1 Adjoint situations

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Proposition 1.1 Given two functors $A \xrightarrow{F} B$, $B \xrightarrow{G} A$, the following are equivalent:

(a) $\exists \eta\colon \mathrm{id}_B \to GF$ and $\epsilon\colon FG \to \mathrm{id}_A$ such that $\forall a \in Ob(A), b \in Ob(B)$ the following two diagrams commute:

$$\begin{tikzcd} A \arrow[r, "F(\eta_b)"] \arrow[dr, "\mathrm{id}_{F(b)}"'] & B \arrow[d, "\epsilon_{F(b)}"] \\ & C \end{tikzcd} \text{ and } \begin{tikzcd} A \arrow[r, "F(\eta_b)"] \arrow[dr, "\mathrm{id}_{F(b)}"'] & B \arrow[d, "\epsilon_{F(b)}"] \\ & C \end{tikzcd}$$

(b) $\forall a \in Ob(A), b \in Ob(B)$ there is a bijection

$$\phi_{a,b}\colon \hom(F(b), a) \to \hom(b, G(a))$$

which is natural in a and b, i.e. for $p\colon a \to a'$:

$$\begin{tikzcd} \hom(F(b), a) \arrow[r] \arrow[d] & \hom(b, G(a)) \arrow[d] \\ \hom(F(b), a') \arrow[r] & \hom(b, G(a')) \end{tikzcd}$$

and for $q\colon b \to b'$:

$$\begin{tikzcd} \hom(F(b'), a) \arrow[r] \arrow[d] & \hom(b', G(a)) \arrow[d] \\ \hom(F(b), a) \arrow[r] & \hom(b, G(a)) \end{tikzcd}$$

Proof: $(a) \implies (b)$:
define

$$\phi_{a,b}\colon \hom(F(b), a) \to \hom(b, G(a))$$

by $g \mapsto G(g) \circ \eta_b$ for $g\colon F(b) \to a$ $\square$

2 Monads and Comonads

2.1 Definition of Monads and Comonads

Definition 2.1 (Monad) A *Monad* (T, μ, η) in a Category $\mathcal{X}$ consists of

- an endofunctor $T : \mathcal{X} \to \mathcal{X}$
- a natural transformation $\eta : \mathrm{id}_{\mathcal{X}} \Rightarrow T$
- a natural transformation $\mu : T^2 \Rightarrow T$

such that the following diagrams commute:

(a)
$$\begin{tikzcd} T^3 \arrow[r] \arrow[d] & T^2 \arrow[d] \\ T^2 \arrow[r] & T \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[dr, "\mathrm{id}_T"'] & T^2 \arrow[d, "\mu"] & T \arrow[l, "T\eta"'] \arrow[dl, "\mathrm{id}_T"] \\ & T & \end{tikzcd}$$

Example 1 (preorder). Recall: A *preorder* $(\mathcal{P}, \leq)$ is a category with $\mathcal{P}$ as objects and a morphism between X and Y iff $X \leq Y$. A functor $T : \mathcal{P} \to \mathcal{P}$ is thus a monotonic function $\mathcal{P} \to \mathcal{P}$ ($x \leq y \implies Tx \leq Ty$). The existence of the natural transformations η is equivalent to

$$x \leq Tx \ \forall x \in \mathcal{P}$$

and the existence of μ is equivalent to

$$T(Tx) \leq Tx \ \forall x \in \mathcal{P}$$

because there is at most one morphism $x \to y$, so the neccessary diagrams commute trivially.
Now suppose $\mathcal{P}$ is a *partial order*, i.e. $x \leq y \leq x \implies x = y \ \forall x, y \in \mathcal{P}$.
Then:

$$x \leq Tx \implies Tx \leq T(Tx)$$
$$T(Tx) \leq Tx \implies Tx = T(Tx)$$

so a Monad T in a partial order $\mathcal{P}$ is a *closure operation* in $\mathcal{P}$, i.e. a monotonic function $T : \mathcal{P} \to \mathcal{P}$ with $x \leq Tx$ and $T(Tx) = Tx \ \forall x \in \mathcal{P}$.
Now every topological space X induces a partial order $\mathcal{P} = (\mathscr{P}(X), \subseteq)$. Here an example for a closure operation is taking the topological closure $A \mapsto \overline{A}$, since it holds for all $A \subseteq X$ that $A \subseteq \overline{A}$ and $\overline{\overline{A}} = \overline{A}$.

