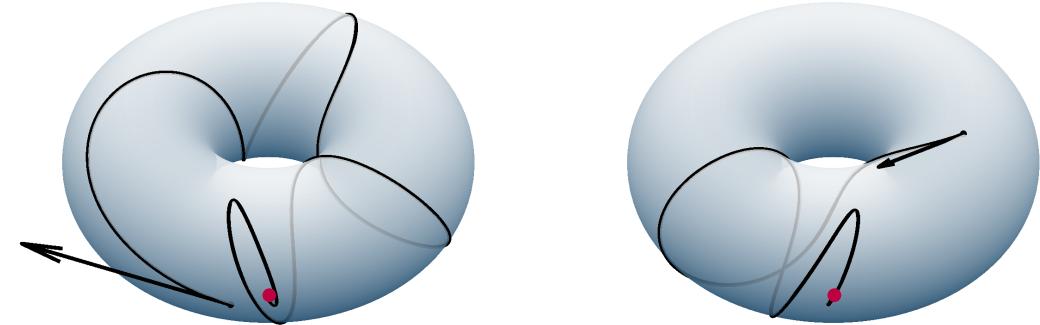


A Compositional Approach to Certifying Almost Global Asymptotic Stability of Cascade Systems

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CASCADES IN CONTROL SYSTEMS

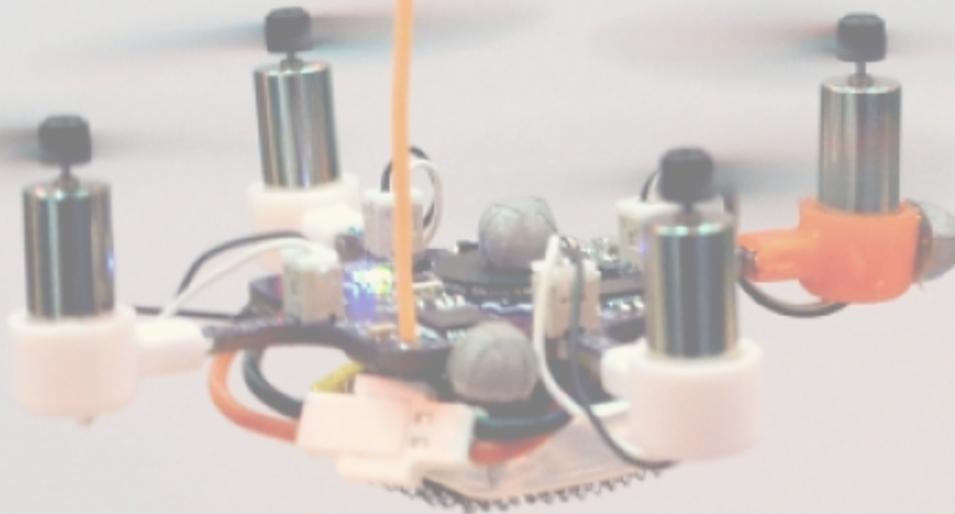
$$\dot{x} = f(x, y),$$

$$\dot{y} = g(y)$$

*cascades often arise
