Does Representation Guarantee Welfare?

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Abstract

A panel satisfies *descriptive representation* when its composition reflects the population. We examine the role of descriptive representation in collective decision making through an optimization lens, asking whether representative panels make decisions that maximize social welfare for the underlying population. Our main results suggest that, in general, representation with respect to *intersections* of two or more features guarantees higher social welfare than that achieved by the status quo of proportionally representing individual features. Moreover, an analysis of real data suggests that representation with respect to pairs of features is feasible in practice. These results have significant implications for the design of *citizens' assemblies*, which are gaining prominence in AI governance.

1 Introduction

A *citizens' assembly* is a group of everyday people who come together to discuss a policy issue, with the goal of informing decision makers [26, 28, 10, 20]. This paradigm has been on the rise around the world, with some of the most prominent assemblies convened by European governments to tackle constitutional questions and climate change. The success of citizens' assemblies is driven, at least in part, by several characteristics.

- *Deliberation:* Assembly members engage in a deliberative process over a period of weeks or months.

- *Random selection:* The assembly is selected through a lottery.

- *Descriptive representation:* The lottery is designed to ensure that the assembly proportionally reflects the population according to multiple features, such as age, gender, ethnicity, level of education, geography, and even viewpoint.

In this paper, we are interested in better understanding the role of descriptive representation. Political theorists have long argued that descriptive representation contributes to the legitimacy of citizens' assemblies and promotes public trust in their recommendations. This thesis is largely supported by experimental evidence [17], but it is disputed by some studies [18].

We steer clear of this (partly normative) debate as we examine descriptive representation through a novel, quantitative lens. Our key research question is this:

> *If a panel satisfies descriptive representation, does it make decisions that optimize social welfare for the underlying population?*

If the answer were a clear "yes," it would provide a new, objective way to justify the importance of descriptive representation. And if the answer were more nuanced (spoiler alert: it is), it would inform the design of future citizens' assemblies that make better decisions or recommendations.

Our approach and results.   There are several things to unpack in our informal research question; we start with *social welfare*. We assume that each individual is identified by their feature values (e.g., state = Massachusetts), and that each *alternative* is associated with benefits for each feature value (e.g., a resident of Massachusetts would derive benefit 0.2). The *utility* of an individual for an alternative is simply these benefits summed over the individual's feature values. A social welfare function maps these individual utilities to an aggregate measure of welfare; for example, *utilitarian social welfare* returns the sum of individual utilities. We focus on the well-established class of *power mean functions* [7, 22], which is parameterized by $p \in [-\infty, 1]$. The class includes utilitarian social welfare ($p = 1$), as well as other prominent social welfare functions such as egalitarian and Nash.

The more subtle definition is that of *descriptive representation*. Ideally, one would want every intersection of features to be proportionally represented on the panel. In practice, citizens' assemblies ask for what we call *1-representation:* The proportion of each feature value on the panel (approximately) matches the population. One of our main insights is to extend this requirement to m-representation, which requires that each intersection of up to m features be proportionally represented on the panel.

With this terminology in hand, we can refine our research question:

> *How well does the p-mean welfare of a m-representative panel estimate the p-mean welfare of the underlying population, for different values of p and m?*

The reader may have noticed an ostensible gap between the earlier goal of *making good decisions* and the current goal of *accurately estimating welfare*; as we show, the latter goal implies the former in our model. But if our goal is simply to estimate welfare, why do we even need a deliberative process — can we not just poll the population? The answer is that we view the utilities of individuals in the population as the utilities they would have under the same conditions as a citizens' assembly, after being informed about the topic and engaging in a deliberative process. These utilities are unknowable, of course, and indeed our guarantees do not require access to them.

In Section 3, we investigate the relation between the status quo of 1-representation and social welfare. Our first technical observation is that for utilitarian social welfare ($p = 1$), if a panel is 1-representative, its (mean) social welfare for any alternative is equal to the (mean of the) population's social welfare for the same alternative. Therefore, a 1-representative panel that seeks to maximize utilitarian social welfare will select an alternative that is optimal for the population (by the same measure). However, for $p < 1$, we prove that this is not the case: there are populations and sets of alternatives such that no 1-representative panel chooses the alternative that maximizes the p-mean welfare of the population. In Theorem 8 we determine the worst-case *accuracy* — that is, the worst-case difference between the p-mean welfare of the panel estimate and the p-mean welfare of the underlying population — of 1-representative panels.

In Section 4 we proceed to study 2-representative panels. In Theorem 9, we prove that 2-representation allows for significantly better worst-case guarantees for accuracy, compared to 1-representation. Our proof crucially relies on the fact that finding the minimum/maximum welfare a m-representative panel can have for an alternative (the difference between this maximum and minimum is the object we want to bound) boils down to solving a *moment problem* [25]. To get explicit bounds, we leverage the fact that extreme points of our moment problem are probability measures (corresponding to populations) supported on a discrete set of size at most 3 [19], for the case of 2-representative panels.

Finally, in Section 5 we experimentally show that it is possible to select 2-representative panels in real-world instances. We analyze data from four different citizens' assemblies: one from a Western European country and three from Australian states. Our experiments show that (approximate) 2-representative panels exist with the current panel sizes for two of the datasets, and would exist by picking a slightly smaller panel for a third dataset. In the final dataset, (approximate) 2-representation is infeasible; we attribute this fact to the very small and skewed pool of volunteers.

Combined, our theoretical and experimental results provide a clear, prescriptive message: It is practical to select panels that are 2-representative and, by doing so, the quality of the decision the panel makes (as measured by its p-mean welfare for the entire population) significantly increases.

Relevance and related work. There is a fast-growing body of work on algorithms for selecting and facilitating citizens' assemblies [12, 3, 4, 8, 2, 14, 16], which includes several papers presented at NeurIPS [11, 13, 9]. More broadly, the NeurIPS community has been increasingly interested in all aspects of *computational social choice* [5]; for example, foundational results on the *method of equal shares* — which has recently been deployed for participatory budgeting elections in several European cities — were first published at NeurIPS [23].

In addition, some of the most prominent technology companies have been experimenting with citizens' assemblies as a tool for AI governance. In particular, Meta has organized a large-scale deliberation on AI policy [6]. Google DeepMind has explored related ideas and their researchers have published prominent work on the use of large language models for mediating deliberation in citizens' assemblies [27].

2 Preliminaries

We consider the problem of a population of individuals selecting a representative panel to make a decision on its behalf. Every individual is represented using a set $\mathcal{F}$ of features (e.g., gender, age, geographical region, and so on), where each feature $f \in \mathcal{F}$ can take d_f different values; we denote by V_f, $|V_f| = d_f$, the set of values that feature $f \in \mathcal{F}$ can take. Every individual is represented by a *type* y, an assignment of values to the features in $\mathcal{F}$. For any $f \in \mathcal{F}$, let $y(f, v)$ be the indicator function for y having value v for feature f. That is, $y(f, v) = 1$ if and only if the type y has value v for feature f, else $y(f, v) = 0$. We say that two types are distinct if there exists at least one feature f for which they have a different value. We write $\mathbf{y} = (y(f, v))_{v \in V_f, f \in \mathcal{F}} \in \{0, 1\}^d$, where $d = \sum_{f \in \mathcal{F}} d_f$, to represent the vector containing $y(f, v)$ for all $v \in V_f$ and $f \in \mathcal{F}$.

A *population* $\mathcal{P} = \{(y_1, q_1), ..., (y_T, q_T)\}$ is a set of T distinct types of individuals $\mathcal{I} = \{y_1, ..., y_T\}$, where individuals of type y_i make up a fraction $q_i \in (0, 1]$ of the population. We call q_i the *population share* of type y_i. It holds that $\sum_{i=1}^{T} q_i = 1$. We denote by n the total number of individuals in the population. Thus, $q_i \cdot n \in \mathbb{N}$ for all $i \in [T]$.

Example 1. Consider a situation where $\mathcal{F} = \{age, gender\}$, with $V_{age} = \{15\text{-}29, 30+\}$ and $V_{gender} = \{Male, Female, Non\text{-}binary\}$. Therefore, $d_{age} = 2$ and $d_{gender} = 3$. Consider an individual who is *30+* and *Male*. Their type, y_1, is such that, for example, $y_1(age, 30+) = 1$ and $y_1(gender, Non\text{-}binary) = 0$. The corresponding vector is $\mathbf{y_1} = (0, 1; 1, 0, 0)$: they have the second value for the first feature and the first value for the second feature. Finally, consider a population $\mathcal{P} = \{(y_1, 0.4), (y_2, 0.4), (y_3, 0.2)\}$, meaning that 40% of the population has type y_1, 40% has type y_2, and 20% has type y_3.

Panels and representation. A *panel* $\mathcal{C} = \{(y_1, r_1), ..., (y_T, r_T)\}$ in a population $\mathcal{P}$ specifies, for each type $y_i \in \mathcal{I}$, its fraction $r_i \in [0, 1]$ in the panel. It holds that $\sum_{i=1}^{T} r_i = 1$. We denote by k the total number of individuals in a panel (thus, $r_i \cdot k \in \mathbb{N}$ for all i). We assume that the panel is a subset of the population $\mathcal{P}$, and therefore $r_i \cdot k \leq q_i \cdot n$ for all $i \in [T]$.

We are interested in panels that are *representative* of the population. A panel $\mathcal{C}$ in a population $\mathcal{P}$ is *1-representative* if for every feature $f \in \mathcal{F}$, and every value $v \in V_f$, the fraction of individuals in $\mathcal{C}$ with value v is equal to the fraction in $\mathcal{P}$. That is, for all $f \in \mathcal{F}$ and $v \in V_f$

$$\sum_{y_i \in \mathcal{I}} q_i \, y_i(f, v) = \sum_{y_i \in \mathcal{I}} r_i \, y_i(f, v),$$

or, more concisely, $\sum_{y_i \in \mathcal{I}} q_i \mathbf{y_i} = \sum_{y_i \in \mathcal{I}} r_i \mathbf{y_i}$. This is essentially equivalent to the notion of representation considered in prior work on panel selection.

More generally, we define an m-representative panel as follows:

Definition 2. For any $m \in \{1, ..., |\mathcal{F}|\}$, we say that a panel $\mathcal{C}$ in a population $\mathcal{P}$ is *m-representative* if for every set of $\ell \leq m$ features $F \subseteq \mathcal{F}$, $|F| = \ell$ and every possible ℓ-tuple of values $(v_f)_{f \in F} \in \bigtimes_{f \in F} V_f$, the fraction of individuals in $\mathcal{C}$ with values v_f is equal to the fraction in $\mathcal{P}$. That is, for all such F and $(v_f)_{f \in F}$,

$$\sum_{y^i \in \mathcal{I}} q_i \prod_{f \in F} y_i(f, v_f) = \sum_{y^i \in \mathcal{I}} r_i \prod_{f \in F} y_i(f, v_f).$$

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Note that $\prod_{f \in F} y_i(f, v_f) = 1$ if and only if type y_i has value v_f for all $f \in F$.

Alternatives and utility.   Panels are asked to decide on an *alternative*, from a given set of alternatives $\mathcal{A}$. For an alternative $a \in \mathcal{A}$ and a feature $f \in \mathcal{F}$, let $a(f, v) \in \mathbb{R}_{\geq 0}$ be the benefit that an individual with value v for feature f gets from a. We define $\mathbf{a} = (a(f, v))_{v \in V_f, f \in \mathcal{F}}$ analogously to $\mathbf{y}$.

We assume that the *utility* of an individual for an alternative a is the sum of all these feature benefits, i.e., the utility of an individual of type $y_i \in \mathcal{I}$ for alternative a is

$$u_i(a) = \mathbf{y} \cdot \mathbf{a} = \sum_{f \in \mathcal{F}} \sum_{v \in V_f} y_i(f, v) a(f, v).$$

Without loss of generality, we assume throughout the paper that utilities are non-negative and that the utility of every individual for every alternative is at most 1. For a given set of alternatives $\mathcal{A}$ and population $\mathcal{P}$, we define $u_{\min} = \min_{y_i \in \mathcal{I}} \min_{a \in \mathcal{A}} u_i(a)$ to be the minimum utility of any individual in the population for any alternative in $\mathcal{A}$. We have that $u_{\min} \in [0, 1]$.

