1: Computing the Canonical Divisor for $\mathbb{P}^2$.

We know that Pic$\mathbb{P}^2 = \mathbb{Z} = \langle L \rangle$ and that the canonical divisor is $W = -3L$. We will show how to compute this using differential 2-forms.

Let $\mathbb{P}^2$ have homogeneous coordinates X, Y, Z, and let the function field $K = k(x, y)$ where $x = X/Z$ and $y = Y/Z$. Let $\omega = \mathsf{d}x \wedge \mathsf{d}y \in \Omega^2_{\mathbb{P}^2}$. We will compute

$$\operatorname{div}(\omega) = \sum_{i \geq 1} i \bigcup_{\substack{P \in \mathbb{P}^2 \\ \operatorname{ord}_\omega(P) = i}} P.$$

We claim that $\operatorname{ord}_\omega(P) = -3$ for $P \in \mathfrak{Z}(Z)$ and $\operatorname{ord}_\omega(P) = 0$ elsewhere. Let us first pick the point $Q = (1, 0, 0)$. We know that $\mathfrak{m}_Q = (x^{-1}, yx^{-1})$. *Note that $yx^{-1} \notin (x^{-1})$ because $y \notin \mathcal{O}_{\mathbb{P}^2,Q}$.* Then $\operatorname{ord}_\omega(Q) = -\operatorname{ord}_f(Q)$ where $f = \dfrac{\mathsf{d}x^{-1} \wedge \mathsf{d}yx^{-1}}{\mathsf{d}x \wedge \mathsf{d}y}$. Note $\mathsf{d}x^{-1} = -x^{-2}\mathsf{d}x$ and $\mathsf{d}yx^{-1} = x^{-1}\mathsf{d}y + y\mathsf{d}x^{-1} = x^{-1}\mathsf{d}y - x^{-2}y\mathsf{d}x$. Therefore the numerator simplifies to $(-x^{-2}\mathsf{d}x) \wedge (x^{-1}\mathsf{d}y - x^{-2}y\mathsf{d}x) = -x^{-2}\mathsf{d}x \wedge x^{-1}\mathsf{d}y = -x^{-3}(\mathsf{d}x \wedge \mathsf{d}y)$.

Therefore f simplifies to $-x^{-3}$. We know that $\operatorname{ord}_{-x^{-3}}(1, 0, 0) = 3$ and so $\operatorname{ord}_\omega(Q) = -3$. This remains true for all $P \in \mathfrak{Z}(Z)$. And for other $P \in \mathbb{P}^2 \setminus \mathfrak{Z}(Z) = \mathfrak{D}_+(Z)$, we have $\operatorname{ord}_\omega(P) = 0$.

Let $R = (a, b, 1) \in \mathfrak{D}_+(Z)$ Then $\mathfrak{m}_R = (x - a, y - b)$. Then $\operatorname{ord}_\omega(R) = -\operatorname{ord}_g(R)$ where $g = \dfrac{\mathsf{d}(x-a) \wedge \mathsf{d}(y-b)}{\mathsf{d}x \wedge \mathsf{d}y} = \dfrac{\mathsf{d}x \wedge \mathsf{d}y}{\mathsf{d}x \wedge \mathsf{d}y} = 1$. Then $\operatorname{ord}_g(R) = 0$ so $\operatorname{ord}_\omega(R_= 0$.

Thus we know that $\operatorname{div}(\omega) = -3\mathfrak{Z}(Z) \cong -3L$. This method also scales to higher dimensions. In general, the canonical divisor for $\mathbb{P}^n$ is $\mathcal{O}(-n-1)$.

2: Why $\mathcal{L}(D) = \mathcal{O}(\deg D)$ and $\mathcal{L}(-D) = \mathcal{I}(D)$ in $\mathbb{P}^n$.

