23.01

A Note About This Chapter

- Last chapter was fairly brutal
  - Physics is really about making hard problems easy with abstraction
    - This chapter will be less labor-intensive
- Through *symmetry*, we can skip parts of problems
  - Like how we only summed the vertical components of $d\vec{E}$ in 22.04

Gauss' Law

- **Gauss' law** = a law that relates net charge of a volume enclosed by a closed surface and the $\vec{E}$ field about that closed surface
  - Discovered by Carl Friedrich Gauss
    - Lived from 1777 until 1855
- Imagine a particle of positive charge q
  - Now superimpose a sphere centered at the particle
    - The surface of the sphere is called a **Gaussian surface**
    - The $\vec{E}$ vectors around the surface point radially outwards
      - Because the particle is *positive*
    - Those same vectors are said to **pierce** the surface of the sphere
- The essential utility of **Gauss' Law** is that we can infer things about the net charge of an object by examining the $\vec{E}$ field about its outer surface
  - Or, equivalently, we can use the net charge to infer information about the $\vec{E}$ about the object's outer surface

Electric Flux

- **Electric flux** = a metric of *how much* the $\vec{E}$ field *pierces* the Gaussian surface
  - The symbol for **electric flux** is ϕ
- The best way to learn about this is to just do a bunch of examples
- The ϕ is
  - Positive if $\vec{E}$ pierces outward
  - Zero if $\vec{E}$ is parallel to the differential area
  - Negative if $\vec{E}$ pierces inward

Electric Flux On a Flat Surface in a Uniform $\vec{E}$ Field

- Imagine we had a uniform $\vec{E}$ field
  - Now superimpose a flat surface of area A
    - Orient it along with yz-plane with its center point at the origin
  - Denote the angle that the uniform $\vec{E}$ vectors make with the x-axis as θ
  - Then, we can imagine splitting the $\vec{E}$ vectors into two components
    - One that *directly* pierces the surface
      - Directly perpendicular to the surface
      - This vector is the **electric flux** for any given differential area
    - One that doesn't pierce the surface at all
      - Directly parallel to the surface
- We can define the magnitude of the electric flux in a subarea of A as

$$d\phi = |\vec{E}|cos(\theta)$$

- This is valid, but there is a more elegant solution
  - This value can calculated with a **dot product**

$$d\phi = \vec{E} \cdot d\vec{A}$$

- where $d\vec{A}$ is a vector perpendicular to the surface with a magnitude equal to the area of the surface
- At some points, the $\vec{E}$ field may pierce *into* the surface and in other points, it may pierce outwards
  - In order to find the **net electric flux**, we use integration

$$\phi = \oint \vec{E} \cdot d\vec{A}$$

23.02

Gauss' Law

- **Gauss' law** = a mathematical model that relates **net flux(ϕ)** and enclosed charge
- Mathematically it looks like this

$$\epsilon_0 \Sigma\phi = \Sigma q$$

- Or, substituting the definition of net flux, we get

$$\epsilon_0 \oint \left(\vec{E} \cdot d\vec{A} \right) = \Sigma q$$

- The charge of Σq determines whether the flux is *inwards* or *outwards*
  - If Σq is *positive*, $\Sigma \phi$ points outward
  - If Σq is *negative*, $\Sigma \phi$ points inward
  - If Σq is zero, $\Sigma \phi$ is a zero
- The interesting thing about Gauss' law is that charges external to the enclosed volume do not affect the net flux
  - Think about that: *if you put a charged particle right up against the barrier, the field lines would change but the net flux wouldn't*

Deriving Coulomb's Law with Gauss' Law

- Recall **Coulomb's law**

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

- This can actually be proven using Gauss' law
- Imagine we had a particle of point q
  - Now superimpose a gaussian sphere that envelops that particle
    - We can use the integral form of Gauss' law to set up an equation

$$\epsilon_0 \oint \left(\vec{E} \cdot d\vec{A} \right) = \Sigma q$$

- A property of dot products is that

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|cos(\theta)$$

- where θ is the angle between $\vec{a}$ and $\vec{b}$
- Using that fact, we can rewrite our equation as

$$\epsilon_0 \oint \left(|\vec{E}||d\vec{A}|cos(\theta) \right) = \Sigma q$$

- Since our problem is basically a one-particle problem, we know that $\vec{E}$ will radiate outwards perpendicular to concecntric spheres
  - As such, $\vec{E}$ and $d\vec{A}$ actually point in the same direction
    - So, θ is zero

$$\epsilon_0 \oint \left(|\vec{E}||d\vec{A}|cos(0) \right) = \Sigma q$$

$$\epsilon_0 \oint \left(|\vec{E}||d\vec{A}|(1) \right) = \Sigma q$$

$$\epsilon_0 \oint \left(|\vec{E}||d\vec{A}| \right) = \Sigma q$$

- At this point, we can rewrite Σq as just q, since our gaussian sphere only contains that one particle