Example 3 (Example 1 continued)**.** Consider a set of alternatives $\mathcal{A} = \{a_1, a_2\}$, where $a_1(\textit{age, 15-29}) = 0$, $a_1(\textit{age, 30+}) = 0.5$, $a_1(\textit{gender, Male}) = 0.3$, $a_1(\textit{gender, Female}) = 0.5$, $a_1(\textit{gender, Non-binary}) = 0$, $a_2(\textit{age, 15-29}) = 0.7$, $a_2(\textit{age, 30+}) = 0.3$, $a_2(\textit{gender, Male}) = 0$, $a_2(\textit{gender, Female}) = 0$, and $a_2(\textit{gender, Non-binary}) = 0.3$. In short, $\mathbf{a_1} = (0, 0.5; 0.3, 0.5, 0)$ and $\mathbf{a_2} = (0.7, 0.3; 0, 0, 0.3)$.

Then, since y_1 has values *30+* and *Male*, we have that

$$u_1(a_1) = \mathbf{y_1} \cdot \mathbf{a_1} = a_1(\textit{age, 30+}) + a_1(\textit{gender, Male}) = 0.5 + 0.3 = 0.8,$$

while $u_1(a_2) = 0.3 + 0 = 0.3$. Assume that y_2, y_3 are such that also $u_2(a_1)$, $u_2(a_2)$, $u_3(a_1)$, $u_3(a_2) \geq 0.3$; therefore, $u_{\min} = 0.3$.

Panel and population welfare.   We assume that the welfare of a population $\mathcal{P}$ for an alternative a is the p-mean of individuals' utilities, for some $p \in \mathbb{R}$, $p \leq 1$. That is, $u_{\mathcal{P}, p}(a) = (\sum_{y_i \in \mathcal{I}} q_i u_i(a)^p)^{1/p}$. For $p = 1$ this is the standard arithmetic mean ($u_{\mathcal{P}, 1}(a) = \sum_{y_i \in \mathcal{I}} q_i u_i(a)$), corresponding to utilitarian social welfare, for $p = 0$ it becomes the geometric mean ($u_{\mathcal{P}, 0}(a) = \prod_{y_i \in \mathcal{I}} u_i(a)^{q_i}$), corresponding to Nash social welfare, and as p approaches $-\infty$ it becomes the minimum ($u_{\mathcal{P}, -\infty}(a) = \min_{y_i \in \mathcal{I}} u_i(a)$), corresponding to egalitarian social welfare. Analogously, we define the utility of a panel $\mathcal{C}$ for an alternative a, $u_{\mathcal{C}, p}$, to be the p-mean of the utilities of the individuals in the panel.

Objectives.   There are several natural ways to measure whether a panel makes good decisions, in the sense of representing the opinion of the population well. A natural choice, for example, would be to ask that the social welfare of the panel is an accurate estimate of the social welfare of the population, for every alternative; then, naturally, optimizing for the panel approximately optimizes for the population. Another choice would be to ask that the panel orders alternatives in a similar way as the population would, i.e., if the panel prefers an alternative a over an alternative a', the population cannot have a high welfare for a' and a low welfare for a.

Formally, given $\epsilon > 0$, we can define the two notions as follows.

- *Welfare based:* We say that a panel $\mathcal{C}$ is ε-accurate with respect to p for welfare estimation if $|u_{\mathcal{C}, p}(a) - u_{\mathcal{P}, p}(a)| \leq \varepsilon$ for all $a \in \mathcal{A}$.
- *Pairwise comparisons:* We say $\mathcal{C}$ is ε-accurate with respect to p for pairwise comparisons if $u_{\mathcal{C}, p}(a) > u_{\mathcal{C}, p}(a')$ implies that $u_{\mathcal{P}, p}(a) \geq u_{\mathcal{P}, p}(a') - \varepsilon$ for all $a, a' \in \mathcal{A}$.

These two metrics are closely related: the former implies the latter.

Theorem 4. *Let $\mathcal{A}$ be a set of alternatives, and let $\mathcal{C}$ be a panel in a population $\mathcal{P}$. Then, for all $\varepsilon \geq 0$ and $p \leq 1$, we have that if $\mathcal{C}$ is ε-accurate with respect to p for welfare estimation, it is 2ε-accurate with respect to p for pairwise comparisons.*

The other direction approximately holds as well. If a committee is not ε-representative for welfare estimation for a set of alternatives $\mathcal{A}$, then it is not ε-representative for pairwise comparisons on $\mathcal{A} \cup \{a\}$ for an additional alternative a.

Theorem 5. *Let $\mathcal{A}$ be a set of alternatives, and let $\mathcal{C}$ be a panel in a population $\mathcal{P}$. Then, for all $\varepsilon \geq 0$ and $p \leq 1$, we have that if $\mathcal{C}$ is not ε-accurate with respect to p and $\mathcal{A}$ for welfare estimation, there exists an alternative a' such that $\mathcal{C}$ is not ε-accurate with respect to p and $\mathcal{A} \cup \{a'\}$ for pairwise comparisons. Furthermore, adding a' to $\mathcal{A}$ doesn't change $u_{\min}$.*

In light of the tight connection between the two notions implied by Theorems 4 and 5 (whose proofs are deferred to Appendix A.1), in the remainder of the paper, we focus on accuracy with respect to welfare estimation. We therefore refer to a panel that is ε-accurate for welfare estimation simply as ε-accurate.

3 Representation to Single Features

In this section, we prove tight upper and lower bounds on the worst-case accuracy of 1-representative panels, for different values of p.

We start by proving that, for $p = 1$, a 1-representative panel always has the same utility as the population for every alternative (i.e., it is ε-accurate for $\varepsilon = 0$).

Theorem 6. *Let $\mathcal{A}$ be a set of alternatives, $\mathcal{P}$ be population and $\mathcal{C}$ be a 1-representative panel in $\mathcal{P}$. Then, for all $a \in \mathcal{A}$, $u_{\mathcal{C},1}(a) = u_{\mathcal{P},1}(a)$. That is, $\mathcal{C}$ is 0-accurate with respect to $p = 1$.*

Proof. For $p = 1$, the welfare of the panel and population is a weighted sum of the utilities of the corresponding individuals. And, an individual's utility is the sum of the alternative's benefit $a(f, v)$ for their feature values $v \in V_f$, $f \in F$. By switching the order of summation, we have

$$u_{\mathcal{C},1}(a) = \sum_{y_i \in \mathcal{I}} r_i u_i(a) = \sum_{y_i \in \mathcal{I}} r_i(\mathbf{a} \cdot \mathbf{y_i}) = \mathbf{a} \cdot \sum_{y_i \in \mathcal{I}} r_i \mathbf{y_i} = \mathbf{a} \cdot \sum_{y_i \in \mathcal{I}} q_i \mathbf{y_i},$$

where the last equality follows from the fact that the panel is 1-representative. Continuing our derivation,

$$u_{\mathcal{C},1}(a) = \mathbf{a} \cdot \sum_{y_i \in \mathcal{I}} q_i \mathbf{y_i} = \sum_{y_i \in \mathcal{I}} q_i u_i(a) = u_{\mathcal{P},1}(a),$$

and the theorem follows. $\square$

Unfortunately, Theorem 6 does not hold for $p < 1$. When $p < 1$, the population and panel welfare functions become non-linear in $\mathbf{a}$, and therefore we cannot factor out $\mathbf{a}$ like we did in the proof of Theorem 6. In fact, as we show next, for any $p < 1$, it is possible that, in the worst case, all 1-representative panels (other than the entire population) perform poorly at welfare estimation for some alternative, and choose alternatives that are suboptimal for the population.

Example 7. In this example we will show that for all $u_{\min} \in [0, 1)$, there exists a population $\mathcal{P}$ and a set of alternatives $\mathcal{A}$, such that $\min_{y_i \in \mathcal{I}} \min_{a \in \mathcal{A}} u_i(a) = u_{\min}$, and such that for all $p < 1$, no 1-representative panel chooses the alternative that maximizes the p-mean welfare of $\mathcal{P}$.

Assume that there are two features, f_v and f_w, that can take values v_1, v_2 and w_1, w_2 respectively. Thus, there are four possible types of individuals, represented in the left part of Table 1. Let $\mathcal{P}$ be the population where each type is present in exactly one individual. Therefore, the only two 1-representative panels are $\mathcal{C}_1 = \{((1, 0; 1, 0), 1/2), ((0, 1; 0, 1), 1/2)\}$ and $\mathcal{C}_2 = \{((0, 1; 1, 0), 1/2), ((1, 0; 0, 1), 1/2)\}$.

Consider the following set of 3 alternatives: $\mathcal{A} = \{a_1, a_2, a_3\}$, where $\mathbf{a_1} = (u_{\min}/2, 1/2, 1/2, u_{\min}/2)$, $\mathbf{a_2} = (u_{\min}/2, 1/2, u_{\min}/2, 1/2)$, and $\mathbf{a_3} = ((1+3u_{\min})/8, (5-u_{\min})/8, (1+3u_{\min})/8, (1+3u_{\min})/8)$. The utility of each type for each alternative are shown in Table 1.

For any $p < 1$, we can see that the first panel, $\mathcal{C}_1$, gets the highest utility from a_1, while the second panel, $\mathcal{C}_2$, gets the highest utility from a_2, since the p-mean of x and $(1 + u_{\min}) - x$ for $p < 1$ is uniquely maximized when $x = (1 + u_{\min}) - x = (1+u_{\min})/2$. However, overall, the population gets the highest utility from alternative a_3. This fact is algebraically tedious to verify; we defer the calculations to Appendix A.2.

The main point of Example 7 is that any 1-representative panel in $\mathcal{P}$, $\mathcal{C}_1$ or $\mathcal{C}_2$, overestimates the welfare of some alternative, a_1 or a_2, respectively, quite badly. For example, for $u_{\min} = 0$ and any

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Table 1: The population and alternatives in Example 7

panels	values	y	a_1	a_2	a_3
	types		alternatives & utilities		
C_1	v_1, w_1	(1,0;1,0)	$(1+u_{\min})/2$	$u_{\min}$	$(1+3u_{\min})/4$
	v_2, w_2	(0,1;0,1)	$(1+u_{\min})/2$	1	$(3+u_{\min})/4$
C_2	v_1, w_2	(1,0;0,1)	$u_{\min}$	$(1+u_{\min})/2$	$(1+3u_{\min})/4$
	v_2, w_1	(0,1;1,0)	1	$(1+u_{\min})/2$	$(3+u_{\min})/4$

$p \le 0$, both a_1 and a_2 have a welfare of 0 with respect to $\mathcal{P}$, but a welfare of $1/2$ for the panel that chooses them. However, alternative a_3 gives the highest welfare to the population (for $u_{\min} = 0$ and $p \le 0$, the population's utility is a number between $1/4$ and $\sqrt{3}/4 \approx 0.43$).

We now state the main theorem of this section, giving a tight bound on the worst-case ε-accuracy of a 1-representative panel.

Theorem 8. *Let $\mathcal{A}$ be a set of alternatives, $\mathcal{P}$ be any population and $\mathcal{C}$ be any 1-representative panel in $\mathcal{P}$, so that $u_{\min} \in [0,1)$. Then, for all $p < 1$, $\mathcal{C}$ is ε-accurate with respect to p, where*

$$
\varepsilon = \begin{cases}
\frac{1-p}{p}\left(p \cdot \frac{1-u_{\min}}{1-u_{\min}{}^p}\right)^{\frac{1}{1-p}} + \frac{u_{\min}-u_{\min}{}^p}{1-u_{\min}{}^p} & \text{for } p > 0 \vee (p < 0 \wedge u_{\min} > 0) \\[2mm]
1 + \frac{1-u_{\min}}{\ln(1/u_{\min})}\left(\ln\left(\frac{1-u_{\min}}{\ln(1/u_{\min})}\right) - 1\right) & \text{for } p = 0 \wedge u_{\min} > 0 \\[2mm]
1 & \text{for } p \le 0 \wedge u_{\min} = 0
\end{cases}.
$$

Furthermore, for all $p < 1$ and $u_{\min} \in [0,1)$, for any $\varepsilon' < \varepsilon$, there exists a 1-representative panel $\mathcal{C}$ such that $\mathcal{C}$ is not ε'-accurate with respect to p.

