1 Monads and Comonads

1.1 Definition of Monads and Comonads

A central notion in algebra is that of a *monoid*, that is, a set M equipped with a map $\mu\colon M \times M \to M$; $(a,b) \mapsto a \cdot b$ (often called *multiplication*) and an element $e \in M$ such that the following two axioms hold:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \text{for all } a,b,c \in M. \qquad \text{(associativity)}$$
$$e \cdot a = a \cdot e = a \quad \text{for all } a \in M \qquad \text{(identity element)}$$

We can give an equivalent definition in terms of maps and commuting diagrams as follows: A *monoid* is a set M together with two functions

$$\mu\colon M \times M \to M, \quad e\colon \{*\} \to M$$

such that the following diagrams commute:

$$\begin{tikzcd} M \times M \times M \arrow[r, "\mathrm{id} \times \mu"] \arrow[d, "\mu \times \mathrm{id}"] & M \times M \arrow[d, "\mu"] \\ M \times M \arrow[r, "\mu"] & M \end{tikzcd} \qquad \text{and} \qquad \begin{tikzcd} \{*\} \times M \arrow[r, "e \times \mathrm{id}"] \arrow[dr, "l"'] & M \times M \arrow[d, "\mu"] & M \times \{*\} \arrow[l, "\mathrm{id} \times e"'] \arrow[dl, "r"] \\ & M & \end{tikzcd}$$

where id is the identity on m, and l and r are the canonical bijections

$$l\colon \{*\} \times M \to M; \; l(*,m) = m$$
$$r\colon M \times \{*\} \to M; \; r(m,*) = m.$$

Explicitly, the first diagram means that for all $a,b,c \in M$:

$$(a \cdot b) \cdot c = a \cdot (b \cdot c) \quad \text{for all } a,b,c \in M.$$

which is verbatim the associativity axiom, the second diagram means that for all $m \in M$:

$$e(*) \cdot m = l(*,m) = m = r(m,*) = m \cdot e(*)$$

which is clearly the identity element axiom for the element $e(*)$. This motivates the following definition: monoid/monad/ monoid object

Definition 1.1 (Monad). A *Monad* (T, μ, η) in a category $\mathcal{X}$ consists of

- an endofunctor $T\colon \mathcal{X} \to \mathcal{X}$
- a natural transformation $\eta\colon \mathrm{id}_{\mathcal{X}} \Rightarrow T$
- a natural transformation $\mu\colon T^2 \Rightarrow T$

such that the following diagrams commute:

$$\begin{tikzcd} T^3 \arrow[r, Rightarrow, "T\mu"] \arrow[d, Rightarrow, "\mu T"'] & T^2 \arrow[d, Rightarrow, "\mu"] \\ T^2 \arrow[r, Rightarrow, "\mu"] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, Rightarrow, "\eta T"] \arrow[dr, Rightarrow, "\mathrm{id}_T"'] & T^2 \arrow[d, Rightarrow, "\mu"] & T \arrow[l, Rightarrow, "T\eta"'] \arrow[dl, Rightarrow, "\mathrm{id}_T"] \\ & T & \end{tikzcd} \tag{1.1}$$

In terms of components, this means that for every object x of $\mathcal{X}$ the following diagrams commute:

$$\begin{tikzcd} T(T(Tx)) \arrow[r, "T(\mu_x)"] \arrow[d, "\mu_{Tx}"'] & T(Tx) \arrow[d, "\mu_x"] \\ T(Tx) \arrow[r, "\mu_x"] & Tx \end{tikzcd} \qquad \begin{tikzcd} Tx \arrow[r, "\eta_{Tx}"] \arrow[dr, "\mathrm{id}_{Tx}"'] & T(Tx) \arrow[d, "\mu_x"] & Tx \arrow[l, "T(\eta_x)"'] \arrow[dl, "\mathrm{id}_{Tx}"] \\ & Tx & \end{tikzcd} \tag{1.2}$$