First note Pic$\mathbb{P}^n = \mathbb{Z} = \langle H \rangle$. Let us first observe $\mathcal{L}(-H)$. Note that $\mathcal{L}(-H)$ has no global sections. This is because there is no function in the function field K can have an order of at least 1 at H without having a negative order elsewhere. Similarly, $\mathcal{O}(-1)$ has no global sections. Let H be determined by the function f, so $f \in \mathcal{O}(1)$. Then we can define an isomorphism $\varphi : \mathcal{L}(-H) \to \mathcal{O}_{\mathbb{P}^n}(-1)$ by $\varphi(U)(g) = g/f$. Note that for $g \in \mathcal{L}(-H)$, we know that for all points $P \in U$,

$$\operatorname{ord}_g(P) \geq \begin{cases} 1 & P \in H \\ 0 & P \notin H \end{cases}.$$

And we don't care about $\operatorname{ord}_g(P)$ for $P \notin U$. Since $g \in \mathcal{L}(-H)(U) \subseteq K(U)$ we know $\deg g = 0$. Since $\deg f = 1$, we know $\deg(g/f) = -1$. But also to check $g/f \in \mathcal{O}(-1)(U)$ we will need to make sure g/f does not have a negative order at any point in U. We know that g does not have negative order anywhere in U, and also that g has order at least 1 along H since $g \in \mathcal{L}(-H)(U)$. Additionally, f only has order 1 along H, and order 0 everywhere else, so g/f cannot have negative order anywhere. Therefore $g/f \in \mathcal{O}(-1)(U)$.

In general, we can define an isomorphism $\varphi : \mathcal{L}(nH) \to \mathcal{O}(n)$ via $\varphi(U)(g) = f^n g$.

In general, $\mathcal{L}(D)$ is a little different from $\mathcal{O}$ because since it is a subspace of K, we are allowed to have functions with poles at points in our U, but these functions still must fulfill certain *order requirements*. In $\mathcal{O}$, these ord requirements are simply that $\operatorname{ord}(P) \geq 0$ for all $P \in U$, but for $\mathcal{L}(D)$ they are different (and sometimes allow negative order at certain points in our U) and depend specifically on the divisor. But like $\mathcal{O}$, we throw out any ord requirements for points not in U.

Now why $\mathcal{L}(-D) = \mathcal{I}(D)$, for D an effective Cartier divisor. We know that every function $g \in \mathcal{L}(-D)(U)$ has order at least $D(P)$ for every point $P \in U$. Since D is effective, this $\mathcal{L}(-D)$ is actually a subsheaf of $\mathcal{O}$ as all of the ord requirements are positive. Specifically, the ord requirements are exactly what they need to be to give us $\mathcal{I}(D)$. For example, the ideal of a conic in $\mathbb{P}^2$ is $\mathcal{L}(-2L) = \mathcal{O}(-2)$.

3: Why $\mathcal{O}(n)$ is invertible.

The short answer is $\mathcal{O}(n)$ is invertible because $\mathcal{O}(n) \otimes \mathcal{O}(-n) = \mathcal{O}$ as the product of a degree-n polynomial and a degree-$-n$ polynomial is a degree-0 polynomial.

The more interesting question is why $\mathcal{O}(n)$ is a locally free sheaf of rank 1. Let S be a graded ring and let $X = \operatorname{Proj} S$. From Hartshorne (II.5.12): Let $f \in S_1$ and consider the restriction $\mathcal{O}_X(n)|_{\mathfrak{D}_+(f)}$. By (5.11) this is isomorphic to $S(n)^{\sim}_{(f)}$ on $\operatorname{Spec} S_{(f)}$. We will show that this restriction is free of rank 1. Indeed, $S(n)_{(f)}$ is a free $S_{(f)}$-module of rank 1. For $S_{(f)}$ is the group of elements of degree 0 in S_f, and $S(n)_{(f)}$ is the group of elements of degree n in S_f. We obtain an isomorphism of one to the other by sending s to $f^n s$. This makes sense, for any $n \in \mathbb{Z}$, because f is invertible in S_f. Now since S is generated by S_1 as an S_0-algebra, X is covered by open sets $\mathfrak{D}_+(f)$ for $f \in S_1$. Hence $\mathcal{O}(n)$ is invertible.