in hierarchical control:*

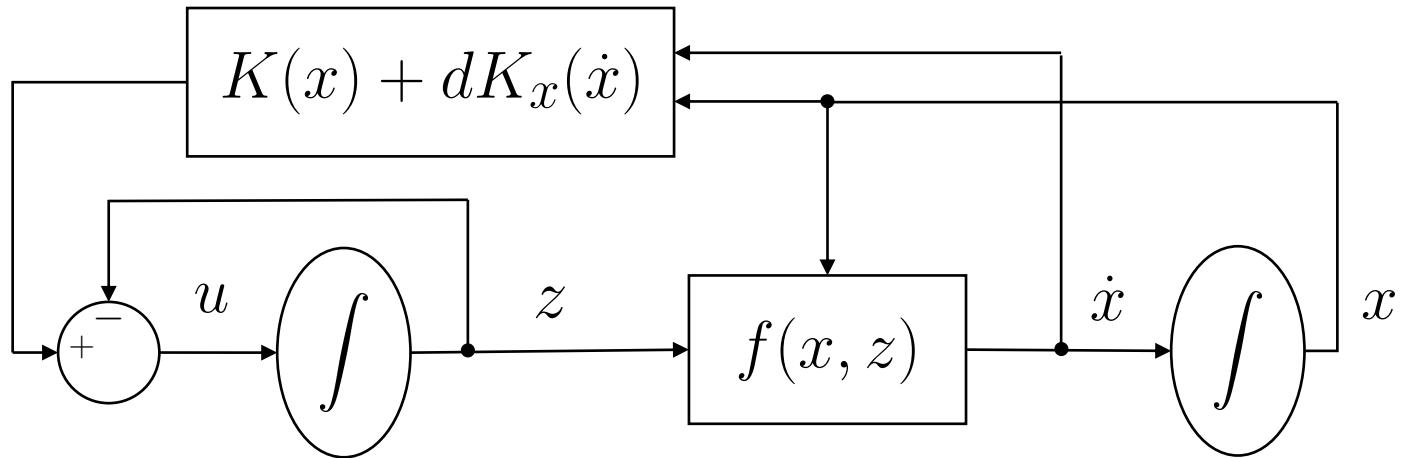
ATTITUDE
DYNAMICS

POSITION
DYNAMICS

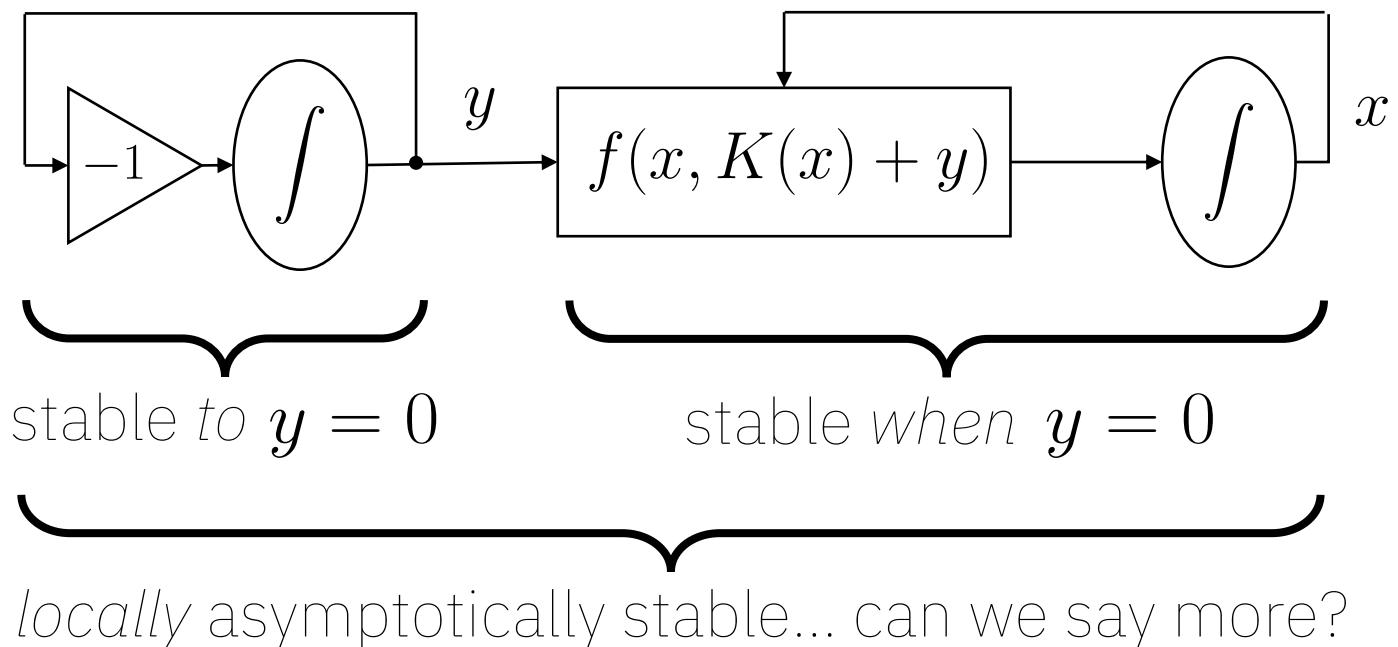


Motivation: Hierarchical Control

Thinking of z as an input,
suppose we know that
 $\dot{x} = f(x, K(x))$ is stable...



Letting $y = z - K(x)$
yields a cascade...



