23.01

A Note About This Chapter

- Last chapter was fairly brutal
  - Physics is really about making hard problems easy with abstraction
    - This chapter will be less labor-intensive
- Through *symmetry*, we can skip parts of problems
  - Like how we only summed the vertical components of $d\vec{E}$ in 22.04

Gauss' Law

- **Gauss' law** = a law that relates net charge of a volume enclosed by a closed surface and the $\vec{E}$ field about that closed surface
  - Discovered by Carl Friedrich Gauss
    - Lived from 1777 until 1855
- Imagine a particle of positive charge q
  - Now superimpose a sphere centered at the particle
    - The surface of the sphere is called a **Gaussian surface**
    - The $\vec{E}$ vectors around the surface point radially outwards
      - Because the particle is *positive*
    - Those same vectors are said to **pierce** the surface of the sphere
- The essential utility of **Gauss' Law** is that we can infer things about the net charge of an object by examining the $\vec{E}$ field about its outer surface
  - Or, equivalently, we can use the net charge to infer information about the $\vec{E}$ about the object's outer surface

Electric Flux

- **Electric flux** = a metric of *how much* the $\vec{E}$ field *pierces* the Gaussian surface
  - The symbol for **electric flux** is ϕ
- The best way to learn about this is to just do a bunch of examples
- The ϕ is
  - Positive if $\vec{E}$ pierces outward
  - Zero if $\vec{E}$ is parallel to the differential area
  - Negative if $\vec{E}$ pierces inward

Electric Flux On a Flat Surface in a Uniform $\vec{E}$ Field

- Imagine we had a uniform $\vec{E}$ field
  - Now superimpose a flat surface of area A
    - Orient it along with yz-plane with its center point at the origin
  - Denote the angle that the uniform $\vec{E}$ vectors make with the x-axis as θ
  - Then, we can imagine splitting the $\vec{E}$ vectors into two components
    - One that *directly* pierces the surface
      - Directly perpendicular to the surface
      - This vector is the **electric flux** for any given differential area
    - One that doesn't pierce the surface at all
      - Directly parallel to the surface
- We can define the magnitude of the electric flux in a subarea of A as

$$d\phi = |\vec{E}|cos(\theta)$$

- This is valid, but there is a more elegant solution
  - This value can calculated with a **dot product**

$$d\phi = \vec{E} \cdot d\vec{A}$$

- where $d\vec{A}$ is a vector perpendicular to the surface with a magnitude equal to the area of the surface
- At some points, the $\vec{E}$ field may pierce *into* the surface and in other points, it may pierce outwards
  - In order to find the **net electric flux**, we use integration

$$\phi = \oint \vec{E} \cdot d\vec{A}$$

23.02

Gauss' Law

- **Gauss' law** = a mathematical model that relates **net flux(ϕ)** and enclosed charge
- Mathematically it looks like this

$$\epsilon_0 \Sigma\phi = \Sigma q$$

- Or, substituting the definition of net flux, we get

$$\epsilon_0 \oint \left(\vec{E} \cdot d\vec{A} \right) = \Sigma q$$

- The charge of Σq determines whether the flux is *inwards* or *outwards*
  - If Σq is *positive*, $\Sigma \phi$ points outward
  - If Σq is *negative*, $\Sigma \phi$ points inward
  - If Σq is zero, $\Sigma \phi$ is a zero
- The interesting thing about Gauss' law is that charges external to the enclosed volume do not affect the net flux
  - Think about that: *if you put a charged particle right up against the barrier, the field lines would change but the net flux wouldn't*

