# Digital Topologies on Graphs

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**Summary.** In this chapter we focus on the relationship between graph theory and topology. Topologies on vertices of graphs became much more essential in topology, with the development of computer science, especially with the development of computer graphic and image analysis. Digital topology is the study of the topological properties of digital images. In most of the literature a digital image has been endowed with a graph model; the vertices being the points of the image, and the edges giving the connectivity between the points. This has led to the investigation of topology on graph [7–12]. We study compatible topologies on graphs, (here compatibility is to be understood as connectivity). We describe some properties of these particular topological spaces. We discuss the relation between $T_0$-spaces (which play an important role) and other compatible topologies on graph. We develop some applications to digital geometry. Other results related to compatible topologies on graphs is developed.

**Key words:** Graph theory, Topology, Computer sciences, Image processing

## 1 Introduction

Because data structures in computer sciences are enumerable, the only set which can be used in this case are *discrete* or *digital*. Roughly speaking discrete or digital is used in this chapter as opposed to *continuous*. For instance the space $\mathbb{R}^n$, $n \geq 1$ is where we do both continuous geometry and continuous topology, while the space $\mathbb{Z}^n$, $n \geq 1$ is an example of a space where we do both digital geometry and digital topology. Graphs have particular significance in computational sciences because of their presence in applications such as solid modeling, molecular biology, computer graphic, image analysis, etc. The most popular approach to define a discrete analog of the topologies of the Euclidean space is the *graph-theoretic* approach. Actually a *digital d-space* is the set of d-tuple of the real Euclidean d-space having integer coordinates. Such a point is called a *digital* point. Moreover a digital space is equipped with a graph structure based on the local adjacency relations. So the graph

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theoretic approach gives directly the connectedness but it is difficult to handle some topological concepts such as continuity, compacity and so on. Hence some important problems arise:

- What are the topological or geometrical properties of a discrete set?
- What does connected component mean for a digital set?
- What does continuous mapping mean between two discrete spaces?
- Which other properties has a digital set?

*Discrete topological spaces*, can be defined as a topological space such that any point has a smallest neighborhood. These types of topological spaces were first study by Alexanfroff [1]. Some applications of these topologies have been developed, where topological spaces are used to model discrete situations:

- In "Graphs, topologies and simple games" J. M. Bilbao [2] uses discrete topological spaces on a finite set to study the existence of connected coalitions in a simple game. Thanks to the digital topology he gives some sufficient conditions for the existence of winning coalitions.
- Baik and Miller [3] give a topological approach for testing equivalence in heterogeneous relational databases.
- By introducing "pretopology" M. Brissaud [4] models preference structures in economy.

We can find other applications in data structures, logics, complexity theory....
Any digital image can be interpreted as a graph, (it is "embedded" in a discrete space) whose vertices are the pixels (geometric points and grey level intensity) and whose edges define nearness and connectedness. Because a digital image can be viewed as a discrete set, hence a graph, it is interesting to study the problems enumerate below. For this we have to define a notion of compatibility between graphs and topologies. The natural way to define the compatible topologies on graphs is the connectedness: Let $G = (V, E)$ be a graph and let $\mathcal{T}$ be a topology on $V$, $\mathcal{T}$ is called a *compatible topology* on $V$ if it satisfies the following conditions:

(a) For every connected induced subgraph $G(V')$, $V'$ is connected for $\mathcal{T}$.
(b) For each $V' \subseteq V$ connected set for $\mathcal{T}$, the induced subgraph $G(V')$ is a connected graph.

Another important problem in image processing is the *digitalization*: let a subset $A$ of $\mathbb{R}^n$, $n \geq 1$ and let $f : \mathbb{R}^n \longrightarrow \mathbb{Z}^n$ be a map which associates to $A$ a discrete set $f(A)$. This map is called a *digitizer*. From this definition some questions arise about the continuity of $f$ or $f^{-1}$, the structural properties of $f$, etc. Consequently compatible topologies have been intensively studied and there are a lot of contributions by many authors [5–15].

In the first part of this chapter we investigate compatible topologies on graphs. We study the relation between connectedness and Alexandroff space. We show that any locally finite Alexandroff topology can be "embedded" into a $T_0$-locally finite Alexandroff topology such that both have the same connected set. We use these results to characterize the graphs which have compatible topologies on the set of vertices. We introduce an example of compatible topology on a bipartite graph and we study exhaustively this one. In a second part, to illustrate the first part, we give some applications to digital plane and digital spaces; others applications of compatible topologies will be given.

## 2 Definitions

All graphs are finite or infinite, undirected without isolated vertices. We consider that these graphs are simple (graphs without no loop or multiple edge). We denote them $G = (V; E)$. Given a graph $G$, we denote the *neighborhood* of a vertex $x$ by $\Gamma(x)$, i.e. the set formed by all the vertices adjacent to $x$:

$$\Gamma(x) = \{y \in V, \{x, y\} \in E\}$$

The number of neighbors of $x$ is the *degree* of $x$ (denoted by $dx$). For all $x \in V$, if $dx$ is finite one will say that $G$ is a *locally finite graph*. A *chain* (or *path*) from $x_0$ to $x_k$ is a sequence of distinct vertices: $V = \{x_0, x_1, \ldots, x_k\}$ such that $\{x_0x_1, x_1x_2, \ldots, x_{k-1}x_k\} \subseteq E$, where $x_ix_{i+1}$, stand for the edge $\{x_i, x_{i+1}\}$. The number of edges is the *length* of this chain. A graph is *connected* if for all $x, y \in V$ there exists a chain from $x$ to $y$. A *cycle* is a chain such that the first vertex and the last vertex are the same. We denote a cycle with a length equal to $n$ by $C_n$. Let $C_n$ be a cycle, a *chord* of $C_n$ is an edge linking two nonconsecutive vertices of $C_n$. A *circle* is a cycle without chord. A graph $G' = (V'; E')$ is a *subgraph* of $G$ when it is a graph satisfying $V' \subseteq V$ and $E' \subseteq E$. If $V' = V$ then $G'$ is a *spanning subgraph*. An *induced subgraph* (generated by $A$) $G(A) = (A; U)$, with $A \subseteq V$ and $U \subseteq E$ is a subgraph such that for $x, y \in A$: $\{x, y\} \in E$ implies $\{x, y\} \in U$. An *orientation* of $G = (V; E)$ is a preorder, (reflexive and transitive relation) on its vertices such that $\{x; y\} \in E$ if and only if $x < y$ or $y < x$. A simple undirected graph $G = (V; E)$ is a *comparability graph* if there exists an orientation of $G$. A graph $G = (V, E)$ is *bipartite* if $V = V_1 \cup V_2$ with $V_1 \cap V_2 = \emptyset$ and every edge joins a vertex of $V_1$ to a vertex $V_2$. We denote a bipartite graph by $G = (V_1, V_2; E)$.

A *topology* on a set $X$ is a nonempty collection $\mathcal{T}$ of subsets of $X$, called *open*, such that any union of open sets is open, any finite intersection of open sets is open, and both $X$ and the empty set are open. A set together with a topology on it is called a *topological space*. We denote a topological space by $(X, \mathcal{T})$.

Let $(X, \mathcal{T}_1)$ and $(X, \mathcal{T}_2$ be two topological spaces. If any open set of $\mathcal{T}_1$ is a an open set of $\mathcal{T}_2$ one will say that $\mathcal{T}_1$ is more thin than $\mathcal{T}_2$ and one will denote that by

$\mathcal{T}_1 \leq \mathcal{T}_2$. A *neighborhood* of $x \in X$ is a subset $V$ of $X$ containing an open subset which contains the point $x$. The set of neighborhoods of a point $x$ will be denoted by $\mathcal{V}(x)$, or $\mathcal{V}_{\mathcal{T}}(x)$.