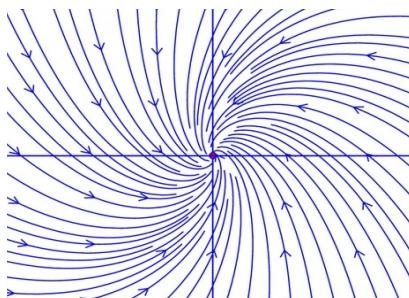
Global Asymptotic Stability of Nonlinear Cascades

$$\begin{aligned}\dot{x} &= f(x, y), \\ \dot{y} &= g(y)\end{aligned}$$

assume globally asymptotically stable when $y = 0$
assume globally asymptotically stable to $y = 0$

When is the combined nonlinear cascade **globally asymptotically stable**?

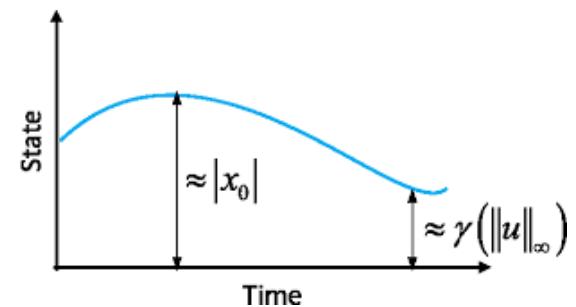
GLOBAL ASYMPTOTIC
STABILITY OF SUBSYSTEMS
AND BOUNDEDNESS



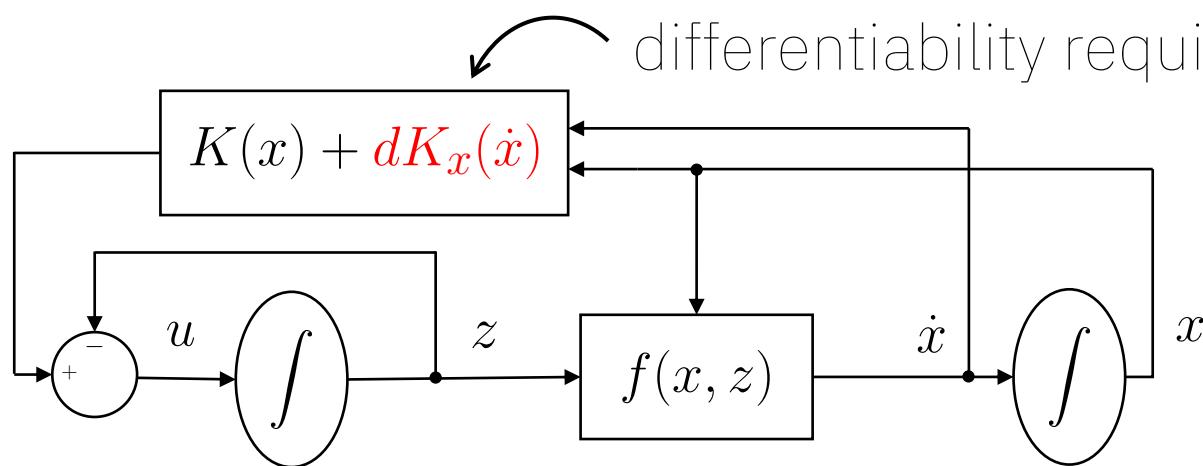
TIME SCALE
SEPARATION
BETWEEN SUBSYSTEMS

$$|\dot{x}| \ll |\dot{y}|$$

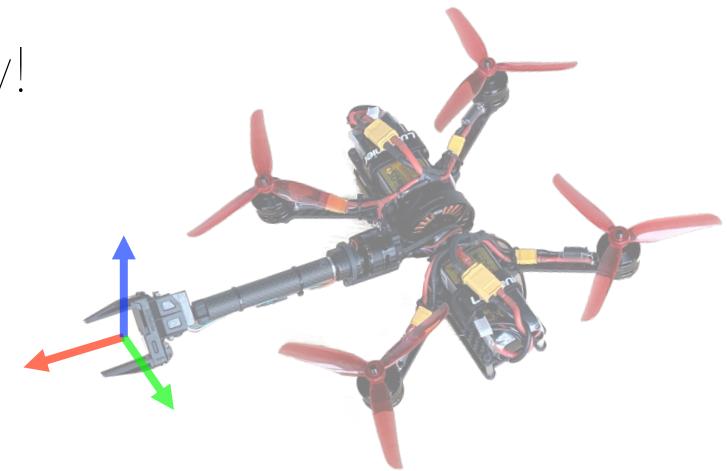
DISTURBANCE
ROBUSTNESS (ISS)
OF OUTER LOOP



Motivation: Geometric Control of Robotic Systems



HIERARCHICAL CONTROLLER



ROBOTIC SYSTEM

For hierarchical control, we want **continuous outer loop feedback** (our intuition is that z evolves continuously, so $K(x)$ should too).