The proof of Theorem 8 is deferred to Appendices A.3.3 and A.4. The first part of the theorem (the upper bounds on ε) follows as a special case of the analysis of the m-representative case in Appendix A.3 that also gives the main results of the following section for 2-representative panels.

4 Representation to Tuples of Features

In this section, we prove bounds on the accuracy of m-representative panels. We start by showing that 2-representation enables significantly better worst-case ε-accuracy compared to 1-representation.

Theorem 9. *Let $\mathcal{A}$ be a set of alternatives, $\mathcal{P}$ be a population and $\mathcal{C}$ be any 2-representative panel in $\mathcal{P}$. Then, for all $p \le 1$ and all $u_{\min} \in (0,1)$, or $p \in (0,1]$ and $u_{\min} = 0$, $\mathcal{C}$ is ε-accurate with respect to p, where*[1]

$$
\varepsilon = \max_{\kappa_1, \kappa_2} \quad \left(\frac{\kappa_2 - \kappa_1^2}{\kappa_2 - 2\kappa_1 + 1} + \frac{(1-\kappa_1)^2}{\kappa_2 - 2\kappa_1 + 1}\left(\frac{\kappa_1 - \kappa_2}{1 - \kappa_1}\right)^p\right)^{\frac{1}{p}}
$$
$$
- \left(\frac{(\kappa_1 - u_{\min})^2}{\kappa_2 - 2u_{\min}\kappa_1 + u_{\min}{}^2}\left(\frac{\kappa_2 - u_{\min}\kappa_1}{\kappa_1 - u_{\min}}\right)^p + \frac{\kappa_2 - \kappa_1^2}{\kappa_2 - 2u_{\min}\kappa_1 + u_{\min}{}^2}u_{\min}{}^p\right)^{\frac{1}{p}}
$$
$$
\text{s.t.} \quad u_{\min}{}^2 < \kappa_1^2 < \kappa_2 \quad \text{and} \quad \frac{\kappa_2 + u_{\min}}{1 + u_{\min}} < \kappa_1 < 1
$$

The optimization program in Theorem 9 has two variables, κ_1 and κ_2. We call those the *moments*:

Definition 10. *For $i = 0, ..., |\mathcal{F}|$, the i-th moment of a population $\mathcal{P}$ for an alternative a is $\kappa_i = \sum_{y_j \in \mathcal{I}} q_j \cdot (u_j(a))^i$. The i-th moment of a panel $\mathcal{C}$ or alternative a is $\kappa_i = \sum_{y_j \in \mathcal{I}} r_j \cdot (u_j(a))^i$.*

Our proof has the following high-level steps. First, we prove that the first m moments of an m-representative panel $\mathcal{C}$ in $\mathcal{P}$ are the same as the moments of $\mathcal{P}$ for any alternative. That is, for all $a \in \mathcal{A}$ and any integer $1 \le \ell \le m$, $\sum_{y_i \in \mathcal{I}} q_i \cdot (u_i(a))^\ell = \sum_{y_i \in \mathcal{I}} r_i \cdot (u_i(a))^\ell$.

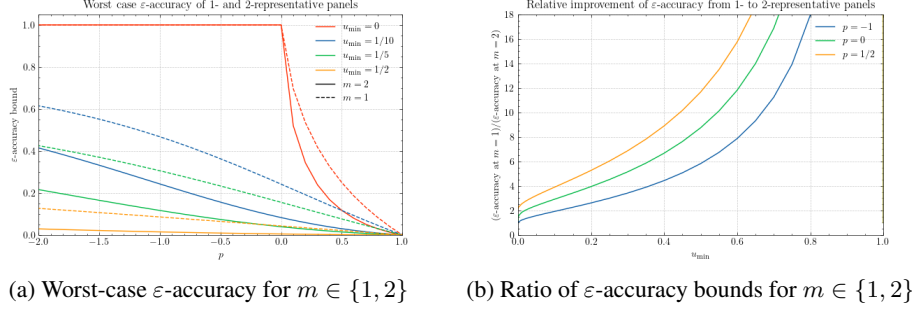[1]We state the objective for $p = 0$, which is the limit of the above objective as $p \to 0$, in Appendix A.3.

(a) Worst-case ε-accuracy for $m \in \{1, 2\}$ (b) Ratio of ε-accuracy bounds for $m \in \{1, 2\}$

Figure 1: Numerical comparisons for ε-accuracy bounds for $m \in \{1, 2\}$.

Second, given this fact, we can write a mathematical program for the minimum and maximum utility a panel or population with fixed first m moments $\kappa_1, \ldots, \kappa_m$ (note that always $\kappa_0 = 1$) can have. Concretely, by having a variable x_j for the utility of a type $y_j \in \mathcal{I}$, given m moments, we can ensure feasibility by asking that $\sum_{y_j \in \mathcal{I}} q_j \cdot (x_j)^i = \kappa_i$, for all $i \in \{0, 1, ..., m\}$ (and $x_j \in [u_{\min}, 1], q_j \geq 0$). Then, by maximizing and minimizing $(\sum_{y_j \in \mathcal{I}} q_j \cdot (x_j)^p)^{1/p}$, we get bounds on the welfare of a panel or population with moments $\kappa_0, \ldots, \kappa_m$.

Third, the relaxation of this program is a *moment problem* [25, 24], hence our choice of terminology in Definition 10; it is known that the extreme points are probability measures (distributions) supported on a discrete set of size at most $m + 1$ [19]. This additional structure allows for explicit bounds on the maximum and minimum utility of a panel/population as a function of $\kappa_1, \ldots, \kappa_m$, which we obtain for $m = 2$. The difference of these quantities is our bound on the ε-accuracy of a 2-representative panel. The feasible region, as we prove, is derived from the set of all κ_1 and κ_2 that can be the first two moments of a population $\mathcal{P}$ for an alternative $a \in \mathcal{A}$. We defer details to Appendix A.3.

For greater m, solving the relaxed optimization program analytically to obtain explicit bounds on the maximum and minimum utility of a panel/population as a function of $\kappa_1, \ldots, \kappa_m$ becomes increasingly difficult. And, even if such bounds were obtained for $m > 2$, the overall bound on ε in the style of Theorem 9 would be an even less tractable optimization program in m variables; it is unclear if its value could even be numerically calculated.

In fact, we believe that already the solution to the optimization program in Theorem 9 does not admit a closed-form solution in terms of p and $u_{\min}$. However, for a fixed p and $u_{\min}$, we can numerically solve this optimization program. We plot these results in Figure 1. In particular, Figure 1a shows the tight worst-case ε-accuracy of 1-representative panels as dashed lines and the upper bound on worst-case ε-accuracy of 2-representative panels as solid lines, for 4 different values of $u_{\min}$, with p varying along the horizontal axis. We find that 2-representative panels are ε-accurate in the worst case for significantly smaller ε. For example, for $u_{\min} = 1/5$ and Nash welfare ($p = 0$), 2-representation gives 0.039 accuracy, while 1-representation only gives 0.155 accuracy. Figure 1b shows the ratio of these bounds: the worst-case ε for 1-representative panels divided by the bound on the worst-case ε for 2-representative panels, for different values of p, with $u_{\min}$ varying along the horizontal axis. We can see that the relative accuracy increases quite drastically, especially when p is not far into the negatives and $u_{\min}$ is far enough away from 0. If utilities are generally high, this ratio is the largest: for $u_{\min} = 4/5$ (corresponding, after scaling, to utilities in $[8, 10]$) and Nash welfare, accuracy increases by 2700% for 2-representative panels over 1-representative panels.

Given the positive result in Theorem 9, we turn our attention to lower bounds on ε-accuracy for m-representative panels. Observe that, when $p \leq 0$, it is necessary that we bound the minimum utility of an individual, $u_{\min}$, away from zero. Otherwise, one could imagine a very popular alternative with utility close to 1 for everyone but a single person y^*, who has utility 0. The population's utility (for $p \leq 0$) for this alternative is 0, but any panel that doesn't include y^* has utility close to 1 for this alternative (and therefore chooses it). We formalize this idea in Theorem 11.

Theorem 11. *There exists a population $\mathcal{P}$ and a set of alternatives $\mathcal{A}$, such that $\min_{y_i \in \mathcal{I}} \min_{a \in \mathcal{A}} u_i(a) = u_{\min} = 0$, and a m-representative panel $\mathcal{C}$ in $\mathcal{P}$, such that $\mathcal{C}$ is not ε-accurate with respect to any $p \leq 0$ for any $\varepsilon < \frac{|\mathcal{F}| - m}{|\mathcal{F}|}$.*

As $|\mathcal{F}|$ becomes large, this lower bound goes to 1, for any fixed m. Since $u_{\mathcal{C},p}(a), u_{\mathcal{P},p}(a) \in [0,1]$ for all $a \in \mathcal{A}$, 1 is also a trivial upper bound. We prove Theorem 11 in Appendix A.5.

5 Empirical Analysis

We have shown that 2-representative panels give significantly better worst-case guarantees than 1-representative panels. However, this result only has practical impact if it is possible to select 2-representative panels in real-world instances. Our next goal is to show that this is indeed the case.

Selecting a panel. Selecting a citizens' assembly in practice commonly has three stages. First, a large number of individuals receive an invitation to participate in the panel. Second, some fraction of these individuals indicate their willingness to participate. We call this group the *pool of volunteers*. Finally, a panel of the desired size is chosen from the pool of volunteers.

It is a common occurrence in practice that individuals with certain features are more likely to accept the invitation to participate than others. Therefore, the composition of the pool of volunteers often is fairly different from the underlying population. If the size of the pool is too small, it may not be possible to select a panel of a desired size that is 2-representative (or even 1-representative).

Data. We analyze data from four citizens' assemblies: a nationwide panel from a Western European country (which we denote EUR1) and three state-wide panels from Australian states (which we denote AUS1, AUS2, and AUS3). Detailed information about the datasets can be found in Appendix A.6. All actual panels were chosen to be approximately 1-representative of the underlying population.

For all four panels, we obtained data on the intersections between feature values of the underlying population. For EUR1, we used the European Social Survey [21], which collected data on a wide variety of features and their intersections, including those used in EUR1. Responses are re-weighted in the data to account for sampling bias. For AUS1, AUS2, and AUS3, we used data from the 2021 Australian Census [1]. Through their online data tools, we obtained data on the intersections of individuals' features. We excluded a feature capturing the level of climate concern from EUR1 since the number of volunteers with no concern at all was too low to even form a 1-representative committee of the desired size. In AUS1 and AUS2, we had to exclude a feature capturing an individual's ownership of their residence since these statistics are stored in a separate dataset on dwellings, so we weren't able to obtain this feature's intersection with other features. For more details on the datasets and data cleaning, we again refer to Appendix A.6

Experiments. For each m between 1 and the number of features considered in the panel, $|\mathcal{F}|$, we found the largest size of an m-representative panel using individuals from the pool of volunteers. For a given m and desired panel size k, we checked for the existence of an m-representative panel of size k using an integer linear program. Each person in the pool of volunteers corresponds to one binary variable, encoding whether this person is in the panel or not. The constraints on the variables are the size of the panel, k, and having to be m-representative to the underlying population data. We used Gurobi, run on a 14-inch MacBook Pro (2023) with Apple M3 Pro chip, to check whether the program is feasible, i.e., whether an m-representative panel exists.

It is worth noting that exact 1-representation or m-representation for any m is (essentially) never attainable for a fixed desired panel size k, as the desired number of individuals in the panel may not be an integer. We thus consider a panel to be m-representative in practice if for any ℓ-tuple of feature values, $\ell \in [m]$, the number of people with these feature values on the panel is equal to the desired fractional number, up to rounding it up or down. Since the quotas from the underlying populations are only ever estimates with some error, this small "wiggle room" around the quotas is justified. It is intuitive that the larger the panel is, the more difficult it is to satisfy m-representation. At the extremes, a panel of size 1 is always m-representative (up to rounding) because the quotas are smaller than 1. And a panel that includes the entire pool of volunteers is rarely even 1-representative, because the pool of volunteers is skewed due to self-selection bias.

Results. Our results are shown in Figures 2a to 2d. The main takeaway is that 2-representation is generally feasible in practice.

