Teorija grafov - Zapiski predavanj

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October 17, 2024

Contents

1 Introduction

A graph is defined as $G = (V, E)$. $n = |V|$ is the number of vertices, $m = |E|$ is the number of edges. We also denote them as $V(G), n(G)$, $E(G), m(G)$. $\delta(G)$ is the minimum degree of a vertex in G, $\Delta(G)$ is the maximum degree. $G[C]$ represents the induced subgraph of G on the vertex set C.

2 Independence, matching, covers

Definition. *The set of vertices $S \subseteq V$ is an **independent set** if $G(S)$ contains no edges. (No two vertices in the independent set are adjacent)*

The independence number $\alpha(G)$ is the size of the maximum independent set.

Definition. *The set of vertices $T \subseteq V$ is a **vertex cover** if $\forall e \in E$ $T \cap e \neq \emptyset$. (All edges have at least one endpoint in the vertex cover)*

The vertex cover number $\beta(G)$ is the size of the minimum vertex cover.

Definition. *A **matching** is a set of edges $M \subseteq E$ such that $\forall e, f \in M$ $e \neq f$ $e \cap f \neq \emptyset$. (No two edges share a vertex)*

The matching number $\alpha'(G)$ is the size of the maximum matching.

Definition. *An **edge cover** is a set of edges $C \subseteq E$ such that $\forall v \in V$ $\exists e \in C$ $v \in e$. (All vertices are covered by at least one edge from C)*

The edge cover number $\beta'(G)$ is the size of the minimum edge cover. Some graphs have no edge covers, for example graphs with isolated vertices.

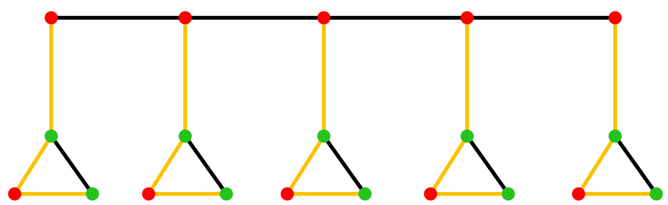


Figure 1: G from example

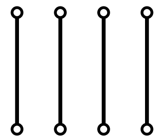
Example. $\alpha(G) = 8$
$h(G) = 20$
$\beta(G) = 12 \to$ *complement of vertex set*
$\alpha'(G) = 10$ *maximum for* α' *is* $\frac{h(G)}{2}$
$\beta'(G) = 10$

Observations

- $\alpha(G) + \beta(G) = |V|$ (the size of the maximum independent set plus the size of the minimum vertex cover is equal to the number of vertices)

  Proof. For every independent set S, the complement $\overline{S}$ is a vertex cover and vice versa. □

  $\alpha'(G) \leq \beta(G)$ (the size of the maximum matching is less than or equal to the size of the minimum vertex cover)

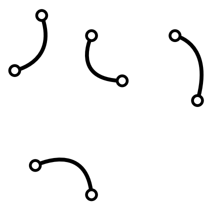
  

  Proof. Every edge in a maximum matching must be covered by different vertices in the vertex cover. □

- $\alpha(G) \leq \beta'(G)$ (the size of the maximum independent set is less than or equal to the size of the minimum edge cover)

  Proof. Every vertex in a maximum independent set must be covered by different edges in the edge cover. □

- if G has no isolated vertices: $\alpha'(G) \leq \frac{n}{2} \leq \beta(G)$

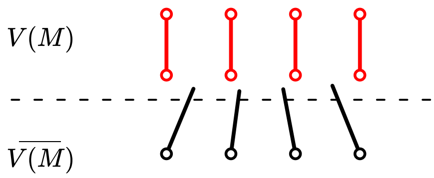
  

3 Matchings

Theorem (Galloi's theorem). *If G has no isolated vertices, then $\alpha'(G) + \beta'(G) = n(G)$.*

Proof. (1) $\beta'(G) + \alpha'(G) \leq |V(G)|$

Take a maximum matching M; $M = \alpha'(G)$. For every vertex not covered in



M ($\overline{V(M)}$), we can take an incident edge and add them to M. We get a set of edges R, which covers every vertex in G.