Fact. If $x \in X \not\cong \mathbb{R}^n$ and f, K are continuous, then the stability of $\dot{x} = f(x, K(x))$ is **no better than almost global**.

Robotic systems evolve on **non-Euclidean manifolds** (e.g. \mathbb{S}^1 , $SO(3)$, $SE(3)$).

question: if the subsystems of a cascade are
almost globally asymptotically stable,
when can we say the same about the combined system?
in other words: how can we certify almost global
asymptotic stability in a **compositional** manner, in
order to design **verifiable hierarchical controllers**?

Simple Example System

$$x = (\theta, \dot{\theta}) \in T\mathbb{S}^1$$

$$\ddot{\theta} = -(\sin \theta + \dot{\theta}) \cos 2\phi,$$

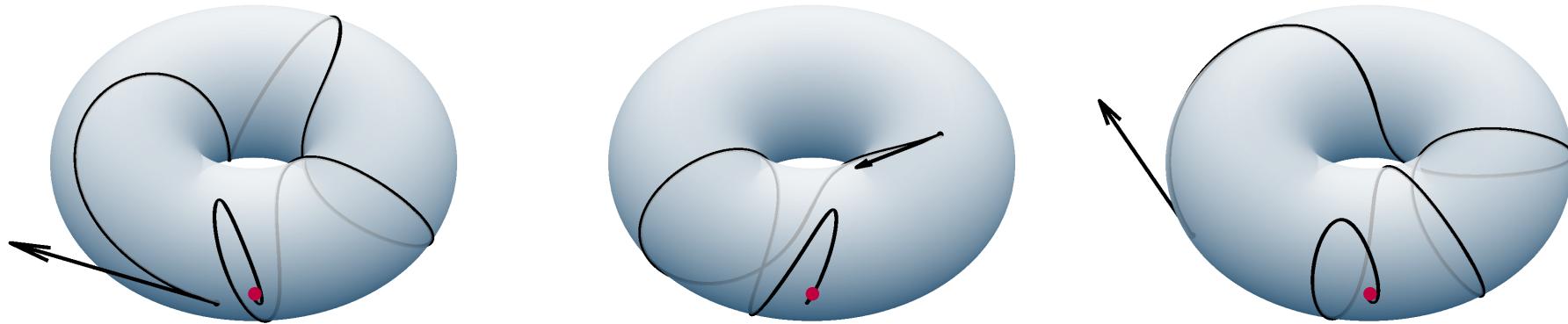
$$y = (\phi, \dot{\phi}) \in T\mathbb{S}^1$$

$$\ddot{\phi} = -(\sin \phi + \dot{\phi}) \leftarrow \begin{matrix} damped \\ pendulum \end{matrix}$$

$$\ddot{\theta} = -(\sin \theta + \dot{\theta}) \text{ when } \phi = 0$$

$$\ddot{\theta} = +(\sin \theta + \dot{\theta}) \text{ when } \phi = \frac{\pi}{2}$$