In the EUR1 and AUS1 datasets, the solid lines — showing the maximum size of an m-representative panel — lie above the "desired panel size" used in practice. For AUS2, the largest 2-representative panel is of size 278 — slightly below the desired panel size of 328. Since the number 328 does not have special significance,[2] our results suggest that slightly contracting the panel in order to boost its degree of representation would be a good tradeoff. By contrast, in AUS3, the largest 2-representative panel is of size 31, which is 69% of the desired panel size of 45. The pool of volunteers for AUS3 only contains 509 individuals — significantly fewer than the other assemblies, which all have more than 1000 volunteers. We conclude that an investment in the recruitment of volunteers may make a greater degree of representation possible.
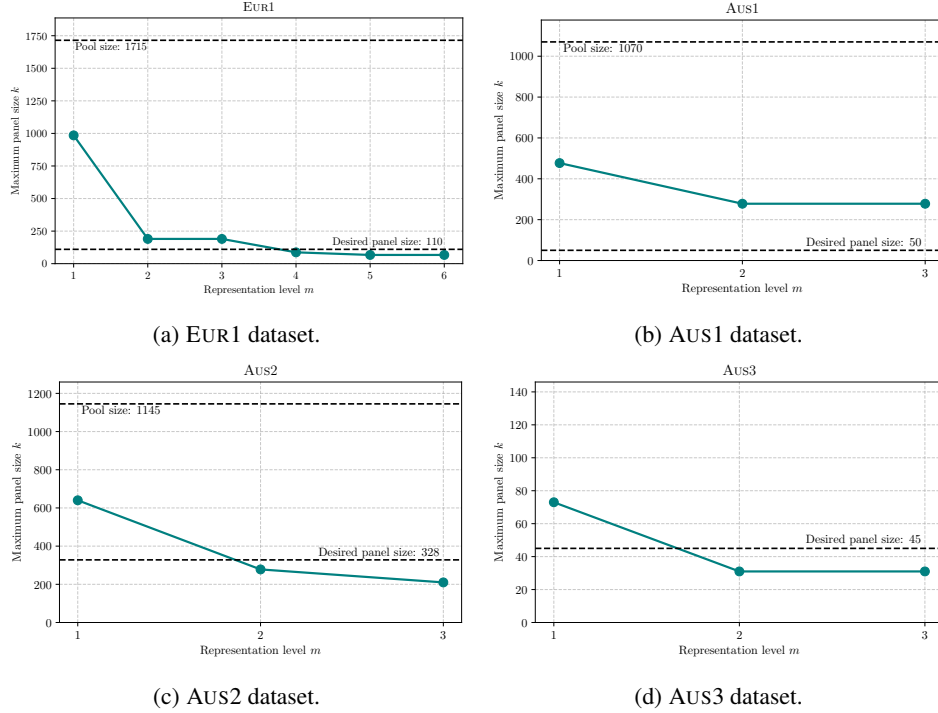

(a) EUR1 dataset.

(b) AUS1 dataset.

(c) AUS2 dataset.

(d) AUS3 dataset.

Figure 2: The maximum size of m-representative panel possible for $m \in [|\mathcal{F}|]$, shown in teal. The dashed lines show the size of the pool of volunteers and the panel size used in practice.

6   Discussion

We conclude by discussing two points that may contribute to a fuller understanding of our approach and its limitations.

Throwing fairness into the mix.   As mentioned in Section 1, citizens' assemblies are randomly selected. In practice, this is done by computing a lottery such that each panel in its support is 1-representative. A commonly used algorithm optimizes the lottery to maximize a fairness objective: the minimum probability of selecting any volunteer [12]. The same algorithm can be directly applied when the underlying notion of representation is m-representation for $m > 1$ instead of $m = 1$. However, this may come at a cost to the fairness objective, because the feasible set shrinks as the constraints become stronger. While we focused on the relation between representation and welfare, future work could explore the tradeoffs between representation, welfare and fairness.

Empirically estimating welfare gains?   The reader may have noticed that, despite access to real data from citizens' assemblies, our empirical analysis does not evaluate our main theoretical claim: a greater degree of representation contributes to social welfare. The reason is that we do not have access

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[2]It likely started out as a "round" number and decreased when selected panelists dropped out.

to real utilities, and synthetic methods for generating them may seem contrived. Let us reiterate, however, that the lack of access to utilities is not a barrier to applying our results, as we argue for a greater degree of representation with respect to known features, which leads to improved social welfare with respect to *unknown* utilities.

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A Technical Appendices and Supplementary Material

A.1 Proofs missing from Section 2

Proof of Theorem 4. Let $a, a' \in \mathcal{A}$ be any two alternatives such that $u_{\mathcal{C},p}(a) > u_{\mathcal{C},p}(a')$. Since the panel is ε-accurate for welfare estimation, we have that

$$u_{\mathcal{P},p}(a) \geq u_{\mathcal{C},p}(a) - \varepsilon > u_{\mathcal{C},p}(a') - \varepsilon \geq u_{\mathcal{P},p}(a') - 2\varepsilon,$$

which implies that $\mathcal{C}$ is 2ε-accurate for pairwise comparisons. $\square$

Proof of Theorem 5. Since $\mathcal{C}$ is not ε-accurate, with respect to p and A, for welfare estimation, there exists an alternative $a \in \mathcal{A}$ such that $|u_{\mathcal{C},p}(a) - u_{\mathcal{P},p}(a)| > \varepsilon$. Let $\delta > 0$ be any real number such that $\delta < |u_{\mathcal{C},p}(a) - u_{\mathcal{P},p}(a)| - \varepsilon$.

If $u_{\mathcal{C},p}(a) > u_{\mathcal{P},p}(a)$, consider an alternative a' such that $a'(f,v) = (u_{\mathcal{P},p}(a) + \varepsilon + \delta)/|\mathcal{F}|$ for all $f \in \mathcal{F}$, $v \in V_f$. Then, $u_y(a') = u_{\mathcal{P},p}(a) + \varepsilon + \delta$ for all $y \in \mathcal{I}$, and therefore also $u_{\mathcal{P},p}(a') = u_{\mathcal{C},p}(a') = u_{\mathcal{P},p}(a) + \varepsilon + \delta$ (where we used the fact that, if all individuals have the same utility x for an alternative, the p-mean of their utilities is also x). We have that $u_{\mathcal{C},p}(a) > u_{\mathcal{P},p}(a) + \varepsilon + \delta = u_{\mathcal{C},p}(a')$ but $u_{\mathcal{P},p}(a) < u_{\mathcal{P},p}(a) + \delta = u_{\mathcal{P},p}(a') - \varepsilon$, which contradicts the fact that $\mathcal{C}$ is ε-accurate for pairwise comparisons. Similarly, if $u_{\mathcal{C},p}(a) < u_{\mathcal{P},p}(a)$, consider an alternative a' such that $a'(f,v) = (u_{\mathcal{P},p}(a) - \varepsilon - \delta)/|\mathcal{F}|$ for all $f \in \mathcal{F}$, $v \in V_f$. Then, $u_y(a') = u_{\mathcal{P},p}(a) - \varepsilon - \delta$ for all $y \in \mathcal{I}$, and therefore $u_{\mathcal{P},p}(a') = u_{\mathcal{C},p}(a') = u_{\mathcal{P},p}(a) - \varepsilon - \delta$. We have that $u_{\mathcal{C},p}(a) < u_{\mathcal{P},p}(a) - \varepsilon - \delta = u_{\mathcal{C},p}(a')$ but $u_{\mathcal{P},p}(a) - \varepsilon > u_{\mathcal{P},p}(a) - \varepsilon - \delta = u_{\mathcal{P},p}(a')$, a contradiction to $\mathcal{C}$ being ε-accurate for pairwise comparisons.

Lastly, we note that the alternatives a' defined above don't make any individual's utility be more than 1 or less then $u_{\min}$. We know that $u_{\mathcal{C},p}(a), u_{\mathcal{P},p}(a) \in [u_{\min}, 1]$, so since $\min(u_{\mathcal{C},p}(a), u_{\mathcal{P},p}(a)) \leq u_y(a') \leq \max(u_{\mathcal{C},p}(a), u_{\mathcal{P},p}(a))$, we also have $u_y(a') \in [u_{\min}, 1]$ for all $y \in \mathcal{I}$. $\square$

A.2 Calculations for Example 7

We first prove a short technical lemma to simplify the calculations.

Lemma 12. *Let $p < 1$ and $c_1, c_2, x \in \mathbb{R}$ so that $c_2 > c_1 \geq x \geq 0$ if $p \in (0,1)$ and $c_2 > c_1 > x \geq 0$ if $p \leq 0$. Then for $p \neq 0$,*

$$\frac{d}{dx}\left(\frac{1}{4}((c_1 - x)^p + (c_1 + x)^p + (c_2 - x)^p + (c_2 + x)^p)\right)^{1/p} \leq 0,$$

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and equality holds only if $x = 0$. Furthermore,

$$\frac{d}{dx}\left(((c_1 - x)(c_1 + x)(c_2 - x)(c_2 + x))^{1/4}\right) \le 0,$$

and again equality holds only if $x = 0$.

Proof. We'll first denote $f(x) = \frac{1}{4}((c_1 - x)^p + (c_1 + x)^p + (c_2 - x)^p + (c_2 + x)^p)$ and consider the case $p \ne 0$. We can see that

$$\frac{d}{dx}(f(x))^{1/p} = \frac{1}{p}(f(x))^{1/p-1} \cdot \frac{1}{4}p\left((c_1 + x)^{p-1} - (c_1 - x)^{p-1} + (c_2 + x)^{p-1} - (c_2 - x)^{p-1}\right)$$

The two p's cancel. Since all of the summand in $f(x)$ are non-negative (positive if $p \le 0$) and $(c_2 + x)$ is positive, we have that $f(x) > 0$, so $(f(x))^{1/p-1} > 0$. It follows that the sign of $\frac{d}{dx}(f(x))^{1/p}$ is the same as the sign of $(c_1 + x)^{p-1} - (c_1 - x)^{p-1} + (c_2 + x)^{p-1} - (c_2 - x)^{p-1}$. Since $p < 1$, we know that the function $g(x) = x^{p-1}$ is strictly decreasing in x, so $(c_1 + x)^{p-1} \le (c_1 - x)^{p-1}$ and $(c_2 + x)^{p-1} \le (c_2 - x)^{p-1}$, with equality only if $x = 0$. The first part of the lemma follows.

For the second part of the lemma, we denote

$$f(x) = (c_1 - x)(c_1 + x)(c_2 - x)(c_2 + x) = (c_1^2 - x^2)(c_2^2 - x^2)$$

and get that

$$\frac{d}{dx}(f(x))^{1/4} = \frac{1}{4}(f(x))^{1/4-1}\left(4x^3 - 2x(c_1^2 + c_2^2)\right).$$

All the terms in the fourth root are positive, so we have that $f(x) > 0$ and thus $(f(x))^{1/4-1} > 0$. It follows that the sign of $\frac{d}{dx}(f(x))^{1/4}$ is the same as the sign of $\left(4x^3 - 2x(c_1^2 + c_2^2)\right)$. If $x = 0$, we are done. Else, since $c_2 \ge c_1 > x > 0$, we have that $4x^3 - 2x(c_1^2 + c_2^2) < 4x^3 - 2x(x^2 + x^2) = 0$. The second part of the lemma follows. $\qquad\square$

To show that the population indeed gets the strictly highest utility from a_3 for any $u_{\min} \in [0, 1)$ as claimed in Example 7, we let

$$h(x) = \begin{cases} \left(\frac{1}{4}((c_1 - x)^p + (c_1 + x)^p + (c_2 - x)^p + (c_2 + x)^p)\right)^{1/p} & \text{if } p \ne 0 \\ ((c_1 - x)(c_1 + x)(c_2 - x)(c_2 + x))^{1/4} & \text{if } p = 0 \end{cases},$$

where $c_1 = \frac{1+3u_{\min}}{4}$ and $c_2 = \frac{3+u_{\min}}{4}$. Note that we can express the welfare of the population for the alternatives in $\mathcal{A}$ using h:

$$u_{\mathcal{P},p}(a_1) = u_{\mathcal{P},p}(a_2) = h\left(\frac{1 - u_{\min}}{4}\right), \quad u_{\mathcal{P},p}(a_3) = h(0).$$

By Lemma 12, we know that h is strictly decreasing on $[0, c_1)$ for $p \le 0$ and on $[0, c_1]$ for $p \in (0, 1)$. For $p \in (0, 1)$ and any $u_{\min} \in [0, 1)$ we have that $c_1 \ge \frac{1-u_{\min}}{4}$, and for $p \le 0$ and $u_{\min} \in (0, 1)$ we have that $c_1 > \frac{1-u_{\min}}{4}$, so we get that $h(0) > h\left(\frac{1-u_{\min}}{4}\right)$. Thus, $u_{\mathcal{P},p}(a_3) > u_{\mathcal{P},p}(a_1), u_{\mathcal{P},p}(a_2)$.

