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Highlights

- An adaptive cracking particle method has been developed for thermoelastic fracture problems.
- Crack discontinuities are handled by the visibility criterion rather than enrichment functions, which can avoid extra unknowns and ill-conditioned system stiffness matrix.
- Thermal gradients around crack tips are captured by dense groups of particles, which are generated by an h-adaptivity approach.
- Abilities to model multiple crack propagation under both mechanical and thermal loadings.

Thermoelastic fracture modelling in 2D by an adaptive cracking particle method without enrichment functions

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Abstract

Thermoelastic fracture of brittle materials is a key concern for many types of engineering structures such as aerospace components and pressure vessels. These problems present difficulties in modelling as they involve coupling of the effects of thermal and mechanical loadings, and usually material behaviours are strongly discontinuous at crack locations, and highly nonlinear during crack propagation. Most current numerical methods applied to these problems use enrichment functions to model discontinuities of temperature and stresses at cracks, however, these enrichments bring extra unknowns into the model and can lead to numerical difficulties in attempting solutions. In this paper, an adaptive cracking particle method is developed for thermoelastic fracture. The method is meshless, so crack discontinuities can be introduced by modifying the influence domains of particles, rather than through external enrichments. Another benefit is an easy implementation of *h*-adaptivity, since no transition is required for different densities of particles, and it is easy to build dense groups of particles around crack tips. The results demonstrate that this new method can provide the same level of accuracy as others that employ enrichment functions, but using fewer degrees of freedom. A number of examples show the flexibility of the new method to model a range of thermoelastic problems in 2D including multiple cracks and crack propagation.

Keywords: thermal-mechanical coupling, adaptivity, cracking particle method, meshless, crack propagation

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1. Introduction

The study of thermoelastic fracture mechanics is of great importance in the design of structures under coupled mechanical and thermal loadings, including aerospace components, turbines and nuclear vessels. When heat travels across a crack, thermal gradients are set up in the neighbourhood of the crack tip, as well as thermal stresses [1]. The thermal stress concentration generated around the crack tip can lead to crack propagation and sudden failure without a change in the mechanical loading. For a crack in a thermoelastic solid, Sih (1962) [1] found that the local character of thermal stresses at the crack tip is of the same nature as the field from mechanical loading, and therefore Williams' expansions [2] for the asymptotic crack tip displacement field are applicable for thermally-induced stresses. In simplified studies of steady heat flux, crack surfaces can be considered to be either fully insulated or perfectly conductive [3, 4]. This is clearly an additional concern in thermoelastic problems, which is distinct from purely mechanical situations. More complex models exist, e.g. a partial insulation crack model has been developed in [5, 6], where the cracked zone is considered to contain a medium capable of conducting heat, e.g. air inside the crack opening domain acting as a thermal conduction medium. The partial insulation coefficient developed in these papers has a nonlinear relationship to the applied mechanical loading and to heat flux [6]. In recent decades, cracking under steady thermal loading has been studied using conventional numerical methods, and much of the development is similar to the development of non-thermal computational fracture. Examples of early numerical analysis of thermoelastic fracture mechanics use the finite element method (FEM) as in [7, 8], where discontinuities of both temperature and displacement at cracks are modelled by element interfaces, so the issue of remeshing during crack propagation remains [9]. The boundary element method (BEM) is an alternative (as in [10]), but meets dilemmas for modelling thermal nonlinearity [11]. Much success has been achieved via the use of enrichment functions within finite elements and meshless methods to model thermal fracture following their success for the purely mechanical case, for instance, in the extended finite element method (XFEM) [4, 12, 13], the element free Galerkin method (EFGM) [14–17], the numerical manifold method (NMM) [18, 19], and the isogeometric analysis method (IGA) [20]. These methods have been used to solve thermoelastic fracture problems including dynamic cracks [13], ductile cracks [21], interface cracks at bimaterial bodies [14] and fatigue fracture in railway brake discs [22]. The

1 scaled boundary finite element method (SBFEM) has been used for thermal fracture with a series of power
 2 functions to handle the stress concentrations at a crack tip [23, 24], however remeshing cannot be avoided for
 3 modelling crack propagation [25, 26]. In the XFEM, the benefits of using high order crack tip enrichments to
 4 improve accuracy for thermoelastic problems have been shown to be greater than for purely mechanical elastic
 5 problems, as mentioned in Zamani et al. (2010) [12]. However, the use of these enrichment functions brings extra
 6 unknowns to the problem, as both thermal and mechanical fields require a set of enrichment terms to model crack
 7 discontinuities in temperature and displacement respectively. This increase in the number of unknowns and
 8 can lead to numerical difficulties such as ill-conditioned linear systems requiring solutions [27], where possible
 9 solutions are obtained either using quasi-orthogonalization of enrichment functions [28, 29] or decomposing the
 10 system stiffness matrix to remove the diagonal zeros [30]. Alternatively, meshless methods can model crack
 11 discontinuities by disabling the influence between nodes on the two sides of the crack, which can avoid the use
 12 of enrichment functions, an advantage over element-based methods. No extra unknowns are introduced into
 13 the model, therefore meshless methods are suitable for crack problems under separate or combined thermal
 14 and mechanical loadings. Barbieri [31, 32] developed approaches with enriched weight functions to handle crack
 15 discontinuities in meshless methods, however tracking the geometry of cracks is required to build the enrichment
 16 functions, e.g. calculating the distance to crack tips, which is computationally expensive for multiple cracks
 17 and 3D crack problems.

18 In this paper, a novel weak form-based meshless method developed by the authors [33] is extended to
 19 thermoelastic problems in 2D. This method models cracks using “cracking particles” which are meshless nodes
 20 with certain properties. The discontinuity created by the presence of a crack is modelled using a simple visibility
 21 criterion which also handles temperature discontinuities at cracks, and no enrichment functions are needed.
 22 Accurate modelling of sharp thermal stress gradients around crack tips is achieved by an h -adaptivity approach,
 23 which generates dense groups of particles around crack tips and provides good control of the overall number
 24 of degrees of freedom. The paper is organised as follows. A brief review of current methods using enrichment
 25 functions for thermal discontinuities is included in Section 2. Then, an alternative to these enrichments is
 26 presented in Section 3 using the cracking particle method (CPM), where the h -adaptivity approach for a

1 temperature field is introduced. The method is applied to several problems to demonstrate its abilities and is
 2 shown to be suitable for modelling multiple crack propagation under thermal loadings.

3 **2. Review of enriched numerical approaches for cracks under thermal loadings**

4 External enrichment functions have been widely used in numerical methods including the XFEM [4, 12, 13]
 5 and the EFGM [14–17] to model the temperature discontinuities at cracks. Generally, two different types of
 6 cracks are studied in steady heat flux problems namely adiabatic and isothermal cracks, and the two cases result
 7 in different enrichment functions, which are applied according to the boundary conditions of the problem. The
 8 former assumes that both temperature and displacement fields are ideally discontinuous at the crack surface,
 9 while in the latter case, crack surfaces are perfectly thermally conductive and are maintained at a specific
 10 temperature, although the cracks still block the heat flux, and displacement is still discontinuous across the
 11 cracks. Partially insulated cracks are in between these two cases but are not covered in this paper.

12 *2.1. Adiabatic cracks*

13 In an adiabatic crack, a node located to one side of the crack has no influence on the other side in the
 14 temperature approximation. The heat flux is singular at the crack tip with the singularity $1/\sqrt{r}$ as in [1], where
 15 r is the distance to the crack tip, and is similar to the stress singularity at the crack tip for mechanical loading.
 16 The leading terms of the asymptotic expansion for temperature T and heat flux \mathbf{q} near to an adiabatic crack
 17 tip are given in Yosibash (1996) [34] as

$$T = -\frac{K_T}{k_T} \sqrt{\frac{2r}{\pi}} \sin \frac{\theta}{2}, \quad \mathbf{q} = \frac{K_T}{\sqrt{2\pi r}} [\sin \frac{\theta}{2}, \cos \frac{\theta}{2}]^T, \quad (1)$$

18 where K_T is the thermal stress intensity factor, k_T is the thermal conductivity, and r and θ are local polar
 19 coordinates centred at the crack tip. Due to the local character of temperature in Equation (1), the temperature
 20 approximation is

$$T^h(\mathbf{x}) = \sum_{i=1}^n \Phi_i(\mathbf{x}) T_i + \sum_{i=1}^{n_1} \Phi_i(\mathbf{x}) H(\mathbf{x}) b_i + \sum_{i=1}^{n_c} \Phi_i(\mathbf{x}) \sqrt{r} \sin\left(\frac{\theta}{2}\right) a_i, \quad (2)$$

21 where $\Phi_i(\mathbf{x})$ is a shape function, $T^h(\mathbf{x})$ is the temperature approximation, \mathbf{x} is a vector of coordinates i.e.
 22 $\mathbf{x} = [x, y]$ in 2D, T_i is the temperature of the i th particle, $H(\mathbf{x})$ is the sign function, a_i and b_i are extra

- 1 unknowns, and n , n_1 and n_c are the numbers of nodes with influence domains covering \mathbf{x} , cut by the crack and
 2 including the crack tip respectively.

3 *2.2. Isothermal cracks*

- 4 The isothermal case differs from the adiabatic case in the angular variation of the temperature field. The
 5 leading terms of the asymptotic expansion near to an isothermal crack tip [4] become

$$T = -\frac{K_T}{k_T} \sqrt{\frac{2r}{\pi}} \cos \frac{\theta}{2}, \quad \mathbf{q} = \frac{K_T}{\sqrt{2\pi r}} [\cos \frac{\theta}{2}, \sin \frac{\theta}{2}]^T. \quad (3)$$

- 6 Analogically, the temperature approximation becomes

$$T^h(\mathbf{x}) = \sum_{i=1}^n \Phi_i(\mathbf{x}) T_i + \sum_{i=1}^{n_1} \Phi_i(\mathbf{x}) H(\mathbf{x}) b_i + \sum_{i=1}^{n_c} \Phi_i(\mathbf{x}) \sqrt{r} \cos\left(\frac{\theta}{2}\right) a_i. \quad (4)$$

- 7 The sign function $H(\mathbf{x})$ in the XFEM is not applicable for an isothermal crack, since the temperature on the
 8 crack surface is continuous (but not the heat flux). $H(\mathbf{x})$ should be continuous across the crack surface with a
 9 discontinuous derivative, have a maximum value along the crack surface and become zero away from the crack.
 10 These properties can be delivered using the normal level set function to replace $H(\mathbf{x})$ as described in Duflot
 11 (2008) [4] as

$$H'(\mathbf{x}) = \sum_{i=1}^{n_1} \Phi_i(\mathbf{x}) |\phi_i| - |\phi(\mathbf{x})|, \quad (5)$$

- 12 where $\phi(\mathbf{x})$ is the level set function describing the distance to the crack surface.

- 13 To summarise, in simple problems involving temperature fields interacting with existing cracks, two types of
 14 cracks have been developed by assuming the crack to be thermally conductive or not, which leads to different
 15 types of temperature distributions. This brings challenges to numerical methods using external enrichment
 16 functions, which should be modified accordingly to capture the discontinuities caused by cracks. As indicated
 17 above and demonstrated in Equations (2) and (4), extra unknowns are required by these enrichment functions
 18 for both thermal and mechanical fields. However, an alternative meshless approach can be taken that entirely
 19 avoids the use of enrichment functions as described in the following section.