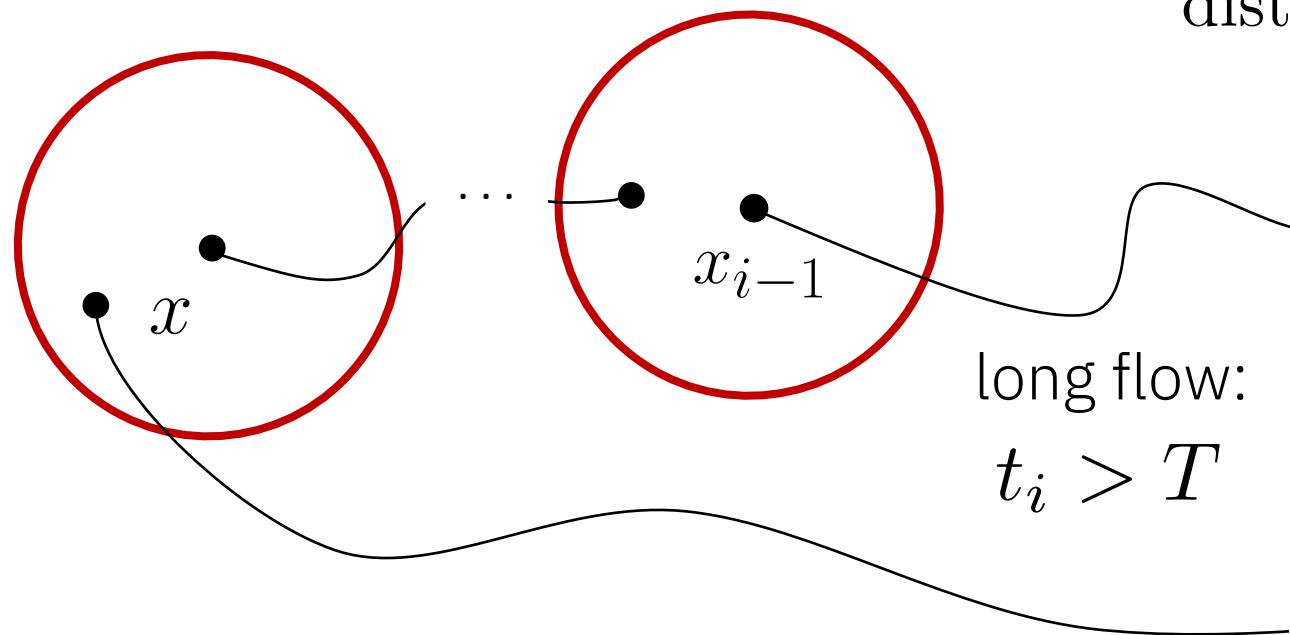
subsystems are almost globally asymptotically stable.... is the full system?



- NO time scale separation!
- NO disturbance robustness (almost ISS)!
- NO global asymptotic stability!

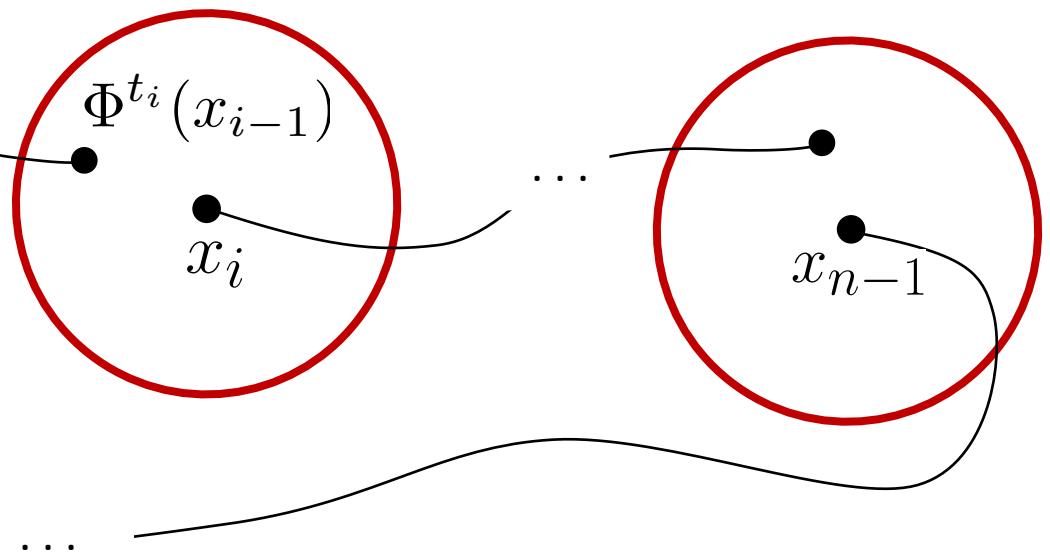
Background: the Chain Recurrent Set of a Dynamical System

closed (ε, T) -chain:



short jumps:

$$\text{dist}(\Phi^{t_i}(x_{i-1}), x_i) < \varepsilon$$



x is **chain recurrent** if there exists a closed (ε, T) -chain at x for all $\varepsilon, T > 0$.

e.g. EQUILIBRIA, PERIODIC ORBITS, NON-WANDERING POINTS

Gradient-Like Dynamical Systems

A system is called **gradient-like** if all its **chain recurrent points** are **equilibria**.

