Hidden Markov Models

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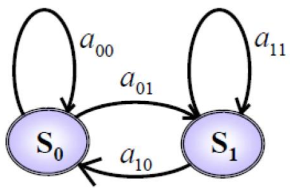
- Hidden Markov models
- Problem 1: Scoring and evaluation
- Problem 2: Decoding
- Problem 3: Training

Markov model

- stochastic model
- used to model a random system that changes state according to a transition rule that depends only on the current state
- Characterized by a set of N states

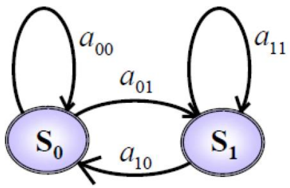


$$S = \{S_0, S_1, \cdots S_N\}$$

and transition probabilities from one state to another a_{ij}

Markov model

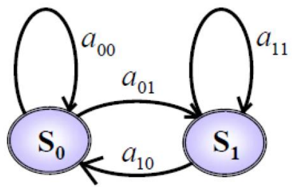
- the current state of the system depends only on the previous state, not on the sequence before that
- the state of the system at time t is q_t
- the transition probability depends only on the previous state:



$$P(q_t = S_j \mid q_{t-1} = S_i, q_{t-2} = S_k \cdots) = P(q_t = S_j \mid q_{t-1} = S_i)$$
$$a_{ij} = P(q_t = S_j \mid q_{t-1} = S_i)$$

Markov model properties

$$\mathrm{A} = \begin{bmatrix} a_{00} & a_{01} \\ a_{10} & a_{11} \end{bmatrix}$$
$$a_{ij} \geq 0 \quad \forall \, \mathrm{i, j}$$
$$\sum_{j=0}^{N} a_{ij} = 1 \quad \forall \, \mathrm{i}$$

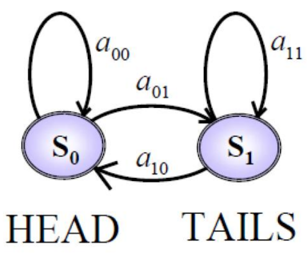


Stochastic matrix:
- each entry is non-negative
- rows sum up to 1

Example 1: Single fair coin observable process

- Observable: the output of the process is a set of states
- Outcomes:
  – Head – State 0
  – Tails – State 1
- Observed outcomes uniquely define a state sequence: HHHTTTHHTT → 0001110011
- Transition probabilities:

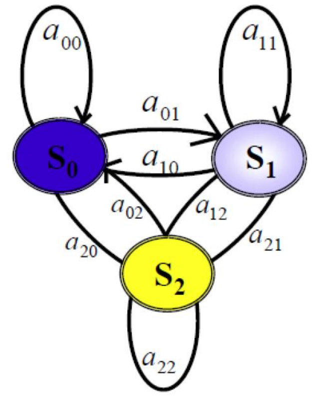
$$A = \begin{bmatrix} 0.5 & 0.5 \\ 0.5 & 0.5 \end{bmatrix}$$



Example 2: Observable Markov model of weather

- Outcomes:
  – State 0 – Rainy
  – State 1 – Cloudy
  – State 2 – Sunny
- State transition probabilities:

$$A = \{a_{ij}\} = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$$



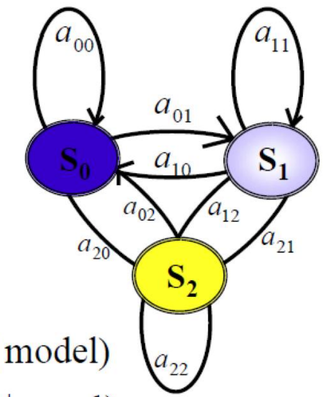
Example 2: Observable Markov model of weather

- What is the probability that the weather for 8 consecutive days is sun, sun, sun, rain, rain, sun, cloudy, sun ?

- Representing the information:
  – Observation sequence is:
    O = {sun, sun, sun, rain, rain, sun, cloudy, sun}
  – Corresponds to state sequence:
    S = {2, 2, 2, 0, 0, 2, 1, 2}
  – We need to calculate P(O | model)
    P(O | model) = P(S={2, 2, 2, 0, 0, 2, 1, 2} | model)

Example 2: Observable Markov model of weather

$$A = \{a_{ij}\} = \begin{bmatrix} 0.4 & 0.3 & 0.3 \\ 0.2 & 0.6 & 0.2 \\ 0.1 & 0.1 & 0.8 \end{bmatrix}$$