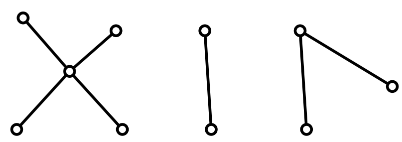
$$\begin{aligned}
|R| &= |M| + |\overline{V(M)}| = |M| + (|V(G)| - 2|M|) \\
&= |V(G)| - |M| \\
\beta'(G) &\leq |R| = |V(G)| - \alpha'(G) \\
\beta'(G) &+ \alpha'(G) \leq |V(G)|
\end{aligned}$$

(2) $\beta'(G) + \alpha'(G) \geq |V(G)|$

Lemma. *Let C be a minimum edge cover. For every edge in C, at least one of its endpoints is covered only once by C.*

Proof. Suppose $uv \in G$ and u and v are covered by other edges in C. $C' = C\ \{uv\}$ is also an edge cover and $|C'| < |C|$ which is a contradiction. □

Because of this, we can see that $G[C]$ is a star forest (for all minimal edge covers). $G[C]$ consists of k components: $|C| = |V(G)| - k$. A matching is obtained by chosing



one edge from every star component of $G[C]$, the resulting matching has k edges ($|M| = k$)

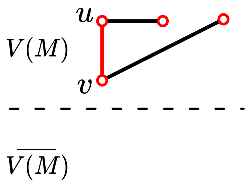
$$\alpha'(G) \geq |M| = k \geq |V(G)| - |C| = |V(G)| - \beta'(G)$$

$$\alpha'(G) + \beta'(G) \geq |V(G)|$$

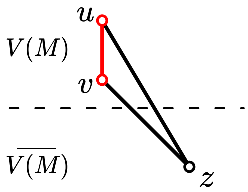
□

Structure of the maximum matching M. For each $uv \in M$ one of these holds:

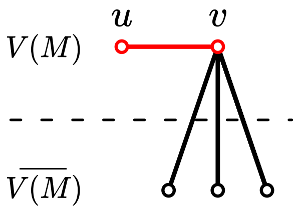
(1)
$$N(u) \cap \overline{V(M)} = \emptyset$$
$$N(v) \cap \overline{V(M)} = \emptyset$$



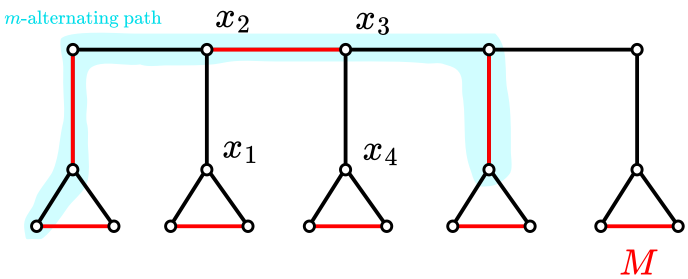
(2)
$$N(u) \cap \overline{V(M)} \neq \emptyset$$
$$N(v) \cap \overline{V(M)} \neq \emptyset$$



(3)
$$N(u) \cap \overline{V(M)} \neq \emptyset \quad \text{or} \quad N(u) \cap \overline{V(M)} = \emptyset$$
$$N(v) \cap \overline{V(M)} = \emptyset \qquad N(v) \cap \overline{V(M)} \neq \emptyset$$



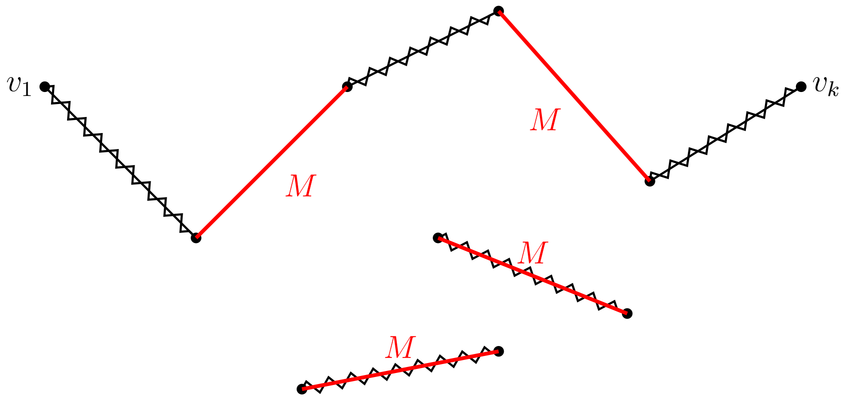
Definition. *Let M be a matching. A path $v_1u_1v_2u_2\ldots v_ku_k(v_{k+1})$ is an **m-altering path** if the edges along the path alternate between M and $\overline{M} = E\ M$*



Definition. *An m-alternating path is m-augmenting if both ends of the path are uncovered by M*

For example $x_1x_2x_3x_4$ in the above figure. This is important because $M' = M \setminus \{x_2x_3\} \cup \{x_1x_2\}$ is a larger matching.

