1 Adjunctions

Adjunctions are a way to compare categories. Two categories $\mathbf{A}$ and $\mathbf{B}$ are said to be *isomorphic*, denoted $\mathbf{A} \cong \mathbf{B}$, if there exist functors $G\colon \mathbf{A} \longrightarrow \mathbf{B}, F\colon \mathbf{B} \longrightarrow \mathbf{A}$ with $FG = \mathrm{id}_A, GF = \mathrm{id}_B$. This condition is too strict to provide us with many examples, which is why there is a different notion: $\mathbf{A}$ and $\mathbf{B}$ are said to be *equivalent*, denoted $\mathbf{A} \simeq \mathbf{B}$, if $FG \cong \mathrm{id}_A, GF \cong \mathrm{id}_B$ via natural isomorphisms $\alpha\colon FG \longrightarrow \mathrm{id}_A$, $\beta\colon GF \longrightarrow \mathrm{id}_B$. Equivalent categories are essentially the same, all categorical properties, like for example initial objects are preserved under equivalence. But there is an even less strict relation between categories, which was first introduced by Daniel Kan in [Kan58] and which formalizes the notion of a free object: consider the categories $\mathbf{Set}$ and $\mathbf{Vect_K}$ for a fixed field K. The two categories can't be equivalent, since $\mathbf{Vect_K}$ has a zero object while $\mathbf{Set}$ doesn't. But there still is a connection between them:

A vector space is a set with additional structure and linear maps are maps of sets which respect these structures. A different way to say this is that there is a forgetful functor $U\colon \mathbf{Vect_K} \longrightarrow \mathbf{Set}$. We can also go in the opposite direction, because there is a "natural" way to make a set X into a vector space:

- We form the set FX of all formal linear combinations of elements of X, i.e. all elements of the form $\sum_{i=1}^n a_i x_i$

- we define addition and scalar multiplication by:

$$\lambda \cdot \Big(\sum_{i=1}^n a_i x_i\Big) := \sum_{i=1}^n (\lambda \cdot a_i) \cdot x_i$$

$$\sum_{i=1}^n a_i x_i + \sum_{i=1}^n b_i x_i := \sum_{i=1}^n (a_i + b_i) x_i$$

  (note that we can assume linear combinations in FX to have the same length, since we can just add 0.)

This gives a vector space, the *free vector space over* X which has X as a basis. Now the universal property of a vector space states that each map $X \longrightarrow U(W)$ extends uniquely to a map $F(X) \longrightarrow W$ and every map $F(X) \longrightarrow W$ gives a map $X \longrightarrow U(W)$ by restriction. This amounts to a bijection

$$\mathrm{Hom}_{\mathbf{Vect_K}}(F(X), W) \cong \mathrm{Hom}_{\mathbf{Set}}(X, U(W))$$

which is also natural in a sense we will discuss later. The functors F and U form the first example and one of the best known examples of an adjunction.

1.1 Definition of Adjunctions

We will start by giving two equivalent definitions of an adjunction, where the first one is especially useful when it comes to monads, while the second, more standard one, is easier to find examples.

Proposition 1.1 *Given two functors* $\mathbf{B} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{A}$ *the following are equivalent:*

(a) *There are natural transformations* $\eta\colon \mathrm{id}_B \Rightarrow GF$ *and* $\varepsilon\colon FG \Rightarrow \mathrm{id}_A$ *such that for all objects* a *of* $\mathbf{A}$, b *of* $\mathbf{B}$ *the following two diagrams commute:*

$$\begin{tikzcd} F(b) \arrow[r, "F(\eta_b)"] \arrow[dr, "\mathrm{id}_{F(b)}"'] & FGF(b) \arrow[d, "\varepsilon_{F(b)}"] \\ & F(b) \end{tikzcd} \qquad \begin{tikzcd} G(a) \arrow[r, "\eta_{G(a)}"] \arrow[dr, "id_{G(a)}"'] & GFG(a) \arrow[d, "G(\varepsilon_a)"] \\ & G(a) \end{tikzcd} \qquad \text{(triangle identity)}$$

(b) *There is a bijection*

$$\phi_{a,b}\colon \mathrm{Hom}_{\mathbf{A}}(F(b), a) \cong \mathrm{Hom}_{\mathbf{B}}(b, G(a))$$

for all objects a *of* $\mathbf{A}$ *and* b *of* $\mathbf{B}$*, which is natural in* a *and* b*.*

Naturality here means that for $p\colon a \to a'$ and for $q\colon b \to b'$ the following two diagrams commute:

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{A}}(F(b), a) \arrow[r, "\phi_{a,b}"] \arrow[d, "p\circ\_"] & \mathrm{Hom}_{\mathbf{B}}(b, G(a)) \arrow[d, "G(p)\circ\_"] \\ \mathrm{Hom}_{\mathbf{A}}(F(b), a') \arrow[r, "\phi_{a',b}"] & \mathrm{Hom}_{\mathbf{B}}(b, G(a')) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Hom}_{\mathbf{A}}(F(b'), a) \arrow[r, "\phi_{a,b'}"] \arrow[d, "\_\circ F(q)"] & \mathrm{Hom}_{\mathbf{B}}(b', G(a)) \arrow[d, "\_\circ q"] \\ \mathrm{Hom}_{\mathbf{A}}(F(b), a) \arrow[r, "\phi_{a,b}"] & \mathrm{Hom}_{\mathbf{B}}(b, G(a)) \end{tikzcd}$$

Proof: $(a) \Rightarrow (b)$: Define

$$\phi_{a,b}\colon \mathrm{Hom}_{\mathbf{A}}(F(b), a) \to \mathrm{Hom}_{\mathbf{B}}(b, G(a)) \quad \text{by} \quad \phi_{a,b}(g) = G(g) \circ \eta_b\colon b \to G(a)$$

$$\psi_{a,b}\colon \mathrm{Hom}_{\mathbf{B}}(b, G(a)) \to \mathrm{Hom}_{\mathbf{A}}(F(b), a) \quad \text{by} \quad \psi_{a,b}(f) = \varepsilon_a \circ F(f)\colon F(b) \to a$$

for $g\colon F(b) \to a$, $f\colon b \to G(a)$.

Claim : $\phi \circ \psi = \mathrm{id}$

Proof of claim. Let $f\colon b \to G(a)$.

$$\begin{aligned} \phi(\psi(f)) &= \phi(\varepsilon_a \circ F(f)) && \text{(Definition of } \psi) \\ &= G(\varepsilon_a \circ F(f)) \circ \eta_b && \text{(Definition of } \phi) \\ &= G(\varepsilon_a) \circ G(F(f)) \circ \eta_b && \text{(Functoriality of } G) \\ &= G(\varepsilon_a) \circ \eta_{G(a)} \circ f && \text{(Naturality of } \eta) \\ &= \mathrm{id}_{G(a)} \circ f = f && \text{(right triangle identity)} \end{aligned}$$

//

Claim : $\psi \circ \phi = \mathrm{id}$

Proof of claim.

$$\begin{aligned}
\psi(\phi(g)) &= \psi(G(g) \circ \eta_b) && \text{(Definition of } \phi) \\
&= \varepsilon_a \circ F(G(g) \circ \eta_b) && \text{(Definition of } \psi) \\
&= \varepsilon_a \circ F(G(g)) \circ F(\eta_b) && \text{(Functoriality of } F) \\
&= g \circ \varepsilon_{F(b)} \circ F(\eta_b) && \text{(Naturality of } \varepsilon) \\
&= g \circ \mathrm{id}_{F(b)} = g && \text{(left triangle identity)}
\end{aligned}$$

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Claim : $\phi_{a,b}$ is natural in a.

Proof of claim. Let $p\colon a \longrightarrow a'$. Then by functoriality of G we have:

$$G(p) \circ \phi_{a,b}(g) = G(p) \circ G(g) \circ \eta_b = G(p \circ g) \circ \eta_b = \phi_{a',b}(p \circ g).$$

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Claim : $\phi_{a,b}$ is natural in b.

Proof of claim. Let $q\colon b \longrightarrow b'$. Then by functoriality of G and naturality of η we have:

$$\phi_{a,b}(g \circ F(q)) = G(g \circ F(q)) \circ \eta_b = G(g) \circ GF(q) \circ \eta_b = G(g) \circ \eta_{b'} \circ q = \phi_{a,b'}(g) \circ q$$

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$(a) \Leftarrow (b)$: Define

$$\eta\colon id_{\mathrm{B}} \Rightarrow GF \quad \text{by} \quad \eta_b \coloneqq \phi_{F(b),b}(\mathrm{id}_{F(b)})\colon b \longrightarrow GF(b)$$

$$\varepsilon\colon FG \Rightarrow id_{\mathrm{A}} \quad \text{by} \quad \varepsilon_a \coloneqq \psi_{a,G(a)}(\mathrm{id}_{G(a)})\colon FG(a) \longrightarrow a$$

Claim : η is a natural transformation.

Proof of claim. For $q\colon b \longrightarrow b'$ we need to show that

$$\begin{array}{ccc}
b & \xrightarrow{\quad q \quad} & b' \\
\downarrow{\scriptstyle \eta_b} & & \downarrow{\scriptstyle \eta_{b'}} \\
GF(b) & \xrightarrow{GF(q)} & GF(b')
\end{array}$$

commutes. But using the naturality of ϕ (applied to $q\colon b \longrightarrow b'$ and $F(q)\colon Fb \longrightarrow Fb'$), we get

$$\begin{aligned}
\eta_{b'} \circ q = \phi(\mathrm{id}_{Fb'}) \circ q = \phi(\mathrm{id}_{Fb'} \circ F(q)) &= \phi(F(q) \circ \mathrm{id}_{Fb}) \\
= GF(q) \circ \phi(\mathrm{id}_{F(b)}) &= GF(q) \circ \eta_b.
\end{aligned}$$

//

Claim : ε is a natural transformation.

Proof of claim. For $p\colon a \to a'$ we need to show that

$$\begin{tikzcd} FGa \arrow[r, "FG(p)"] \arrow[d, "\varepsilon_a"] & FGa' \arrow[d, "\varepsilon_{a'}"] \\ a \arrow[r, "p"] & a' \end{tikzcd}$$

commutes. Since ϕ is natural, ψ is natural as well, meaning that the diagrams above where ϕ is replaced by ψ commute. Using the naturality of ψ in a (applied to $p\colon a \to a'$) and b (applied to $G(p)\colon Ga \to Ga'$), we get

$$\begin{aligned} p \circ \varepsilon_a = p \circ \psi(\mathrm{id}_{Ga}) &= \psi(G(p) \circ \mathrm{id}_{Ga}) = \psi(\mathrm{id}_{Ga'} \circ G(p)) \\ &= \psi(\mathrm{id}_{Ga'}) \circ FG(p) = \varepsilon_{a'} \circ FG(p). \end{aligned}$$

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Claim : η and ε satisfy the triangle identities.

Proof of claim.

$$\mathrm{id}_{F(b)} = \psi(\phi(\mathrm{id}_{F(b)})) = \psi(\eta_b) = \psi(\eta_b \circ \mathrm{id}_b) = \psi(\mathrm{id}_b) \circ F(\eta_b) = \varepsilon_{F(b)} \circ F(\eta_b)$$

$$\mathrm{id}_{G(a)} = \phi(\psi(\mathrm{id}_{G(a)})) = \phi(\varepsilon_a) = \phi(\mathrm{id}_a \circ \varepsilon_a) = G(\varepsilon_a) \circ \phi(\mathrm{id}_a) = G(\varepsilon_a) \circ \eta_a$$

using the definitions of η, ε and the naturality of ϕ and ψ. //

□

Definition (Adjunction)**.** Let $\mathbf{A}$ and $\mathbf{B}$ be categories. Then functors $F\colon \mathbf{B} \to \mathbf{A}$, $G\colon \mathbf{A} \to \mathbf{B}$ are said to form an *adjunction between* $\mathbf{A}$ *and* $\mathbf{B}$, if F and G satisfy the equivalent conditions of 1.1. In this case F is called *left-adjoint* to G and G is called *right-adjoint* to F.

We will denote the adjunction either by $F \dashv_\phi G\colon \mathbf{B} \rightleftarrows \mathbf{A}$ or by $F \,{}^{\eta}_{\varepsilon}\!\dashv\, G\colon \mathbf{B} \rightleftarrows \mathbf{A}$, sometimes even just $F \dashv G$ or $\begin{tikzcd} \mathbf{B} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{A} \arrow[l, bend left, "G"] \end{tikzcd}$, depending on the context. The natural transformations η and ε are called *unit* respectively *counit* of the adjunction.

The following is another reason why adjunctions are a very useful concept:

Remark (proven for example in [Rie17, Chapter 4.5])**.** Let $F \dashv G$ be an adjunction. Then

(a) G preserves limits

(b) F preserves colimits.

1.2 Examples

Example 1 (free-forgetful adjunction)**.** The example from the introduction is an example of a so called *free-forgetful adjunction*, where a functor is left-adjoint to a forgetful functor. These adjunctions arise for almost any algebraic structure: groups, semigroups, rings, monoids, etc. For example the *free group* F_S of a set S can be constructed as the set of all reduced words in the alphabet $S \cup S^{-1}$, with the group multiplication being concatenation followed by reduction. This construction is again functorial and enjoys a universal property, which can be reformulated as

$$\mathrm{Hom}_{\mathbf{Grp}}(F_S, G) \cong \mathrm{Hom}_{\mathbf{Set}}(S, U(G))$$

for every set S and every group G, where U is the forgetful functor $\mathbf{Grp} \to \mathbf{Set}$. Thus we again have an adjunction $F \dashv U \colon \mathbf{Set} \rightleftarrows \mathbf{Grp}$.

Example 2 (Tensor-Hom-Adjunction)**.** If A is a ring, then by the universal property of the tensor product of A-modules, for a fixed A-module N, there is a natural bijection

$$\mathrm{Hom}_{\mathbf{A}}(M \otimes_A N, P) \cong \mathrm{Hom}_{\mathbf{A}}(M, \mathrm{Hom}_{\mathbf{A}}(N, P))$$

for all A-modules M and P. This implies that the tensor-product is right-exact, since as a left-adjoint, it preserves colimits, so in particular cokernels.

Example 3 (Galois connection)**.** Let $(\mathbf{A}, \leq)$ and $(\mathbf{B}, \leq)$ be two partially ordered sets. A *monotone Galois connection* consists of two monotone functions $F \colon \mathbf{B} \to \mathbf{A}$ and $G \colon \mathbf{A} \to \mathbf{B}$ such that $\forall a \in \mathbf{A}, b \in \mathbf{B}$, we have

$$Fb \leq a \iff b \leq Ga \tag{$*$}$$

An *antitone Galois connection* consists of two antitone (order-reversing) functions $F \colon \mathbf{B} \to \mathbf{A}$ and $G \colon \mathbf{A} \to \mathbf{B}$ such that $\forall a \in \mathbf{A}, b \in \mathbf{B}$, we have

$$a \leq Fb \iff b \leq Ga \tag{$**$}$$

Now every partially ordered set $(\mathbf{P}, \leq)$ can be viewed as a category with objects the elements of $\mathbf{P}$ and a morphism between x and y if and only if $x \leq y$. Then if F and G are monotone, they are functors and $(*)$ can be restated as

$$\mathrm{Hom}_{\mathbf{A}}(Fb, a) \cong \mathrm{Hom}_{\mathbf{B}}(b, Ga)$$

thus a monotone Galois connection froms and adjunction between the categories $\mathbf{A}$ and $\mathbf{B}$ (the naturality diagrams alle commute, because there is at most one morphism between objects). For the antitone case, $(**)$ gives us

$$\begin{aligned} &\mathrm{Hom}_{\mathbf{A}}(a, Fb) \cong \mathrm{Hom}_{\mathbf{B}}(b, Ga) \\ \iff &\mathrm{Hom}_{\mathbf{A}^{\mathrm{op}}}(Fb, a) \cong \mathrm{Hom}_{\mathbf{B}}(b, Ga) \end{aligned}$$

and since F and G are antitone, we can view them as (covariant) functors $\mathbf{B} \rightleftarrows \mathbf{A}^{\mathbf{op}}$. Thus an antitone Galois connection forms an adjunction between $\mathbf{A}^{\mathbf{op}}$ and $\mathbf{B}$. Galois connections appear all over mathematics, here are a few examples:

1. (Convex sets): Let $\mathbf{C} := \{U \subseteq \mathbb{R}^n \mid U \text{ is convex}\}$ and $\mathbf{S} := \{V \subseteq \mathbb{R}^n\}$, both partially ordered by inclusion. Define two monotonic functions $F\colon \mathbf{S} \to \mathbf{C}$ and $G\colon \mathbf{C} \to \mathbf{S}$ by $F(V) = \langle V\rangle_c$, where $\langle\cdot\rangle_c$ denotes the convex hull operator, and $G(U) = U$. Then we have
$$\langle V\rangle_c \subseteq U \iff V \subseteq U$$
hence a monotone Galois connection.