3. A cracking particle method requiring no enrichment functions

While finite element methods such as the XFEM have received much attention for modelling fracture problems, meshless methods using only nodes (or particles) in problem discretisation have also been prominent since they remove the need for element connectivity to be generated, and have major advantages when it comes to adaptivity. In meshless methods, connectivity between nodes is achieved by defining overlapping domains adjacent to the nodes over which they have influence, i.e. domains of influence or support. The most well-known weak form-based meshless method is the EFGM from which the cracking particle method was developed. The CPM models a crack path using directional segments that are centred at nodes, as shown in Figure 1 (a-b). It is a highly flexible approach able to model multiple cracks and branching. Discontinuities at cracks are simulated by using the so-called visibility criterion. This applies a “line-of-sight” check to alter the influence domain of a node if it overlaps with a crack as presented in Figure 1 (c) and so removes the need for any enrichment to do this job. The issue of having to deal with problems in which there are strong stress singularities at crack tips, which is dealt with in enriched approaches by altering the basis functions, is dealt with here by an h -adaptivity approach allowing a very fine discretisation around a crack tip. This will be shown to work satisfactorily later in the paper. The CPM was originally presented by Rabczuk and Belytschko (2004) [35] and has since been applied to model 3D cracks [36–38], dynamic fracture [39], ductile fracture [40], shear bands [41, 42] and multiple cracks [43]. Here the CPM is applied to thermoelastic fracture problems, where only the adiabatic crack is studied since there is little difference in the visibility criterion process for checking connectivities of particles between the adiabatic and isothermal situations. An isothermal crack differs in that its surface is maintained at a specified temperature, which can be handled with methods for essential boundary conditions such as Lagrange multipliers [14], widely used in meshless methods to impose essential boundary conditions [44].

3.1. Displacement and temperature approximation

In the CPM, the displacement field is approximated by the moving least squares (MLS) approach [44] using a set of sampling points (and their locations). In the same fashion, the temperature field can be approximated

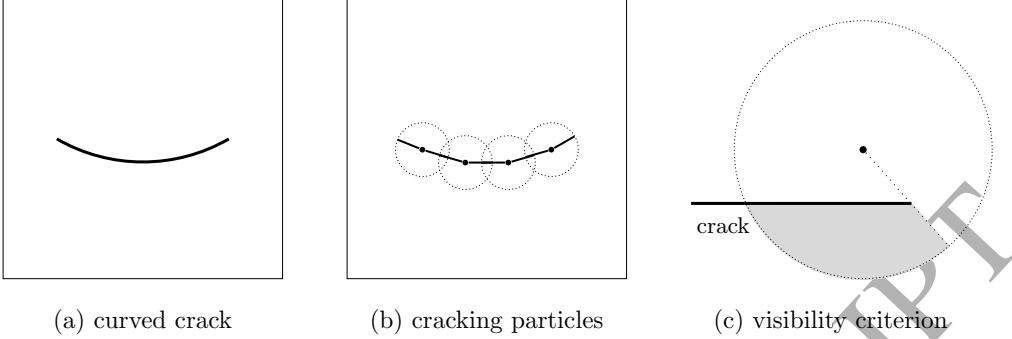


Figure 1: Crack description by the cracking particle method.

25 with the same MLS approach in [14, 16] as

$$u^h(\mathbf{x}) = \sum_{i=1}^n \Phi_i(\mathbf{x}) u_i, \quad T^h(\mathbf{x}) = \sum_{i=1}^n \Phi_i(\mathbf{x}) T_i, \quad (6)$$

1 where $u^h(\mathbf{x})$ and $T^h(\mathbf{x})$ are approximations of displacement and temperature respectively, u_i is the nodal value
2 of displacement at the i th particle which can be the displacement along x or y axis, T_i is the nodal value for
3 temperature and $\Phi_i(\mathbf{x})$ is the shape function obtained by the MLS approximation.

4 3.2. Weak form for mechanical and thermal governing equations

5 The CPM uses the EFGM to discretise the problem domain, and the latter for elastic problems is well known,
6 so the details will not be repeated here but can be found in [33, 44]. The following includes the discretised
7 governing equations for the CPM, to demonstrate the similarities between the modelling of the mechanical
8 and the thermal fields. It is notable that the shape functions of the EFGM cannot meet the Kronecker delta
9 property, and essential boundary conditions cannot be imposed directly, but through methods such as the
10 Lagrange multiplier approach [44]. The weak form of the equilibrium equation for the mechanical field is

$$\int_{\Omega} \delta \boldsymbol{\epsilon}^T \cdot \boldsymbol{\sigma} d\Omega - \int_{\Gamma_u} \delta \boldsymbol{\lambda}^T \cdot (\mathbf{u} - \bar{\mathbf{u}}) d\Gamma_u - \int_{\Gamma_u} \delta \mathbf{u}^T \cdot \boldsymbol{\lambda} d\Gamma_u = \int_{\Omega} \delta \mathbf{u}^T \cdot \mathbf{b} d\Omega + \int_{\Gamma_t} \delta \mathbf{u}^T \cdot \bar{\mathbf{t}} d\Gamma_t, \quad (7)$$

11 and the discretised governing equation is

$$\begin{bmatrix} \mathbf{K} & \mathbf{G} \\ \mathbf{G}^T & \mathbf{0} \end{bmatrix} \begin{Bmatrix} \mathbf{u} \\ \boldsymbol{\lambda} \end{Bmatrix} = \begin{Bmatrix} \mathbf{f} \\ \mathbf{R} \end{Bmatrix}, \quad (8)$$

12 where \mathbf{u} and $\boldsymbol{\lambda}$ are vectors of displacement and Lagrange multipliers respectively, $\boldsymbol{\sigma}$ and $\boldsymbol{\epsilon}$ are vectors of Cauchy
 1 stress and strain respectively using Voigt notation, $\bar{\mathbf{u}}$ is a vector of displacement constraints, \mathbf{b} is a vector of
 2 body forces and $\bar{\mathbf{t}}$ is a vector of external forces. \mathbf{K} and \mathbf{G} are stiffness matrices for displacement and Lagrange
 3 multipliers respectively, \mathbf{f} is a vector of force acting at each degree of freedom, and \mathbf{R} is a vector of constraint
 4 at displacement boundaries. \mathbf{K} , \mathbf{G} , \mathbf{f} and \mathbf{R} are calculated in [33] as

$$\mathbf{K}_{ij} = \int_{\Omega} \mathbf{B}_i^T \mathbf{D} \mathbf{B}_j d\Omega, \quad \mathbf{G}_{ik} = - \int_{\Gamma_u} \Phi_i N_k d\Gamma_u, \quad (9a)$$

$$\mathbf{f}_i = \int_{\Gamma_t} \Phi_i \bar{\mathbf{t}} d\Gamma_t + \int_{\Omega} \Phi_i \mathbf{b} d\Omega + \int_{\Omega} \mathbf{B}_i^T \mathbf{D} \boldsymbol{\epsilon}_T d\Omega, \quad \mathbf{R}_k = - \int_{\Gamma_u} N_k \bar{\mathbf{u}} d\Gamma_u, \quad (9b)$$

6 where

$$\mathbf{B}_i = \begin{bmatrix} \Phi_{i,x} & 0 \\ 0 & \Phi_{i,y} \\ \Phi_{i,y} & \Phi_{i,x} \end{bmatrix}, \quad \mathbf{D} = \frac{\bar{E}}{1-\bar{\nu}} \begin{bmatrix} 1 & \bar{\nu} & 0 \\ \bar{\nu} & 1 & 0 \\ 0 & 0 & \frac{1-\bar{\nu}}{2} \end{bmatrix}, \quad (9c)$$

$$\bar{E} = \begin{cases} E & \text{plane stress,} \\ \frac{E}{1-\nu^2} & \text{plane strain,} \end{cases} \quad \bar{\nu} = \begin{cases} \nu & \text{plane stress,} \\ \frac{\nu}{1-\nu} & \text{plane strain.} \end{cases} \quad (9d)$$

8 N_k is the shape function for Lagrange multipliers constructed by Lagrangian interpolation. In Equation (9a)
 9 the subscripts i, j do not stand for the components of the matrix but for particles i and j , similar rule for i, k .
 10 The coupling between the mechanical and thermal fields is through the assumption that a material is deformed
 11 by the temperature change due to thermal expansion, e.g. the thermal strain $\boldsymbol{\epsilon}_T$ in 2D using Voigt notation is

$$\boldsymbol{\epsilon}_T = \alpha_T (T - T_0) [1, 1, 0]^T, \quad (10)$$

12 where T_0 is the reference temperature before thermal loading and α_T is the coefficient of thermal expansion.
 13 The constitutive model in the mechanical field is therefore modified to include thermal effects, as

$$\boldsymbol{\sigma} = \mathbf{D}(\boldsymbol{\epsilon} - \boldsymbol{\epsilon}_T). \quad (11)$$

14 The force vector in the governing equation in Equation (9b) is different from the purely mechanical situations,
 15 to take the thermal effects into account.

¹⁶ Similarly to the mechanical field, the weak form for the conservation of thermal flows from [14] is

$$\int_{\Omega} \delta \mathbf{q}^T \cdot \mathbf{q} d\Omega - \int_{\Gamma_T} \delta \lambda(T - \bar{T}) d\Gamma_T - \int_{\Gamma_T} \delta T \lambda d\Gamma_T = \int_{\Omega} \delta T \bar{q} d\Omega + \int_{\Gamma_q} \delta T Q d\Gamma_q, \quad (12)$$

¹ where \bar{q} is the constraint of thermal flux with $\mathbf{q} \cdot \mathbf{n} = \bar{q}$ and Q is the heat source. The discretised equation is

$$\begin{bmatrix} \hat{\mathbf{K}} & \hat{\mathbf{G}} \\ \hat{\mathbf{G}}^T & 0 \end{bmatrix} \begin{Bmatrix} \mathbf{T} \\ \hat{\lambda} \end{Bmatrix} = \begin{Bmatrix} \hat{\mathbf{f}} \\ \hat{\mathbf{R}} \end{Bmatrix}, \quad (13)$$

² where the hat is used for terms in the thermal field, distinct from the corresponding terms in the mechanical
³ governing equations. The “stiffness” matrix $\hat{\mathbf{K}}$ for the temperature, the external load $\hat{\mathbf{f}}$, $\hat{\mathbf{G}}$ and $\hat{\mathbf{R}}$ are calculated
⁴ respectively in [14] as

$$\hat{\mathbf{K}}_{ij} = \int_{\Omega} \begin{bmatrix} \Phi_{i,x} \\ \Phi_{i,y} \end{bmatrix}^T \begin{bmatrix} k_T & 0 \\ 0 & k_T \end{bmatrix} \begin{bmatrix} \Phi_{i,x} \\ \Phi_{i,y} \end{bmatrix} d\Omega, \quad \hat{\mathbf{f}}_i = \int_{\Gamma_q} \Phi_i \bar{q} d\Gamma_q + \int_{\Omega} \Phi_i Q d\Omega, \quad (14a)$$

$$\hat{\mathbf{G}}_{ik} = - \int_{\Gamma_u} \Phi_i N_k d\Gamma_T, \quad \hat{\mathbf{R}}_k = - \int_{\Gamma_u} N_k \bar{T} d\Gamma_T, \quad (14b)$$

⁵ where k_T is the thermal conductivity.

