Teorija grafov - Zapiski predavanj

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Contents

1 Introduction

A graph is defined as $G = (V, E)$. $n = |V|$ is the number of vertices, $m = |E|$ is the number of edges. We also denote them as $V(G), n(G)$, $E(G), m(G)$. $\delta(G)$ is the minimum degree of a vertex in G, $\Delta(G)$ is the maximum degree. $G[C]$ represents the induced subgraph of G on the vertex set C.

2 Independence, matching, covers

Definition. *The set of vertices $S \subseteq V$ is an* ***independent set*** *if $G(S)$ contains no edges. (No two vertices in the independent set are adjacent)*

The independence number $\alpha(G)$ is the size of the maximum independent set.

Definition. *The set of vertices $T \subseteq V$ is a* ***vertex cover*** *if $\forall e \in E$ $T \cap e \neq \emptyset$. (All edges have at least one endpoint in the vertex cover)*

The vertex cover number $\beta(G)$ is the size of the minimum vertex cover.

Definition. *A* ***matching*** *is a set of edges $M \subseteq E$ such that $\forall e, f \in M$ $e \neq f$ $e \cap f \neq \emptyset$. (No two edges share a vertex)*

The matching number $\alpha'(G)$ is the size of the maximum matching.

Definition. *An* ***edge cover*** *is a set of edges $C \subseteq E$ such that $\forall v \in V$ $\exists e \in C$ $v \in e$. (All vertices are covered by at least one edge from C)*

The edge cover number $\beta'(G)$ is the size of the minimum edge cover. Some graphs have no edge covers, for example graphs with isolated vertices.

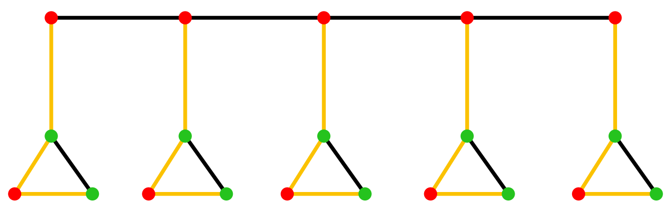


Figure 1: G from example

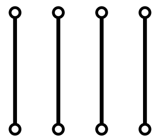
Example. $\alpha(G) = 8$
$h(G) = 20$
$\beta(G) = 12 \rightarrow$ *complement of vertex set*
$\alpha'(G) = 10$ *maximum for* α' *is* $\frac{h(G)}{2}$
$\beta'(G) = 10$

Observations

- $\alpha(G) + \beta(G) = |V|$ (the size of the maximum independent set plus the size of the minimum vertex cover is equal to the number of vertices)

  Proof. For every independent set S, the complement $\overline{S}$ is a vertex cover and vice versa. $\square$

  $\alpha'(G) \leq \beta(G)$ (the size of the maximum matching is less than or equal to the size of the minimum vertex cover)

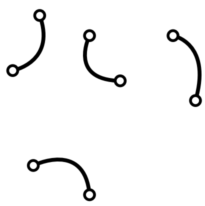
  

  Proof. Every edge in a maximum matching must be covered by different vertices in the vertex cover. $\square$

- $\alpha(G) \leq \beta'(G)$ (the size of the maximum independent set is less than or equal to the size of the minimum edge cover)

  Proof. Every vertex in a maximum independent set must be covered by different edges in the edge cover. $\square$

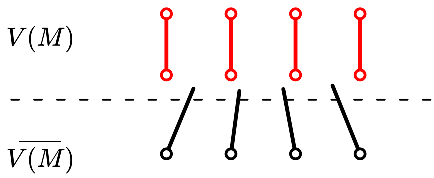
- if G has no isolated vertices: $\alpha'(G) \leq \frac{n}{2} \leq \beta(G)$

Theorem (Galloi's theorem)**.** *If G has no isolated vertices, then $\alpha'(G) + \beta'(G) = n(G)$.*

Proof. (1) $\beta'(G) + \alpha'(G) \le |V(G)|$

Take a maximum matching M; $M = \alpha'(G)$. For every vertex not covered in



M ($\overline{V(M)}$), we can take an incident edge and add them to M. We get a set of edges R, which covers every vertex in G.