π_i : initial state probability

$\pi_i = P(q_1 = i)$

$$P(O \mid \text{model}) = P(S = \{2,2,2,0,0,2,1,2\} \mid \text{model})$$
$$= P(q_1 = 2) P(q_2 = 2 \mid q_1 = 2) \cdots P(q_8 = 2 \mid q_7 = 1)$$
$$= \pi_2 \cdot a_{22} \cdot a_{22} \cdot a_{20} \cdot a_{00} \cdot a_{02} \cdot a_{21} \cdot a_{12}$$
$$= \pi_2 \cdot (0.8)^2 (0.1) \cdot (0.4) \cdot (0.3) \cdot (0.1) \cdot (0.2)$$

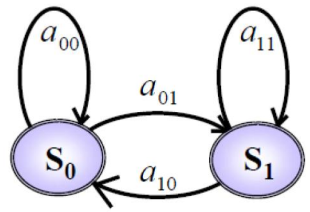
Hidden Markov models

- Observations are a probabilistic function of state
- The underlying sequence of states is not observable (it is hidden)
- Outputs are independent – observations are dependent only on the state that generated them, not on eachother

Example: Two coins observable Markov process

- States: coins
- Observations:
  - Head
  - Tail
  - Each state can generate each observation with a certain probability
- The observed outcomes do not uniquely define state sequence
- Transition probabilities:

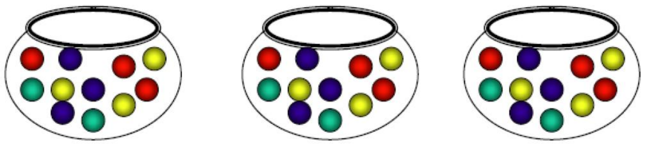
$$A = \begin{bmatrix} a_{00} & 1-a_{00} \\ 1-a_{11} & a_{11} \end{bmatrix}$$



$P(H) = P_1$ $\quad P(H) = P_2$

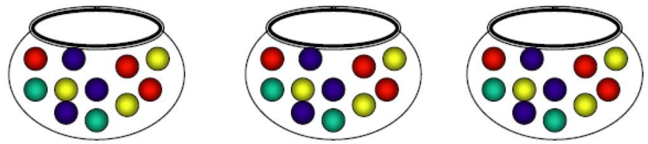
$P(T) = 1 - P_1$ $\quad P(T) = 1 - P_2$

Example: urn and ball model



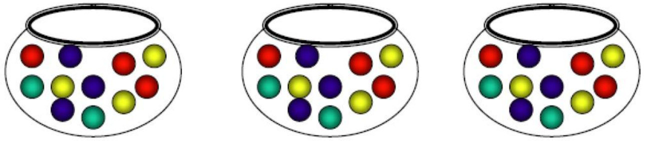
- 3 urns with 4 different color balls
- We do not see the urns, someone is extracting balls from them and tells us the color of each
- Steps:
  1. Select one urn at random
  2. Pick a ball from the urn, tell what color it is
  3. Put ball back to the urn
  4. Select new urn based on a random selection procedure from current urn
  5. Repeat steps 2-4

Example: urn and ball model



- Observations: the colors of the balls
- States: the identity of the urn
- State transitions: the selection process for next urn given current one

Example: urn and ball model



P(R)=0.20	P(R)=0.45	P(R)=0.15
P(G)=0.30	P(G)=0.15	P(G)=0.70
P(B)=0.10	P(B)=0.20	P(B)=0.10
P(Y)=0.40	P(Y)=0.20	P(Y)=0.05

- Urns contain different ratio of colours
- Observation sequence: R B Y Y G B Y G R ..
- The observation sequence (individual colors) do not reveal the state (which urn it comes from)

Discrete symbol observation HMM

- A set of N states

$$S = \{S_0, S_1, \cdots S_N\}$$

- Transition probabilities

$$a_{ij} = P(q_t = S_j \mid q_{t-1} = S_i)$$

- A set of M observation symbols

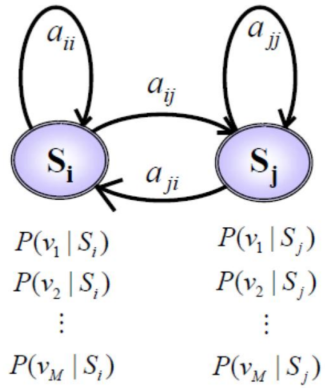
$$V = \{v_1, v_2, \cdots v_M\}$$

- Probability distribution (state j symbol k)

