1 Adjoint situations

Moreover, adjunctions provide us with many (technically even all) examples of monads and comonads, as we will later see.

einleitender Satz

Proposition 1.1 *Given two functors* $\mathbf{A} \underset{F}{\overset{G}{\rightleftarrows}} \mathbf{B}$ *the following are equivalent:*

(a) $\exists \eta\colon \mathrm{id}_B \Rightarrow GF$ *and* $\varepsilon\colon FG \Rightarrow \mathrm{id}_A$ *natural transformations such that* $\forall a \in Ob(A), b \in Ob(B)$ *the following two diagrams commute:*

$$\begin{tikzcd} F(b) \arrow[r, "F(\eta_b)"] \arrow[dr, "\mathrm{id}_{F(b)}"'] & FGF(b) \arrow[d, "\varepsilon_{F(b)}"] \\ & F(b) \end{tikzcd} \qquad \begin{tikzcd} G(a) \arrow[r, "\eta_{G(a)}"] \arrow[dr, "id_{G(a)}"'] & GFG(a) \arrow[d, "G(\varepsilon_a)"] \\ & G(a) \end{tikzcd} \qquad \text{(triangle identity)}$$

(b) $\forall a \in Ob(A), b \in Ob(B)$ *there is a bijection*

$$\phi_{a,b}\colon \mathrm{Hom}_{\mathbf{A}}(F(b), a) \longrightarrow \mathrm{Hom}_{\mathbf{B}}(b, G(a))$$

which is natural in a and b, which means that for $p\colon a \longrightarrow a'$:

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{A}}(F(b), a) \arrow[r, "\phi_{a,b}"] \arrow[d, "p\circ\_"] & \mathrm{Hom}_{\mathbf{B}}(b, G(a)) \arrow[d, "G(p)\circ\_"] \\ \mathrm{Hom}_{\mathbf{A}}(F(b), a') \arrow[r, "\phi_{a',b}"] & \mathrm{Hom}_{\mathbf{B}}(b, G(a')) \end{tikzcd}$$

and for $q\colon b \longrightarrow b'$:

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{A}}(F(b'), a) \arrow[r, "\phi_{a,b'}"] \arrow[d, "\_\circ F(q)"] & \mathrm{Hom}_{\mathbf{B}}(b', G(a)) \arrow[d, "\_\circ q"] \\ \mathrm{Hom}_{\mathbf{A}}(F(b), a) \arrow[r, "\phi_{a,b}"] & \mathrm{Hom}_{\mathbf{B}}(b, G(a)) \end{tikzcd}$$

commute.

Proof: $(a) \implies (b)$:
define

$$\phi_{a,b}\colon \mathrm{Hom}_{\mathbf{A}}(F(b), a) \longrightarrow \mathrm{Hom}_{\mathbf{B}}(b, G(a))$$

by

$$\phi_{a,b}(g) = G(g) \circ \eta_b\colon b \longrightarrow G(a)$$

for $g\colon F(b) \longrightarrow a$ and define

$$\psi_{a,b}\colon \mathrm{Hom}_{\mathbf{B}}(b, G(a)) \longrightarrow \mathrm{Hom}_{\mathbf{A}}(F(b), a)$$

by

$$\psi_{a,b}(f) = \varepsilon_a \circ F(f)\colon F(b) \longrightarrow a$$

for $f\colon b \longrightarrow G(a)$.

Claim 1. $\psi \circ \phi = id$

Proof of claim 1. Let $f : b \to G(a)$.

$$\begin{aligned}
\phi(\psi(f)) &= \phi(\varepsilon_a \circ F(f)) && \text{(Definition of } \psi\text{)}\\
&= G(\varepsilon_a \circ F(f)) \circ \eta_b && \text{(Definition of } \phi\text{)}\\
&= G(\varepsilon_a) \circ G(F(f)) \circ \eta_b && \text{(Functoriality of } G\text{)}\\
&= G(\varepsilon_a) \circ \eta_{G(a)} \circ f && \text{(Naturality of } \eta\text{)}\\
&= \mathrm{id}_{G(a)} \circ f && \text{(right triangle identity)}\\
&= f
\end{aligned}$$

//

□

Remark 1.2. Let $F \dashv G$ be an adjoint situation, i.e. F is left-adjoint to G and G is right-adjoint to F. Then

1. G preserves limits
2. F preserves colimits.

those are DUAL adjunctions!

Example 1 (Galois connection). blablabla examples include:

1. (Fundamental theorem of Galois theory): this example.
2. (Algebraic geometry): that example.