⁶ Here mechanical and thermal governing equations are considered to be uncoupled as in [14, 45] with an
⁷ implicit assumption of an infinite propagation speed of the heat wave. The temperature and displacement fields
⁸ are approximated using the same set of particles for the problem discretisation. Comparing Equations (9a)
⁹ and (14a), the stiffness matrices for temperature and displacement can be assembled in one loop, and shape
¹⁰ functions and their derivatives are calculated once at each Gauss point (in the background integration cells used
¹¹ in the EFGM), so the computational expense of including the temperature field is not high. The whole process
¹² is demonstrated in Algorithm 1.

¹⁴ 3.3. Adaptivity

¹⁵ Although crack-tip enrichments have been used in the XFEM [4, 12] and the EFGM [16, 17] to approximate
¹⁶ the high stress gradients generated around the crack tip, there are still problems in the integration of the system
¹⁷ stiffness, where a large number of integration points (or elements) are required in the local zone of the crack
¹⁸ tip. The adaptivity approach is suitable for crack problems, because it can provide an optimised distribution of

Algorithm 1 Solve the temperature and displacement field

-
- 1: **for** $i=1:n_g$ % Loop over each Gauss point **do**
 - 2: Find all particles with support covering to the Gauss point i
 - 3: Calculate shape functions through the MLS approximation
 - 4: Assemble global stiffness matrix for the displacement field using Equation (9a)
 - 5: Assemble global stiffness matrix for the temperature field using Equation (14a)
 - 6: **end for**
 - 7: Solve the temperature field using Equation (13)
 - 8: Use the solution of temperatures as input, obtain the displacement field using Equation (8)
-

19 particles so both accuracy around the crack tip and calculation efficiency can be maintained. Many adaptivity
 1 approaches have been developed in the last two decades and reviews can be found in [33, 46]. The *a posteriori*
 2 h -adaptivity approach, developed by the authors specifically for the CPM [33], is here extended to thermal
 3 situations as below. Of note is the fact that the approach both refines and coarsens the discretisation, leading
 4 to high computational efficiency.

5 The adaptivity approach uses a *recovery-based* error estimator

$$\|E_g\| = \left\{ \frac{1}{2} \int_{\Omega} (\boldsymbol{\sigma}^p - \boldsymbol{\sigma}^h)^T \mathbf{D}^{-1} (\boldsymbol{\sigma}^p - \boldsymbol{\sigma}^h) d\Omega \right\}^{\frac{1}{2}}, \quad (15)$$

6 where $\boldsymbol{\sigma}^h$ and $\boldsymbol{\sigma}^p$ are calculated stress and “projected” stress respectively, written in Voigt notation. $\boldsymbol{\sigma}^p$ is an
 7 approximation of the exact stress in [47], as

$$\boldsymbol{\sigma}^p = \sum_{k=1}^m \Psi_k(\mathbf{x}) \boldsymbol{\sigma}^h(\mathbf{x}_k), \quad (16)$$

8 where $\Psi_k(\mathbf{x})$ are shape functions calculated by the MLS approximation using smaller supports than those for
 9 calculating $\Phi_i(\mathbf{x})$, and m is the number of particles with the smaller supports covering \mathbf{x} . $\Psi_k(\mathbf{x})$ are discontinuous
 10 at the cracks which are handled by the visibility criterion mentioned above. To consider the thermal effects
 11 on the problem error approximation, $\boldsymbol{\sigma}^h$ and $\boldsymbol{\sigma}^p$ in the CPM for thermoelastic problems are modified using
 12 Equation (11) to

$$\boldsymbol{\sigma}^h = \mathbf{D} \sum_{i=1}^n \left(\mathbf{B}_i \mathbf{u}_i - \Phi_i \alpha_T (T_i - T_0) [1, 1, 0]^T \right), \quad (17a)$$

13

$$\boldsymbol{\sigma}^p = \sum_{k=1}^m \left\{ \Psi_k \mathbf{D} \sum_{i=1}^n \left(\mathbf{B}_i \mathbf{u}_i - \Phi_i \alpha_T (T_i - T_0) [1, 1, 0]^T \right) \right\}. \quad (17b)$$

- 1 The approach for refining and coarsening particles in [33] is also applicable to thermal situations, and here a
2 brief summary is included. The relative global error η_g is evaluated using

$$\eta_g = \frac{\|E_g\|}{\|U\|}, \quad \text{with } \|U\| = \left\{ \frac{1}{2} \int_{\Omega} (\boldsymbol{\sigma}^h)^T \mathbf{D}^{-1} \boldsymbol{\sigma}^h d\Omega \right\}^{1/2}. \quad (18)$$

- 3 When the global error η_g is larger than the target error η_t , the adaptive approach is enabled. The local error
4 $\|E_i\|$ of a background cell Ω_i is estimated using Equation (15) with Ω_i replacing Ω . The relative local error is

$$\eta_i = \frac{\|E_i\|}{\|U\| / \sqrt{n_{\text{cell}}}}, \quad (19)$$

- 5 where n_{cell} is the number of all background cells. Two thresholds have been used in [33] to refine or coarsen
6 particles, as

$$\begin{cases} \eta_i > 3\eta_t & \text{to be refined,} \\ \eta_i < \frac{1}{3}\eta_t & \text{to be coarsened.} \end{cases} \quad (20)$$

- 7 When a cell i has $\eta_i > 3\eta_t$, it is divided into four smaller cells and five particles are added; when four cells have
8 $\eta_i < \frac{1}{3}\eta_t$, they are combined to a larger cell and five particles are deleted, so the density of particles is reduced
9 in areas where they are not needed, leading to computational efficiencies.

10 3.4. Interaction integration for cracks under thermal loadings

- 11 The thermal stress intensity factors are calculated by the interaction integral [48]. Compared to the inter-
12 action integration in mechanical situations [48], the difference for thermal situations is the inclusion of thermal
13 terms as in [12, 14, 48]. The interaction integral is the superposition of two states for the J-integral: the real
14 state (as state 1) which contains σ_{ij} , ε_{ij} , u_i , T_i ; and the auxiliary state (as state 2) which excludes T_i , while
15 the other terms are predefined as in [48]. The interaction integral is defined as

$$I^{(1,2)} = \int_A \left((\sigma_{ij}^{(1)} u_{j,1}^{(2)} + \sigma_{ij}^{(2)} u_{j,1}^{(1)} - W^{(1,2)} \delta_{1i}) q_{,i} + q \bar{\alpha}_T \sigma_{kk}^{(2)} T_{,1}^{(1)} \right) dA, \quad i, j, k \in \{1, 2\}, \quad (21)$$

- 16 where index notation is used with the Einstein summation. The superscripts (1) and (2) represent the terms
17 from state 1 and state 2 respectively, $W^{(1,2)}$ is the interaction strain energy density between the two states, as

$$W^{(1,2)} = \sigma_{ij}^{(1)} \varepsilon_{ij}^{(2)} = \sigma_{ij}^{(2)} \varepsilon_{ij}^{(1)}. \quad (22)$$

18 Considering the relationship between the SIFs and the interaction integral, the SIFs for a mixed-mode crack
 1 are obtained by applying the specific auxiliary state, as in [12].

2 4. Numerical examples

3 Several numerical examples are presented in this section to demonstrate the performance of the proposed
 4 method. The first is an inclined crack under thermal loading, where mixed-mode fracture is considered. The
 5 second example contains a curved crack to test the abilities of the proposed method for cracks with complex
 6 geometries. Crack propagation is included in the third example, where the behaviours of a crack in a cruciform
 7 shaped plate under mechanical and thermal loadings are compared. The next example is extended from the
 8 third with one more crack, and is used to explore the performance of the proposed method for multiple cracks.

9 **A penny-shaped crack problem under thermal loading is included to demonstrate the ability of the proposed
 10 method for 3D crack problems.** Unless stated otherwise, all examples are under plane strain assumptions with
 11 linear elastic material properties: Young's modulus $E = 200$ GPa, Poisson's ratio $\nu = 0.3$, thermal conductivity
 12 $k_T = 100$ W/(m°C) and coefficient of thermal expansion $\alpha_T = 10^{-5}$ °C⁻¹, combinations of which are used to
 13 normalise the results, and the reference environment temperature is zero. For all examples, a circular influence
 14 domain is used for every particle, and the ratio of the radius to the average particle distance is $d_s = 2.2$. A 4th
 15 order spline function is used as the weight function in the MLS approximation (where a discussion of different
 16 support sizes and weight functions can be found in [49]). Rigid body translation and rotation are fixed as in
 17 [50] by setting

$$\int_{\Omega} u_x d\Omega = 0, \quad \int_{\Omega} u_y d\Omega = 0, \quad \int_{\Omega} \frac{\partial u_x}{\partial y} - \frac{\partial u_y}{\partial x} d\Omega = 0, \quad (23)$$

18 where u_x and u_y are horizontal and vertical displacements respectively. These boundary conditions are imposed
 19 on the weak form by Lagrange multipliers with only three extra unknowns.

20 4.1. Inclined central crack

21 An inclined central crack is considered in a rectangular plate as shown in Figure 2. The configuration of this
 22 problem is $w = 0.1$ m, $h = 0.5w$, $a = 0.3w$. The top of the plate is at a high temperature $\bar{T} = 100$ °C, while

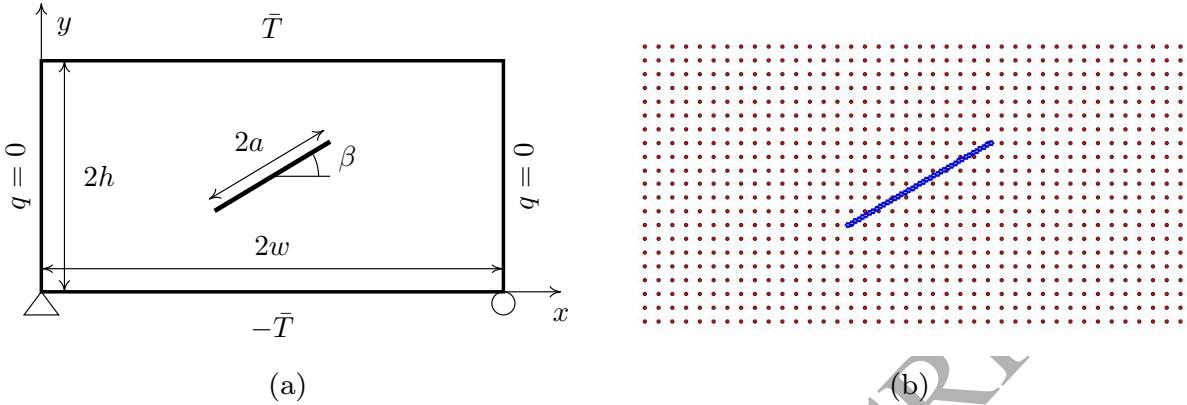


Figure 2: An inclined central crack under thermal loading: (a) configuration; (b) initial particle arrangement, with blue points for cracking particles.