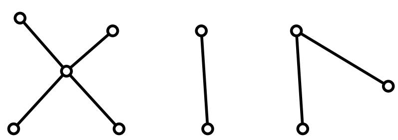
$$\begin{aligned}|R| &= |M| + |\overline{V(M)}| = |M| + (|V(G)| - 2|M|)\\ &= |V(G)| - |M|\\ \beta'(G) &\le |R| = |V(G)| - \alpha'(G)\\ \beta'(G) &+ \alpha'(G) \le |V(G)|\end{aligned}$$

(2) $\beta'(G) + \alpha'(G) \ge |V(G)|$

Lemma. *Let C be a minimum edge cover. For every edge in C, at least one of its endpoints is covered only once by C.*

Proof. Suppose $uv \in G$ and u and v are covered by other edges in C. $C' = C\ \{uv\}$ is also an edge cover and $|C'| < |C|$ which is a contradiction. □

Because of this, we can see that $G[C]$ is a star forest (for all minimal edge covers). $G[C]$ consists of k components: $|C| = |V(G)| - k$. A matching is obtained by chosing



one edge from every star component of $G[C]$, the resulting matching has k edges ($|M| = k$)

$$\alpha'(G) \ge |M| = k \ge |V(G)| - |C| = |V(G)| - \beta'(G)$$

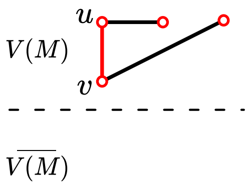
$$\alpha'(G) + \beta'(G) \ge |V(G)|$$

□

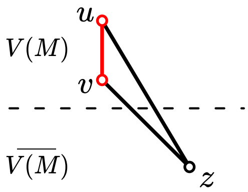
3 Matchings

Structure of the maximum matching M. For each $uv \in M$ one of these holds:

(1)
$$N(u) \cap \overline{V(M)} = \emptyset$$
$$N(v) \cap \overline{V(M)} = \emptyset$$

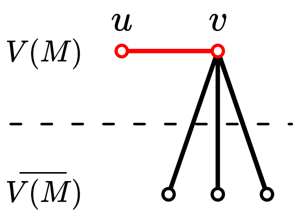


(2)
$$N(u) \cap \overline{V(M)} \neq \emptyset$$
$$N(v) \cap \overline{V(M)} \neq \emptyset$$

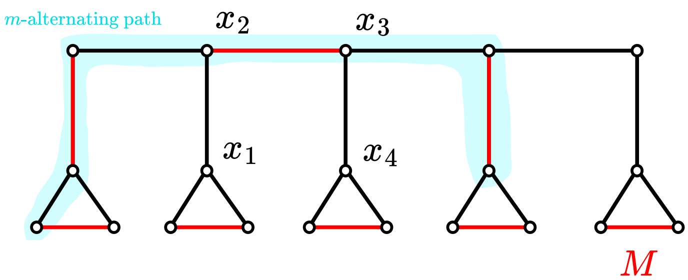


(3) $N(u) \cap \overline{V(M)} \neq \emptyset$ or $N(u) \cap \overline{V(M)} = \emptyset$

$N(v) \cap \overline{V(M)} = \emptyset$ $\quad$ $N(v) \cap \overline{V(M)} \neq \emptyset$



Definition. *Let M be a matching. A path $v_1u_1v_2u_2 \ldots v_ku_k(v_{k+1})$ is an **m-altering path** if the edges along the path alternate between M and $\overline{M} = E\ M$*



Definition. *An m-alternating path is m-augmenting if both ends of the path are uncovered by M*

For example $x_1x_2x_3x_4$ in the above figure. This is important because $M' = M \setminus \{x_2x_3\} \cup \{x_1x_2\}$ is a larger matching.