Suppose we have a topology on a set $X$, and a collection of neighborhoods $\mathcal{V}' = (V_i')_{i \in I}$ of a point $x \in X$ such that any neighborhood of $x$ contains an element of $\mathcal{V}'$. Then $\mathcal{V}'$ is called a *fundamental system of neighborhood* of $x$. Suppose we have a topology on a set $X$, and a collection $\mathcal{O}$ of open sets such that every open set is a union of members of $\mathcal{O}$: then $\mathcal{O}$ is called a *base* for the topology and elements of $\mathcal{O}$ are called *basic open sets*.

A family $\mathcal{U} = \{U_i\}$ of (open) subsets of $X$ is an (open) *covering* if each element in $X$ belongs to at least one $U_i \in \mathcal{U}$. The *closure* of $A \subseteq X$ is $\phi(A) = \{x \in X; \forall\ V \in \mathcal{V}(x)\ V \cap A \neq \emptyset\}$.

$(X, \mathcal{P}(X))$ is a topological space called *discrete topological space*, we will denote it by $(X, \mathcal{D})$. The set $\{X, \emptyset\} = \mathcal{G}$ defines a topology on $X$, we call it *trivial topology*.

Let $V'$ be a subset of $X$ and let $\mathcal{T}$ be a topology on $X$; the collection of sets $\mathcal{T}' = \{V' \cap U; U \in \mathcal{T}\}$ defines a topology on $X'$ called *induced topology*. We will call *subspace* of $X$ this topological space.

A topological space $(X, \mathcal{T})$ is *connected* if $X$ cannot be expressed as the union of two disjoint nonempty open sets. A *connected subset* of $X$ is a connected subspace of $X$. The *connected component* $C(x)$ of $x$ is the biggest connected subset of $X$ containing $x$. $\{C(x), x \in X\}$ is a closed partition of $X$.

$X$ is *totally disconnected* if $C(x) = \{x\}$ for every $x \in X$. A topological space $X$ is *path connected*, if given any two points $a$ and $b$ in $X$, there exists a continuous path between them, that is a continuous map $\gamma : [0,1] \longrightarrow X$ such that $\gamma(0) = a$ and $\gamma(1) = b$, where $[0,1]$ is equipped with the usual topology.

A topological space $X$ is *locally connected* if there exists a fundamental system of connected neighborhood for every $x \in X$. An *Alexandroff topology* is one in which every intersection of open sets is open. So in an Alexandroff space any point $x$ has a smallest neighborhood denoted by $\mathcal{N}(x)$, or $\mathcal{N}_{\mathcal{T}}(x)$: it is a open set.

A topological space $(X, \mathcal{T})$ is a $T_0$-*space* if, for any distinct points $x, y \in X$ if $x \in \phi(\{y\})$, then $y \notin \phi(\{x\})$.

Let $G = (V; E)$ be a graph and let $\mathcal{T}$ be a topology on $V$, $\mathcal{T}$ is called a *compatible topology* on $G$ if it satisfies the following conditions:

(a) For every connected induced subgraph $G'(V')$, $V'$ is connected for $\mathcal{T}$.
(b) For each $V' \subseteq V$ connected set for $\mathcal{T}$, $G(V')$ is a connected graph.

## 3 Connectivity and Alexandroff Spaces

We give first a preliminary lemma.

**Lemma 1** *Let* $(X, \mathcal{T})$ *be a topological space,*

*(1)* $\{x, y\}$ *is connected if and only if* $y \in \phi(\{x\})$ *or* $x \in \phi(\{y\})$.

*(2) If $(X, \mathcal{T})$ is connected then for any open covering $\mathcal{U}$, one has the following property: for any $x, y \in X$ either there exists $V_1 \in \mathcal{U}$ such that $x, y \in V_1$, or there exists $n \geq 2$ and $V_1, V_2, \ldots, V_n \in \mathcal{U}$ such that $x \in V_1 \setminus V_2, y \in V_n \setminus V_{n-1}$ and satisfying: $V_i \cap V_j \neq \emptyset$ if and only if $|i - j| \leq 1$.*

*Moreover if $(X, \mathcal{T})$ is an Alexandroff space then:*

*(a) $\{x, y\}$ is connected if and only if $x \in \mathcal{N}(y)$ or $y \in \mathcal{N}(x)$.*
*(b) For any $x$, if $x \in A \subseteq \mathcal{N}(x)$, then $A$ is connected; so $(X, \mathcal{T})$ is locally connected. In particular for any $x$, $C(x)$ is a set both open and closed.*
*(c) If $X$ is connected, then $X$ is path connected.*

*Proof.* (1) Let us suppose that $\{x, y\}$ is nonconnected. There exists $O_1, O_2 \in \mathcal{T}$ with $O_1' = O_1 \cap \{x, y\}$ and $O_2' = O_2 \cap \{x, y\}$ such that $O_1' \cap O_2' = \emptyset$ and $\{x, y\} = O_1' \cup O_2'$. So $y \notin O_1$ (for example) and $O_1$ is an open neighborhood of $x$ in $X$. Consequently $x \notin \phi(\{y\})$. In the same way $x \notin O_2$, so $O_2$ is an open neighborhood of $y$, that leads to $y \notin \phi(\{x\})$.

Suppose that $x \notin \phi(\{y\})$ and $y \notin \phi(\{x\})$. There exists an open neighborhood $V$ of $x$ (respectively, open neighborhood $W$ of $y$) such that $y \notin V$ (respectively, $x \notin W$). So $\{x, y\} = (V \cap \{x, y\}) \cup (W \cap \{x, y\})$ and $(V \cap \{x, y\}) \cap (W \cap \{x, y\}) = \emptyset$.

(2) See [16], ex 11, p 188.

Suppose now that $(X, \mathcal{T})$ is an Alexandroff space.

(a) Because $y \in \mathcal{N}(x) \Longleftrightarrow x \notin \phi(y)$ and (1).
(b) $A = \bigcup_{y \in A} \{x, y\}$ and any $\{x, y\}$ is connected by $(a)$.
(c) It is well known that if $(X, \mathcal{T})$ is locally connected then $C(x)$ is open.

Suppose that $X$ is connected. Take the open covering $\mathcal{U} = \{\mathcal{N}(x), x \in X\}$. Let us suppose $u, v \in X$; $X$ being connected we have $u \in \mathcal{N}(x_0), \ldots, \mathcal{N}(x_i), \ldots, \mathcal{N}(x_n) \ni v$, with $\mathcal{N}(x_i) \cap \mathcal{N}(x_j) \neq \emptyset$ if and only if $|i - j| \leq 1$. So it is sufficient to prove that if $x \in \mathcal{N}(y)$ then there exists a path from $x$ to $y$: $\gamma(t) = x$ if $0 \leq t < 1$ and $\gamma(1) = y$ is suitable because $[0, 1] \subseteq \gamma^{-1}(\mathcal{N}(x))$ and $[0, 1] \subseteq \gamma^{-1}(\mathcal{N}(y))$. $\square$

### 3.1 Generation of Alexandroff Spaces

If $\mathcal{T}$ is an Alexandroff topology then it is easy to see that the map $x \longmapsto \mathcal{N}(x)$ satisfies

(a) $x \in \mathcal{N}(x)$.
(b) $y \in \mathcal{N}(x)$ involves $\mathcal{N}(y) \subseteq \mathcal{N}(x)$.

Conversely if we have a map $\mathcal{N}$ verifying the conditions (a), (b), then there exists an Alexandroff topology $\mathcal{T}$ where the open sets containing $x$ are defined in the following way: $\{Y, \mathcal{N}(x) \subseteq Y\}$. $\mathcal{T}$ is the Alexandroff topology associated to $\mathcal{N}$. The following assertion characterizes generated Alexandroff space.