$$b_j(k) = P(o_t = v_k \mid q_t = j)$$

- Initial state distribution

$$\pi = \{\pi_i\} = P(q_1 = i)$$

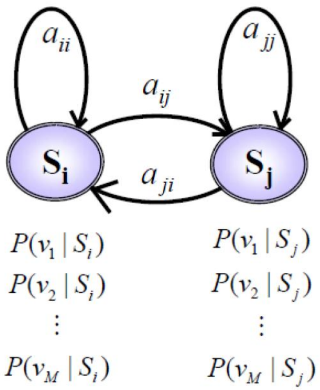


Discrete symbol observation HMM

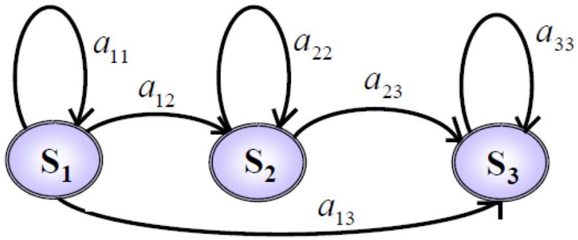
Specification of an HMM:

- Two model parameters, N and M
  - Number of states N
  - Number of symbols M
- Three probability measures A, B, π
  - Transition probability matrix A
  - State probability distribution B
  - Initial state distribution π

$$\lambda = (A, B, \pi)$$



Left-to-right HMM

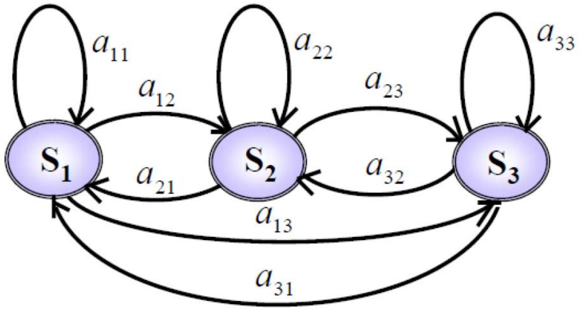


$$\mathrm{A} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}$$

$$a_{ij} = 0 \qquad j < i$$

$$\pi_i = \begin{cases} 0, & i \neq 1 \\ 1, & i = 1 \end{cases}$$

Ergodic HMM



$$a_{ij} > 0 \qquad \forall i, \forall j$$

HMM as a symbol generator

An HMM with parameters N, M, A, B, and π can generate an observation sequence:

$$O = \{ o_1, o_2, o_3, .. o_T \}$$

1. Choose initial state $q_1 = i$ from the initial state distribution π
2. Set $t=1$
3. Choose $o_T = v_k$ according to distribution $b_j(k)$
4. Transition to state $q_{t+1} = j$ according to state transition probability a_{ij}
5. Set $t=t+1$
6. Repeat from step 3

HMM as a symbol generator

Time t	1	2	3	4	5	6	7	8	…	T
State	q_1	q_2	q_3	q_4	q_5	q_6	q_7	q_8	…	q_T
Observation	o_1	o_2	o_3	o_4	o_5	o_6	o_7	o_8	…	o_T

Think of the HMM as generating the observation sequence as it transitions from state to state

HMM problems

- Problem 1: Scoring and evaluation
  - How to compute efficiently the probability of an observation sequence O given the model λ ? (How to calculate P(O| λ))
- Problem 2: Decoding
  - Given an observation sequence O and a model λ, how do we determine the corresponding state sequence q that best explains how the observations were generated?
- Problem 3: Training
  - How to adjust the parameters λ={A,B, π} to maximize the probability of generating a given observation sequence?

    (How to maximize P(O| λ))

Problem 1: Scoring and evaluation

- Given an observation sequence O = { o_1, o_2, o_3, .. o_T }

we want to compute the probability of generating it $P(O|\lambda)$

- We assume a sequence of states q = { q_1, q_2, q_3, .. q_T }
- Decompose the problem by summing over all possible state sequences:

$$P(\mathbf{O}\mid\lambda)=\sum_{\text{all q}}P(\mathbf{O}\mid q,\lambda)P(q\mid\lambda)$$

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Likelihood of generating the
observed symbol sequence
given the assumed state sequence

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Likelihood of generating the
observed symbol sequence
given the assumed state sequence

How likely it is for the
system to go through the
given sequence of states

Problem 1: Scoring and evaluation

- Probability of the observation sequence given the state sequence:

$$P(\mathbf{O}\mid q,\lambda)=\prod_{t=1}^{T}p(\mathbf{o}_t\mid q_t,\lambda)=b_{q_1}(\mathbf{o}_1)\cdot b_{q_2}(\mathbf{o}_2)\cdots b_{q_T}(\mathbf{o}_T)$$