$$\epsilon_0 \oint \left(|\vec{E}||d\vec{A}| \right) = q$$

- Now, the direction of $\vec{E}$ clearly changes from point to point on the gaussian sphere
  - However, the *magnitude* does not change
    * So, we can pull it out of the integral

$$\epsilon_0 |\vec{E}| \oint |d\vec{A}| = q$$

- Now, surface integrals, which is what that $\oint$ symbol denotes, aren't in the scope of this course
  - Really, all you need to know is that they integrate a function over every point on a surface(in this case, the surface area of the sphere)
  - If the function your surface integrates is just 1, then the surface integral returns the surface area
    * So, really, our surface integral just returns the surface area of our sphere
      · Which, if you remember from geometry is

$$SA = 4\pi r^2$$

- Substituting that, we get

$$\epsilon_0 |\vec{E}| \left(4\pi r^2\right) = q$$

- Then, some simple algebra gets us to

$$|\vec{E}| = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

23.03

Gauss' Law and the Behavior of Conductors

- Gauss' law can actually be used to explain phenomenon regarding conductors with excess charge
  - For example, in a conductor with excess charge, the free electrons will disperse themselves amongst the outer surface of the object

        * This kind of makes intuitive sense, as like charges repel, and that permutation ensures maximum distance between particles

- We can demonstrate this fact through Gauss' law
    - Imagine we had a chunk of copper with excess charge q hanging from an insulating thread
    - Now superimpose a Gaussian surface that is *just* inside of the outer surface of the copper
    - Now, if we assume there is no current *inside* the copper, we can deduce that $\vec{E}$ is zero among all points inside of the surface
        * This is because, in order for there to be current, there must be a non-zero force pushing electrons around
            · Which cannot exist without a nonzero $\vec{E}$ field
    - If we make that assumption, then we can use Gauss' law

$$\epsilon_0 \oint \left(\vec{E} \cdot d\vec{A}\right) = \Sigma q$$

$$\epsilon_0 \oint \left(\vec{0} \cdot d\vec{A}\right) = \Sigma q$$

- Now, any dot product between a vector and the zero vector($\vec{0}$) is just equal to 0(the scalar this time)

$$\epsilon_0 \left(0\right) = \Sigma q$$

$$0 = \Sigma q$$

$$\Sigma q = 0$$

- As such, our excess charge q *cannot* exist *inside* of the chunk of copper
    - Rather, it must exist on the outer surface of it
- We can demonstrate similar properties with different conductor shapes

Gauss' Law and a Conductor with a Cavity

- The same line of reasoning can be used on a conductor with a cavity
- Imagine a chunk of copper with excess charge q hanging from an insulating thread
    - Now, without changing the charge, remove some material from the core of the material
        * The result is like a tennis ball with thick material; hollow inside but solid on the exterior
- Now, making that same assumption that there is no internal current, we can form a Gaussian surface just inside of the very exterior of the object

- And, we can conclude the flux is zero, since there cannot be any net flux field if there is no current
  - Then, we use Gauss' law to conclude the charge enclosed by that Gaussian surface is zero
    - As such, the charge must only reside on the very outer surface of the object

Gauss' Law and a Vanishing Conductor

- Now, imagine you had the hollow chunk of copper from the previous example
  - We know the charge carriers would distribute themselves along the outer surface of the object
  - Now, imagine we expanded the hollow core until the copper conductor simply didn't exist
    - For the purposes of visualization, also assume that the charge carriers didn't move during the process
  - At the very instant where that last shell of copper disappears, the $\vec{E}$ field does not change
    - This is because the $\vec{E}$ field is set up by charges, not by conductors
- The lesson here is that charged particles will try to space themselves as far from one another along the outer surface of an object
  - Not only that, but the particles are practically limited in mobility by the size of the conductor
    - If the conductor were instantly made larger, the particles would bubble up to the outer surface–this time farther apart from one another

Gauss' Law and Surface Charge Density on Non-spherical conductors

- Recall that the symbol for surface charge density is σ
- In any spherical conductor, electrostatic equilibrium will be attained the σ not changing over the surface
  - This makes sense, because of the nature of the sphere's symmetry
- However, in a non-spherical conductor, things get *interesting*
  - Imagine we had a non-spherical conductor and we selected a differential circular area along that surface
    - Label the differential area dA
      - Note that this is distinct from $d\vec{A}$–a vector; dA just represents the patch of area along the surface

- Now, imagine creating a Gaussian cylinder whose bases are parallel to dA
  - Since dA is assumed to be