**Theorem 1** *Let $(X,\mathcal{T})$ be a topological space, and $\mathcal{N}(x)=\bigcap_{V\in\mathcal{V}(x)} V$, $x\in X$:*

*(i) $\mathcal{N}$ produces an Alexandroff topology on $X$, denoted by $\mathcal{AT}$.*
*(ii) $\mathcal{AT}=\inf\{\mathcal{T}',\mathcal{T}'$ Alexandroff topology and $\mathcal{T}\le\mathcal{T}'\}$ ($\mathcal{AT}$ is the Alexandroff topology generated by $\mathcal{T}$).*
*(iii) $\mathcal{T}$ is $T_0 \Longleftrightarrow \mathcal{AT}$ is $T_0 \Longleftrightarrow$ the map $\mathcal{N}$ is injective.*
*(iv) The connected sets with two elements are the same for $\mathcal{T}$ and $\mathcal{AT}$.*
*(v) $\phi_{\mathcal{T}}(\{x\})=\phi_{\mathcal{AT}}(\{x\})$ for all $x\in X$.*
*(vi) $\mathcal{AT}$ can have fewer connected sets than $\mathcal{T}$.*

*Proof.* (i) It is obvious that $x\in\mathcal{N}(x)$. Let us show that if $y\in\mathcal{N}(x)$ then $\mathcal{N}(y)\subseteq\mathcal{N}(x)$: let $t\in\mathcal{N}(y)$: for every $W\in\mathcal{V}(x)$, $W$ open set, we have $y\in W$ (because $y\in\mathcal{N}(x)$); $W$ being open set is a neighborhood of $y$, hence $t\in W$ so $t\in\mathcal{N}(x)$.

(ii) One has $\mathcal{T}\le\mathcal{AT}$ because if $V\in\mathcal{V}_{\mathcal{T}}(x)$ then $V\supseteq\mathcal{N}(x)$, so $V\in\mathcal{V}_{\mathcal{AT}}(x)$. Moreover if $\mathcal{T}\le\mathcal{T}'$, $\mathcal{T}'$ an Alexandroff topology, for all $V\in\mathcal{V}_{\mathcal{T}}(x)$ one has $V\supseteq\mathcal{N}_{\mathcal{T}'}(x)$; consequently $\mathcal{N}(x)\supseteq\mathcal{N}_{\mathcal{T}'}(x)$, so $\mathcal{AT}\le\mathcal{T}'$.

(iii) If $y\in\mathcal{N}(x)$ then for all $V\in\mathcal{V}_{\mathcal{T}}(x)$, $y\in V$. $\mathcal{T}$ being $T_0$ there exists $W\in\mathcal{V}(y)$ such that $x\notin W$, so $x\notin\mathcal{N}(y)$.

(iv) If $\{x,y\}$ is a connected set for $\mathcal{AT}$, it is a connected set for $\mathcal{T}$ because $\mathcal{T}\le\mathcal{AT}$.

If $\{x,y\}$ is a nonconnected set for $\mathcal{AT}$, we have $x\notin\phi_{\mathcal{AT}}(\{y\})$ and $y\notin\phi_{\mathcal{AT}}(\{x\})$, so $y\notin\mathcal{N}(x)$ and $x\notin\mathcal{N}(y)$. Consequently there exists $V\in V_{\mathcal{T}}(x)$ such that $y\notin V$ and there exists $W\in V_{\mathcal{T}}(y)$ such that $x\notin W$, that leads to $x\notin\phi_{\mathcal{T}}(\{y\})$ and $y\notin\phi_{\mathcal{T}}(\{x\})$, and $\{x,y\}$ is a nonconnected set for $\mathcal{T}$.

(v) Easy from the fact that $y\in\phi_{\mathcal{T}}(\{x\})$ if and only if $x\in\bigcap_{V\in V_{\mathcal{T}}(y)} V$.

(vi) For instance if $\mathcal{T}$ is the usual topology on $\mathbb{R}$ then $\mathcal{AT}$ is the discrete topology. □

Theorem 2 gives more precisions about the generation of Alexandroff spaces.

**Theorem 2** *Let $(X,\mathcal{T})$ be a Alexandroff space, there exists $\mathcal{T}'$ such that:*

*(i) $\mathcal{T}\le\mathcal{T}'$ and $\mathcal{T}'$ is an $T_0$ Alexandroff space.*
*(ii) $\mathcal{T}$ and $\mathcal{T}'$ have the same connected sets.*
*(iii) $\mathcal{T}'$ is a minimal element of the set:*
*$\mathcal{M}=\{\mathcal{S},\ \mathcal{S}$ is an $T_0$ Alexandroff topology on $X,\ \mathcal{T}\le\mathcal{S},$ and $\mathcal{S},\mathcal{T}$ have the same connected subsets$\}$.*

*Proof.* (i) Let us define the map $\mathcal{N}:X\longrightarrow\mathcal{P}(X)$, with $\mathcal{N}(x)$ the smallest neighborhood of $x$. For $V\in\mathcal{P}(X)$ one denotes $\mathcal{N}^{-1}(V)=\{x,\mathcal{N}(x)=V\}$. So $X=\bigcup_{V\in\mathcal{P}(X)}\mathcal{N}^{-1}(V)$ is a partition of $X$. For every nonempty element of this partition:

– If $\mathcal{N}^{-1}(V)=\{x\}$ one has $\mathcal{N}(x)=V$, and one choose $\mathcal{N}'(x)=V$.
– If $\#(\mathcal{N}^{-1}(V))\ge 2$, by the well-ordered axiom, $V$ can be well ordered, so one takes the smallest element $\omega$ of $V$ and one sets $\mathcal{N}'(\omega)=V$. Now suppose

that $\mathcal{N}'(x)$ is defined for any $x < \alpha$, $(x, \alpha \in V)$, one takes $\mathcal{N}'(\alpha) = V \setminus \{t, t < \alpha\}$.

It is obvious that $x \in \mathcal{N}'(x) \subseteq \mathcal{N}(x)$, for all $x \in X$.

Let us suppose $y \in \mathcal{N}'(x)$, either $\mathcal{N}'(x) = \mathcal{N}(x)$ and $\mathcal{N}'(y) \subseteq \mathcal{N}'(x)$, or $\mathcal{N}'(x) \subsetneq \mathcal{N}(x) = V$, $\mathcal{N}'(x) = V \setminus \{t, t < x\}$, $V$ being well ordered, we have $y \geq x$. Consequently $\mathcal{N}'(y) = V \setminus \{t, t < y\} \subseteq \mathcal{N}'(x) = V \setminus \{t, t < x\}$. That is to say $\mathcal{N}'(y) \subseteq \mathcal{N}'(x)$. One can conclude that $\mathcal{N}'$ generates an Alexandroff topology $\mathcal{T}'$, and $\mathcal{T} \leq \mathcal{T}'$. Now suppose $x \neq y$:

- If $\mathcal{N}(x) \neq \mathcal{N}(y)$ then $x \notin \mathcal{N}(y)$ or $y \notin \mathcal{N}(x)$; "a fortiori" $y \notin \mathcal{N}'(x)$ or $x \notin \mathcal{N}'(y)$.
- If $\mathcal{N}(x) = \mathcal{N}(y) = V$ then $\mathcal{N}'(x) = V \setminus \{t, t < x\}$ and $\mathcal{N}'(y) = V \setminus \{t, t < y\}$. Moreover $x < y$ or $y < x$: Consequently $y \notin \mathcal{N}'(x)$ or $x \notin \mathcal{N}'(y)$.

So $(X, \mathcal{T}')$ is a $T_0$ Alexandroff space and $\mathcal{T} \leq \mathcal{T}'$.

(ii) Let us show that $\mathcal{T}$ and $\mathcal{T}'$ have the same connected sets. It is obvious that any connected set for $\mathcal{T}'$ is a connected set of $\mathcal{T}$.