Theorem. *Let G be a graph and M a matching in G. If there exists an M-augmenting path then M is not a maximum matching.*

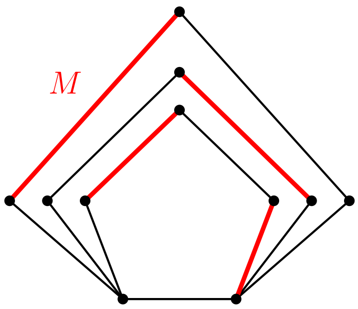


Proof. Let P be an M-augmenting path. $P = v_1v_2\dots v_k$, because it is an augmenting path $v_1v_2 \notin M$ $v_{k-1}v_k \notin M \Rightarrow |E(P) \cap \overline{M}| = |E(P) \cap M| + 1$ Let $M' = M \Delta E(P)$, M' is a matching in G (Marked as zigzag lines in the graph).
M' is a matching

- For vertices outside P the edges are the same in M and M'.
- v_1, v_k were uncovered by M so they are now covered by M'.
- For internal vertices: v_i is covered once in M. This edge is missing in M', therefore v_i is covered by another edge in M'.

$\Rightarrow$ M' is a matching, $|M'| = |M| + 1$, so M is not a maximum matching. □

König-Egerváry graphs:



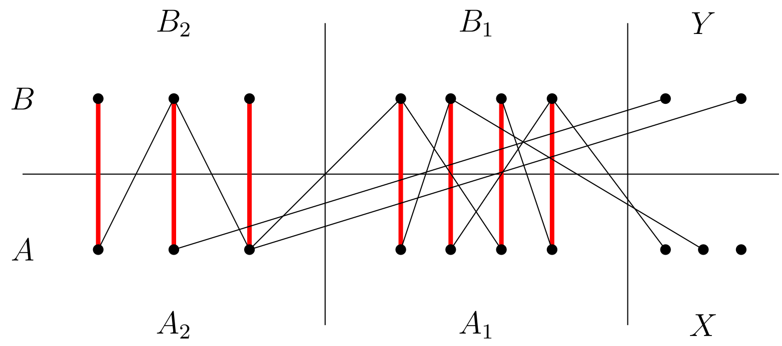
König-Egerváry have $\alpha'(G) = \beta(G)$. In the example: $\alpha'(G) = 5 = \beta(G)$, G is not bipartite.

Theorem (Königs theorem)**.** *Let G be a bipartite graph, then the following holds:*

(a) $\alpha'(G) = \beta(G)$

(b) let M be a matching in G and a M-augmenting path does not exist, then M is a maximum matching.

Proof. Let G be a bipartite graph on sets A, B. Suppose that M is a matching such that an M-augmenting path in G does not exist (such an M exists).



- M is marked in red
- x, y are the sets of uncovered vertices in A, B ($X = A \setminus V(M)$, $Y = B \setminus V(M)$).
- B_1 is the set of vertices in B that are reachable from X with an M-alternating path.
- A_1 is the set of vertices in A that are reachable from X with an M-alternating path.
- $B_2 = B \setminus (B_1 \cup Y)$
- $A_2 = A \setminus (A_1 \cup X)$

$A \to B$ edges are in $\overline{M}$
$B \to A$ edges are in M

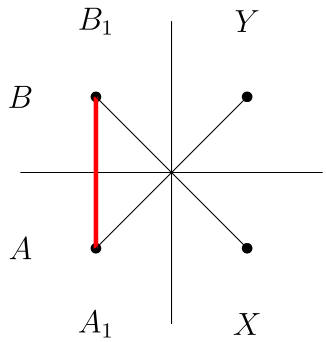
$$A_1 = \{a \in A \mid \exists b \in B_1 : ab \in M\}, \quad A_2 = \{a \in A \mid \exists b \in B_2 : ab \in M\}$$

$$|A_1| = |B_1|; \quad |A_2| = |B_2|; \quad |A_1| + |A_2| = |M|$$

We can prove that the following statements hold:

- There are no edges between X and Y
  Lets propose $x \in X, y \in Y, xy \in E$. This is an M-alternating path of length 1. It is also an M-augmenting path, because its endpoints are not covered by M. This is a contradiction, because we assumed that there are no M-augmenting paths.
- There are no edges between X and B_2
  An edge $x \in X, b \in B2, xb \in E$ is an M-alternating path from X to B_2 which contradicts the definition of B_1 and B_2.

