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NOTE

Digital Topology on Graphs

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The possibility or impossibility of topologization is shown for certain graphs which could represent a digital space. This includes a new proof of the known impossibility of topologizing the digital plane while retaining 8-connectivity. © 1996 Academic Press, Inc.

1. INTRODUCTION

A large number of publications are concerned with the problem of topologizing discrete structures. There are mainly two reasons for these efforts: Whenever spatial relations are modeled on a computer, a discrete topology is needed which resembles an ordinary topology as close as possible in the sense of implicitly containing as much "intrinsic" spatial information as possible. The other reason for dealing with discrete topological structures is the fact that topology is a powerful tool leading to such useful notions as continuity, connectivity, and homotopy.

Alexandroff and Hopf [1] introduced topology for discrete structures. This topology was constructed to suit the theory of cell-complexes. When applied to other digital structures, it lacks some essential properties which are desirable for certain applications. In picture processing, for example, one wants a topology which is translation invariant. This is not the case for the Alexandroff topology.

In this paper the following problem is investigated: Is it possible for a given graph G with a set V of vertices to introduce a topology on V, by declaring certain subsets of V as "open," so that a subset of V is topologically connected if and only if it is connected in G (i.e., if the corresponding subgraph of G is connected)?

The results we arrrive at are not new and this particular problem has been the subject of a number of articles, though often in a more specialized context. It is a generalization of the problem of constructing a topology on a digital space which retains one of the standard notions of digital connectivity (i.e., "4-connectivity," "8-connectivity"). In 1970 Marcus and Wyse [5] defined a topology on $\mathbb{Z}^n$ in which any subset is topologically connected if and only if it is $2n$-connected. In 1978 Chassery [2] proved this topology to be the only one on $\mathbb{Z}^2$ compatible with 4-connectivity. He further proved that there doesn't exist a topology on $\mathbb{Z}^2$ which retains the 8-connectivity. A much simpler proof of this latter fact was given quite recently by Latecki [4]. As a by-product of our investigations, a different proof which is extremely simple can be given for this assertion. At the end of the article, we will mention a further proof in which the *Alexandroff specialization* relation is used.

2. A TOPOLOGY ON A GRAPH

A graph $G = (V, E)$ is defined to consist of a set V of vertices or points, and a set $E \subseteq \binom{V}{2}$ of edges. For an edge $\{x, y\} \in E$, $x, y \in V$, we just write xy. G is a *connected* graph, if for every pair $x, y \in V$ there exists a finite sequence of vertices $v_1, \ldots, v_n \in V$, so that $xv_1, v_1v_2, \ldots, v_ny \in E$. If $V = \emptyset$, G is called *empty*. If V consists of a finite number n of points, which can be enumerated in such a way that $V = \{v_1, \ldots, v_n\}$ and $E = \{v_1v_2, \ldots, v_{n-1}v_n, v_nv_1\}$, then G is called a *circle*.

For a subset $V' \subseteq V$ and for $E' := \{xy \in E \mid x, y \in V'\}$ the graph $G' := (V', E')$ is called the *induced subgraph* of G. So, an induced subgraph retains all the edges of G which join vertices from V'. The subgraph induced by any $V' \subseteq V$ is denoted as $G[V']$. If G' is an induced subgraph of G, we write $G' \sqsubseteq G$.

If we want to denote a point set V of a specific graph G, we write $V(G)$. For edges, we use $E(G)$ accordingly.

For two graphs, $G_1 \cup G_2$ is defined as $(V(G_1) \cup V(G_2), E(G_1) \cup E(G_2))$.

DEFINITION 2.1. Let $G = (V, E)$ be a graph. Let O be a topology on V. O is called a topology on G if it satisfies the following conditions:

(1) For each connected $G' \sqsubseteq G$, $V(G')$ is connected in O.

(2) For each $V' \subseteq V$ connected in O, $G[V']$ is a connected graph.

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THEOREM 2.1. *Let G be a graph with a topology O. For every $H \sqsubseteq G$, the topology restricted to $V(H)$, $O|_{V(H)}$, is a topology on H.*

Proof. For every $H' \sqsubseteq H$ we have $O|_{V(H')} = O|_{V(H)}|_{V(H')}$. So *each subset* of $V(H)$ is connected in $O|_{V(H)}$ if and only if it is connected in O. And $H' \sqsubseteq G$ is connected if and only if $V(H') \subseteq V(H)$ is connected in O. Thus (1) and (2) follow.

