ASYMPTOTIC RESURGENCES FOR IDEALS OF POINTS IN $\mathbb{P}^2$

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Abstract. We show that two ostensibly different asymptotic definitions for the resurgence introduced by Guardo, Harbourne and Van Tuyl in 2013 are the same, under certain conditions. We also ...

1. Introduction

Let k be a field and let $(0) \neq I \subsetneq k[\mathbb{P}^N]$ be a homogeneous ideal; thus $I = \oplus I_t$, where I_t is the k vector space span of all homogeneous polynomials of degree t in I. The resurgence $\rho(I)$ was introduced in [BH1] to study the containment problem, for symbolic powers $I^{(m)}$ of I compared to ordinary powers I^r. While $I^r \subseteq I^{(m)}$ holds if and only if $m \leq r$, determining for which m and r $I^{(m)} \subseteq I^r$ holds is much more subtle. It is known that $I^{(rN)} \subseteq I^r$ [ELS, HoHu]. The resurgence gives some notion of how small the ratio m/r can be and still be sure to have $I^{(m)} \subseteq I^r$; specifically,

$$\rho(I) = \sup\left\{\frac{m}{r} : I^{(m)} \not\subseteq I^r\right\}.$$

A case of particular interest is that of ideals of *fat points*. Given distinct points $p_1, \ldots, p_s \in \mathbb{P}^N$ and nonnegative integers m_i, let $Z = m_1p_1 + \cdots + m_sp_s$ denote the scheme defined by the ideal

$$I(Z) = \cap_{i=1}^s (I(p_i)^{m_i}) \subseteq k[\mathbb{P}^N],$$

where $I(p_i)$ is the ideal generated by all homogeneous polynomials vanishing at p_i. Then Z is called a scheme of fat points, and symbolic powers of I in this situation take the form $I(Z)^{(m)} = I(mZ) = \cap_{i=1}^s (I(p_i)^{mm_i})$.

Subsequently, [GHVT] introduced two asymptotic notions of the resurgence. The first is

$$\widehat{\rho}(I(Z)) = \sup\left\{\frac{m}{r} : I^{(mt)} \not\subseteq I^{rt}, t \gg 0\right\}.$$

The second is

$$\rho'(I(Z)) = \limsup_t \rho(I, t),$$

where $\rho(I, t) = \sup\left\{\frac{m}{r} : I^{(m)} \not\subseteq I^r, m \geq t, r \geq t\right\}$.

As [GHVT] notes, we have

$$\widehat{\rho}(I(Z)) \leq \rho'(I(Z)) \leq \rho(I(Z)).$$

Examples show that $\widehat{\rho}(I(Z)) < \rho(I(Z))$ can occur. For example, let $I = (x(y^n - z^n), y(z^n - x^n), z(x^n - y^n)) \subset \mathbb{C}[x, y, z] = \mathbb{C}[\mathbb{P}^2]$, so $I = I(Z)$ where Z is a certain set of $n^2 + 3$ points. Then by [DHNSST, Theorem 2.1], we have $\widehat{\rho}(I(Z)) = (n + 1)/n < 3/2 = \rho(I(Z))$.

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The paper [DFMS] reinterprets $\widehat{\rho}(I)$ in a way that provides in some cases, at least in principle, a computational determination of $\widehat{\rho}(I)$. In cases for which there is an m such that all powers of $I(mZ)$ are symbolic, [D] provides an algorithm for computing $\rho(I(Z))$ arbitrarily accurately. Together, it should thus in some cases be possible to verify by computation that $\widehat{\rho}(I(Z)) < \rho(I(Z))$. We also study the question of whether Denkert's approach can be used to compute $\widehat{\rho}(I(Z))$ arbitrarily accurately when there is an m such that all powers of $I(mZ)$ are symbolic.

However, no attention has been given to $\rho'(I(Z))$. In particular, no examples are known where either $\widehat{\rho}(I(Z)) < \rho'(I(Z))$ or $\rho'(I(Z)) < \rho(I(Z))$. Here we show that if there is an m such that all powers of $I(mZ)$ are symbolic, then $\widehat{\rho}(I(Z)) = \rho'(I(Z))$. As a corollary, we thus obtain examples with $\rho'(I(Z)) < \rho(I(Z))$. This also raises the question of whether $\widehat{\rho}(I(Z)) = \rho'(I(Z))$ is always true.

The main technique for knowing whether there is an m such that $I(mZ)$ are symbolic, and if so finding it, is [HaHu, Proposition 3.5]. When $Z = p_1 + \cdots + p_s$, so $I(Z)$ is a radical ideal, this result states that $I^(mZ)^t = I(mtZ)$ for all $t \geq 1$ if $\alpha_m(I(Z))\beta_m(I(Z)) = m^2 s$, where $\alpha_m(I(Z)) = \alpha(I(mZ)) = \min\{t : I(Z)_t \neq 0\}$ and $\beta_m(I(Z))$ is the least t such that the base locus of $I(Z)_t$ is zero dimensional.