Under mild assumptions, all the following are gradient-like systems:

1. GRADIENT SYSTEMS

$$\dot{q} = -\text{grad}_\kappa V(q)$$

Riemannian metric ↗ ↘ *cost function*

2. DISSIPATIVE MECHANICAL SYSTEMS

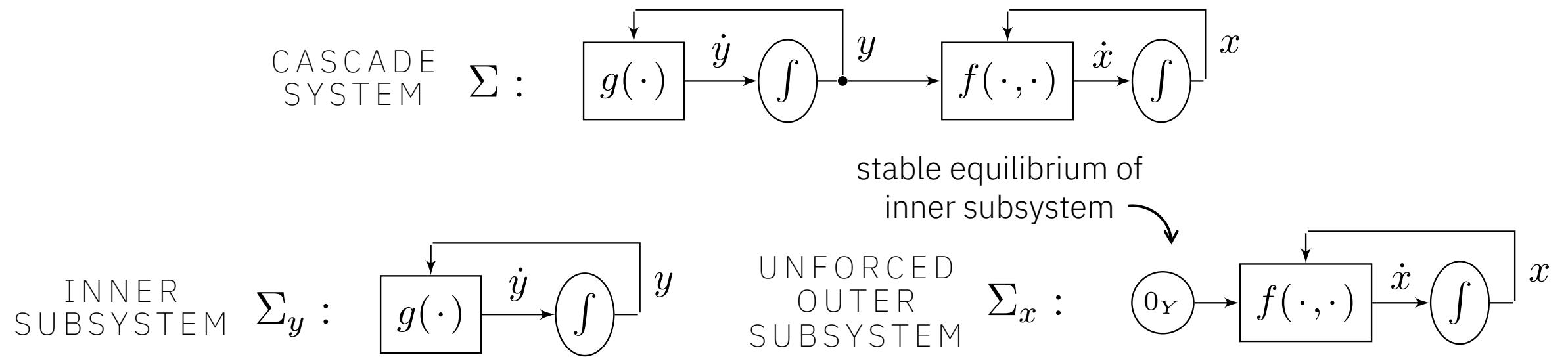
$$\nabla_{\dot{q}} \dot{q} = -\text{grad}_\kappa V(q) - \kappa^\sharp \circ \nu^b(\dot{q})$$

kinetic energy metric ↗ ↘ *strict Rayleigh dissipation (damping)*
 ↑ ↗
 potential energy

3. GLOBALLY ASYMPTOTICALLY STABLE SYSTEMS

4. SYSTEMS w/ A DECREASING LYAPUNOV FUNCTION

Main Result: Almost Global Asymptotic Stability of Cascades



Theorem (Welde, Kvalheim, and Kumar). Suppose that Σ_x and Σ_y are almost globally asymptotically stable, and 0_Y and all chain recurrent points of Σ_x are hyperbolic equilibria. Then, Σ is almost globally asymptotically stable and locally exponentially stable as long as all forward trajectories are bounded.

(Some of these assumptions can be relaxed; here we state a simpler result for clarity.)

Sketch of Proof for Main Result

Theorem (Welde, Kvalheim, and Kumar). Suppose that Σ_x and Σ_y are almost globally asymptotically stable, and 0_Y and all chain recurrent points of Σ_x are hyperbolic equilibria. Then, Σ is almost globally asymptotically stable and locally exponentially stable as long as all forward trajectories are bounded.

(Some of these assumptions can be relaxed; here we state a simpler result for clarity.)

Sketch of the Proof:

- For each converging initial condition $y(0)$, $\dot{x} = f(x, y(t))$ generates an asymptotically autonomous semiflow with limit semiflow $\dot{x} = f(x, 0_Y)$
- Bounded trajectories of asymptotically autonomous semiflows converge to the chain recurrent set of the limit semiflow (Mischaikow, Smith and Thieme)
- Thus, each $(x(t), y(t))$ converges to some hyperbolic equilibrium $(x^*, 0_Y)$
- By the stable manifold theorem, almost no solutions converge to unstable $(x^*, 0_Y)$

Generalization to Upper Triangular Systems

Corollary (*Welde, Kvalheim, and Kumar*). Consider an upper triangular system

$$\begin{aligned} x \left\{ \begin{aligned} \dot{x}_1 &= f_1(x_1, x_2, \dots, x_n), \\ \dot{x}_2 &= f_2(x_2, \dots, x_n), \\ &\vdots \\ \dot{x}_n &= f_n(x_n), \end{aligned} \right. \quad \left. \right\} n-1 \text{ systems} \end{aligned}$$

