

23.01

A Note About This Chapter

- Last chapter was fairly brutal
 - Physics is really about making hard problems easy with abstraction
 - * This chapter will be less labor-intensive
- Through *symmetry*, we can skip parts of problems
 - Like how we only summed the vertical components of $d\vec{E}$ in 22.04

Gauss' Law

- **Gauss' law** = a law that relates net charge of a volume enclosed by a closed surface and the \vec{E} field about that closed surface
 - Discovered by Carl Friedrich Gauss
 - * Lived from 1777 until 1855
- Imagine a particle of positive charge q
 - Now superimpose a sphere centered at the particle
 - * The surface of the sphere is called a **Gaussian surface**
 - * The \vec{E} vectors around the surface point radially outwards
 - Because the particle is *positive*
 - * Those same vectors are said to **pierce** the surface of the sphere
- The essential utility of **Gauss' Law** is that we can infer things about the net charge of an object by examining the \vec{E} field about its outer surface
 - Or, equivalently, we can use the net charge to infer information about the \vec{E} about the object's outer surface

Electric Flux

- **Electric flux** = a metric of *how much* the \vec{E} field *pierces* the Gaussian surface
 - The symbol for **electric flux** is ϕ
- The best way to learn about this is to just do a bunch of examples
- The ϕ is
 - Positive if \vec{E} pierces outward
 - Zero if \vec{E} is parallel to the differential area
 - Negative if \vec{E} pierces inward

Electric Flux On a Flat Surface in a Uniform \vec{E} Field

- Imagine we had a uniform \vec{E} field
 - Now superimpose a flat surface of area A
 - * Orient it along with yz-plane with its center point at the origin
 - Denote the angle that the uniform \vec{E} vectors make with the x-axis as θ
 - Then, we can imagine splitting the \vec{E} vectors into two components
 - * One that *directly* pierces the surface
 - Directly perpendicular to the surface
 - This vector is the **electric flux** for any given differential area
 - * One that doesn't pierce the surface at all
 - Directly parallel to the surface
- We can define the magnitude of the electric flux in a subarea of A as

$$d\phi = |\vec{E}| \cos(\theta)$$

- This is valid, but there is a more elegant solution
 - This value can be calculated with a **dot product**

$$d\phi = \vec{E} \cdot d\vec{A}$$

- where $d\vec{A}$ is a vector perpendicular to the surface with a magnitude equal to the area of the surface
- At some points, the \vec{E} field may pierce *into* the surface and in other points, it may pierce outwards
 - In order to find the **net electric flux**, we use integration

$$\phi = \oint \vec{E} \cdot d\vec{A}$$

23.02

Gauss' Law

- **Gauss' law** = a mathematical model that relates **net flux**(ϕ) and enclosed charge
- Mathematically it looks like this

$$\epsilon_0 \Sigma \phi = \Sigma q$$

- Or, substituting the definition of net flux, we get

$$\epsilon_0 \oint \left(\vec{E} \cdot d\vec{A} \right) = \Sigma q$$

- The charge of Σq determines whether the flux is *inwards* or *outwards*
 - If Σq is *positive*, $\Sigma \phi$ points outward
 - If Σq is *negative*, $\Sigma \phi$ points inward
 - If Σq is zero, $\Sigma \phi$ is a zero
- The interesting thing about Gauss' law is that charges external to the enclosed volume do not affect the net flux
 - Think about that: *if you put a charged particle right up against the barrier, the field lines would change but the net flux wouldn't*

Deriving Coulomb's Law with Gauss' Law

- Recall **Coulomb's law**

$$\vec{E} = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

- This can actually be proven using Gauss' law
- Imagine we had a particle of point q
 - Now superimpose a gaussian sphere that envelops that particle
 - * We can use the integral form of Gauss' law to set up an equation

$$\epsilon_0 \oint \left(\vec{E} \cdot d\vec{A} \right) = \Sigma q$$

- A property of dot products is that

$$\vec{a} \cdot \vec{b} = |\vec{a}| |\vec{b}| \cos(\theta)$$

- where θ is the angle between \vec{a} and \vec{b}
- Using that fact, we can rewrite our equation as

$$\epsilon_0 \oint \left(|\vec{E}| |d\vec{A}| \cos(\theta) \right) = \Sigma q$$

- Since our problem is basically a one-particle problem, we know that \vec{E} will radiate outwards perpendicular to concentric spheres
 - As such, \vec{E} and $d\vec{A}$ actually point in the same direction
 - * So, θ is zero

$$\epsilon_0 \oint \left(|\vec{E}| |d\vec{A}| \cos(0) \right) = \Sigma q$$

$$\epsilon_0 \oint \left(|\vec{E}| |d\vec{A}| (1) \right) = \Sigma q$$

$$\epsilon_0 \oint \left(|\vec{E}| |d\vec{A}| \right) = \Sigma q$$

- At this point, we can rewrite Σq as just q , since our gaussian sphere only contains that one particle

