Let V_4 be the plane $\mathbb{P}^2$ blown up at four general points. Then $\mathrm{Pic}(V_4) = \langle \ell, e_1, e_2, e_3, e_4 \rangle$. Then the (-2)-curves are generated by

$$\begin{aligned} \alpha_0 &= \ell - e_1 - e_2 - e_3, \\ \alpha_1 &= e_1 - e_2, \\ \alpha_2 &= e_2 - e_3, \\ \alpha_3 &= e_3 - e_4. \end{aligned}$$

Denote by $O(\mathrm{Pic}(V_4))$ the automorphisms of $\mathrm{Pic}(V_4)$ as a $\mathbb{Z}$-module that preserve inner product. Then

$$\Phi : \mathrm{Aut}(V_4) \to O(\mathrm{Pic}(V_4))$$

defined by $\Phi(\sigma) = (\sigma^{-1})^*$ is a group homomorphism.

For $1 \leq i \leq 4$, define

$$\begin{aligned} s_i : &\mathrm{Pic}(V_4) \to \mathrm{Pic}(V_4) \\ &x \mapsto x + (x.\alpha_i)\alpha_i. \end{aligned}$$

Then the Weyl group of V_4 is

$$W = \langle s_0, s_1, s_2, s_3 \rangle \leq O(\mathrm{Pic}(V_4)).$$

To prove this, let $x, y \in \mathrm{Pic}(V_4)$. Then

$$\begin{aligned} s_i(x).s_i(y) &= (x + (x.\alpha_i)\alpha_i).(y + (y.\alpha_i)\alpha_i) \\ &= x.y + (x.\alpha_i)(y.\alpha_i) + (x.\alpha_i)(y.\alpha_i) + (x.\alpha_i)(y.\alpha_i)\underbrace{(\alpha_i.\alpha_i)}_{-2} = x.y. \end{aligned}$$

As a side note, we can express each s_i as a matrix on $\mathbb{Z}^4 \cong \langle \alpha_0, \alpha_1, \alpha_2, \alpha_3 \rangle$ where

$$s_0 = \begin{pmatrix} -1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$s_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$s_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

$$s_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & -1 \end{pmatrix}.$$

The α_i's form a root system of A_4 given by using the α_i's as vertices and adjoining them with an edge if their intersection product is not zero. We have

$$\alpha_1 \text{ ——— } \alpha_2 \text{ ——— } \alpha_3 \text{ ——— } \alpha_0$$

as our A_4 Dynkin diagram. It is known that the A_n root system induces a Weyl group of S_{n+1} (Bourbaki, Groupes et Algèbres de Lie), so

$$W \cong S_5.$$

We will show that $\Phi : \mathrm{Aut}(V_4) \to W$ is an isomorphism.

Surjective: Let $\Phi(\sigma) = \mathrm{id}$. Then $\sigma(e_i) = e_i$ for all i. Hence $p \cdot \sigma \cdot p^{-1} \in \mathrm{PGL}(2)$ fixes each blowup point P_i (here p is the blowdown map $p : V_4 \to \mathbb{P}^2$). So by the four-point theorem, $p\sigma p^{-1} = \mathrm{id}$ so $\sigma = \mathrm{id}$.

Injective: Fix $w \in W$. Consider $w(e_i)$ for each i. Take a unique member $f_i \in |w(e_i)|$, so f_i is a curve on V_4 with self-intersection -1. Then $f_1, \ldots, f_4$ each have self-intersection -1 and intersection product 0 with each other.

Let q be the contraction morphism of $f_1, \ldots, f_4$, $q : V_4 \to \mathbb{P}^2$. Then let $Q_i = q(f_i) \in \mathbb{P}^2$, so the points $Q_1, \ldots, Q_4$ are in general position in $\mathbb{P}^2$. So we can find $\tau \in \mathrm{PGL}(2)$ such that $\tau(P_i) = Q_i$. So there exists a $\sigma \in \mathrm{Aut}(V_4)$ where $q\sigma = \tau p$, as below:

$$\begin{tikzcd} V_4 \arrow[r, "\sigma"] \arrow[d, "p"] & V_4 \arrow[d, "q"] \\ \mathbb{P}^2 \arrow[r, "\tau"] & \mathbb{P}^2 \end{tikzcd}$$

So $\Phi(\sigma) = w$.

The only thing missing is showing that $\mathrm{im}\,\Phi \subseteq W$. Let I be the set of sequences $(f_0, \ldots, f_4)$ with $f_i \in \mathrm{Pic}V_4$ such that $f_0.f_0 = 1$, $f_i.f_i = -1$, $f_0.K = -3$, $f_i.K = -1$, $f_i.f_j = 0$.

Then W act transitively on I (??). Then for $\sigma \in \mathrm{Aut}(V_4)$,

$$(\Phi(\sigma)(\ell), \ldots \Phi(\sigma)(e_4)) \in I.$$

Since W acts transitively on I, there is a $w \in W$ such that $w \cdot (\ell, \ldots, e_4) = (\Phi(\sigma)(\ell), \ldots \Phi(\sigma)(e_4))$. So $w(\ell) = \Phi(\sigma)(\ell)$ and $w(e_i) = \Phi(\sigma)(e_i)$. Since $\{\ell, e_1, \ldots, e_4\}$ form a basis of $\mathrm{Pic}(V_4)$, $w = \Phi(\sigma)$. Thus $\mathrm{im}\,\Phi \subseteq W$.