There remains the case $u_{\min} = 0$ and $p \le 0$ that we already treat in Example 7 but repeat here for the sake of completeness. In that case, both a_1 and a_2 have a welfare of 0 for $\mathcal{P}$ since one individual in $\mathcal{P}$ has utility 0. Alternative a_3 has a welfare of at least $\frac{1}{4} > 0$ for $\mathcal{P}$ (using that the p-mean of a list of numbers is at least as big as the smallest number).

A.3   Proof of Theorem 9

Let's restate Theorem 9, including the bound in the case $p = 0$. Note that the case $p = 0$ is the same as the general case, replacing the weighted p-means by weighted geometric means.

Theorem 9 (restated). *Let $\mathcal{A}$ be a set of alternatives, $\mathcal{P}$ be a population and $\mathcal{C}$ be any 2-representative panel in $\mathcal{P}$. Then, for all $p \le 1$ and all $u_{\min} \in (0,1)$, or $p \in (0,1]$ and $u_{\min} = 0$, $\mathcal{C}$ is ε-accurate with respect to p, where for $p \neq 0$,*

$$\varepsilon = \max_{\kappa_1, \kappa_2} \quad \left(\frac{\kappa_2 - \kappa_1^2}{\kappa_2 - 2\kappa_1 + 1} + \frac{(1 - \kappa_1)^2}{\kappa_2 - 2\kappa_1 + 1} \left(\frac{\kappa_1 - \kappa_2}{1 - \kappa_1} \right)^p \right)^{\frac{1}{p}}$$

$$- \left(\left(\frac{(\kappa_1 - u_{\min})^2}{\kappa_2 - 2u_{\min}\kappa_1 + u_{\min}^2} \left(\frac{\kappa_2 - u_{\min}\kappa_1}{\kappa_1 - u_{\min}} \right)^p + \frac{\kappa_2 - \kappa_1^2}{\kappa_2 - 2u_{\min}\kappa_1 + u_{\min}^2} u_{\min}^p \right)^{\frac{1}{p}} \right.$$

$$\text{s.t.} \quad u_{\min}^2 < \kappa_1^2 < \kappa_2 \quad \text{and} \quad \frac{\kappa_2 + u_{\min}}{1 + u_{\min}} < \kappa_1 < 1,$$

and for $p = 0$,

$$\varepsilon = \max_{\kappa_1, \kappa_2} \quad \left(\frac{\kappa_1 - \kappa_2}{1 - \kappa_1} \right)^{\left(\frac{(1 - \kappa_1)^2}{\kappa_2 - 2\kappa_1 + 1} \right)}$$

$$- \left(\frac{\kappa_2 - u_{\min}\kappa_1}{\kappa_1 - u_{\min}} \right)^{\left(\frac{(\kappa_1 - u_{\min})^2}{\kappa_2 - 2u_{\min}\kappa_1 + u_{\min}^2} \right)} \cdot (u_{\min})^{\left(\frac{\kappa_2 - \kappa_1^2}{\kappa_2 - 2u_{\min}\kappa_1 + u_{\min}^2} \right)}$$

$$\text{s.t.} \quad u_{\min}^2 < \kappa_1^2 < \kappa_2 \quad \text{and} \quad \frac{\kappa_2 + u_{\min}}{1 + u_{\min}} < \kappa_1 < 1.$$

In order to prove Theorem 9, we first establish our general framework for bounding the ε-accuracy of m-representative panels in Appendix A.3.1. We then turn to the cases $m = 1$ and $m = 2$ (Appendices A.3.3 and A.3.4), where we analytically solve some of the optimization problems from the general framework for m-representative panel. We use these to prove Theorem 9 and the upper bound in Theorem 8.

A.3.1 Properties of m-representative panels

We start by proving Lemma 13, a property of m-representative panels that is crucial for the proof of Theorem 9.

Lemma 13. *Let $\mathcal{A}$ be a set of alternatives, $\mathcal{P}$ be a population, and $\mathcal{C}$ be an m-representative panel in $\mathcal{P}$. Then, for any $a \in \mathcal{A}$, the first m moments of $\mathcal{C}$ and $\mathcal{P}$ for a are the same. That is, for any integer $1 \le \ell \le m$,*

$$\sum_{y_i \in \mathcal{I}} q_i(u_i(a))^\ell = \sum_{y_i \in \mathcal{I}} r_i(u_i(a))^\ell.$$

Proof of Lemma 13. First, note that by definition a panel that is m-representative is also ℓ-representative, for $\ell \le m$. Thus, it suffices to show that the equation in Lemma 13 holds for $\ell = m$.

The key insight is that $\sum_{y_j \in \mathcal{I}} q_j(u_j(a))^m$ is a polynomial in $\{y_i(f, v)\}_{f \in \mathcal{F}, v \in V_f}$ of degree m. This allows us to use the definition of being m-representative to "switch" the q_j in any term of the polynomial to r_j. Formally, we get that

$$\sum_{y_j \in \mathcal{I}} q_j(u_j(a))^m = \sum_{y_j \in \mathcal{I}} q_j \left(\sum_{f \in \mathcal{F}} \sum_{v \in V_f} a(f, v) y_j(f, v) \right)^m$$

$$= \sum_{y_j \in \mathcal{I}} q_j \sum_{\substack{(f_1, \ldots, f_m) \\ \in \mathcal{F}^m}} \sum_{\substack{(v_1, \ldots, v_m) \\ \in V_{f_1} \times \ldots \times V_{f_m}}} \prod_{i=1}^{m} a(f_i, v_i) y_j(f_i, v_i)$$

$$= \sum_{\substack{(f_1, \ldots, f_m) \\ \in \mathcal{F}^m}} \sum_{\substack{(v_1, \ldots, v_m) \\ \in V_{f_1} \times \ldots \times V_{f_m}}} \left(\prod_{i=1}^{m} a(f_i, v_i) \right) \left(\sum_{y_j \in \mathcal{I}} q_j \prod_{i=1}^{m} y_j(f_i, v_i) \right).$$

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and analogously for the panel

$$\sum_{y_j \in \mathcal{I}} r_j (u_j(a))^m = \sum_{\substack{(f_1,...,f_m) \\ \in \mathcal{F}^m}} \sum_{\substack{(v_1,...,v_m) \\ \in V_{f_1} \times ... \times V_{f_m}}} \left(\prod_{i=1}^{m} a(f_i, v_i) \right) \left(\sum_{y_j \in \mathcal{I}} r_j \prod_{i=1}^{m} y_j(f_i, v_i) \right). \quad (1)$$

We conclude the proof by arguing that for any $(f_1, ..., f_m) \in \mathcal{F}^m$ and $v_1, ..., v_m \in V_{f_1} \times ... \times V_{f_m}$, it holds that

$$\sum_{y_j \in \mathcal{I}} q_j \prod_{i=1}^{m} y_j(f_i, v_i) = \sum_{y_j \in \mathcal{I}} r_j \prod_{i=1}^{m} y_j(f_i, v_i).$$

Let's first assume there exist two indices $i \neq i'$ such that $f_i = f_{i'}$, but $v_i \neq v_{i'}$. In this case, we know both sides of Equation (1) are zero since no type $y_j \in \mathcal{I}$ can have $y_j(f, v_i) = y_j(f, v_{i'}) = 1$ for $v_i \neq v_{i'}$ (i.e., each feature takes exactly one value per type), where $f = f_i = f_{i'}$.

The case where whenever $f_i = f_{i'}$, also $v_i = v_{i'}$ remains. We know that $y_j(f_i, v_i) = y_j(f_{i'}, v_{i'}) \in \{0, 1\}$ for all $y_j \in \mathcal{I}$, and thus $y_j(f_i, v_i) = y_j(f_i, v_i) y_j(f_{i'}, v_{i'})$. Let $Z = \{f \in (f_1, ..., f_m)\}$ be the set of the unique features in $(f_1, ..., f_m)$, $|Z| \leq m$. We get that for any $(v_f)_{f \in Z} \in \bigtimes_{f \in Z} V_f$, it holds that

$$\sum_{y_j \in \mathcal{I}} q_j \prod_{i=1}^{m} y_j(f_i, v_i) = \sum_{y_j \in \mathcal{I}} q_j \prod_{f \in Z} y_j(f, v_f) = \sum_{y_j \in \mathcal{I}} r_j \prod_{f \in Z} y_j(f, v_f) = \sum_{y_j \in \mathcal{I}} r_j \prod_{i=1}^{m} y_j(f_i, v_i),$$

where the second equality follows from the definition of m-representation (see Definition 2), and the fact that $|Z| \leq m$. $\qquad \square$

From Lemma 13, we have that the first m moments of an m-representative panel $\mathcal{C}$ in $\mathcal{P}$ are the same as the moments of $\mathcal{P}$. Also note that the 0-th moment is always identical, $\kappa_0 = \sum_{y_j \in \mathcal{I}} q_j = 1$. We'll now use moments to bound the utility an m-representative panel can have.

Lemma 14. *Let $u_{\min} \in [0, 1]$ and $p < 1$. Let $\mathcal{P}$ be a population (resp., let $\mathcal{C}$ be a panel) with moments $\kappa_0 = 1, \kappa_1, ..., \kappa_m$. Then for any alternative a such that $u_i(a) \in [u_{\min}, 1]$ for all $y_i \in \mathcal{I}$, it holds that*

$$\text{UMIN}(\kappa_1, ..., \kappa_m; p, u_{\min}) \leq u_{\mathcal{P},p}(a), u_{\mathcal{C},p}(a) \leq \text{UMAX}(\kappa_1, ..., \kappa_m; p, u_{\min}),$$

where[3] $\quad \text{UMIN}(\kappa_1, ..., \kappa_m; p, u_{\min}) \quad$ *and* $\quad \text{UMAX}(\kappa_1, ..., \kappa_m; p, u_{\min}) \quad$ *are,* *respectively,*

$$\min \left(\sum_{i=0}^{m} q_i x_i^p \right)^{\frac{1}{p}} \qquad\qquad \max \left(\sum_{i=0}^{m} q_i x_i^p \right)^{\frac{1}{p}}$$

$$\text{s.t.} \quad \sum_{i=0}^{m} q_i x_i^j = \kappa_j \quad \text{for all } j \in \{0, ..., m\} \qquad \text{s.t.} \quad \sum_{i=0}^{m} q_i x_i^j = \kappa_j \quad \text{for all } j \in \{0, ..., m\}$$

$$q_i \geq 0 \quad \text{for all } i \in \{0, ..., m\} \qquad\qquad q_i \geq 0 \quad \text{for all } i \in \{0, ..., m\}$$

$$x_i \in [u_{\min}, 1] \quad \text{for all } i \in \{0, ..., m\}. \qquad x_i \in [u_{\min}, 1] \quad \text{for all } i \in \{0, ..., m\}.$$

Proof. We'll prove the theorem for any population $\mathcal{P}$. The proof for a panel $\mathcal{C}$ is identical.

We know that for any population $\mathcal{P}$ with moments $\kappa_0, \kappa_1, ..., \kappa_m$ it holds that the utility $u_{\mathcal{P},p}(a)$ is upper and lower bounded by maximizing and minimizing, respectively, the following optimization

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[3] To keep the notation concise, we'll write the following proofs for a general p-mean and don't explicitly state the special case $p = 0$ whenever the corresponding definitions or statements follow immediately when the p-mean is replaced by a geometric mean.

problem

$$\max / \min \quad \left(\sum_{y_j \in \mathcal{I}} q_j \cdot (u_j)^p \right)^{\frac{1}{p}}$$

$$\text{s.t.} \quad \sum_{y_j \in \mathcal{I}} q_j \cdot (u_j)^i = \kappa_i \quad \text{for all } i \in \{0, 1, ..., m\}$$

$$u_j \in [u_{\min}, 1] \quad \text{for all } y_j \in \mathcal{I}$$

since the feasible region is exactly the region of all populations with moments as given, and the utilities are allowed to take any value within $[u_{\min}, 1]$.