Deriving Coulomb's Law with Gauss' Law

- Recall **Coulomb's law**

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

- This can actually be proven using Gauss' law
- Imagine we had a particle of point q
  - Now superimpose a gaussian sphere that envelops that particle
    - We can use the integral form of Gauss' law to set up an equation

$$\epsilon_0 \oint \left(\vec{E} \cdot d\vec{A} \right) = \Sigma q$$

- A property of dot products is that

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|cos(\theta)$$

- where θ is the angle between $\vec{a}$ and $\vec{b}$
- Using that fact, we can rewrite our equation as

$$\epsilon_0 \oint \left(|\vec{E}||d\vec{A}|cos(\theta) \right) = \Sigma q$$

- Since our problem is basically a one-particle problem, we know that $\vec{E}$ will radiate outwards perpendicular to concecntric spheres
  - As such, $\vec{E}$ and $d\vec{A}$ actually point in the same direction
    - So, θ is zero

$$\epsilon_0 \oint \left(|\vec{E}||d\vec{A}|cos(0) \right) = \Sigma q$$

$$\epsilon_0 \oint \left(|\vec{E}||d\vec{A}|(1) \right) = \Sigma q$$

$$\epsilon_0 \oint \left(|\vec{E}||d\vec{A}| \right) = \Sigma q$$

- At this point, we can rewrite Σq as just q, since our gaussian sphere only contains that one particle

$$\epsilon_0 \oint \left(|\vec{E}||d\vec{A}|\right) = q$$

- Now, the direction of $\vec{E}$ clearly changes from point to point on the gaussian sphere
  - However, the *magnitude* does not change
    - So, we can pull it out of the integral

$$\epsilon_0 |\vec{E}| \oint |d\vec{A}| = q$$

- Now, surface integrals, which is what that $\oint$ symbol denotes, aren't in the scope of this course
  - Really, all you need to know is that they integrate a function over every point on a surface(in this case, the surface area of the sphere)
  - If the function your surface integrates is just 1, then the surface integral returns the surface area
    - So, really, our surface integral just returns the surface area of our sphere
      - Which, if you remember from geometry is

$$SA = 4\pi r^2$$

- Substituting that, we get

$$\epsilon_0 |\vec{E}| \left(4\pi r^2\right) = q$$

- Then, some simple algebra gets us to

$$|\vec{E}| = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

23.03

Gauss' Law and the Behavior of Conductors

- Gauss' law can actually be used to explain phenomenon regarding conductors with excess charge
  - For example, in a conductor with excess charge, the free electrons will disperse themselves amongst the outer surface of the object

        - This kind of makes intuitive sense, as like charges repel, and that permutation ensures maximum distance between particles

- We can demonstrate this fact through Gauss' law
    - Imagine we had a chunk of copper with excess charge q hanging from an insulating thread
    - Now superimpose a Gaussian surface that is *just* inside of the outer surface of the copper
    - Now, if we assume there is no current *inside* the copper, we can deduce that $\vec{E}$ is zero among all points inside of the surface
        - This is because, in order for there to be current, there must be a non-zero force pushing electrons around
            - Which cannot exist without a nonzero $\vec{E}$ field
    - If we make that assumption, then we can use Gauss' law

$$\epsilon_0 \oint \left(\vec{E} \cdot d\vec{A}\right) = \Sigma q$$

$$\epsilon_0 \oint \left(\vec{0} \cdot d\vec{A}\right) = \Sigma q$$

- Now, any dot product between a vector and the zero vector($\vec{0}$) is just equal to 0(the scalar this time)

$$\epsilon_0 \left(0\right) = \Sigma q$$

$$0 = \Sigma q$$

$$\Sigma q = 0$$

- As such, our excess charge q *cannot* exist *inside* of the chunk of copper
    - Rather, it must exist on the outer surface of it
- We can demonstrate similar properties with different conductor shapes

Gauss' Law and a Conductor with a Cavity

- The same line of reasoning can be used on a conductor with a cavity
- Imagine a chunk of copper with excess charge q hanging from an insulating thread
    - Now, without changing the charge, remove some material from the core of the material
        - The result is like a tennis ball with thick material; hollow inside but solid on the exterior
- Now, making that same assumption that there is no internal current, we can form a Gaussian surface just inside of the very exterior of the object

- And, we can conclude the flux is zero, since there cannot be any net flux field if there is no current
    - Then, we use Gauss' law to conclude the charge enclosed by that Gaussian surface is zero
        - As such, the charge must only reside on the very outer surface of the object