Example 2 (Coproduct $\dashv \Delta \dashv$ Product). this.

Example 3 (free-forgetful adjunction). that.

Example 4 (Tensor-Hom-Adjunction). There is a natural bijection

$$\mathrm{Hom}_{\mathbf{A}}(M \otimes_A N, P) \cong \mathrm{Hom}_{\mathbf{A}}(M, \mathrm{Hom}_{\mathbf{A}}(N, P))$$

This implies that the tensor-product is right-exact, since it preserves cokernels.

2 Monads and Comonads

2.1 Definition of Monads and Comonads

A central notion in algebra is that of a *monoid*, that is, a set M equipped with a map $\mu\colon M \times M \longrightarrow M$; $(a, b) \mapsto a \cdot b$ (often called *multiplication*) and an element $e \in M$ such that the following two axioms hold:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \text{for all } a, b, c \in M. \qquad \text{(associativity)}$$
$$e \cdot a = a \cdot e = a \quad \text{for all } a \in M \qquad \text{(identity element)}$$

We can give an equivalent definition in terms of maps and commuting diagrams as follows: A *monoid* is a set M together with two functions

$$\mu\colon M \times M \to M, \qquad e\colon \{*\} \to M$$

such that the following diagrams commute:

$$\begin{tikzcd} M \times M \times M \arrow[r, "\mathrm{id}\times\mu"] \arrow[d, "\mu\times\mathrm{id}"] & M \times M \arrow[d, "\mu"] \\ M \times M \arrow[r, "\mu"] & M \end{tikzcd} \qquad \begin{tikzcd} \{*\} \times M \arrow[r, "e\times\mathrm{id}"] \arrow[dr, "l"'] & M \times M \arrow[d, "\mu"] & M \times \{*\} \arrow[l, "\mathrm{id}\times e"'] \arrow[dl, "r"] \\ & M & \end{tikzcd}$$

where id is the identity on m, and l and r are the canonical bijections

$$l\colon \{*\} \times M \longrightarrow M;\ l(*, m) = m$$
$$r\colon M \times \{*\} \longrightarrow M;\ r(m, *) = m.$$

Explicitly, the first diagram means that for all $a, b, c \in M$:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \text{for all } a, b, c \in M.$$

which is verbatim the associativity axiom, the second diagram means that for all $m \in M$:

$$e(*) \cdot m = l(*, m) = m = r(m, *) = m \cdot e(*)$$

which is clearly the identity element axiom for the element $e(*)$. This motivates the following definition: monoid/monad/ monoid object

Definition 2.1 (Monad). A *Monad* (T, μ, η) in a category $\mathcal{X}$ consists of

- an endofunctor $T\colon \mathcal{X} \to \mathcal{X}$
- a natural transformation $\eta\colon\ \mathrm{id}_{\mathcal{X}} \Rightarrow T$
- a natural transformation $\mu\colon T^2 \Rightarrow T$

such that the following diagrams commute:

$$\begin{tikzcd} T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\ T^2 \arrow[r, Rightarrow, "\mu"] & T \end{tikzcd}$$

(unitality)

$$\begin{tikzcd} T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "\mathrm{id}_T"'] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "T\eta"'] \arrow[dl, Rightarrow, "\mathrm{id}_T"] \\ & T & \end{tikzcd}$$

(associativity)

In terms of components, unitality and associativity mean that for every object x of $\mathcal{X}$ the following diagrams commute:

$$\begin{tikzcd} T(T(Tx)) \arrow[r, "T(\mu_x)"] \arrow[d, "\mu_{Tx}"'] & T(Tx) \arrow[d, "\mu_x"] \\ T(Tx) \arrow[r, "\mu_x"] & Tx \end{tikzcd}$$

(unitality)

$$\begin{tikzcd} Tx \arrow[r, "\eta_{Tx}"] \arrow[dr, "\mathrm{id}_{Tx}"'] & T(Tx) \arrow[d, "\mu_x"] & Tx \arrow[l, "T(\eta_x)"'] \arrow[dl, "\mathrm{id}_{Tx}"] \\ & Tx & \end{tikzcd}$$

(associativity)