- There are no edges between A_1 and Y
  Suppose that $a \in A_1, y \in Y, ay \in E$.



  There exists an M-alternating path from X to B_1:

$$v_1 v_2 \dots v_k,\ v_1 \in X,\ v_k \in B_1$$

  We can now add an edge from v_k to a from the matching M and ay not from M. The path

$$v_1 v_2 \dots v_k a y$$

  is an M-alternating path and because v_1 and y are not covered by M it is also an M-augmenting path.

- There are no edges between A_1 and B_2
  If we begin an M-alternating path in X and end in A_1 we get a path:

$$x v_2 \dots v_{k-1} a,\ x \in X,\ a \in A_1$$

  Now we can add the edge $ab, b \in B_1, ab \notin M$. We know that $v_{k-1}a \in M$ so $x_1 v_2 \dots v_{k-1} ab$ is an M-alternating path. B_1 is therefore reachable from X with an M-alternating path. This a contradiction with the definition of B_1 and B_2.

$T = B_1 \cup A_2$ is a vertex cover
We only need to prove that all edges from $A = X \cup A_1 \cup A_2$ are covered by the vertex cover because the graph is a bipartite graph.

- all edges from X are covered by a vertex in B_1
- all edges from A_1 are covered by a vertex in B_1
- all edges from A_2 are covered by a vertex in A_2

Now we can finish the proof:

$$|T| = |B_1| + |A_2| = |A_1| + |A_2| = |M|$$

$$\beta(G) \leq |T| = |M| \leq \underbrace{\alpha'(G) \leq \beta(G)}_{\text{proven}}$$

From this we can conclude:

(a) $\alpha'(G) = \beta(G)$

(b) $|M| = \alpha'(G)$ therefore M is a maximum matching.

$\square$

Corollary. *If G is a bipartite graph then $\alpha(G) = \beta'(G)$.*

Proof.

$$\alpha(G) = n(G) - \beta(G) \overset{\text{Königs}}{\overset{\downarrow}{=}} n(G) - \alpha'(G) \overset{\text{Gallois}}{\overset{\downarrow}{=}} \beta'(G)$$

$\square$

Definition (HC)**.** *G is a bipartite on the sets A, B. **Halls condition (HC)** holds for A if*

$$\forall S \subseteq A: \quad |S| \leq |N(G)|$$

where $N(S) = \cup_{x \in S} N(x)$ is the set of neighbors of S.

Theorem (Halls theorem)**.** *G bipartite on A, B. There exists a matching that covers $A \iff$ HC holds for A.*

Proof.

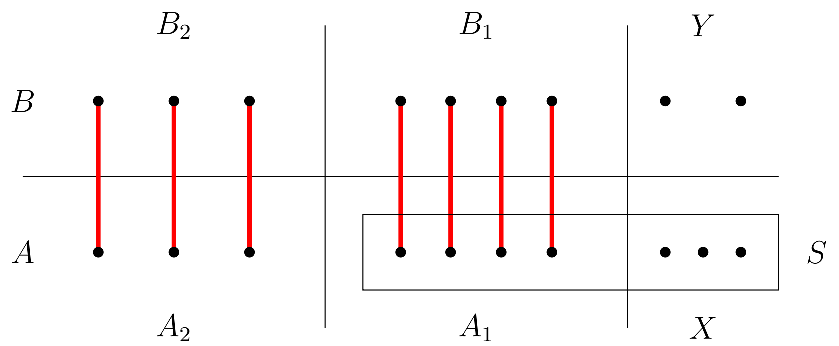
(⇒) Let M be the matching that covers A. $\forall S \subseteq A$ we can take the pairs matched by M:

$$B_S = \{v \in B \mid \exists u \in S : uv \in M\}$$

We know that $|S| = |B_S|$ and $B_S \subseteq N(S) \Rightarrow |S| = |B_S| \leq |N(S)|$. Therefore HC holds for A.