3. THE TOPOLOGY OF BIPARTITE GRAPHS

Let G^b be a connected, *bipartite* graph $G^b = (V, E)$, V containing at least three points. Thus, V is the union of two nonempty, disjoint sets V_A, V_B, and *every edge* in E only joins vertices from V_A with vertices from V_B.

We now characterize two topologies on the point set V by describing for each point $x \in V$ its *topological neighbourhood*, i.e., the smallest open set $U_x \in O$ in which this point is contained. Because of being the smallest open set, each U_x and all of its subsets containing x are connected. Every $U \in O$, $U \neq \emptyset$, results from a union of certain U_x.

Let N_x denote the set of all points in V adjacent to x.

$$O_1\colon U_x := \{x\} \quad \forall x \in V_A, \quad U_x := \{x\} \cup N_x \quad \forall x \in V_B$$

$$O_2\colon U_x := \{x\} \cup N_x \quad \forall x \in V_A, \quad U_x := \{x\} \quad \forall x \in V_B.$$

Only on graphs with $E(G) = \emptyset$ are those topologies equivalent. They are not even necessarily homeomorphic.

LEMMA 3.1. *O_1 and O_2 are topologies on G^b.*

Proof. Let O be either O_1 or O_2.

(1) Let $G' \sqsubseteq G^b$ be a connected graph. For any two adjacent points x, y, $U_x \cap U_y$ is always nonempty. So, for any nonempty, disjoint V_1, V_2, with $V(G') = V_1 \cup V_2$, the intersection of the smallest open sets O_1, $\mathrm{O}_2 \in O|_{V(G')}$ containing V_1 and V_2, respectively, is also nonempty, and $V(G')$ is connected in O.

(2) Let $G' \sqsubseteq G^b$ be a disconnected graph. G' is the union of two nonempty (not necessarily connected) induced subgraphs C and D, where $V(C)$ does not contain any points adjacent to $V(D)$. So

$$\bigcup_{x \in V(C)} U_x \cap V(D) = \emptyset = V(C) \cap \bigcup_{x \in V(D)} U_x.$$

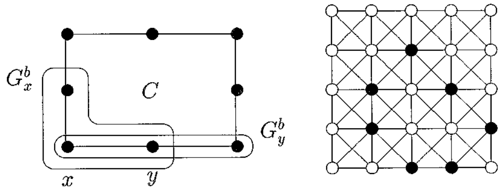

FIG. 1. Bipartite graphs generated by x, y in a circle C (left). Seven-point circle in an 8-grid (right).

Because of

$$V(C) = V(G') \cap \left(\bigcup_{x \in V(C)} U_x \right) \quad \text{and}$$

$$V(D) = V(G') \cap \left(\bigcup_{x \in V(D)} U_x \right)$$

we have $V(C), V(D) \in O|_{V(G')}$, and $V(G')$ is disconnected in O.

Let O be *any* topology on G^b. The following Lemmas hold $\forall x \in V$:

LEMMA 3.2. $\{x\} \subseteq U_x \subseteq \{x\} \cup N_x$.

Otherwise, for at least one x' there would exist $y \in V \backslash (\{x'\} \cup N_{x'})$, with $\{x', y\} \subseteq U_{x'}$ being connected in O.

This contradicts $G^b[\{x', y\}]$ being a disconnected graph.

LEMMA 3.3. $U_x = \{x\} \vee U_x = \{x\} \cup N_x$.

Otherwise, for at least one x', $\{x'\} \neq U_{x'}$, there would exist $y \in N_{x'} \backslash U_{x'}$. Because of G^b being bipartite, $N_{x'} \cap N_y = \emptyset$.

So, either $U_{x'} \cap U_y = \emptyset$ (contradicts $\{x', y\}$ being connected in O), or $U_{x'} \cap U_y = \{x'\} = U_{x'}$ (contradicts $\{x'\} \neq U_{x'}$).

LEMMA 3.4. *All $y \in N_x$ satisfy*: $U_x = \{x\} \Leftrightarrow U_y = \{y\} \cup N_y$.