4: Why the Exceptional Curve has Negative Self-Intersection.

We will begin with Hartshorne's proof, which is more scheme-theoretic. Then we will proceed with a more geometrically-minded investigation.

Hartshorne (II.7.13). Let X be a noetherian scheme, $\mathcal{I}$ a coherent sheaf of ideals, and let $\pi : \widetilde{X} \to X$ by the blowing up of $\mathcal{I}$. Then the inverse image ideal sheaf $\widetilde{\mathcal{I}} = \pi^{-1}\mathcal{I} \cdot \mathcal{O}_{\widetilde{X}}$ is an invertible sheaf on $\widetilde{X}$.

Proof. Since $X = \mathbf{Proj}\mathcal{S}$, where $\mathcal{S} = \bigoplus_{d\geq 0} \mathcal{I}^d$, it comes equipped with a natural invertible sheaf $\mathcal{O}(1)$. For any open affine $U \subseteq X$, this sheaf $\mathcal{O}(1)$ on $\operatorname{Proj}\mathcal{S}(U)$ is the sheaf associated to the graded $\mathcal{S}(U)$-module $\mathcal{S}(U)(1) = \bigoplus_{d\geq 1} \mathcal{I}^d(U)$. But this is clearly equal to the ideal $\mathcal{I} \cdot \mathcal{S}(U)$ generated by $\mathcal{I}$ in $\mathcal{S}(U)$, so we see that the inverse image ideal sheaf $\widetilde{\mathcal{I}} = \pi^{-1}\mathcal{I} \cdot \mathcal{O}_X$ is in fact equal to $\mathcal{O}_{\widetilde{X}}(1)$. Hence it is an invertible sheaf. $\square$

Then (V.1.4.1) gives the definition of C^2 as $\deg_C \mathcal{N}_{C/X}$, where $\mathcal{N}_{C/X}$ is the dual of the sheaf $\mathcal{I}(C) \otimes \mathcal{O}_C \cong \mathcal{I}(C)/\mathcal{I}(C)^2$ ($\mathcal{O}_C$ is like $\mathcal{O}_X/\mathcal{I}(C)$). Note that the dual of $\mathcal{I}(E) = \mathcal{O}(1)$ is $\mathcal{O}(-1)$. So $E^2 = -1$.

Related note, given any ring A and ideal $\mathfrak{a}$ of A, we have $\mathfrak{a}/\mathfrak{a}^2 \cong \mathfrak{a} \otimes_A A/\mathfrak{a}$. The simple tensor $a \otimes b + \mathfrak{a}$ will be associated to $ab + \mathfrak{a}^2$. Even more generally, for a ring A, an ideal $\mathfrak{a}$

and an A-module M, $M/\mathfrak{a}M \cong M \otimes_A A/\mathfrak{a}$.

From Wikipedia's article on Divisor (algebraic geometry): When D is smooth, $\mathcal{N}_{D/X} = \mathcal{O}_D(D)$, which is $\mathcal{L}(D)\otimes\mathcal{O}_D = \mathcal{O}_X(D)\otimes\mathcal{O}_D$. This is the dual of $\mathcal{L}(-D)\otimes\mathcal{O}_D = \mathcal{I}(D)\otimes\mathcal{O}_D = \mathcal{I}(D)/\mathcal{I}(D)^2$, which lines up with what we already knew.

Now for the more geometric point of view. Let X be a surface and let P be the point we are blowing up. Let C be a line through P and let D be a line not containing P. Let $\pi : \widetilde{X} \to X$ be the blowup map.