2. (Algebraic geometry): Let k be an algebraically closed field, $n \in \mathbb{N}$. Then for $\mathcal{R} := \{I \subseteq k[X_1,\dots,X_n]\}$ and $\mathcal{A} := \{M \subseteq \mathbb{A}^n(k)\}$ there are inclusion-reversing maps $\mathcal{R} \underset{\mathcal{I}}{\overset{\mathcal{V}}{\rightleftarrows}} \mathcal{A}$ where $\mathcal{V}(I)$ is the *zero set* of I and $\mathcal{I}(M)$ is the *vanishing ideal* of M (*Hilbert's Nullstellensatz* deals precisely with this correspondence). In particular we have
$$M \subseteq \mathcal{V}(I) \iff I \subseteq \mathcal{I}(M)$$
hence an antitone Galois connection.

3. (Galois theory): Let L/K be a field extension, let $\mathbf{A} := \{K \subseteq M \subseteq L \mid M \text{ is a field}\}$ and $\mathbf{B} := \{G \subseteq Gal(L/K)\}$ where $Gal(L/K)$ is the group of automorphisms on L that fix K. For a subgroup $G \subseteq Gal(L/K)$ let L^G be the *fixed field* of G. Then there is an antitone Galois connection given by
$$E \mapsto Gal(L/E),\ G \mapsto L^G.$$

Example 4 (Coproduct $\dashv \Delta \dashv$ Product)**.** Even universal construction such as (co-)products can be seen as adjunctions: For a category $\mathbf{C}$ that has finite products and coproducts, we have the *diagonal functor* $\Delta\colon \mathbf{C} \to \mathbf{C}\times\mathbf{C}$, $\Delta X = (X,X)$, $\Delta(f) = (f,f)$ as well as the *product functor* $Prod\colon \mathbf{C}\times\mathbf{C} \to \mathbf{C}$, $Prod(X,Y) = X \times Y$, $Prod((f,g)\colon (X_1,Y_1) \to (X_2,Y_2)) = f \times g\colon X_1 \times Y_1 \to X_2 \times Y_2$ and the *coproduct functor Coprod*$\colon \mathbf{C}\times\mathbf{C} \to \mathbf{C}$, $Prod(X,Y) = X \sqcup Y$, $Coprod((f,g)\colon (X_1,Y_1) \to (X_2,Y_2)) = f \sqcup g\colon X_1 \sqcup Y_1 \to X_2 \sqcup Y_2$. Then the universal property of the product states that
$$\mathrm{Hom}_{\mathbf{C}\times\mathbf{C}}((A,A),(X,Y)) \cong \mathrm{Hom}_{\mathbf{C}}(A, X\times Y)$$
for every $A \in \mathbf{C}$, which shows $\Delta \dashv Prod$. Analogous one sees that $Coprod \dashv \Delta$.

2 Monads and Comonads

Monads were first introduced in 1958 under the name *standard construction* or *construction fondamentale* by Roger Godement in [God58, Appendix, 3.], where he used them for applications in sheaf cohomology. They were also used in algebraic topology and homotopy theory, for example in [Hub61]. In the early category theory-literature monads were called *triples*, other names were *monoid, dual standard construction* and *triad.* The name *monad* first appeared in [Bén67], the exact reason for this name being unclear today, although it surely inspired by *monoids*, which monads are related to. Monads are closely connected to adjunctions, as we will explore in this chapter, besides giving lots of examples, with many interesting examples coming from [Per21]. In computer science, monads play an important role in functional programming. This chapter is based on [Mac98, Chapter VI], which is the standard resource for first learning about monads and comonads. Some of the proofs are taken from [HST14, Chapter II.3] instead. Another great exposition is [Per21, Chapter 5].

2.1 Definition of Monads and Comonads

A central notion in algebra is that of a *monoid*, that is, a set M equipped with a map $\mu\colon M \times M \to M$; $(a, b) \mapsto a \cdot b$ (often called *multiplication*) and an element $e \in M$ such that the following two axioms hold:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \text{for all } a, b, c \in M. \qquad \text{(associativity)}$$

$$e \cdot a = a \cdot e = a \quad \text{for all } a \in M \qquad \text{(identity element)}$$

We can give an equivalent definition in terms of maps and commuting diagrams as follows: A *monoid* is a set M together with two functions

$$\mu\colon M \times M \to M, \quad e\colon \{*\} \to M$$

such that the following diagrams commute:

$$\begin{tikzcd} M \times M \times M \arrow[r, "\mathrm{id} \times \mu"] \arrow[d, "\mu \times \mathrm{id}"] & M \times M \arrow[d, "\mu"] \\ M \times M \arrow[r, "\mu"] & M \end{tikzcd} \qquad \begin{tikzcd} \{*\} \times M \arrow[r, "e \times \mathrm{id}"] \arrow[dr, "l"'] & M \times M \arrow[d, "\mu"] & M \times \{*\} \arrow[l, "\mathrm{id} \times e"'] \arrow[dl, "r"] \\ & M & \end{tikzcd}$$

where id is the identity on m, and l and r are the canonical bijections

$$l\colon \{*\} \times M \to M;\ l(*, m) = m$$
$$r\colon M \times \{*\} \to M;\ r(m, *) = m.$$

Explicitly, the first diagram means that for all $a, b, c \in M$:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \text{for all } a, b, c \in M.$$

which is verbatim the associativity axiom, the second diagram means that for all $m \in M$:

$$e(*) \cdot m = l(*, m) = m = r(m, *) = m \cdot e(*)$$

which is clearly the identity element axiom for the element $e(*)$. This motivates the following definition:

Definition (monad). A *monad* (T, η, μ) in a category $\mathbf{X}$ consists of

- an endofunctor $T\colon \mathbf{X} \to \mathbf{X}$
- a natural transformation $\eta\colon \mathrm{id}_{\mathbf{X}} \Rightarrow T$
- a natural transformation $\mu\colon T^2 \Rightarrow T$

such that the following diagrams commute:

$$\begin{tikzcd} T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\ T^2 \arrow[r, Rightarrow, "\mu"] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "\mathrm{id}_T"'] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "T\eta"'] \arrow[dl, Rightarrow, "\mathrm{id}_T"] \\ & T & \end{tikzcd}$$

(associativity) (unitality)

In terms of components, associativity and unitality mean that for every object x of $\mathbf{X}$ the following diagrams commute:

$$\begin{tikzcd} T(T(Tx)) \arrow[r, "T(\mu_x)"] \arrow[d, "\mu_{Tx}"'] & T(Tx) \arrow[d, "\mu_x"] \\ T(Tx) \arrow[r, "\mu_x"] & Tx \end{tikzcd} \qquad \begin{tikzcd} Tx \arrow[r, "\eta_{Tx}"] \arrow[dr, "\mathrm{id}_{Tx}"'] & T(Tx) \arrow[d, "\mu_x"] & Tx \arrow[l, "T(\eta_x)"'] \arrow[dl, "\mathrm{id}_{Tx}"] \\ & Tx & \end{tikzcd}$$

(associativity) (unitality)

Remark. In general, if $\mathbf{A}, \mathbf{B}, \mathbf{C}, \mathbf{D}$ are categories, $G\colon \mathbf{A} \to \mathbf{B}$, $S, S'\colon \mathbf{B} \to \mathbf{C}$ and $F\colon \mathbf{C} \to \mathbf{D}$ are functors and $\sigma\colon S \to S'$ is a natural transformation, then

$$F\sigma G\colon F \circ S \circ G \to F \circ S' \circ G,$$

defined by $(F\sigma G)_x \coloneqq F(\sigma_{Gx})$ is again a natural transformation. In particular, if we take F or G to be identity functors, we get the definition of $T\mu$ and μT above.

Example 5 (Partial order). Recall: A *partial order* $(\mathbf{P}, \leq)$ is a category with $\mathbf{P}$ as objects and a morphism between X and Y iff $X \leq Y$. A functor $T\colon \mathbf{P} \to \mathbf{P}$ is a monotone function $\mathbf{P} \to \mathbf{P}$. Now the existence of the natural transformation η is equivalent to $x \leq Tx$ and the existence of μ is equivalent to $T(Tx) \leq Tx$ $\forall x \in \mathbf{P}$ because there is at most one morphism $x \to y$, so the neccessary diagrams commute trivially. Then:

$$\begin{aligned} x \leq Tx &\implies Tx \leq T(Tx) \\ T(Tx) \leq Tx &\implies Tx = T(Tx) \end{aligned}$$

so a monad T in a partial order $\mathbf{P}$ is a *closure operation* in $\mathbf{P}$, i.e. a monotone function $T\colon \mathbf{P} \to \mathbf{P}$ with $x \leq Tx$ and $T(Tx) = Tx\ \forall x \in \mathbf{P}$.
Now every topological space X induces a partial order $\mathbf{P} = (\mathscr{P}(X), \subseteq)$. Here an example for a closure operation is taking the topological closure $A \mapsto \overline{A}$, since it holds for all $A \subseteq X$ that $A \subseteq \overline{A}$ and $\overline{\overline{A}} = \overline{A}$.

Example 6 (M-action monad)**.** Let $(M, \cdot, 1)$ be a monoid. Then for each set X we can form the set $X \times M$ and for a map $f\colon X \to Y$ we have a map $f \times \mathrm{id}_M\colon X \times M \to Y \times M; (x, m) \mapsto (f(x), m)$. This is functorial and the functor canonically has the structure of a monad, induced by the monoid structure of M.

- The unit η is defined by $\eta_X\colon X \to X \times M; x \mapsto (x, 1)$
- The multiplication μ is defined by $\mu_X\colon X \times M \times M \to X \times M; (x, m, n) \mapsto (x, m \cdot n)$

These are clearly natural maps and the monad axioms follow directly from the monoid axioms for M, if we look at the corresponding diagrams:

$$\begin{tikzcd} X \times M \times M \times M \arrow[r, "\mu_X \times \mathrm{id}_M"] \arrow[d, "\mu_{X \times M}"] & X \times M \times M \arrow[d, "\mu_X"] \\ X \times M \times M \arrow[r, "\mu_X"] & X \times M \end{tikzcd} \qquad \begin{tikzcd} X \times M \arrow[r, "\eta_X \times M"] \arrow[dr, "\mathrm{id}_{X \times M}"'] & X \times M \times M \arrow[d, "\mu_X"] & X \times M \arrow[l, "\eta_X \times \mathrm{id}_M"'] \arrow[dl, "\mathrm{id}_{X \times M}"] \\ & X \times M & \end{tikzcd}$$

The associativity axiom means that $(m \cdot n) \cdot k = m \cdot (n \cdot k)$ which is just the associativity axiom for the monoid M, while unitality means that $1 \cdot m = m = m \cdot 1$ which holds by the identity element axiom for M. We will call this monad on **Set** the *M-action monad*, the reason for this name will be clear once we look at it's algebras, see Section 2.2.

Example 7 (Maybe monad)**.** The *Maybe monad* $Y\colon \mathbf{Set} \to \mathbf{Set}$ is defined by $X \mapsto X \sqcup \{*\}$ where $f\colon X \to Y$ gets mapped to the function $Y(f)\colon X \sqcup \{*\} \to Y \sqcup \{*\}$ which maps x to $f(x)$ and $*$ to $*$. Unit and multiplication are defined to be the natural maps

- $\eta_X\colon X \to X \sqcup \{*\}; x \mapsto x$
- $\mu_X\colon X \sqcup \{*_1\} \sqcup \{*_2\} \to X \sqcup \{*\}; x \mapsto x, *_1 \mapsto *, *_2 \mapsto *$

An easy computation shows that the monads axioms are indeed satisfied, the reason being that in both diagrams, $x \in X$ is always mapped to itself, while all extra points $*_1, *_2, *_3$ get mapped to the extra point $*$.

A *comonad* in $\mathbf{A}$ is a monad in $\mathbf{A}^{op}$. Explicitly:

Definition (comonad). A *comonad* (L, ε, ω) in a Category $\mathbf{A}$ consists of

- an endofunctor $L\colon \mathbf{A} \to \mathbf{A}$
- a natural transformation $\varepsilon\colon L \Rightarrow \mathrm{id}_{\mathbf{A}}$
- a natural transformation $\omega\colon L \Rightarrow L^2$

such that the following diagrams commute:

$$\begin{tikzcd} L \arrow[r, Rightarrow, "\omega"] \arrow[d, Rightarrow, "\omega"'] & L^2 \arrow[d, Rightarrow, "L\omega"] \\ L^2 \arrow[r, Rightarrow, "\omega L"] & L^3 \end{tikzcd}$$

(coassociativity)

$$\begin{tikzcd} & L \arrow[dl, Rightarrow, "\mathrm{id}_L"'] \arrow[d, Rightarrow, "\omega"] \arrow[dr, Rightarrow, "\mathrm{id}_L"] & \\ L & L^2 \arrow[l, Rightarrow, "\varepsilon L"'] \arrow[r, Rightarrow, "L\varepsilon"] & L \end{tikzcd}$$

(counitality)

In terms of components, this means that for every object x of $\mathbf{A}$ the following diagrams commute:

$$\begin{tikzcd} Lx \arrow[r, "\omega_x"] \arrow[d, "\omega_x"'] & L(Lx) \arrow[d, "L(\omega_x)"] \\ L(Lx) \arrow[r, "\omega_{Lx}"] & L(L(Lx)) \end{tikzcd}$$

(coassociativity)

$$\begin{tikzcd} & Lx \arrow[dl, "\mathrm{id}_{Lx}"'] \arrow[d, "\omega_x"] \arrow[dr, "\mathrm{id}_{Lx}"] & \\ Lx & L(Lx) \arrow[l, "\varepsilon_{Lx}"'] \arrow[r, "L(\varepsilon_x)"] & Lx \end{tikzcd}$$

(counitality)

Example 8 (Reader comonad). Let E be a set. Define a functor $C_E\colon \mathbf{Set} \to \mathbf{Set}$ by $C_E(X) = X \times E$ and, given $f\colon X \to Y$, $C_E(f) = f \times \mathrm{id}_E\colon X \times E \to Y \times E$. We can view E as " extra information" and give C_E a comonadic structure as follows:

- the counit $\varepsilon_X\colon X \times E \to X; (x, e) \mapsto x$ "forgets the extra information"
- the comultiplication $\omega_X\colon X \times E \to X \times E \times E; (x, e) \mapsto (x, e, e)$ "copies the extra information".

Now the comonad axioms say that the following diagrams have to commute:

$$\begin{tikzcd} X \times E \arrow[r, "\omega_X"] \arrow[d, "\omega_X"'] & X \times E \times E \arrow[d, "\omega_X \times \mathrm{id}_E"] \\ X \times E \times E \arrow[r, "\omega_{X \times E}"] & X \times E \times E \times E \end{tikzcd}$$

$$\begin{tikzcd} & X \times E \arrow[dl, "\mathrm{id}_{X \times E}"'] \arrow[d, "\omega_X"] \arrow[dr, "\mathrm{id}_{X \times E}"] & \\ X \times E & X \times E \times E \arrow[l, "\varepsilon_{X \times E}"'] \arrow[r, "\varepsilon_X \times \mathrm{id}_E"] & X \times E \end{tikzcd}$$

The first diagram commutes, because for a tuple (x, e, e), copying the second or third element produces the same tuple. The second diagram commutes, because copying the extra information and the deleting either one of the copies gives the same result. The resulting comonad $(C_E, \varepsilon, \omega)$ on **Set** is called the *reader comonad*. Note that as a functor, it is almost the same as the *M-action comonad*, but we gave it kind of a dual structure.

We now consider another example of a comonad; the *free monoid ring comonad*.

Definition (monoid ring)**.** Let R be a ring and let G be a monoid. The *monoid ring* of G over R, denoted $R[G]$ or RG is the set of formal finite sums $\sum_{g\in G} r_g \cdot g$ with addition and multiplication defined by:

$$\Big(\sum_{g\in G} r_g \cdot g\Big) + \Big(\sum_{g\in G} s_g \cdot g\Big) := \sum_{g\in G} (r_g + s_g) \cdot g$$
$$\Big(\sum_{g\in G} r_g \cdot g\Big) \cdot \Big(\sum_{g\in G} s_g \cdot g\Big) := \sum_{g\in G} \Big(\sum_{k\cdot l=g} r_k \cdot s_l\Big) \cdot g$$

Example 9. $R = \mathbb{R}, G = \{x^n \mid n \in \mathbb{N}\} \implies RG = \mathbb{R}[X]$

Remark. $R[G]$ together with the ring homomorphism $\alpha\colon R \to R[G]; r \mapsto r \cdot 1$ and the monoid homomorphism $\beta\colon G \to R[G]; g \mapsto 1 \cdot g$ enjoys the following universal property:

$$\alpha(r) \cdot \beta(g) = \beta(g) \cdot \alpha(r) \quad \forall r \in R, g \in G$$

and if (S, α', β') is another such triple with $\alpha'(r) \cdot \beta'(g) = \beta'(g) \cdot \alpha'(r) \quad \forall r \in R, g \in G$, there is a unique ring homomorphism $\gamma\colon R[G] \to S$ such that the following diagram commutes:

$$\begin{tikzcd} & S & \\ R \arrow[ur, "\alpha'"] \arrow[r, "\alpha"'] & R[G] \arrow[u, dotted, "\gamma"] & G \arrow[ul, "\beta'"'] \arrow[l, "\beta"] \end{tikzcd}$$

Here, γ is defined by $\sum_{g\in G} r_g \cdot g \mapsto \sum_{g\in G} \alpha'(r_g) \cdot \beta'(g)$.

