The following is notes from Frank Gounelas' and Alexis Kouvidakis' paper *Geometry of lines on a cubic fourfold*.

## 1. Introduction

Let $X \subseteq \mathbb{P}^5_{\mathbb{C}}$ be a smooth cubic fourfold and $F \subseteq \mathrm{Gr}(2,6)$ its Fano scheme of lines. The geometry of these varieties has received a lot of attention since the beginning of the 20th century. Consider the universal family of lines with the two projections

$$\begin{tikzcd} I \arrow[r, "p"] \arrow[d, "q"] & X \\ F & \end{tikzcd} .$$

From Math Stack Exchange:

$$U = \{(L, P) \in \mathrm{Gr}(1, \mathbb{P}^r) \times \mathbb{P}^r : P \in L\}$$

is the "universal line" from Kock and Vainsencher. The projection map $\pi : U \to \mathrm{Gr}(1, \mathbb{P}^r)$ is the universal family of lines, which means that for every family of lines $f : X \to T$ (this means $X$ is closed in $T \times \mathbb{P}^r$, $f = \pi|_X$ is the restriction and $f^{-1}(t) \subsetneq \{t\} \times \mathbb{P}^r = \mathbb{P}^r$ is a line for all $t \in T$) there exists exactly one morphism $\varrho_f : T \to \mathrm{Gr}(1, \mathbb{P}^r)$ such that $X = \varrho_f^* U$. Put another way, this tells you that $(\mathrm{Gr}(1, \mathbb{P}^r), \pi)$ represents the functor

$$T \mapsto \{\text{families of lines in } \mathbb{P}^r \text{ parametrized by } T\}.$$

In another way, $\pi$ is the final object in the category of families of lines, where such families are described above, and morphisms between them are cartesian squares. Another name used is *tautological family*: this reflects the fact that if you look at the fiber of $\pi$ over a line $[\ell] \in \mathrm{Gr}(1, \mathbb{P}^r)$ you get exactly

$$\pi^{-1}([\ell]) = \ell \subsetneq \{[\ell]\} \times \mathbb{P}^r = \mathbb{P}^r.$$

From Hartshorne: A *geometrically ruled surface* or simply *ruled surface* is a surface $X$, together with a surjective morphism $\pi : X \to C$ to a nonsingular curve $C$, such that the fiber $X_y$ is isomorphic to $\mathbb{P}^1$ for each $y \in C$, and such that $\pi$ admits a section (i.e., a morphism $\sigma : C \to X$ such that $\pi \circ \sigma = \mathrm{id}_C$). That is, $X$ is a big ol' curve of lines.

It turns out that the morphism $p$ is a fibration of (2,3)-complete intersections in $\mathbb{P}^3$ associated to the geometry of $X$, and all but finitely many fibers are genus 4 curves. As such, each fiber $C_x = p^{-1}(x) \subseteq I$, which parametrizes lines in $X$ through $x$, comes equipped with two $g^1_3$ linear systems by restricting the rulings of the corresponding quadric. The aim of this paper is to study the fibrations $p, q$ and the geometry and intersection theory of various geometric loci in each variety.

A *fibration* is a continuous map $p : E \to B$ satisfying the homotopy lifting property with respect to any space. The most basic form of this property is given a point $e \in E$ and a path $[0,1] \to B$ in $B$ starting at $p(e)$, the path can be lifted up to a path in $E$ starting at $e$. One generally also assumes the lifting of additional structures (including "higher homotopies") in $B$ which, in particular, imply that the path lifting is unique up to homotopy. The *fibers* are subspaces of $E$ that are preimages of points $b \in B$. If $B$ is path connected, it is a consequence of the definition that the fibers of two different points $b_1$ and $b_2$ in $B$ are homotopy equivalent. Therefore, one usually speaks of "the fiber" $F$.

An algebraic variety $V$ is a complete intersection if the number of generators of the ideal associated to $V$ is equal to the codimension of $V$. A complete intersection has a *multidegree*, written as a tuple of the degrees of the defining hypersurfaces. For example, $V(x_0^2 + x_1^2 + x_2^2 + x_3^2 + x_4x_5, x_4^4 + x_5^4 - 2x_0x_1x_2x_3) \subseteq \mathbb{P}^5$ is a threefold complete intersection of type $(2,4)$.