If $\{x, y\}$ is connected for $\mathcal{T}$, it is equivalent to say that $y \in \mathcal{N}(x)$ or $x \in \mathcal{N}(y)$ (from Lemma 1). Without losing generality, suppose $y \in \mathcal{N}(x)$. We have two cases:

- If $\mathcal{N}'(x) = \mathcal{N}(x)$ then $\{x, y\}$ is a connected set for $\mathcal{T}'$.
- If $\mathcal{N}'(x) \subsetneq \mathcal{N}(x) = V$ then $\mathcal{N}'(x) = V \setminus \{t, t < x\}$. If $y \in \mathcal{N}'(x)$ then $\{x, y\}$ will be a connected set for $\mathcal{T}'$. If $y \notin \mathcal{N}'(x)$ then $y < x$, consequently $x \in \mathcal{N}'(y)$ and $\{x, y\}$ is a connected set for $\mathcal{T}'$.

Let us now suppose that $C$ is a connected set for $\mathcal{T}$, from the proof of Lemma 1, it is path connected. Let us suppose $x, y \in C$ with $y \in \mathcal{N}(x)$. From above $\{x, y\}$ is a connected set for both topologies, so $\{x, y\}$ is a connected set for $\mathcal{T}'$. From lemma 1 one can conclude that $C$ is a connected set for $\mathcal{T}'$.

(iii) If $\mathcal{S} \in \mathcal{M}$ and $\mathcal{S} \leq \mathcal{T}'$ one has $\mathcal{N}_{\mathcal{T}'}(x) \subseteq \mathcal{N}_{\mathcal{S}}(x) \subseteq \mathcal{N}_{\mathcal{T}}(x)$; and if $y \in \mathcal{N}_{\mathcal{S}}(x)$ necessarily $x \notin \mathcal{N}_{\mathcal{S}}(y)$, a fortiori $x \notin \mathcal{N}_{\mathcal{T}'}(y)$; but $y \in \mathcal{N}_{\mathcal{S}}(x)$ implies $\{x, y\}$ is connected for $\mathcal{S}$, hence $\{x, y\}$ is connected for $\mathcal{T}'$, so since $x \notin \mathcal{N}_{\mathcal{T}'}(y)$, necessarily $y \in \mathcal{N}_{\mathcal{T}'}(x)$: consequently $\mathcal{N}_{\mathcal{S}}(x) \subseteq \mathcal{N}_{\mathcal{T}'}(x)$. □

**Proposition 1** *Let $(X, \mathcal{T})$ be a $T_0$-Alexandroff space, $\mathcal{T}$ is maximal in the set of topologies on $X$ for which the connected sets are the same as for $\mathcal{T}$.*

*Proof.* Let us suppose that $\mathcal{T} < \mathcal{T}'$, there is an open set U for $\mathcal{T}'$ which is not an open set for $\mathcal{T}$. So there exists $x \in X$ such that $U \in \mathcal{V}_{\mathcal{T}'}(x)$ and $U \notin \mathcal{V}_{\mathcal{T}}(x)$. Consequently $U \not\supseteq \mathcal{N}(x)$ and there exists $y \in \mathcal{N}(x)$ such that $y \notin U$:
$\{x, y\}$ is a connected set for $\mathcal{T}$.

$\mathcal{T}$ is a $T_0$ space, so $x \notin \mathcal{N}(y)$ which means that $y \notin \phi_{\mathcal{T}}(x)$, consequently $y \notin \phi_{\mathcal{T}'}(x)$. Moreover $y \notin U$ implies $x \notin \phi_{\mathcal{T}'}(y)$ therefore $\{x, y\}$ is not a connected set for $\mathcal{T}'$. □

## 4 Dual Alexandroff Topologies

Let $(X, \mathcal{T})$ be an Alexandroff space. Because any intersection of open sets is an open set, the closed sets of $\mathcal{T}$ are the open sets of a topology on $X$. We denote this topology by $\tilde{\mathcal{T}}$, and we will call it the *dual topology* of $\mathcal{T}$. If $\mathcal{T} = \tilde{\mathcal{T}}$ one say that $\mathcal{T}$ is *self-dual*.

Theorem 3 provides us with a way of building dual Alexandroff topologies.

**Theorem 3** *Let $\sigma : X \longrightarrow \mathcal{P}(X)$ be a map verifying:*

- $\forall\, x \in X$, $x \in \sigma(x)$.
- $y \in \sigma(x)$ *implies* $\sigma(y) \subseteq \sigma(x)$.

*There exists two Alexandroff topologies associated with $\sigma$:*
*$\mathcal{T}$ for which $\mathcal{N}_{\mathcal{T}}(x) = \sigma(x)$, and $\mathcal{T}'$ for which $\phi_{\mathcal{T}'}(\{x\}) = \sigma(x)$.*

- *$\mathcal{T}$ and $\mathcal{T}'$ are dual: $\mathcal{T}' = \tilde{\mathcal{T}}$.*
- *$\mathcal{T}$ is $T_0$ if and only if $\tilde{\mathcal{T}}$ is $T_0$ if and only if $\sigma$ is injective.*
- *$\mathcal{T}$ and $\tilde{\mathcal{T}}$ have the same connected sets, hence these topologies have the same connected components.*

*Proof.* Let $\mathcal{T}$ the Alexandroff topology associated to $\sigma$. The map $\psi$, (see Sect. 3.1): $\mathcal{P}(X) \longrightarrow \mathcal{P}(X)$ defined by $\psi(A) = \bigcup_{x \in A} \sigma(x)$ is a closure operator. Hence this closure operator generates a topology $\mathcal{T}'$.

Let us suppose that $U$ an open set for $\mathcal{T}$. For all $x \in U$ $\sigma(x) \subseteq U$, so $U = \bigcup_{x \in U} \sigma(x) = \psi(U)$ and $U$ is a closed set for $\mathcal{T}'$. If $F$ is a closed set for $\mathcal{T}'$ we have $\psi(F) = F = \bigcup_{x \in F} \sigma(x)$. Consequently $x \in F$ implies that $\sigma(x) \subseteq F$. Hence $F$ is a open set for $\mathcal{T}$. So $\mathcal{T}' = \tilde{\mathcal{T}}$.

Suppose that $\mathcal{T}$ is $T_0$, so $y \in \mathcal{N}(x)$ implies $x \notin \mathcal{N}(y)$, but $\mathcal{N}(x) = \sigma(x)$, so $y \in \sigma(x)$ means that $x \notin \sigma(y)$ and $\tilde{\mathcal{T}}$ is $T_0$. The converse can be obtained in the same way. $\mathcal{T}$ and $\tilde{\mathcal{T}}$ have the same connected sets because the open sets for $\mathcal{T}$ are the closed sets for $\mathcal{T}'$. $\square$

Proposition 2 characterizes self-dual topologies.

**Proposition 2** *If $\mathcal{T} = \tilde{\mathcal{T}}$ then $B = \{\mathcal{N}(x)\}_{x \in X}$ is a partition of $X$.*

*If $(A_i)_{i \in I}$ is a partition of $X$, $B = \{A_i, i \in I\}$ is a base of a self-dual Alexandroff topology $\mathcal{T} = \tilde{\mathcal{T}}$.*

*Moreover $X$ is connected if and only if $\mathcal{T} = \mathcal{G}$.*

*Proof.* Let us suppose $\mathcal{T} = \tilde{\mathcal{T}}$, $\mathcal{N}(x)$ is the smallest open set containing $x$ and we have $\mathcal{N}(x) = \phi(\{x\})$. If $t \in \mathcal{N}(x) \cap \mathcal{N}(y)$ then $\mathcal{N}(t) \subseteq \mathcal{N}(x)$; $t \in \phi(\{x\})$ implies $x \in \mathcal{N}(t)$ therefore $\mathcal{N}(x) \subseteq \mathcal{N}(t)$. Likewise $\mathcal{N}(y) \subseteq \mathcal{N}(t)$: $B$ is a partition of $X$.