Example 10. Let S be a ring, G be a monoid. Since there is a unique ring homomorphism $\mathbb{Z} \to S$, each monoid homomorphism $G \to S$ induces a unique ring homomorphism $\mathbb{Z}G \to S$ such that the following commutes:

$$\begin{tikzcd} G \arrow[r] \arrow[dr] & S \\ & \mathbb{Z}G \arrow[u] \end{tikzcd}$$

Now if H is another monoid and $f\colon G \to H$ a monoid morphism, $G \xrightarrow{f} H \to \mathbb{Z}H$ is a monoid homomorphism, hence it extends uniquely to $f\colon \mathbb{Z}G \to \mathbb{Z}H$, $\sum_{g\in G} r_g \cdot g \mapsto \sum_{g\in G} r_g \cdot f(g)$.
In this way, the free monoid ring construction over $\mathbb{Z}$ is functorial.

Let $G\colon \mathbf{CRing} \to \mathbf{CMon}$, $(R, +, \cdot) \mapsto (R, \cdot)$ be the forgetful functor and let $F\colon \mathbf{CMon} \to \mathbf{CRing}$ be the functor $M \mapsto \mathbb{Z}M$. Then the composition $\mathbb{Z}[\_] := F \circ G\colon \mathbf{CRing} \to \mathbf{CRing}$ is the functor $R \mapsto \mathbb{Z}R$, which we call the *free monoid ring functor*.

Claim : $\mathbb{Z}[\_]$ is a comonad on **CRing**.

Proof: Define the counit and comultiplication by

$$\begin{aligned} \varepsilon_R\colon \mathbb{Z}R &\to R \\ \sum_{r\in R} a_r \cdot [r] &\mapsto \sum_{r\in R} a_r \cdot r \end{aligned} \qquad\qquad \begin{aligned} \omega_R\colon \mathbb{Z}R &\to \mathbb{Z}\mathbb{Z}R \\ \sum_{r\in R} a_r \cdot [r] &\mapsto \Big[\sum_{r\in R} a_r \cdot [r]\Big] \end{aligned}$$

those are clearly natural and the following diagrams commute:

$$\begin{tikzcd}
\mathbb{Z}R \arrow[r, "\omega_x"] \arrow[d, "\omega_x"'] & \mathbb{Z}\mathbb{Z}R \arrow[d, "L(\omega_x)"] \\
\mathbb{Z}\mathbb{Z}R \arrow[r, "\omega_{Lx}"] & \mathbb{Z}\mathbb{Z}\mathbb{Z}R
\end{tikzcd}
\qquad
\begin{tikzcd}
 & \mathbb{Z}R \arrow[dl, "\mathrm{id}_{Lx}"'] \arrow[d, "\omega_x"] \arrow[dr, "\mathrm{id}_{Lx}"] & \\
\mathbb{Z}R & \mathbb{Z}\mathbb{Z}R \arrow[l, "\varepsilon_{Lx}"'] \arrow[r, "L(\varepsilon_x)"] & \mathbb{Z}R
\end{tikzcd}$$

□

Remark. We can define a variation of this, by setting $\underline{\mathbb{Z}}R := \mathbb{Z}R/(0)$ where $(0) = \{r \cdot 0 \mid r \in \mathbb{Z}R\}$ is the ideal generated by $0 \in R$.

Lemma 2.1 *Let T, T' be endofunctors on a category $\mathbf{X}$ and let $\delta\colon T \longrightarrow T'$ be a natural transformation. Then for every object x in $\mathbf{X}$, the following diagram commutes:*

$$\begin{tikzcd}
T(Tx) \arrow[r, "T(\delta_x)"] \arrow[d, "\delta_{Tx}"] & T(T'x) \arrow[d, "\delta_{T'x}"] \\
T'(Tx) \arrow[r, "T'(\delta_x)"] & T'(T'x)
\end{tikzcd}$$

this means

$$\delta T' \circ T\delta = T'\delta \circ \delta T \colon T^2 \Rightarrow (T')^2.$$

We denote this natural transformation by $\delta \otimes \delta$, since this is actually the monoidal product of morphisms in the monoidal category of endofunctors on $\mathbf{X}$.

Proof: $\delta_x\colon Tx \longrightarrow T'x$ is a morphism. Since $\delta\colon T \Rightarrow T'$ is natural transformation, the square commutes. □

Definition (Morphism of monads)**.** Let $\mathbf{X}$ be a category, let (T, η, μ) and (T', η', μ') be monads in $\mathbf{X}$. We say that a natural transformation $\delta\colon T \Rightarrow T'$ is a *morphism of monads* if it preserves the unit and the multiplication, i.e. the following diagrams commute:

$$\begin{tikzcd}
\mathrm{id}_X \arrow[r, Rightarrow, "\eta"] \arrow[dr, Rightarrow, "\eta'"'] & T \arrow[d, Rightarrow, "\delta"] \\
 & T'
\end{tikzcd}
\qquad
\begin{tikzcd}
T^2 \arrow[r, Rightarrow, "\mu"] \arrow[d, Rightarrow, "\delta\otimes\delta"'] & T \arrow[d, Rightarrow, "\delta"] \\
T'^2 \arrow[r, Rightarrow, "\mu'"] & T'
\end{tikzcd}$$

(unit-preserving) (multiplication-preserving)

Definition (Morphism of comonads)**.** Let $\mathbf{A}$ be a category, let (L, ε, ω) and $(L', \varepsilon', \omega')$ be comonads in $\mathbf{A}$. We say that a natural transformation $\delta\colon L \Rightarrow L'$ is a *morphism of monads* if it preserves the counit and the comultiplication, i.e. the following diagrams commute:

$$\begin{tikzcd} L \arrow[r, Rightarrow, "\delta"] \arrow[dr, Rightarrow, "\varepsilon"'] & L' \arrow[d, Rightarrow, "\varepsilon'"] \\ & \mathrm{id}_A \end{tikzcd} \qquad\qquad \begin{tikzcd} L \arrow[r, Rightarrow, "\omega"] \arrow[d, Rightarrow, "\delta"] & L^2 \arrow[d, Rightarrow, "\delta\otimes\delta"] \\ L' \arrow[r, Rightarrow, "\omega'"] & L'^2 \end{tikzcd}$$

(counit-preserving) (comultiplication-preserving)

Example 11. Consider the *subsingletons monad* $\mathbb{P}^1\colon \mathbf{Set} \to \mathbf{Set}$, which assigns to each set X the set of subsets of X containing *at most* one element, so an element of $\mathbb{P}^1(X)$ is either $\varnothing$ or a singleton $\{x\}$. For a function $f\colon X \to Y$, the induced function maps $\varnothing$ to $\varnothing$ and $\{x\}$ to $\{f(x)\}$ (compare this to the power set functor). If we define the unit η' by

$$\eta'_X\colon X \to \mathbb{P}^1(X); x \mapsto \{x\}$$

and the multiplication μ' by

$$\mu'_X\colon \mathbb{P}^1(\mathbb{P}^1(X)) \to \mathbb{P}^1(X); \{\{x\}\} \mapsto \{x\}, \{\varnothing\} \mapsto \varnothing, \varnothing \mapsto \varnothing$$

then the resulting monad looks really similar to the *Maybe monad*. This is not a coincidence: the map

$$\delta_X\colon X \sqcup \{*\} \to \mathbb{P}^1(X); x \mapsto \{x\}, * \mapsto \varnothing$$

gives a natural isomorphism $Y \Rightarrow \mathbb{P}^1$ which is an isomorphism of monads. Indeed: δ is clearly a natural bijection. It is left to show that the following diagrams commute for every set X:

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\eta'_X"'] & X \sqcup \{*\} \arrow[d, "\delta_x"] \\ & \mathbb{P}^1(X) \end{tikzcd} \qquad\qquad \begin{tikzcd} X \sqcup \{*_1\} \sqcup \{*_2\} \arrow[r, "\mu_X"] \arrow[d, "(\delta\otimes\delta)_X"] & X \sqcup \{*_1\} \arrow[d, "\delta_X"] \\ \mathbb{P}^1(\mathbb{P}^1(X)) \arrow[r, "\mu'_X"] & \mathbb{P}^1(X) \end{tikzcd}$$

The left diagram commutes, since we have $\delta_X(\eta_X(x)) = \delta_X(x) = \{x\} = \eta'_X(x)$. For the right diagram, first note that $(\delta \otimes \delta)_X = \mathbb{P}^1(\delta_X) \circ \delta_{X \sqcup \{*\}}$ is the map

$$\begin{aligned} x &\mapsto \{x\} &&\mapsto \{\{x\}\} \\ *_1 &\mapsto \{*_1\} &&\mapsto \{\varnothing\} \\ *_2 &\mapsto \varnothing &&\mapsto \varnothing \end{aligned}$$

so we compute that going the lower way resp. the upper way are the following maps:

$$\begin{aligned} x &\mapsto \{\{x\}\} &&\mapsto \{x\} \\ *_1 &\mapsto \{\varnothing\} &&\mapsto \varnothing \\ *_2 &\mapsto \varnothing &&\mapsto \varnothing \end{aligned} \qquad\qquad \begin{aligned} x &\mapsto x &&\mapsto \{x\} \\ *_1 &\mapsto *_1 &&\mapsto \varnothing \\ *_2 &\mapsto *_1 &&\mapsto \varnothing \end{aligned}$$

which are equal. This proves that δ is an isomorphism of monads.

The following theorem gives us a way to create many examples of monads and comonads. It was first proven in [Hub61].

Theorem 2.2 (Every adjunction induces a monad and a comonad) *Let* $F \overset{\eta}{\underset{\varepsilon}{\dashv}} G\colon \mathbf{B} \rightleftarrows \mathbf{A}$ *be an adjunction. Then* $(GF, \eta, G\varepsilon F)$ *is a monad on* B *and* $(FG, \varepsilon, F\eta G)$ *is a comonad on* A*, which we call the monad respectively comonad induced by the adjunction.*

Proof: We have to show that the first of the following diagrams commutes, but by removing G from the left and F from the right, it suffices to show that the right diagram commutes.

$$\begin{tikzcd}
GFGFGF \arrow[r, Rightarrow, "GFG\varepsilon F"] \arrow[d, Rightarrow, "G\varepsilon FGF"] & GFGF \arrow[d, Rightarrow, "G\varepsilon F"] \\
GFGF \arrow[r, Rightarrow, "G\varepsilon F"] & GF
\end{tikzcd}
\qquad
\begin{tikzcd}
FGFG \arrow[r, Rightarrow, "FG\varepsilon"] \arrow[d, Rightarrow, "\varepsilon FG"] & FG \arrow[d, Rightarrow, "\varepsilon"] \\
FG \arrow[r, Rightarrow, "\varepsilon"] & \mathrm{id}_B
\end{tikzcd}$$

The second diagram now commutes by 2.1. To show unitality we need to show that the following diagram commutes.

$$\begin{tikzcd}
GF \arrow[r, Rightarrow, "\eta GF"] \arrow[dr, Rightarrow, "\mathrm{id}_{GF}"'] & GFGF \arrow[d, Rightarrow, "G\varepsilon F"] & GF \arrow[l, Rightarrow, "GF\eta"'] \arrow[dl, Rightarrow, "\mathrm{id}_{GF}"] \\
& GF &
\end{tikzcd}$$

but this is essentially the diagrams stating the left and right triangle identity for the adjunction after applying F respectively G. The proof that $(FG, \varepsilon, F\eta G)$ is a comonad on A is dual. ◻

Example 12 (Galois connection). Recall the examples from the first section:

- (Convex sets): The monotone Galois connection $\mathbf{S} \underset{G}{\overset{F}{\rightleftarrows}} \mathbf{C}$, where F is the convex hull operator and the G is the forgetful map, induces the closure operator $GF = \langle\cdot\rangle_c\colon S \to S$, which is indeed monotone. The induced interior operator is $FG = \mathrm{id}_{\mathbf{C}}$.
- (Algebraic geometry): Note that for the antitone Galois connection $\mathcal{R} \underset{\mathcal{I}}{\overset{\mathcal{V}}{\rightleftarrows}} \mathcal{A}$ we get an induced *co*monad on $\mathcal{A}^{op}$, which is just a monad on $\mathcal{A}$. Therefore the connection induces two closure operators, $\mathcal{V}\mathcal{I} = \overline{(\_)}, \mathcal{I}\mathcal{V} = \sqrt{(\_)}$, where $\overline{(\_)}$ is the *Zariski closure* and $I \mapsto \sqrt{I}$ is taking the *radical ideal* of I.

Example 13 (Abelianization). The functor $(\_)^{ab}\colon \mathbf{Grp} \to \mathbf{Grp}$, $G^{ab} = G/[G:G]$, where $[G:G] = \langle\{a^{-1}b^{-1}ab \mid a, b \in G\}\rangle$ is a monad, induced by the adjunction $(\_)^{ab} \dashv U\colon \mathbf{Grp} \rightleftarrows \mathbf{AbGrp}$, where U is the forgetful functor. The unit of the adjunction is given by $\eta_G\colon G \to G/[G:G]; g \mapsto [g]$ and since for every abelian group N we have $[N:N] = \langle 1\rangle$, the counit is given by $\varepsilon_N\colon N/\langle 1\rangle \to N$; $[n] \mapsto n$, it is then easy to check that the triangle identities are satisfied.

Now that we know that every adjunction induces a monad, one may ask, if the converse is true, that is if every monad is induced by an adjunction. This can be reformulated to asking if the following category is non-empty:

Definition (Category of T-inducing adjunctions)**.** Let $\mathbf{X}$ be a category, $T = (T, \eta, \mu)$ a monad on X. The *category of T-inducing adjunctions*, denoted $Adj(T)$, has as objects adjunctions $F \overset{\eta}{\underset{\varepsilon}{\dashv}} G\colon \mathbf{X} \rightleftarrows \mathbf{Y}$ for some category $\mathbf{Y}$ (Note that the unit η is fixed for every adjunction). A morphism between two adjunctions $F \overset{\eta}{\underset{\varepsilon}{\dashv}} G\colon \mathbf{X} \rightleftarrows \mathbf{Y}$ and $F' \overset{\eta}{\underset{\varepsilon'}{\dashv}} G'\colon \mathbf{X} \rightleftarrows \mathbf{Y}'$ is a functor $H\colon \mathbf{Y} \to \mathbf{Y}'$ making the following diagram commute:

$$\begin{tikzcd}
 & Y \arrow[dd, "H"] \arrow[dl, bend left, "G"] \\
X \arrow[ur, bend left, "F"] \arrow[dr, bend left, "F'"] \arrow[ur, phantom, "\perp"] \arrow[dr, phantom, "\perp"] & \\
 & Y' \arrow[ul, bend left, "G'"]
\end{tikzcd}$$

where a diagram of this form is said to commute if the F-diagram and the G-diagram commute, i.e. we have $HF = F'$ and $G'H = G$.

We will see that indeed there exist objects in this category and there are even multiple ways to induce a given monad T. The first one is a construction called the *Eilenberg-Moore-Category* due to S. Eilenberg and J. Moore in [EM65].

2.2 The Eilenberg-Moore-Category of a Monad

Definition (Eilenberg-Moore-Category)**.** Let $T = (T, \eta, \mu)$ be a monad in a category $\mathbf{X}$. A T*-algebra* is a pair (x, h) where x is an object of $\mathbf{X}$ and $h\colon Tx \to x$ is a morphism such that the following diagrams commute:

$$\begin{tikzcd}
T^2x \arrow[r, "Th"] \arrow[d, "\mu_x"] & Tx \arrow[d, "h"] \\
Tx \arrow[r, "h"] & x
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
x \arrow[r, "\eta_x"] \arrow[dr, "id_x"'] & Tx \arrow[d, "h"] \\
 & x
\end{tikzcd}$$

We call h the *structure map* of (x, h). A *morphism of* T*-algebras* $f\colon (x, h) \to (x', h')$ is an arrow $f\colon x \to x'$ such that

$$\begin{tikzcd}
Tx \arrow[r, "Tf"] \arrow[d, "h"] & Tx' \arrow[d, "h'"] \\
x \arrow[r, "f"] & x'
\end{tikzcd}$$

commutes. The set of all T-algebras together with their morphisms clearly form a category, which is called the *Eilenberg-Moore-category* and denoted by $\mathbf{X}^T$.

Example 14 (M-action monad)**.** A T_M-algebra is a set X together with a map $h\colon X \times M \to X$ such that

$$\begin{tikzcd}
X \times M \times M \arrow[r, "h \times \mathrm{id}_M"] \arrow[d, "\mu_X"] & X \times M \arrow[d, "h"] \\
X \times M \arrow[r, "h"] & X
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X \arrow[r, "\eta_x"] \arrow[dr, "id_x"'] & X \times M \arrow[d, "h"] \\
 & X
\end{tikzcd}$$

commute. If we denote $h(x,m)$ by $(x.m)$, this means that $(x.m).n = x.(m \cdot n)$ and $x.1 = 1$. So T_M-algebras are nothing but sets equipped with a right M-action.

Example 15 (Partial order)**.** If $T\colon \mathbf{P} \to \mathbf{P}$ is a closure operator viewed as a monad, then a T-algebra is an object $x \in \mathbf{P}$ with $Tx \leq x$. Since $x \leq Tx$, it follows that $x = Tx$, which means that a T-algebra is just a *closed element* in $\mathbf{P}$. In particular, if we consider $A \mapsto \overline{A}$ in a topological space, the T-algebras are exactly the closed sets.