$$\epsilon_0 \oint \left(|\vec{E}| |d\vec{A}| \right) = q$$

- Now, the direction of \vec{E} clearly changes from point to point on the gaussian sphere
 - However, the *magnitude* does not change
 - * So, we can pull it out of the integral

$$\epsilon_0 |\vec{E}| \oint |d\vec{A}| = q$$

- Now, surface integrals, which is what that \oint symbol denotes, aren't in the scope of this course
 - Really, all you need to know is that they integrate a function over every point on a surface (in this case, the surface area of the sphere)
 - If the function your surface integrates is just 1, then the surface integral returns the surface area
 - * So, really, our surface integral just returns the surface area of our sphere
 - Which, if you remember from geometry is

$$SA = 4\pi r^2$$

- Substituting that, we get

$$\epsilon_0 |\vec{E}| (4\pi r^2) = q$$

- Then, some simple algebra gets us to

$$|\vec{E}| = \frac{1}{4\pi\epsilon_0} \frac{q}{r^2}$$

23.03

Gauss' Law and the Behavior of Conductors

- Gauss' law can actually be used to explain phenomenon regarding conductors with excess charge
 - For example, in a conductor with excess charge, the free electrons will disperse themselves amongst the outer surface of the object

- * This kind of makes intuitive sense, as like charges repel, and that permutation ensures maximum distance between particles
- We can demonstrate this fact through Gauss' law
 - Imagine we had a chunk of copper with excess charge q hanging from an insulating thread
 - Now superimpose a Gaussian surface that is *just* inside of the outer surface of the copper
 - Now, if we assume there is no current *inside* the copper, we can deduce that \vec{E} is zero among all points inside of the surface
 - * This is because, in order for there to be current, there must be a non-zero force pushing electrons around
 - Which cannot exist without a nonzero \vec{E} field
 - If we make that assumption, then we can use Gauss' law

$$\epsilon_0 \oint (\vec{E} \cdot d\vec{A}) = \Sigma q$$

$$\epsilon_0 \oint (\vec{0} \cdot d\vec{A}) = \Sigma q$$

- Now, any dot product between a vector and the zero vector($\vec{0}$) is just equal to 0(the scalar this time)

$$\epsilon_0 \oint (\vec{0}) = \Sigma q$$

$$0 = \Sigma q$$

$$\Sigma q = 0$$

- As such, our excess charge q *cannot* exist *inside* of the chunk of copper
 - Rather, it must exist on the outer surface of it
- We can demonstrate similar properties with different conductor shapes

Gauss' Law and a Conductor with a Cavity

- The same line of reasoning can be used on a conductor with a cavity
- Imagine a chunk of copper with excess charge q hanging from an insulating thread
 - Now, without changing the charge, remove some material from the core of the material
 - * The result is like a tennis ball with thick material; hollow inside but solid on the exterior
- Now, making that same assumption that there is no internal current, we can form a Gaussian surface just inside of the very exterior of the object

- And, we can conclude the flux is zero, since there cannot be any net flux field if there is no current
 - * Then, we use Gauss' law to conclude the charge enclosed by that Gaussian surface is zero
 - As such, the charge must only reside on the very outer surface of the object

Gauss' Law and a Vanishing Conductor

- Now, imagine you had the hollow chunk of copper from the previous example
 - We know the charge carriers would distribute themselves along the outer surface of the object
 - Now, imagine we expanded the hollow core until the copper conductor simply didn't exist
 - * For the purposes of visualization, also assume that the charge carriers didn't move during the process
 - At the very instant where that last shell of copper disappears, the \vec{E} field does not change
 - * This is because the \vec{E} field is set up by charges, not by conductors
- The lesson here is that charged particles will try to space themselves as far from one another along the outer surface of an object
 - Not only that, but the particles are practically limited in mobility by the size of the conductor
 - * If the conductor were instantly made larger, the particles would bubble up to the outer surface—this time farther apart from one another

Gauss' Law and Surface Charge Density on Non-spherical conductors

- Recall that the symbol for surface charge density is σ
- In any spherical conductor, electrostatic equilibrium will be attained the σ not changing over the surface
 - This makes sense, because of the nature of the sphere's symmetry
- However, in a non-spherical conductor, things get *interesting*
 - Imagine we had a non-spherical conductor and we selected a differential circular area along that surface
 - * Label the differential area dA
 - Note that this is distinct from $d\vec{A}$ —a vector; dA just represents the patch of area along the surface