Since $\sum_{y_j \in \mathcal{I}} q_j = 1$ (this is enforced above by the moment constraint for $i = 0$), we can interpret the q_j and u_j in the above program as a probability measure over $[u_{\min}, 1]$, placing probability mass q_j on u_j. We can use this observation to relax the constraints of the above program by replacing the q_i and u_i with any probability measure on $[u_{\min}, 1]$. Note that this is essentially equivalent to allowing n, the individuals in the population, go to infinity. Thus, we get that $u_{\mathcal{P},p}(a)$ is upper and lower bounded by maximizing and minimizing, respectively, the following optimization problem

$$\max / \min \quad \left(\int_{u_{\min}}^{1} x^p d\mu(x) \right)^{\frac{1}{p}}$$

$$\text{s.t.} \quad \int_{u_{\min}}^{1} x^i d\mu(x) = \kappa_i \quad \text{for all } i \in \{0, 1, ..., m\}$$

$$\mu(x) \geq 0 \quad \text{for all } x \in [u_{\min}, 1].$$

Let M be the feasible region of the optimization program(s) above (noting that continuous, discrete, and mixed probability measures are all in M). If we take the μ corresponding to some population, we see that the moments of the population are identical to the moments of the probability distribution $\mu \in M$.

Problems of the above type, where a space of probability measures over a bounded interval is defined by their moments, are called moment problems [25]. It is known that the space of such probability measures is convex, with the extreme points being distributions supported on a discrete set of size at most $m + 1$, where m is the number of moments [19, Theorem 2.1 for a general $u_{\min}$, Proposition 3.2 for $u_{\min} = 0$].

It remains to show that the objective function of the above program takes its extreme values at the extreme points of the constraint set. To see this, let $\mu_1, \mu_2 \in M$ be any two distributions, and for some $t \in [0, 1]$, let $\mu = t\mu_1 + (1 - t)\mu_2 \in M$ be a convex combination of μ_1, μ_2. Then, the objective function (i.e. p-mean) of μ is

$$\left(\int_{u_{\min}}^{1} x^p d\mu \right)^{\frac{1}{p}} = \left(t \int_{u_{\min}}^{1} x^p d\mu_1 + (1 - t) \int_{u_{\min}}^{1} x^p \mu_2 \right)^{\frac{1}{p}}$$

$$= \left(t \left(\int_{u_{\min}}^{1} x^p d\mu_1 \right)^{\frac{1}{p}p} + \left((1 - t) \int_{u_{\min}}^{1} x^p \mu_2 \right)^{\frac{1}{p}p} \right)^{\frac{1}{p}}.$$

In particular, the objective function for μ is the p-mean of the objective functions for μ_1 and μ_2, weighted with t and $1 - t$, respectively, and as such, always between the two.

Thus, we conclude that the above optimization program takes its maximum and minimum values at extreme points of the constraint set, which is discrete distributions with support size at most $m + 1$. Applying this yields the optimization programs in the theorem statement. $\square$

Lemma 14 gives us an upper and lower bound on the utility of a population or panel with given first m-moments. We can use Lemma 14 to prove the following lemma.

Lemma 15. *Let $u_{\min} \in [0, 1]$ and $p < 1$. Let $\mathcal{P}$ be a population and let $\mathcal{C}$ be a m-representative panel in $\mathcal{P}$. Let a be any alternative such that $u_j(a) \in [u_{\min}, 1]$ for all $y_j \in \mathcal{I}$. Then $\mathcal{C}$ is ε-accurate*

with

$$\varepsilon = \max \quad \text{UMAX}(\kappa_1, ..., \kappa_m; p, u_{\min}) - \text{UMIN}(\kappa_1, ..., \kappa_m; p, u_{\min})$$

$$\text{s.t.} \quad [\kappa_{i+j}]_{i,j=0}^{m/2}, [(1 + u_{\min})\kappa_{i+j+1} - \kappa_{i+j+2} - u_{\min}\kappa_{i+j}]_{i,j=0}^{m/2-1} \succeq 0 \quad \textit{if } m \equiv 0 \mod 2$$

$$[\kappa_{i+j+1} - u_{\min}\kappa_{i+j}]_{i,j=0}^{(m-1)/2}, [\kappa_{i+j} - \kappa_{i+j+1}]_{i,j=0}^{(m-1)/2} \succeq 0 \qquad \textit{if } m \equiv 1 \mod 2$$

where $[M_{i,j}]_{i,j=0}^{d-1}$ is a $d \times d$ matrix with entry $M_{i,j}$ at position (i,j) and $M \succeq 0$ means that M is positive-semidefinite.

Proof. Lemma 14 implies that for a given sequence of moments $\kappa_1, ..., \kappa_m$ it holds that

$$|u_{\mathcal{C},p}(a) - u_{\mathcal{P},p}(a)| \leq \text{UMAX}(\kappa_1, ..., \kappa_m; p, u_{\min}) - \text{UMIN}(\kappa_1, ..., \kappa_m; p, u_{\min}).$$

Thus, $\mathcal{C}$ is guaranteed to be ε-accurate with

$$\varepsilon = \max \quad \text{UMAX}(\kappa_1, ..., \kappa_m; p, u_{\min}) - \text{UMIN}(\kappa_1, ..., \kappa_m; p, u_{\min})$$

$$\text{s.t.} \quad \kappa_1, ..., \kappa_m \text{ are the moments of a population (or panel) and alternative, given } u_{\min}$$

We can relax the constraints on $\kappa_1, ..., \kappa_m$, similarly to the proof of Lemma 14, by only requiring $\kappa_1, ..., \kappa_m$ to be the moments of a probability measure μ on $[u_{\min}, 1]$. Determining whether such a probability measure μ exists given moments and a bounded interval is known as the one-dimensional truncated moment problem on a bounded interval; we refer to [24] for a concise and contemporary proof that the constraints in the lemma statement are necessary and sufficient for $\kappa_1, ..., \kappa_m$ to be the moments of a probability measure μ on $[u_{\min}, 1]$. $\qquad \square$

A.3.2 Solving for UMIN and UMAX

The optimization program in Lemma 15 is by itself not overly insightful since there are two further optimization programs nested in it, from the definitions of $\text{UMAX}(\kappa_1, ..., \kappa_m; p, u_{\min})$ and $\text{UMIN}(\kappa_1, ..., \kappa_m; p, u_{\min})$. In this section, we describe a general framework for getting closed form solutions for UMAX and UMIN, which we apply in subsequent sections for $m = 1$ and $m = 2$.

We start by noting that the feasible regions of $\text{UMAX}(\kappa_1, ..., \kappa_m; p, u_{\min})$ and $\text{UMIN}(\kappa_1, ..., \kappa_m; p, u_{\min})$ are identical. There are $m + 1$ moment conditions, in the form of $m + 1$ equality constraints, all linear in the $m + 1$ q_i's. W.l.o.g. we assume that $x_0 > x_1 > ... > x_m$ (we don't need to consider equality since it is equivalent to setting a q_i to 0), so we get that the $m + 1$ moment equality constraints for fixed x_i are linearly independent since the coefficients are x_i^j for $j = 0, ..., m$ and the x_i are distinct. We can thus solve uniquely for the q_i to get

$$q_i = \frac{1}{\prod_{j \neq i}(x_i - x_j)} \sum_{j=0}^{m} (-1)^j \kappa_{m-j} e_j(x_{-i}), \tag{2}$$

where e_j is the j-th elementary symmetric polynomial and x_{-i} are all $x_0, \ldots, x_m$ except x_i:

$$e_j(x_{-i}) = e_j(x_0, ..., x_{i-1}, x_{i+1}, ..., x_m) = \sum_{\substack{0 \leq i_1 < ... < i_j \leq m \\ i_1, ..., i_j \neq i}} x_{i_1}...x_{i_j}.$$

Defining the q_i as a function of $x = (x_0, ..., x_m)$, we can rewrite the optimization problems with only $m + 1$ instead of $2m + 2$ variables so that

$$\text{UMIN}(\kappa_1, ..., \kappa_m; p, u_{\min}) = \qquad\qquad \text{UMAX}(\kappa_1, ..., \kappa_m; p, u_{\min}) =$$

$$\min \left(\sum_{i=0}^{m} q_i(x) x_i^p \right)^{\frac{1}{p}} \qquad\qquad\qquad \max \left(\sum_{i=0}^{m} q_i(x) x_i^p \right)^{\frac{1}{p}}$$

$$\text{s.t.} \quad q_i(x) \geq 0 \quad \text{for all } i \in \{0, ..., m\} \qquad \text{s.t.} \quad q_i(x) \geq 0 \quad \text{for all } i \in \{0, ..., m\}$$

$$1 \geq x_0 > x_1 > \cdots > x_m \geq u_{\min}, \qquad\qquad\quad 1 \geq x_0 > x_1 > \cdots > x_m \geq u_{\min}.$$

To get an analytical solution for this optimization problem, we note that for $m = 1$ and $m = 2$, the derivative of the objective with respect to each variable is either non-negative or non-positive on the entire feasible region.

Lemma 16. *Let $p \leq 1$ and $u_{\min} \in (0,1)$, or $p \in (0,1]$ and $u_{\min} = 0$. Let $m \in \{1,2\}$ and $1 \geq x_0 > x_1 > \cdots > x_m \geq u_{\min}$. Then for all $x = (x_0, \ldots, x_m)$ so that $q_j(x) \geq 0$ for all $j \in \{0, ..., m\}$, it holds that*

$$\frac{\partial}{\partial x_i} \left(\sum_{i=0}^{m} q_i(x) x_i^p \right)^{\frac{1}{p}} \begin{cases} \geq 0 & \text{if } i \equiv m \pmod 2 \\ \leq 0 & \text{if } i \not\equiv m \pmod 2 \end{cases},$$

for all $i = 0, ..., m$, where the q_i are a function of $x_0, ..., x_m$ as defined in Equation (2).

We conjecture that Lemma 16 also holds for $m > 2$ and can be proven almost analogously to our proof for $m = 2$. For our results, we only investigate 1-representative and 2-representative panels and prove Lemma 16 for the respective m in the corresponding sections. We then apply it to solve explicitly for UMIN and UMAX.

A.3.3 1-representative panels

We'll first consider the case $m = 1$. We prove Lemma 16 for $m = 1$ and then use it to solve for UMIN and UMAX. By Lemma 15, the proof of the first, "upper bound", part of Theorem 8 will follow (the proof of the second, "lower bound", part can be found in Appendix A.4).

Proof of Lemma 16, $m = 1$. Recall that w.l.o.g. we assume $x_0 > x_1$. We can solve for q_i as in Equation (2) to get

$$q_0 = \frac{\kappa_1 - x_1}{x_0 - x_1} \qquad\qquad q_1 = 1 - q_0 = \frac{x_0 - \kappa_1}{x_0 - x_1}.$$

We'll now first treat the case $p \neq 0$ and return to $p = 0$ at the end.