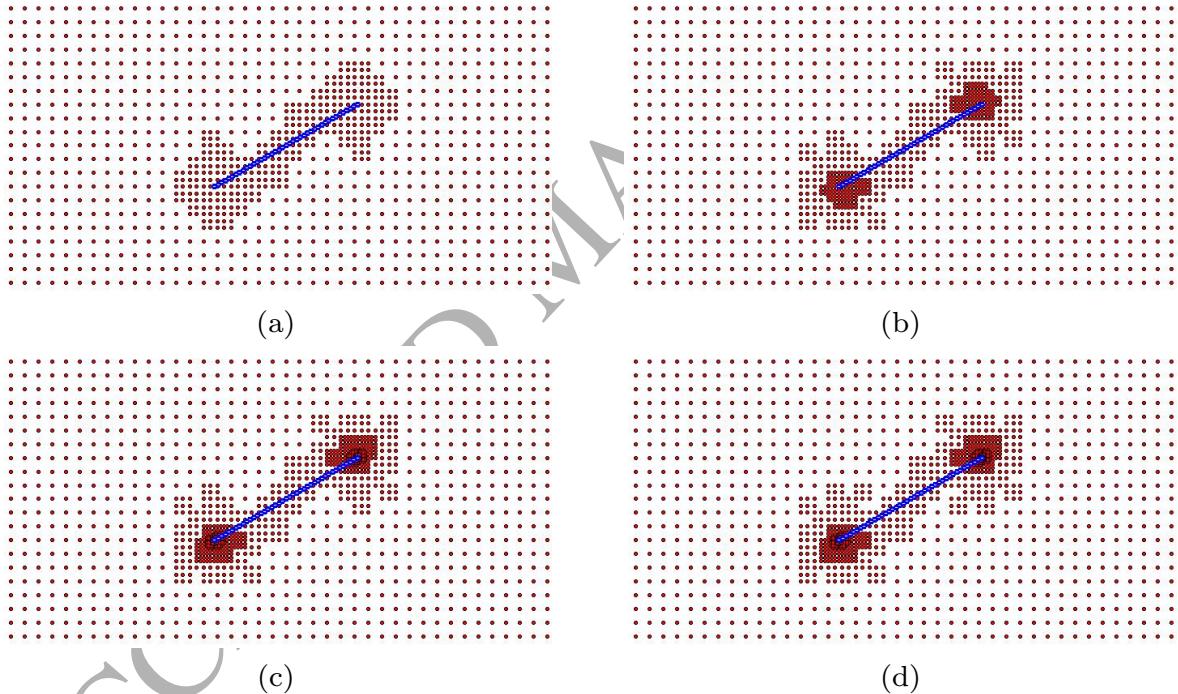


Figure 3: Adaptive particle arrangements for the inclined central crack ($\beta = 30^\circ$) under thermal loading: (a) step 1; (b) step 2; (c) step 4; (d) step 7.

the bottom is at a low temperature $-\bar{T}$. The heat flux on the two sides of the plate is zero. The problem is initially discretised by 21×41 particles with 41 particles (shown in blue) along the crack as in Figure 2 (b). The target error for the adaptivity approach is $\eta_t = 0.06$, and adaptive particle arrangements are given in Figure 3, where particle refinement is executed automatically around the two crack tips. Figures 4 (a-c)

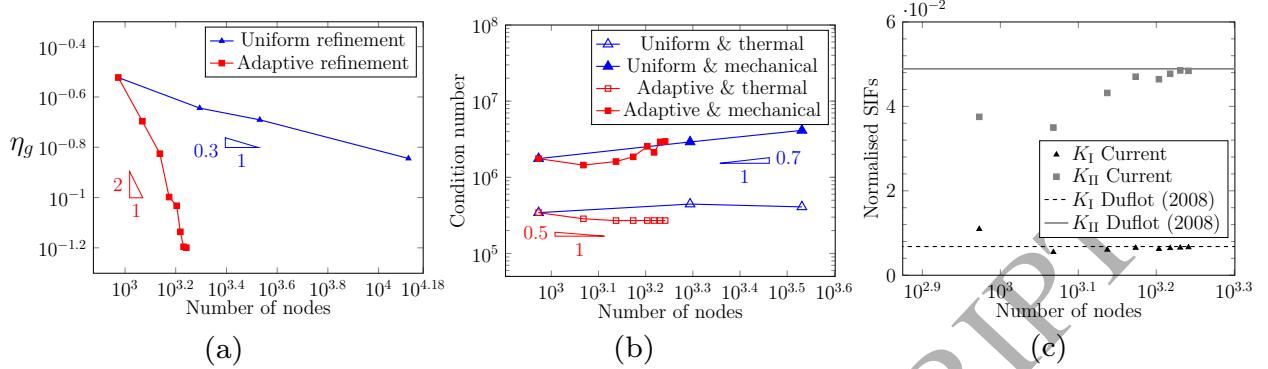


Figure 4: Adaptive results for the inclined central crack under thermal loading: (a) convergence rate for error; (b) condition number and (c) SIFs.

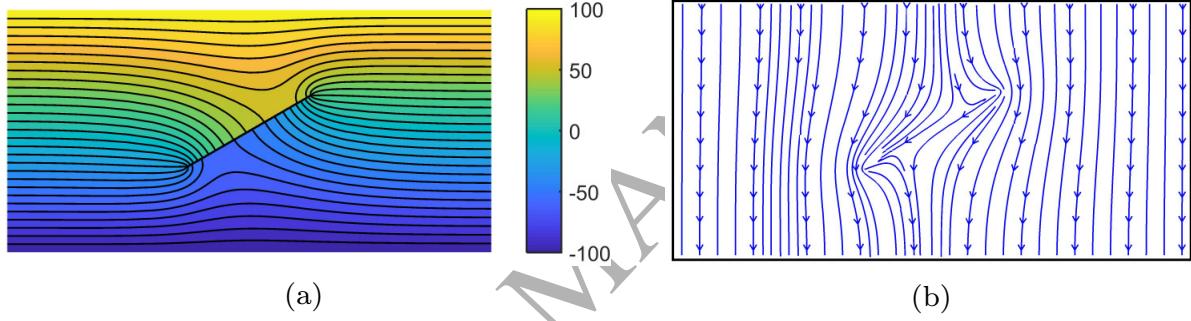


Figure 5: Thermal results of the inclined central crack: (a) temperature profile ($^{\circ}\text{C}$); (b) heat flux.

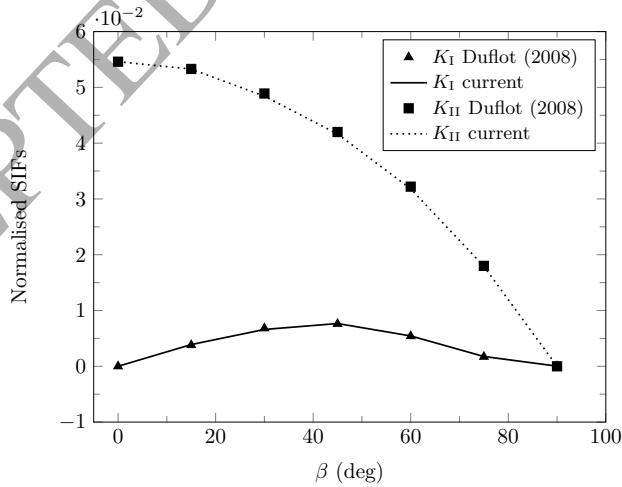


Figure 6: Normalised stress intensity factors for various crack inclinations.

illustrate the results during the adaptive steps, where it is shown that the convergence rate of error for the adaptivity approach is much higher than that for uniform refinement, and results for the SIFs converge to the

Table 1: Validation of calculated SIFs for the inclined crack under thermal loading

	CPM		HXFEM [12]		XFEM [4]		BEM [10]	
θ	K'_I	K'_{II}	K'_I	K'_{II}	K'_I	K'_{II}	K'_I	K'_{II}
0°	0.00000	0.05532	0.00000	0.05471	0.00000	0.05460	0.00000	0.05400
30°	0.00664	0.04894	0.00670	0.04873	0.00680	0.04890	0.00640	0.04800
60°	0.00561	0.03200	0.00544	0.03209	0.00540	0.03220	0.00490	0.03200

reference values during adaptive steps. Figures 4 (b) presents the second norm condition number of the thermal and mechanical system “stiffness” matrix in Equations (13) and (8) respectively, where adaptive refinement results in smaller condition numbers than uniform refinement. The SIFs at one crack tip are normalised by

$$K'_n = \frac{K_n}{\alpha_T \bar{T} E} \cdot \frac{h}{w\sqrt{2w}}, \quad n \in \{I, II\}. \quad (24)$$

and in Table 1 are compared with the results from other methods, including the XFEM from Duflot [4], the XFEM with high order enrichment functions (marked as HXFEM) from Zamani et al. [12] and the BEM from Prasad et al. [10]. The proposed method gives results with the same level of accuracy as using enrichment functions, and with adaptivity the maximum number of particles here is 1744 versus 1891 rectangular elements in the HXFEM [12], and 7000 triangular elements in the XFEM [4]. Figures 5 (a-b) illustrate the temperature distribution and heat flux in the problem domain. The calculated SIFs for the crack with various inclinations are presented in Figure 6 and good agreement with the results from [4] is achieved.

4.2. Curved central crack

The second example comprises an arc-shaped crack in a square domain as shown in Figure 7, the configuration of which is $w = 0.2$ m, $2a = 0.1w$, $\beta = \pi/4$. The plate is under a constant and upward heat flux $\bar{q} = 10^4$ W/m², where the bottom of the plate is at a high temperature and the top is at a low temperature. There is an analytical solution for this problem from Chen and Hasebe [51] when this problem is considered in an infinite

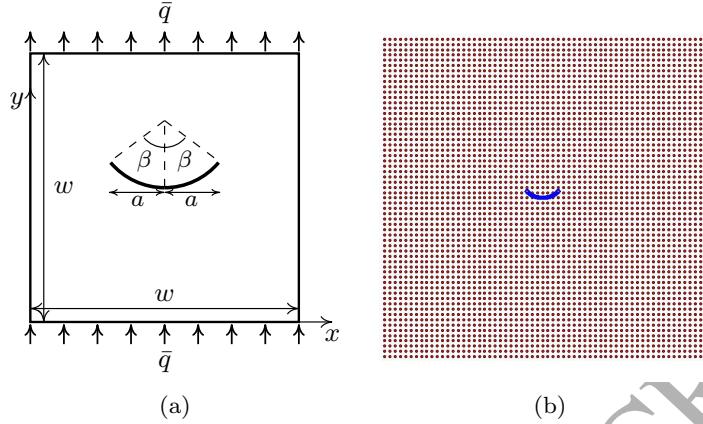


Figure 7: A curved crack under constant heat flux: (a) configuration; (b) initial particle arrangement where cracking particles are in blue.