- Now, imagine creating a Gaussian cylinder whose bases are parallel to dA
 - Since dA is assumed to be infinitesimally small, we can assume that it is essentially flat
 - You can imagine that the cylinder encompasses some enclosed charge q
 - * As we have seen in the previous sections, the charge inhabits the surface of the object we are studying which is within our cylinder
 - Then, since dA is a differential area, you can imagine that all the \vec{E} vectors are *perpendicular* to dA
 - Then, we can set up Gauss' law, this time in its non-integral form

$$\epsilon_0 \Sigma \phi = \Sigma q$$

- In order to evaluate $\Sigma \phi$, we can split the cylinder into three surfaces
 - The inner base — B_1
 - The outer base — B_2
 - The curved side — S
- Since B_1 resides inside of the conductor, it experiences no flux
- Since S is essentially perpendicular to dA , the \vec{E} vectors don't pierce S , and it contributes no flux
- Thus, all of the flux comes from B_2 , the base of the cylinder that lies outside of the conductor
 - We can define σ as follows

$$\sigma = \frac{\Sigma q}{dA}$$

- Using that definition, we can rewrite Gauss' law

$$\epsilon_0 \Sigma \phi = \sigma dA$$

- Flux can be evaluated just by multiplying the magnitude of the piercing component and the area
 - And, since we demonstrated that flux only comes from B_2 , we can do that quite easily
 - * Note that we assume dA is equal in area to B_2 , but this is a safe assumption because they are differential areas

$$\Sigma \phi = |\vec{E}| dA$$

- Using that definition, we can again rewrite Gauss' law

$$\epsilon_0 |\vec{E}| dA = \sigma dA$$

- Then, we can cancel out the dA 's from both sides

$$\epsilon_0 |\vec{E}| = \sigma$$

- and isolate $|\vec{E}|$

$$|\vec{E}| = \frac{\sigma}{\epsilon_0}$$

- This formula gives us the magnitude of the \vec{E} field *just* outside of the conductor
 - gg well played m8
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23.04

Using Gauss' Law to Find \vec{E} Near an Infinite Charged Rod

- Imagine you had a cylindrical rod that extended infinitely in both directions
 - Also imagine the linear charge density(λ) is constant over the object
- We can use Gauss' Law to determine \vec{E} for a point P whose distance from the charged rod is r
 - In order to do this, superimpose a Gaussian cylinder with the charged rod intersecting the circular bases at their center

- We can use Gauss' Law to determine some interesting things about the situation

$$\epsilon_0 \oint (\vec{E} \cdot d\vec{A}) = \Sigma q$$

- Using the definition of dot product, we get

$$\epsilon_0 \oint (|\vec{E}| |d\vec{A}| \cos(\theta)) = \Sigma q$$

$$\epsilon_0 \oint (|\vec{E}| \cos(\theta) |d\vec{A}|) = \Sigma q$$

- Now, we can split our cylinder into three surfaces: two bases and one curved side
 - The two bases are parallel to \vec{E} , so they contribute no flux
 - * Therefore, all of the flux must come from the curved side
- Now, we must consider whether $|\vec{E}|$ is constant
 - Because the rod is infinite in both directions, moving up or down while staying the same distance to the rod will have no difference

* Thus, $|\vec{E}|$ is constant, and we can pull it out of our integral

$$\epsilon_0 |\vec{E}| \oint (\cos(\theta) |d\vec{A}|) = \Sigma q$$

- Now, since the way our Gaussian surface is set up, θ is always equal to 0

$$\epsilon_0 |\vec{E}| \oint (\cos(0) |d\vec{A}|) = \Sigma q$$

$$\epsilon_0 |\vec{E}| \oint ((1) |d\vec{A}|) = \Sigma q$$

$$\epsilon_0 |\vec{E}| \oint (|d\vec{A}|) = \Sigma q$$

- Now, like in a previous section, we've encountered a surface integral whose integrating function is just one

- This will just return the surface area of our surface

- * Note that we have dismissed the two bases early on, so this is only the surface area of the curved wall

$$SA = 2\pi rh$$

- Kind of like how $\int dx$ just returns x

- Substituting that, we get

$$\epsilon_0 |\vec{E}| (2\pi rh) = \Sigma q$$

- Now, Σq is just the enclosed charge within the cylinder

- Since we know the linear charge density(λ), we can just multiply that by the height of the cylinder

$$\Sigma q = \lambda h$$

- Substituting that, we get

$$\epsilon_0 |\vec{E}| (2\pi rh) = \lambda h$$

$$\epsilon_0 |\vec{E}| (2\pi r) = \lambda$$

- Isolating $|\vec{E}|$, we get

$$|\vec{E}| = \frac{\lambda}{2\pi\epsilon_0 r}$$

- And, with that, we have a nice little formula for \vec{E} at a point P outside of a infinite charged rod

- This class sure has a lot of formulas

23.05

Gauss' Law and Planar Symmetry

- Imagine we had an infinite, nonconducting plane with excess positive charge
 - Also assume the surface charge density(σ) is constant throughout the plane
- Let's imagine we wanted to find \vec{E} at a point P that is d units from the plane
 - We can set up Gauss' Law in non-integral form

$$\epsilon_0 \Sigma \phi = q_{enc}$$

- Our Gaussian surface is going to be a cylinder perpendicular to the plane
 - The curved wall is parallel to \vec{E} , so it contributes no flux
 - Instead, the flux comes from the two bases
 - * If \vec{E} is constant over the two bases, we can find the flux just by multiplying the surface area with $|\vec{E}|$
 - \vec{E} actually *is* constant over the end caps, because each point is the same distance from the charge source

$$\Sigma \phi = (|\vec{E}|A) + (|\vec{E}|A)$$