Let's calculate the derivatives of the objective without the outer p-th root:

$$\frac{\partial}{\partial x_0}(q_0 x_0^p + q_1 x_1^p) = p x_0^{p-1} \frac{\kappa_1 - x_1}{x_0 - x_1} + x_0^p \frac{x_1 - \kappa_1}{(x_0 - x_1)^2} + x_1^p \frac{\kappa_1 - x_1}{(x_0 - x_1)^2}$$

$$= \frac{\kappa_1 - x_1}{x_0 - x_1}\left(p x_0^{p-1} - \frac{x_0^p - x_1^p}{x_0 - x_1}\right)$$

$$= q_0 \left(p x_0^{p-1} - \frac{x_0^p - x_1^p}{x_0 - x_1}\right),$$

$$\frac{\partial}{\partial x_1}(q_0 x_0^p + q_1 x_1^p) = q_1 \left(p x_1^{p-1} - \frac{x_0^p - x_1^p}{x_0 - x_1}\right).$$

To determine the sign of the two expressions in parenthesis, we let $f(x) = x^p$ and note that $\frac{df}{dx}(x_0) = p x_0^{p-1}$ and $\frac{df}{dx}(x_1) = p x_1^{p-1}$ while $\frac{x_0^p - x_1^p}{x_0 - x_1}$ is the difference quotient of f for $[x_1, x_0]$. For $p \in (0,1)$, f is concave, so we get $p x_0^{p-1} < \frac{x_0^p - x_1^p}{x_0 - x_1} < p x_1^{p-1}$; for $p < 0$, f is convex, so $p x_0^{p-1} > \frac{x_0^p - x_1^p}{x_0 - x_1} > p x_1^{p-1}$. Since $q_0, q_1 \geq 0$, we get that

$$\frac{\partial}{\partial x_i}(q_0 x_0^p + q_1 x_1^p) \begin{cases} \leq 0 & \text{if } (i = 0 \wedge p \in (0,1)) \vee (i = 1 \wedge p < 0) \\ \geq 0 & \text{if } (i = 1 \wedge p \in (0,1)) \vee (i = 0 \wedge p < 0) \end{cases}.$$

To finish the proof, we note that

$$\frac{\partial}{\partial x_0}(q_0 x_0^p + q_1 x_1^p)^{\frac{1}{p}} = \frac{1}{p}(q_0 x_0^p + q_1 x_1^p)^{\frac{1}{p}-1} \cdot \frac{\partial}{\partial x_0}(q_0 x_0^p + q_1 x_1^p),$$

so if $p < 0$ the sign switches when considering the objective function instead of the objective function without the p-th root. This implies the lemma for $p \neq 0$.

Let's now turn to the case $p = 0$. We note that the sign of the objective doesn't change when taking the logarithm, so it suffices to show that

$$\frac{\partial}{\partial x_i}(q_0 \log(x_0) + q_1 \log(x_1)) \begin{cases} \leq 0 & \text{if } i = 0 \\ \geq 0 & \text{if } i = 1 \end{cases}.$$

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Taking the derivatives, we get

$$\frac{\partial}{\partial x_0}\left(q_0 \log(x_0) + q_1 \log(x_1)\right) = q_0 \left(\frac{1}{x_0} - \frac{\log(x_0) - \log(x_1)}{x_0 - x_1}\right),$$

$$\frac{\partial}{\partial x_1}\left(q_0 \log(x_0) + q_1 \log(x_1)\right) = q_1 \left(\frac{1}{x_1} - \frac{\log(x_0) - \log(x_1)}{x_0 - x_1}\right).$$

If we let $f(x) = \log(x)$, again, the expressions in the parenthesis is the difference between the derivative of f at x_0 and x_1, respectively, minus the difference quotient of f for $[x_1, x_0]$. Since f is concave, the lemma for $p = 0$ follows. $\qquad\square$

We now apply Lemma 16 to solve for UMIN and UMAX.

Lemma 17. *Let $p \leq 1$ and $u_{\min} \in (0,1)$, or $p \in (0,1]$ and $u_{\min} = 0$. Let $\kappa_1 \in (u_{\min}, 1)$. Then* $\text{UMAX}(\kappa_1; p, u_{\min}) = \kappa_1$ *and*

$$\text{UMIN}(\kappa_1; p, u_{\min}) = \begin{cases} \left(\frac{\kappa_1 - u_{\min}}{1 - u_{\min}} + \frac{1 - \kappa_1}{1 - u_{\min}} \cdot u_{\min}{}^p\right)^{\frac{1}{p}} & \text{if } p \neq 0 \\ (u_{\min})^{\frac{1 - \kappa_1}{1 - u_{\min}}} & \text{if } p = 0 \end{cases}$$

Proof. W.l.o.g. we assume $x_0 > x_1$. Thus, the constraints $q_i \geq 0$ are equivalent to $\kappa_1 \geq x_1$ and $x_0 \geq \kappa_1$. We solve for UMIN and UMAX by finding the minimum and maximum of their defining optimization program from Lemma 14.

Finding the maximum: By Lemma 16, we know that the objective (non-strictly) increases as x_0 decreases. The only lower bounds on x_0 are $x_0 \geq \kappa_1$ and $x_0 > x_1$, so we know there is a maximum either at $x_0 = \kappa_1$ (where the first lower bound is tight) or for $x_1 \geq \kappa_1$ (the only case in which x_0 can't be tight to the first lower bound since $x_0 > x_1$). Since κ_1 is the weighted mean of x_0 and x_1, we know that $x_1 \leq \kappa_1$, so the second case becomes $x_1 = \kappa_1$. These two cases, $x_0 = \kappa_1$ and $x_1 = \kappa_1$, imply, respectively, $q_1 = 0$ and $q_0 = 0$, so the other variable doesn't affect the objective. We thus get that a maximum is attained at

$$x_0 = \kappa_1, \quad q_0 = 1, \quad x_1 \in [u_{\min}, \kappa_1), \quad q_1 = 0 \quad \text{or} \quad x_0 \in (\kappa_1, 1], \quad q_0 = 0, \quad x_1 = \kappa_1, \quad q_1 = 1.$$

In both cases, the objective value is κ_1.

Finding the minimum: By Lemma 16, we know that the objective (non-strictly) decreases as x_0 increases. The only upper bound on x_0 is $x_0 \leq 1$. Similarly, we know that the objective (non-strictly) decreases as x_1 decreases. The only lower bound on x_1 is $x_1 \geq u_{\min}$. Thus, we know that the objective takes its minimum value at

$$x_0 = 1, \qquad q_0 = \frac{\kappa_1 - u_{\min}}{1 - u_{\min}}, \qquad x_1 = u_{\min}, \qquad q_1 = \frac{1 - \kappa_1}{1 - u_{\min}}.$$

Plugging these values into the objective gives the lemma statement. $\qquad\square$

We know have the tools to prove the first part of the main theorem, Theorem 8, the upper bound on ε-accuracy of a 1-representative panel.

Proof of Theorem 8, upper bound. We first consider the case $p \leq 0$ and $u_{\min} = 0$. Since $u_{\mathcal{C},p}(a), u_{\mathcal{P},p}(a) \in [0,1]$ for all $a \in \mathcal{A}$, we know that $|u_{\mathcal{C},p}(a) - u_{\mathcal{P},p}(a)| \leq 1$ for all $a \in \mathcal{A}$. Thus, $\mathcal{C}$ is 1-accurate.

Let's now consider the case $p < 1$ and $u_{\min} \in (0,1)$ or $p \in (0,1)$ and $u_{\min} = 0$. By Lemma 15, we know that any 1-representative panel $\mathcal{C}$ is ε-accurate with

$$\varepsilon = \max \quad \text{UMAX}(\kappa_1; p, u_{\min}) - \text{UMIN}(\kappa_1, \kappa_2; p, u_{\min})$$

$$\text{s.t.} \quad [\kappa_{i+j+1} - u_{\min}\kappa_{i+j}]_{i,j=0}^0, [\kappa_{i+j} - \kappa_{i+j+1}]_{i,j=0}^0 \succeq 0$$

where $[M_{i,j}]_{i,j=0}^{d-1}$ is a $d \times d$ matrix with entry $M_{i,j}$ at position (i,j) and $M \succeq 0$ means that M is positive-semi-definite. Since both matrices are 1×1 and $\kappa_0 = 1$, the constraints are equivalent to $\kappa_1 \geq u_{\min}$ and $1 \geq \kappa_1$.

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If both constraints are strict, we know the values of UMIN and UMAX by Lemma 17. There remains the case when at least one of the two inequalities is not strict (recall Lemma 14 for the definitions of UMAX and UMIN in this case). Let's again assume w.l.o.g. that $x_0 > x_1$. Since κ_1 is the weighted mean of x_0 and x_1 (with weights q_0 and q_1, respectively), and both $x_0, x_1 \in [u_{\min}, 1]$, we know that $\kappa_1 = u_{\min}$ implies $x_1 = u_{\min}, q_0 = 0$, so $\text{UMIN}(u_{\min}; p, u_{\min}) = \text{UMAX}(u_{\min}; p, u_{\min}) = u_{\min}$. Similarly, $\kappa_1 = 1$ implies $x_0 = 1, q_1 = 0$, so $\text{UMIN}(1; p, u_{\min}) = \text{UMAX}(1; p, u_{\min}) = 1$.

In either case, if one of the two constraints is not tight, $\text{UMAX}(\kappa_1, \kappa_2; p, u_{\min}) - \text{UMIN}(\kappa_1, \kappa_2; p, u_{\min}) = 0$. Since there always exists a point at which all constraints are strictly followed, and $\text{UMAX}(\kappa_1, \kappa_2; p, u_{\min}) - \text{UMIN}(\kappa_1, \kappa_2; p, u_{\min}) \geq 0$ at any such point, as follows immediately from their definition, we can restrict the constraints to being strict, to get that any 1-representative panel is ε-accurate for $p \neq 0$ with

$$\max_{\kappa_1} \quad \kappa_1 - \left(\frac{\kappa_1 - u_{\min}}{1 - u_{\min}} + \frac{1 - \kappa_1}{1 - u_{\min}} \cdot u_{\min}^p \right)^{\frac{1}{p}}$$
$$\text{s.t.} \quad u_{\min} < \kappa_1 < 1,$$

and for $p = 0$ with

$$\max_{\kappa_1} \quad \kappa_1 - u_{\min}^{\frac{1 - \kappa_1}{1 - u_{\min}}}$$
$$\text{s.t.} \quad u_{\min} < \kappa_1 < 1.$$

We'll now solve these two optimization programs, starting with the case $p \neq 0$. We let $f(\kappa_1) = \kappa_1 - \left(\frac{\kappa_1 - u_{\min}}{1 - u_{\min}} + \frac{1 - \kappa_1}{1 - u_{\min}} \cdot u_{\min}^p \right)^{\frac{1}{p}}$ be the objective of the optimization program. We take the first derivative of f, to get

$$\frac{df}{d\kappa_1} = 1 - \frac{1}{p} \left(\frac{\kappa_1 - u_{\min}}{1 - u_{\min}} + \frac{1 - \kappa_1}{1 - u_{\min}} \cdot u_{\min}^p \right)^{\frac{1}{p} - 1} \frac{1 - u_{\min}^p}{1 - u_{\min}}.$$

Setting this equal to zero, we get that

$$p \frac{1 - u_{\min}}{1 - u_{\min}^p} = \left(\frac{\kappa_1 - u_{\min}}{1 - u_{\min}} + \frac{1 - \kappa_1}{1 - u_{\min}} \cdot u_{\min}^p \right)^{\frac{1}{p} - 1}.$$

We know that both the left hand side as well as everything within the parenthesis on the right hand side is positive for $\kappa_1 \in (u_{\min}, 1)$, so we can raise the above equality to the $(1 - p)/p$-th power without introducing or loosing solutions for $\kappa_1 \in (u_{\min}, 1)$. We do so and solve for κ_1 to get

$$\kappa_1 = \frac{1 - u_{\min}}{1 - u_{\min}^p} \left(p \cdot \frac{1 - u_{\min}}{1 - u_{\min}^p} \right)^{\frac{p}{1 - p}} + \frac{u_{\min} - u_{\min}^p}{1 - u_{\min}^p}. \tag{3}$$

Taking the second derivative of f, we see that for $\kappa_1 \in [u_{\min}, 1]$,

$$\frac{d^2 f}{(d\kappa_1)^2} = -\frac{1}{p} \left(\frac{1}{p} - 1 \right) \left(\frac{\kappa_1 - u_{\min}}{1 - u_{\min}} + \frac{1 - \kappa_1}{1 - u_{\min}} \cdot u_{\min}^p \right)^{\frac{1}{p} - 2} \left(\frac{1 - u_{\min}^p}{1 - u_{\min}} \right)^2 \leq 0,$$

so f is concave on $[u_{\min}, 1]$. We can see that $f(u_{\min}) = f(1) = 0$, and we know from above that $f(x) > 0$ for some $x \in (u_{\min}, 1)$. Thus, we know that f has an extreme point on $[u_{\min}, 1]$, which we know has to be at κ_1 as specified in Equation (3), and f is maximized on $(u_{\min}, 1)$ for this κ_1.