Definition 2.2 (Comonad) A *Comonad* (L, ε, ω) in a Category $\mathcal{A}$ consists of

- an endofunctor $L : \mathcal{A} \to \mathcal{A}$
- a natural transformation $\varepsilon : L \Rightarrow \mathrm{id}_{\mathcal{A}}$
- a natural transformation $\omega : L \Rightarrow L^2$

such that the following diagrams commute:

(a)

$$\begin{tikzcd} L \arrow[r, "L\omega"] \arrow[d, "\omega L"'] & L^2 \arrow[d, "L\omega"] \\ L^2 \arrow[r, "\omega L"] & L^3 \end{tikzcd} \quad \text{and} \quad \begin{tikzcd} L & L^2 \arrow[l, "\varepsilon L"'] \arrow[r, "L\varepsilon"] & L \\ & L \arrow[ul, "\mathrm{id}_L"] \arrow[u, "\omega"'] \arrow[ur, "\mathrm{id}_L"'] & \end{tikzcd}$$

Definition 2.3 (Morphism of monads) Let $\mathcal{X}$ be a category, let (T, η, μ) and (T', η', μ') be monads in $\mathcal{X}$. We say that a natural transformation $\delta \colon T \implies T'$ is a *morphism of monads* if it preserves the unit and the multiplication, i.e. the following diagrams commute:

$$\begin{tikzcd} \mathrm{id}_x \arrow[r, "\eta_x"] \arrow[dr, "\eta'_x"'] & Tx \arrow[d, "\delta_x"] \\ & T'x \end{tikzcd}$$

$$\begin{tikzcd} T^2x \arrow[r, "\mu_x"] \arrow[d, "\delta T' \circ T\delta"'] & Tx \arrow[d, "\delta_x"] \\ T'^2 \arrow[r, "\mu'_x"] & T'x \end{tikzcd}$$

show that the other composition is the same(siehe iPad)

Definition 2.4 (Morphism of comonads)

definition

3 Witt vectors

Construction of the witt vectors

Definition 3.1 (truncation set) Let $\mathbb{N}$ be the set of positive integers and let $S \subseteq \mathbb{N}$ be a subset with the property that $\forall n \in \mathbb{N}$: if d is a divisor of n, then $d \in S$. We then say that S is a *truncation set.*

As a set, we define the *big Witt ring* $\mathbb{W}_S(A)$ to be A^S, we will give it a unique ring structure, such that the *ghost map* is a ring homomorphism.

Definition 3.2 (ghost map) We define $w\colon \mathbb{W}_S(A) \to A^S$ by $(a_n)_{n\in S} \mapsto (w_n)_{n\in S}$ where

$$w_n = \sum_{d|n} d a_d^{n/d}$$

Lemma 3.3 (Dwork) Suppose that for every prime number p there exists a ring homomorphism $\phi_p\colon A \to A$ with the property that $\phi_p(a) \equiv a^p$ modulo pA. Then for every sequence $x = (x_n)_{n\in S}$, the following are equivalent:

(i) The sequence x is in the image of the ghost map $w\colon \mathbb{W}_S(A) \to A^S$.

(ii) For every prime number p and every $n \in S$ with $v_p(n) \geqslant 1$,

$$x_n \equiv \phi_p(x_{n/p}) \qquad \text{modulo } p^{v_p(n)}A.$$

Proof: ($\Rightarrow$) Suppose x is in the image of the ghost map, that means there is a sequence $a = (a_n)_{n\in S}$ such that $x_n = w_n(a)$ for all $n \in S$. We calculate:

$$\phi(x_{n/p}) = \phi(w_{n/p}(a)) = \phi(\sum_{d|n/p} d a_d^{n/pd}) = \sum_{d|n/p} d \cdot \phi(a_d^{n/pd})$$

since ϕ is a ring homomorphism and $d \in \mathbb{N}$.