Recall that there are two types of points $[\ell] \in F$, i.e., lines $\ell \subseteq X$, depending on the decomposition of the normal bundle $N_{\ell/X}$. The generic line is call *of first type*, whereas the is a surface $S \subseteq F$ parametrizing those of *second type*.

From Hartshorne: Let $Y$ be a nonsingular subvariety of a nonsingular variety $X$ over $k$. The locally free sheaf $\mathcal{I}/\mathcal{I}^2$ we call the *conormal sheaf* of $Y$ in $X$. Its dual $\mathcal{N}_{Y/X} = \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{I}/\mathcal{I}^2, \mathcal{O}_Y)$ is called the *normal sheaf* of $Y$ in $X$. It is locally free of rank $r = \operatorname{codim}(Y, X)$.

Our first goal is to further expand on the geometry of the curves $C_x$, which we achieve by studying properties of the fibration $p$ on $I$ but also the induced restricted fibration over $S \subseteq F$. To name one example, we obtain the following as a combination of Proposition 3.4, Corollary 3.6.

**Theorem A.** For $x \in X \subseteq \mathbb{P}^5$ a general cubic fourfold, $C_x$ is Brill-Noether general and the morphism

$$p|_{q^{-1}(S)} q^{-1}(S) \to W \subseteq X$$

is birational to its image $W$, an irreducible divisor in $X$.

In Remark 3.7 we also extend the second part of this result to arbitrary cubic hypersurfaces, which seems to have been expected but was missing from the literature.

Moving now to geometric constructions on $F$, Voisin defined a map

$$\phi : F \dashrightarrow F$$

taking a general point $[\ell]$ corresponding to a line $\ell \subseteq X$ to the point $[\ell']$, where there is a unique plane $\Pi_\ell = \mathbb{P}^2$ so that $X \cap \Pi_\ell = 2\ell + \ell'$. This map was also studies in Ame11 and if $X$ does not contain any planes, is resolved by a single blowup of the locus $S$. In section 4 we further analyse the geometry of this map as well as various related geometric constructions, such as the natural map $F \dashrightarrow X$ taking $[\ell] \mapsto \ell \cap \ell'$, by interpreting the residual line $[\ell']$ in terms of the ramification points of the $g_3^1$'s on $C_x$ and their conjugates(?).

As an application of the analysis above, in Section 5 we define the loci

$$R, N \subseteq I$$

which are the closure of the locus of ramification points (resp. triple ramification points) of the two $g^1_3$'s on the smooth fibres $C_x$. The locus $R \subseteq I$ is of particular interest as it is birational to $F$ and contains the family of second type lines $q^{-1}(S)$. In Section 6, we compute the classes of $R, N$ in the Chow group of $I$ in terms of the tautological line bundle $\ell = \mathcal{O}_I(1)$, $g = q^* H_F$ the pullback of the class of the Plücker polarization and $c = q^* c_2(\mathcal{U}_F)$ where $\mathcal{U}_F$ is the restriction of the universal rank 2 subbundle of the Grassmannian to $F$.

**Theorem B.** If $X \subseteq \mathbb{P}^5$ is a general cubic fourfold, then the classes $[R] \in \mathrm{CH}^1(I)$ and $[N] \in \mathrm{CH}^2(I)$ are given as follows

$$\begin{aligned} [R] &= 4g + \ell \\ [N] &= 4\ell^2 - 4\ell g + 25c. \end{aligned}$$

Our next aim is to introduce the locus of fixed points of $\phi$

$$V := \mathrm{Fix}(\phi) \subseteq F,$$

which to our knowledge has so far eluded study, consisting of lines for which there is a $\Pi_\ell = \mathbb{P}^2$ so that $X \cap \Pi_\ell = 3\ell$. We will call this locus of *triple lines* and it consists of both first and second type lines. We study the geometry of this locus in Section 7.

Lots of so on and so forth and then.....

## 2. Background and notation:

For a vector bundle $E$ we denote by $\mathbb{P}(E) = \mathrm{Proj}(\mathrm{Sym}(E^*))$, so that projective space parametrizes one-dimensional subspaces. We denote by $\mathrm{Gr}(k, n)$ the space of $k$-dimensional subspaces of $\mathbb{C}^n$ with universal bundle $\mathcal{U}$ of rank $k$ and universal quotient bundle $\mathcal{Q}$ of rank $n - k$. We will denote by $\sigma_I$ the standard Schubert cycles for an index $I$ so that, e.g., $\sigma_i = c_i(\mathcal{Q})$ for $i \geq 1$.