(⇐) We have to prove that there does not exist a matching M that covers $A \Rightarrow$ HC does not hold for A

Take the maximum matching in G. Lets draw the same picture for G as in the proof of Königs theorem.



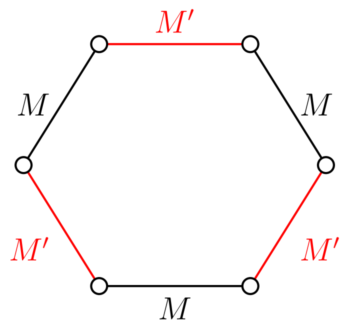
The maximum matching does not cover $A \Rightarrow X \neq \emptyset$. Consider $S = A_1 \cup X$. We have seen that all neighbors of S are in B_1 (in Königs) so $N(S) = B_1$, $|S| = |A_1| + |X|$ and $|N(S)| = |B_1| = |A_1|$ so $|S| > |N(S)|$ which means that HC does not hold for A.

$\square$

Theorem. *Let M be a matching in G. There exists an M-augmenting path $\iff$ M is not a maximum matching.*

Proof. ($\Rightarrow$) this was proven in the previous lecture

($\Leftarrow$) M is a matching, not maximum. Then there exists a M' matching such that $|M'| > |M|$. Consider $M \Delta M'$ (symmetric difference). Let $G' = G[M \Delta M']$. In G', the maximum degree $\Delta(G') \leq 2$. G' contains only paths and cycles. If The component is a cycle:

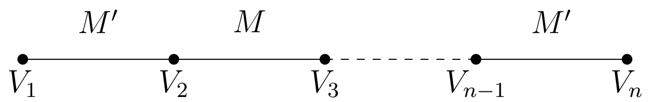


M and M' edges alternate in the cycle, therefore there are the same number of edges from M and M' in the cycle.
If The component is a path:

- path of even length, then M,M'-edges alternate, therefore there are the same number of edges from M and M' in the path.
- path of odd length, then there is one more edge from M' than from M or vice versa.

$$|M'| > |M| \Rightarrow |M' \setminus M| > |M \setminus M'|$$

$\Rightarrow$ there exists a component G'_1 in G' such that there are more edges from M' than from M. This component is a path of odd length that starts and ends with an edge from M'.



The following holds:

- G'_1 is an M-alternating path
- v_1 is not covered by M.
- in G' $deg(v_1) = 1$ (because it is a path component), therefore no edge from $M \setminus M'$ is incident to v_1.
- v_1 is not incident to any edge from $M \cap M'$ since v_1 is already covered by an edge from M'.

In conclusion v_1, v_k are not covered by M so G'_1 is an alternating path with uncovered endpoints and is therefore an M-augmenting path.

□

Algorithms for the above theorems:

- "Blossom algorithms" - to find an M-augmenting path and improve the matching if possible. The best known algorithm is $O(n\sqrt{n})$.
- We can determine the matching number and the edge cover number in polynomial time.

Definition (Oddity). *$o(G)$: number of odd components in G*

Theorem (Tuttes theorem). *G has a perfect matching* $\iff \forall S \subseteq V(G)\ o(G-S) \leq |S|$

$o(G-S) \leq |S|$ is called the Tutte condition.

Proof. ($\Rightarrow$) M is a perfect matching in G. For every $S \subseteq V(G)$.