Claim 1. $\sum_{d|n/p} d \cdot \phi(a_d^{n/pd}) \equiv \sum_{d|n/p} d \cdot a_d^{n/d} \qquad mod\ p^{v_p(n)}A.$

Proof (Proof of claim 1): //

Claim 2. $\sum_{d|n/p} d \cdot a_d^{n/d} \equiv \sum_{d|n} d \cdot a_d^{n/d} \qquad mod\ p^{v_p(n)}A$

Proof (Proof of claim 2): //

so we get

$$\phi(x_{n/p}) \equiv \sum_{d|n} d \cdot a_d^{n/d} = w_n(a) = x_n \qquad mod\ p^{v_p(n)}A.$$

($\Leftarrow$) Let $(x_n)_{n\in S}$ be a sequence such that $x_n \equiv \phi_p(x_{n/p}) \qquad mod\ p^{v_p(n)}A\ \forall p$ prime, $n \in S, v_p(n) \geqslant 1$. Define $(a_n)_{n\in S}$ with $w_n(a) = x_n$ as follows:

$$a_1 := x_1$$

and if a_d has been chosen for all $d \mid n$ such that $w_d(a) = x_d$ we see that

$$\begin{aligned} x_n &\equiv \phi_p(x_{n/p}) \qquad mod\ p^{v_p(n)}A \\ &= \phi_p(\sum_{d|n/p} d \cdot a_d^{n/pd}) \\ &= \sum_{d|n/p} d \cdot \phi(a_d^{n/pd}) \end{aligned}$$

□ finish proof

We will often need the following

Lemma 3.4 if A is a torsion-free ring, the ghost map is injective.

Now we can finish the construction of the Witt vectors:

Theorem 3.5 There exists a unique ring structure such that the ghost map

$$w : \mathbb{W}_S(A) \longrightarrow A^s$$

is a natural transformation of functors from rings to rings.

Proof: □

Corollary 3.6 $w_n \colon \mathbb{W}_S(A) \longrightarrow A$ is a natural transformation for all $n \in S$.

Proposition 3.7 $\mathbb{W}_S$ is a functor **CRing** $\longrightarrow$ **CRing**.

The Verschiebung, Frobenius and Teichmüller maps

Definition 3.8 (Restriction map) If $T \subseteq S$ are two truncation sets, the *restriction from S to T*

$$R^S_T \colon \mathbb{W}_S(A) \longrightarrow \mathbb{W}_T(A)$$

is a natural ring homomorphism.

If $S \subseteq \mathbb{N}$ is a truncation set, then

$$S/n := \{d \in \mathbb{N} \mid nd \in S\}$$

is again a truncation set.

Definition 3.9 (Verschiebung) Define

$$V_n \colon \mathbb{W}_{S/n} \longrightarrow \mathbb{W}_S(A);\ V_n((a_d)_{d \in S/n})_m := \begin{cases} a_d, & \text{if } m = n \cdot d \\ 0, & \text{else} \end{cases}$$

which is called the *n-th Verschiebung map*. Furthermore define

$$\widetilde{V}_n \colon A^{S/n} \longrightarrow A^S;\ \widetilde{V}_n((x_d)_{d \in S/n})_m := \begin{cases} n \cdot x_d, & \text{if } m = n \cdot d \\ 0, & \text{else} \end{cases}$$

Lemma 3.10 The Verschiebung map V_n is additive.

Proof:

Claim.
$$\begin{tikzcd} \mathbb{W}_{S/n}(A) \arrow[r, "w"] \arrow[d, "V_n"] & A^{S/n} \arrow[d, "\widetilde{V_n}"] \\ \mathbb{W}_S(A) \arrow[r, "w"] & A^S \end{tikzcd}$$
commutes.

Proof (Proof of claim): //

□

The comonad structure of witt vectors

We will need the following lemma:

Lemma 3.11 Let $m \in \mathbb{Z}$. If m is a non-zero divisor in A, then it is a non-zero divisor in $\mathbb{W}_S(A)$ as well.

