23.01

A Note About This Chapter

- Last chapter was fairly brutal
  - Physics is really about making hard problems easy with abstraction
    * This chapter will be less labor-intensive
- Through *symmetry*, we can skip parts of problems
  - Like how we only summed the vertical components of $d\vec{E}$ in 22.04

Gauss' Law

- **Gauss' law** = a law that relates net charge of a volume enclosed by a closed surface and the $\vec{E}$ field about that closed surface
  - Discovered by Carl Friedrich Gauss
    * Lived from 1777 until 1855
- Imagine a particle of positive charge q
  - Now superimpose a sphere centered at the particle
    * The surface of the sphere is called a **Gaussian surface**
    * The $\vec{E}$ vectors around the surface point radially outwards
      · Because the particle is *positive*
    * Those same vectors are said to **pierce** the surface of the sphere
- The essential utility of **Gauss' Law** is that we can infer things about the net charge of an object by examining the $\vec{E}$ field about its outer surface
  - Or, equivalently, we can use the net charge to infer information about the $\vec{E}$ about the object's outer surface

Electric Flux

- **Electric flux** = a metric of *how much* the $\vec{E}$ field *pierces* the Gaussian surface
  - The symbol for **electric flux** is ϕ
- The best way to learn about this is to just do a bunch of examples
- The ϕ is
  - Positive if $\vec{E}$ pierces outward
  - Zero if $\vec{E}$ is parallel to the differential area
  - Negative if $\vec{E}$ pierces inward

Electric Flux On a Flat Surface in a Uniform $\vec{E}$ Field

- Imagine we had a uniform $\vec{E}$ field
  - Now superimpose a flat surface of area A
    - Orient it along with yz-plane with its center point at the origin
  - Denote the angle that the uniform $\vec{E}$ vectors make with the x-axis as θ
  - Then, we can imagine splitting the $\vec{E}$ vectors into two components
    - One that *directly* pierces the surface
      - Directly perpendicular to the surface
      - This vector is the **electric flux** for any given differential area
    - One that doesn't pierce the surface at all
      - Directly parallel to the surface
- We can define the magnitude of the electric flux in a subarea of dA as

$$d\phi = |\vec{E}| dAcos(\theta)$$

- This is valid, but there is a more elegant solution
  - This value can calculated with a **dot product**

$$d\phi = \vec{E} \cdot d\vec{A}$$

- where $d\vec{A}$ is a vector perpendicular to the surface with a magnitude equal to the area of the differential area
- At some points, the $\vec{E}$ field may pierce *into* the surface and in other points, it may pierce outwards
  - In order to find the **net electric flux**, we use integration

$$\phi = \oint \vec{E} \cdot d\vec{A}$$

23.02

Gauss' Law

- **Gauss' law** = a mathematical model that relates **net flux(ϕ)** and enclosed charge
- Mathematically it looks like this

$$\epsilon_0 \Sigma\phi = \Sigma q$$

- Or, substituting the definition of net flux, we get

$$\epsilon_0 \oint \left(\vec{E} \cdot d\vec{A} \right) = \Sigma q$$

- The charge of Σq determines whether the flux is *inwards* or *outwards*
  - If Σq is *positive*, $\Sigma \phi$ points outward
  - If Σq is *negative*, $\Sigma \phi$ points inward
  - If Σq is zero, $\Sigma \phi$ is a zero
- The interesting thing about Gauss' law is that charges external to the enclosed volume do not affect the net flux
  - Think about that: *if you put a charged particle right up against the barrier, the field lines would change but the net flux wouldn't*