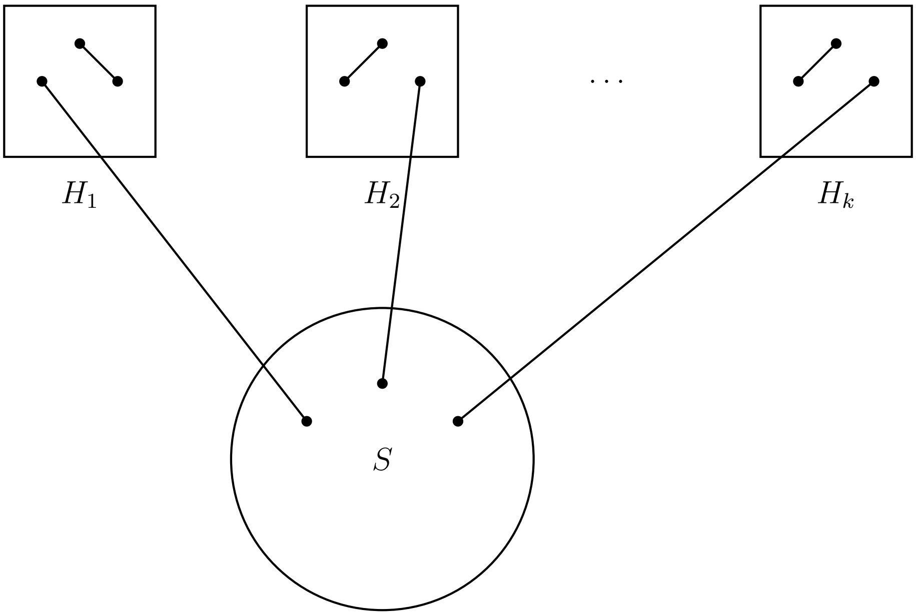


Figure 2: Graph G showing S and H_i which are odd components not in S

if H_1 is an odd component then there exists at least one M-edge between $V(H_i)$ and S. This holds for every odd component. Therefore there exists at least $o(G-S)$ M-edges between S and $\overline{S}$. As M is a matching these edges are incident do different vertices from S. Therefore $o(G-S) \leq |S|$ (Tuttes condition holds).

($\Leftarrow$) If the Tuttes condition holds then there exists a perfect matching. Suppose that the lemma does not hold. Then there exists a counter example F on n vertices.

(1) F satisfies the Tuttes condition

(2) There does not exist a perfect matching in F

We can add edges to F while it remains a counter example, until we get a graph G. G is then the largest counter example.

Lemma. *$n(G)$ is even*

Proof. if we take $S = \emptyset$, then

$$|S| = 0 \geq o(G - s) = o(G) \geq 0 \quad \Rightarrow \quad o(G) = 0$$

Because the number of odd components is zero, n is even. $\square$

Lemma. *$\forall e \in E(\bar{(G)})$ $G' = G + e$ has a perfect matching*

Proof.
- $G + e$ is not a counter example (because of the maximality of G).
- $G + e$ satisfies TC: For every $S \subseteq V(G + e)$ $o(G + e - S) \leq |S|$.
  When adding an edge e to G we have 4 cases:
  (a) If the edge e is inside S or inside a component of $G - S$, then the number of odd components does not change.
  (b) If the edge e is between two even components of $G - S$, then the number of odd components does not change.
  (c) If the edge e is between odd-even components of $G - S$, then the number of odd components does not change.
  (d) If the edge e is between two odd components of $G - S$, then the number of odd components decreases by two (the two odd components become an even component).

  Therefore G' satisfies the Tuttes condition:

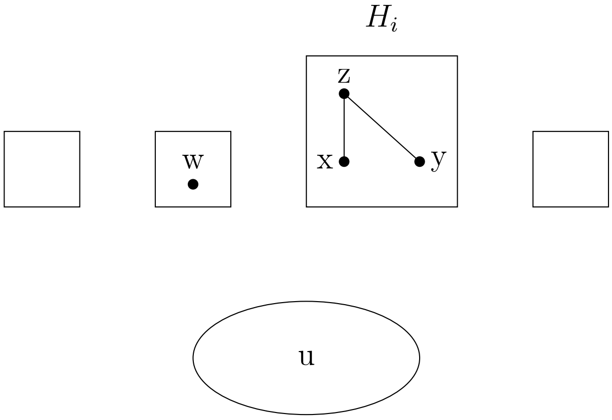
$$|S| \geq o(G - S) \geq o(G' - S)$$

G' is not a counter example, therefore (2) does not hold and there exists a perfect matching in G'. $\square$

To end the proof we need to show that G has a perfect matching. Let u be the set of universal vertices ($deg(v) = n-1$) in G. $G[u]$ is a complete graph (fully connected). We can construct a perfect matching for different cases:

1. If every H_i induces a complete graph, then we can construct a matching M
   - If H_i is an even component then then we can cover every vertex with M-edges inside H_i.
   - If H_i is an odd component then we cover all but one vertex with M-edges inside H_i.
   - The remaining $o(G - u)$ vertices fro the odd components can be covered with M-edges between u and the components using $o(G - u)$ different vertices from u. This can be done as all edges are present in G between u and $\overline{u}$ and $o(G - u) \leq |u|$.
   - If some vertices from u are uncovered then we define a matching on them. This is possible because $G[u]$ is a complete graph, $n(G)$ is even and M covered an even number of vertices so far.