Example 16 (Maybe monad)**.** The category of Y-algebras is isomorphic to the category of pointed spaces $\mathbf{Set}_*$. Indeed: by definition, a Y-algebra is a pair (X,h) where $h\colon X \sqcup \{*\} \to X$ and since

$$\begin{tikzcd} X \arrow[r, "\eta_X"] \arrow[dr, "\mathrm{id}_X"'] & X \sqcup \{*\} \arrow[d, "h"] \\ & X \end{tikzcd}$$

commutes, $h|_X = \mathrm{id}_X$. Now define $F\colon \mathbf{Set}^Y \to \mathbf{Set}_*$ by

$$(X,h) \mapsto (X, h(*))$$
$$f\colon (X,h) \to (Y,i) \mapsto f\colon (X,h(*)) \to (Y,i(*))$$

and define $G\colon \mathbf{Set}_* \to \mathbf{Set}^Y$ by

$$(X,x) \mapsto (X, \mathrm{id}_X^x)$$
$$f\colon (X,x) \to (Y,y) \mapsto f\colon (X,\mathrm{id}_X^x) \to (Y,\mathrm{id}_Y^y)$$

where $\mathrm{id}_X^x\colon X \sqcup \{*\} \to X$ is the identity on X and maps $*$ to x. Then an easy computation shows that F and G are well-defined functors with $GF = \mathrm{id}_{\mathbf{Set}^Y}$ and $FG = \mathrm{id}_{\mathbf{Set}_*}$.

Example 17 (Free-forgetful adjunction)**.** The category of algebras for the free-forgetful adjunction $\mathbf{Set} \rightleftarrows \mathbf{Grp}$ is $\mathbf{Grp}$.
Sketch of proof: For every group G we can define $h\colon UFG \to G$ to be the evaluation map which maps a reduced word to the actual evaluated expression in G, then this gives a UF-algebra. Conversely, let S be a set together with a structure map $h\colon UFS \to S$ and denote the group multiplication in FS by $\star$. Then we can define a canonical group structure on S as follows:
- $1_S := h(e)$ where e is the empty word
- for $a, b \in S$, set $a \cdot b := h(a \star b)$.

Since h is a structure map, this satisfies the group axioms. One can check that UF-algebra morphisms are exactly those maps preserving multiplication, i.e. group homomorphisms. In a similar way, one can show that the algebras of the free-forgetful adjunction $\mathbf{Set} \rightleftarrows \mathbf{Vect}_\mathbf{K}$ are vector spaces.

Theorem 2.3 (Every monad is defined by its T-algebras) *Let (T,η,μ) be a monad in a category $\mathbf{X}$. Then there is an adjunction $F^T \dashv G^T : \mathbf{X} \rightleftarrows \mathbf{X}^\mathbf{T}$ such that the monad induced by this adjunction is (T,η,μ). We call this the Eilenberg-Moore-adjunction.*

Proof: • Define $F^T : \mathbf{X} \to \mathbf{X}^T$ by

$$\begin{tikzcd} x \arrow[r, mapsto] \arrow[d, "f"] & (Tx, \mu_x) \arrow[d, "Tf"] \\ x' \arrow[r, mapsto] & (Tx', \mu_{x'}) \end{tikzcd}$$

(Tx, μ_x) is indeed a T-algebra, since μ_x is an arrow $T^2x \to Tx$ and the diagrams

$$\begin{tikzcd} T^3x \arrow[r, "T(\mu_x)"] \arrow[d, "\mu_{Tx}"] & T^2x \arrow[d, "\mu_x"] \\ T^2x \arrow[r, "\mu_x"] & Tx \end{tikzcd} \qquad \begin{tikzcd} Tx \arrow[r, "\eta_{Tx}"] \arrow[dr, "id_{Tx}"'] & T^x \arrow[d, "\mu_x"] \\ & Tx \end{tikzcd}$$

are just the commuting diagrams for the associativity respectively left unitality axioms from the definition of a monad.
$Tf\colon (Tx, \mu_x) \to (Tx', \mu_{x'})$ is indeed a morphism of T-algebras, since the commutativity of

$$\begin{tikzcd} T^2x \arrow[r, "T^2(f)"] \arrow[d, "\mu_x"] & T^2x' \arrow[d, "\mu_{x'}"] \\ Tx \arrow[r, "T(f)"] & Tx' \end{tikzcd}$$

is given by naturality of μ. The functoriality of F^T follows from the functoriality of T.

• Define $G^T : \mathbf{X}^\mathbf{T} \to \mathbf{X}$ by

$$\begin{tikzcd} (x,h) \arrow[r, mapsto] \arrow[d, "f"] & x \arrow[d, "f"] \\ (x',h') \arrow[r, mapsto] & x' \end{tikzcd}$$

so G is just the forgetful functor.
Claim : $G^T \circ F^T = T$ and $F^T G^T(x,h) = (Tx, \mu_x)$.

Proof of claim. Let $x \in \mathbf{X}$. Then $G^T(F^T(x)) = G^T(Tx, \mu_x) = Tx$. Now let $f\colon x \to y$. Then $G^T(F^T(f)) = G^t(Tf) = Tf$. Finally, $F^T G^T(x,h) = F^T(x) = (Tx, \mu_x)$. //

• So we can set

$$\eta^T := \eta\colon \mathrm{id}_\mathbf{X} \Rightarrow G^T F^T$$

as the unit and we can define the counit $\varepsilon^T : F^T G^T \Rightarrow \mathrm{id}_{\mathbf{X}^T}$ by

$$\varepsilon^T_{(x,h)} := h\colon (Tx, \mu_x) \to (x,h).$$

h is a morphism of T-algebras because (x,h) is a T-algebra, since both statements mean that the left of the following two diagrams commutes.

$$\begin{tikzcd} T^2x \arrow[r, "Th"] \arrow[d, "\mu"] & Tx \arrow[d, "h"] \\ T \arrow[r, "h"] & x \end{tikzcd} \qquad \begin{tikzcd} Tx \arrow[r, "Tf"] \arrow[d, "h"] & Tx' \arrow[d, "h'"] \\ x \arrow[r, "f"] & x' \end{tikzcd}$$

ε^T is natural, because if $f\colon (x,h) \to (x',h')$ is a morphism of T-algebras, naturality means that the right diagram above commutes, but this is exactly the definition of f being a morphism of T-algebras.

- To show the triangle identity, we have to show that

$$\begin{tikzcd} Tx \arrow[r, "T\eta_x"] \arrow[dr, "id_{Tx}"'] & T^2x \arrow[d, "\mu_x"] \\ & Tx \end{tikzcd} \qquad \begin{tikzcd} x \arrow[r, "\eta_x"] \arrow[dr, "id_x"'] & Tx \arrow[d, "h"] \\ & x \end{tikzcd}$$

commute, but the first diagram commutes by the right unitality law for the monad T, the second one commutes, since (x,h) is a T-algebra.

- The induced monad of the adjunction now has unit η^T and multiplication $\mu^T = G^T\varepsilon^T F^T$. But $G^T F^T = T$ and $\eta^T = \eta$ is already shown and $\mu^T_x = (G^T\varepsilon^T)_{F^T x} = (G^T\varepsilon^T)_{(Tx,\mu_x)} = G^T(\mu_x) = \mu_x$.

□

Theorem 2.4 (Comparison of adjunctions with algebras) *Let $F \overset{\eta}{\underset{\varepsilon}{\dashv}} G\colon \mathbf{X} \rightleftarrows \mathbf{A}$ be an adjunction, $T = (GF, \eta, G\eta F)$ the monad it defines in $\mathbf{X}$. Then there is unique functor $K\colon \mathbf{A} \to \mathbf{X^T}$ such that the following diagram commutes:*

$$\begin{tikzcd} & \mathbf{A} \arrow[dl, bend left, "G"] \arrow[dd, "K"] \\ \mathbf{X} \arrow[ur, bend left, "F"] \arrow[ur, phantom, "\perp"] \arrow[dr, bend left, "F^T"] \arrow[dr, phantom, "\perp"] & \\ & \mathbf{X^T} \arrow[ul, bend left, "G^T"] \end{tikzcd}$$

In other words, the Eilenberg-Moore-adjunction $F^T \dashv G^T : \mathbf{X} \rightleftarrows \mathbf{X^T}$ is a terminal object in $Adj(T)$.

Proof: Existence: For $f\colon a \to a'$ in A we define K by:

$$Ka = (Ga, G(\varepsilon_a))$$
$$Kf = Gf\colon (Ga, G(\varepsilon_a)) \to (Ga', G(\varepsilon_{a'}))$$

We have to show that this is well-defined.

Claim : Ka is a T-algebra.

Proof of claim. Ka is the morphism $GFGa \overset{G(\varepsilon_a)}{\to} Ga$ and we need to show that the following diagrams commute:

$$\begin{tikzcd} GFGFGa \arrow[r, "GFG(\varepsilon_a)"] \arrow[d, "G(\varepsilon_{FGa})"] & GFGa \arrow[d, "G(\varepsilon_a)"] \\ GFGa \arrow[r, "G(\varepsilon_a)"] & Ga \end{tikzcd} \qquad \begin{tikzcd} Ga \arrow[r, "\eta_{Ga}"] \arrow[dr, "\mathrm{id}_{Ga}"'] & GFGa \arrow[d, "G(\varepsilon_a)"] \\ & Ga \end{tikzcd}$$

The second diagram is just one of the triangle identity for the adjunction. The first diagram is the image under G of:

$$\begin{tikzcd} FGFGa \arrow[r, "FG(\varepsilon_a)"] \arrow[d, "\varepsilon_{FGa}"] & FGa \arrow[d, "\varepsilon_a"] \\ FGa \arrow[r, "\varepsilon_a"] & a \end{tikzcd}$$

which commutes by 2.1. //

Claim : Kf is a morphism of T-algebras.

Proof of claim. We have to show that the first of the following two diagrams commutes:

$$\begin{tikzcd} GFGa \arrow[r, "G(\varepsilon_a)"] \arrow[d, "GFG(f)"] & Ga \arrow[d, "G(f)"] \\ GFGa' \arrow[r, "G(\varepsilon_{a'})"] & Ga' \end{tikzcd} \qquad \begin{tikzcd} FGa \arrow[r, "\varepsilon_a"] \arrow[d, "FG(f)"] & a \arrow[d, "f"] \\ FGa' \arrow[r, "\varepsilon_{a'}"] & a' \end{tikzcd}$$

but the first diagram is the image of the second diagram under G, which commutes by naturality of $\varepsilon\colon FG \Rightarrow \mathrm{id}_A$. //

Functoriality of K follows from the functoriality of G. For the commutativity of the diagram in the statement, let $f\colon a \to a'$ and $g\colon x \to x'$ be morphisms. Then the G-diagram commutes, since we have:

$$G^T Ka = G^T(Ga, G(\varepsilon_a)) = Ga$$
$$G^T K(f) = G^T(Gf) = Gf$$

and for the F-diagram we compute:

$$KFx = (Gfx, G(\varepsilon_{Fx})) = (Tx, \mu_x) = F^T x$$
$$KF(g) = GF(g) = T(g) = F^T(g)$$

Uniqueness: $G^T K = G \implies$ for $f\colon a \to a'$, $K(f) = G(f)$ (viewed as a morphism of algebras) and Ka has to be of the form (Ga, h) for some structure map h. We will show that the commutativity of the two diagrams implies $h = G(\varepsilon_a)$. For that we need the following

Claim : $K(\varepsilon_a) = \varepsilon^T_{Ka}$ for all $a \in \mathbf{A}$.

Proof of claim. Denote by $\phi, \psi, \phi^T, \psi^T$ the natural Hom-isomorphisms from the adjunctions $F \dashv G$ respectively $F^T \dashv G^T$. The left of the following diagrams commutes, since going the upper way maps g to $G(g) \circ \eta_x$, while going the lower way maps it to $G^T K(g) \circ \eta^T_x = G(g) \circ \eta_x$.

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{A}}(Fx, a) \arrow[r, "\phi_{a,x}", "\cong"'] \arrow[d, "K"] & \mathrm{Hom}_{\mathbf{X}}(x, Ga) \arrow[dd, equal] \\ \mathrm{Hom}_{\mathbf{X}^T}(KFx, Ka) \arrow[d, equal] & \\ \mathrm{Hom}_{\mathbf{X}^T}(F^T x, Ka) \arrow[r, "\phi^T_{Ka,x}", "\cong"'] & \mathrm{Hom}_{\mathbf{X}}(x, G^T Ka) \end{tikzcd} \qquad \begin{tikzcd} \mathrm{Hom}_{\mathbf{A}}(Fx, a) \arrow[d, "K"] & \mathrm{Hom}_{\mathbf{X}}(x, Ga) \arrow[l, "\psi_{a,x}"', "\cong"] \arrow[dd, equal] \\ \mathrm{Hom}_{\mathbf{X}^T}(KFx, Ka) \arrow[d, equal] & \\ \mathrm{Hom}_{\mathbf{X}^T}(F^T x, Ka) & \mathrm{Hom}_{\mathbf{X}}(x, G^T Ka) \arrow[l, "\psi^T_{Ka,x}"', "\cong"] \end{tikzcd}$$

This means that the right diagram, which is just the inverses, commutes as well. Now let $x := Ga$ and chase id_{Ga} in the the right diagram to see that $K(\varepsilon_a) = \varepsilon^T_{Ka}$. //

By definition of ε^T, it follows that

$$G(\varepsilon_a) = K(\varepsilon_a) = \varepsilon^T_{Ka} = h.$$

□

2.3 The Kleisli Category of a Monad

There is another way to induce a monad by an adjunction, which was introduced by Heinrich Kleisli in [Kle65]:

Definition (Kleisli category). Let $\mathbf{X}$ be a category, $T = (T, \eta, \mu)$ be a monad in $\mathbf{X}$. The *Kleisli category* $\mathbf{X}_T$ is defined by

- objects the same as in $\mathbf{X}$, but we relabel x to x_T for all $x \in \mathbf{X}$.
- for $x_T, y_T \in \mathbf{X}_T$, $f : x \to Ty$ is a morphism which we denote by $f^b : x_T \to y_T$.
- composition will be denoted by $\bullet$ for distinction and is defined by

$$g^b \bullet f^b := (\mu_z \circ Tg \circ f)^b : x_T \to z_T$$

  for $f^b : x_T \to y_T$, $g^b : y_T \to z_T$.
  This is indeed again a morphism: $x \xrightarrow{f} Ty \xrightarrow{Tg} T^2 z \xrightarrow{\mu_z} Tz$

Claim : This defines a category.

Proof of claim. associativity: Let $x_T \xrightarrow{f^b} y_T \xrightarrow{g^b} z_T \xrightarrow{h^b} w_T$ be objects and morphisms in the Kleisli category.

$$\begin{aligned} (h^b \bullet g^b) \bullet f^b &= (\mu_w \circ Th \circ g)^b \bullet f^b \\ &= (\mu_w \circ T(\mu_w \circ Th \circ g) \circ f)^b \\ &= (\mu_w \circ T\mu_w \circ T^2 h \circ Tg \circ f)^b. \end{aligned}$$

Now the associativity axiom for the monad T states that

$$\begin{tikzcd} T(T(Tw)) \arrow[r, "T(\mu_w)"] \arrow[d, "\mu_{Tw}"'] & T(Tw) \arrow[d, "\mu_w"] \\ T(Tw) \arrow[r, "\mu_w"] & Tw \end{tikzcd}$$

commutes, hence

$$(\mu_w \circ T\mu_w \circ T^2h \circ Tg \circ f)^b = (\mu_w \circ \mu_{Tw} \circ T^2h \circ Tg \circ f)^b$$

By naturality of μ, the diagram

$$\begin{tikzcd} T^2z \arrow[r, "T^2h"] \arrow[d, "\mu_z"] & T^3w \arrow[d, "\mu_{Tw}"] \\ Tz \arrow[r, "Th"] & T^2w \end{tikzcd}$$

commutes, so it follows that

$$\begin{aligned}(\mu_w \circ \mu_{Tw} \circ T^2h \circ Tg \circ f)^b &= (\mu_w \circ Th \circ \mu_z \circ Tg \circ f)^b \\ &= h^b \bullet (g^b \bullet f^b)\end{aligned}$$

identity axiom: Let $f^b : x_T \longrightarrow y_T$ be a morphism.

$$f^b \bullet (\eta_x)^b = (\mu_x \circ Tf \circ \eta_x)^b = (\mu_x \circ \eta_{Ty} \circ f)^b = (\mathrm{id}_{Ty} \circ f)^b = f^b$$

where the second equality follows from the naturality of η and the third equality is due to the left unitality law for T.

$$(\eta_y)^b \bullet f^b = (\mu_y \circ T\eta_y \circ f)^b = (id_{Ty} \circ f)^b = f^b$$

where the second equality is due to the right unitality law for T. This proves that for $x_T \in \mathbf{X}_T$ we have $\mathrm{id}_{x_T} = (\eta_x)^b \in \mathrm{Hom}_{\mathbf{X}_T}(x_T, x_T)$ □

Example 18 (M-action monad)**.** Let M be a monoid, written additively. A Kleisli morphism for the M-action monad T_M is a map $k\colon X \longrightarrow Y \times M$. It can be interpreted as a process producing not only an output $y \in Y$ but also counting a cost $m \in M$ of this process. Now if we have two Kleisli morphisms

$$\begin{aligned} k &\colon X \longrightarrow Y \times M, x \mapsto (k_0(x), m) \\ h &\colon Y \longrightarrow Z \times M, y \mapsto (h_0(y), n) \end{aligned}$$

the Kleisli composition is defined to be $X \xrightarrow{k} Y \times M \xrightarrow{h \times \mathrm{id}_M} Z \times M \times M \xrightarrow{\mu_x} Z \times M$ which maps x to $(h_0(k_0(x)), m + n)$. So this composition allows us to model the assumption that the cost of executing two processes one after another is the sum of the two costs.

