variety via $\begin{pmatrix}1&0&0&0&0&0\\0&0&0&0&0&1\\0&0&1&0&0&0\\0&1&0&0&0&0\\0&0&0&0&1&0\\0&0&0&1&0&0\end{pmatrix}$.

We can look in the Grassmannian of planes in $\mathbb{P}^5$. $\mathfrak{Gr}(3,6)$ is 9-dimensional and embeds in $\mathbb{P}^{19}$ via Plücker. According to Macaulay2, the degree of the Plücker embedding of $\mathfrak{Gr}(3,6)\hookrightarrow\mathbb{P}^{19}$ is 42.

```
i1:  loadPackage "SchurRings"
```

```
i2:  S=schurRing(QQ,s,3)

i3:  s_1^9
```

The coefficient of the $s_{3,3,3}$ is 42. The $\mathbb{P}^2 \times \mathbb{P}^1$ is some kind of curve in $\mathfrak{Gr}(3,6)$. Is it a curve of degree 3?

Or maybe... we use the Chow rings. $A(\mathbb{P}^2) \cong \mathbb{Z}[L]/(L^3)$ and $A(\mathbb{P}^1) \cong \mathbb{Z}[P]/(P^2)$, and $A(\mathbb{P}^5) \cong \mathbb{Z}[H]/(H^6)$. So we have the function of rings

$$\mathbb{Z}[H]/(H^6) \to \mathbb{Z}[L]/(L^3) \otimes \mathbb{Z}[P]/(P^2)$$

where $A^0(\mathbb{P}^2 \times \mathbb{P}^1)$ is generated by $1 \otimes 1$, A^1 is generated by $L \otimes 1$ and $1 \otimes P$, A^2 is generated by $L^2 \otimes 1$ and $L \otimes P$, and A^3 is generated by $L^2 \otimes P$. Identifying $\ell = L \otimes 1$ and $p = 1 \otimes P$, we have $A(\mathbb{P}^2 \times \mathbb{P}^1) \cong \mathbb{Z}[\ell, p]/(\ell^3, p^2)$. Then for consistency let's write $h = H$ and we have

$$f : \mathbb{Z}[h]/(h^6) \to \mathbb{Z}[\ell, p]/(\ell^3, p^2)$$

where $h \mapsto a\ell + bp$, $h^2 \mapsto a^2\ell^2 + 2ab\ell p$, $h^3 \mapsto 3a^2b\ell^2p$, and $h^4 \mapsto 0$. We can find a and b through a few tests: $f(h)p = a\ell p$. The class p represents a general plane $\mathbb{P}^2 \times \{*\} \subseteq \mathbb{P}^2 \times \mathbb{P}^1$. Then ℓp is the class of a line contained in a specific section $L \times \{*\}$.

Maybeee... we can construct a specific h whose intersection with the Segre variety is a $\mathbb{P}^2 \times \{*\} \cup \mathbb{P}^1 \times \mathbb{P}^1$. Yeah, take the two classes $\mathbb{P}^2 \times \{*\}, \mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^2 \times \mathbb{P}^1$: they meet at the line $\mathbb{P}^1 \times \{*\}$. The $\mathbb{P}^1 \times \mathbb{P}^1$ is contained in a $\mathbb{P}^3$ which must necessarily contain the line $\mathbb{P}^1 \times \{*\}$. Then it is only a matter of including one of the points of $\mathbb{P}^2 \times \{*\}$ not shared by the $\mathbb{P}^1 \times \mathbb{P}^1$, and that extra point extends the $\mathbb{P}^3$ to a $\mathbb{P}^4$, which is the class h. So in fact $f(h) = \ell + p$, and so $h^3 = 3\ell^2 p$, so the degree of the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^1 \hookrightarrow \mathbb{P}^5$ is 3.

Similarly, the degree of the Segre embedding $\mathbb{P}^2 \times \mathbb{P}^2 \hookrightarrow \mathbb{P}^8$ is $(\ell_1 + \ell_2)^4 = 6\ell_1^2\ell_2^2$.

In general, the degree of the Segre embedding $\mathbb{P}^n \times \mathbb{P}^m \hookrightarrow \mathbb{P}^N$ is $\binom{n+m}{n}$?

Update: 4 March 2025 According to Hirschfeld, (affine) k-dimensional spreads exists in $\mathbb{F}_q^n$ iff $k|n$.

Following Gorla's presentation on spreads, we find another construction.

Let $n, k \in \mathbb{Z}_+$, $p = x^k + p_{k-1}x^{k-1} + \cdots + p_0 \in \mathbb{F}_q[x]$ irreducible. Let

$$P = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ \vdots & \ddots & & & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ -p_0 & -p_1 & \cdots & -p_{k-2} & -p_{k-1} \end{pmatrix} \in \mathrm{GL}_k(\mathbb{F}_q)$$

be the companion matrix of p. (That is, $p(P) = 0$ and $\chi_P(\lambda) = p(\lambda)$).