Deriving Coulomb's Law with Gauss' Law

- Recall **Coulomb's law**

$$\vec{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}$$

- This can actually be proven using Gauss' law
- Imagine we had a particle of point q
  - Now superimpose a gaussian sphere that envelops that particle
    - We can use the integral form of Gauss' law to set up an equation

$$\epsilon_0 \oint \left(\vec{E} \cdot d\vec{A} \right) = \Sigma q$$

- A property of dot products is that

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|cos(\theta)$$

- where θ is the angle between $\vec{a}$ and $\vec{b}$
- Using that fact, we can rewrite our equation as

$$\epsilon_0 \oint \left(|\vec{E}||d\vec{A}|cos(\theta) \right) = \Sigma q$$

- Since our problem is basically a one-particle problem, we know that $\vec{E}$ will radiate outwards perpendicular to concecntric spheres
  - As such, $\vec{E}$ and $d\vec{A}$ actually point in the same direction
    - So, θ is zero

$$\epsilon_0 \oint \left(|\vec{E}||d\vec{A}|cos(0) \right) = \Sigma q$$

$$\epsilon_0 \oint \left(|\vec{E}||d\vec{A}|(1) \right) = \Sigma q$$

$$\epsilon_0 \oint \left(|\vec{E}||d\vec{A}| \right) = \Sigma q$$

- At this point, we can rewrite Σq as just q, since our gaussian sphere only contains that one particle

$$\epsilon_0 \oint \left(|\vec{E}||d\vec{A}| \right) = q$$

- Now, the direction of $\vec{E}$ clearly changes from point to point on the gaussian sphere
  - However, the *magnitude* does not change
    - So, we can pull it out of the integral

$$\epsilon_0 |\vec{E}| \oint |d\vec{A}| = q$$

- Now, surface integrals, which is what that $\oint$ symbol denotes, aren't in the scope of this course
  - Really, all you need to know is that they integrate a function over every point on a surface(in this case, the surface area of the sphere)
  - If the function your surface integrates is just 1, then the surface integral returns the surface area
    - So, really, our surface integral just returns the surface area of our sphere
      - Which, if you remember from geometry is

$$SA = 4\pi r^2$$

- Substituting that, we get

$$\epsilon_0 |\vec{E}| \left(4\pi r^2 \right) = q$$

- Then, some simple algebra gets us to

$$|\vec{E}| = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

23.03

Gauss' Law and the Behavior of Conductors

- Gauss' law can actually be used to explain phenomenon regarding conductors with excess charge
  - For example, in a conductor with excess charge, the free electrons will disperse themselves amongst the outer surface of the object

        * This kind of makes intuitive sense, as like charges repel, and that permutation ensures maximum distance between particles

- We can demonstrate this fact through Gauss' law
    - Imagine we had a chunk of copper with excess charge q hanging from an insulating thread
    - Now superimpose a Gaussian surface that is *just* inside of the outer surface of the copper
    - Now, if we assume there is no current *inside* the copper, we can deduce that $\vec{E}$ is zero among all points inside of the surface
        * This is because, in order for there to be current, there must be a non-zero force pushing electrons around
            · Which cannot exist without a nonzero $\vec{E}$ field
    - If we make that assumption, then we can use Gauss' law

$$\epsilon_0 \oint \left(\vec{E} \cdot d\vec{A}\right) = \Sigma q$$

$$\epsilon_0 \oint \left(\vec{0} \cdot d\vec{A}\right) = \Sigma q$$

- Now, any dot product between a vector and the zero vector($\vec{0}$) is just equal to 0(the scalar this time)

$$\epsilon_0 \left(0 \right) = \Sigma q$$

$$0 = \Sigma q$$

$$\Sigma q = 0$$

- As such, our excess charge q *cannot* exist *inside* of the chunk of copper
    - Rather, it must exist on the outer surface of it
- We can demonstrate similar properties with different conductor shapes

Gauss' Law and a Conductor with a Cavity

- The same line of reasoning can be used on a conductor with a cavity
- Imagine a chunk of copper with excess charge q hanging from an insulating thread
    - Now, without changing the charge, remove some material from the core of the material
        * The result is like a tennis ball with thick material; hollow inside but solid on the exterior
- Now, making that same assumption that there is no internal current, we can form a Gaussian surface just inside of the very exterior of the object

    - And, we can conclude the flux is zero, since there cannot be any net flux field if there is no current
        * Then, we use Gauss' law to conclude the charge enclosed by that Gaussian surface is zero
            · As such, the charge must only reside on the very outer surface of the object