- Probability of the state sequence:

$$P(q\mid\lambda)=\pi_{q_1}(a_{q_1q_2})\cdot(a_{q_2q_3})\cdots(a_{q_{T-1}q_T})$$

- Using the chain rule:

$$P(\mathbf{O}\mid\lambda)=\sum_{\text{all q}}P(\mathbf{O}\mid q,\lambda)P(q\mid\lambda)$$

$$=\sum_{\text{all q}}\pi_{q_1}b_{q_1}(\mathrm{o}_1)a_{q_1q_2}b_{q_2}(\mathrm{o}_2)\cdots a_{q_{T-1}q_T}b_{q_T}(\mathrm{o}_T)$$

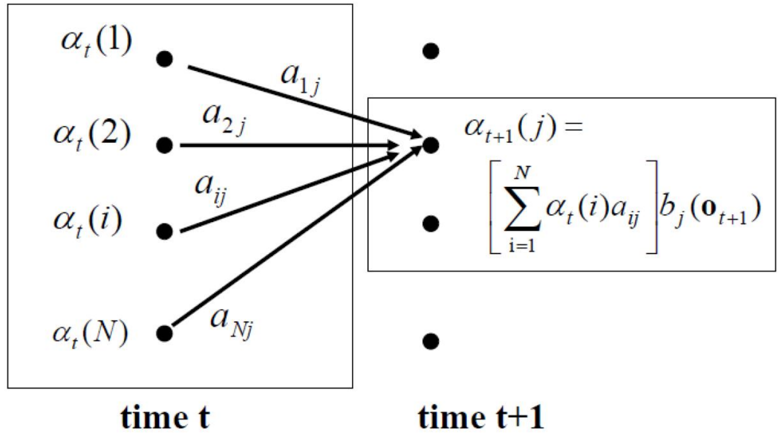
Not practical to compute!

Forward algorithm

- Define the probability of seeing observations o_1 to o_T, and ending in state i, given HMM λ:

$$\alpha_t(i) = P(\mathbf{o}_1\mathbf{o}_2\ldots\mathbf{o}_t, q_t = i \mid \lambda)$$

Forward algorithm



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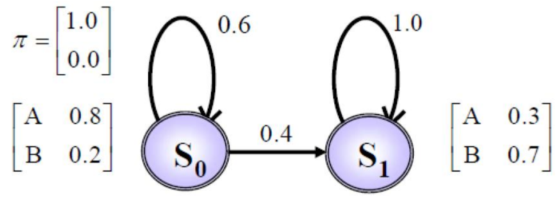
$$\alpha_t(i) = P(\mathbf{o}_1\mathbf{o}_2\ldots\mathbf{o}_t, q_t = i \mid \lambda)$$

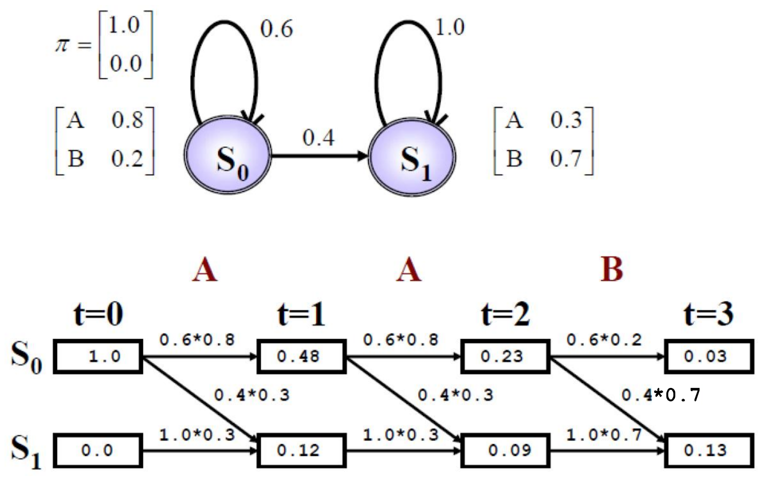
- Initialization: $\alpha_0(i) = \pi_i$
- Induction: $\alpha_{t+1}(j) = \left[\sum_{i=1}^{N} \alpha_t(i) a_{ij}\right] b_j(\mathbf{o}_{t+1})$
- Termination: $P(\mathbf{O} \mid \lambda) = \sum_{i=1}^{N} \alpha_T(i)$

Forward algorithm example



- Given this HMM with discrete observations A and B, what is the probability of generating the sequence {A,A,B}?
- We need to calculate P(O = {A,A,B} | λ)