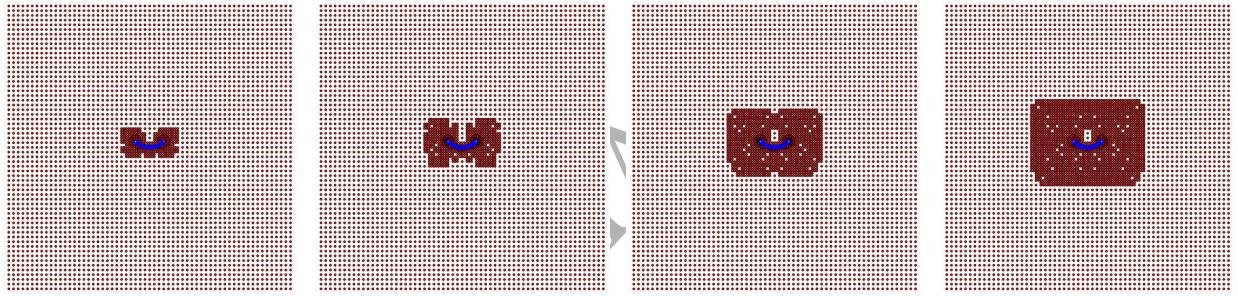


Figure 8: Adaptive particle arrangements for the curved crack problem: (a) step 2; (b) step 4; (c) step 6; (d) step 8.

plate, as

$$K_I = \frac{3 + \cos \beta}{4} \left(\sin \frac{\beta}{2} + \frac{4(1 - \cos \beta)}{(3 - \cos \beta) \cos \beta} \cos \frac{\beta}{2} \right) \frac{E \alpha \bar{q}}{k_T} a \sqrt{\pi a}, \quad (25a)$$

$$K_{II} = \frac{3 + \cos \beta}{4} \left(\cos \frac{\beta}{2} - \frac{4(1 - \cos \beta)}{(3 - \cos \beta) \sin \beta} \sin \frac{\beta}{2} \right) \frac{E \alpha \bar{q}}{k_T} a \sqrt{\pi a}. \quad (25b)$$

For the sake of data analysis, the SIFs are normalised as

$$K'_n = \frac{K_n}{\alpha \bar{q} E} \cdot \frac{k_T}{a \sqrt{\pi a}}, \quad n \in \{I, II\}. \quad (26)$$

- Since the ratio between the size of the crack and the plate is $2a/w = 1/10$, the boundaries of this finite domain are far from the crack and the analytical solutions are applicable to this problem. The plate is initially discretised with 61×61 particles, from which the adaptivity approach proceeds. The curvature of the crack is modelled by

Table 2: Normalised SIFs for a curved crack under thermal loading

	CPM		HXFEM [12]		Analytical [51]	
θ	K'_I	K'_{II}	K'_I	K'_{II}	K'_I	K'_{II}
30°	0.71745	0.79217	0.73962	0.82085	0.71904	0.80793
45°	0.97096	0.58499	0.97630	0.57770	0.97338	0.59995
60°	1.12586	0.34761	1.13546	0.33238	1.13750	0.35363

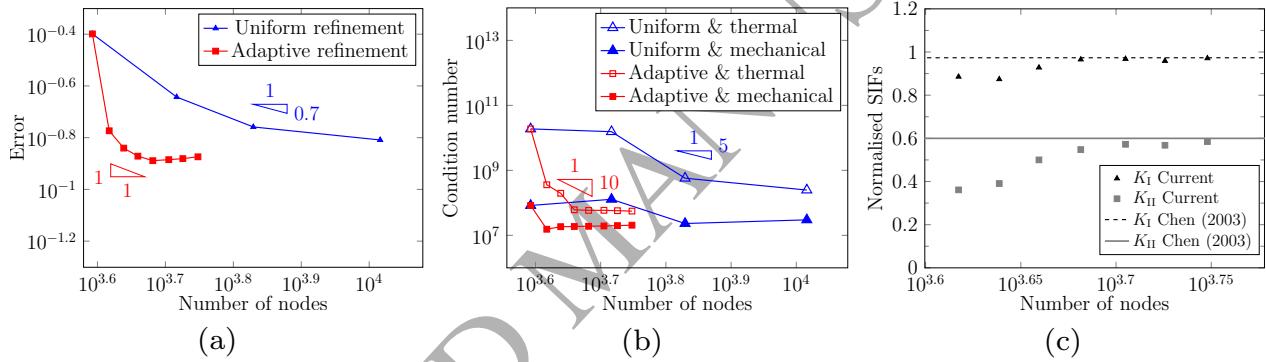
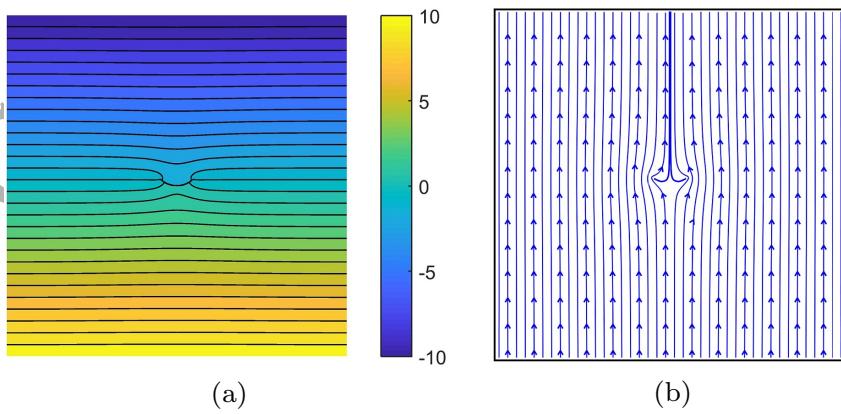


Figure 9: Adaptive results for the curved crack under thermal loading: (a) convergence rate for error; (b) condition number and (c) SIFs, where analytical results are from Chen and Hasebe [51]

Figure 10: Thermal results for the curved crack problem: (a) temperature profile ($^{\circ}\text{C}$); (b) heat flux.

6 41 cracking particles (with straight segments connecting them), which are in blue in Figure 7 (b).

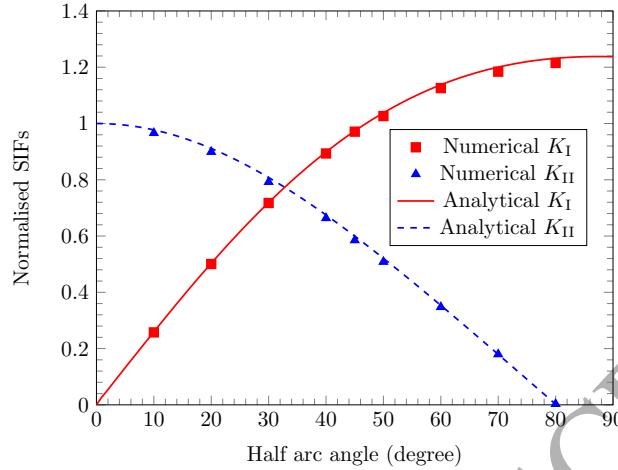


Figure 11: Normalised stress intensity factors for the curved crack with various values of the half arc angle.

The adaptivity approach uses a target error of $\eta_t = 0.12$ and adaptive particle distributions are shown in Figure 8. The local zone containing the crack is refined and two “masses” of particles are generated around the two crack tips. Particle refinement is also executed at both the top and bottom where the heat flux boundary condition is applied. Results including global error, the condition number of system stiffness matrix and SIFs are presented in Figure 9, where adaptive refinement shows lower errors and condition numbers than uniform refinement and good agreements of SIFs with analytical solutions are evident. The results of temperature profile and heat flux distribution are given in Figure 10, where it is shown that the crack disturbs the temperature profile at the centre of the plate and the heat flux travels encircling the crack surface. SIFs are calculated and compared with the results from Zamani et al. [12] using the HXFEM in Table 2. The same level of accuracy is obtained by the proposed method with 8391 particles compared with more than 10^4 nodes in the HXFEM [12]. For the curved crack with various curvatures, the proposed method delivers accurate SIFs compared to the analytical solution from Equation (25), as given in Figure 11. It has also been shown that, even with uniformly distributed particles, the proposed method is capable of capturing the stress gradients at the crack tip of a curved crack.

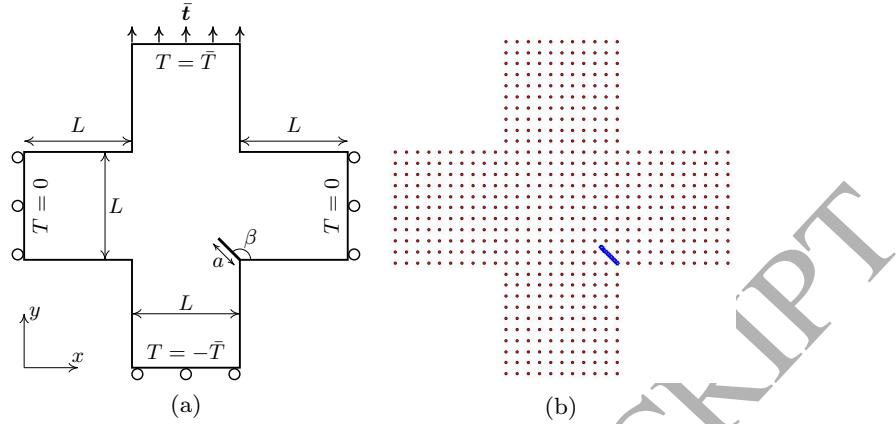


Figure 12: A crack in a cruciform shaped plate: (a) configuration; (b) initial particle arrangement (cracking particles are in blue).

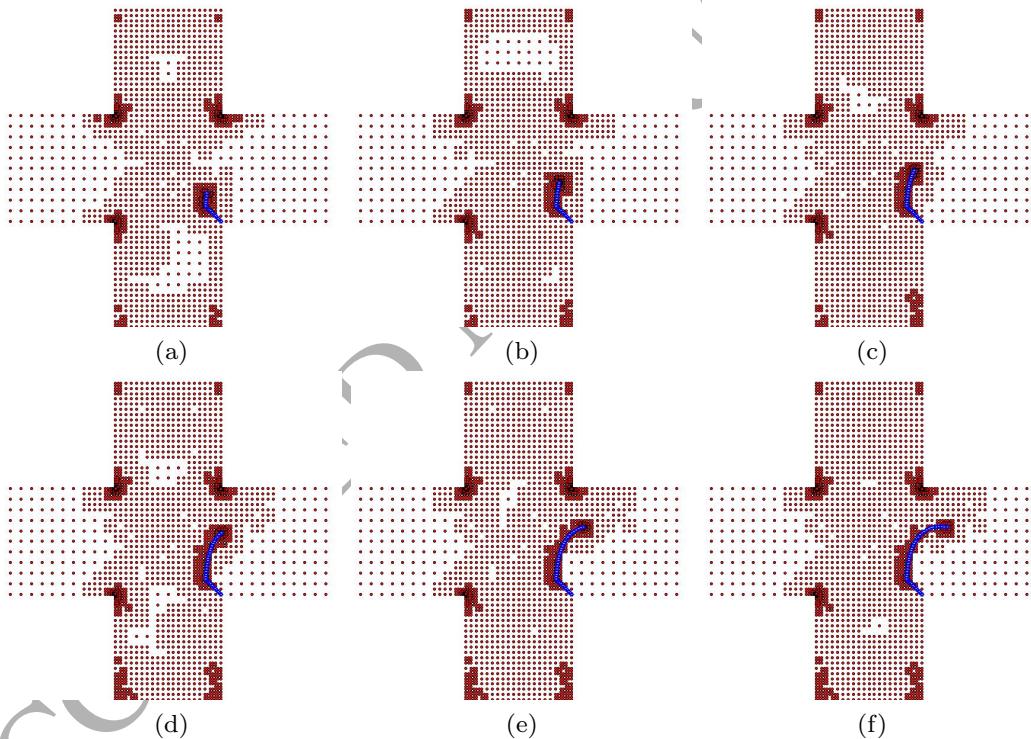


Figure 13: Adaptive particle arrangements of crack propagation steps for the cruciform shaped plate problem: (a-f) step=5, 10, 15, 20, 25 and 30.