Gauss' Law and a Vanishing Conductor

- Now, imagine you had the hollow chunk of copper from the previous example
    - We know the charge carriers would distribute themselves along the outer surface of the object
    - Now, imagine we expanded the hollow core until the copper conductor simply didn't exist
        * For the purposes of visualization, also assume that the charge carriers didn't move during the process
    - At the very instant where that last shell of copper disappears, the $\vec{E}$ field does not change
        * This is because the $\vec{E}$ field is set up by charges, not by conductors
- The lesson here is that charged particles will try to space themselves as far from one another along the outer surface of an object
    - Not only that, but the particles are practically limited in mobility by the size of the conductor
        * If the conductor were instantly made larger, the particles would bubble up to the outer surface–this time farther apart from one another

Gauss' Law and Surface Charge Density on Non-spherical conductors

- Recall that the symbol for surface charge density is σ
- In any spherical conductor, electrostatic equilibrium will be attained the σ not changing over the surface
    - This makes sense, because of the nature of the sphere's symmetry
- However, in a non-spherical conductor, things get *interesting*
    - Imagine we had a non-spherical conductor and we selected a differential circular area along that surface
        * Label the differential area dA
            · Note that this is distinct from $d\vec{A}$–a vector; dA just represents the patch of area along the surface

- Now, imagine creating a Gaussian cylinder whose bases are parallel to dA
  - Since dA is assumed to be infinitesimally small, we can assume that it is essentially flat
  - You can imagine that the cylinder encompasses some enclosed charge q
    * As we have seen in the previous sections, the charge inhabits the surface of the object we are studying which is within our cylinder
  - Then, since dA is a differential area, you can imagine that all the $\vec{E}$ vectors are *perpendicular* to dA
  - Then, we can set up Gauss' law, this time in its non-integral form

$$\epsilon_0 \Sigma\phi = \Sigma q$$

- In order to evaluate $\Sigma\phi$, we can split the cylinder into three surfaces
  - The inner base — B_1
  - The outer base — B_2
  - The curved side — S
- Since B_1 resides inside of the conductor, it experiences no flux
- Since S is essentially perpendicular to dA, the $\vec{E}$ vectors don't pierce S, and it contributes no flux
- Thus, all of the flux comes from B_2, the base of the cylinder that lies outside of the conductor
  - We can define σ as follows

$$\sigma = \frac{\Sigma q}{dA}$$

- Using that definition, we can rewrite Gauss' law

$$\epsilon_0 \Sigma\phi = \sigma dA$$

- Flux can be evaluated just by multiplying the magnitude of the piercing component and the area
  - And, since we demonstrated that flux only comes from B_2, we can do that quite easily
    * Note that we assume dA is equal in area to B_2, but this is a safe assumption because they are differential areas

$$\Sigma\phi = |\vec{E}|dA$$

- Using that definition, we can again rewrite Gauss' law

$$\epsilon_0 |\vec{E}| dA = \sigma dA$$

- Then, we can cancel out the dA's from both sides

$$\epsilon_0 |\vec{E}| = \sigma$$

- and isolate $|\vec{E}|$

$$|\vec{E}| = \frac{\sigma}{\epsilon_0}$$

- This formula gives us the magnitude of the $\vec{E}$ field *just* outside of the conductor
- gg well played m8

23.04

Using Gauss' Law to Find $\vec{E}$ Near an Infinite Charged Rod

- Imagine you had a cylindrical rod that extended infinitely in both directions
  - Also imagine the linear charge density(λ) is constant over the object
- We can use Gauss' Law to determine $\vec{E}$ for a point P whose distance from the charged rod is r
  - In order to do this, superimpose a Gaussian cylinder with the charged rod intersecting the circular bases at their center