This raises the second question of whether a version of this result holds for fat points Z; i.e., when $I(Z)$ is not radical. For $Z = m_1p_1 + \cdots + m_sp_s$, the example at the end of [BH2] suggests that it might be true that all powers of $I(mZ)$ are symbolic if $\alpha_m(I(Z))\beta_m(I(Z)) = m_1^2 + \cdots + m_s^2$.

Theorem 1.1. *Let Z be a fat point subscheme of $\mathbb{P}^N$ with an integer c such that $I(cZ)^t = I(ctZ)$ for all $t \geq 1$. Then $\widehat{\rho}(I(Z)) = \rho'(I(Z))$.*

Proof. Let b be a rational such that $\widehat{\rho}(I(Z)) < c/b$. Pick any integer $d > 0$ such that db is an integer. Since $\widehat{\rho}(I(Z)) < cd/(db) = c/b$, we have by definition of $\widehat{\rho}(I(Z))$ that $I(cdtZ) \subseteq I(Z)^{dbt}$ for $t \gg 0$, Note that $I(cdZ)^t = I(cZ)^{dt} = I(cdtZ)$ for all $t \geq 1$. Hence by [GHVT, Theorem 1.2(3)] we have $\rho'(I(Z)) \leq cdt/(dbt) = c/b$. Since this holds for all b with $c/b > \widehat{\rho}(I(Z))$, we have $\widehat{\rho}(I(Z)) \geq \rho'(I(Z))$. But by [GHVT, Theorem 1.2(1)] we have $\widehat{\rho}(I(Z)) \leq \rho'(I(Z))$, hence $\widehat{\rho}(I(Z)) = \rho'(I(Z))$. □

Example 1.2. Let $Z = p_1 + \cdots + p_s$ be general points of $\mathbb{P}^2$. Then $I(m_sZ)^t = I(m_stZ)$ for all $t \geq 1$ where $m_1 = 1$, $m_2 = 1$, $m_3 = 2$, $m_4 = 1$, $m_5 = 2$, $m_6 = 10$, $m_7 = 24$ and $m_8 = 102$. [WE SHOULD PUT IN DETAILS.]

Also, by [BH2], we have have $I(tZ) = I(Z)^t$ for $t \geq 1$ when $Z = (d-1)p_1 + p_2 + \cdots + p_s$ with $s = 2d$. Thus in each case we have $\widehat{\rho}(I(Z)) = \rho'(I(Z))$, but for these we also have $\rho'(I(Z)) = \rho(I(Z))$. Since we always have $1 \leq \widehat{\rho}(I(Z)) \leq \rho'(I(Z)) \leq \rho(I(Z))$ and since whenever all powers of $I(Z)$ are symbolic we have $\rho(I(Z)) = 1$, we know $\widehat{\rho}(I(Z)) = \rho'(I(Z)) = \rho(I(Z)) = 1$ for $Z = p_1$, $Z = p_1 + p_2$, $Z = p_1 + \cdots + p_4$ and $Z = (d-1)p_1 + p_2 + \cdots + p_{2d}$. This leaves the case of Z_s consisting of $s = 5, 6, 7, 8$ general points. But by [BH2] [WE SHOULD PUT IN MORE DETAILS] we have $\widehat{\rho}(I(Z_s)) = (s+1)/s = \rho(I(Z))$ for $s = 5, 7$, $\widehat{\rho}(I(Z_6)) = 5/4 = \rho(I(Z))$ and $\widehat{\rho}(I(Z_8)) = 17/12 = \rho(I(Z))$.

Example 1.3. For the ideal $I(Z) = (x(y^n - z^n), y(z^n - x^n), z(x^n - y^n))$ where by [DHNSST] we know $\rho(I(Z)) < \rho(I(Z))$, computations show $\alpha(mZ)\beta(mZ) = m^2|Z|$ for $n = m = 3$. Hence we have $\widehat{\rho}(I(Z)) = \rho'(I(Z)) < \rho(I(Z))$. We should put in more details for the $n = m = 3$ case and then l ook at $n > 3$ for this example, and look at the other example in [DHNSST] of Z with $\widehat{\rho}(I(Z)) < \rho(I(Z))$ to see if there is an m such that $I(mZ)^t = I(mtZ)$

for all $t > 0$. Then we would know that $\widehat{\rho}(I(Z)) = \rho'(I(Z)) < \rho(I(Z))$ for more than just one example.

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