Throughout, $X \subseteq \mathbb{P}^5$ will be a smooth cubic fourfold with $H_X = \mathcal{O}_X(1)$ and $F \subseteq \mathrm{Gr}(2, 6)$ the Fano scheme of lines contained in $X$ which is a hyperkähler fourfold. We denote by $\mathcal{U}_F, \mathcal{Q}_F$ the restrictions of $\mathcal{U}, \mathcal{Q}$ to $F$. The universal family of lines sits in the diagram

$$\begin{array}{ccc} \mathbb{P}(\mathcal{U}) & \xrightarrow{p} & \mathbb{P}^5 \\ \big\downarrow{\scriptstyle q} & & \\ \mathrm{Gr}(2,6) & & \end{array}$$

and note that we have

$$p^* \mathcal{O}_{\mathbb{P}^5}(1) \cong \mathcal{O}_{\mathbb{P}(\mathcal{U})}(1).$$

We use the same notation $p, q$ for the induced maps on

$$I := \mathbb{P}(\mathcal{U}_F)$$

the universal family of lines on $F$. We will often denote by $I_Y$ this universal family restricted to some subset $Y \subseteq F$ or $Y \subseteq X$, and will continue using $p, q$ for the corresponding restricted morphisms.

Denote by $H = c_1(\mathcal{U}^*)$ or $H_F$ the Plücker ample line bundle on $F$. The subvariety $F \subseteq \mathrm{Gr}(2,6)$ is given by the section of the rank four bundle $\mathrm{Sym}^3\mathcal{U}^* \cong q_* p^* \mathcal{O}_{\mathbb{P}^5}(3)$ - in fact it is the section induced, under this isomorphism, by $f \in k[x_0, \ldots, x_5]_3$ whose vanishing is $X$ so its cohomology class in the Grassmannian is given by $c_4(\mathrm{Sym}^3\mathcal{U}^*)$ which can be computed as follows:

$$\begin{aligned}[F] &= 18c_1(\mathcal{U}^*)^2 c_2(\mathcal{U}^*) + 9c_2(\mathcal{U}^*)^2 \\ &= 18\sigma_1^2\sigma_{1,1} + 9\sigma_{1,1}^2 \\ &= 27\sigma_2^2 - 9\sigma_1\sigma_3 - 18\sigma_4.\end{aligned}$$

Consider now the morphism $p : I \to X$. We denote by

$$C_x := p^{-1}(x)$$

the fiber over $x$, which parametrizes the lines in $X$ containing $x$. If $X$ is general, then $C_x$ is 1-dimensional for all $x$, whereas for arbitrary $X$ there are only finitely many points in $X$ where the fibre can be two dimensional. $C_x$ embeds in $F$ via $q$ and can also be realized as the $(2,3)$-complete intersection in $\mathbb{P}^3$ formed by the intersection points of the lines through $x$ with $T_xX \cap A$, where $A$ is a hyperplane not containing $x$: for $R_xX \cap T_xX \cap X \cap A$. As such, if it is 1-dimensional, it is of arithmetic genus 4 and has two $g_3^1$'s, counted with multiplicities, namely the restrictions of the rulings of the quadric $R_xX \cap T_xX \cap A$.

Blah blah blah...

## 3. The curve of lines through a point:

**Lemma 3.1.** Let $X \subseteq \mathbb{P}^5$ be a smooth cubic fourfold and $x \in X$ such that $C_x = p^{-1}(x)$ is 1-dimensional.

(1) If $\ell$ is a line of first type through $x$, then $C_x$ is smooth at $[\ell]$.

(2) If $\ell$ is a line of second type through $x$, then $C_x$ is singular at the point $[\ell]$ and the singularity has a 2-dimensional tangent cone (like the singularity of an almost-torus).