Forward algorithm example



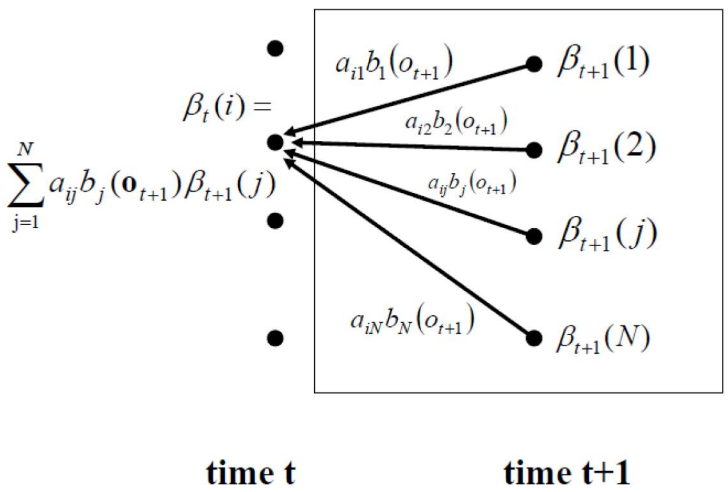
P(O = {A,A,B} | λ } = 0.03 + 0.13 = 0.16

Backward algorithm

- Define the probability of seeing observations o_{t+1} to o_T, given state *i* at time *t* and HMM λ:

$$\beta_t(i) = P(\mathbf{o}_{t+1}\mathbf{o}_{t+2}\ldots\mathbf{o}_T, q_t = i \mid \lambda)$$

Backward algorithm



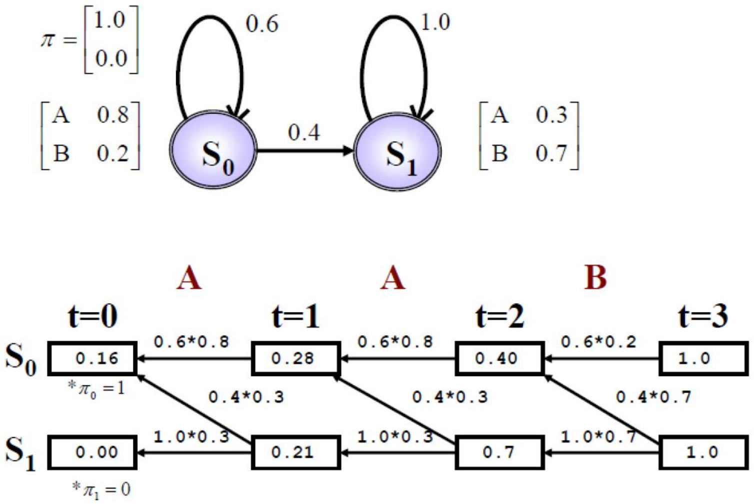
Backward algorithm

- Define the probability of seeing observations o_{t+1} to o_T, given state *i* at time *t* and HMM λ:

$$\beta_t(i) = P(\mathbf{o}_{t+1}\mathbf{o}_{t+2}\ldots\mathbf{o}_T, q_t = i \mid \lambda)$$

- Initialization: $\beta_T(i) = 1$
- Induction: $$\beta_t(i) = \sum_{j=1}^{N} a_{ij} b_j(\mathbf{o}_{t+1})\beta_{t+1}(j)$$ $t = T-1, T-2, \ldots, 1$ $1 \le i \le N$
- Termination: $$P(\mathbf{O} \mid \lambda) = \sum_{i=1}^{N} \pi_i \beta_0(i)$$