where for all $i = 1, 2, \dots, n$, the unforced system

$$\dot{x}_i = f_i(x_i, 0_{i+1}, 0_{i+2}, \dots, 0_n)$$

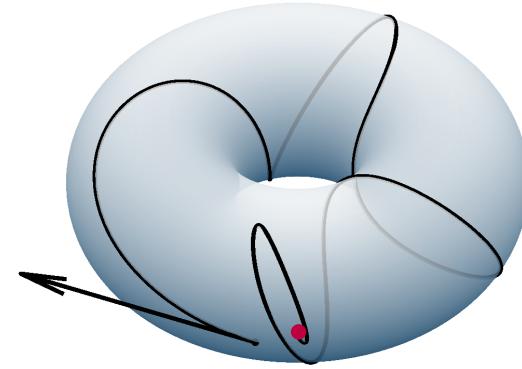
is almost globally asymptotically stable with respect to $\mathbf{0}_i \in X_i$ and all chain recurrent points are hyperbolic equilibria. Then, the full system is almost globally asymptotically stable and locally exponentially stable with respect to $(0_1, 0_2, \dots, 0_n) \in X_1 \times X_2 \times \dots \times X_n$ if all its forward trajectories are bounded.

Proof: by induction!

Revisiting to the Simple Example System

$$\ddot{\theta} = -(\sin \theta + \dot{\theta}) \cos 2\phi,$$

$$\ddot{\phi} = -(\sin \phi + \dot{\phi}) \quad \leftarrow \begin{matrix} \\ damped \\ pendulum \end{matrix}$$



In fact, the system $\ddot{\phi} = -(\sin \phi + \dot{\phi})$ is dissipative mechanical for the kinetic energy and damping $\kappa = \nu = d\phi \otimes d\phi$ and potential $V : \mathbb{S}^1 \rightarrow \mathbb{R}$, $\phi \mapsto 1 - \cos \phi$, so it is gradient-like i.e. all chain recurrent points are equilibria (and hyperbolic).

Theorem (Koditschek). A dissipative mechanical system with a strict Rayleigh dissipation and a polar Morse potential is almost globally asymptotically stable and locally exponentially stable.

Thus, our main result implies that boundedness of this system's forward trajectories will suffice for almost global asymptotic stability!

Boundedness of Cascades on Riemannian Manifolds

Theorem

Suppose

I.

I.
 con-
 ful-

III.
son
are

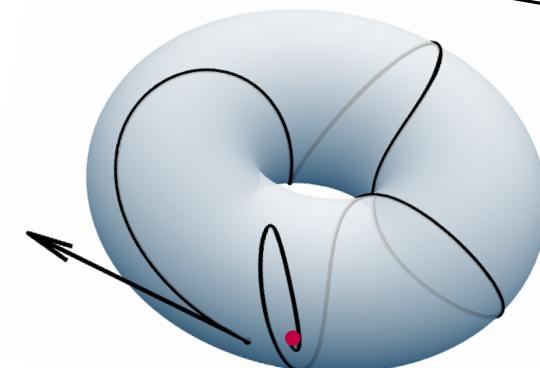
(Lin and Kumar).

INNER SH

BOUNDED!

CASCADE IS ALMOST GLOBALLY
ASYMPTOTICALLY STABLE!

$$\begin{aligned}\ddot{\theta} &= -(\sin \theta + \dot{\theta}) \cos 2\phi, \\ \ddot{\phi} &= -(\sin \phi + \dot{\phi}),\end{aligned}$$