- We can use Gauss' Law to determine some interesting things about the situation

$$\epsilon_0 \oint \left(\vec{E} \cdot d\vec{A}\right) = \Sigma q$$

- Using the definition of dot product, we get

$$\epsilon_0 \oint \left(|\vec{E}||d\vec{A}|cos(\theta)\right) = \Sigma q$$

$$\epsilon_0 \oint \left(|\vec{E}|cos(\theta)|d\vec{A}|\right) = \Sigma q$$

- Now, we can split our cylinder into three surfaces: two bases and one curved side
  - The two bases are parallel to $\vec{E}$, so they contribute no flux
    - Therefore, all of the flux must come from the curved side
- Now, we must consider whether $|\vec{E}|$ is constant
  - Because the rod is infinite in both directions, moving up or down while staying the same distance to the rod will have no difference

    * Thus, $|\vec{E}|$ is constant, and we can pull it out of our integral

$$\epsilon_0|\vec{E}| \oint \left(cos(\theta)|d\vec{A}|\right) = \Sigma q$$

- Now, since the way our Gaussian surface is set up, θ is always equal to 0

$$\epsilon_0|\vec{E}| \oint \left(cos(0)|d\vec{A}|\right) = \Sigma q$$

$$\epsilon_0|\vec{E}| \oint \left((1)|d\vec{A}|\right) = \Sigma q$$

$$\epsilon_0|\vec{E}| \oint \left(|d\vec{A}|\right) = \Sigma q$$

- Now, like in a previous section, we've encountered a surface integral whose integrating function is just one
  - This will just returns the surface area of our surface
    * Note that we have dismissed the two bases early on, so this is only the surface area of the curved wall

$$SA = 2\pi rh$$

  - Kind of like how $\int dx$ just returns x
- Substituting that, we get

$$\epsilon_0|\vec{E}|\left(2\pi rh\right) = \Sigma q$$

- Now, Σq is just the enclosed charge within the cylinder
  - Since we know the linear charge density(λ), we can just multiply that by the height of the cylinder

$$\Sigma q = \lambda h$$

- Substituting that, we get

$$\epsilon_0|\vec{E}|\left(2\pi rh\right) = \lambda h$$

$$\epsilon_0|\vec{E}|\left(2\pi r\right) = \lambda$$

- Isolating $|\vec{E}|$, we get

$$|\vec{E}| = \frac{\lambda}{2\pi\epsilon_0 r}$$

- And, with that, we have a nice little formula for $\vec{E}$ at a point P outside of a infinite charged rod
  - This class sure has a lot of formulas

23.05

Gauss' Law and Planar Symmetry

- Imagine we had an infinite, nonconducting plane with excess positive charge
  - Also assume the surface charge density(σ) is constant throughout the plane
- Let's imagine we wanted to find $\vec{E}$ at a point P that is d units from the plane
  - We can set up Gauss' Law in non-integral form

$$\epsilon_0 \Sigma\phi = q_{enc}$$

- Our Gaussian surface is going to be a cylinder perpendicular to the plane
  - The curved wall is parallel to $\vec{E}$, so it contributes no flux
  - Instead, the flux comes from the two bases
    * If $\vec{E}$ is constant over the two bases, we can find the flux just by multiplying the surface area with $|\vec{E}|$
      · $\vec{E}$ actually *is* constant over the end caps, because each point is the same distance from the charge source

$$\Sigma\phi = \big(|\vec{E}|A\big) + \big(|\vec{E}|A\big)$$

$$\Sigma\phi = 2|\vec{E}|A$$

- The two bases both have positive flux, because the source charge is positive, and $\vec{E}$ points outward
- Substituting that into Gauss' law, we get

$$\epsilon_0\big(2|\vec{E}|A\big) = q_{enc}$$

- And, solving for $|\vec{E}|$

$$|\vec{E}| = \frac{q_{enc}}{2A\epsilon_0}$$

- But remember that

$$\sigma = \frac{q}{A}$$

- Making that substitution, we get

$$|\vec{E}| = \frac{\sigma}{2\epsilon_0}$$

- And with that, we have a formula for $|\vec{E}|$ just outside of the positively charged infinite sheet
  - Isn't physics *so useful?*
    * I know this can be boring but bear with it

23.06

Proving Shell Theorems with Gauss' Law

- As it turns out, we can use this tool to prove the two shell theorems

First Shell Theorem

> A charged particle outside a shell with charge uniformly distributed on its surface is attracted or repelled as if the shell's charge were concentrated as a particle at the shell's center