Proof (Proof of claim):

$$0 \longrightarrow A \xrightarrow{V_n} \mathbb{W}_S(A) \xrightarrow{R^S_T} W_T(A) \longrightarrow 0$$

which we can extend to the following commutative diagram:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\cdot m"] & \mathbb{W}_S(A) \arrow[r] \arrow[d, "\cdot m"] & \mathbb{W}_T(A) \arrow[r] \arrow[d, "\cdot m"] & 0 \\ 0 \arrow[r] & A \arrow[r] & \mathbb{W}_S(A) \arrow[r] & \mathbb{W}_T(A) \arrow[r] & 0 \end{tikzcd}$$

finish

//

Definition 3.12 $\mathbb{W}(A) := \mathbb{W}_{\mathbb{N}}(A)$

For the construction of a natural transformation $\mathbb{W}(A) \to \mathbb{W}(\mathbb{W}(A))$ we want to use Lemma 3.3 again. Hence we first show:

Lemma 3.13 Let p be a prime number, let A be any ring. Then the ring homomorphism $F_p \colon \mathbb{W}(A) \to \mathbb{W}(A)$ satisfies $F_p(a) \equiv a^p \; mod \; pA$.

Proposition 3.14 There exists a unique natural transformation

$$\Delta \colon \mathbb{W}(A) \to \mathbb{W}(\mathbb{W}(A))$$

such that $w_n(\Delta(a)) = F_n(A)$ for all $a \in A, n \in \mathbb{N}$.

Theorem 3.15 The functor $\mathbb{W}(\_) \colon \mathbf{CRing} \to \mathbf{CRing}$ together with the natural transformations $\Delta \colon \mathbb{W} \Rightarrow \mathbb{W}^2$, $w_1 \colon \mathbb{W} \Rightarrow \mathrm{id}_{\mathbf{CRing}}$ form a comonad.

Proof:

Claim.
$$\begin{tikzcd} \mathbb{W}(A) \arrow[r, "\Delta_A"] \arrow[d, "\Delta_A"] \arrow[dr, phantom, "\#"] & \mathbb{W}(\mathbb{W}(A)) \arrow[d, "\mathbb{W}(\Delta_A)"] \\ \mathbb{W}(\mathbb{W}(A)) \arrow[r, "\Delta_{\mathbb{W}(A)}"] & \mathbb{W}(\mathbb{W}(\mathbb{W}(A))) \end{tikzcd}$$
commutes.

Proof (Proof of claim): evaluating the ghost coordinates leads to:

$$\begin{tikzcd} \mathbb{W}(A) \arrow[r, "\Delta_A"] \arrow[d, "\Delta_A"] \arrow[rr, dotted, bend left, "F_A"] & \mathbb{W}(\mathbb{W}(A)) \arrow[r, dotted, "w"] \arrow[d, "\mathbb{W}(\Delta_A)"] & \mathbb{W}(A)^{\mathbb{N}} \arrow[d, dotted, "\Delta_A^{\mathbb{N}}"] \\ \mathbb{W}(\mathbb{W}(A)) \arrow[r, "\Delta_{\mathbb{W}(A)}"] \arrow[rr, dotted, bend right, "F_{\mathbb{W}_A}"'] & \mathbb{W}(\mathbb{W}(\mathbb{W}(A))) \arrow[r, blue, "w"] & \mathbb{W}(\mathbb{W}(A))^{\mathbb{N}} \end{tikzcd}$$

which simplifies to

$$\begin{tikzcd} \mathbb{W}(A) \arrow[r, "F_A"] \arrow[d, "\Delta_A"] & \mathbb{W}(A)^{\mathbb{N}} \arrow[d, "\Delta_A^{\mathbb{N}}"] \\ \mathbb{W}(\mathbb{W}(A)) \arrow[r, "F_{\mathbb{W}(A)}"] & \mathbb{W}(\mathbb{W}(A))^{\mathbb{N}} \end{tikzcd}$$

now it suffices to show for an arbitrary n that the following diagram commutes:

$$\begin{tikzcd} \mathbb{W}(A) \arrow[r, "F_{n_A}"] \arrow[d, "\Delta_A"] & \mathbb{W}(A) \arrow[d, "\Delta_A"] \\ \mathbb{W}(\mathbb{W}(A)) \arrow[r, "F_{n_{\mathbb{W}(A)}}"] & \mathbb{W}(\mathbb{W}(A)) \end{tikzcd}$$

evaluating the ghost coordinates again, keeping in mind that by Lemma 9, $w\colon \mathbb{W}(\mathbb{W}(A)) \to \mathbb{W}(A)^{\mathbb{N}}$ is injective as well, we get