Theorem 2.5 *Let (T, η, μ) be a monad in a category $\mathbf{X}$. Then there is an adjunction $F_T \dashv G_T : \mathbf{X} \rightleftarrows \mathbf{X_T}$ such that the monad induced by this adjunction is (T, η, μ). We call this the Kleisli-adjunction.*

Proof: • Define $F_T : \mathbf{X} \to \mathbf{X_T}$ by

$$\begin{aligned} x &\mapsto x_T \\ f : x \to y &\mapsto (Tf \circ \eta_x)^b : x_T \to y_T \end{aligned}$$

Then $F_T(\mathrm{id}_x) = (\eta_x)^b$, which is the identity on x_T. Now

$$F_T(g \circ f) = (T(g \circ f) \circ \eta_x)^b = (Tg \circ Tf \circ \eta_x)^b$$

$$\begin{aligned} F_T(g) \bullet F_T(f) &= (Tg \circ \eta_y)^b \bullet (Tf \circ \eta_x)^b && \text{(Definition of } F^T\text{)} \\ &= (\mu_z \circ T(Tg \circ \eta_y) \circ Tf \circ \eta_x)^b && \text{(Definition of Kleisli composition)} \\ &= (\mu_z \circ T^2 g \circ T\eta_y \circ Tf \circ \eta_x)^b && \text{(Functoriality of T)} \\ &= (Tg \circ \mu_z \circ T\eta_y \circ Tf \circ \eta_x)^b && \text{(Naturality of } \mu\text{)} \\ &= (Tg \circ Tf \circ \eta_x)^b && \text{(right unitality law for } T\text{)} \end{aligned}$$

This proves that F_T is a functor.

• Define $G_T : \mathbf{X_T} \to \mathbf{X}$ by

$$\begin{aligned} x_T &\mapsto Tx \\ f^b : x_T \to y_T &\mapsto \mu_y \circ Tf : Tx \to Ty \end{aligned}$$

Then $G_T(\mathrm{id}_{x_T}) = G_T(\eta_x^b) = \mu_x \circ T\eta_x = \mathrm{id}_x$ by the right unitality law for T. Now we compute that

$$\begin{aligned} G_T(g^b \bullet f^b) = G_T((\mu_z \circ Tg \circ f)^b) &= \mu_z \circ T\mu_z \circ T^2 g \circ Tf \\ G_T(g^b) \circ G_T(f^b) &= \mu_z \circ Tg \circ \mu_z \circ Tf \end{aligned}$$

so it suffices to show that left of the following diagrams commutes:

$$\begin{tikzcd} Tx \arrow[r, "Tf"] \arrow[d, "Tf"] & T^2 y \arrow[r, "T^2 g"] & T^3 z \arrow[r, "T\mu_z"] & T^2 z \arrow[d, "\mu_z"] \\ T^2 y \arrow[r, "\mu_y"] & Ty \arrow[r, "Tg"] & T^2 z \arrow[r, "\mu_z"] & Tz \end{tikzcd} \qquad \begin{tikzcd} Tx \arrow[r, "Tf"] \arrow[d, "Tf"] \arrow[dr, phantom, "(1)"] & T^2 y \arrow[r, "T^2 g"] \arrow[d, dotted, "\mu_y"] \arrow[dr, phantom, "(2)"] & T^3 z \arrow[r, "T\mu_z"] \arrow[d, dotted, "\mu_{Tz}"] \arrow[dr, phantom, "(3)"] & T^2 z \arrow[d, "\mu_z"] \\ T^2 y \arrow[r, "\mu_y"] & Ty \arrow[r, "Tg"] & T^2 z \arrow[r, "\mu_z"] & Tz \end{tikzcd}$$

But we can fill this in to get the right diagram, where (1) commutes trivially, (2) by naturality of μ and (3) by the associativity of T.

Claim : $G_T F_T = T$.

Proof of claim.

$$
\begin{aligned}
G_T(F_T(x)) &= G_T(x_T) = Tx \\
G_T(F_T(f)) &= G_T((Tf \circ \eta_x)^b) = \mu_y \circ T^2 f \circ T\eta_x \\
&= \mu_y \circ T\eta_y \circ Tf && \text{(naturality of } \eta\text{)} \\
&= Tf && \text{(unitality of } T\text{)}
\end{aligned}
$$

//

- We now set the unit and counit to be

$$
\begin{aligned}
&\eta_x \colon x \longrightarrow Tx \\
&\varepsilon_x = \mathrm{id}_{Tx}^b \colon (Tx)_T \longrightarrow x_T
\end{aligned}
$$

We need to show that η and ε satisfy the triangle identities:

$$
\begin{tikzcd}
F_T x \arrow[r, "F_T(\eta_x)"] \arrow[dr, "\mathrm{id}_{F_T x}"'] & F_T G_T F_T x \arrow[d, "\varepsilon_{F_T x}"] \\
& F_T x
\end{tikzcd}
\qquad
\begin{tikzcd}
G_T x_T \arrow[r, "\eta_{G_T x_T}"] \arrow[dr, "\mathrm{id}_{G_T x_T}"'] & G_T F_T G_T x_T \arrow[d, "G_T(\varepsilon_x)"] \\
& G_T x_T
\end{tikzcd}
$$

the left diagram commutes, since we have

$$
\begin{aligned}
\varepsilon_{F_T x} \bullet F_T(\eta_x) &= (\mathrm{id}_{Tx})^b \bullet (T\eta_x \circ \eta_x)^b = (\mu_x \circ T(\mathrm{id}_{Tx}) \circ T\eta_x \circ \eta_x)^b \\
&= (\mu_x \circ T\eta_x \circ \eta_x)^b = (\eta_x)^b = \mathrm{id}_{F_T x}
\end{aligned}
$$

using the right unitality of T. The right diagram commutes, since we have

$$
G_T(\varepsilon_x) \circ \eta_{G_T x_T} = \mu_x \circ T(\mathrm{id}_{Tx}) \circ \eta_{Tx} = \mu_x \circ \eta_{Tx} = \mathrm{id}_{Tx}
$$

using the left unitality of T. The only thing left to show is that $\mu = G_T \varepsilon F_T$:

$$
G_T(\varepsilon_{F_T x}) = G_T(\varepsilon_{x_T}) = G_T(\mathrm{id}_{Tx}^b) = \mu_x \circ T(\mathrm{id}_{Tx}) = \mu_x
$$

□

Theorem 2.6 (Comparison of adjunctions with the Kleisli-construction) *Let* $F \underset{\varepsilon}{\overset{\eta}{\dashv}} G \colon \mathbf{X} \rightleftarrows \mathbf{A}$ *be an adjunction,* $T = (GF, \eta, G\eta F)$ *the monad it defines in* $\mathbf{X}$*. Then there is a unique functor* $L \colon \mathbf{X_T} \to \mathbf{A}$ *such that the following diagram commutes:*

$$\begin{tikzcd}
 & \mathbf{X_T} \arrow[dd, "L"] \arrow[dl, bend left, "G_T"] \\
\mathbf{X} \arrow[ur, bend left, "F_T"] \arrow[dr, bend left, "F"] & \\
 & \mathbf{A} \arrow[ul, bend left, "G"]
\end{tikzcd}$$

In other words, the Kleisli-adjunction $F_T \dashv G_T \colon \mathbf{X} \rightleftarrows \mathbf{X_T}$ *is an initial object in* $Adj(T)$*.*

Proof: • Define L by

$$\begin{aligned} x_T &\mapsto Fx \\ f^b \colon x_t \to y_t &\mapsto \varepsilon_{Fy} \circ F(f) \colon Fx \to FGFy \to Fy \end{aligned}$$

- $L(\mathrm{id}_{x_t}) = L(\eta_x^b) = \varepsilon_{Fx} \circ F(\eta_x) = \mathrm{id}_{Fx}$ by the triangle identity of the adjunction.
- Let $x_T \xrightarrow{f} y_T \xrightarrow{g} z_T$.

$$\begin{aligned} L(g \bullet f) &= \varepsilon_{Fz} \circ F(\mu_z \circ GF(g) \circ f) = \varepsilon_{Fz} \circ F(\mu_z) \circ FGF(g) \circ F(f) \\ L(g) \circ L(f) &= \varepsilon_{Fz} \circ Fg \circ \varepsilon_{Fy} \circ F(f) \end{aligned}$$

  so we have to show that the following diagram commutes:

$$\begin{tikzcd}
Fx \arrow[r, "F(f)"] \arrow[d, "F(f)"] & FGFy \arrow[r, "FGF(g)"] & FGFGFz \arrow[r, "F(\mu_z)"] & FGFz \arrow[d, "\varepsilon_{Fz}"] \\
FGFy \arrow[r, "\varepsilon_{Fy}"] & Fy \arrow[r, "F(g)"] & FGFz \arrow[r, "\varepsilon_{Fz}"] & Fz
\end{tikzcd}$$

  but we can fill in the counit morphisms:

$$\begin{tikzcd}
Fx \arrow[r, "F(f)"] \arrow[d, "F(f)"] \arrow[dr, phantom, "(1)"] & FGFy \arrow[r, "FGF(g)"] \arrow[d, dotted, "\varepsilon_{Fy}"] \arrow[dr, phantom, "(2)"] & FGFGFz \arrow[r, "F(\mu_z)"] \arrow[d, dotted, "\varepsilon_{FGFz}"] \arrow[dr, phantom, "(3)"] & FGFz \arrow[d, "\varepsilon_{Fz}"] \\
FGFy \arrow[r, "\varepsilon_{Fy}"] & Fy \arrow[r, "F(g)"] & FGFz \arrow[r, "\varepsilon_{Fz}"] & Fz
\end{tikzcd}$$

  and now (1) commutes trivially, while (2) and (3) commute by naturality of ε since $\mu_z = G(\varepsilon_{Fz})$.
- Commutativity of the diagrams: Let $f \colon x \to y$.

$$\begin{aligned} GLx_T &= GFx = Tx = G_T x_T \\ GL(f^b) &= G(\varepsilon_{Fy} \circ F(f)) = G(\varepsilon_{Fy}) \circ GF(f) = \mu_y \circ T(f) = G_T(F^b) \end{aligned}$$

$$
\begin{aligned}
LF_T x &= Lx_T = Fx \\
LF_T(f) &= L((T(f) \circ \eta_x)^b) = \varepsilon_{Fy} \circ F(T(f) \circ \eta_x) = \varepsilon_{Fy} \circ FGF(f) \circ F(\eta_x) \\
&= F(f) \circ \varepsilon_{Fx} \circ F(\eta_x) && \text{(naturality of } \varepsilon\text{)} \\
&= F(f) \circ \mathrm{id}_{Fx} = F(f) && \text{(triangle identity)}
\end{aligned}
$$

- Uniqueness: Let L' be another functor making the diagrams commute. Then since F_T is surjective on objects we have

  $$L'F_T = F \Rightarrow L'x_T = Fx = Lx_T \text{ for all objects } x_T$$

  and for $f^b : x_T \to y_T$ we can precompose with the identity to see that

  $$
  \begin{aligned}
  L'(f^b) = L'(f^b) \circ \mathrm{id}_{L'x} &= L'(f^b) \circ \varepsilon_{Fx} \circ F(\eta_x) && \text{(triangle identity)} \\
  &= \varepsilon_{Fy} \circ FGL'(f^b) \circ F(\eta_x) = \varepsilon_{Fy} \circ FG_T(f^b) \circ F(\eta_x) \\
  &= \varepsilon_{Fy} \circ F(\mu_y \circ T(f)) \circ F(\eta_x) = \varepsilon_{Fy} \circ F(\mu_y) \circ FGF(f) \circ F(\eta_x) \\
  &= \varepsilon_{Fy} \circ F(\mu_y) \circ F(\eta_{GFy}) \circ F(f) && \text{(naturality of } \eta\text{)} \\
  &= \varepsilon_{Fy} \circ F(\mu_y \circ \eta_{GFy}) F(f) \\
  &= \varepsilon_{Fy} \circ F(f) = L(f^b) && \text{(triangle identity)}
  \end{aligned}
  $$

□

2.4 Co-Eilenberg-Moore and Co-Kleisli

Since a comonad is just a monad in the dual category, the definitions of the last two chapters can be dualized to get the *co-Eilenberg-Moore category* and the *co-Kleisli-category*. Every theorem that was proven in the last two chapters can also be dualized, the proofs being the dual ones. Due to space limitations, I will not restate the definitions and statements, but instead give an example for a co-Eilenberg-Moore category:

Example 19 (Topology). Recall that a monad in a partial order $(\mathbf{P}, \leq)$ is a closure operator. Dually, a comonad L in $\mathbf{P}$ is an *interior operation*, that is a monotone function $L : \mathbf{P} \to \mathbf{P}$ with $Lx \leq x$ and $L(Lx) = Lx \; \forall x \in \mathbf{P}$. Now if X is a set equipped with a topology, we can again consider the partial order $(\mathcal{P}(X), \subseteq)$. Then the topological interior operator $U \mapsto \mathring{U}$ is a comonad on $\mathcal{P}(X)$. A coalgebra is a set U with $U \subseteq \mathring{U}$, which implies $U = \mathring{U}$, so the coalgebras are exactly the open sets.

3 Witt vectors

The goal of this section is to give a very important example of a comonad: the Witt vector construction is a functor **CRing** $\to$ **CRing** which is used frequently in several mathematical fields, especially Number Theory and Algebraic Geometry. Historically, Witt vectors have been introduced by Ernst Witt in [Wit37], who defined what is today called *p-typical Witt vectors* while studying cyclic algebras of degree p^n. The ring structure on the Witt vectors is highly unintuitive and the whole construction is rather complicated, which is why this section starts with a rigorous, detailed and self-contained introduction to the topic. We will define the p-typical Witt vectors as well as the *big Witt vectors*, which are due to [Car67]. This is essentially an elaboration of [Hes08] (some of the material is also covered in [Hes15]), making the proofs as seamless as possible, while only stating what is needed for proving the final theorem. For different expositions to Witt vectors, consider [Rab14], [Ser79]. The most complete account of Witt vectors that I know of is [Haz09].

3.1 Construction of the Witt Vectors

Definition (truncation set). Let $\mathbb{N}$ be the set of positive integers and let $S \subseteq \mathbb{N}$ be a subset with the property that $\forall n \in S$: if d is a divisor of n, then $d \in S$. We then say that S is a *truncation set.*

Now let S be a truncation set. As a set, we define the *Witt ring* $\mathbb{W}_S(A)$ to be A^S, and we will give it a unique ring structure such that the *ghost map* is a ring homomorphism. Furthermore, if $f : A \to B$ is a ring homomorphism, we define $\mathbb{W}_S(f) \colon \mathbb{W}_S(A) \to \mathbb{W}_S(B)$ to be the function which applies f componentwise, that is $(a_n)_{n\in S} \mapsto (f(a_n))_{n\in S}$. This construction will turn out to be functorial and we will see that the Witt vector functor admits a comonadic structure.

Definition (ghost map). We define $w \colon \mathbb{W}_S(A) \to A^S$ by $(a_n)_{n\in S} \mapsto (w_n)_{n\in S}$ where

$$w_n = \sum_{d|n} d a_d^{n/d}$$

For $a \in \mathbb{W}_S(A)$, we call $(w_n(a))_n = (w_n)_n$ the *ghost coordinates* of a.

Recall that for every prime number p, we have the *p-adic valuation map*:

Definition (p-adic valuation). $v_p \colon \mathbb{Z} \to \mathbb{N} \cup \{\infty\}$ is defined by

$$v_p(n) = \begin{cases} \max\{k \in \mathbb{N} : p^k \mid n\} & \text{if } n \neq 0 \\ \infty & \text{if } n = 0 \end{cases}$$

Lemma 3.1 *Let A be a ring, $a, b \in A$, $v \in \mathbb{N}$, and p a prime number. Then:*

$$a \equiv b \mod pA \implies a^{p^v} \equiv b^{p^v} \mod p^{v+1}A.$$

Proof: We can write $a = b + p\varepsilon$ for some $\varepsilon \in A$, then by the binomial theorem we get:

$$a^{p^v} = (b + p\varepsilon)^{p^v} = \sum_{i=0}^{p^v} \binom{p^v}{i} b^{p^v - i} (p\varepsilon)^i = b^{p^v} + \sum_{i=1}^{p^v} \binom{p^v}{i} b^{p^v - i} p^i \varepsilon^i.$$

Claim : for every $1 \leq i \leq p^v$: $v_p\left(\binom{p^v}{i}\right) = v - v_p(i)$.

