The radio emission model

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The downstream electron spectrum

Test particle regime power-law distribution: Diffusive shock acceleration (DSA) theory predicts (in its simplest incarnation, i.e. without reaction effects, 'test particle' regime) that the electron distribution function, $dn_{\mathrm{e,rel}}/dp$, is a power-law of the electron momentum, p,

$$\frac{\mathrm{d}n_{\mathrm{e,rel}}}{\mathrm{d}\mathbf{p}}(p) = \frac{1}{4\pi}\frac{n_{\mathrm{e}}\, f_{\mathrm{rel}}}{(m_{\mathrm{e}}c)^3}\left(\frac{p}{m_{\mathrm{e}}c}\right)^{-s} \qquad \text{with} \qquad s = \frac{3r}{r-1}, \tag{1}$$

where r is the compression ratio at the shock front [1]. Note, the spectral slope, s is in 3D momentum space, i.e. $s > 4$.

Momentum cut-off: We introduce a cut-off in the momentum distribution of the form

$$(1 - p/p_{\mathrm{max}})^x, \tag{2}$$

where x has to be positive, $x > 0$. We follow [2] and choose for the exponent $(s-4)$, note again, s is given in 3D momentum space, hence $s > 4$. The electron distribution function becomes

$$\frac{\mathrm{d}n_{\mathrm{e,rel}}}{\mathrm{d}\mathbf{p}}(p) = \frac{1}{4\pi}\frac{n_{\mathrm{e}}\, f_{\mathrm{rel}}}{(m_{\mathrm{e}}c)^3}\left(\frac{p}{m_{\mathrm{e}}c}\right)^{-s}\left(1 - \frac{p}{p_{\mathrm{max}}}\right)^{s-4}. \tag{3}$$

The momentum cut-off is particular important for the normalization, n_0, when the spectrum is 'soft', i.e. $s \to 4$.

Number of relativistic electrons: The total number of relativistic electrons is given via

$$n_{\mathrm{e,rel}} = \int \mathrm{d}\mathbf{p}\,\frac{\mathrm{d}n_{\mathrm{e}}}{\mathrm{d}p} = \int_{p_{\mathrm{min}}}^{\infty}\mathrm{d}p\; 4\pi p^2\;\frac{\mathrm{d}n_{\mathrm{e}}}{\mathrm{d}p} \tag{4}$$

Kinetic energy content: The kinetic energy density stored in particles with momentum larger p_{min} is given by

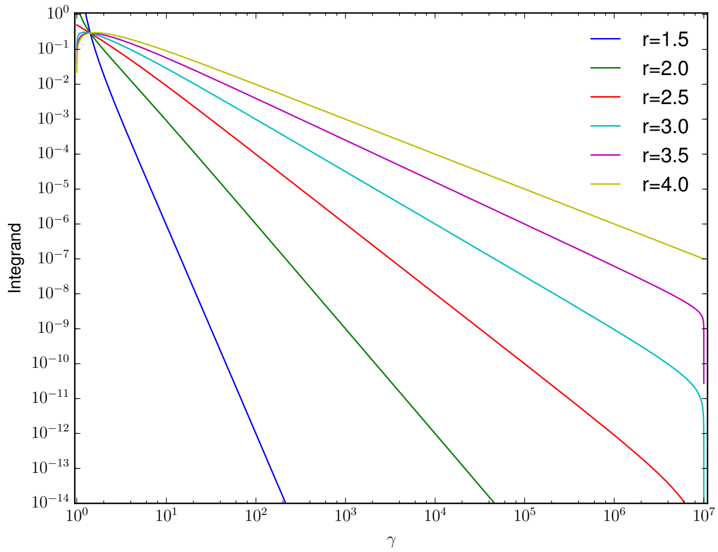
$$\varepsilon_{\mathrm{kin,rel}}(> p_{\mathrm{min}}) = \int_{p_{\mathrm{min}}}^{\infty}\mathrm{d}p\; 4\pi p^2\;\frac{\mathrm{d}n_{\mathrm{e}}}{\mathrm{d}p}\; E_{\mathrm{kin}}(p). \tag{5}$$

We now insert the expression for the momentum distribution as given above, and then change the integration variable to γ