Example 5 (preorder). Recall: A *preorder* $(\mathcal{P}, \leq)$ is a category with $\mathcal{P}$ as objects and a morphism between X and Y iff $X \leq Y$. A functor $T\colon \mathcal{P} \to \mathcal{P}$ is thus a monotonic function $\mathcal{P} \to \mathcal{P}$ ($x \leq y \implies Tx \leq Ty$). The existence of the natural transformations η is equivalent to

$$x \leq Tx \ \forall x \in \mathcal{P}$$

and the existence of μ is equivalent to

$$T(Tx) \leq Tx \ \forall x \in \mathcal{P}$$

because there is at most one morphism $x \to y$, so the neccessary diagrams commute trivially.
Now suppose $\mathcal{P}$ is a *partial order*, i.e. $x \leq y \leq x \implies x = y \ \forall x, y \in \mathcal{P}$.
Then:

$$\begin{aligned} x \leq Tx &\implies Tx \leq T(Tx) \\ T(Tx) \leq Tx &\implies Tx = T(Tx) \end{aligned}$$

so a Monad T in a partial order $\mathcal{P}$ is a *closure operation* in $\mathcal{P}$, i.e. a monotonic function $T\colon \mathcal{P} \to \mathcal{P}$ with $x \leq Tx$ and $T(Tx) = Tx \ \forall x \in \mathcal{P}$.
Now every topological space X induces a partial order $\mathcal{P} = (\mathscr{P}(X), \subseteq)$. Here an example for a closure operation is taking the topological closure $A \mapsto \overline{A}$, since it holds for all $A \subseteq X$ that $A \subseteq \overline{A}$ and $\overline{\overline{A}} = \overline{A}$.

Definition 2.2 (Comonad). A *Comonad* (L, ε, ω) in a Category $\mathcal{A}$ consists of

- an endofunctor $L\colon \mathcal{A} \to \mathcal{A}$
- a natural transformation $\varepsilon\colon L \Rightarrow \mathrm{id}_{\mathcal{A}}$
- a natural transformation $\omega\colon L \Rightarrow L^2$

such that the following diagrams commute:

$$\begin{tikzcd} L \arrow[r, Rightarrow, "L\omega"] \arrow[d, Rightarrow, "\omega L"'] & L^2 \arrow[d, Rightarrow, "L\omega"] \\ L^2 \arrow[r, Rightarrow, "\omega L"] & L^3 \end{tikzcd}$$

(counitality)

$$\begin{tikzcd} & L \arrow[dl, Rightarrow, "\mathrm{id}_L"'] \arrow[d, Rightarrow, "\omega"] \arrow[dr, Rightarrow, "\mathrm{id}_L"] & \\ L & L^2 \arrow[l, Rightarrow, "\varepsilon L"'] \arrow[r, Rightarrow, "L\varepsilon"] & L \end{tikzcd}$$

(coassociativity)

In terms of components, this means that for every object x of $\mathcal{A}$ the following diagrams commute:

$$\begin{tikzcd} Lx \arrow[r, "L(\omega_x)"] \arrow[d, "\omega_{Lx}"'] & L(Lx) \arrow[d, "L(\omega_x)"] \\ L(Lx) \arrow[r, "\omega_{Lx}"] & L(L(Lx)) \end{tikzcd}$$

(counitality)

$$\begin{tikzcd} & Lx \arrow[dl, "\mathrm{id}_{Lx}"'] \arrow[d, "\omega_x"] \arrow[dr, "\mathrm{id}_{Lx}"] & \\ Lx & L(Lx) \arrow[l, "\varepsilon_{Lx}"'] \arrow[r, "L(\varepsilon_x)"] & Lx \end{tikzcd}$$

(coassociativity)

Lemma 2.3 *For every object x in $\mathcal{X}$, the following diagram commutes:*

$$\begin{tikzcd} T(Tx) \arrow[r, "T(\delta_x)"] \arrow[d, "\delta_{Tx}"] & T(T'x) \arrow[d, "\delta_{T'x}"] \\ T(T'x) \arrow[r, "T'(\delta_x)"] & T'(T'x) \end{tikzcd}$$

this means

$$\delta T' \circ T\delta = T'\delta \circ \delta T \colon T^2 \Rightarrow (T')^2.$$

Proof: $\delta \colon T \Rightarrow T'$ is natural. □ finish proof

Definition 2.4 (Morphism of monads). Let $\mathcal{X}$ be a category, let (T, η, μ) and (T', η', μ') be monads in $\mathcal{X}$. We say that a natural transformation $\delta \colon T \Rightarrow T'$ is a *morphism of monads* if it preserves the unit and the multiplication, i.e. the following diagrams commute:

$$\begin{tikzcd} \mathrm{id}_x \arrow[r, "\eta_x"] \arrow[dr, "\eta'_x"'] & Tx \arrow[d, "\delta_x"] \\ & T'x \end{tikzcd} \qquad \begin{tikzcd} T^2x \arrow[r, "\mu_x"] \arrow[d, "\delta T' \circ T\delta"'] & Tx \arrow[d, "\delta_x"] \\ T'^2x \arrow[r, "\mu'_x"] & T'x \end{tikzcd}$$

Definition 2.5 (Morphism of comonads).