Let $B = \{A_i, i \in I\}$ be a partition: obviously $B$ is a base of one topology denoted by $\mathcal{T}$. Moreover if $x \in A_i$, $A_i$ is the smallest open set containing $x$. So $\mathcal{T}$ is an Alexandroff topology and $\mathcal{N}(x) = A_i$.

Let us now show that $\phi(\{x\}) = \mathcal{N}(x)$.

$A_i$ is a closed set, consequently $\phi(\{x\}) \subseteq A_i = \mathcal{N}(x)$. We have for all $y \in \mathcal{N}(x), \mathcal{N}(y) = \mathcal{N}(x)$, $x \in \mathcal{N}(y)$ and $y \in \phi(\{x\})$. Consequently $\phi(\{x\}) = \mathcal{N}(x)$ and $\mathcal{T} = \tilde{\mathcal{T}}$.

Any $\mathcal{N}(x)$ is both a closed and open set, so $X$ is a connected set if and only if for every $x \in X, \mathcal{N}(x) = X$ if and only if $\mathcal{T} = \mathcal{G}$. $\square$

**Examples**

The trivial topology is self-dual, it is connected. The discrete topology is also self-dual, it is totally disconnected if $cardX \geq 2$

We give now some results on the topological lattice of dual topologies on the set $X$.

**Proposition 3** *Let* $(X, \mathcal{T})$ *be an Alexandroff space.*

– $\sup(\mathcal{T}, \tilde{\mathcal{T}})$ *is the self-dual Alexandroff topology associated with the partition* $\{A(x)\}_{x\in X}$ *with* $A(x) = \{y, \mathcal{N}(y) = \mathcal{N}(x)\}$.
 *Moreover* $\mathcal{T}$ *is* $T_0$*, if and only if* $\sup(\mathcal{T}, \tilde{\mathcal{T}}) = \mathcal{D}$*, (*$\mathcal{D}$ *being the discrete topology).*
– $\inf(\mathcal{T}, \tilde{\mathcal{T}})$ *is the self-dual Alexandroff topology associated with the partition* $\{C(x), x \in X\}$ *where* $C(x)$ *is the connected component of* $x$ *for* $\mathcal{T}$*; so* $\mathcal{T}$ *is connected if and only if* $\inf(\mathcal{T}, \tilde{\mathcal{T}}) = \mathcal{G}$*, (*$\mathcal{G}$ *being the trivial topology).*

*Proof.* Let us $\mathcal{S} = \sup(\mathcal{T}, \tilde{\mathcal{T}})$; for any $x \in X, \mathcal{N}(x) \cap \phi(\{x\}) = A(x)$ is an open neighborhood of $x$ for $\mathcal{S}$. Moreover $\mathcal{N}(x) \cap \phi(\{x\}) = A(x)$ implies that $A(x) = \{y, \mathcal{N}(y) = \mathcal{N}(x)\}$, (because any neighborhood of $x$ is a neighborhood of $y$ and conversely). So $\{A(x)\}_{x\in X}$ is a partition of $X$ and from the last proposition we have a self-dual Alexandroff topology $\mathcal{U}$ associated with it. We have $\mathcal{T} \leq \mathcal{U}$ because $A(x) \subseteq \mathcal{N}(x)$ for all $x \in X$. $\mathcal{N}(x)$ is a neighborhood of $x$ for $\mathcal{U}$, therefore $\mathcal{T} \leq \mathcal{U}$, hence $\tilde{\mathcal{T}} \leq \tilde{\mathcal{U}} = \mathcal{U}$; which means that $\mathcal{S} \leq \mathcal{U}$. $A(x)$ is an open set of $\mathcal{S}$ and since $\mathcal{U} \leq \mathcal{S}$, so $\mathcal{S} = \mathcal{U}$.

$\mathcal{T}$ is $T_0$ if and only if $\mathcal{N}$ is injective if and only if $A(x) = \{x\}$ if and only if $\mathcal{S} = \mathcal{D}$.

Let $\mathcal{C}$ be the self-dual topology associated with the partition $\{C(x), x \in X\}$; each $C(x)$ is both open and closed for $\mathcal{T}$ (from Theorem 3 or Lemma 1), so $C(x)$ is open set for $\tilde{\mathcal{T}}$, hence $\mathcal{C} \leq \mathcal{T}$ and $\mathcal{C} \leq \tilde{\mathcal{T}}$.

Assume now $\mathcal{E} \leq \mathcal{T}$ and $\mathcal{E} \leq \tilde{\mathcal{T}}$; every open set $U$ for $\mathcal{E}$ is an open and closed set for $\mathcal{T}$. So for all $x \in U$ we have $C(x) \subseteq U$, (because $C(x)$ is a subset of any open–closed set containing $x$). As a consequence $U = \bigcup_{x\in U} C(x)$ is an open set for $\mathcal{C}$. So $\mathcal{E} \leq \mathcal{C}$. $\square$

In Sect. 5 we study compatible topologies on a graph $G = (V; E)$.

## 5 Compatible Topologies on Graph

Recall, [15], that a topology $\mathcal{T}$ on the set of vertices $V$ of a graph $G = (V; E)$ is *compatible* if the connected subspaces of $(V, \mathcal{T})$ are the same as the connected induced subgraphs of $G = (V; E)$.

Some preliminary results are given by:

**Lemma 2** *Let $\mathcal{T}$ be a compatible topology on G, we have the following properties:*

*(a) $\phi(\{x\}) \cup \bigcap_{V\in\mathcal{V}(x)} V = \{x\} \cup \Gamma(x)$.*
*(b) $\mathcal{T}$ is $T_0$ if and only if $\phi(\{x\}) \cap \bigcap_{V\in\mathcal{V}(x)} V = \{x\}$.*
*(c) If $\mathcal{T}$ is an Alexandroff topology then $\phi(\{x\}) \cup \mathcal{N}(x) = \{x\} \cup \Gamma(x)$ and $\{x\} \cup \Gamma(x)$ is a connected neighborhood of $x$, and $\mathcal{T}$ is locally connected.*

*Proof.* (a) Let us suppose that $y \in \phi(\{x\})$, $x \neq y$, so $\{x, y\}$ is a connected set for $\mathcal{T}$. Hence $\{x, y\} \in E$ and $y \in \{x\} \cup \Gamma(x)$.

Let us suppose that $y \in \bigcap_{V\in\mathcal{V}(x)} V$, one has for all $V \in \mathcal{V}(x)$, $y \in V$, so $x \in \phi(\{y\})$ and $y \in \{x\} \cup \Gamma(x)$.

Now suppose that $y \in \Gamma(x)$, $\{x, y\} \in E$ and $\{x, y\}$ is a connected set for $\mathcal{T}$, from Lemma 1: $y \in \phi(\{x\})$, or $x \in \phi(\{y\})$: consequently for all $V \in \mathcal{V}(x)$, $y \in V$ and $y \in \bigcap_{V\in\mathcal{V}(x)} V$.

(b) Suppose that $\mathcal{T}$ is $T_0$. Let us suppose $y \in \phi(\{x\}) \cap \bigcap_{V\in\mathcal{V}(x)} V$, so $x \in \phi(\{y\})$ and $y \in \phi(\{x\})$, consequently $x = y$ and $\phi(\{x\}) \cap \bigcap_{V\in\mathcal{V}(x)} V = \{x\}$.

(c) Indeed $\mathcal{N}(x) = \bigcap_{V\in\mathcal{V}(x)} V$. □

An important result can be inferred from Theorem 4. The assertion (b) is to be found in [15].

**Theorem 4** *Let $\mathcal{T}$ be a compatible topology on G, we have the following properties:*

*(a) If $\mathcal{T}$ is an Alexandroff topology then $\{x\} \cup \Gamma(x)$ is a neighborhood of $x$.*
*(b) If G is locally finite then $\{x\} \cup \Gamma(x)$ is a neighborhood of $x$.*

*Proof.* For $x \in X$ the connected component of $X \setminus \Gamma(x)$ are $\{x\}$ and $(C_i)_{i\in I}$. $\{x\} \cup C_i$ is not a connected set (by construction), so $x \notin \phi(C_i)$. From this, there exists $V_i \in \mathcal{V}(x)$ such that $V_i \cap C_i = \emptyset$. Let us set $W := \bigcap_{i\in I} V_i$, one has for all $i \in I$, $W \cap C_i = \emptyset$ so $W \subseteq \{x\} \cap \Gamma(x)$.