Gauss' Law and a Vanishing Conductor

- Now, imagine you had the hollow chunk of copper from the previous example
    - We know the charge carriers would distribute themselves along the outer surface of the object
    - Now, imagine we expanded the hollow core until the copper conductor simply didn't exist
        - For the purposes of visualization, also assume that the charge carriers didn't move during the process
    - At the very instant where that last shell of copper disappears, the $\vec{E}$ field does not change
        - This is because the $\vec{E}$ field is set up by charges, not by conductors
- The lesson here is that charged particles will try to space themselves as far from one another along the outer surface of an object
    - Not only that, but the particles are practically limited in mobility by the size of the conductor
        - If the conductor were instantly made larger, the particles would bubble up to the outer surface–this time farther apart from one another

Gauss' Law and Surface Charge Density on Non-spherical conductors

- Recall that the symbol for surface charge density is σ
- In any spherical conductor, electrostatic equilibrium will be attained the σ not changing over the surface
    - This makes sense, because of the nature of the sphere's symmetry
- However, in a non-spherical conductor, things get *interesting*
    - Imagine we had a non-spherical conductor and we selected a differential circular area along that surface
        - Label the differential area dA
            - Note that this is distinct from $d\vec{A}$–a vector; dA just represents the patch of area along the surface

- Now, imagine creating a Gaussian cylinder whose bases are parallel to dA
  - Since dA is assumed to be infinitesimally small, we can assume that it is essentially flat
  - You can imagine that the cylinder encompasses some enclosed charge q
    - As we have seen in the previous sections, the charge inhabits the surface of the object we are studying which is within our cylinder
  - Then, since dA is a differential area, you can imagine that all the $\vec{E}$ vectors are *perpendicular* to dA
  - Then, we can set up Gauss' law, this time in its non-integral form

$$\epsilon_0 \Sigma\phi = \Sigma q$$

- In order to evaluate $\Sigma\phi$, we can split the cylinder into three surfaces
  - The inner base — B_1
  - The outer base — B_2
  - The curved side — S
- Since B_1 resides inside of the conductor, it experiences no flux
- Since S is essentially perpendicular to dA, the $\vec{E}$ vectors don't pierce S, and it contributes no flux
- Thus, all of the flux comes from B_2, the base of the cylinder that lies outside of the conductor
  - We can define σ as follows

$$\sigma = \frac{\Sigma q}{dA}$$

- Using that definition, we can rewrite Gauss' law

$$\epsilon_0 \Sigma\phi = \sigma dA$$

- Flux can be evaluated just by multiplying the magnitude of the piercing component and the area
  - And, since we demonstrated that flux only comes from B_2, we can do that quite easily
    - Note that we assume dA is equal in area to B_2, but this is a safe assumption because they are differential areas

$$\Sigma\phi = |\vec{E}|dA$$

- Using that definition, we can again rewrite Gauss' law

$$\epsilon_0 |\vec{E}|dA = \sigma dA$$

- Then, we can cancel out the dA's from both sides

$$\epsilon_0 |\vec{E}| = \sigma$$

- and isolate $|\vec{E}|$
$$|\vec{E}| = \frac{\sigma}{\epsilon_0}$$
- This formula gives us the magnitude of the $\vec{E}$ field *just* outside of the conductor
- gg well played m8

23.04

Using Gauss' Law to Find $\vec{E}$ Near an Infinite Charged Rod

- Imagine you had a cylindrical rod that extended infinitely in both directions
  - Also imagine the linear charge density(λ) is constant over the object
- We can use Gauss' Law to determine $\vec{E}$ for a point P whose distance from the charged rod is r
  - In order to do this, superimpose a Gaussian cylinder with the charged rod intersecting the circular bases at their center

- We can use Gauss' Law to determine some interesting things about the situation
$$\epsilon_0 \oint \left(\vec{E} \cdot d\vec{A}\right) = \Sigma q$$
- Using the definition of dot product, we get
$$\epsilon_0 \oint \left(|\vec{E}||d\vec{A}|cos(\theta)\right) = \Sigma q$$
$$\epsilon_0 \oint \left(|\vec{E}|cos(\theta)|d\vec{A}|\right) = \Sigma q$$
- Now, we can split our cylinder into three surfaces: two bases and one curved side
  - The two bases are parallel to $\vec{E}$, so they contribute no flux
    - Therefore, all of the flux must come from the curved side
- Now, we must consider whether $|\vec{E}|$ is constant
  - Because the rod is infinite in both directions, moving up or down while staying the same distance to the rod will have no difference