3 Witt vectors

Construction of the witt vectors

Recall that for every prime number p, we have the *p-adic valuation map*:

Definition 3.1 (p-adic valuation). $v_p\colon \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$ is defined by

$$v_p(n) = \begin{cases} \max\{k \in \mathbb{N} : p^k \mid n\} & \text{if } n \neq 0 \\ \infty & \text{if } n = 0 \end{cases}$$

Definition 3.2 (truncation set). Let $\mathbb{N}$ be the set of positive integers and let $S \subseteq \mathbb{N}$ be a subset with the property that $\forall n \in \mathbb{N}$: if d is a divisor of n, then $d \in S$. We then say that S is a *truncation set.*

As a set, we define the *big Witt ring* $\mathbb{W}_S(A)$ to be A^S, we will give it a unique ring structure, such that the *ghost map* is a ring homomorphism.

Definition 3.3 (ghost map). We define $w\colon \mathbb{W}_S(A) \to A^S$ by $(a_n)_{n\in S} \mapsto (w_n)_{n\in S}$ where

$$w_n = \sum_{d|n} d a_d^{n/d}$$

The core of the construction is contained in the following Lemma:

Lemma 3.4 (Dwork) *Suppose that for every prime number p there exists a ring homomorphism $\phi_p\colon A \to A$ with the property that $\phi_p(a) \equiv a^p$ modulo pA. Then for every sequence $x = (x_n)_{n\in S}$, the following are equivalent:*

(i) *The sequence x is in the image of the ghost map $w\colon \mathbb{W}_S(A) \to A^S$.*

(ii) *For every prime number p and every $n \in S$ with $v_p(n) \geqslant 1$,*

$$x_n \equiv \phi_p(x_{n/p}) \qquad \textit{modulo } p^{v_p(n)}A.$$

Proof: ($\Rightarrow$) Suppose x is in the image of the ghost map, that means there is a sequence $a = (a_n)_{n\in S}$ such that $x_n = w_n(a)$ for all $n \in S$. We calculate:

$$\phi(x_{n/p}) = \phi(w_{n/p}(a)) = \phi(\sum_{d|n/p} d a_d^{n/pd}) = \sum_{d|n/p} d \cdot \phi(a_d^{n/pd})$$

since ϕ is a ring homomorphism and $d \in \mathbb{N}$.

Claim 1.

$$\sum_{d|n/p} d \cdot \phi(a_d^{n/pd}) \equiv \sum_{d|n/p} d \cdot a_d^{n/d} \qquad \textit{mod } p^{v_p(n)}A.$$

Proof of claim 1. //

Claim 2.

$$\sum_{d|n/p} d \cdot a_d^{n/d} \equiv \sum_{d|n} d \cdot a_d^{n/d} \qquad \textit{mod } p^{v_p(n)}A$$

Proof of claim 2. //

so we get

$$\phi(x_{n/p}) \equiv \sum_{d|n} d \cdot a_d^{n/d} = w_n(a) = x_n \qquad mod\ p^{v_p(n)}A.$$

$(\Leftarrow)$ Let $(x_n)_{n\in S}$ be a sequence such that $x_n \equiv \phi_p(x_{n/p}) \qquad mod\ p^{v_p(n)}A\ \forall p$ prime, $n \in S, v_p(n) \geqslant 1$. Define $(a_n)_{n\in S}$ with $w_n(a) = x_n$ as follows:

$$a_1 := x_1$$

and if a_d has been chosen for all $d \mid n$ such that $w_d(a) = x_d$ we see that

$$\begin{aligned} x_n &\equiv \phi_p(x_{n/p}) \qquad mod\ p^{v_p(n)}A \\ &= \phi_p(\sum_{d|n/p} d \cdot a_d^{n/pd}) \\ &= \sum_{d|n/p} d \cdot \phi(a_d^{n/pd}) \end{aligned}$$

□ finish proof

We will often need the following

Lemma 3.5 *if A is a torsion-free ring, the ghost map is injective.*

Now we can finish the construction of the Witt vectors:

Theorem 3.6 *There exists a unique ring structure such that the ghost map*

$$w : \mathbb{W}_S(A) \to A^s$$

is a natural transformation of functors from rings to rings.

