Teorija grafov - Zapiski predavanj

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Contents

1 Introduction

A graph is defined as $G = (V, E)$. $n = |V|$ is the number of vertices, $m = |E|$ is the number of edges. We also denote them as $V(G), n(G), E(G), m(G)$. $\delta(G)$ is the minimum degree of a vertex in G, $\Delta(G)$ is the maximum degree. $G[C]$ represents the induced subgraph of G on the vertex set C.

2 Independence, matching, covers

Definition. *The set of vertices $S \subseteq V$ is an **independent set** if $G(S)$ contains no edges. (No two vertices in the independent set are adjacent)*

The independence number $\alpha(G)$ is the size of the maximum independent set.

Definition. *The set of vertices $T \subseteq V$ is a **vertex cover** if $\forall e \in E \ T \cap e \neq \emptyset$. (All edges have at least one endpoint in the vertex cover)*

The vertex cover number $\beta(G)$ is the size of the minimum vertex cover.

Definition. *A **matching** is a set of edges $M \subseteq E$ such that $\forall e, f \in M \ e \neq f \ e \cap f \neq \emptyset$. (No two edges share a vertex)*

The matching number $\alpha'(G)$ is the size of the maximum matching.

Definition. *An **edge cover** is a set of edges $C \subseteq E$ such that $\forall v \in V \ \exists e \in C \ v \in e$. (All vertices are covered by at least one edge from C)*

The edge cover number $\beta'(G)$ is the size of the minimum edge cover. Some graphs have no edge covers, for example graphs with isolated vertices.
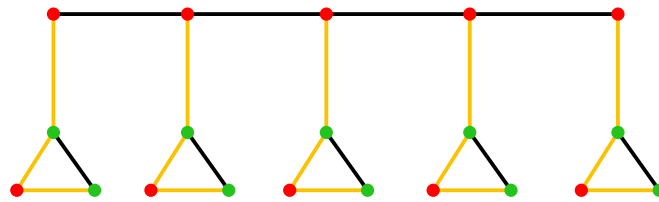

Figure 1: G from example

Example. $\alpha(G) = 8$
$h(G) = 20$
$\beta(G) = 12 \rightarrow$ *complement of vertex set*
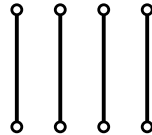$\alpha'(G) = 10$ *maximum for α' is* $\frac{h(G)}{2}$
$\beta'(G) = 10$

Observations

- $\alpha(G) + \beta(G) = |V|$ (the size of the maximum independent set plus the size of the minimum vertex cover is equal to the number of vertices)

  Proof. For every independent set S, the complement $\overline{S}$ is a vertex cover and vice versa. □

  $\alpha'(G) \le \beta(G)$ (the size of the maximum matching is less than or equal to the size of the minimum vertex cover)
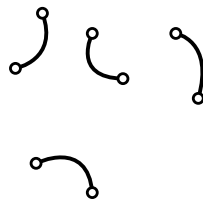

  Proof. Every edge in a maximum matching must be covered by different vertices in the vertex cover. □

- $\alpha(G) \le \beta'(G)$ (the size of the maximum independent set is less than or equal to the size of the minimum edge cover)

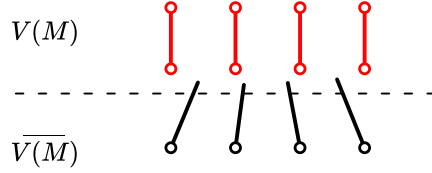 *Proof.* Every vertex in a maximum independent set must be covered by different edges in the edge cover. □

- if G has no isolated vertices: $\alpha'(G) \le \frac{n}{2} \le \beta(G)$

Theorem (Galloi's theorem)**.** *If G has no isolated vertices, then $\alpha'(G) + \beta'(G) = n(G)$.*

Proof.   (1) $\beta'(G) + \alpha'(G) \leq |V(G)|$

Take a maximum matching M; $M = \alpha'(G)$. For every vertex not covered in

$V(M)$

$\overline{V(M)}$

M $(\overline{V(M)})$, we can take an incident edge and add them to M. We get a set of edges R, which covers every vertex in G.

$$|R| = |M| + |\overline{V(M)}| = |M| + (|V(G)| - 2|M|)$$
$$= |V(G)| - |M|$$
$$\beta'(G) \leq |R| = |V(G)| - \alpha'(G)$$
$$\beta'(G) + \alpha'(G) \leq |V(G)|$$

(2) $\beta'(G) + \alpha'(G) \geq |V(G)|$

Lemma. *Let C be a minimum edge cover. For every edge in C, at least one of its endpoints is covered only once by C.*

Proof. Suppose $uv \in G$ and u and v are covered by other edges in C. $C' = C \: \{uv\}$ is also an edge cover and $|C'| < |C|$ which is a contradiction. $\square$

Because of this, we can see that $G[C]$ is a star forest (for all minimal edge covers). $G[C]$ consists of k components: $|C| = |V(G)| - k$. A matching is obtained by chosing
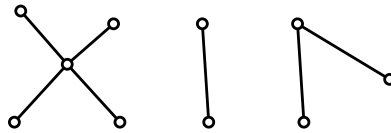
one edge from every star component of $G[C]$, the resulting matching has k edges $(|M| = k)$

$$\alpha'(G) \geq |M| = k \geq |V(G)| - |C| = |V(G)| - \beta'(G)$$
$$\alpha'(G) + \beta'(G) \geq |V(G)|$$

$\square$

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