¹⁴ 4.3. Crack propagation in a cruciform shaped plate

¹ Crack propagation under thermal loading is considered in the third example. The crack is located at the
² bottom-right corner of a cruciform shaped plate as shown in Figure 12 and the configuration is $L = 0.1$ m, $a =$

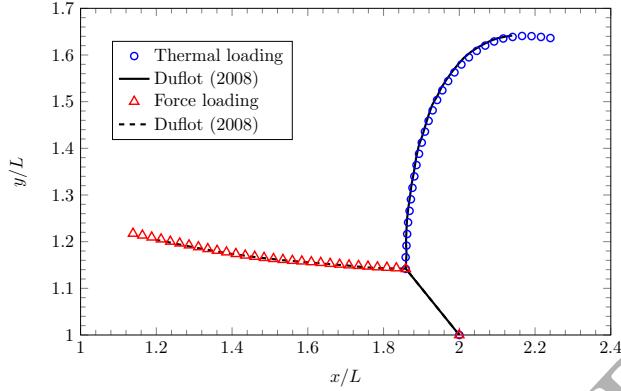


Figure 14: Comparison of crack growth in the cruciform shaped plate between under thermal loading and force loading.

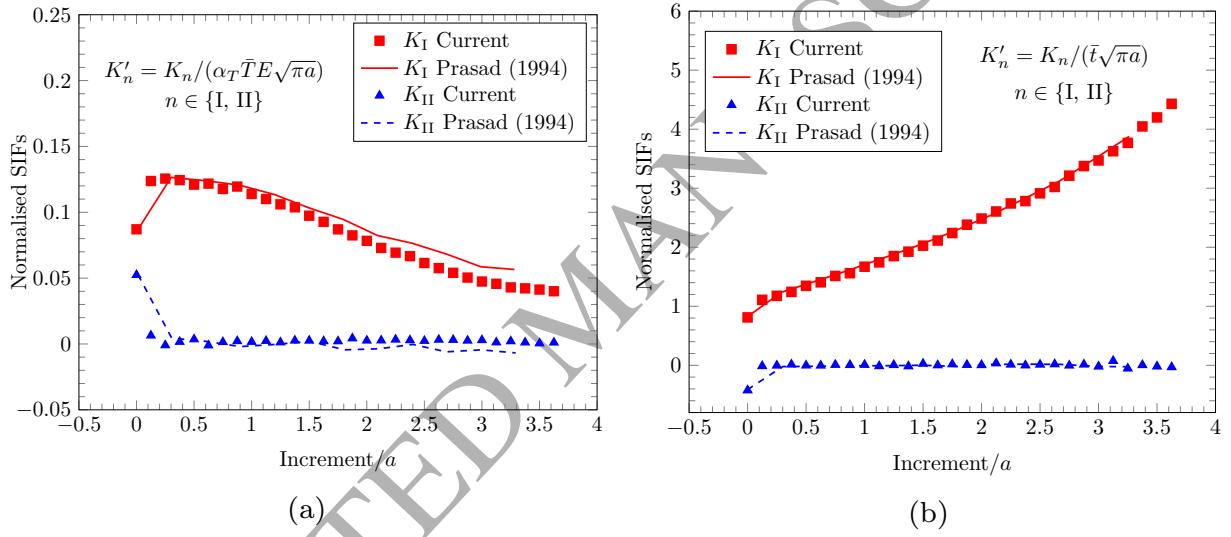


Figure 15: Normalised SIFs during crack propagation for the cruciform shaped plate problem: (a) thermal loading; (b) force loading.

³ $0.2L$ and $\beta = 3\pi/4$. The cruciform shaped plate is constrained in displacement at the bottom and both left and
¹ right sides. Two loading situations are considered: Case 1, thermal loading with $\bar{T} \neq 0, \bar{t} = 0$; Case 2, mechanical
² loading with $\bar{T} \neq 0, \bar{t} \neq 0$. The crack propagation direction is determined by the maximum circumferential
³ stress criterion and the crack increment is set as $a/8$. A wide range of alternative crack propagation criteria are
⁴ available, and surveys can be found in the literature, e.g. [52, 53], however the maximum circumferential stress
⁵ criterion used here has been shown to be both easy to implement and robust [54].

⁶ The adaptivity approach is defined with target error $\eta_t = 0.04$ and starts from the initial particle arrangement
⁷ shown in Figure 12 (b). Adaptive particle arrangements during the crack propagation for Case 1 are given in

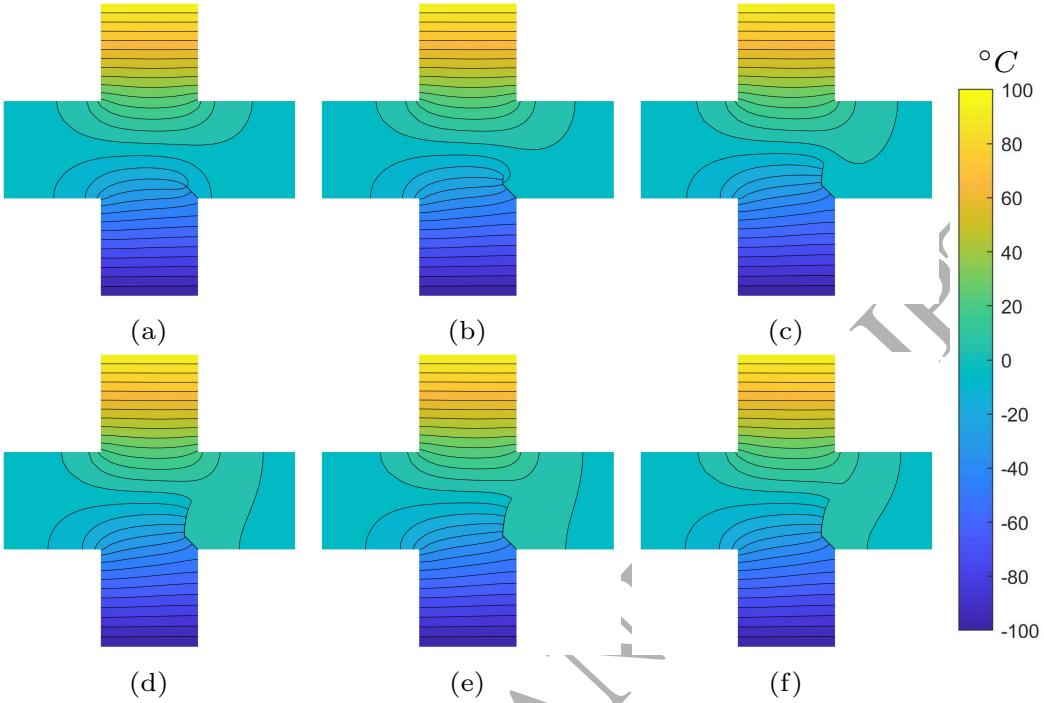


Figure 16: Temperature distribution during crack propagation steps for the cruciform shaped plate: (a-f) step=1, 5, 10, 15, 20 and 30 respectively.

8 Figure 13, where particle distribution is refined at the three corners of the cruciform shaped plate and the crack
 1 tip. The crack propagates upward and then turns right after 15 propagation steps. The crack growth predicted
 2 by the proposed method is compared with the results from Duflot [4] using the XFEM, where good agreement
 3 is found and similar accuracy is obtained for Case 2 as shown in Figure 14. Comparing the crack growth in
 4 the two cases, the crack propagates right under thermal loading while the mechanical loading leads the crack
 5 propagation to the left. The SIFs during crack propagation steps in the two cases are calculated, which match
 6 well with the results from Prasad et al. [55] as shown in Figure 15. The temperature profile during crack
 7 propagation steps can be seen in Figure 16, where the crack path changes the temperature distribution on the
 8 two sides of the crack. The thermal flux is also affected by the crack path as given in Figure 17, where the flux
 9 lines travel along rather than across the crack.

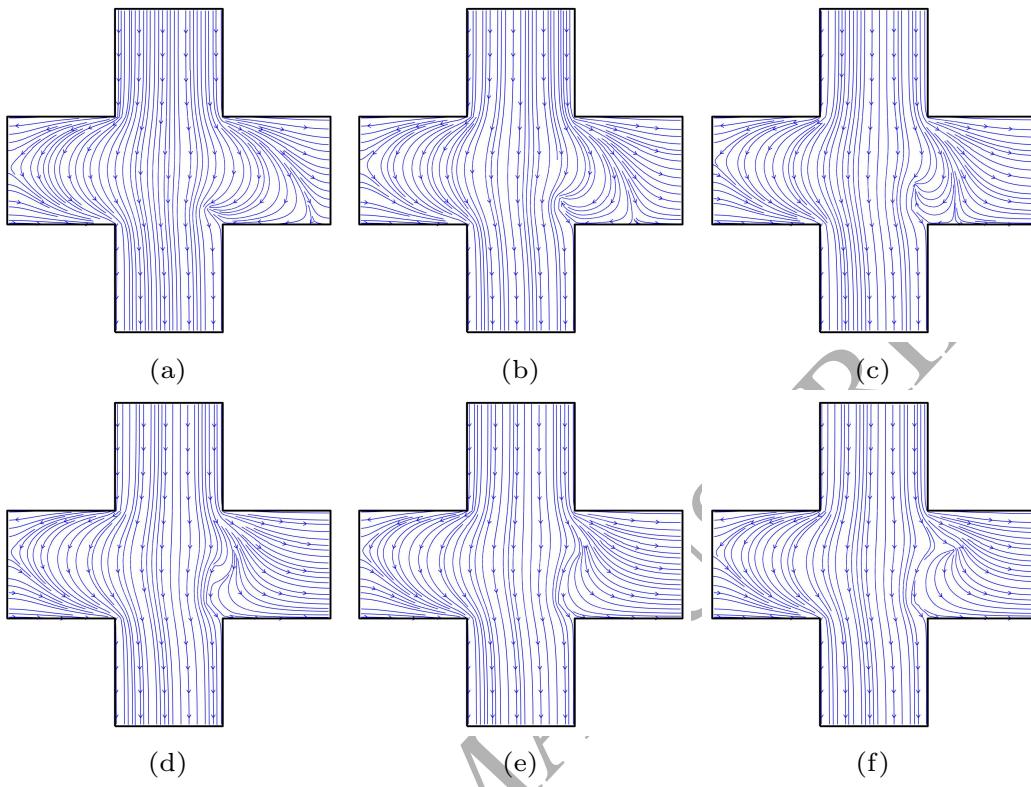


Figure 17: Thermal flux during crack propagation steps for the cruciform shaped plate: (a-f) step=1, 5, 10, 15, 20 and 30.