PROOF: □

Corollary 3.7 *$w_n : \mathbb{W}_S(A) \to A$ is a natural ring homomorphism for all $n \in S$.*

Proposition 3.8 *$\mathbb{W}_S$ is a functor* **CRing** $\to$ **CRing**.

The Verschiebung, Frobenius and Teichmüller maps

We have various operations on witt vectors that are of interest.

Definition 3.9 (Restriction map). If $T \subseteq S$ are two truncation sets, the *restriction from S to T*

$$R_T^S : \mathbb{W}_S(A) \to \mathbb{W}_T(A)$$

is a natural ring homomorphism.

If $S \subseteq \mathbb{N}$ is a truncation set, $n \in \mathbb{N}$, then

$$S/n := \{d \in \mathbb{N} \mid nd \in S\}$$

is again a truncation set.

Definition 3.10 (Verschiebung). Define

$$V_n\colon \mathbb{W}_{S/n} \longrightarrow \mathbb{W}_S(A);\ V_n((a_d)_{d\in S/n})_m := \begin{cases} a_d, & \text{if } m = n\cdot d \\ 0, & \text{else} \end{cases}$$

which is called the *n-th Verschiebung map*. Furthermore define

$$\widetilde{V_n}\colon A^{S/n} \longrightarrow A^S;\ \widetilde{V_n}((x_d)_{d\in S/n})_m := \begin{cases} n\cdot x_d, & \text{if } m = n\cdot d \\ 0, & \text{else} \end{cases}$$

Lemma 3.11 *The Verschiebung map V_n is additive.*

Proof:

Claim.
$$\begin{tikzcd} \mathbb{W}_{S/n}(A) \arrow[r, "w"] \arrow[d, "V_n"] & A^{S/n} \arrow[d, "\widetilde{V_n}"] \\ \mathbb{W}_S(A) \arrow[r, "w"] & A^S \end{tikzcd}$$
commutes.

Proof of claim. //

□

Define $\widetilde{F_n}\colon A^S \longrightarrow A^{S/n}$ by $\widetilde{F_n}((x_m)_{m\in S})_d = x_{nd}$.

Lemma 3.12 (Frobenius homomorphism) *There exists a unique natural ring homomorphism*

$$F_n\colon \mathbb{W}_S(A) \to \mathbb{W}_{S/n}(A)$$

such that the diagram

$$\begin{tikzcd} \mathbb{W}_S(A) \arrow[r, "w"] \arrow[d, "F_n"] & A^S \arrow[d, "\widetilde{F_n}"] \\ \mathbb{W}_{S/n}(A) \arrow[r, "w"] & A^{S/n} \end{tikzcd}$$

commutes.

remark und definition haben andere font

We call F_n the *nth Frobenius homomorphism*. The commutativity of the diagram above is equivalent to commutativity of the following diagram for every $d \in S/n$:

$$\begin{tikzcd} \mathbb{W}_S(A) \arrow[d, "F_n"] \arrow[dr, "w_{nd}"] & \\ \mathbb{W}_{S/n}(A) \arrow[r, "w_d"'] & A \end{tikzcd}$$

Proof of Lemma 3.12. easy □

Lemma 3.13 *Let $n, m \in \mathbb{N}$. Then*

$$F_n \circ F_m = F_{nm}.$$

Proof: □

Definition 3.14 (teichmüller representative). The *teichmüller representative* is the map

$$\tau\colon A \to \mathbb{W}_S(A)$$

defined by

$$(\tau(a))_m = \begin{cases} a, & \text{if } m = 1 \\ 0, & \text{else} \end{cases}$$

Lemma 3.15 *The teichmüller map is multiplicative.*

Proof: The map $\widetilde{\tau}\colon A \to A^S;\ (\widetilde{\tau})_n = a^n$ is multiplicative and there is a commutative diagram

$$\begin{tikzcd} & A \arrow[dl, "\tau"'] \arrow[dr, "\widetilde{\tau}"] & \\ \mathbb{W}_S(A) \arrow[rr, "w"] & & A^S. \end{tikzcd}$$

Indeed, $w_n(\tau(a)) = w_n((a, 0, 0, \dots)) = a^n$ by definition of w_n.