Proof of claim. First, note that $v_p(p^v - i) = v_p(i)$. (Indeed: write $i = p^{v_p(i)} \cdot k$ for some $k \in \mathbb{Z}, p \nmid k$. Then $p^v - i = p^v - p^{v_p(i)} \cdot k = p^{v_p(i)} \cdot (p^{v - v_p(i)} - k)$, hence $p^{v_p(i)} \mid p^v - i$. But $p^{v_p(i)+1} \nmid p^v - i$, since $p \nmid k$.)
Now we can apply the p-adic valuation to the following equality:

$$\begin{aligned}
i! \cdot \binom{p^v}{i} &= p^v \cdot (p^v - 1) \cdot \ldots \cdot (p^v - (i-1)) \\
\implies v_p\left(i! \cdot \binom{p^v}{i}\right) &= v_p(p^v \cdot (p^v - 1) \cdot \ldots \cdot (p^v - (i-1))) \\
\iff v_p(i!) + v_p\left(\binom{p^v}{i}\right) &= v_p(p^v) + v_p(p^v - 1) + \ldots + v_p(p^v - (i-1)) \\
\iff v_p(i!) + v_p\left(\binom{p^v}{i}\right) &= v + v_p((i-1)!) \\
\iff v_p\left(\binom{p^v}{i}\right) &= v + v_p((i-1)!) - v_p(i!) \\
\iff v_p\left(\binom{p^v}{i}\right) &= v + v_p\left(\frac{(i-1)!}{i!}\right) \\
\iff v_p\left(\binom{p^v}{i}\right) &= v - v_p(i)
\end{aligned}$$

where we use the multiplicativity of the p-adic valuation. //

It follows that

$$v_p\left(\binom{p^v}{i} \cdot p^i\right) = v - v_p(i) + i \geq v + 1$$

which means that those summands vanish mod $p^{v+1}A$. □

The core of the construction is contained in the following Lemma:

Lemma 3.2 (Dwork) *Suppose that for every prime number p there exists a ring homomorphism $\phi_p \colon A \to A$ with the property that $\phi_p(a) \equiv a^p$ modulo pA. Then for every sequence $x = (x_n)_{n\in S}$, the following are equivalent:*

(i) *The sequence x is in the image of the ghost map $w \colon \mathbb{W}_S(A) \to A^S$.*

(ii) *For every prime number p and every $n \in S$ with $v_p(n) \geqslant 1$,*

$$x_n \equiv \phi_p(x_{n/p}) \qquad \text{modulo } p^{v_p(n)}A.$$

Proof: ($\Rightarrow$) Suppose x is in the image of the ghost map, that means there is a sequence $a = (a_n)_{n\in S}$ such that $x_n = w_n(a)$ for all $n \in S$. We calculate:

$$\phi(x_{n/p}) = \phi(w_{n/p}(a)) = \phi(\sum_{d|n/p} da_d^{n/pd}) = \sum_{d|n/p} d \cdot \phi(a_d^{n/pd})$$

since ϕ is a ring homomorphism and $d \in \mathbb{N}$. Now

$$\begin{aligned}\sum_{d|n/p} d \cdot \phi(a_d^{n/pd}) &\equiv \sum_{d|n/p} d \cdot a_d^{n/d} && \bmod p^{v_p(n)}A && \text{(Eq.1)}\\ &\equiv \sum_{d|n} d \cdot a_d^{n/d} && \bmod p^{v_p(n)}A && \text{(Eq.2)}\end{aligned}$$

so we get

$$\phi(x_{n/p}) \equiv \sum_{d|n} d \cdot a_d^{n/d} = w_n(a) = x_n \qquad \bmod p^{v_p(n)}A.$$

Proof of (Eq.1). First, note that

$$x \equiv y \qquad \bmod p^m A \implies dx \equiv dy \qquad \bmod p^{m+v_p(d)}A \qquad (*)$$

for all $m \in \mathbb{N}, d \in \mathbb{Z}$. Now we can write $n/pd = p^\alpha \cdot N$ for some $N \in \mathbb{Z}$, $p \nmid N$, $\alpha = v_p(n/pd)$. Now by the assumptions of the lemma we get that $\phi_p(a_d^N) \equiv a_d^{p\cdot N} \bmod pA$, so we can calculate:

$$\phi_p(a_d^{n/pd}) \overset{\text{def.}}{=} \phi_p(a_d^{p^\alpha \cdot N}) = \phi_p(a_d^N)^{p^\alpha} \equiv a_d^{(p\cdot N)^{p^\alpha}} \qquad \bmod p^{\alpha+1}A$$

using Lemma 3.1 for the last congruence. Now $(*)$ and the fact that

$$a_d^{(p\cdot N)^{p^\alpha}} = a_d^{p\cdot N\cdot p^\alpha} \overset{\text{def.}}{=} a_d^{p\cdot n/pd} = a_d^{n/d}$$

gives us

$$d \cdot \phi_p(a_d^{n/pd}) \equiv d \cdot a_d^{n/d} \qquad \bmod p^{\alpha+1+v_p(d)}$$

But

$$\alpha + 1 + v_p(d) \overset{\text{def.}}{=} v_p(n/pd) + 1 + v_p(d) = v_p(n/d) + v_p(d) = v_p(n)$$

so it follows that for every d

$$d \cdot \phi_p(a_d^{n/pd}) \equiv d \cdot a_d^{n/d} \qquad \text{mod } p^{v_p(n)}$$

which implies (1). □

Proof of (Eq.2). It suffices to show that if $d \mid n$, $d \nmid n/p$, the term $d \cdot a_d^{n/d}$ vanishes mod $p^{v_p(n)}A$. But in this case, $v_p(d) = v_p(n)$, hence $d \equiv 0 \text{ mod } p^{v_p(n)}A$. □

($\Leftarrow$) Let $(x_n)_{n \in S}$ be a sequence such that $x_n \equiv \phi_p(x_{n/p}) \qquad mod\; p^{v_p(n)}A\; \forall p$ prime, $n \in S, v_p(n) \geqslant 1$. Define $(a_n)_{n \in S}$ with $w_n((a_n)_{n \in S}) = x_n$ as follows:

$$a_1 \coloneqq x_1$$

and if a_d has been chosen for all $d \mid n$ such that $w_d(a) = x_d$ we see that for every prime $p \mid n$:

$$\begin{aligned} x_n &\equiv \phi_p(x_{n/p}) \qquad mod\; p^{v_p(n)}A \\ &= \phi_p\Big(\sum_{d \mid n/p} d \cdot a_d^{n/pd}\Big) \\ &= \sum_{d \mid n/p} d \cdot \phi(a_d^{n/pd}) \end{aligned}$$

because ϕ_p is a ring homomorphism. Using our previous calculations, we see that

$$\begin{aligned} \sum_{d \mid n/p} d \cdot \phi(a_d^{n/pd}) &\overset{(Eq.1)}{\equiv} \sum_{d \mid n/p} d \cdot a_d^{n/d} \qquad \text{mod } p^{v_p(n)}A \\ &\overset{(Eq.2)}{\equiv} \sum_{d \mid n} d \cdot a_d^{n/d} \qquad \text{mod } p^{v_p(n)}A \\ &\equiv \sum_{d \mid n, d \neq n} d \cdot a_d^{n/d} \qquad \text{mod } p^{v_p(n)}A \end{aligned}$$

In conclusion:

$$p^{v_p(n)} \mid \left(x_n - \sum_{d \mid n, d \neq n} d \cdot a_d^{n/d} \right)$$

for all $p \mid n$. But this implies that

$$n \mid \left(x_n - \sum_{d \mid n, d \neq n} d \cdot a_d^{n/d} \right)$$

hence $\exists a_n \in A$ such that

$$x_n = \sum_{d \mid n, d \neq n} d \cdot a_d^{n/d} + n \cdot a_n = \sum_{d \mid n} d \cdot a_d^{n/d}.$$

□

We will often need the following

Lemma 3.3 *If A is a torsion-free ring, the ghost map is injective.*

Proof: Let $a = (a_n)_{n\in S}$ such that $w(a) = 0$. This means $w_n = 0$ for all $n \in S$. We will prove by induction, that $a_n = 0$ for all $n \in S$. First, $a_1 = w_1 = 0$. And if $a_d = 0$ for all $d \in S, d < n$ we see that

$$0 = w_n = \sum_{d|n} d \cdot a_d^{n/d} = n \cdot a_n$$

and since A is torsion-free, this implies $a_n = 0$. □

Now we can finish the construction of the Witt vectors:

Theorem 3.4 *There exists a unique ring structure on $\mathbb{W}_S(A)$ such that the ghost map*

$$w : \mathbb{W}_S(A) \to A^s$$

is a natural transformation of functors from rings to rings.

Proof: Step 1: Let $A = \mathbb{Z}[a_n, b_n \mid n \in S]$. Consider the unique ring homomorphism

$$\phi_p : A \to A;\ a_n \mapsto a_n^p,\ b_n \mapsto b_n^p$$

ϕ_p satisfies that $\phi_p(f) \equiv f^p$ modulo pA (Indeed: it suffices to show that $\overline{\phi_p(f)} = \overline{f^p}$ in $\mathbb{F}_p[a_n, b_n \mid n \in S]$, which is apparent). Let $a = (a_n)_n, b = (b_n)_n$.

Claim : $w(a) + w(b)$, $w(a) \cdot w(b)$ and $-w(a)$ are in the image of the ghost map.

Proof of claim. Since we can use Lemma 3.2 , it suffices to show that for all prime p, for all $n \in S$ with $p \mid n$:

$$\begin{aligned} w_n(a) + w_n(b) &\equiv \phi_p(w_{n/p}(a) + w_{n/p}(b)) && \mod p^{v_p(n)}A \\ w_n(a) \cdot w_n(b) &\equiv \phi_p(w_{n/p}(a) \cdot w_{n/p}(b)) && \mod p^{v_p(n)}A \\ -w_n(a) &\equiv \phi_p(-w_{n/p}(a)) && \mod p^{v_p(n)}A \end{aligned}$$

but since $w_n(a)$ and $w_n(b)$ are both in the image of the ghost map, we know that $w_n(a) \equiv \phi_p(w_{n/p}(a)) \mod p^{v_p(n)}A$ and $w_n(b) \equiv \phi_p(w_{n/p}(b)) \mod p^{v_p(n)}A$. The claim now follows using the fact that ϕ_p is a ring homomorphism and that congruence is compatible with addition and multiplication. //

It follows there are sequences $S = (S_n)_{n\in S}$, $P = (P_n)_{n\in S}$ and $I = (I_n)_{n\in S}$ of polynomials such that

$$w(S) = w(a) + w(b),\ w(P) = w(a) \cdot w(b),\ w(I) = -w(a)$$

Since A is torsion-free, the ghost map is injective by 3.3 and hence, these polynomials are unique.

Step 2: Now let A' be any ring (Remember that $A = \mathbb{Z}[a_n, b_n \mid n \in S]$). Let $a' = (a'_n)_{n\in S}$, $b' = (b'_n)_{n\in S}$ be two vectors in $\mathbb{W}_S(A')$. Then there is a unique ring homomorphism

$$e\colon A \longrightarrow A';\ a_n \mapsto a'_n,\ b_n \mapsto b'_n$$

such that $\mathbb{W}_S(e)(a) = a'$ and $\mathbb{W}_S(e)(b) = b'$ We define:

$$\begin{aligned} a' + b' &:= \mathbb{W}_S(e)(S) = (S_n(a'_1, \dots, a'_n, b'_1, \dots, b'_n))_{n\in S} \\ a' \cdot b' &:= \mathbb{W}_S(e)(P) = (P_n(a'_1, \dots, a'_n, b'_1, \dots, b'_n))_{n\in S} \\ -a' &:= \mathbb{W}_S(e)(I) = (I_n(a'_1, \dots, a'_n))_{n\in S} \end{aligned}$$

where e commutes with integer polynomials, since it is a ring homomorphism. This is the unique way to define the ring structure on $\mathbb{W}_S(A')$, since functoriality of $\mathbb{W}$ forces $\mathbb{W}_S(e)$ to be a ring homomorphism.

Claim : These operations make $\mathbb{W}_S(A')$ into a ring.

Proof of claim. Suppose first that A' is torsion-free, then the ghost map is injective and hence the ring axioms are satisfied. For the general case, choose a surjective ring homomorphism $g\colon A'' \longrightarrow A'$ from a torsion-free ring A''(For example, one could take A'' to be $\mathbb{Z}A'$). Then $\mathbb{W}_S(g)\colon \mathbb{W}_S(A'') \longrightarrow \mathbb{W}_S(A')$ is again surjective, and since the ring axioms are satisfied on the left-hand side, they are satisfied on the right-hand side. //

Claim : $w\colon \mathbb{W}_S(A) \longrightarrow A^S$ is a natural ring homomorphism.

w is natural, because for $f\colon A \longrightarrow B$:

$$\begin{tikzcd} \mathbb{W}_S(A) \arrow[r, "\mathbb{W}_S(f)"] \arrow[d, "w"] & \mathbb{W}_S(B) \arrow[d, "w"] \\ A^S \arrow[r, "f^S"] & B^S \end{tikzcd}$$

commutes since f is a ring homomorphism, hence commutes with the integer polynomials w_n. To show that w is a ring homomorphism, let $a', b' \in \mathbb{W}_S(A')$. Then:

$$\begin{aligned} w(a' + b') &= w(\mathbb{W}_S(e)(S)) = e^S(w(S)) = e^S(w(a) + w(b)) \\ &= e^S(w(a)) + e^S(w(b)) = w(a') + w(b') \end{aligned}$$

and analogously $w(a' \cdot b') = w(a') \cdot w(b')$. □

Corollary 3.5 *$w_n\colon \mathbb{W}_S(A) \longrightarrow A$ is a natural ring homomorphism for all $n \in S$.*

Lemma 3.6 *The zero element in $\mathbb{W}_S(A)$ is given by $(0, 0, 0, \dots)$ and the unit in $\mathbb{W}_S(A)$ is given by $(1, 0, 0, \dots)$.*

Proof: (For better readability, this proof assumes $S = \mathbb{N}$, but the general proof is exactly the same.) Suppose first that $A = \mathbb{Z}[a_n, b_n \mid n \in \mathbb{N}]$. Let $a = (a_n)_n$ be a Witt vector. Then:

$$w((0,0,0,\dots)) = (0,0,0,\dots)$$

since $w_n(0,0,0,\dots) = 0$ for all n.

$$w((1,0,0,\dots)) = (1,1,1,\dots)$$

since $w_n(1,0,0,\dots) = 1^n = 1$ for all n. By injectivity of the ghost map, the claim follows, because $(0,0,0,\dots)$ and $(1,1,1,\dots)$ are the zero element respectively the unit in $A^{\mathbb{N}}$. In the general case: For A' any ring, $(a'_n)_n \in \mathbb{W}_S(A')$, $(a'_n)_n + (0,0,\dots)$ is defined as $(S_1(a'_1, 0), S_2(a'_1, a'_2, 0, 0), \dots)$ and since $(S_1(a_1, 0), S_2(a_1, a_2, 0, 0), \dots) = (a_1, a_2, \dots) \in \mathbb{Z}[a_n, b_n \mid n \in \mathbb{N}]$, these polynomial equations still hold if we plug in a different sequence. The same reasoning show that $(1, 0, \dots)$ is the unit. □

Proposition 3.7 $\mathbb{W}_S(\_)$ *is a functor* **CRing** $\to$ **CRing**.

Proof: $\mathbb{W}_S(\mathrm{id}) = \mathrm{id}$ and $\mathbb{W}_S(g \circ f) = \mathbb{W}_S(g) \circ W_S(f)$ are clear, since $\mathbb{W}_S(\_)$ on morphisms is identical with the countable product functor $(\_)^{\mathbb{N}}$. All that is left to show is that for a ring homomorphism $f\colon A \to B$, $\mathbb{W}_S(f)\colon W_S(A) \to W_S(B)$ is again a ring homomorphism.

$$\mathbb{W}_S(f)(1,0,\dots) = (f(1), f(0), \dots) = (1, 0, \dots)$$

Now let $x = (x_n)_n, y = (y_n)_n$ be two Witt vectors.

$$\begin{aligned}
\mathbb{W}_S(f)(x+y) &= \mathbb{W}_S(f)(S_n(x_1,\dots,x_n,y_1,\dots,y_n))_n \\
&= (f(S_n(x_1,\dots,x_n,y_1,\dots,y_n)))_n \\
&= (S_n(f(x_1),\dots,f(x_n),f(y_1),\dots,f(y_n)))_n \\
&= \mathbb{W}_S(f)(x) + \mathbb{W}_S(f)(y)
\end{aligned}$$

where f commutes with integer polynomials since it is a ring homomorphism. An identical computation shows that

$$W_S(f)(x \cdot y) = \mathbb{W}_S(f)(x) \cdot \mathbb{W}_S(f)(y)$$

□

3.2 The Verschiebung, Frobenius and Teichmüller Maps

We have various operations on Witt vectors that are of interest.

Definition (Restriction map). If $T \subseteq S$ are two truncation sets, the *restriction from S to T*

$$R^S_T\colon \mathbb{W}_S(A) \to \mathbb{W}_T(A)$$

is a natural ring homomorphism. This follows from the fact that for the polynomials used to define addition and multiplication in the Witt vector ring we have $S_n, P_n \in \mathbb{Z}[a_1,\dots,a_n,b_1,\dots,b_n]$(see the proof of Dwork's lemma, $(\Leftarrow)$).