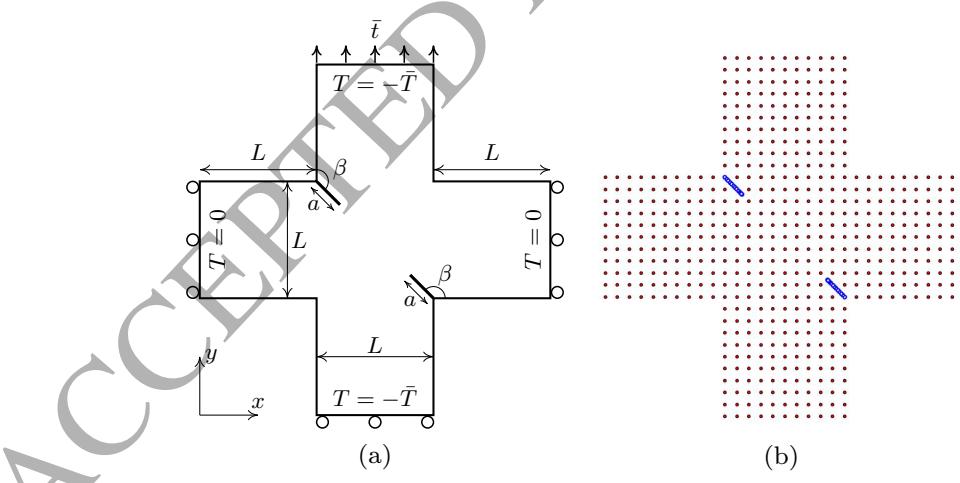


Figure 18: Two cracks in a cruciform shaped plate: (a) configuration; (b) initial particle arrangement.

4.4. Two cracks in a cruciform shaped plate

The fourth example matches the third problem with one extra crack as shown in Figure 18, and other aspects including displacement boundary conditions and material properties are identical. There are also two loading

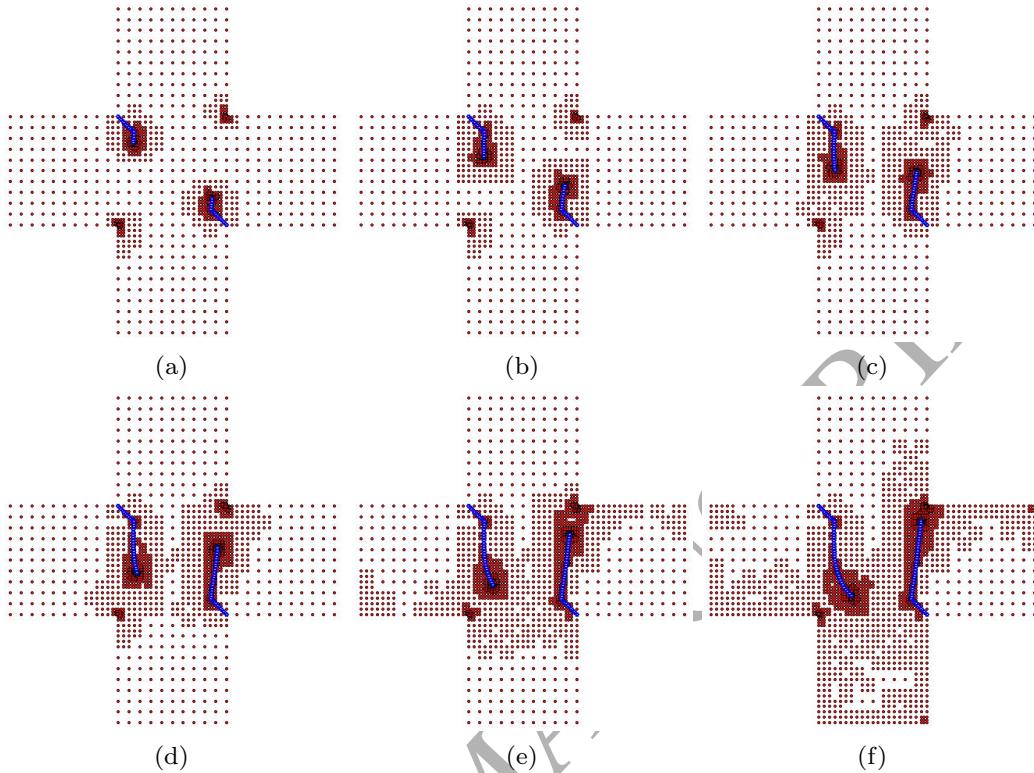


Figure 19: Adaptive particle arrangements for the two cracks in a cruciform shaped plate: (a-f) step=5, 10, 15, 20, 25 and 30.

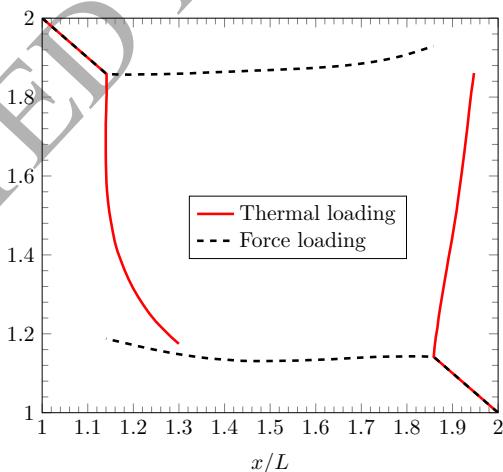


Figure 20: Propagation of two cracks in the cruciform shaped plate under thermal loading and force loading.

- 3 conditions involved, which are thermal loading and mechanical loading, and the difference is in Case 1 where
- 1 the top and bottom sides of the plate are loaded by a low temperature $-\bar{T}$.
- 2 Adaptive particle distributions for the propagation of the two cracks in Case 1 are given in Figure 19, where

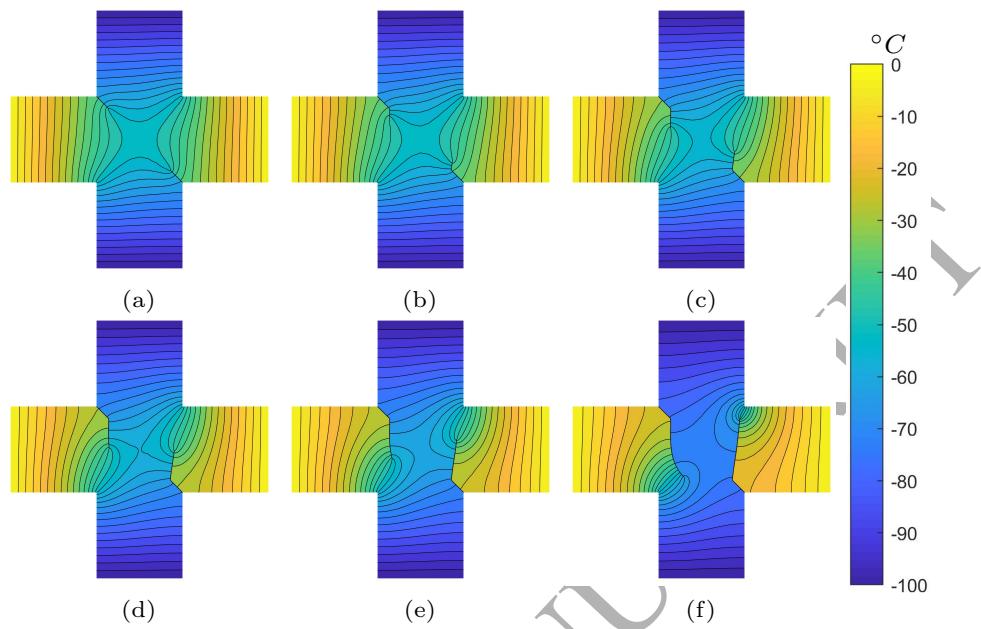


Figure 21: Temperature distribution during crack propagation steps for two cracks in the cruciform shaped plate: (a-f) step=1, 5, 10, 15, 20 and 30.

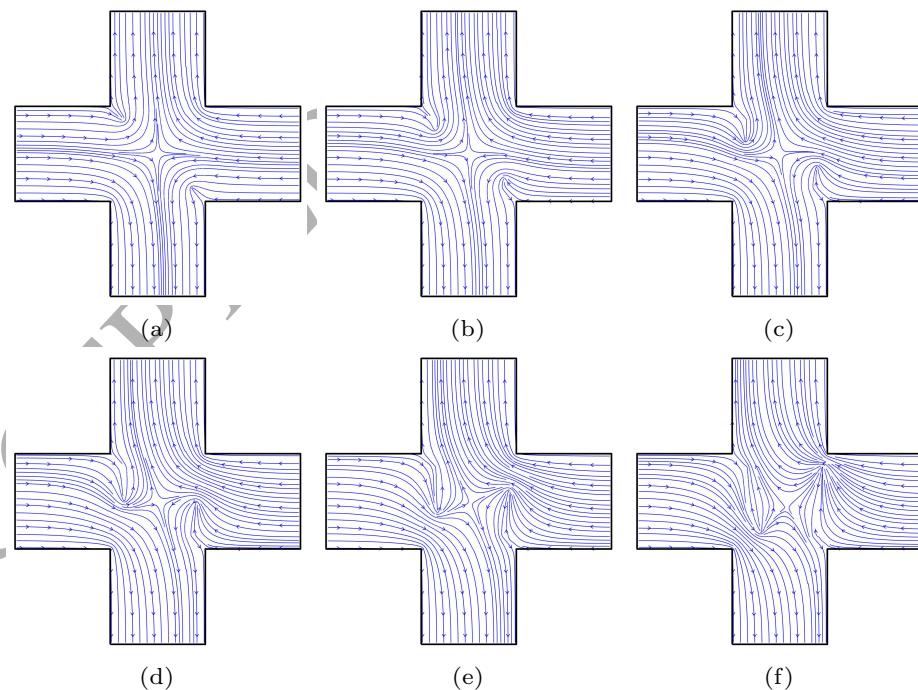


Figure 22: Thermal flux during crack propagation steps for two cracks in the cruciform shaped plate: (a-f) step=1, 5, 10, 15, 20 and 30.

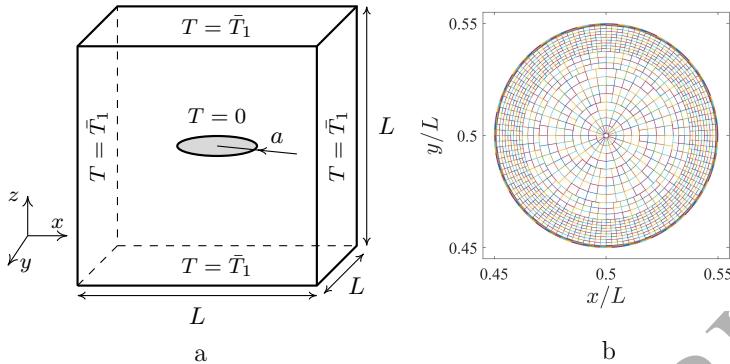


Figure 23: Configuration of a penny-shaped crack in a large medium with high temperature at all surfaces: (a) configuration and (b) integration cells to impose boundary conditions at crack surfaces.

again there are two “masses” of particles generated around the two crack tips which travel with the two crack tips during the crack propagation process. The crack on the right propagates straight upwards towards the upper-right corner, while the left crack descends and turns right after 20 propagation steps. The crack growths for both cases are given in Figure 20, where it is shown that the two cracks propagate vertically under the thermal loading here, while the mechanical loading leads to horizontal crack propagation. The temperature profile and heat flux during crack propagation steps for the two cracks are presented in Figures 21 and 22, where the considerable influence of the presence of the two cracks on both temperature distribution and heat flux direction is evident. No analytical solution has been found for this particular problem yet, and this example is used here to test the ability of the proposed methodology for modelling multiple crack propagation. The original results are included in Table 3, which can be used for comparison by researchers wishing to test their own methods.