□

The comonad structure of witt vectors

We will need the following lemma:

Lemma 3.16 *Let $m \in \mathbb{Z}$. If m is a non-zero divisor in A, then it is a non-zero divisor in $\mathbb{W}_S(A)$ as well.*

Proof:

$$0 \longrightarrow A \xrightarrow{V_n} \mathbb{W}_S(A) \xrightarrow{R^S_T} W_T(A) \longrightarrow 0$$

which we can extend to the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\cdot m"] & \mathbb{W}_S(A) \arrow[r] \arrow[d, "\cdot m"] & \mathbb{W}_T(A) \arrow[r] \arrow[d, "\cdot m"] & 0 \\ 0 \arrow[r] & A \arrow[r] & \mathbb{W}_S(A) \arrow[r] & \mathbb{W}_T(A) \arrow[r] & 0 \end{tikzcd}$$

□ finish

Corollary 3.17 *If A is torsion-free, then $\mathbb{W}_S(A)$ is torsion-free as well.*

Definition 3.18. $\mathbb{W}(A) := \mathbb{W}_{\mathbb{N}}(A)$

For the construction of a natural transformation $\mathbb{W}(A) \to \mathbb{W}(\mathbb{W}(A))$ we want to use Lemma 3.4 again. Hence we first show:

Lemma 3.19 *Let p be a prime number, let A be any ring. Then the ring homomorphism $F_p\colon \mathbb{W}(A) \to \mathbb{W}(A)$ satisfies $F_p(a) \equiv a^p \bmod pA$.*

Proposition 3.20 *There exists a unique natural transformation*

$$\Delta\colon \mathbb{W}(A) \to \mathbb{W}(\mathbb{W}(A))$$

such that $w_n(\Delta(a)) = F_n(A)$ *for all* $a \in A, n \in \mathbb{N}$.

Recall that by 3.7, $w_1 \colon \mathbb{W}(A) \to A;\ (a_n)_{n\in\mathbb{N}} \mapsto a_1$ is a natural transformation $\mathbb{W} \Rightarrow \mathrm{id}_{\mathbf{CRing}}$.

Theorem 3.21 *The functor* $\mathbb{W}(\_)\colon \mathbf{CRing} \to \mathbf{CRing}$ *together with the natural transformations* $\Delta\colon \mathbb{W} \Rightarrow \mathbb{W}^2$, $w_1\colon \mathbb{W} \Rightarrow \mathrm{id}_{\mathbf{CRing}}$ *form a comonad* $(\mathbb{W}, w_1, \Delta)$.

Proof:

Claim.
$$\begin{tikzcd} \mathbb{W}(A) \arrow[r, "\Delta_A"] \arrow[d, "\Delta_A"] \arrow[dr, phantom, "\#"] & \mathbb{W}(\mathbb{W}(A)) \arrow[d, "\mathbb{W}(\Delta_A)"] \\ \mathbb{W}(\mathbb{W}(A)) \arrow[r, "\Delta_{\mathbb{W}(A)}"] & \mathbb{W}(\mathbb{W}(\mathbb{W}(A))) \end{tikzcd}$$
commutes.

Proof of claim. evaluating the ghost coordinates leads to:

$$\begin{tikzcd} \mathbb{W}(A) \arrow[r, "\Delta_A"] \arrow[d, "\Delta_A"] \arrow[rr, bend left, dotted, "F_A"] & \mathbb{W}(\mathbb{W}(A)) \arrow[r, dotted, "w"] \arrow[d, "\mathbb{W}(\Delta_A)"] & \mathbb{W}(A)^{\mathbb{N}} \arrow[d, dotted, "\Delta_A^{\mathbb{N}}"] \\ \mathbb{W}(\mathbb{W}(A)) \arrow[r, "\Delta_{\mathbb{W}(A)}"] \arrow[rr, bend right, dotted, "F_{\mathbb{W}_A}"'] & \mathbb{W}(\mathbb{W}(\mathbb{W}(A))) \arrow[r, "w"] & \mathbb{W}(\mathbb{W}(A))^{\mathbb{N}} \end{tikzcd}$$

which by 3.20 simplifies to

$$\begin{tikzcd} \mathbb{W}(A) \arrow[r, "F_A"] \arrow[d, "\Delta_A"] & \mathbb{W}(A)^{\mathbb{N}} \arrow[d, "\Delta_A^{\mathbb{N}}"] \\ \mathbb{W}(\mathbb{W}(A)) \arrow[r, "F_{\mathbb{W}(A)}"] & \mathbb{W}(\mathbb{W}(A))^{\mathbb{N}} \end{tikzcd}$$

now it suffices to show for an arbitrary n that the following diagram commutes:

$$\begin{tikzcd} \mathbb{W}(A) \arrow[r, "F_{n_A}"] \arrow[d, "\Delta_A"] & \mathbb{W}(A) \arrow[d, "\Delta_A"] \\ \mathbb{W}(\mathbb{W}(A)) \arrow[r, "F_{n_{\mathbb{W}(A)}}"] & \mathbb{W}(\mathbb{W}(A)) \end{tikzcd}$$

evaluating the ghost coordinates again, keeping in mind that by 3.17 and 3.5, $w\colon \mathbb{W}(\mathbb{W}(A)) \to \mathbb{W}(A)^{\mathbb{N}}$ is injective as well, we get

$$\begin{tikzcd} \mathbb{W}(A) \arrow[r, "F_{n_A}"] \arrow[d, "\Delta_A"] & \mathbb{W}(A) \arrow[d, "\Delta_A"] \arrow[dd, bend left, dotted, "F_A"] \\ \mathbb{W}(\mathbb{W}(A)) \arrow[r, "F_{n_{\mathbb{W}(A)}}"] \arrow[d, dotted, "w"] & \mathbb{W}(\mathbb{W}(A)) \arrow[d, "w"] \\ \mathbb{W}(A)^{\mathbb{N}} \arrow[r, dotted, "\widetilde{F_n}_{\mathbb{W}(A)}"] & \mathbb{W}(A)^{\mathbb{N}} \end{tikzcd}$$

using the fact that $\begin{tikzcd} \mathbb{W}(\mathbb{W}(A)) \arrow[d, dotted, "w"] \arrow[dr, dotted, "w_{nm}"] & \\ \mathbb{W}(A)^{\mathbb{N}} \arrow[r, dotted, "\widetilde{F_n}_{\mathbb{W}(A)}"] & \mathbb{W}(A)^{\mathbb{N}} \end{tikzcd}$ commutes, we can simplify the situation to

$$\begin{tikzcd} \mathbb{W}(A) \arrow[r, "F_n"] \arrow[d, "\Delta_A"] \arrow[dr, dotted, "F_{nm}"] & \mathbb{W}(A) \arrow[d, "F_m"] \\ \mathbb{W}(\mathbb{W}(A)) \arrow[r, "w_{nm}"] & \mathbb{W}(A) \end{tikzcd}$$

which can again be simplified to

$$\begin{tikzcd} \mathbb{W}(A) \arrow[r, "F_n"] \arrow[dr, "F_{nm}"'] & \mathbb{W}(A) \arrow[d, "F_m"] \\ & \mathbb{W}(A) \end{tikzcd}$$

now this commutes by ???, hence we are finished. //

Claim. $\begin{tikzcd} \mathbb{W}(A) \arrow[d, "\Delta_A"'] \arrow[dr, "\mathrm{id}_{\mathbb{W}(A)}"] & \\ \mathbb{W}(\mathbb{W}(A)) \arrow[r, "\mathbb{W}(w_1)"'] & \mathbb{W}(A) \end{tikzcd}$ *commutes.*

Proof of claim. evaluate the ghost coordinates:

$$\begin{tikzcd} \mathbb{W}(A) \arrow[d, "\Delta_A"'] \arrow[dr, "\mathrm{id}_{\mathbb{W}(A)}"] \arrow[dd, dotted, bend right, "F"'] & \\ \mathbb{W}(\mathbb{W}(A)) \arrow[r, "\mathbb{W}(w_1)"'] \arrow[d, dotted, "w"] & \mathbb{W}(A) \arrow[d, "w"] \\ \mathbb{W}(A)^{\mathbb{N}} \arrow[r, dotted, "w_1^{\mathbb{N}}"] & A^{\mathbb{N}} \end{tikzcd}$$

we can then simplify to

$$\begin{tikzcd} \mathbb{W}(A) \arrow[d, "F"'] \arrow[dr, "w"] & \\ \mathbb{W}(A)^{\mathbb{N}} \arrow[r, "w_1^{\mathbb{N}}"'] & A^{\mathbb{N}} \end{tikzcd}$$

now it suffices to show for all n that

$$\begin{tikzcd} \mathbb{W}(A) \arrow[d, "F_n"'] \arrow[dr, "w_n"] & \\ \mathbb{W}(A) \arrow[r, "w_1"'] & A \end{tikzcd}$$

commutes, which is true by ??? ($\varepsilon = w_1$).