If $S \subseteq \mathbb{N}$ is a truncation set, $n \in \mathbb{N}$, then $S/n := \{d \in \mathbb{N} \mid nd \in S\}$ is again a truncation set.

Definition (Verschiebung). Define

$$V_n \colon \mathbb{W}_{S/n} \longrightarrow \mathbb{W}_S(A);\ V_n((a_d)_{d \in S/n})_m := \begin{cases} a_d, & \text{if } m = n \cdot d \\ 0, & \text{else} \end{cases}$$

which is called the *n-th Verschiebung map*. Furthermore define

$$\widetilde{V_n} \colon A^{S/n} \longrightarrow A^S;\ \widetilde{V_n}((x_d)_{d \in S/n})_m := \begin{cases} n \cdot x_d, & \text{if } m = n \cdot d \\ 0, & \text{else} \end{cases}$$

Lemma 3.8 *The Verschiebung map V_n is additive.*

Proof:

Claim :
$$\begin{array}{ccc} \mathbb{W}_{S/n}(A) & \xrightarrow{w} & A^{S/n} \\ \downarrow{\scriptstyle V_n} & & \downarrow{\scriptstyle \widetilde{V_n}} \\ \mathbb{W}_S(A) & \xrightarrow{w} & A^S \end{array}$$
commutes.

Proof of claim. Let $a = (a_d)_{d \in S/n} \in \mathbb{W}_{S/n}(A)$. Let $m \in S$.

- case 1: $m \neq n \cdot d\ \forall d \in S/n$: Then $\widetilde{V_n}(w(a))_m = (\widetilde{V_n}(w_d)_{d \in S/n})_m = 0$ and

$$w(V_n(a))_m = \sum_{k \mid m, k = nd} k \cdot a_d^{m/k} = 0$$

  because if there would be $k \mid m, k = nd$, this would mean that $m = k \cdot d' = n \cdot d \cdot d'$ for $d, d' \in S/n$ and then $d \cdot d' \mid m$ which is a contradiction to case 1.

- case 2: $m = n \cdot d$ for some $d \in S/n$:

$$\widetilde{V_n}(w(a))_m = (\widetilde{V_n}(w_d)_{d \in S/n})_m = n \cdot w_d = n \cdot \sum_{k \mid d} k \cdot a_k^{d/k}.$$

$$\begin{aligned} w(V_n(a))_m = w_m(V_n(a)) &= \sum_{k \mid nd} k \cdot (V_n(a))_k^{nd/k} \\ &= \sum_{k \mid nd, k = nd_k} k \cdot a_{d_k}^{nd/k} = n \cdot \sum_{k \mid nd, k = nd_k} d_k \cdot a_{d_k}^{nd/nd_k} \\ &= n \cdot \sum_{k \mid nd, k = nd_k} d_k \cdot a_{d_k}^{d/d_k} = n \cdot \sum_{k \mid d} k \cdot a_k^{d/k} \end{aligned}$$

  because $nd_k \mid nd \iff d_k \mid d$ for $d_k, d, n \in \mathbb{N}$.

//

$\widetilde{V}_n$ is obviously additive, so assume now that A is torsion-free. Then the ghost map is injective, so it is enough to check that $w(V_n(a+b)) = w(V_n(a)+V_n(b))$ for $a, b \in \mathbb{W}_{S/n}$. Since

$$\begin{tikzcd} \mathbb{W}_{S/n}(A) \arrow[r, "w"] \arrow[d, "V_n"] & A^{S/n} \arrow[d, "\widetilde{V}_n"] \\ \mathbb{W}_S(A) \arrow[r, "w"] & A^S \end{tikzcd}$$

commutes, we calculate:

$$\begin{aligned} w(V_n(a+b)) &= \widetilde{V}_n(w(a+b)) = \widetilde{V}_n(w(a)+w(b)) \\ &= \widetilde{V}_n(w(a)) + \widetilde{V}_n(w(b)) = w(V_n(a)) + w(V_n(b)) = w(V_n(a)+V_n(b)) \end{aligned}$$

For the general case, choose a surjective ring homomorphism $g\colon A \to A'$, where A is torsion-free. Then the diagram

$$\begin{tikzcd} \mathbb{W}_{S/n}(A) \arrow[r, two heads, "\mathbb{W}_{S/n}(g)"] \arrow[d, "V_n"] & \mathbb{W}_{S/n}(A') \arrow[d, "V_n"] \\ \mathbb{W}_S(A) \arrow[r, two heads, "\mathbb{W}_S(g)"] & \mathbb{W}_S(A') \end{tikzcd}$$

clearly commutes and since $\mathbb{W}_{S/n}(g)$ is surjective, there are $x, y \in \mathbb{W}_{S/n}(A)$ such that $\mathbb{W}_{S/n}(g)(x) = a$, $\mathbb{W}_{S/n}(g)(y) = b$. Then

$$\begin{aligned} V_n(a+b) &= V_n(\mathbb{W}_{S/n}(g)(x+y)) = \mathbb{W}_S(g)(V_n(x+y)) \\ &= \mathbb{W}_S(g)(V_n(x)) + \mathbb{W}_S(g)(V_n(y)) = V_n(\mathbb{W}_{S/n}(g)(x)) + V_n(\mathbb{W}_{S/n}(g)(y)) \\ &= V_n(a) + V_n(b) \end{aligned}$$

□

Next, we will introduce the *frobenius homomorphism*, which will play an important rule in the proof of the comonadic structure of $\mathbb{W}$ as well. For that, first define $\widetilde{F}_n\colon A^S \to A^{S/n}$ by $\widetilde{F}_n((x_m)_{m\in S}) = (x_{nm})_{m\in S/n}$.

> **Lemma 3.9** (Frobenius homomorphism) *There exists a unique natural ring homomorphism*
>
> $$F_n\colon \mathbb{W}_S(A) \to \mathbb{W}_{S/n}(A)$$
>
> *such that the diagram*
>
> $$\begin{tikzcd} \mathbb{W}_S(A) \arrow[r, "w"] \arrow[d, "F_n"] & A^S \arrow[d, "\widetilde{F}_n"] \\ \mathbb{W}_{S/n}(A) \arrow[r, "w"] & A^{S/n} \end{tikzcd}$$
>
> *commutes.*

We call F_n the *nth Frobenius homomorphism*. The commutativity of the diagram above is equivalent to commutativity of the following diagram for every $d \in S/n$:

$$\begin{tikzcd} \mathbb{W}_S(A) \arrow[d, "F_n"] \arrow[dr, "w_{nd}"] & \\ \mathbb{W}_{S/n}(A) \arrow[r, "w_d"'] & A \end{tikzcd}$$

Proof of Lemma 3.9. We construct F_n similar to the construction of the ring operations on $\mathbb{W}_S(A)$ using Lemma 3.2 again. So let A be the polynomial ring $\mathbb{Z}[a_i \mid i \in S]$, let $a = (a_i)_{i \in S}$ and let ϕ_p be the unique ring homomorphism $a_i \mapsto a_i^p$. It satisfies $\phi_p(a) \equiv a^p \bmod pA$ (compare the proof of 3.4). Then Lemma 3.2 shows that the sequence $\widetilde{F}_n(w(a)) \in A^{S/n}$ is in the image of a unique element

$$F_n(a) = (f_{n,d}(a))_{d \in S/n}$$

by the ghost map, where the $f_{n,d}$ are integer polynomials with indeterminates a_i. (Indeed: we have

$$\phi_p((\widetilde{F}_n(w(a)))_{m/p}) = \phi_p((w_{nm/p})) = \sum_{k \mid nm/p} k \cdot a_k^{nm/k}$$
$$\widetilde{F}_n(w(a))_m = w_{nm} = \sum_{k \mid nm} k \cdot a_k^{nm/k}$$

and both sums are congruent mod $p^{v_p(m)}A$.) If A' is any ring and if $a' = (a'_i)_{i \in S}$ is a vector in $\mathbb{W}_S(A)$, then we define

$$F_n(a') := \mathbb{W}_{S/n}(e_{a'})(F_n(a)) = (f_{n,d}(a'))_{d \in S/n}$$

where $e_{a'} : A \longrightarrow A'$ is the unique ringhomomorphism that maps a to a'. Now since $\widetilde{F}_n$ is clearly a ring homomorphism, we can argue similar as in the proof of Lemma 3.8 to show that F_n is a ring homomorphism. F_n is natural, since for a ring homomorphism $f : A' \longrightarrow B'$ the diagram

$$\begin{tikzcd} \mathbb{W}_S(A') \arrow[r, "\mathbb{W}_S(f)"] \arrow[d, "F_n"] & \mathbb{W}_S(B') \arrow[d, "F_n"] \\ \mathbb{W}_{S/n}(A') \arrow[r, "\mathbb{W}_{S/n}(f)"] & \mathbb{W}_{S/n}(B') \end{tikzcd}$$

commutes, because f commutes with integer polynomials, as it is a ring homomorphism. Lastly, uniqueness of F_n follows from naturality, since for $a' \in A'$, the following diagram has to commute:

$$\begin{tikzcd} \mathbb{W}_S(A) \arrow[r, "\mathbb{W}_S(e_{a'})"] \arrow[d, "F_n"] & \mathbb{W}_S(A') \arrow[d, "F_n"] \\ \mathbb{W}_{S/n}(A) \arrow[r, "\mathbb{W}_{S/n}(e_{a'})"] & \mathbb{W}_{S/n}(A') \end{tikzcd}$$

□

Note that for $n, m \in \mathbb{N}$ we have $(S/n)/m = S/nm$ by definition.

Lemma 3.10 *Let $n, m \in \mathbb{N}$. Then*

$$F_n \circ F_m = F_{nm}.$$

Proof: We have $\widetilde{F}_n \circ \widetilde{F}_m = \widetilde{F}_{nm}$, since

$$\widetilde{F}_n(\widetilde{F}_m(x_d)_{d\in S}) = \widetilde{F}_n((x_{md})_{d\in S/m}) = (x_{nmd})_{d\in S/nm} = \widetilde{F}_{nm}((x_d)_{d\in S}).$$

Now suppose that A is torsion-free, which means that the ghost map is injective. We have the following commutative diagram:

$$\begin{tikzcd}
\mathbb{W}_S(A) \arrow[r, hook, "w"] \arrow[d, "F_n"] & A^S \arrow[d, "\widetilde{F}_n"] \\
\mathbb{W}_{S/n}(A) \arrow[r, hook, "w"] \arrow[d, "F_m"] & A^{S/n} \arrow[d, "\widetilde{F}_m"] \\
\mathbb{W}_{S/nm}(A) \arrow[r, hook, "w"] & A^{S/nm}
\end{tikzcd}$$

and then $w \circ (F_n \circ F_m) = \widetilde{F}_n \circ \widetilde{F}_m \circ w = \widetilde{F}_{nm} \circ w = w \circ (F_{nm})$ which implies $F_n \circ F_m = F_{nm}$, since w is injective, hence a mono. Now, for the general case choose $g\colon A \to A'$ surjective, then we have the following commuting diagram:

$$\begin{tikzcd}
\mathbb{W}_S(A) \arrow[r, two heads, "\mathbb{W}(g)"] \arrow[d, "F_n"] & \mathbb{W}_S(A') \arrow[d, "F'_n"] \\
\mathbb{W}_{S/n}(A) \arrow[r, two heads, "\mathbb{W}(g)"] \arrow[d, "F_m"] & \mathbb{W}_{S/n} \arrow[d, "F'_m"] \\
\mathbb{W}_{S/nm}(A) \arrow[r, two heads, "\mathbb{W}(g)"] & \mathbb{W}_{S/nm}(A')
\end{tikzcd}$$

and then $F'_n \circ F'_m \circ \mathbb{W}(g) = \mathbb{W}(g) \circ F_n \circ F_m = \mathbb{W}(g) \circ F_{nm} = F'_{nm} \circ \mathbb{W}(g)$ which implies $F'_n \circ F'_m$ since $\mathbb{W}(g)$ is surjective, hence an epi. □

Lemma 3.11 $F_1 = \mathrm{id}\colon W_S(A) \to W_S(A)$.

Proof: clearly, $\widetilde{F}_1 = \mathrm{id}_{A^S}$, now if A is torsion-free, the claim follows, and in the general case we can argue as before. □

Definition (teichmüller representative). The *teichmüller representative* is the map

$$\tau\colon A \to \mathbb{W}_S(A)$$

defined by

$$(\tau(a))_m = \begin{cases} a, & \text{if } m = 1 \\ 0, & \text{else} \end{cases}$$

Lemma 3.12 *The teichmüller map is multiplicative.*

Proof: The map $\widetilde{\tau}\colon A \to A^S$; $(\widetilde{\tau}(a))_n = a^n$ is multiplicative and there is a commutative diagram

$$\begin{tikzcd} & A \arrow[dl, "\tau"'] \arrow[dr, "\widetilde{\tau}"] & \\ \mathbb{W}_S(A) \arrow[rr, "w"] & & A^S. \end{tikzcd}$$

Indeed, $w_n(\tau(a)) = w_n((a, 0, 0, \dots)) = a^n$ by definition of w_n.

□

3.3 The Comonad Structure of Witt Vectors

We will need the following lemma:

Lemma 3.13 *Let $m \in \mathbb{Z}$. If m is a non-zero divisor in A, then it is a non-zero divisor in $\mathbb{W}_S(A)$ as well.*

Proof: We can assume that S is finite, since $\mathbb{W}_S(A)$ is the inverse limit of all $\mathbb{W}_T(A)$ where T is a finite sub-truncation set of S, together with the fact that the inverse limit is left-exact, hence preserves injectivity. We will prove the Lemma by induction over $|S|$. If $S = \emptyset$, the statement is trivial, so let $|S| = 1$, this means that $S = \{n\}$ for some $n \in \mathbb{N}$, but then $\mathbb{W}_{\{n\}}(A) \cong \mathbb{W}_{\{1\}}(A) = A$ via V_n. Now for the induction step, let $n \in S$ be maximal and let $T = S - \{n\}$. Then $S/n = \{1\}$ and therefore we have a short exact sequence of $\mathbb{Z}$-modules

$$0 \longrightarrow A \xrightarrow{V_n} \mathbb{W}_S(A) \xrightarrow{R_T^S} W_T(A) \longrightarrow 0$$

since V_n maps a to $(0, \dots, a)$ and R_T^S forgets the last coordinate. We can extend the sequence to the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\cdot m"] & \mathbb{W}_S(A) \arrow[r] \arrow[d, "\cdot m"] & \mathbb{W}_T(A) \arrow[r] \arrow[d, "\cdot m"] & 0 \\ 0 \arrow[r] & A \arrow[r] & \mathbb{W}_S(A) \arrow[r] & \mathbb{W}_T(A) \arrow[r] & 0 \end{tikzcd}$$

Now m being a non-zero divisor is equivalent to $\cdot m$ being injective, but if the two outer vertical maps are injective, applying the snake lemma yields that the middle map has to be injective, too. □

Corollary 3.14 *If A is torsion-free, then $\mathbb{W}_S(A)$ is torsion-free as well.*

Definition (p-typical and big Witt vectors). For a prime p, the set $P := \{1, p, p^2, \dots\}$ is a truncation set. The ring $\mathbb{W}_P(A)$ is called the *p-typical Witt vectors*, the ring $\mathbb{W}_n(A) := \mathbb{W}_{\{1,p,p^2,\dots,p^n\}}(A)$ is called the *p-typical Witt vectors of length n*. In most of the literature, elements in those two rings are indexed by their exponent. We define the *big Witt vectors* to be $\mathbb{W}(A) := \mathbb{W}_{\mathbb{N}}(A)$

For the construction of a natural transformation $\mathbb{W}(A) \to \mathbb{W}(\mathbb{W}(A))$ we want to use Lemma 3.2 again. Hence we first show:

Lemma 3.15 *Let p be a prime number, let A be any ring. Then the ring homomorphism $F_p\colon \mathbb{W}(A) \to \mathbb{W}(A)$ satisfies $F_p(a) \equiv a^p$ mod pA.*

Proof: Suppose first, that $A = \mathbb{Z}[a_1, a_2, \dots]$ and let $a = (a_1, a_2, \dots)$. Since

$$\begin{aligned} & F_p(a) \equiv a^p && \bmod p\mathbb{W}(A) \\ \iff & F_p(a) - a^p \equiv 0 && \bmod p\mathbb{W}(A) \\ \iff & F_p(a) - a^p \in p\mathbb{W}(A) \end{aligned}$$

it suffices to show there exists $b \in \mathbb{W}(A)$ such that $F_p(a) - a^p = p \cdot b$. By Lemma 3.13, this element is unique. Applying the ghost map gives us:

$$w_n(F_p(a) - a^p) = w_n(F_p(a)) - w_n(a)^p = w_{pn}(a) - w_n(a)^p = \sum_{d|pn} d \cdot a_d^{pn/d} - (\sum_{d|n} d \cdot a_d^{n/d})^p$$

using Lemma 3.9. This is now congruent to 0 mod pA: modulo p, $x \mapsto x^p$ is a ring homomorphism, so the second summand is congruent to $\sum_{d|n} d \cdot a_d^{np/d}$. Now if $d \mid pn$, $d \nmid n$, then $p \mid n$, which shows that the two summands are congruent. It follows that there exists $x = (x_n)_{n \in \mathbb{N}}$ such that

$$p \cdot x_n = w_n(F_p(a) - a^p) \iff x_n = \frac{1}{p} \cdot w_n(F_p(a) - a^p) \qquad \text{(Eq.3)}$$