10

11 4.5. Penny-shaped crack

12 A penny shaped crack problem is included to test the proposed method in 3D crack problems with the
 13 isothermal crack assumption, as presented in Figure 23. The radius of the crack is $a = 0.01\text{m}$, and the size of
 14 the cube is $L = 20a$. The surfaces of the crack are at low temperature, $T = 0$, while all surfaces of the cube
 15 are subject to a high temperature, $T = \bar{T}_1 = 100^\circ\text{C}$. The cube is large enough so that the analytical solutions
 16 by Das [56] and Murakami et al. [57] for a penny-shaped crack in an infinite medium under thermal load is

Table 3: Crack propagation paths for the two cracks in the cruciform shaped plate under thermal loading

steps	crack 1		crack 2		steps	crack 1		crack 2	
	x/L	y/L	x/L	y/L		x/L	y/L	x/L	y/L
0	2.000	1.000	1.000	2.000	16	1.909	1.513	1.151	1.484
1	1.859	1.142	1.141	1.858	17	1.912	1.538	1.155	1.459
2	1.861	1.166	1.141	1.833	18	1.915	1.563	1.159	1.435
3	1.863	1.191	1.142	1.808	19	1.918	1.588	1.165	1.410
4	1.867	1.216	1.141	1.783	20	1.921	1.612	1.172	1.387
5	1.869	1.241	1.140	1.758	21	1.923	1.637	1.180	1.363
6	1.873	1.266	1.140	1.733	22	1.926	1.662	1.189	1.339
7	1.876	1.290	1.140	1.708	23	1.929	1.687	1.199	1.316
8	1.880	1.315	1.139	1.683	24	1.931	1.712	1.210	1.294
9	1.884	1.340	1.139	1.658	25	1.934	1.737	1.222	1.272
10	1.887	1.365	1.140	1.633	26	1.936	1.762	1.235	1.251
11	1.891	1.389	1.140	1.608	27	1.939	1.786	1.250	1.231
12	1.895	1.414	1.141	1.583	28	1.941	1.811	1.266	1.211
13	1.899	1.439	1.143	1.559	29	1.944	1.836	1.282	1.193
14	1.902	1.464	1.145	1.534	30	1.947	1.861	1.300	1.175
15	1.906	1.488	1.147	1.509					

¹⁷ applicable here. SIFs at the crack tip are normalised by

$$K'_I = \frac{K_I(1-\nu)}{\alpha_T E \bar{T}_1 \sqrt{a/\pi}}. \quad (27)$$

¹ The error of the mode I SIF from the analytical value K_I^0 is measured as

$$e(K_I) = \sum_{j=1}^{n_c} \frac{|K_I^j - K_I^0|}{n_c K_I^0}, \quad (28)$$

² where n_c is the number of particles at the crack front. Temperature constraints at the crack surfaces are

³ imposed by Lagrange multipliers using Equation (14b), where shape functions N_k are calculated by the MLS

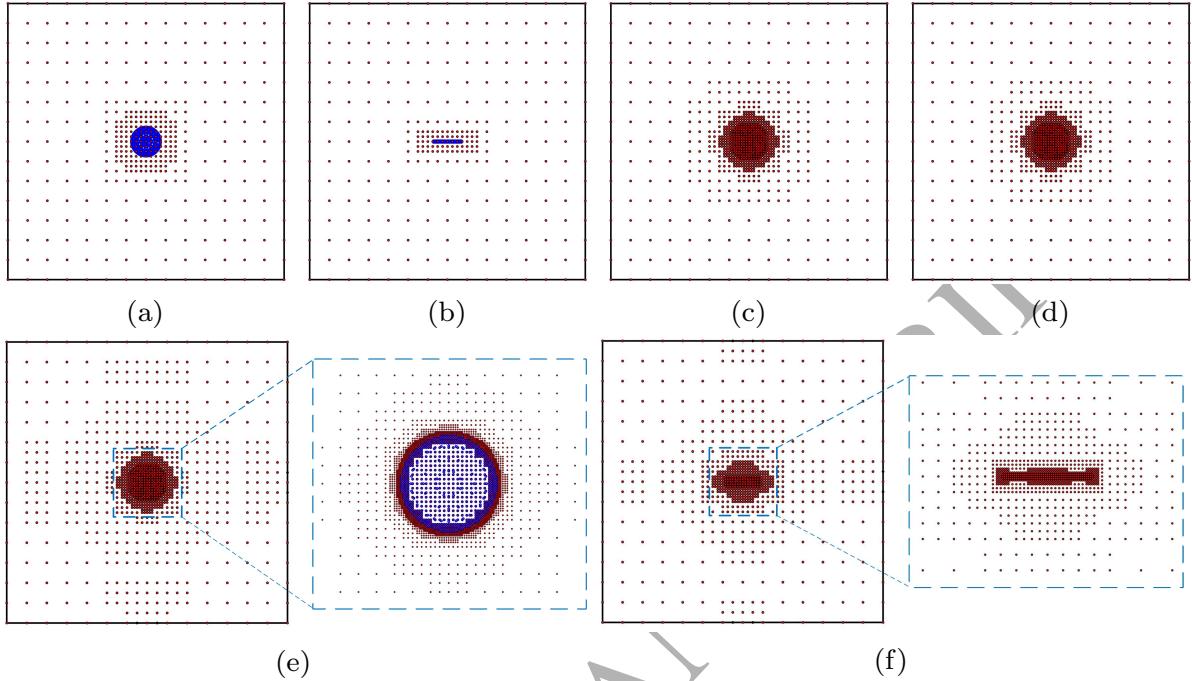


Figure 24: Adaptive refinement steps for the penny-shaped crack under thermal loading: (a) step 1 xy view; (b) step 1 xz view; (c) step 3 xy view; (d) step 3 xz view; (e) step 5 xy view; (f) step 5 xz view. Cracking particles are marked in blue.

approximation over particles on the crack surfaces and the integration cells are given in Figure 23 (b). Crack discontinuities are handled using the visibility criterion, as detailed in the 3D version of the CPM described in [38], and no other special handling is required. The cube is initially discretised with $15 \times 15 \times 15$ particles with refined particles around the crack, as shown in Figure 24 (a). Adaptive refinement steps using $\eta_t = 0.04$ are presented in Figure 24, where dense particles were generated around the crack front to capture the stress and temperature gradients. Results during adaptive steps are given in Figure 25, where adaptive refinements show better convergence than uniform refinements in global error and SIF error, while no significant differences are found for the condition numbers of the system stiffness matrix in Figure 25 (c). The SIFs agree well with the analytical solutions [57] and results from the XFEM [58] and the BEM [59] as shown in Figure 26 (a), and the temperature distribution inside the cube is presented in Figure 26 (b).

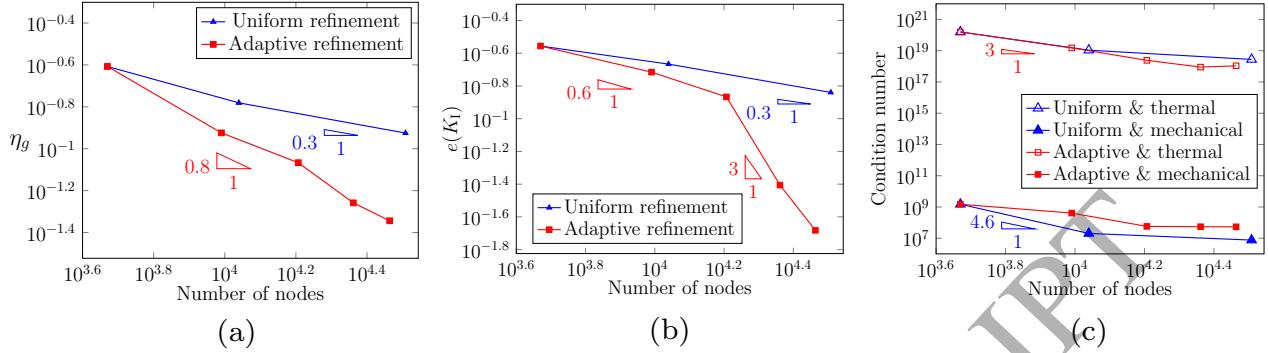


Figure 25: Adaptive results for the penny-shaped crack under thermal loading: (a) convergence rate for global error; (b) error of SIF; (c) condition number.

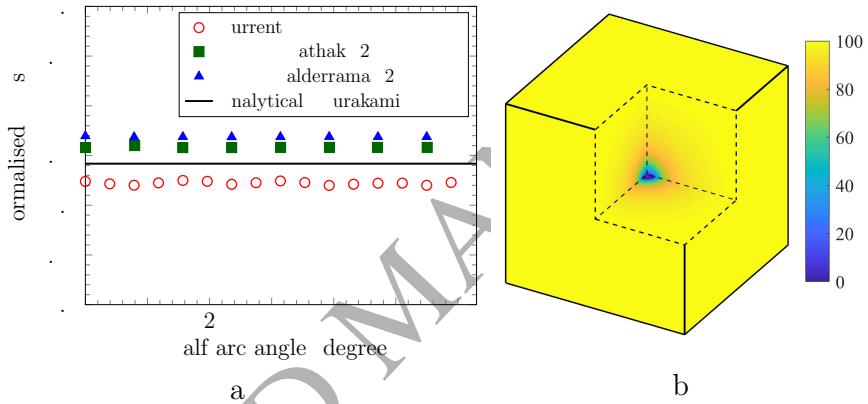


Figure 26: Results at the last adaptive step for the penny-shaped crack problem: (a) SIF and (b) temperature ($^{\circ}\text{C}$), where a fraction of the cube (marked by dashed lines) is omitted to illustrate the temperature distribution inside.

5. Summary

An adaptive CPM for modelling thermoelastic fracture in 2D has been developed in this paper, where enrichment functions, which have been widely used in traditional numerical methods including the XFEM and the EFGM, are not required. The discontinuities of temperature and displacement at a crack are modelled by the CPM using the visibility criterion, and stress gradients are captured using a high density of particles around the crack tip, which is controlled by an adaptivity approach. The mixed-mode stress intensity factors under thermal loadings are calculated by the interaction integral and the proposed method has been shown to provide results at the same level of accuracy as the XFEM with high order enrichments while using fewer particles than the number of nodes in the standard XFEM. The proposed methodology has therefore shown its potential to

9 handle the crack propagation of multiple cracks under thermal loading.

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 3 which this work formed a part.

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