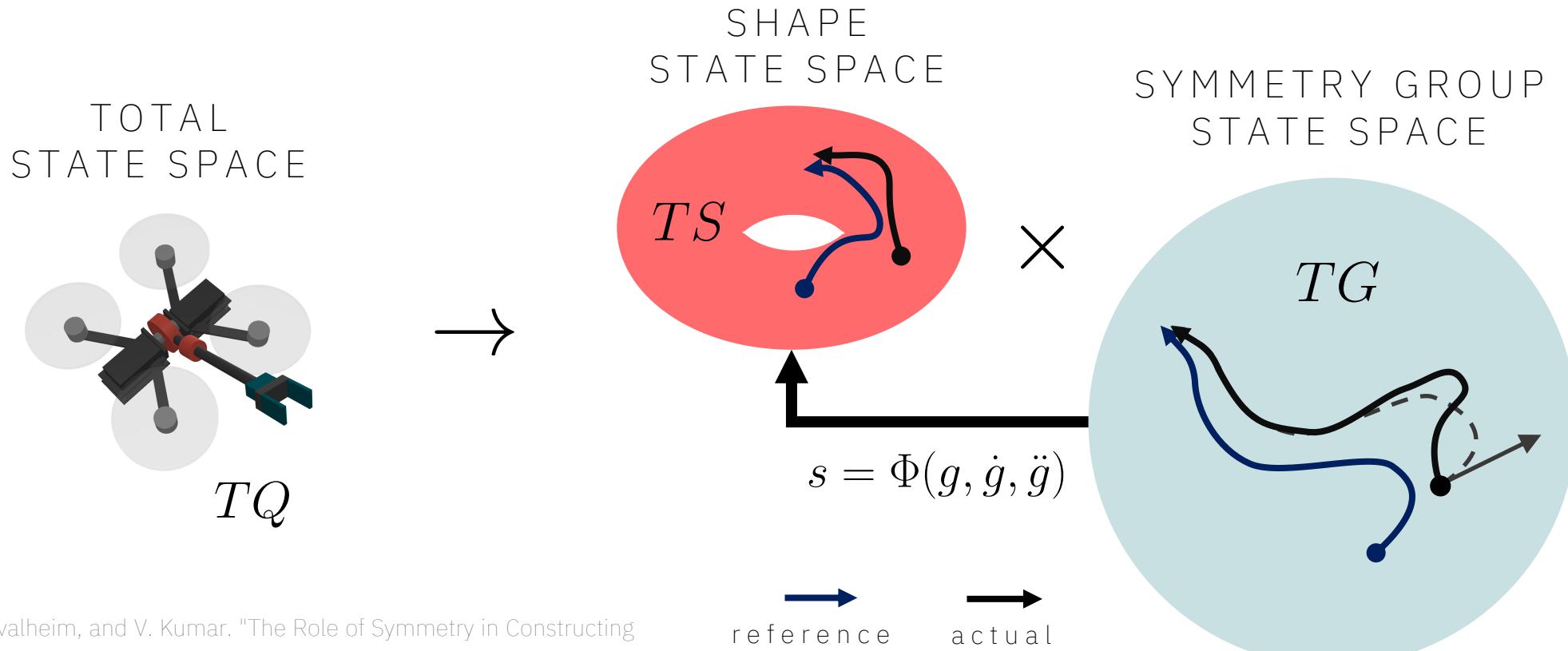


Then any trajectory with y starting in the basin of attraction of 0_Y is bounded in forward time.



Sketch of Future Work

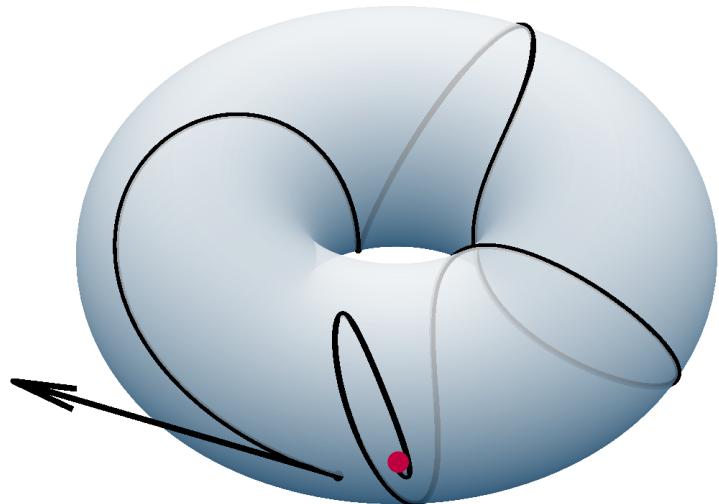
question: can we use these **compositional** stability certificates to synthesize tracking controller for a class of **underactuated robotic systems?**



In Summary

1. We give compositional sufficient conditions for almost global asymptotic stability of cascade and upper triangular systems of arbitrary size.
2. Our results constitute an almost global extension of classic global results
 - a. Classic Result: GAS + GAS + Bounded \Rightarrow GAS
 - b. Our Result: aGAS + aGAS + Bounded + “Hyperbolic Gradient-Like” \Rightarrow aGAS
 - c. Note that for GAS systems, the only chain recurrent point is the stable equilibrium!
 - d. Boundedness criteria is the Riemannian analog of Euclidean “linear growth” criteria
3. Are there more general ways to show boundedness? Further work is needed.
4. We are pursuing applications in the control of underactuated robotic systems
5. Can we extend the approach to time-varying systems?

THANKS FOR LISTENING! QUESTIONS?



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