$$\begin{aligned}
\varepsilon_{\mathrm{kin,rel}}(>p_{\mathrm{min}}) &= n_{\mathrm{e}}\, f_{\mathrm{rel}} \int_{p_{\mathrm{min}}}^{p_{\mathrm{max}}}\mathrm{d}p\; p^2\;\frac{1}{(m_{\mathrm{e}}c)^3}\;\left(\frac{p}{m_{\mathrm{e}}c}\right)^{-s}\left(1 - \frac{p}{p_{\mathrm{max}}}\right)^{s-4} E_{\mathrm{kin}}(p) && (6)\\
&= n_{\mathrm{e}}\, f_{\mathrm{rel}} \int_{\gamma_{\mathrm{min}}}^{\gamma_{\mathrm{max}}}\mathrm{d}\gamma\;\frac{\mathrm{d}p}{\mathrm{d}\gamma}\;\frac{1}{m_{\mathrm{e}}c}\;(\gamma\beta)^{-(s-2)}\;\left[1 - \left(\frac{\gamma^2-1}{\gamma_{\mathrm{max}}^2-1}\right)^{1/2}\right]^{s-4}(\gamma-1)\, m_{\mathrm{e}}c^2 && (7)\\
&= n_{\mathrm{e}}\, f_{\mathrm{rel}}\, m_{\mathrm{e}}c^2 \int_{\gamma_{\mathrm{min}}}^{\gamma_{\mathrm{max}}}\mathrm{d}\gamma\;\beta\,(\gamma\beta)^{2-s}\left[1 - \left(\frac{\gamma^2-1}{\gamma_{\mathrm{max}}^2-1}\right)^{1/2}\right]^{s-4}(\gamma-1) && (8)\\
&= n_{\mathrm{e}}\, f_{\mathrm{rel}}\, m_{\mathrm{e}}c^2 \int_{\gamma_{\mathrm{min}}}^{\gamma_{\mathrm{max}}}\mathrm{d}\gamma\;\left(1 - \frac{1}{\gamma^2}\right)^{\frac{1}{2}}(\gamma^2-1)^{\frac{2-s}{2}}\left[1 - \left(\frac{\gamma^2-1}{\gamma_{\mathrm{max}}^2-1}\right)^{1/2}\right]^{s-4}(\gamma-1) && (9)\\
&= n_{\mathrm{e}}\, f_{\mathrm{rel}}\, m_{\mathrm{e}}c^2\; I(s,\gamma_{\mathrm{min}},\gamma_{\mathrm{max}}), && (10)
\end{aligned}$$

where $I(s,\gamma_{\mathrm{min}},\gamma_{\mathrm{max}})$ denotes the integral. Hence, the normalization is given via

$$n_{\mathrm{e}}\, f_{\mathrm{rel}} = \frac{\varepsilon_{\mathrm{kin,rel}}(>\gamma_{\mathrm{min}})}{m_{\mathrm{e}}c^2}\frac{1}{I(s,\gamma_{\mathrm{min}},\gamma_{\mathrm{max}})} \tag{11}$$



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γ
r=1.5
r=2.0
r=2.5
r=3.0
r=3.5
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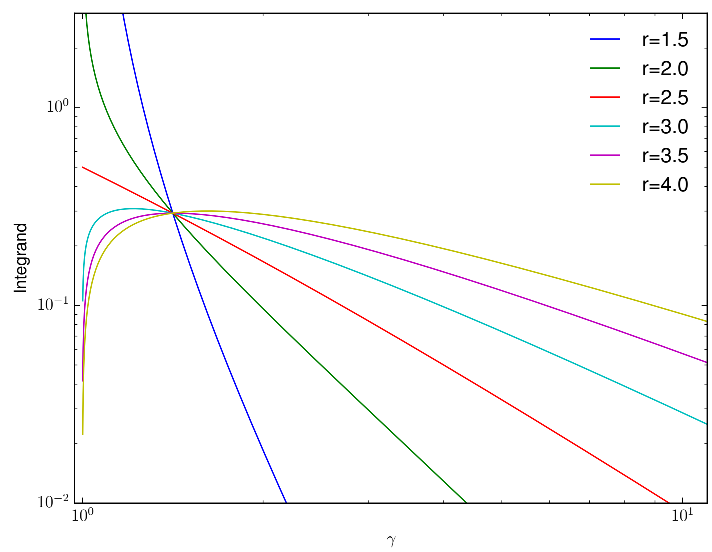

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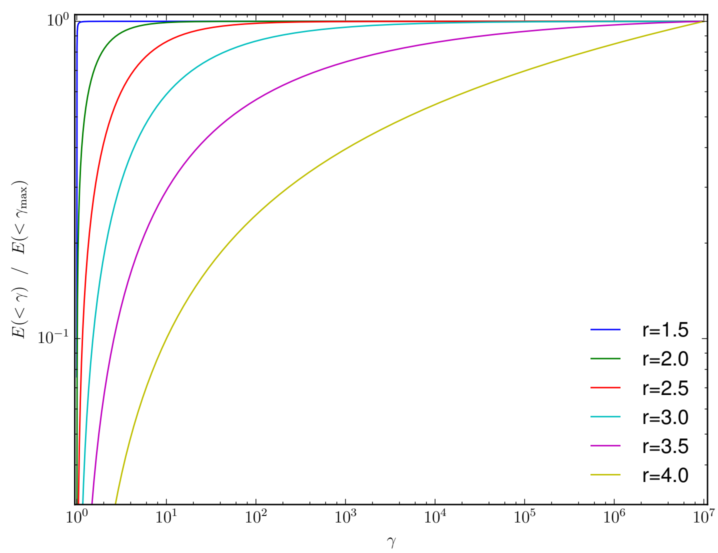