Then $\mathcal{S} = \{\mathrm{rowsp}\begin{pmatrix} 0_k & \cdots & 0_k & I_k & A_{i+1} & \cdots & A_{n/k} \end{pmatrix} : 1 \le i \le n/k, A_j \in \mathbb{F}_q[P]\}$ is a subset of $\mathfrak{Gr}(k, n)$ and a spread of $\mathbb{F}_q^n$, where 0_k is the $k \times k$ zero-matrix, I_k is the $k \times k$ identity.

Example 1. Let's test this against an easy example: $\mathbb{F}_2^2$. Let $p = x^2 + x + 1$ and let

$$P = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$$

be the companion matrix. Then $\mathbb{F}_2[P] = \{0, I_2, P, P + I_2\}$. Here $k = 2$ and $n = 4$. Then $\mathcal{S}$ comprises

$$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \end{pmatrix}$$

are five two-dimensional subspaces of $\mathbb{F}_2^4$ (lines in $\mathbb{P}_2^3$) that compose a spread.

The idea behind the construction is to pad out the left side with as many 0_k's as needed (possibly none) and then have exactly one I_k followed by whatever A_j's you've got. When $k = n/2$ (for $n \in 2\mathbb{Z}$) does this construction agree with the Hopf spread?

Gorla's paper mentions a similar but distinct contruction or partial spreads.

Example 16 from Gorla's paper: We construct a parital spread code of length 7 and dimension 2 over the binary field $\mathbb{F}_2$. Let $(q, k, n) := (2, 2, 7)$ and observe that $n \equiv 1 \mod k$. Hence we have $r = 1$. Take the irreducible monic polynomials $p := x^2 + x + 1$, $p' = x^3 + x + 1$

in $\mathbb{F}_2[x]$ of degree k and $k+r$, respectively. The conpansion matrices of p and p' are as follows:

$$P := M(p) = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix},\ P' := M(p') = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}.$$

The elements of $\mathcal{C}_2(2,7;p,p')$ are the rowspaces of all the matrices of the following forms:

$$\begin{pmatrix} 1 & 0 & \multirow{2}{*}{A_1} & \multirow{2}{*}{$A_{(2)}$} \\ 0 & 1 & & \end{pmatrix},\ \begin{pmatrix} 0 & 0 & 1 & 0 & \multirow{2}{*}{$B_{(2)}$} \\ 0 & 0 & 0 & 1 & \end{pmatrix},\ \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix},$$

where A_1 is any matrix in $\mathbb{F}_2[P]$ and $A_{(2)}$ and $B_{(2)}$ denote the last two rows of any $A, B \in \mathbb{F}_2[P']$. It can be checked that $\mathcal{C}_2(2,7;p,p')$ has $2^2 \cdot 2^3 + 2^3 + 1 = 41$ elements, which covers $41 * 3 = 123$ points of $\mathbb{P}^6_2$, whereas $\mathbb{P}^6_2$ has 127 points. This creates a partial spread of planes in the vector space $\mathbb{F}_2^7$, or a partial spread of lines in the projective space $\mathbb{P}^6_{\mathbb{F}_2}$. Note that $\mathbb{P}^6$ cannot have a complete spread because 6 is odd.

My own examples: since 6 is divisible by both 2 and 3, $\mathbb{F}_q^6$ contains both $k=2$- and $k=3$-spreads. That is, $\mathbb{P}^5$ contains spreads of $\mathbb{P}^1$'s and spreads of $\mathbb{P}^2$'s.

Example 2. We are already familiar with spreads of planes in $\mathbb{P}^5$. For this example let $q=2$. We can use a degree $k=3$-extension x^3+x+1. This has companion matrix $P = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. The matrix P happens to be a primitive root and so

$$\begin{aligned} \mathbb{F}_2[P] &= \{0, I_3, P, P^2, P^3, P^4, P^5, P^6\} \\ &= \left\{0, I_3, P, \begin{pmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}, \begin{pmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}, \begin{pmatrix} 1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right\}. \end{aligned}$$

Now we can cover $\mathbb{P}^5$ with 9 planes as follows:

$$\begin{pmatrix} 0 & I_3 \end{pmatrix}, \begin{pmatrix} I_3 & 0 \end{pmatrix}, \begin{pmatrix} I_3 & I_3 \end{pmatrix}, \begin{pmatrix} I_3 & P \end{pmatrix}, \begin{pmatrix} I_3 & P^2 \end{pmatrix}, \begin{pmatrix} I_3 & P^3 \end{pmatrix}, \begin{pmatrix} I_3 & P^4 \end{pmatrix}, \begin{pmatrix} I_3 & P^5 \end{pmatrix}, \begin{pmatrix} I_3 & P^6 \end{pmatrix}$$

where for example

$$\begin{pmatrix} I_3 & P^6 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \end{pmatrix}$$

is a plane containing the three points given by the rows.