$$\begin{tikzcd} \mathbb{W}(A) \arrow[r, "F_{n_A}"] \arrow[d, "\Delta_A"] & \mathbb{W}(A) \arrow[d, "\Delta_A"] \arrow[dd, dotted, bend left, "F_A"] \\ \mathbb{W}(\mathbb{W}(A)) \arrow[r, "F_{n_{\mathbb{W}(A)}}"] \arrow[d, dotted, "w"] & \mathbb{W}(\mathbb{W}(A)) \arrow[d, blue, "w"] \\ \mathbb{W}(A)^{\mathbb{N}} \arrow[r, dotted, "\widetilde{F_n}_{\mathbb{W}(A)}"] & \mathbb{W}(A)^{\mathbb{N}} \end{tikzcd}$$

using the fact that
$$\begin{tikzcd} \mathbb{W}(\mathbb{W}(A)) \arrow[d, dotted, "w"] \arrow[dr, dotted, "w_{nm}"] & \\ \mathbb{W}(A)^{\mathbb{N}} \arrow[r, dotted, "\widetilde{F_n}_{\mathbb{W}(A)}"] & \mathbb{W}(A)^{\mathbb{N}} \end{tikzcd}$$
commutes, we can simplify the situation to

$$\begin{tikzcd} \mathbb{W}(A) \arrow[r, "F_n"] \arrow[d, "\Delta_A"] \arrow[dr, dotted, "F_{nm}"] & \mathbb{W}(A) \arrow[d, "F_m"] \\ \mathbb{W}(\mathbb{W}(A)) \arrow[r, "w_{nm}"] & \mathbb{W}(A) \end{tikzcd}$$

which can again be simplified to

$$\begin{tikzcd} \mathbb{W}(A) \arrow[r, "F_n"] \arrow[dr, "F_{nm}"'] & \mathbb{W}(A) \arrow[d, "F_m"] \\ & \mathbb{W}(A) \end{tikzcd}$$

now this commutes by ???, hence we are finished. //

Claim. $\begin{tikzcd} \mathbb{W}(A) \arrow[d, "\Delta_A"'] \arrow[dr, "\mathrm{id}_{\mathbb{W}(A)}"] & \\ \mathbb{W}(\mathbb{W}(A)) \arrow[r, "\mathbb{W}(\varepsilon_A)"'] & \mathbb{W}(A) \end{tikzcd}$ *commutes.*

Proof (Proof of claim): evaluate the ghost coordinates:

$$\begin{tikzcd} \mathbb{W}(A) \arrow[d, "\Delta_A"'] \arrow[dr, "\mathrm{id}_{\mathbb{W}(A)}"] \arrow[dd, dotted, bend right, "F"'] & \\ \mathbb{W}(\mathbb{W}(A)) \arrow[r, "\mathbb{W}(\varepsilon_A)"'] \arrow[d, dotted, "w"] & \mathbb{W}(A) \arrow[d, blue, "w"] \\ \mathbb{W}(A)^{\mathbb{N}} \arrow[r, dotted, "\varepsilon_A^{\mathbb{N}}"] & A^{\mathbb{N}} \end{tikzcd}$$

we can then simplify to

$$\begin{tikzcd} \mathbb{W}(A) \arrow[d, "F"'] \arrow[dr, "w"] & \\ \mathbb{W}(A)^{\mathbb{N}} \arrow[r, "\varepsilon_A^{\mathbb{N}}"'] & A^{\mathbb{N}} \end{tikzcd}$$

now it suffices to show for all n that

$$\begin{tikzcd} \mathbb{W}(A) \arrow[d, "F_n"'] \arrow[dr, "w_n"] & \\ \mathbb{W}(A) \arrow[r, "\varepsilon_A"'] & A \end{tikzcd}$$

commutes, which is true by ??? ($\varepsilon = w_1$).