*Proof.* Let $\ell$ be a line of first type in $X$. We may assume that it is given by the equations $x_2 = x_3 = x_4 = x_5 = 0$. Then the equation of $X$ may take the form:

$$F = x_2x_0^2 + x_3x_0x_1 + x_4x_1^2 + x_0Q_0(x_2, x_3, x_4, x_5) \quad (1)$$

$$+x_1Q_1(x_2, x_3, x_4, x_5) + P(x_2, x_3, x_4, x_5) = 0. \quad (2)$$

Since $x \in \ell$ we may assume that $x = [1, a, 0, 0, 0, 0] \in \ell$ (the case $x = [0, 1, 0, 0, 0, 0]$ is treated similarly). To find $C_x$, we write the equation as

$$x_2 x_0^2 + x_3 x_0(x_1 - ax_0) + ax_3 x_0^2 + x_4(x_1 - ax_0)^2 + 2ax_0 x_4(x_1 - ax_0) \\ + a^2 x_0^2 x_4 + x_0 Q_0 + (x_1 - ax_0)Q_1 + ax_0 Q_1 + P = 0.$$

Putting $x_0 = 1$ and $x_1' = x_1 - a$ we get

$$[x_2 + ax_3 + a^2 x_4] + [x_3 x_1' + 2ax_4 x_1' + Q_0 + aQ_1] + [x_4(x_1')^2 + x_1' Q_1 + P] = 0.$$

Then the lines through $x = (0, 0, 0, 0, 0) \in \mathbb{A}^5$ (in the $x_1', x_2, x_3, x_4, x_5$ coordinates) are determined by their slopes in $\mathbb{P}^4$. they are parametrised by the following curve which is given by the system

$$\begin{aligned} &x_2 + ax_3 + a^2 x_4 = 0 \\ &T_2 = x_3 x_1' + 2ax_4 x_1' + Q_0(x_2, x_3, x_4, x_5) + aQ_1(x_2, x_3, x_4, x_5) = 0 \\ &T_3 = x_4(x_1')^2 + x_1' Q_1(x_2, x_3, x_4, x_5) + P(x_2, x_3, x_4, x_5) = 0. \end{aligned}$$

The points $[x_1' : x_2 : x_3 : x_4 : x_5] \in \mathbb{P}^4$ of this curve correspond to the slopes of the lines through $x$. The line $\ell$ has slope $\langle 1, 0, 0, 0, 0 \rangle$. Substituting $x_2$ from the 1st equation, $T_2 = 0, T_3 = 0$ become equations in the variables $x_1', x_3, x_4, x_5$ in $\mathbb{P}^3$. The point in $C_x$ which corresponds to the line $\ell$ is then $[1 : 0 : 0 : 0]$ (in the $x_1', x_3, x_4, x_5$-coordinates). At this point the gradients of the surfaces $T_2, T_3$ are $\langle 0, 1, 2a, 0 \rangle$ and $\langle 0, 0, 1, 0 \rangle$ respectively. Hence the intersection is transversal at that point and the curve $C_x$ is smooth at the point $[\ell]$.

Now let $\ell$ be a line of second type, given by $x_2 = x_3 = x_4 = x_5 = 0$. Then the equation of $X$ may take the form

$$F = x_2 x_0^2 + x_3 x_1^2 + x_0 Q_0(x_2, x_3, x_4, x_5) + x_1 Q_1(x_2, x_3, x_4, x_5) \\ + P(x_2, x_3, x_4, x_5) = 0.$$

A similar calculation shows that with center the point $x = [1 : a : 0 : 0 : 0 : 0] \in \ell$ (the case $x = [0 : 1 : 0 : 0 : 0 : 0]$ is treated similarly) the equation of $X$ becomes (by putting $x_0 = 1$ and $x_1 - a = x_1'$)

$$[x_2 + a^2 x_3] + [2ax_3 x_1' + Q_0 + aQ_1] + [x_3(x_1')^2 + x_1' Q_1 + P] = 0.$$

Then $C_x$ is the intersection of two surfaces in a 3-dimensional projective space with gradients $\langle 0, 2a, 0, 0 \rangle$ and $\langle 0, 1, 0, 0 \rangle$ respectively. Hence the intersection is not transversal and $C_x$ is singular. □

Geometrically one can see this as follows. Take $x \in \ell \subseteq X$. The tangent space to the line $\ell$ in $X$ is spanned by $\ell$ and the tangent cone to $C_x$ at the point $[\ell]$. This space has dimension 2 (i.e., $[\ell]$ is a line of first type) if and only if $C_x$ is smooth that $[\ell]$. It has dimension 3 (i.e., $[\ell]$ is a line of second type) if and only if $C_x$ has a singularity at $[\ell]$ (with tangent cone of dimension two). When we consider $C_x$ as a curve in $F$ (i.e., as $q(p^{-1}(x))$), the singular points correspond to the intersection points with the surface $S$. When $x \in W$ this intersection is empty and the curve $C_x$ is smooth.