Example 3. We can also cover $\mathbb{P}^5$ in lines by using a $k=2$-degree extension. For example, take the irreducible polynomial $x^2+x+1 \in \mathbb{F}_2[x]$. Then the companion matrix is $P = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$. Then we can form

$$\mathbb{F}_2[P] = \{0, I_2, P, P^2\} = \left\{0, I_2, P, \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\right\}.$$

Thus we can cover $\mathbb{P}^5$ in 21 lines of the form

$$\begin{gathered}
\begin{pmatrix}0 & 0 & I_2\end{pmatrix}, \begin{pmatrix}0 & I_2 & 0\end{pmatrix}, \begin{pmatrix}0 & I_2 & I_2\end{pmatrix}, \\
\begin{pmatrix}0 & I_2 & P\end{pmatrix}, \begin{pmatrix}0 & I_2 & P^2\end{pmatrix}, \begin{pmatrix}I_2 & 0 & 0\end{pmatrix}, \\
\begin{pmatrix}I_2 & 0 & I_2\end{pmatrix}, \begin{pmatrix}I_2 & 0 & P\end{pmatrix}, \begin{pmatrix}I_2 & 0 & P^2\end{pmatrix}, \\
\begin{pmatrix}I_2 & I_2 & 0\end{pmatrix}, \begin{pmatrix}I_2 & I_2 & I_2\end{pmatrix}, \begin{pmatrix}I_2 & I_2 & P\end{pmatrix}, \\
\begin{pmatrix}I_2 & I_2 & P^2\end{pmatrix}, \begin{pmatrix}I_2 & P & 0\end{pmatrix}, \begin{pmatrix}I_2 & P & I_2\end{pmatrix}, \\
\begin{pmatrix}I_2 & P & P\end{pmatrix}, \begin{pmatrix}I_2 & P & P^2\end{pmatrix}, \begin{pmatrix}I_2 & P^2 & 0\end{pmatrix}, \\
\begin{pmatrix}I_2 & P^2 & I_2\end{pmatrix}, \begin{pmatrix}I_2 & P^2 & P\end{pmatrix}, \begin{pmatrix}I_2 & P^2 & P^2\end{pmatrix},
\end{gathered}$$

where for example

$$\begin{pmatrix}I_2 & P^2 & P^2\end{pmatrix} = \begin{pmatrix}1 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0\end{pmatrix}$$

is the line containing the two points given by the rows.

Example 4. Since $n = 9$ is divisible by $k = 3$, we can create a spread of $\mathbb{P}^2$'s inside $\mathbb{P}^8$.

Let $q = 2$ again. Then a cubic extension of $\mathbb{F}_2$ is under the irreducible polynomial $x^3 + x + 1$. This has companion matrix $P = \begin{pmatrix}0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 0\end{pmatrix}$. The matrix P happens to be a primitive root and so

$$\begin{aligned}
\mathbb{F}_2[P] &= \{0, I_3, P, P^2, P^3, P^4, P^5, P^6\} \\
&= \left\{0, I_3, P, \begin{pmatrix}0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1\end{pmatrix}, \begin{pmatrix}1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1\end{pmatrix}, \begin{pmatrix}0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1\end{pmatrix}, \begin{pmatrix}1 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0\end{pmatrix}, \begin{pmatrix}1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0\end{pmatrix}\right\}.
\end{aligned}$$

Now we can cover $\mathbb{P}^8$ with 73 planes as follows:

$$\begin{gathered}
\begin{pmatrix}0 & 0 & I_3\end{pmatrix} \text{ (1 plane)}, \\
\begin{pmatrix}0 & I_3 & A \in \mathbb{F}_2[P]\end{pmatrix} \text{ (8 planes)}, \\
\begin{pmatrix}I_3 & B \in \mathbb{F}_2[P] & C \in \mathbb{F}_2[P]\end{pmatrix} \text{ (64 planes)},
\end{gathered}$$

where for example

$$\begin{pmatrix}I_3 & P^3 & P^4\end{pmatrix} = \begin{pmatrix}1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 & 1 & 0 & 1\end{pmatrix}$$

is the plane containing the three points given by the rows.

I wonder if replacing A with A^T yields a new spread or the same spread? Perhaps a similar deal with the regulus and its opposite.