(a) If $\mathcal{T}$ is an Alexandroff topology then $W \in \mathcal{V}(x)$ and $\{x\} \cap \Gamma(x)$ is a neighborhood of $x$.
(b) If $G$ is locally finite then $\#(I) \leq \#(\Gamma(\Gamma(x))) < \aleph_0$, so $I$ is a finite set. Consequently $W \in \mathcal{V}(x)$ and $\{x\} \cup \Gamma(x)$ is a neighborhood of $x$. □

**Corollary 1** *If G is locally finite then $\mathcal{T}$ is an Alexandroff topology.*

*Proof.* Because for all $x$, $(V \cap (\{x\} \cup \Gamma(x)))_{V\in\mathcal{V}(x)}$ is a fundamental neighborhood system of $x$ with a finite number of elements, (the number of subsets of $\{x\} \cup \Gamma(x)$ at most), consequently the intersection of these elements is the smallest neighborhood of $x$. □

**Problem 1.** Which are the compatible topologies such that $\{x\} \cup \Gamma(x)$ is neighborhood of $x$?

The following result can be found in [14, 15]. The proof given here is shorter.

**Theorem 5** *Let $G = (V, E)$ be a graph, the following properties are equivalent:*

*(i) $G$ has a compatible topology $\mathcal{T}$.*
*(ii) $G$ is a comparability graph.*

*Proof.* Under hypothesis (i), and from Theorem 1 there exists a compatible $T_0$-Alexandroff topology, $\mathcal{A}_0$, such that $\mathcal{T} \leq \mathcal{A}_0$. Let $\leq$ the binary relation defined by $x \leq y$ if and only if $y \in \phi(\{x\})$. This relation is a partial order relation, so

- If $\{x, y\} \in E$, then $\{x, y\}$ is a connected set for $\mathcal{A}_0$, (by (iv) from Theorem 1) consequently $y \in \phi(\{x\})$ or $x \in \phi(\{y\})$, and $x \leq y$ or $y \leq x$.
- If $x \leq y$, $y \in \phi(\{x\})$, so $\{x, y\}$ is a connected set for $\mathcal{T}$, so for $\mathcal{A}_0$, and $\{x, y\} \in E$.

Under hypothesis (ii), let $\leq$ be a preorder on $V$ verifying: $[x \leq y$ or $y \leq x]$ if and only if $\{x, y\} \in E$. To this preorder one can associate an Alexandroff topology $\mathcal{A}$ defined by: $\phi(\{x\}) = \{y \in V, x \leq y\}$. So $\{x, y\}$ is a connected set for $\mathcal{A}$ if and only if $y \in \phi(\{x\})$ or $x \in \phi(\{y\})$, if and only if $y \leq x$ or $x \leq y$. $\square$

Proposition 4 links compatible topologies and generated Alexandroff topologies.

**Proposition 4** *Let $\mathcal{T}$ be a compatible topology, we denote by $\mathcal{A}$ the Alexandroff topology generated by $\mathcal{T}$ and the $T_0$ Alexandroff topology by $\mathcal{A}_0$. We have the following properties:*

- *$\mathcal{A}$, $\tilde{\mathcal{A}}$ and $\mathcal{A}_0$, $\tilde{\mathcal{A}}_0$ are compatible.*
- *$\sup\{\mathcal{A}_0, \tilde{\mathcal{A}}_0\} = \mathcal{D}$ (not compatible if $\#(E) > 1$).*
- *$\inf\{\mathcal{A}, \tilde{\mathcal{A}}\} = \inf\{\mathcal{A}_0, \tilde{\mathcal{A}}_0\} = \mathcal{G}$. (not compatible if $\#(E) > 2$).*

*Proof.* Obvious from Theorems 1–3. $\square$

**Proposition 5** *If $\mathcal{T}$ is a $T_0$ compatible Alexandroff topology, it is maximal in the compatible topologies.*

*Proof.* Obvious from Proposition 1. $\square$

**Proposition 6** *Let $G = (V, W; A)$ be a locally finite connected bipartite graph, one has:*

*(a) If $\#(A) = 1$ there are three compatible topologies on $V$: $\mathcal{G}$, $\mathcal{A} = \{\emptyset, V, V \cup W\}$ and $\tilde{\mathcal{T}} = \{\emptyset, W, V \cup W\}$.*
*(b) If $\#(A) \geq 2$ there are two compatible topologies:*
  *(1) the Alexandroff topology $\mathcal{A}$ associated with $\mathcal{N}_{\mathcal{A}}(x) = \{x\}$ for $x \in V$ and $\mathcal{N}_{\mathcal{A}}(x) = \{x\} \cup \Gamma(x)$ for $x \in W$.*
  *(2) the dual Alexandrov topology $\tilde{\mathcal{A}}$ with $\mathcal{N}_{\tilde{\mathcal{A}}}(x) = \{x\}$ for $x \in W$ and $\mathcal{N}_{\tilde{\mathcal{A}}}(x) = \{x\} \cup \Gamma(x)$ for $x \in V$.*

*These two topologies are $T_0$.*

*Proof.* (a) Obvious.
(b) By Corollary 1 any topology $\mathcal{U}$ on the set of vertices of $G$ is an Alexandroff topology. From Theorem 4, $\{x\} \cup \Gamma(x)$ is a neighborhood of $x$. Hence $\{x\} \cup \Gamma(x)$

contains an open set containing $x$; the graph being a bipartite graph and $\mathcal{U}$ being a compatible topology this open set is either $\{x\} \cup \Gamma(x)$ or $\{x\}$. By connectivity, if $x \in V$ and $\mathcal{N}(x) = \{x\}$ then $\mathcal{N}(y) = \{y\}$ for all $y \in V$. This topologies are $T_0$. Let us $x \neq y$ and $\{z, y\} \in E$: if $x \in \phi(y)$ then $y \notin \phi(x)$, otherwise $x, y, z$ should be an odd cycle. $\square$

To illustrate this, we give some examples.

## 6 Examples

Let $G = (V; E)$ be the bipartite graph defined by: $V = \mathbb{Q} \cup \{\infty\}$ and $\{q, y\} \in E$ if and only if $q \in \mathbb{Q}$ and $y = \infty$ ($\mathbb{Q}$ being the set of rational numbers).

### 6.1 Construction of Alexandroff Topologies

Let $\mathcal{A}_\infty$ be the Alexandroff topology defined by: $\mathcal{N}(\infty) = \{\infty\}$ and $\mathcal{N}(q) = \{q, \infty\}$. So $\phi(\{\infty\}) = V$ (because for all $q \in \mathbb{Q}$, $V \in \mathcal{V}(q)$ implies $\infty \in V$, it is equivalent to $\mathbb{Q} \subset \phi(\{\infty\})$); $\phi(\{q\}) = \{q\}$. The open set are $\emptyset$ and the subsets $A \subseteq V$ such that $\infty \in A$.

It goes without saying that $\mathcal{A}_\infty$ is a $T_0$ topology.

$V$ is a connected set for $\mathcal{A}_\infty$ because $\{\infty\}$ is connected and $\phi(\{\infty\}) = V$.

### 6.2 Construction of $\tilde{\mathcal{A}}_\infty$

$\phi(\{q\}) = \{q\}$ and $\phi(\{\infty\}) = V$, so $\tilde{\mathcal{N}}(q) = \{q\}$ and $\tilde{\mathcal{N}}(\infty) = V$. Consequently the open sets of $\tilde{\mathcal{A}}_\infty$ are: $\emptyset$, $V$ and the subsets $A \subseteq \mathbb{Q}$.

