1 Adjoint situations

einleitender Satz

1.0.1 Proposition Given two functors $A \xrightarrow{F} B$, $B \xrightarrow{G} A$, the following are equivalent:

(a) $\exists \eta : id_B \to GF$ and $\epsilon : FG \to id_A$ such that $\forall a \in Ob(A), b \in Ob(B)$ the following two diagrams commute:

$$\begin{tikzcd} A \arrow[r, "F(\eta_b)"] \arrow[dr, "\mathrm{id}_{F(b)}"'] & B \arrow[d, "\epsilon_{F(b)}"] \\ & C \end{tikzcd} \text{ and } \begin{tikzcd} A \arrow[r, "F(\eta_b)"] \arrow[dr, "\mathrm{id}_{F(b)}"'] & B \arrow[d, "\epsilon_{F(b)}"] \\ & C \end{tikzcd}$$

(b) $\forall a \in A, b \in B$ there is a bijection

$$\phi_{a,b} : \hom(F(b), a) \to \hom(b, G(a))$$

which is natural in a and b, i.e. for $p : a \to a'$:

$$\begin{tikzcd} \hom(F(b), a) \arrow[r] \arrow[d] & \hom(b, G(a)) \arrow[d] \\ \hom(F(b), a') \arrow[r] & \hom(b, G(a')) \end{tikzcd}$$

and for $q : b \to b'$:

$$\begin{tikzcd} \hom(F(b'), a) \arrow[r] \arrow[d] & \hom(b', G(a)) \arrow[d] \\ \hom(F(b), a) \arrow[r] & \hom(b, G(a)) \end{tikzcd}$$

Beweis. $(a) \implies (b)$:
define

$$\phi_{a,b} : \hom(F(b), a) \to \hom(b, G(a))$$

by $g \mapsto G(g) \circ \eta_b$ for $g : F(b) \to a$ □