   Therefore G has a perfect matching.
2. If there exists a non-complete component H_i, then we can find a pair of vertices x, y such that $dist(x, y) = 2$.



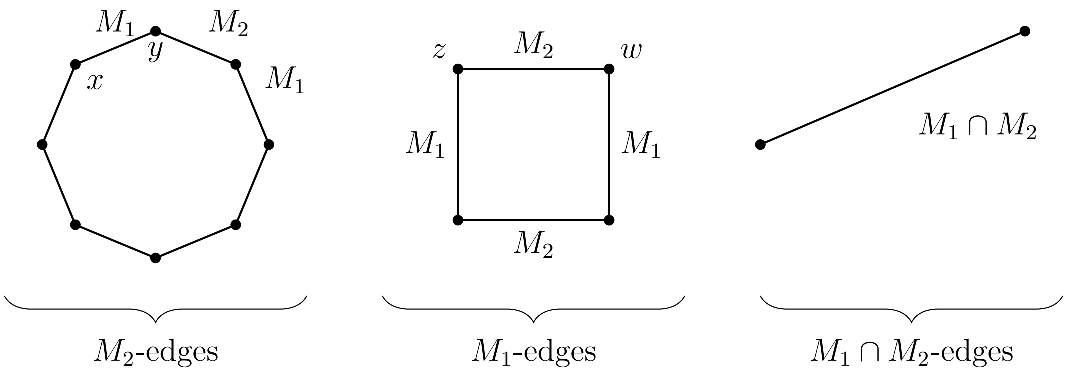
We can define z as the shared neighbor of x and y. There exists a vertex w such that w is not connected to z. This vertex exists because z is not a universal vertex. We define 2 expansions (both have a perfect matching because of lemma 2):

- $G_1 = G + xy \to M_1$
- $G_2 = G + zw \to M_2$

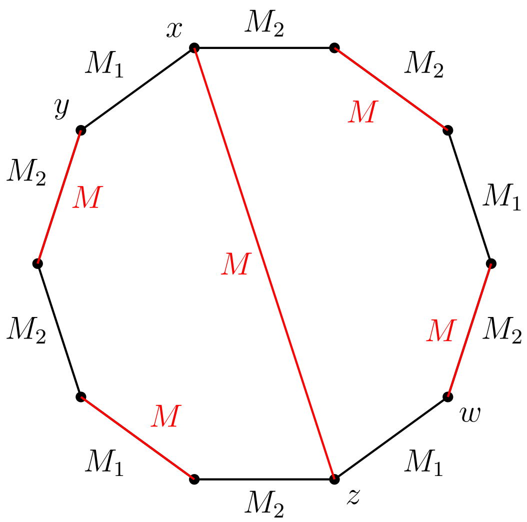
M_1 contains xy, otherwise M_1 is a perfect matching in G. M_2 contains zw because of the same reason. Consider $M_1 \Delta M_2$. First remove the edges from $M_1 \cup M_2$ (isolated vertices). For the remaining edges all vertices are of degree 2, therefore $\exists v$ such that v is connected to one M_1-edge and one M_2-edge.

Every non-isolated vertex from $G[M_1 \Delta M_2]$ is of degree 2, therefore the graph is composed of only cycles. Because M_1 and M_2 alternate, these are even cycles. $xy \in M_1$ and $xy \notin M_2 \Rightarrow xy \in M_1 \Delta M_2$

- If xy and zw belong to different components in $G[M_1 \Delta M_2]$.



  $M = M_1 \Delta M_2$ is a perfect matching in G:
  - $M \subseteq E(G)$ $xy \notin M$, $zw \notin M$
  - Every vertex is covered exactly once by M
- If xy and zw belong to the same component, without loss of generality along the cycle the edges are ordered: $zw \ldots xy \ldots$ (yx is a symmetric case)



$\square$

Borge-Tutte formula: A maximum matching in G leaves exactly

$$\max_{S \subseteq V(G)} \{o(G - S) - |S|\}$$

uncovered vertices. Equivalently:

$$\alpha'(G) = \frac{1}{2}\left(n - \max_{S \subseteq V(G)} \{o(G - S) - |S|\}\right)$$