$E(<\gamma)\ /\ E(<\gamma_{\max})$
γ
r=1.5
r=2.0
r=2.5
r=3.0
r=3.5
r=4.0

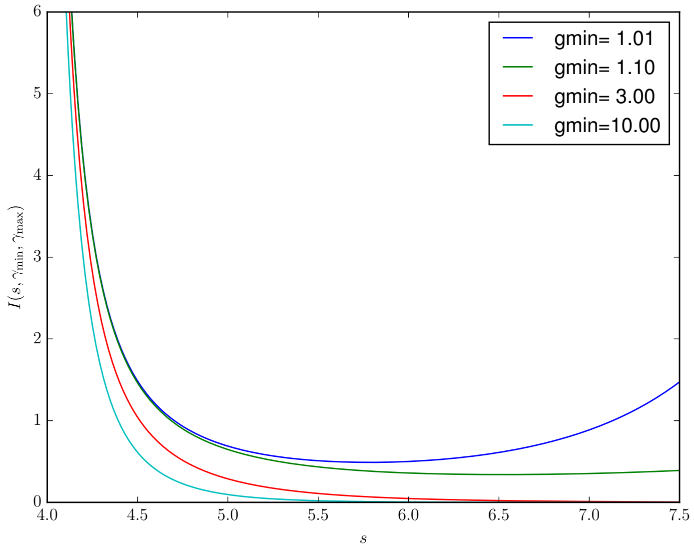

Figure 1: last panel: $\gamma_{\max} = 10^8$

Energy dissipation

In Hoeft & Brüggen (2007), Eq. 15, we found that the fraction of downstream internal energy density which has dissipated just at the shock is given by

$$u_{\mathrm{d}}\rho_{\mathrm{d}} - r^{\gamma} u_{\mathrm{u}}\rho_{\mathrm{u}} = u_{\mathrm{d}}\rho_{\mathrm{d}} \frac{q-1}{q}, \tag{12}$$

where q is the ratio between downstream and upstream entropy. For Mach numbers in the range of interest, namely 2 to 3, also the q is in the range 2 to 3.

The energy density is given by

$$\rho u = \varepsilon = \frac{n_{\mathrm{all}}}{n_{\mathrm{b}}} n_{\mathrm{b}} \frac{f}{2} k_{\mathrm{B}} T \tag{13}$$

Efficiency: As next step we assume that a fraction of this energy, ξ_e, is used to accelerate electrons to relativistic particles as introduced above.

Normalization: Finally, we can derive the normalization for the relativistic electron distribution

$$\xi_{\mathrm{e}} \frac{q-1}{q} \frac{n_{\mathrm{all}}}{n_{\mathrm{b}}} n_{\mathrm{b}} \frac{f}{2} k_{\mathrm{B}} T = n_{\mathrm{e}} f_{\mathrm{rel}} m_{\mathrm{e}} c^2 \, I(s, \gamma_{\min}, \gamma_{\max}) \tag{14}$$

Hence,

$$f_{\mathrm{rel}} = \xi_{\mathrm{e}} \frac{q-1}{q} \frac{n_{\mathrm{all}}}{n_{\mathrm{b}}} \frac{n_{\mathrm{b}}}{n_{\mathrm{e}}} \frac{f}{2} \frac{k_{\mathrm{B}} T}{m_{\mathrm{e}} c^2} \tag{15}$$

The thermal distribution

possible Maxwell-Jütter distribution ? [1]

$$f_{\mathrm{MJ}} = \frac{1}{4\pi m^3 c^3 \theta K_2(1/\theta)} \exp\left(-\frac{\gamma(p)}{\theta}\right) \tag{16}$$

with

$$\theta = \frac{kT}{mc^2}, \qquad \gamma(p) = \sqrt{1 + \left(\frac{p}{mc}\right)^2} \tag{17}$$

[1] https://en.wikipedia.org/wiki/Maxwell-Jüttner_distribution

Basic relation from special relativity

The energy-momentum relation

$$E^2 = (pc)^2 + (m_0 c^2)^2 \tag{18}$$

The kinetic energy is given as

$$E_{\mathrm{kin}} = \sqrt{p^2 c^2 + m_0^2 c^4} - m_0 c^2 \tag{19}$$

The relativistic momentum is given by

$$p = mv = \gamma m_0 \beta c, \tag{20}$$

with

$$\beta = \frac{v}{c} \qquad \text{and} \qquad \gamma = \frac{1}{\sqrt{1-\beta^2}} \tag{21}$$

Hence

$$\beta^2 = 1 - \frac{1}{\gamma^2} \tag{22}$$

The kinetic energy as a function of γ becomes

$$E_{\mathrm{kin}} = \sqrt{\gamma^2 \beta^2 m_0^2 c^4 + m_0^2 c^4} - m_0 c^2 = (\sqrt{(\gamma^2 - 1) + 1} - 1)\, m_0 c^2 = (\gamma - 1)\, m_0 c^2 \tag{23}$$

References

1. Drury, 1983, SSRv 36
2. Ensslin & Brüggen, 2002, MNRAS 331