Backward algorithm example



Problem 1: Scoring and evaluation

- Solution: two ways of calculating P(O| λ)
  - Forward algorithm

$$P(\mathbf{O} \mid \lambda) = \sum_{i=1}^{N} \alpha_T(i)$$

  - Backward algorithm

$$P(\mathbf{O} \mid \lambda) = \sum_{i=1}^{N} \pi_i \beta_0(i)$$

Problem 2: Decoding

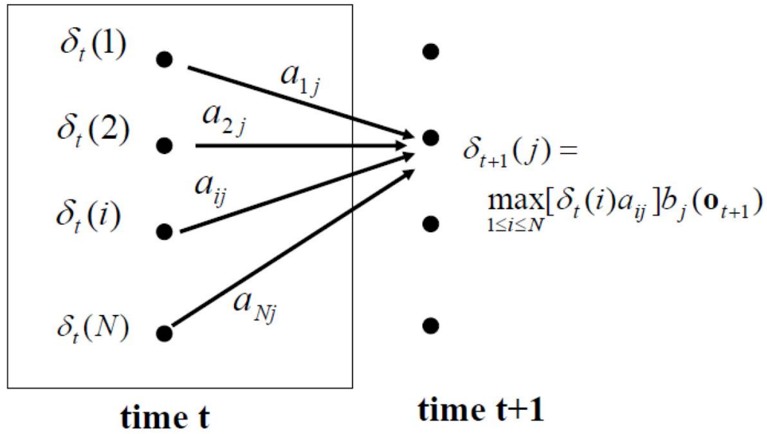
- Given an observation sequence O = { o_1, o_2, o_3, .. o_T }, and a model λ, how do we find the best sequence of states q = { q_1, q_2, q_3, .. q_T } which maximizes P(O,q| λ)?
- Define the highest probably state sequence that accounts for observations o_1 to o_t and ends in state i at time t:

$$\delta_t(i) = \max_{q_1 q_2 \cdots q_{t-1}} P(q_1 q_2 \cdots q_{t-1}, q_t = i, \mathbf{o}_1 \mathbf{o}_2 \ldots \mathbf{o}_t \mid \lambda)$$

- At next transition:

$$\delta_{t+1}(j) = [\max_i \delta_t(i) a_{ij}] \cdot b_j(o_{t+1})$$

Viterbi algorithm



Viterbi algorithm

- Initialization:

$$\delta_1(i) = \pi_i b_i(\mathbf{o}_1)$$

$$\psi_1(i) = 0$$

- Recursion:

$$\delta_t(j) = \max_{1 \le i \le N}[\delta_{t-1}(i)a_{ij}]b_j(\mathbf{o}_t)$$

$$\psi_t(j) = \arg\max_{1 \le i \le N}[\delta_{t-1}(i)a_{ij}]$$

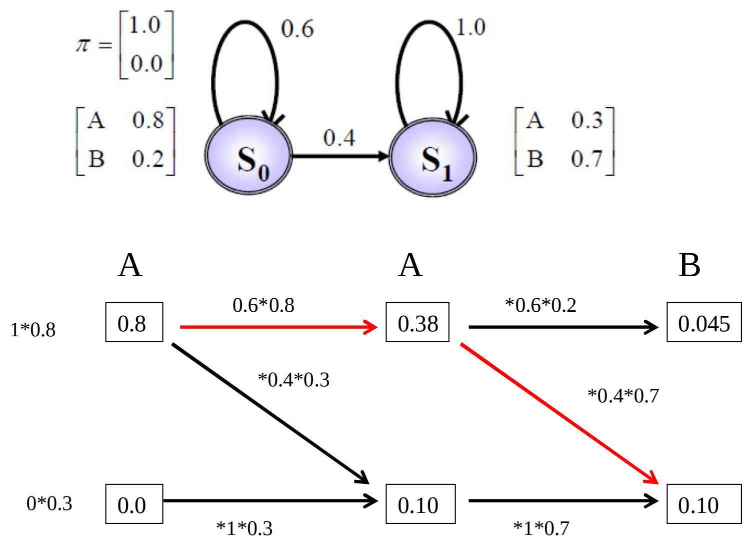
- Termination:

$$P^* = \max_{1 \le i \le N}\left[\delta_T(i)\right]$$

$$q_T^* = \arg\max_{1 \le i \le N}\left[\delta_T(i)\right]$$

- Path back-tracing: $q_t^* = \psi_{t+1}(q_{t+1}^*)$

Viterbi algorithm example



Viterbi algorithm in log-domain

$$\tilde{\pi}_i = \log(\pi_i)$$

$$\tilde{b}_j(o_t) = \log(b_j(o_t))$$

$$\tilde{a}_{ij} = \log(a_{ij})$$

- Same steps are followed in log-domain:

$$\delta_1(i) = \pi_i b_i(\mathbf{o}_1) \longrightarrow \tilde{\delta}_1(i) = \tilde{\pi}_i + \tilde{b}_i(\mathbf{o}_1)$$

Viterbi algorithm in log-domain

- Initialization:

$$\tilde{\delta}_1(i) = \tilde{\pi}_i + \tilde{b}_i(\mathbf{o}_1)$$

$$\psi_1(i) = 0$$

- Recursion:

$$\delta_t(j) = \max_{1 \le i \le N}[\tilde{\delta}_{t-1}(i) + \tilde{a}_{ij}] + \tilde{b}_j(\mathbf{o}_t)$$

$$\psi_t(j) = \arg\max_{1 \le i \le N}[\tilde{\delta}_{t-1}(i) + \tilde{a}_{ij}]$$

