# Conference-Exam on Quantitative Finance: Detailed Topics with Formulas

## Introduction

This document outlines the topics for the upcoming conference-exam on Quantitative Finance. The conference will explore critical aspects of Quantitative Finance divided into three main categories: Mathematical Aspects, Programming Aspects, and Financial Aspects. Each topic is designed to provide a comprehensive understanding and challenge the participants' knowledge and application skills in the respective areas.

## 1 Mathematical Aspects

### 1.1 Local Volatility Model and Fokker-Planck Equation

The Local Volatility Model, conceptualized by Dupire, is pivotal in the pricing of derivative securities. This topic covers the derivation of Dupire's formula, which is essential for understanding the local volatility dynamics in the presence of a drift term. The general form of Dupire's equation with a drift $\mu(S, t)$ is given by:

$$\sigma_{\text{loc}}^2(S, T) = \frac{\frac{\partial C}{\partial T} + \mu(S, T) S \frac{\partial C}{\partial S} + rC}{\frac{1}{2} S^2 \frac{\partial^2 C}{\partial S^2}},$$

where $C(S, T)$ is the price of the European call option, $S$ is the spot price of the underlying asset, $T$ is the time to maturity, $\mu(S, T)$ is the drift term, and $r$ is the risk-free interest rate. This formula adjusts the local volatility to account for the drift in the underlying asset's price, providing a more accurate model in non-neutral market conditions.

We will also explore the associated Fokker-Planck equation used to describe the probability density functions of stochastic processes under the influence of local volatility and drift:

$$\frac{\partial p}{\partial t} = -\frac{\partial}{\partial x}(\mu p) + \frac{1}{2} \frac{\partial^2}{\partial x^2}(\sigma_{\text{loc}}^2 p),$$

where $p(x, t)$ is the probability density function of the underlying asset price $x$ at time $t$. This equation helps in understanding how the probability distribution of the asset price evolves over time under the specified dynamics.

### 1.2 FX Quanto Adjustment and the Radon-Nikodym Derivative

This topic delves into the adjustments required when dealing with foreign exchange derivatives (quanto derivatives) and the application of the Radon-Nikodym derivative in changing measure from one probability space to another:

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \exp\left(-\frac{1}{2}\theta^2 T + \theta W_T\right),$$

where $\theta$ is the market price of risk. Participants will explore the mathematical foundation of the FX quanto adjustment and its significance in pricing cross-currency derivatives.

# 2 Programming Aspects

## 2.1 Using Binomial Trees for Pricing Instruments with American Exercise

This session focuses on implementing binomial trees to price derivatives with American-style exercise features, specifically Bermuda options. Bermuda options are a type of exotic option that can be exercised only on predetermined dates. For this exercise, we will consider a Bermuda option on a stock, say Apple Inc. (AAPL), with the following parameters:

- **Stock Initial Price (S0)**: \$150
- **Strike Price (K)**: \$160
- **Volatility ($\sigma$)**: 20%
- **Risk-free Interest Rate (r)**: 5%
- **Time to Maturity (T)**: 1 year
- **Early Exercise Dates**: Quarterly (i.e., after 3, 6, 9, and 12 months)

The binomial tree model will be used to simulate the possible future prices of AAPL stock at each exercise date. The tree will be constructed with a time step corresponding to the early exercise dates, and the option value will be calculated at each node using the following recursive relationship:

$$V_n = \max\left(\text{Exercise Value}, e^{-r\Delta t}\left(pV_{n+1}^u + (1-p)V_{n+1}^d\right)\right),$$

where $V_n$ is the option value at node $n$, $V_{n+1}^u$ and $V_{n+1}^d$ are the option values at the up and down nodes at the next time step, respectively, $\Delta t$ is the time step (one quarter of a year in this case), and $p$ is the risk-neutral probability calculated as:

$$p = \frac{e^{r\Delta t} - d}{u - d},$$

with $u = e^{\sigma\sqrt{\Delta t}}$ and $d = \frac{1}{u}$ representing the up and down factors, respectively.

This approach allows us to capture the flexibility of the Bermuda option's exercise feature and to analyze how the option's value is affected by the possibility of early exercise at discrete times throughout the life of the option.