So we have: $\sup\{\mathcal{A}_\infty, \tilde{\mathcal{A}}_\infty\} = \mathcal{D}$, because $\mathcal{A}_\infty$ is $T_0$. $\inf\{\mathcal{A}_\infty, \tilde{\mathcal{A}}_\infty\} = \mathcal{G}$, because $V$ is a connected set for $\mathcal{A}_\infty$.

### 6.3 Connected Sets of $\mathcal{A}_\infty$ and $\tilde{\mathcal{A}}_\infty$

By hypothesis $\mathcal{N}(q) = \{q, \infty\}$ is a connected set. Moreover if $\{x, y\}$ is a connected set then $x \in \mathcal{N}(y)$ or $y \in \mathcal{N}(x)$. Consequently, if $x \neq y$ there exists just one possibility $\{x, y\} = \{x, \infty\}$. Any connected set $A$ with $\#(A) > 1$ contains $\infty$. Indeed, let $A$ be a connected set, $A$ is path connected; for $x, y \in A$ so there exists a path from $x$ to $y$, this one contains $\infty$.

### 6.4 Construction of Other Topologies Having the same Connected Sets with two Elements

Let $\mathcal{T}$ be a compatible topology, (not necessarily an Alexandroff topology) on $V$. For this topology the subspace $\mathbb{Q}$ is totally disconnected: indeed, if $A \subseteq \mathbb{Q}$ is a connected set for the induced topology on $\mathbb{Q}$, $A$ is a connected set for the topology $\mathcal{T}$ on $V$. But $\infty \notin A$, so the cardinality of $A$ is less or equal to one.

Conversely if $\mathcal{T}$ is a totally disconnected topology on $\mathbb{Q}$, one can associate two topologies with $\mathcal{T}$:

For the first one topology $\mathcal{T}_\infty$, the set of open sets from this topology is: $\{\emptyset\} \cup \{U \cup \{\infty\},\ U \text{ open set of } \mathcal{T}\}$.
For the second $\mathcal{T}^*_\infty$, the set of open sets from this topology is: $\{V\} \cup \{U,\ U \text{ openset of } \mathcal{T}\}$. (These topologies are not necessarily Alexandroff topologies).

We have a lot of choices for $\mathcal{T}$, for example one can choose the discrete topology $\mathcal{D}$ (which gives $\mathcal{T}_\infty = \mathcal{A}_\infty$ from Sect. 6.1), or the topology $\mathcal{T}_+$ whose an open base is given by $[x, y[$, $x, y \in \mathbb{Q}$, or the topology $\mathcal{T}_-$ whose an open base is given by $]x, y]$, $x, y \in \mathbb{Q}$, or the usual topology $\mathcal{T}_0$ on $\mathbb{Q}$ (associated with the usual distance on $\mathbb{Q}$), or the $p$-adic topology (associated with the $p$-adic distance on $\mathbb{Q}$), etc.

## 7 Applications to Digital Topology

The main focus of digital topology and digital geometry is to determine geometrical and topological properties between the discrete nature of a computational objects and their theoretical representation in terms of continuous space $\mathbb{R}^n$. To use the geometrical and topological notions in digital topology and digital geometry it is necessary to define an analog to the continuous space $\mathbb{R}^n$, $(n > 1)$.

### 7.1 Digital Spaces

Usually there is two ways to define a digital space: Let $x = (x_1,\ x_2,\ x_3,\ \ldots, x_p)$ be a point of $\mathbb{Z}^p$. The $(3^p - 1)$-neighbors of $x$ are all points $y = (y_1,\ y_2,\ y_3,\ \ldots, y_p) \in \mathbb{Z}^p$ such that:

$$\max |x_i - y_i| = 1$$

The discrete space with a dimension equal to $n$, $(n \geq 2)$ defined thanks to the equation below will be denote by $(d_\infty, n)$-space.

In the case where $p = 2$ we obtain the “8-connected” digital plane, see Fig. 1.
The $2p$-neighbors of $x$ are all points $y = (y_1,\ y_2,\ y_3,\ \ldots, y_p) \in \mathbb{Z}^p$ such that:

$$\sum_{i=1}^{p} |x_i - y_i| = 1$$

The discrete space with a dimension equal to $n$, $(n \geq 2)$ defined thanks to the equation below will be denote by $(d_1, n)$-space.

In the case where $p = 2$ we obtain the “4-connected” digital plane, see Fig. 2.

It is well known that an induced graph of a comparability graph is a comparability graph. So if a graph $\Gamma$ contains an induced subgraph which is not a comparability graph then $\Gamma$ is not a comparability graph. In [5] we showed that:

**Proposition 7** *Let $\Gamma = (V; E)$ be a graph with a compatible topology on $V$. For every induced subgraph $H$ the topology restricted to the vertices of $H$ is a compatible topology on $H$.*

*Proof.* From remark above and Theorem 5. □

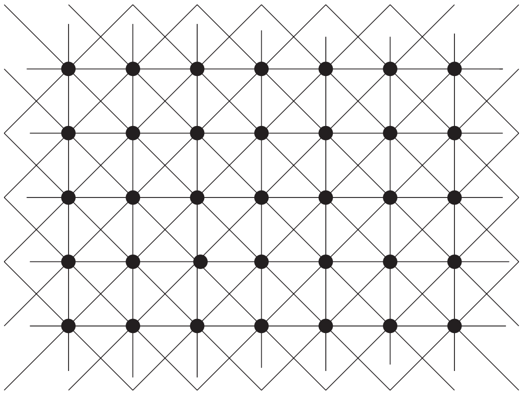

**Fig. 1.** A piece of 8-connected discrete plane

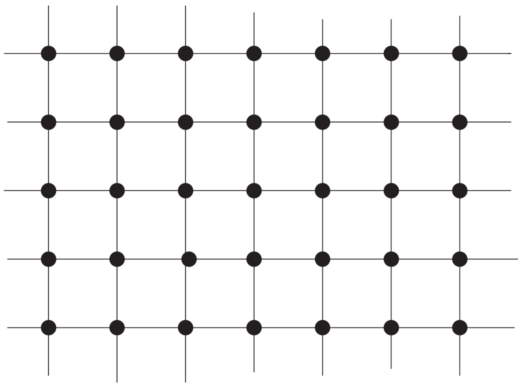

**Fig. 2.** A piece of 4-connected discrete plane

In Fig. 3 we see that the 8-connected digital plane is not a comparability graph because we are able to display an induced subgraph which is not a comparability graph. From Theorem 5 we find a generalization of the well known CHASSERY' theorem [5, 7, 17, 18]:

**Theorem 6** *The ($d_\infty, n$)-space has no compatible topology.*

The next theorem can be found in [5, 17]

**Theorem 7** *The ($d_1, n$)-discrete space has exactly two compatible topologies.*

*Proof.* It is easy to see that the $(d_1, n)$-discrete space is a bipartite graph, (is the Cartesian product of $n$ chains [5]), hence from Proposition 6 the result follow. □

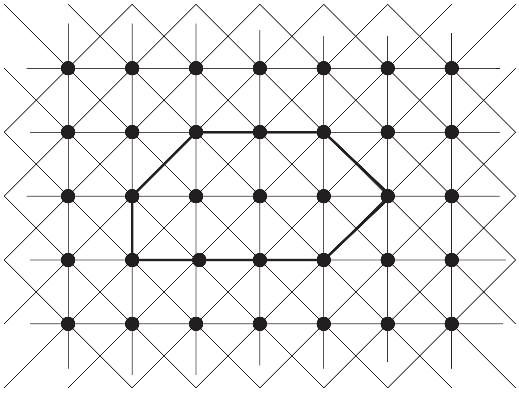

**Fig. 3.** Circle with an odd length

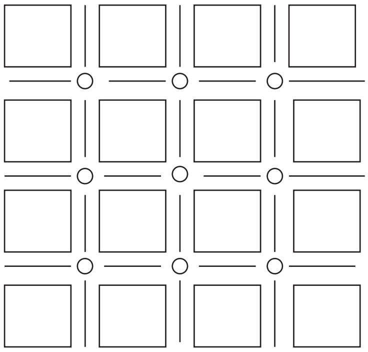

**Fig. 4.** An example of a part of an abstract complex. The points have a dimension equal to 0, the lines are cells with a dimension equal to 1 and the square are cells with a dimension equal to 2

### 7.2 Cell Complexes

An *abstract cell complex*, is a triplet $C = (E, B, Dim)$ where $E$ is a set of abstract elements called *cells*, $B \subseteq E \times E$ is an antisymmetric, irreflexive and transitive binary relation and with a dimension function $DimE \ \longrightarrow I \subseteq \mathbb{N}$ such that $Dim(e) < Dim(e')$ for all pairs $(e, e') \in B$. The plane $\mathbb{R}^2$ can be see as abstract cell complex, see Fig. 4.