- Termination:

$$\tilde{P}^* = \max_{1 \le i \le N}\left[\tilde{\delta}_T(i)\right]$$

$$q_T^* = \arg\max_{1 \le i \le N}\left[\tilde{\delta}_T(i)\right]$$

- Path backtracking: $q_t^* = \psi_{t+1}(q_{t+1}^*)$

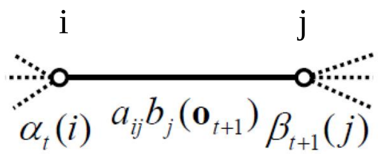
Problem 3: Training

- How do we tune λ to maximize P(O| λ) ?
  - No efficient algorithm to find global optimum
- Baum-Welch algorithm (forward-backward algorithm)
  - Iterative algorithm to find a local optimum
  - Compute probabilities using current model
  - Refine model parameters based on computed values

Forward-backward algorithm

- Define the probability of being in state *i* at time *t* and in state *j* at time *t+1*, given the model and the sequence

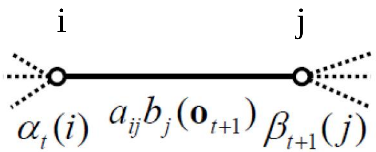
$$\xi_t(i,j) = P(q_t = i, q_{t+1} = j \mid \mathbf{O}, \lambda)$$



Forward-backward algorithm

- Define the probability of being in state *i* at time *t* and in state *j* at time *t+1*, given the model and the sequence

$$\xi_t(i,j) = \frac{\alpha_t(i) a_{ij} b_j(\mathbf{o}_{t+1}) \beta_{t+1}(j)}{P(\mathbf{O} \mid \lambda)} = \frac{\alpha_t(i) a_{ij} b_j(\mathbf{o}_{t+1}) \beta_{t+1}(j)}{\sum_{i=1}^{N} \alpha_t(i) \beta_t(i)}$$



Forward-backward algorithm

- More definitions, based on $\xi_t(i,j)$

$$\gamma_t(i) = \sum_{j=1}^{N} \xi_t(i,j)$$ Probability of being in state i at time t

$$\sum_{t=1}^{T-1} \gamma_t(i)$$ Expected number of transitions from state i in O

$$\sum_{t=1}^{T-1} \xi_t(i,j)$$ Expected number of transitions from state i to state j in O

Computing the model parameters

- Initial state occupancy probability is the **expected number of times in state *i* at time *t=1***

$$\overline{\pi}_i = \gamma_1(i)$$

Computing the model parameters

- transition probability from state i to state j

$$\overline{a}_{ij} = \frac{\text{expected number of transitions from state } i \text{ to state } j}{\text{expected number of transitions from state } i}$$

$$= \frac{\sum_{t=1}^{T-1} \xi_t(i,j)}{\sum_{t=1}^{T-1} \gamma_t(i)}$$

Computing the model parameters

- Probability of observing symbol *k* in state *j*

$$\overline{b}_j(k) = \frac{\text{expected number of times in state } j \text{ and observing kth symbol}}{\text{expected number times in state } j}$$

$$= \frac{\sum_{\substack{t=1 \\ \mathbf{o}_t = v_k}}^{T} \gamma_t(j)}{\sum_{t=1}^{T} \gamma_t(j)} = \frac{\sum_{\substack{t=1 \\ \mathbf{o}_t = v_k}}^{T} \alpha_t(j)\beta_t(j)}{\sum_{t=1}^{T} \alpha_t(j)\beta_t(j)}$$

Forward-backward algorithm iterations

1. Initialize $\lambda = (\mathbf{A}, \mathbf{B}, \pi)$
2. Compute α, β and ξ
3. Estimate $\overline{\lambda} = (\overline{\mathbf{A}}, \overline{\mathbf{B}}, \overline{\pi})$
4. Replace λ with $\overline{\lambda}$
5. Repeat from step 2 until convergence

Constraints:

$$\sum_{i=1}^{N} \overline{\pi}_i = 1$$

$$\sum_{j=1}^{N} \overline{a}_{ij} = 1 \qquad 1 \le i \le N$$

$$\sum_{k=1}^{M} \overline{b}_j(k) = 1 \qquad 1 \le j \le N$$

It can be shown that $\mathrm{P}(\mathbf{O} \mid \overline{\lambda}) > \mathrm{P}(\mathbf{O} \mid \lambda)$ unless $\overline{\lambda} = \lambda$