## 2.2 Longstaff-Schwartz Conditional Monte Carlo Method

In this session, participants will implement the Longstaff-Schwartz algorithm, a least-squares Monte Carlo simulation method, to price American options. This method is particularly useful for options where analytical solutions are difficult to obtain. We will focus on an American put option with the following parameters:

- **Underlying Asset (Stock)**: Microsoft Corp. (MSFT)
- **Strike Price (K)**: \$200
- **Volatility** ($\sigma$): 25%
- **Risk-free Interest Rate (r)**: 3%
- **Time to Maturity (T)**: 1 year

The Longstaff-Schwartz method involves simulating multiple paths for the underlying asset's price using a geometric Brownian motion and then determining the optimal stopping rule (exercise strategy) based on the regression of simulated payoffs. The option value is estimated as:

$$V_t = \max\left(\text{Exercise Value}, \mathbb{E}^{\mathbb{Q}}[V_{t+1}|\mathcal{F}_t]\right),$$

where Exercise Value $= \max(K - S_t, 0)$ for a put option, and $\mathbb{E}^{\mathbb{Q}}[\cdot|\mathcal{F}_t]$ is the conditional expectation under the risk-neutral measure, estimated using least squares regression on the simulated paths.

To provide a comprehensive understanding and validate the effectiveness of the Longstaff-Schwartz method, participants will also implement a binomial tree model for the same American put option. The binomial tree parameters will be set as follows:

- Number of time steps: 50 (to match the granularity of the Monte Carlo simulation)
- Up factor ($u$): $e^{\sigma\sqrt{\Delta t}}$
- Down factor ($d$): $1/u$
- Risk-neutral probability ($p$): $\frac{e^{r\Delta t}-d}{u-d}$

After implementing both methods, participants will compare the results to discuss the accuracy, computational efficiency, and practical applicability of each method. This comparison will highlight the strengths and limitations of the Longstaff-Schwartz method relative to the more traditional binomial tree approach in the context of American option pricing.

## 2.3 Monte Carlo Simulations Using Local Volatility Model

This session involves implementing a local volatility model using the parametric form for a specific structured product, an autocallable note based on a single underlying asset. The details of the instrument are as follows:

- **Underlying Asset**: Tesla Inc. (TSLA)

- **Initial Stock Price (S0)**: \$700
- **Strike Price (K)**: \$700 (at-the-money)
- **Volatility ($\sigma$)**: 30%
- **Risk-free Interest Rate (r)**: 4%
- **Maturity (T)**: 3 years
- **Autocallable Feature**: Possible early redemption after 1 year if the stock price exceeds 110% of the initial price (\$770).

The payoff of the autocallable note is structured as follows:

- If the stock price is above 110% of the initial price at the end of the first year, the note is automatically called, and the investor receives a predetermined fixed return.
- If the note is not called, the final payoff at maturity is based on the performance of the underlying asset. Specifically, the payoff is the maximum of zero or the difference between the stock price at maturity and the strike price, i.e., $\max(0, S_T - K)$.

To price this autocallable note, we will use Monte Carlo simulations with the local volatility model specified by:

$$\sigma_{\text{Local}} = \alpha + \beta\kappa + \gamma\kappa^2,$$

where $\kappa = \log(K/F_t)$ represents the moneyness, and $F_t$ is the forward price of the stock. The parameters $\alpha$, $\beta$, and $\gamma$ will be calibrated using market data.

The Monte Carlo simulation involves generating a large number of possible future paths for the underlying asset's price using the stochastic differential equation:

$$dS_t = rS_t dt + \sigma_{\text{Local}}(S_t, t) S_t dW_t,$$

where $W_t$ is a standard Brownian motion. For each path, the autocallable feature and the final payoff are evaluated to determine the path's contribution to the option's value. The final price of the autocallable note is then estimated as the average discounted payoff across all simulated paths.

This practical session will provide participants with hands-on experience in pricing complex structured products using advanced simulation techniques and understanding the impact of local volatility adjustments on the pricing of options with path-dependent features.