From the abstract cell complex which stands for the plane $\mathbb{R}^n$, $n \geq 1$, we can construct a graph in the following way:

- The set of vertices $V$ is the set of cells.
- Two vertices, $x, y \in V$ form an edge if and only if either $(x, y) \in B$ or $(y, x) \in B$.

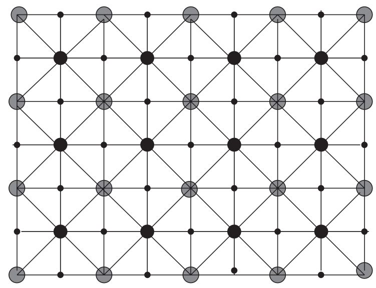

**Fig. 5.** Induced subgraph associated with a part of the cell complex of the plane

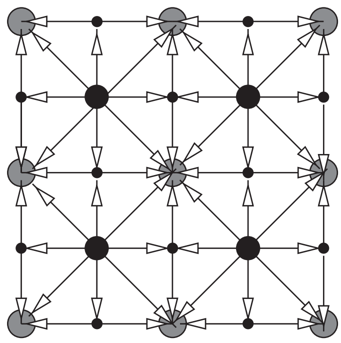

**Fig. 6.** Orientation of graph Fig. 5

By putting a black point on the cells with a dimension equal 2, a small black point on the cells with a dimension equal 1, and a grey point on the cells with a dimension equal 0, we obtain the graph of Fig. 5.

This graph is denoted by $G(C)$; the associate graph with a cell complex. The binary relation $B$ being transitive, graph Fig. 5 is a comparability graph. An orientation is given Fig. 6. We will denote by $H(x)$ the horizontal neighbors of a small black point $x$, and by $V(x)$ the vertical neighbors of a small black point $x$.

**Lemma 3** *Let $\mathcal{T}$ be a $T_0$-Alexandroff compatible topology with $G(C)$, $C$ being the cell complex associated with the plane. For all $x$ black point or grey point we have: either $\mathcal{N}_{\mathcal{T}}(x) = \{x\} \cup \Gamma(x)$ or $\mathcal{N}_{\mathcal{T}}(x) = \{x\}$.*

*Sketch of Proof.* If $\mathcal{N}_{\mathcal{T}}(x) \neq \{x\}$ one can show thank to $T_0$ hypothesis than 6 of the 8 neighbors of $x$ are in $\mathcal{N}_{\mathcal{T}}(x)$ and one can conclude that the two others neighbors are equally in $\mathcal{N}_{\mathcal{T}}(x)$.

Let $\mathcal{T}$ be a $T_0$-Alexandroff compatible topology and let $x$ be a grey point, we have two cases.

(a) We suppose that $\mathcal{N}_{\mathcal{T}}(x) = \{x\}$. One can show that for all grey point $y$ we have $\mathcal{N}_{\mathcal{T}}(y) = \{y\}$, and for all $y$ small black points we have $\mathcal{N}_{\mathcal{T}}(x) = H(x)$. Consequently for all black points $y$ we have $\mathcal{N}_{\mathcal{T}}(x) = \{x\} \cup \Gamma(x)$.

(b) If $\mathcal{N}_{\mathcal{T}}(x) = \{x\} \cup \Gamma(x)$ in the same way we show a similar result.

So from this remark and by applying Theorems 2–5 we have:

**Theorem 8** *There are exactly two $T_0$ Alexandroff compatible topologies, $\mathcal{T}_1$ and $\mathcal{T}_2$ on the graph associated with the cell complex of the plane.*

These two topologies are described in the following way:

– Let $x$ be a black point, $\mathcal{N}_{\mathcal{T}_1}(x) = \{x\} \cup \Gamma(x)$ and $\mathcal{N}_{\mathcal{T}_2}(x) = \{x\}$.
– Let $x$ be a small black points, $\mathcal{N}_{\mathcal{T}_1}(x) = H(x)\, \mathcal{N}_{\mathcal{T}_2}(x) = V(x)$.
– Let $x$ be a grey points $\mathcal{N}_{\mathcal{T}_1}(x) = \{x\}$ and $\mathcal{N}_{\mathcal{T}_2}(x) = \{x\} \cup \Gamma(x)$.

We are now showing that this two topologies are homeomorphic, (but not equal).

**Theorem 9** *The two $T_0$ Alexandroff compatible topologies, $\mathcal{T}_1$ and $\mathcal{T}_2$ are homeomorphic.*

*Proof.* Settle:

– The set of grey points $W = \{(2n, 2m+1), n, m \in \mathbb{Z}\}$.
– The set of small black points $SB = \{(n, m), m + n = 2k, k \in \mathbb{N}^*, n, m \in \mathbb{Z}\}$.
– The set of black points $B = \{(2n+1, 2m), n, m \in \mathbb{Z}\}$.

Let us define the following mapping:

$$\phi \ : (\mathbb{Z}^2; \mathcal{T}_1) \longrightarrow (\mathbb{Z}^2; \mathcal{T}_2)$$

$$(x, y) \longmapsto (y, x)$$

It is a symmetry given Fig. 7. We have to show that

$$\phi(\mathcal{N}_{\mathcal{T}_1}((x, y))) = \mathcal{N}_{\mathcal{T}_2}((y, x)).$$

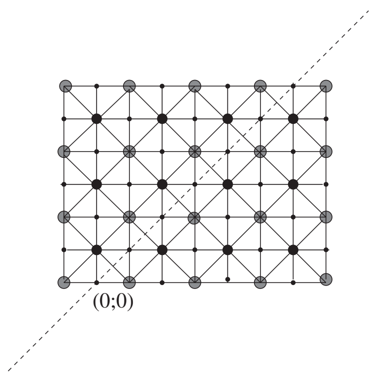


**Fig. 7.** Symmetry of the plane

(1) Let $(x, y)$ be a point of $SB$. Because $x + y = 2k = y + x$ it is easy to see that if $(x, y) \in SB$ then $(y, x) \in SB$. Consequently

$$\phi(\mathcal{N}_{\mathcal{T}_1}((x,y)))=\phi(\{(x+\epsilon,y),\epsilon=0,\pm1\})=\{(y,x+\epsilon=0,\pm1)\}=\mathcal{N}_{\mathcal{T}_2}((y,x)).$$

(2) Let $(x, y) = (2n; 2m + 1) \in W$, hence $(y, x) = (2n + 1; 2m) \in B$, so:

$$\phi(\mathcal{N}_{\mathcal{T}_1}((x,y))) = \phi(\{(x,y)\}) = \{(y,x)\} = \mathcal{N}_{\mathcal{T}_2}((y,x)).$$

Moreover the identity is not an homeomorphism because $\mathcal{N}_{\mathcal{T}_1}((x, y)) \neq \mathcal{N}_{\mathcal{T}_2}((y, x))$.
□

## References


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