We want to show that $x = w(b)$ for some $b \in \mathbb{W}(A)$. Then

$$w(p \cdot b) = p \cdot w(b) = p \cdot x = w(F_p(a) - a^p)$$

which implies by injectivity of w that $p \cdot b = F_p(a) - a^p$. For this, we want to use Lemma 3.2 again. Consider the unique ring homomorphism $\phi_l\colon A \to A$ which maps a_n to a_n^l. By Lemma 3.2 it suffices to show:

$$x_n \equiv \phi_l(x_{n/l}) \qquad \bmod l^{v_l(n)}$$

for all primes l, for all $n \in N$ with $l \mid n$. But this is equivalent to:

$$w_n(F_p(a) - a^p) \equiv \phi_l(w_{n/l}(F_p(a) - a^p)) \qquad \bmod l^{v_l(n)}A \quad \forall l \neq p, \forall n \in l\mathbb{N}$$

and

$$w_n(F_p(a) - a^p) \equiv \phi_p(W_{n/p}(F_p(a) - a^p)) \qquad \bmod p^{v_p(n)+1}A \quad \forall n \in p\mathbb{N}$$

(Using Eq.3 we have for $l = p$:

$$\begin{aligned} x_n \equiv \phi_p(x_{n/p}) \bmod p^{v_p(n)}A \iff & p \cdot x_n \equiv p \cdot \phi_p(x_{n/p}) && \bmod p^{v_p(n)+1}A \\ \overset{Eq.3}{\iff} & w_n(F_p(a) - a^p) \equiv \phi_p(w_{n/p}(F_p(a) - a^p)) && \bmod p^{v_p(n)+1}A \end{aligned}$$

and for $l \neq p$:

$$x_n \equiv \phi_l(x_{n/l}) \bmod l^{v_l(n)}A \iff p \cdot x_n \equiv p \cdot \phi_l(x_{n/l}) \qquad \bmod l^{v_l(n)}A$$
$$\overset{Eq.3}{\iff} w_n(F_p(a) - a^p) \equiv \phi_l(w_{n/l}(F_p(a) - a^p)) \qquad \bmod l^{v_l(n)}A.)$$

For $l \neq p$, the statement follows directly from Lemma 3.2. So now let $l = p$, let $n \in p\mathbb{N}$. Then:

$$\begin{aligned} & w_n(F_p(a) - a^p) - \phi_p(w_{n/p}(F_p(a) - a^p)) \\ &= w_{pn}(a) - w_n(a)^p - \phi_p(w_n(a)) + \phi_p(w_{n/p}(a))^p \\ &= \sum_{d|pn} d \cdot a_d^{pn/d} - (\sum_{d|n} d \cdot a_d^{n/d})^p - \sum_{d|n} d \cdot a_d^{np/d} + (\sum_{d|n/p} d \cdot a_d^{n/d})^p \end{aligned}$$

using Lemma 3.9 for the first equality. Now if $d \mid pn, d \nmid n$, then $v_p(d) = v_p(n) + 1$, hence the first and third summand cancel each other out, and for the second and forth summand, using Eq.2 and 3.1 again we have

$$\sum_{d|n} d \cdot a_d^{n/d} \equiv \sum_{d|n/p} d \cdot a_d^{n/d} \bmod p^{v_p(n)}A \implies (\sum_{d|n} d \cdot a_d^{n/d})^p \equiv (\sum_{d|n/p} d \cdot a_d^{n/d})^p \bmod p^{v_p(n)+1}A$$

which proves the claim. Now in the general case, let $a' \in \mathbb{W}(A')$. Then

$$F_p(a') = \mathbb{W}e(F_p(a)) = \mathbb{W}e(a^p + p \cdot r) = (a')^p + p \cdot \mathbb{W}e(r)$$

for some $r \in A$. □

Proposition 3.16 *There exists a unique natural transformation*

$$\Delta\colon \mathbb{W}(A) \to \mathbb{W}(\mathbb{W}(A))$$

such that $w_n(\Delta(a)) = F_n(a)$ for all $a \in \mathbb{W}(A), n \in \mathbb{N}$.

Proof: By naturality of Δ, we can assume A to be torsion-free. (If A' is an arbitrary ring, then the naturality implies uniqueness in the same way we argued in 3.9.) By applying Corollary 3.14, we get that the ghost map

$$w\colon \mathbb{W}(\mathbb{W}(A)) \to \mathbb{W}(A)^{\mathbb{N}}$$

is injective. Now by Lemma 3.15, $F_p\colon \mathbb{W}(A) \to \mathbb{W}(A)$ satisfies $F_p(a) \equiv a^p \bmod p\mathbb{W}(A)$, hence we can use Lemma 3.2 again and just show that

$$F_n(a) \equiv F_p(F_{n/p}(a)) \qquad \bmod p^{v_p(n)}A$$

for all p prime, $n \in p\mathbb{N}$. But this immediately follows from Lemma 3.10, so there is a unique $\Delta(a) \in \mathbb{W}(\mathbb{W}(A))$ such that $w_n(\Delta(a)) = F_n(a)$. Now Δ is a natural ring homomorphism by construction, arguing as in 3.9. □

Recall that by 3.5, $w_1\colon \mathbb{W}(A) \to A$; $(a_n)_{n\in\mathbb{N}} \mapsto a_1$ is a natural transformation of functors $\mathbb{W} \Rightarrow \mathrm{id}_{\mathbf{CRing}}$.

Theorem 3.17 *The functor* $\mathbb{W}(\_)\colon \mathbf{CRing} \to \mathbf{CRing}$ *together with the natural transformations* $\Delta\colon \mathbb{W} \Rightarrow \mathbb{W}^2$, $w_1\colon \mathbb{W} \Rightarrow \mathrm{id}_{\mathbf{CRing}}$ *form a comonad* $(\mathbb{W}, w_1, \Delta)$.

Proof: By naturality of Δ, we can assume that A is torsion-free, because if A' is an arbitrary ring, to show the associativity axiom, we can choose a torsion-free ring A and $g\colon A \to A'$ surjective as before and then consider the following cube:

$$\begin{tikzcd}
\mathbb{W}(A) \arrow[rr, "\Delta_A"] \arrow[dr, two heads, "\mathbb{W}(g)"'] \arrow[dd, "\Delta_A"'] & & \mathbb{W}(\mathbb{W}(A)) \arrow[dr, "\mathbb{W}(\mathbb{W}(g))"] \arrow[dd, dotted, "\Delta_{\mathbb{W}(A)}"] & \\
& \mathbb{W}(A') \arrow[rr, "\Delta_{A'}"] \arrow[dd, "\Delta_{A'}"'] & & \mathbb{W}(\mathbb{W}(A')) \arrow[dd, "\Delta_{\mathbb{W}(A')}"] \\
\mathbb{W}(\mathbb{W}(A)) \arrow[rr, dotted, "\mathbb{W}(\Delta_A)"] \arrow[dr, "\mathbb{W}(\mathbb{W}(g))"'] & & \mathbb{W}(\mathbb{W}(\mathbb{W}(A))) \arrow[dr, dotted, "\mathbb{W}(\mathbb{W}(\mathbb{W}(g)))"] & \\
& \mathbb{W}(\mathbb{W}(A')) \arrow[rr, "\mathbb{W}(\Delta_{A'})"] & & \mathbb{W}(\mathbb{W}(\mathbb{W}(A')))
\end{tikzcd}$$

Since all the other faces of the cube commute and $\mathbb{W}(g)$ is surjective, the front face has to commute as well. By the same reasoning we get the unitality axiom in the general case.

Claim :
$$\begin{tikzcd}
\mathbb{W}(A) \arrow[r, "\Delta_A"] \arrow[d, "\Delta_A"'] \arrow[dr, phantom, "\#"] & \mathbb{W}(\mathbb{W}(A)) \arrow[d, "\mathbb{W}(\Delta_A)"] \\
\mathbb{W}(\mathbb{W}(A)) \arrow[r, "\Delta_{\mathbb{W}(A)}"] & \mathbb{W}(\mathbb{W}(\mathbb{W}(A)))
\end{tikzcd}$$
commutes.

Proof of claim. evaluating the ghost coordinates leads to:

$$\begin{tikzcd}
\mathbb{W}(A) \arrow[r, "\Delta_A"] \arrow[d, "\Delta_A"'] \arrow[rr, dotted, bend left, "F_A"] & \mathbb{W}(\mathbb{W}(A)) \arrow[r, dotted, "w"] \arrow[d, "\mathbb{W}(\Delta_A)"] & \mathbb{W}(A)^{\mathbb{N}} \arrow[d, dotted, "\Delta_A^{\mathbb{N}}"] \\
\mathbb{W}(\mathbb{W}(A)) \arrow[r, "\Delta_{\mathbb{W}(A)}"] \arrow[rr, dotted, bend right, "F_{\mathbb{W}_A}"'] & \mathbb{W}(\mathbb{W}(\mathbb{W}(A))) \arrow[r, blue, "w"] & \mathbb{W}(\mathbb{W}(A))^{\mathbb{N}}
\end{tikzcd}$$

which by Proposition 3.16 simplifies to the left of the following diagrams, now it suffices to show for an arbitrary n that the right diagram commutes.

$$\begin{tikzcd}
\mathbb{W}(A) \arrow[r, "F_A"] \arrow[d, "\Delta_A"'] & \mathbb{W}(A)^{\mathbb{N}} \arrow[d, "\Delta_A^{\mathbb{N}}"] \\
\mathbb{W}(\mathbb{W}(A)) \arrow[r, "F_{\mathbb{W}(A)}"] & \mathbb{W}(\mathbb{W}(A))^{\mathbb{N}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{W}(A) \arrow[r, "(F_n)_A"] \arrow[d, "\Delta_A"'] & \mathbb{W}(A) \arrow[d, "\Delta_A"] \\
\mathbb{W}(\mathbb{W}(A)) \arrow[r, "(F_n)_{\mathbb{W}(A)}"] & \mathbb{W}(\mathbb{W}(A))
\end{tikzcd}$$

evaluating the ghost coordinates again, keeping in mind that by 3.14 and 3.3, the map $w\colon \mathbb{W}(\mathbb{W}(A)) \to$

$\mathbb{W}(A)^{\mathbb{N}}$ is injective as well, we get

$$\begin{tikzcd}
\mathbb{W}(A) \arrow[r, "(F_n)_A"] \arrow[d, "\Delta_A"] & \mathbb{W}(A) \arrow[d, "\Delta_A"] \arrow[dd, dotted, bend left, "F_A"] \\
\mathbb{W}(\mathbb{W}(A)) \arrow[r, "(F_n)_{\mathbb{W}(A)}"] \arrow[d, dotted, "w"] & \mathbb{W}(\mathbb{W}(A)) \arrow[d, "w"] \\
\mathbb{W}(A)^{\mathbb{N}} \arrow[r, dotted, "(\widetilde{F}_n)_{\mathbb{W}(A)}"] & \mathbb{W}(A)^{\mathbb{N}}
\end{tikzcd}$$

using the fact that $\begin{tikzcd}
\mathbb{W}(\mathbb{W}(A)) \arrow[d, dotted, "w"] \arrow[dr, dotted, "w_{nm}"] & \\
\mathbb{W}(A)^{\mathbb{N}} \arrow[r, dotted, "(\widetilde{F}_n)_{\mathbb{W}(A)}"] & \mathbb{W}(A)^{\mathbb{N}}
\end{tikzcd}$ commutes, we can simplify the situation to the left of the following two diagrams which can again be simplified to the right diagram for every n.

$$\begin{tikzcd}
\mathbb{W}(A) \arrow[r, "F_n"] \arrow[d, "\Delta_A"] \arrow[dr, dotted, "F_{nm}"] & \mathbb{W}(A) \arrow[d, "F_m"] \\
\mathbb{W}(\mathbb{W}(A)) \arrow[r, "w_{nm}"] & \mathbb{W}(A)
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{W}(A) \arrow[r, "F_n"] \arrow[dr, "F_{nm}"'] & \mathbb{W}(A) \arrow[d, "F_m"] \\
& \mathbb{W}(A)
\end{tikzcd}$$

Now this commutes by Lemma 3.10, hence we are finished. //

Claim : $\begin{tikzcd}
\mathbb{W}(A) \arrow[d, "\Delta_A"'] \arrow[dr, "id_{\mathbb{W}(A)}"] & \\
\mathbb{W}(\mathbb{W}(A)) \arrow[r, "\mathbb{W}(w_1)"'] & \mathbb{W}(A)
\end{tikzcd}$ commutes.

Proof of claim. evaluate the ghost coordinates:

$$\begin{tikzcd}
\mathbb{W}(A) \arrow[d, "\Delta_A"'] \arrow[dr, "id_{\mathbb{W}(A)}"] \arrow[dd, dotted, bend right, "F"'] & \\
\mathbb{W}(\mathbb{W}(A)) \arrow[r, "\mathbb{W}(w_1)"'] \arrow[d, dotted, "w"] & \mathbb{W}(A) \arrow[d, "w"] \\
\mathbb{W}(A)^{\mathbb{N}} \arrow[r, dotted, "w_1^{\mathbb{N}}"] & A^{\mathbb{N}}
\end{tikzcd}$$

we can then simplify to the left of the following diagrams.

$$\begin{tikzcd}
\mathbb{W}(A) \arrow[d, "F"'] \arrow[dr, "w"] & \\
\mathbb{W}(A)^{\mathbb{N}} \arrow[r, "w_1^{\mathbb{N}}"'] & A^{\mathbb{N}}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
\mathbb{W}(A) \arrow[d, "F_n"'] \arrow[dr, "w_n"] & \\
\mathbb{W}(A) \arrow[r, "w_1"'] & A
\end{tikzcd}$$

Again it suffices to show that for all n the right of the two diagrams commutes, which is true by Lemma 3.9. //

Claim :
$$\begin{tikzcd} & \mathbb{W}(A) \arrow[dl, "\mathrm{id}_{\mathbb{W}(A)}"'] \arrow[d, "\Delta_A"] \\ \mathbb{W}(A) & \mathbb{W}(\mathbb{W}(A)) \arrow[l, "w_1"] \end{tikzcd}$$
commutes.

Proof of claim. Let $a \in \mathbb{W}(A)$.
$w_1(\Delta_A(a)) = F_1(a) = a$, since $F_1 = \mathrm{id}_{\mathbb{W}(A)}$ by Lemma 3.11. //

This concludes the proof. □

3.4 The Teichmüller Map induces a Morphism of Comonads

Now consider the *teichmüller map* $\tau\colon A \to \mathbb{W}(A); a \mapsto (a, 0, 0, 0, \dots)$. It is multiplicative and preserves the unit, hence it extends uniquely to a natural ring homomorphism

$$\tau_A \colon \mathbb{Z}A \to \mathbb{W}(A)$$

(see Example 10).

Theorem 3.18 *$\tau\colon \mathbb{Z}[\_] \Rightarrow \mathbb{W}(\_)$ is a morphism of comonads.*

Proof: We need to show that the following diagrams commute:

$$\begin{tikzcd} \mathbb{Z}A \arrow[r, "\tau_A"] \arrow[dr, "\varepsilon_A"'] & \mathbb{W}(A) \arrow[d, "(w_1)_A"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} \mathbb{Z}A \arrow[r, "\omega_A"] \arrow[d, "\tau_A"] & \mathbb{Z}\mathbb{Z}A \arrow[d, "\tau\otimes\tau"] \\ \mathbb{W}(A) \arrow[r, "\Delta_A"] & \mathbb{W}(\mathbb{W}(A)) \end{tikzcd}$$

By the universal property of $\mathbb{Z}A$, it suffices to consider elements of the form $[a]$ for $a \in A$. For the first diagram: $w_1(\tau([a])) = a = \varepsilon([a])$. For the second diagram, arguing as before, it suffices to show commutativity after evaluating the ghost coordinates:

$$\begin{tikzcd} \mathbb{Z}A \arrow[r, "\omega_A"] \arrow[d, "\tau_A"] & \mathbb{Z}\mathbb{Z}A \arrow[d, "\tau\otimes\tau"] \\ \mathbb{W}(A) \arrow[r, "\Delta_A"] \arrow[dr, dotted, "F_n"'] & \mathbb{W}(\mathbb{W}(A)) \arrow[d, "w_n"] \\ & \mathbb{W}(A) \end{tikzcd}$$

Note that $F_n(\tau([a])) = \tau([a^n])$ since evaluating the ghost coordinates shows that the equation holds if A is torsion-free (using 3.9), and hence, in general. Using this, we see that $w_n(\tau \otimes \tau(\omega([a]))) = w_n(\tau \otimes \tau([[a]])) = w_n((\tau([a]), 0, \dots)) = \tau([a])^n = (a^n, 0, \dots)$ and $F_n(\tau([a])) = \tau([a^n]) = (a^n, 0, \dots)$. This concludes the proof. □