# 3 Financial Aspects

## 3.1 Counterparty Credit Risk and XVA Adjustments

Counterparty credit risk arises when one of the parties in a financial contract may default on its obligations. This risk is quantified and managed through various valuation adjustments collectively known as XVA. The main components are:

- **CVA (Credit Valuation Adjustment)**: Represents the cost of hedging default risk of the counterparty. It is defined as:

$$\text{CVA} = \mathbb{E}[(1 - R) \cdot (V^+ \cdot \mathbf{1}_{\tau \leq T})],$$

  where $R$ is the recovery rate, $V^+$ is the positive exposure, $\tau$ is the default time, and $T$ is the maturity of the contract.

- **DVA (Debt Valuation Adjustment)**: This adjustment reflects the benefit to a firm when its own credit quality deteriorates, potentially reducing its liabilities upon default. It is calculated similarly to CVA but considers the firm's own default risk.

- **FVA (Funding Valuation Adjustment)**: Accounts for the cost of funding uncollateralized trades. It is the additional cost or benefit that arises from funding the derivative positions:

$$\text{FVA} = \int_0^T (f_u - r) \cdot V_t \, dt,$$

  where $f_u$ is the funding rate and $r$ is the risk-free rate.

- **KVA (Capital Valuation Adjustment)**: Reflects the cost of capital required to maintain the trade, considering regulatory capital requirements:

$$\text{KVA} = \int_0^T e^{-rt} \cdot \text{Cost of Capital} \cdot \text{Economic Capital} \, dt.$$

These adjustments have significant impacts on pricing and risk management in the financial industry, influencing trading strategies, pricing models, and regulatory compliance.

## 3.2 Bootstrapping Default Probabilities Using CDS Instruments

Credit Default Swaps (CDS) are instruments used to transfer the credit risk of fixed income products between parties. Bootstrapping default probabilities from CDS spreads involves constructing a zero-coupon survival curve that represents the probability of default-free survival to any point in time. The process typically involves:

- Calculating the implied hazard rates from CDS spreads using the formula:

$$s = (1 - R) \int_0^T \lambda(t) e^{-\int_0^t \lambda(u) \, du} \, dt,$$

  where $s$ is the CDS spread, $R$ is the recovery rate, and $\lambda(t)$ is the hazard rate at time $t$.

- Using these hazard rates to estimate the cumulative probability of default, $Q(t)$, by:

$$Q(t) = 1 - e^{-\int_0^t \lambda(u) \, du}.$$

This methodology is crucial for pricing and managing credit risk in portfolios, influencing decisions in risk management, investment, and regulatory reporting.

## 3.3 CDOs, Wrong Way Risk, and the 2008 Financial Crisis

Collateralized Debt Obligations (CDOs) are structured financial instruments that pool various debt obligations and then issue tranches with different risk levels to investors. The 2008 financial crisis highlighted significant issues with CDOs, particularly related to wrong way risk—where the exposure to a counterparty is negatively correlated with the credit quality of that counterparty. This section explores:

- The structure of CDOs and the tranching principle, which prioritizes payments to senior tranches before junior tranches.
- How mispricing of risk, excessive leverage, and lack of transparency in CDOs contributed to the financial crisis.
- The concept of wrong way risk, illustrated by the formula:

$$\text{Exposure} \times \text{Probability of Default} \uparrow \text{ as Credit Quality} \downarrow .$$

Additionally, the crisis was exacerbated by the widespread use of the Gaussian Copula model to price CDOs. This mathematical model was used to estimate the probability of default correlations among various assets in a CDO. The Gaussian Copula model assumes a normal distribution of defaults, which underestimated the tail risk and the likelihood of simultaneous defaults during extreme market conditions. The formula for the Gaussian Copula function is:

$$C(u, v; \rho) = \Phi_2(\Phi^{-1}(u), \Phi^{-1}(v); \rho),$$

where $\Phi_2$ is the bivariate normal distribution function, $\Phi^{-1}$ is the inverse of the standard normal cumulative distribution function, and $\rho$ is the correlation parameter.

The crisis was further intensified by the role of mortgage monolines. Originally, monoline insurance companies were designed to insure municipal bonds against default. However, during the housing boom, they expanded into insuring mortgage-backed securities and other complex debt instruments. When the housing market collapsed, these monolines faced enormous claims that they could not cover, leading to significant financial distress. This exposed the financial system to greater risk as the guarantees provided by monolines became unreliable.

This session aims to provide a comprehensive understanding of these complex instruments, the systemic risks they can pose, and the regulatory changes implemented post-crisis to mitigate such risks.