Bezier regression model

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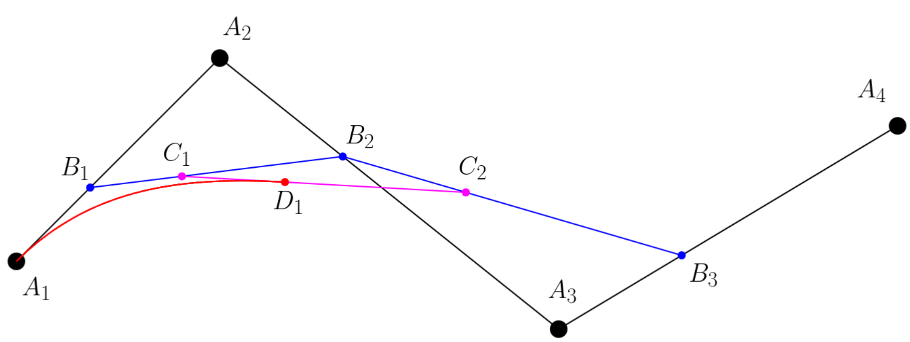
1 Inspiration

The bezier curve is calculated by providing N points. The algorithm calculates the points by changing t from 0 to 1. We can treat this model as kind a bijection. If we want to fit a bezier curve to a function, we can treat this model similarly to a neural network. We can then calculate an approximation using gradient descend.

2 Forward propagation

Here we have an example model at $N = 4$



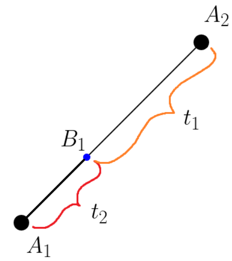
First we have the "forward propagation" - calculating the D_1

$$\vec{D_{n,t}} = P_t(\vec{C_{n,t}}, \vec{C_{n+1,t}}) \tag{1}$$

$$\vec{C_{m,t}} = P_t(\vec{B_{m,t}}, \vec{B_{m+1,t}}) \tag{2}$$

$$\vec{B_{k,t}} = P_t(\vec{A_{k,t}}, \vec{A_{k+1,t}}) \tag{3}$$

The function P takes in 2 vectors and returns a single one. This returns the "midpoint" between the 2 points, with a given coefficient:



Where $t = \frac{t_2}{t_1}$, therefore:

$$P_t(\vec{A_1}, \vec{A_2}) = \vec{A_1} + (\vec{A_2} - \vec{A_1}) * t \tag{4}$$

3 Backpropagation

To fit our bezier curve to a function $f(x)$, we have to calculate an error at each step. Using the mean square error, we can do this by the next equation (here n is determined by how accurate we want our function to be):

$$E = \sum_{t=0}^{1} \frac{1}{2}(f(x) - D_{1,t})^2 \tag{5}$$

The x here is calculated by the same function P using the first and the last point and the taking the x component. If we want to minimize this error, we can calculate the derivatives with respect to the input points $(A_1, A_2, ..., A_n)$, and then "go down the hill" with gradient descend. We have to approximate the y components of the vectors, so I will not be using vector notation from here forward. We can do this by applying the chain rule at each layer:

$$\frac{\partial E}{\partial A_n} = \sum_{t=0}^{1} \sum_{n=1}^{1} \sum_{m=1}^{2} \sum_{k=1}^{3} \frac{\partial E}{\partial D_{1,t}} * \frac{\partial D_{1,t}}{\partial C_{m,t}} * \frac{\partial C_{m,t}}{\partial B_{k,t}} * \frac{\partial B_{k,t}}{\partial A_{i,t}} \tag{6}$$

First lets define the derivatives of our function P

$$\frac{\partial P_t(A_1, A_2)}{\partial A_1} = \frac{\partial \left[A_1 + (A_2 - A_1) * t\right]}{\partial A_1} = 1 - t \tag{7}$$

$$\frac{\partial P_t(A_1, A_2)}{\partial A_2} = \frac{\partial \left[A_1 + (A_2 - A_1) * t\right]}{\partial A_2} = t \tag{8}$$

The first factor in the chain rule is the derivative of (5):

$$\frac{\partial E}{\partial D_{1,t}} = \frac{\partial}{\partial D_{1,t}} \left[\sum_{t_1=0}^{1} \frac{1}{2}(f(x) - D_{1,t_1})^2 \right] = (D_{t,1} - f(x)) \tag{9}$$

The second factor in the chain rule is the derivative of (1): Here there is a difference in the derivative with respect to C_m, if it is the first argument of the function, we use (7), else we use (**??**)

$$\frac{\partial D_{1,t}}{\partial C_{m,t}} = \frac{\partial\left[P_t(C_{n,t}, C_{n+1,t})\right]}{\partial C_{m,t}} = \begin{cases} 1-t & m=n \\ t & m=n+1 \end{cases} \tag{10}$$

All other factors are computed the same. General formula where K_t is the one layer forward of $J_{m,t}$ and isn't the last layer.

$$\frac{\partial K_t}{\partial J_{n,t}} = \frac{\partial\left[P_t(J_{n,t}, J_{n+1,t})\right]}{\partial J_{n,t}} + \frac{\partial\left[P_t(J_{n-1,t}, J_{n,t})\right]}{\partial J_{n,t}} \tag{11}$$