Proof

- Imagine a spherical shell of total charge q, uniform in surface charge density, and with a radius of R
  - Now, superimpose a concentric sphere of a radius larger than R
    - This will be our Gaussian surface
  - Setting up Gauss' Law, we get

$$\epsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{enc}$$

- Since our Gaussian surface is larger than the shell of charge, q_{enc} is just q

$$\epsilon_0 \oint \vec{E} \cdot d\vec{A} = q$$

- Reducing the dot product, we get

$$\epsilon_0 \oint |\vec{E}||d\vec{A}|cos(\theta) = q$$

- Now, we must ask what θ is
  - If we imagine a plane cutting our Gaussian surface in half, we can see that all the charged particles among the shell of charge on each half will cancel each other out in the direction perpendicular to the plane
    - Thus, $\vec{E}$ and $d\vec{A}$ point in the same direction, so θ is zero

$$\epsilon_0 \oint |\vec{E}||d\vec{A}|cos(0) = q$$

$$\epsilon_0 \oint |\vec{E}||d\vec{A}| = q$$

- Now, proving that $|\vec{E}|$ is constant about our Gaussian surface is rather tedious
  - If you'd like, you can prove it by integrating $|d\vec{E}|$ along the entire surface
    - Or you could just take my word for it
  - It is constant, so we can pull it out of the integral

$$\epsilon_0 |\vec{E}| \oint |d\vec{A}| = q$$

- Again, surface integrals where the integrating function is 1 just returns surface area

$$\epsilon_0 |\vec{E}| \left(4\pi r^2\right) = q$$

- Solving for $|\vec{E}|$, we get

$$|\vec{E}| = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

- Now, multiplying both sides by the charge of some particle

$$|\vec{E}| q_1 = \frac{1}{4\pi\epsilon_0} \frac{q q_1}{r^2}$$

- Now, recall the definition of $|\vec{E}|$

$$|\vec{E}| = \frac{|\vec{F}|}{q_{particle}}$$

- Rearranging, we get

$$|\vec{F}| = |\vec{E}| q_{particle}$$

- Making that substitution, we get

$$|\vec{F}| = \frac{1}{4\pi\epsilon_0} \frac{q q_1}{r^2}$$

- Changing variable names, we get

$$|\vec{F}| = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{r^2}$$

- We end up with Coulomb's Law!
  - This means that the force experienced by a particle external to the shell of charge would be identical to if the force were created by a particle at the shell's center of charge q
    - wew lads we proved it

Second Shell Theorem

a charged particle placed inside of a shell that is uniformly charged about its surface will experience no electrostatic force from the shell

Proof

- Imagine a shell of charge q with uniform surface charge density and radius R
  - Now, superimpose, a Gaussian sphere, concentric with the shell of charge, of radius $r < R$
    - We can set up Gauss' Law as follows

$$\epsilon_0 \oint \vec{E} \cdot d\vec{A} = q_{enc}$$

- Simplifying the dot product, we get

$$\epsilon_0 \oint |\vec{E}||d\vec{A}|cos(\theta) = q$$

- Now, the same symmetry exists as before; $\vec{E}$ and $d\vec{A}$ have the same symmetry
  - So, θ is zero

$$\epsilon_0 \oint |\vec{E}||d\vec{A}|cos(0) = q$$

$$\epsilon_0 \oint |\vec{E}||d\vec{A}| = q$$

- Similarly, $|\vec{E}|$ is constant about the surface of our Gaussian surface, so it can be pulled out of the integral

$$\epsilon_0|\vec{E}| \oint |d\vec{A}| = q$$

- Simplifying the surface integral, we get

$$\epsilon_0|\vec{E}|\big(4\pi r^2\big) = q$$

- Now, since our Gaussian sphere is smaller than the shell, it encloses no charge

$$\epsilon_0|\vec{E}|\big(4\pi r^2\big) = 0$$

- The only way for the left side to be zero is if $|\vec{E}|$ is zero or r is zero
  - r is not zero, so $|\vec{E}|$ must be zero!
    - *Yay.*