//

Claim.
$$\begin{tikzcd} & \mathbb{W}(A) \arrow[dl, "\mathrm{id}_{\mathbb{W}(A)}"'] \arrow[d, "\Delta_A"] \\ \mathbb{W}(\mathbb{W}(A)) & \mathbb{W}(A) \arrow[l, "\varepsilon_{\mathbb{W}(A)}"] \end{tikzcd}$$
commutes.

Proof of claim. Let $a \in \mathbb{W}(A)$.
$\varepsilon(\Delta_A(a)) = w_1(\Delta_A(a)) = F_1(a) = a$, since $F_1 = \mathrm{id}_{\mathbb{W}(A)}$. //

This concludes the proof. □

The Teichmüller map induces a morphism of comonads

maybe ins Kapitel Comonaden als Beispiel schieben

We now consider another example of a comonad; the *free monoid comonad.*

Definition 3.22 (monoid ring). Let R be a ring and let G be a monoid. The *monoid ring* of G over R, denoted $R[G]$ or RG is the set of formal finite sums $\sum_{g \in G} r_g \cdot g$ with addition and multiplication defined by:

$$\sum_{g \in G} r_g \cdot g + \sum_{g \in G} s_g \cdot g := \sum_{g \in G} (r_g + s_g) \cdot g$$
$$\sum_{g \in G} r_g \cdot g \cdot \sum_{g \in G} s_g \cdot g := \sum_{g \in G} \Big(\sum_{k \cdot l = g} r_k \cdot s_l \Big) \cdot g$$

Example 6. $R = \mathbb{R}, G = \{x^n \mid n \in \mathbb{N}\} \implies RG = \mathbb{R}[X]$

Remark 3.23. $R[G]$ together with the ring homomorphism $\alpha \colon R \to R[G]; r \mapsto r \cdot 1$ and the monoid homomorphism $\beta \colon G \to R[G]; g \mapsto 1 \cdot g$ enjoys the following universal property:

$$\alpha(r) \cdot \beta(g) = \beta(g) \cdot \alpha(r) \quad \forall r \in R, g \in G$$

and if (S, α', β') is another such triple with $\alpha'(r) \cdot \beta'(g) = \beta'(g) \cdot \alpha'(r) \quad \forall r \in R, g \in G$, there is a unique monoid homomorphism $\gamma \colon R[G] \to S$ such that the following diagram commutes:

$$\begin{tikzcd} & S & \\ R \arrow[ur, "\alpha'"] \arrow[r, "\alpha"'] & R[G] \arrow[u, dotted, "\gamma"] & G \arrow[ul, "\beta'"'] \arrow[l, "\beta"] \end{tikzcd}$$

Here, γ is defined by $\sum_{g \in G} r_g \cdot g \mapsto \sum_{g \in G} \alpha'(r_g) \cdot \beta'(g)$.

Example 7. Let S be a ring, G be a monoid. Since there is a unique ring homomorphism $\mathbb{Z} \to S$, each monoid homomorphism $G \to S$ induces a unique ring homomorphism $\mathbb{Z}G \to S$ such that the following commutes:

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & S \\ & \mathbb{Z}G \arrow[u] \end{tikzcd}$$

Now if H is another monoid and $f \colon G \to H$ a monoid morphism, $G \xrightarrow{f} H \to \mathbb{Z}H$ is a monoid homomorphism, hence it extends uniquely to $f \colon \mathbb{Z}G \to \mathbb{Z}H$, $\sum_{g \in G} r_g \cdot g \mapsto \sum_{g \in G} r_g \cdot f(g)$.
In this way, the free monoid ring construction over $\mathbb{Z}$ is functorial.

Let $G \colon \mathbf{CRing} \to \mathbf{CMon}$, $(R, +, \cdot) \mapsto (R, \cdot)$ be the forgetful functor and let $F \colon \mathbf{CMon} \to \mathbf{CRing}$ be the *free monoid ring functor*, $G \mapsto \mathbb{Z}G$.

Proposition 3.24 *There is an adjoint situation*
$$\begin{tikzcd} \mathbf{CMon} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{CRing} \arrow[l, bend left, "G"] \end{tikzcd}$$

Now consider the *teichmüller map* $\tau\colon A \to \mathbb{W}(A); a \mapsto (a, 0, 0, 0, \dots)$. τ is multiplicative and preserves the unit, hence it extends uniquely to a ring homomorphism

$$\tau\colon \mathbb{Z}A \to \mathbb{W}(A)$$

Theorem 3.25 $\tau\colon \mathbb{Z}A \to \mathbb{W}(A)$ *is a morphism of comonads.*