        * Thus, $|\vec{E}|$ is constant, and we can pull it out of our integral

$$\epsilon_0 |\vec{E}| \oint \big(cos(\theta)|d\vec{A}|\big) = \Sigma q$$

- Now, since the way our Gaussian surface is set up, θ is always equal to 0

$$\epsilon_0 |\vec{E}| \oint \big(cos(0)|d\vec{A}|\big) = \Sigma q$$

$$\epsilon_0 |\vec{E}| \oint \big((1)|d\vec{A}|\big) = \Sigma q$$

$$\epsilon_0 |\vec{E}| \oint \big(|d\vec{A}|\big) = \Sigma q$$

- Now, like in a previous section, we've encountered a surface integral whose integrating function is just one
    - This will just returns the surface area of our surface
        * Note that we have dismissed the two bases early on, so this is only the surface area of the curved wall

$$SA = 2\pi r h$$

    - Kind of like how $\int dx$ just returns x
- Substituting that, we get

$$\epsilon_0 |\vec{E}| \big(2\pi r h\big) = \Sigma q$$

- Now, Σq is just the enclosed charge within the cylinder
    - Since we know the linear charge density(λ), we can just multiply that by the height of the cylinder

$$\Sigma q = \lambda h$$

- Substituting that, we get

$$\epsilon_0 |\vec{E}| \big(2\pi r h\big) = \lambda h$$

$$\epsilon_0 |\vec{E}| \big(2\pi r\big) = \lambda$$

- Isolating $|\vec{E}|$, we get

$$|\vec{E}| = \frac{\lambda}{2\pi\epsilon_0 r}$$

- And, with that, we have a nice little formula for $\vec{E}$ at a point P outside of a infinite charged rod
    - This class sure has a lot of formulas

23.05

Gauss' Law and Planar Symmetry

- Imagine we had an infinite, nonconducting plane with excess positive charge
  - Also assume the surface charge density(σ) is constant throughout the plane
- Let's imagine we wanted to find $\vec{E}$ at a point P that is d units from the plane
  - We can set up Gauss' Law in non-integral form

$$\epsilon_0 \Sigma\phi = q_{enc}$$

- Our Gaussian surface is going to be a cylinder perpendicular to the plane
  - The curved wall is parallel to $\vec{E}$, so it contributes no flux
  - Instead, the flux comes from the two bases
    * If $\vec{E}$ is constant over the two bases, we can find the flux just by multiplying the surface area with $|\vec{E}|$
      · $\vec{E}$ actually *is* constant over the end caps, because each point is the same distance from the charge source

$$\Sigma\phi = \left(|\vec{E}|A\right) + \left(|\vec{E}|A\right)$$

$$\Sigma\phi = 2|\vec{E}|A$$

- The two bases both have positive flux, because the source charge is positive, and $\vec{E}$ points outward
- Substituting that into Gauss' law, we get

$$\epsilon_0\left(2|\vec{E}|A\right) = q_{enc}$$

- And, solving for $|\vec{E}|$

$$|\vec{E}| = \frac{q_{enc}}{2A\epsilon_0}$$

- But remember that

$$\sigma = \frac{q}{A}$$

- Making that substitution, we get

$$|\vec{E}| = \frac{\sigma}{2\epsilon_0}$$

- And with that, we have a formula for $|\vec{E}|$ just outside of the positively charged infinite sheet
  - Isn't physics *so useful?*
    * I know this can be boring but bear with it

23.06

Proving Shell Theorems with Gauss' Law

- As it turns out, we can use this tool to prove the two shell theorems

First Shell Theorem

> A charged particle outside a shell with charge uniformly distributed on its surface is attracted or repelled as if the shell's charge were concentrated as a particle at the shell's center

Proof