Then note $C.D = 1$ since two lines meet at a point. We can call this point Q. Now let us observe the proper transforms of each of these divisors: $\pi^*C = \widetilde{C} + E$ and $\pi^*D = \widetilde{D}$ where $\widetilde{C}$ and $\widetilde{D}$ are the strict transforms of C and D, respectively, and E is the exceptional divisor. Then note that $(\pi^*C).(\pi^*D) = 1$ as π^*C and π^*D still meet at Q. Then $(\widetilde{C} + E).(\widetilde{D}) = \widetilde{C}.\widetilde{D} + E.\widetilde{D} = 1 + 0 = 1$, as D does not contain P and so $\widetilde{D}$ does not meet E. Since C and D are linearly equivalent, their proper transforms must have the same intersection number with E. Thus $0 = (\pi^*C).E = (\widetilde{C} + E).E = \widetilde{C}.E + E^2$. We know that $\widetilde{C}.E = 1$ since $P \in C$, so solving gives $E^2 = -1$.

Note that also the *strict* transforms of C and D become fundamentally different as well. We know $1 = (\pi^*D)^2 = \widetilde{D}^2$, but $1 = (\pi^*C)^2 = (\widetilde{C} + E)^2 = \widetilde{C}^2 + 2\widetilde{C}.E + E^2 = \widetilde{C}^2 + 2 - 1$. Solving gives $\widetilde{C}^2 = 0$. This means that $\widetilde{C}$ will not meet any divisors that are linearly equivalent to it, whereas $\widetilde{D}$ will meet any divisor linearly equivalent to it exactly once. The fact $E^2 = -1$ implies there *is no* other divisor linearly equivalent to it. It is exceptional after all!

5: Why Dimension Goes Down by 1 When We Assign a Base Point.

Let $\mathfrak{d}$ be a linear system. It is associated to some vector subspace V of some vector space $\mathcal{L}(X)$, where $V = \{s \in \mathcal{L}(X) : (s)_0 \in \mathfrak{d}\} \cup \{0\}$. The dimension of $\mathfrak{d}$ is its dimension as a linear projective variety, so $\dim \mathfrak{d} = \dim V - 1$. When we assign a base point P to $\mathfrak{d}$ to get a new linear system $\mathfrak{d}'$, we are only including the divisors in $\mathfrak{d}$ that contain the point P. So we are only looking at a subset of V where $s(P) = 0$. This forms a subspace of dimension 1 less?

Hartshorne (IV.3.1): $|D|$ has no base points if and only if for every point $P \in X$,

$$\dim |D - P| = \dim |D| - 1.$$

(This was only proven for curves) (Also abuse of notation: $D - P$ does not necessarily mean assigning a base point here).

First consider the exact sequence of sheaves

$$0 \to \mathcal{L}(D - P) \to \mathcal{L}(D) \to k(P) \to 0.$$

Taking global sections, we get

$$0 \to \Gamma(X, \mathcal{L}(D - P)) \to \Gamma(X, \mathcal{L}(D)) \to k,$$

so in any case we see that $\dim |D-P|$ is equal to either $\dim |D|$ or $\dim |D|-1$. Furthermore, sending a divisor E to $E + P$ defines a linear map

$$\varphi : |D - P| \to |D|$$

which is injective. Therefore, the dimensions of these two linear systems are equal if and only if φ is surjective. On the other hand, φ is surjective if and only if P is a base point of $|D|$. This proves (IV.3.1).

We know that if P is not already an unassigned base point of $\mathfrak{d}$ that $\mathfrak{d}' \subsetneq \mathfrak{d}$, so $\dim \mathfrak{d}' < \dim \mathfrak{d}$. I think $k(P) = \mathcal{L}(P)$, but the global sections of $\mathcal{L}(P)$ is not k???

Anyway, let V correspond to $\mathfrak{d}$ and let U correspond to $\mathfrak{d}'$. So $U = \{f \in V : f(P) = 0\}$. We get that U is the kernel of the map $\psi : V \to k$ defined by $\psi(f) = f(P)$. Thus $\dim U = \dim V - 1$, so $\dim \mathfrak{d}' = \dim \mathfrak{d} - 1$.