If $U_x = \{x\}$, $U_y = \{y\}$ for any y, then $\{x, y\}$ is disconnected in O (contradicts xy being an edge of G^b).

If $U_x = \{x\} \cup N_x$, $U_y = \{y\} \cup N_y$ for any y, then $U_x \cap U_y = \{x, y\} \in O$ and $U_x = U_y = \{x, y\}$.

It follows from Lemma 3.3 that $N_x = \{y\}$ and $N_y = \{x\}$. And because of G^b being connected, $V(G^b) = \{x, y\}$ (contradiction of $V(G^b)$ having at least three points).

Following directly from Lemma 3.4:

THEOREM 3.5. *Any connected, bipartite graph $G^b = (V, E)$, V consisting of at least three points, has exactly two topologies, namely O_1 and O_2 as defined above.*

4. CONCLUSION

COROLLARY 4.1. *A circle C with an odd number of vertices $n > 3$ has no topology.*

Proof. For every $x \in V(C)$, $G_x^b := G[\{x\} \cup N_x]$ is a three-point, bipartite graph (Fig. 1). Assuming that C has a topology O, on every G_x^b either O_1 or O_2 is induced. For any adjacent points $x, y \in V(C)$ follows:

$$O|_{V(G_x^b)} \cong O_1 \Leftrightarrow O|_{V(G_y^b)} \cong O_2.$$

Otherwise the point set $\{x, y\}$ would be disconnected in O.

Starting with a point $x_0 \in V(C)$ and sequentially tracing the change of $O|_{V(G_x^b)}$ from O_1 to O_2, the odd number of vertices results in the contradiction that both O_1 and O_2 are induced on $G_{x_0}^b$.

Following from the preceding Corollary and from Theorem 2.1:

THEOREM 4.2. *Any graph G containing a circle C with an odd number of points $n > 3$ as an induced subgraph has no topology.*

Especially—this is Chassery's [2] result—there is no topology for the 8-connected digital plane or for the analogous n-dimensional case.

The $2n$-connected $\mathbb{Z}^n$ can be represented by a bipartite graph. In this case O_1 and O_2 are homeomorphic and equivalent to the topology constructed by Marcus and Wyse [5].

Corollary 4.1 and therefore Theorem 4.2 can also be deduced from a 1–1 correspondence of the topologies on a graph G to the relations ρ on $V(G)$ that satisfy:

1. ρ is reflexive and transitive.
2. Whenever u and v are adjacent vertices of G, either $(u, v) \in \rho$ or $(v, u) \in \rho$.
3. Whenever u and v are distinct nonadjacent vertices of G, $(u, v) \notin \rho$.

For a topology O, the corresponding relation ρ_O is given by $(u, v) \in \rho_O$ if and only if u lies in the ρ_O-closure of $\{v\}$. ρ_O is called the *Alexandroff specialization* relation of O. Once the existence of the 1–1 correspondence is validated, it is easily verified that if G has an odd circle with more than three vertices as an induced subgraph then there is no relation on $V(G)$ with properties 1–3. For further information see [3] and [6].

We note, that any graph G can be topologized by a "discrete topology" according to Alexandroff [1] (see also [7], Kapitel VIII, Section 32.5). Here, the topological space consists of $V(G) \cup E(G)$. For every vertex, the set consisting of it and all edges incident to it is defined as open. These open sets generate the topology on G in the usual way by unions and intersections. The disadvantage of this construction lies in the two different types of elements in the topological space. Consequently, translation invariance cannot be expected.

Disregarding all points without any incident edges (where this topology would be equivalent to ours), we can easily model Alexandroff's topology within our theory by introducing new vertices situated in the middle of each edge. We then have obtained a bipartite graph G^b, where $V_A := V(G)$ and V_B consists of the new vertices corresponding to the edges of the old graph G. Thus, Alexandroff's topology on G corresponds to O_2 on G^b. If we define V_A and V_B the other way round, then O_1 is induced on G^b.

Of course, the same is true for the "dual" topology, where every *edge* and its two incident vertices (and all vertices without incident edges) are considered as open.

Furthermore, this proves Alexandroff's topology and its "dual" to be the only topologies on any graph, if the topological space is to consist not only of all vertices, but of all edges too.

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