Example 1 (preorder). Recall: A *preorder* $(\mathcal{P}, \leq)$ is a category with $\mathcal{P}$ as objects and a morphism between X and Y iff $X \leq Y$. A functor $T\colon \mathcal{P} \to \mathcal{P}$ is thus a monotonic function $\mathcal{P} \to \mathcal{P}$ ($x \leq y \implies Tx \leq Ty$). The existence of the natural transformations η is equivalent to

$$x \leq Tx \ \forall x \in \mathcal{P}$$

and the existence of μ is equivalent to

$$T(Tx) \leq Tx \ \forall x \in \mathcal{P}$$

because there is at most one morphism $x \to y$, so the neccessary diagrams commute trivially.
Now suppose $\mathcal{P}$ is a *partial order*, i.e. $x \leq y \leq x \implies x = y \ \forall x, y \in \mathcal{P}$.
Then:

$$x \leq Tx \implies Tx \leq T(Tx)$$
$$T(Tx) \leq Tx \implies Tx = T(Tx)$$

so a Monad T in a partial order $\mathcal{P}$ is a *closure operation* in $\mathcal{P}$, i.e. a monotonic function $T\colon \mathcal{P} \to \mathcal{P}$ with $x \leq Tx$ and $T(Tx) = Tx \ \forall x \in \mathcal{P}$.
Now every topological space X induces a partial order $\mathcal{P} = (\mathscr{P}(X), \subseteq)$. Here an example for a closure operation is taking the topological closure $A \mapsto \overline{A}$, since it holds for all $A \subseteq X$ that $A \subseteq \overline{A}$ and $\overline{\overline{A}} = \overline{A}$.

Definition 1.2 (Comonad). A *Comonad* (L, ε, ω) in a Category $\mathcal{A}$ consists of

- an endofunctor $L\colon \mathcal{A} \to \mathcal{A}$
- a natural transformation $\varepsilon\colon L \Rightarrow \mathrm{id}_{\mathcal{A}}$
- a natural transformation $\omega\colon L \Rightarrow L^2$

such that the following diagrams commute:

$$\begin{tikzcd} L \arrow[r, Rightarrow, "L\omega"] \arrow[d, Rightarrow, "\omega L"'] & L^2 \arrow[d, Rightarrow, "L\omega"] \\ L^2 \arrow[r, Rightarrow, "\omega L"] & L^3 \end{tikzcd} \qquad \begin{tikzcd} & L \arrow[dl, Rightarrow, "\mathrm{id}_L"'] \arrow[d, Rightarrow, "\omega"] \arrow[dr, Rightarrow, "\mathrm{id}_L"] & \\ L & L^2 \arrow[l, Rightarrow, "\varepsilon L"'] \arrow[r, Rightarrow, "L\varepsilon"] & L \end{tikzcd} \tag{1.3}$$

In terms of components, this means that for every object x of $\mathcal{A}$ the following diagrams commute:

$$\begin{tikzcd} Lx \arrow[r, "L(\omega_x)"] \arrow[d, "\omega_{Lx}"'] & L(Lx) \arrow[d, "L(\omega_x)"] \\ L(Lx) \arrow[r, "\omega_{Lx}"] & L(L(Lx)) \end{tikzcd} \qquad \begin{tikzcd} & Lx \arrow[dl, "\mathrm{id}_{Lx}"'] \arrow[d, "\omega_x"] \arrow[dr, "\mathrm{id}_{Lx}"] & \\ Lx & L(Lx) \arrow[l, "\varepsilon_{Lx}"'] \arrow[r, "L(\varepsilon_x)"] & Lx \end{tikzcd} \tag{1.4}$$

Lemma 1.3 *For every object x in $\mathcal{X}$, the following diagram commutes:*

$$\begin{tikzcd} T(Tx) \arrow[r, "T(\delta_x)"] \arrow[d, "\delta_{Tx}"] & T(T'x) \arrow[d, "\delta_{T'x}"] \\ T(T'x) \arrow[r, "T'(\delta_x)"] & T'(T'x) \end{tikzcd}$$

this means

$$\delta T' \circ T\delta = T'\delta \circ \delta T \colon T^2 \Rightarrow (T')^2.$$

Proof: $\delta \colon T \Rightarrow T'$ is natural. □

Definition 1.4 (Morphism of monads). Let $\mathcal{X}$ be a category, let (T, η, μ) and (T', η', μ') be monads in $\mathcal{X}$. We say that a natural transformation $\delta \colon T \Rightarrow T'$ is a *morphism of monads* if it preserves the unit and the multiplication, i.e. the following diagrams commute:

$$\begin{tikzcd} \mathrm{id}_x \arrow[r, "\eta_x"] \arrow[dr, "\eta'_x"'] & Tx \arrow[d, "\delta_x"] \\ & T'x \end{tikzcd}$$

$$\begin{tikzcd} T^2x \arrow[r, "\mu_x"] \arrow[d, "\delta T' \circ T\delta"'] & Tx \arrow[d, "\delta_x"] \\ T'^2x \arrow[r, "\mu'_x"] & T'x \end{tikzcd}$$

show that the other composition is the same(siehe iPad)

Definition 1.5 (Morphism of comonads).

definition