//

Claim. $\begin{tikzcd} & \mathbb{W}(A) \arrow[dl, "\mathrm{id}_{\mathbb{W}(A)}"'] \arrow[d, "\Delta_A"] \\ \mathbb{W}(\mathbb{W}(A)) & \mathbb{W}(A) \arrow[l, "\varepsilon_{\mathbb{W}(A)}"] \end{tikzcd}$ *commutes.*

Proof (Proof of claim): Let $a \in \mathbb{W}(A)$.
$\varepsilon(\Delta_A(a)) = w_1(\Delta_A(a)) = F_1(a) = a$, since $F_1 = \mathrm{id}_{\mathbb{W}(A)}$. //

This concludes the proof. $\square$

The Teichmüller map induces a morphism of comonads

We now consider another example of a comonad; the *free monoid comonad.*

Definition 3.16 (monoid ring) Let R be a ring and let G be a monoid. The *monoid ring* of G over R, denoted $R[G]$ or RG is the set of formal finite sums $\sum_{g\in G} r_g \cdot g$ with addition and multiplication defined by:

$$\sum_{g\in G} r_g \cdot g + \sum_{g\in G} s_g \cdot g := \sum_{g\in G} (r_g + s_g) \cdot g$$
$$\sum_{g\in G} r_g \cdot g \cdot \sum_{g\in G} s_g \cdot g := \sum_{g\in G} (\sum_{k\cdot l=g} r_k \cdot s_l) \cdot g$$

Example 2. $R = \mathbb{R}, G = \{x^n \mid n \in \mathbb{N}\} \implies RG = \mathbb{R}[X]$

Proposition 3.17 $R[G]$ together with the ring homomorphism $\alpha\colon R \to R[G]; r \mapsto r \cdot 1$ and the monoid homomorphism $\beta\colon G \to R[G]; g \mapsto 1 \cdot g$ enjoys the following universal property:

$$\alpha(r) \cdot \beta(g) = \beta(g) \cdot \alpha(r) \quad \forall r \in R, g \in G$$

and if (S, α', β') is another such triple with $\alpha'(r) \cdot \beta'(g) = \beta'(g) \cdot \alpha'(r) \quad \forall r \in R, g \in G$, there is a unique monoid homomorphism $\gamma\colon R[G] \to S$ such that the following diagram commutes:

$$\begin{tikzcd}
 & S & \\
R \arrow[ur, "\alpha'"] \arrow[r, "\alpha"'] & R[G] \arrow[u, dotted, "\gamma"] & G \arrow[ul, "\beta'"'] \arrow[l, "\beta"]
\end{tikzcd}$$

Here, γ is defined by $\sum_{g\in G} r_g \cdot g \mapsto \sum_{g\in G} \alpha'(r_g) \cdot \beta'(g)$.

Example 3. Let S be a ring, G be a monoid. Since there is a unique ring homomorphism $\mathbb{Z} \to S$, each monoid homomorphism $G \to S$ induces a unique ring homomorphism $\mathbb{Z}G \to S$ such that the following commutes:

$$\begin{tikzcd}
G \arrow[r] \arrow[dr] & S \\
 & \mathbb{Z}G \arrow[u]
\end{tikzcd}$$

Now if H is another monoid and $f\colon G \to H$ a monoid morphism, $G \xrightarrow{f} H \to \mathbb{Z}H$ is a monoid homomorphism, hence it extends uniquely to $f\colon \mathbb{Z}G \to \mathbb{Z}H$, $\sum_{g\in G} r_g \cdot g \mapsto \sum_{g\in G} r_g \cdot f(g)$.
In this way, the free monoid ring construction over $\mathbb{Z}$ is functorial.

Let $G\colon$ **CRing** $\to$ **CMon**, $(R, +, \cdot) \mapsto (R, \cdot)$ be the forgetful functor and let $F\colon$ **CMon** $\to$ **CRing** be the *free monoid ring functor*, $G \mapsto \mathbb{Z}G$.

Proposition 3.18 There is an adjoint situation $\begin{tikzcd} \mathbf{CMon} \arrow[r, bend left, "F"] \arrow[r, phantom, "\perp"] & \mathbf{CRing} \arrow[l, bend left, "G"] \end{tikzcd}$

Now consider the *teichmüller map* $\tau\colon A \to \mathbb{W}(A)$. τ is multiplicative and preserves the unit, hence it extends uniquely to a ring homomorphism

$$\tau\colon \mathbb{Z}A \to \mathbb{W}(A)$$

Theorem 3.19 $\tau\colon \mathbb{Z}A \to \mathbb{W}(A)$ is a morphism of comonads.