1 Adjoint situations

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Proposition 1.1 *Given two functors* $\mathbf{A} \underset{F}{\overset{G}{\rightleftarrows}} \mathbf{B}$ *the following are equivalent:*

(a) $\exists \eta\colon \mathrm{id}_B \Rightarrow GF$ *and* $\varepsilon\colon FG \Rightarrow \mathrm{id}_A$ *natural transformations such that* $\forall a \in Ob(A), b \in Ob(B)$ *the following two diagrams commute:*

$$\begin{tikzcd} A \arrow[r, "F(\eta_b)"] \arrow[dr, "\mathrm{id}_{F(b)}"'] & B \arrow[d, "\varepsilon_{F(b)}"] \\ & C \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "F(\eta_b)"] \arrow[dr, "\mathrm{id}_{F(b)}"'] & B \arrow[d, "\varepsilon_{F(b)}"] \\ & C \end{tikzcd} \qquad \text{(triangle identity)}$$

(b) $\forall a \in Ob(A), b \in Ob(B)$ *there is a bijection*

$$\phi_{a,b}\colon \mathrm{Hom}_{\mathbf{A}}(F(b), a) \to \mathrm{Hom}_{\mathbf{B}}(b, G(a))$$

which is natural in a and b, i.e. for $p\colon a \to a'$:

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{A}}(F(b), a) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbf{B}}(b, G(a)) \arrow[d] \\ \mathrm{Hom}_{\mathbf{A}}(F(b), a') \arrow[r] & \mathrm{Hom}_{\mathbf{B}}(b, G(a')) \end{tikzcd}$$

and for $q\colon b \to b'$:

$$\begin{tikzcd} \mathrm{Hom}_{\mathbf{A}}(F(b'), a) \arrow[r] \arrow[d] & \mathrm{Hom}_{\mathbf{B}}(b', G(a)) \arrow[d] \\ \mathrm{Hom}_{\mathbf{A}}(F(b), a) \arrow[r] & \mathrm{Hom}_{\mathbf{B}}(b, G(a)) \end{tikzcd}$$

Proof: $(a) \implies (b)$:
define

$$\phi_{a,b}\colon \mathrm{Hom}(F(b), a) \to \mathrm{Hom}(b, G(a))$$

by $g \mapsto G(g) \circ \eta_b$ for $g\colon F(b) \to a$ □