Mixture Gaussian PDFs

- Probability distribution of the state is a gaussian mixture

$$b_j(\mathbf{o_t}) = \sum_{k=1}^{M} c_{jk} \mathrm{N}(\mathbf{o_t}, \mu_{jk}, \Sigma_{jk}) \qquad \sum_{k=1}^{M} c_{jk} = 1$$

$$c_{jk} \geq 0 \qquad 1 \leq \mathrm{k} \leq \mathrm{M}$$

- Probability of being in state *j* at time *t* with the mixture component *k* accounting for the observation o_t is:

$$\gamma_t(j,k) = \left[\frac{\alpha_t(j)\beta_t(j)}{\sum_{j=1}^{N} \alpha_t(j)\beta_t(j)} \right] \left[\frac{c_{jk} \mathrm{N}(\mathbf{o}_t, \mu_{jk}, \Sigma_{jk})}{\sum_{m=1}^{M} c_{jm} \mathrm{N}(\mathbf{o}_t, \mu_{jm}, \Sigma_{jm})} \right]$$

Parameter update equations for GMM PDF

- Mixture weight and mean:

$$\bar{c}_{jk} = \frac{\sum_{t=1}^{T} \gamma_t(j,k)}{\sum_{t=1}^{T} \sum_{k'=1}^{M} \gamma_t(j,k')} \qquad \bar{\mu}_{jk} = \frac{\sum_{t=1}^{T} \gamma_t(j,k) \cdot \mathbf{o}_t}{\sum_{t=1}^{T} \gamma_t(j,k)}$$

- Transition matrix elements a_{ij} get updates same way as in the case of discrete symbols
- Covariance matrix:

$$\overline{\Sigma}_{jk} = \frac{\sum_{t=1}^{T} \gamma_t(j,k) \cdot (\mathbf{o}_t - \overline{\mu}_{jk})(\mathbf{o}_t - \overline{\mu}_{jk})'}{\sum_{t=1}^{T} \gamma_t(j,k)}$$

Multiple observation sequences

- Variability in producing each sound unit is modeled by estimating HMM parameters from multiple examples of speech, collected form different speakers

- Assume *K* training observation sequences

$$\mathbf{O} = [\mathbf{O}^{(1)}, \mathbf{O}^{(2)}, \ldots, \mathbf{O}^{(K)}]$$

$$\mathbf{O}^{(k)} = \{\mathbf{o}_1^k, \mathbf{o}_2^k, \ldots \mathbf{o}_{T_k}^k\}$$

$$\mathrm{P_k} = \mathrm{P}(\mathbf{O}^{(k)} \mid \lambda)$$

Parameter update for multiple observations

$$\bar{a}_{ij} = \frac{\sum_{k=1}^{K} \frac{1}{P_k} \sum_{t=1}^{T_k - 1} \alpha_t^k(i) a_{ij} b_j(\mathbf{o_{t+1}^{(k)}}) \beta_{t+1}^k(j)}{\sum_{k=1}^{K} \frac{1}{P_k} \sum_{t=1}^{T_k - 1} \alpha_t^k(i) \beta_t^k(i)}$$

$$\bar{b}_j(l) = \frac{\sum_{k=1}^{K} \frac{1}{P_k} \sum_{\substack{t=1 \\ s.t. O_t = v_l}}^{T_k - 1} \alpha_t^k(i) \beta_t^k(i)}{\sum_{k=1}^{K} \frac{1}{P_k} \sum_{t=1}^{T_k - 1} \alpha_t^k(i) \beta_t^k(i)}$$

One-state HMM with M component GMM

- Observation probability is a GMM with M components

$$b(\mathbf{o_t}) = \sum_{k=1}^{M} w_k b_k(\mathbf{o_t}, \mu_k, \Sigma_k)$$

- Probability that o_t is generated by the kth component

$$P(k \mid o_t, \lambda) = \frac{w_k b_k(o_t)}{\sum_{k=1}^{M} w_k b_k(o_t)}$$

Update equations for one-state HMM

Updated (new) parameter estimates

$$\boxed{\overline{w}_k} = \frac{1}{T}\sum_{t=1}^{T} \boxed{P(k \mid o_t, \lambda)}$$

$$\boxed{\overline{u}_k} = \frac{\sum_{t=1}^{T} P(k \mid o_t, \lambda) \cdot o_t}{\sum_{t=1}^{T} P(k \mid o_t, \lambda)}$$

$$\boxed{\overline{\Sigma}_k} = \frac{\sum_{t=1}^{T} P(k \mid o_t, \lambda) \cdot (o_t)^2}{\sum_{t=1}^{T} P(k \mid o_t, \lambda)} - (\overline{u}_k)^2$$

This term is computed using model parameters from previous algorithm iteration.