**Remark 3.2.** For $[\ell] \in S$, $x = [1 : a : 0 : 0 : 0 : 0] \in X$ and $Y_a = T_xX \cap X$, one sees that $Y_a$ contains $\ell$ and has in general two singular points on $\ell$, namely $x$ and $[1 : -a : 0 : 0 : 0 : 0]$. For $a = 0$, $Y_0$ has a nonordinary singularity at $x$ (and the same for $x = [0 : 1 : 0 : 0 : 0 : 0]$). On the other hand, if $[\ell] \in F \setminus S$ and $x \in \ell$ then $Y_x = T_xX \cap X$ does not have a singularity on $\ell$ other than $x$. In particular, if $x \notin W$ then the only singularity of $Y_x$ is at $x$. This is in accordance with [CG72, Definition 6.6], defining lines of first and second type as those for which the image of the dual mapping is a smooth plane conic or a two-to-one covering of the projective line respectively. In fact, in the latter case, $\nabla F([s : t : 0 : 0 : 0 : 0 :]) = \langle 0, 0, s^2, t_2, 0, 0\rangle$ and the fibres of the two-to-one covering are formed by the points $[\pm s : t : 0 : 0 : 0 : 0]$, with ramification points the two points at infinity $[1 : 0 : 0 : 0 : 0 : 0], [0 : 1 : 0 : 0 : 0 : 0]$. Finally note that for a point $x \in \ell \in S$, the singularity of $C_x$ corresponding to $\ell$ has the same type as the section $T_xX \cap X$ has at the point $x'$, the conjugate to $x$ under the Gauss map (in fact $T_xX = T_{x'}X$). Hence if $x'$ is not in the Hessian this singularity is nodal. If there are several second type lines containing $x$, each one induces a singularity as above.

**Lemma 3.3.** We have the following two correspondences

(a) between pairs $(Y, x)$ where $Y \subseteq \mathbb{P}^4$ is a cubic threefold with a singular ordinary double point $x$, and smooth non-hyperelliptic curves of genus 4 with canonical image which lies on a smooth quadric.

(b) between pairs $(Y, x)$ where $Y \subseteq \mathbb{P}^4$ is a cubic threefold with two collinear ordinary double points so that $x$ is one of them, and smooth non-hyperelliptic genus 3 curves with two marked points and with nodal canonical image lying on a smooth quadric.

In both cases the canonical image of the corresponding curve parametrizes the lines in $Y$ through $x$.

*Proof.* The first has appeared many times in the literature, see, e.g., [CG72, p.306-307] and [CML09, §3.1]. The second is a minor modification of this so we briefly sketch the construction. For $(C', p_1, p_2) \in \mathcal{M}_{3,2}^{\mathrm{nhyp}}$, the linear system $|K_{C'} + p_1 + p_2|$ gives a morphism $C' \to \mathbb{P}^3$ with image a sextic curve $C$ with one node $p$. Just as in the classical construction of the canonical embedding of a non-hyperelliptic genus 4 curves, $H^0(\mathbb{P}^3, I_C(2)) = \mathbf{C}$ so there is a unique quadric $Q$ containing $C$, which we may assume is smooth by genericity. Also, the linear system $|I_C(3)|$ induces a birational map $h : \mathbb{P}^3 \dashrightarrow \mathbb{P}^4$ with image a singular cubic threefold $Y$. This can also be realized by blowing up $\mathbb{P}^3$ at the curve $C$ to obtain a variety $\widetilde{Y}$ with one node ($C$ and $\widetilde{Y}$ have the same singularity count and type) lying above the node of $C$, and blowing down the strict transform of the quadric $Q$ to obtain $Y$. The threefold $Y$ will be singular both at the image $x$ of the quadric but also the image of the singular point of $\widetilde{Y}$ and these two must lie on the same line. Projecting from $x$ gives the inverse map to $h$. $\square$

Note that from Lemma 2.2 if $X$ is general, the locus $W \subseteq X$ spanned by lines of second type is an irreducible divisor.