Plugging Equation (3) into f, we get that

$$\max_{u_{\min} < \kappa_1 < 1} f(\kappa_1) = \frac{1 - p}{p} \left(p \cdot \frac{1 - u_{\min}}{1 - u_{\min}^p} \right)^{\frac{1}{1 - p}} + \frac{u_{\min} - u_{\min}^p}{1 - u_{\min}^p},$$

which gives ε from the theorem statement, for $p < 0$ and $u_{\min} \in (0, 1)$ or $p \in (0, 1)$ and $u_{\min} = 0$.

Let's now solve the optimization program in the remaining case $p = 0$ and $u_{\min} \in (0, 1)$. Again, we let $f(\kappa_1) = \kappa_1 - u_{\min}^{\frac{1 - \kappa_1}{1 - u_{\min}}}$ be the objective of corresponding optimization program. We take the first derivative of f, to get

$$\frac{df}{d\kappa_1} = 1 + \frac{1}{1 - u_{\min}} u_{\min}^{\frac{1 - \kappa_1}{1 - u_{\min}}} \ln(u_{\min}).$$

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Setting this equal to zero and solving for κ_1, we get

$$\kappa_1 = 1 - (1 - u_{\min}) \frac{\ln\left(\frac{1}{1-u_{\min}}\right) + \ln\ln\left(\frac{1}{u_{\min}}\right)}{\ln\left(\frac{1}{u_{\min}}\right)} \tag{4}$$

Taking the second derivative of f, we see that for $\kappa_1 \in [u_{\min}, 1]$,

$$\frac{d^2 f}{(d\kappa_1)^2} = -\left(\frac{1}{1 - u_{\min}}\right)^2 u_{\min}^{\frac{1-\kappa_1}{1-u_{\min}}} \ln^2(u_{\min}) \leq 0,$$

so f is concave on $[u_{\min}, 1]$. Similar to before, we can see that $f(u_{\min}) = f(1) = 0$, and we know from above that $f(x) > 0$ for some $x \in (u_{\min}, 1)$. Thus, we know that f has an extreme points on $[u_{\min}, 1]$, which we know has to be at κ_1 as specified in Equation (4), and f is maximized on $(u_{\min}, 1)$ for this κ_1.

Plugging Equation (4) into f, we get that

$$\max_{u_{\min} < \kappa_1 < 1} f(\kappa_1) = 1 + \frac{1 - u_{\min}}{\ln\left(\frac{1}{u_{\min}}\right)}\left(\ln\left(\frac{1 - u_{\min}}{\ln\left(\frac{1}{u_{\min}}\right)}\right) - 1\right),$$

which gives ε from the theorem statement for $p = 0$ and $u_{\min} \in (0, 1)$. $\qquad\square$

A.3.4   2-representative panels

Let's consider $m = 2$. We will first prove Lemma 16 for $m = 2$, which we'll then apply to solve for UMIN and UMAX. Together with Lemma 15 the proof of Theorem 9 will follow.

Proof of Lemma 16, $m = 2$. From Equation (2) we obtain that

$$q_0 = \frac{1}{(x_0 - x_1)(x_0 - x_2)}(\kappa_2 - \kappa_1(x_1 + x_2) + x_1 x_2)$$

$$q_1 = \frac{1}{(x_1 - x_0)(x_1 - x_2)}(\kappa_2 - \kappa_1(x_0 + x_2) + x_0 x_2)$$

$$q_2 = \frac{1}{(x_2 - x_0)(x_2 - x_1)}(\kappa_2 - \kappa_1(x_0 + x_1) + x_0 x_1).$$

We now calculate the derivative of the objective without the outer p-th root. We'll treat the case $p = 0$ separately at the end.

$$\frac{\partial}{\partial x_0}\left(\sum_{i=0}^{2} q_i x_i^p\right) = q_0 p x_0^{p-1} + x_0^p q_0 \frac{(x_1 - x_0) + (x_2 - x_0)}{(x_0 - x_1)(x_0 - x_2)}$$

$$+ q_1 \frac{x_1^p}{(x_1 - x_0)} + \frac{x_1^p(x_2 - \kappa_1)}{(x_1 - x_0)(x_1 - x_2)}$$

$$+ q_2 \frac{x_2^p}{(x_2 - x_0)} + \frac{x_2^p(x_1 - \kappa_1)}{(x_2 - x_0)(x_2 - x_1)}$$

$$= q_0 \left(p x_0^{p-1} - \frac{x_0^p}{x_0 - x_1} - \frac{x_0^p}{x_0 - x_2}\right.$$

$$+ \frac{x_1^p}{(x_1 - x_0)(x_1 - x_2)}\left(\frac{q_1(x_1 - x_2) + (x_2 - \kappa_1)}{q_0}\right)$$

$$+ \left.\frac{x_2^p}{(x_2 - x_0)(x_2 - x_1)}\left(\frac{q_2(x_2 - x_1) + (x_1 - \kappa_1)}{q_0}\right)\right)$$

$$= q_0 \left(p x_0^{p-1} - \frac{x_0^p}{x_0 - x_1} - \frac{x_0^p}{x_0 - x_2}\right.$$

$$\left.+ \frac{x_1^p(x_2 - x_0)}{(x_1 - x_0)(x_1 - x_2)}\left(\frac{(\kappa_2 - \kappa_1(x_0 + x_2) + x_0 x_2) + (x_2 - \kappa_1)(x_1 - x_0)}{\kappa_2 - \kappa_1(x_1 + x_2) + x_1 x_2}\right)\right.$$

$$+ \frac{x_2^p(x_1 - x_0)}{(x_2 - x_0)(x_2 - x_1)}\left(\frac{(\kappa_2 - \kappa_1(x_0 + x_1) + x_0 x_1) + (x_1 - \kappa_1)(x_2 - x_0)}{\kappa_2 - \kappa_1(x_1 + x_2) + x_1 x_2}\right)\Bigg)$$

$$= q_0\left(px_0^{p-1} - \frac{x_0^p - x_1^p \frac{x_0 - x_2}{x_1 - x_2}}{x_0 - x_1} - \frac{x_0^p - x_2^p \frac{x_0 - x_1}{x_2 - x_1}}{x_0 - x_2}\right).$$

To determine the sign of the last line we consider the function $f : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ where $f(x) = x^p$. We let P be the unique degree-2 polynomial for which $P(x_0) = f(x_0)$, $P(x_1) = f(x_1)$, $P(x_2) = f(x_2)$ using the Lagrange basis polynomials:

$$P(x) = x_0^p \frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + x_1^p \frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + x_2^p \frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}.$$

We can now note that

$$\frac{dP}{dx}(x_0) = \frac{x_0^p - x_1^p \frac{x_0 - x_2}{x_1 - x_2}}{x_0 - x_1} + \frac{x_0^p - x_2^p \frac{x_0 - x_1}{x_2 - x_1}}{x_0 - x_2} \qquad\qquad \frac{df}{dx}(x_0) = px_0^{p-1},$$

so that

$$\frac{\partial}{\partial x_0}\left(\sum_{i=0}^{2} q_i x_i^p\right) = q_0\left(\frac{df}{dx}(x_0) - \frac{dP}{dx}(x_0)\right).$$

Analogously, for x_1 and x_2 we obtain

$$\frac{\partial}{\partial x_1}\left(\sum_{i=0}^{2} q_i x_i^p\right) = q_1\left(px_1^{p-1} - \frac{x_1^p - x_0^p \frac{x_1 - x_2}{x_0 - x_2}}{x_1 - x_0} - \frac{x_1^p - x_2^p \frac{x_1 - x_0}{x_2 - x_0}}{x_1 - x_2}\right) = q_1\left(\frac{df}{dx}(x_1) - \frac{dP}{dx}(x_1)\right),$$

$$\frac{\partial}{\partial x_2}\left(\sum_{i=0}^{2} q_i x_i^p\right) = q_2\left(px_2^{p-1} - \frac{x_2^p - x_0^p \frac{x_2 - x_1}{x_0 - x_1}}{x_2 - x_0} - \frac{x_2^p - x_1^p \frac{x_2 - x_0}{x_1 - x_0}}{x_2 - x_1}\right) = q_2\left(\frac{df}{dx}(x_2) - \frac{dP}{dx}(x_2)\right).$$

We now prove that f can intersect in at most 3 points with a degree-2 polynomial. If we let $P(x) = a_2 x^2 + a_1 x^1 + a_0$ for some a_0, a_1, a_2, we get that $\frac{d^2}{dx^2}(f - P) = p(p-1)x^{p-2} - 2a_2$. Since x^{p-2} is monotone and non-negative (and we know $a_2 \neq 0$ since the three points P is interpolating through are not collinear) $\frac{d^2}{dx^2}(f - P)$ has at most one (single) root if a_2 and $p(p-1)$ are either both positive or both negative, and no root otherwise. We conclude that $f - P$ has at most 3 roots, and that $a_2 < 0$ when $p \in (0, 1)$ while $a_2 > 0$ when $p < 0$.

Since we know x_0, x_1, x_2 are 3 distinct roots of $f - P$, we know that all 3 roots are all single, and therefore that which one of f and P is greater 'switches' at each root. In particular, if $p \in (0, 1)$, we know that $a_2 < 0$, so $P(x) < f(x)$ for $x > x_0$ and $x \in (x_2, x_1)$, while $P(x) > f(x)$ for $x < x_2$ and $x \in (x_1, x_0)$. Therefore

$$\frac{df}{dx}(x_0) - \frac{dP}{dx}(x_0) > 0, \qquad \frac{df}{dx}(x_1) - \frac{dP}{dx}(x_1) < 0, \qquad \frac{df}{dx}(x_2) - \frac{dP}{dx}(x_2) > 0.$$

Similarly, for $p < 0$, we know that $a_2 > 0$, so $P(x) > f(x)$ for $x > x_0$ and $x \in (x_2, x_1)$, while $P(x) < f(x)$ for $x < x_2$ and $x \in (x_1, x_0)$. Therefore

$$\frac{df}{dx}(x_0) - \frac{dP}{dx}(x_0) < 0, \qquad \frac{df}{dx}(x_1) - \frac{dP}{dx}(x_1) > 0, \qquad \frac{df}{dx}(x_2) - \frac{dP}{dx}(x_2) < 0.$$

Since $q_0, q_1, q_2 \geq 0$, we get that

$$\frac{\partial}{\partial x_i} \sum_{i=0}^{2} q_i x_i^p \begin{cases} \geq 0 & \text{if } i = 0, 2 \\ \leq 0 & \text{if } i = 1 \end{cases} \text{if } p \in (0,1), \qquad \frac{\partial}{\partial x_i} \sum_{i=0}^{2} q_i x_i^p \begin{cases} \leq 0 & \text{if } i = 0, 2 \\ \geq 0 & \text{if } i = 1 \end{cases} \text{if } p < 0.$$

To finish the proof, we note that

$$\frac{\partial}{\partial x_0}\left(\sum_{i=0}^{2}q_i x_i^p\right)^{\frac{1}{p}} = \frac{1}{p}\left(\sum_{i=0}^{2}q_i x_i^p\right)^{\frac{1}{p}-1}\frac{\partial}{\partial x_0}\left(\sum_{i=0}^{2}q_i x_i^p\right),$$

so if $p < 0$ the sign switches when considering the objective function instead of the objective function without the p-th root. This implies the lemma for $p \neq 0$.

We now turn to the case $p = 0$. Taking the logarithm of objective doesn't change the sign of its derivative, so it suffices to show that

$$\frac{\partial}{\partial x_i}\left(\sum_{i=0}^{2}q_i \ln(x_i)\right)\begin{cases}\geq 0 & \text{if } i \equiv m \pmod 2 \\ \leq 0 & \text{if } i \not\equiv m \pmod 2\end{cases}.$$

Taking the derivative of the objective with respect to x_0, we obtain similarly to the case before that

$$\frac{\partial}{\partial x_0}\left(\sum_{i=0}^{2}q_i \ln(x_i)\right) = q_0\frac{1}{x_0} + \ln(x_0)q_0\frac{(x_1 - x_0) + (x_2 - x_0)}{(x_0 - x_1)(x_0 - x_2)}$$

$$+ q_1\frac{\ln(x_1)}{(x_1 - x_0)} + \frac{\ln(x_1)(x_2 - \kappa_1)}{(x_1 - x_0)(x_1 - x_2)}$$

$$+ q_2\frac{\ln(x_2)}{(x_2 - x_0)} + \frac{\ln(x_2)(x_1 - \kappa_1)}{(x_2 - x_0)(x_2 - x_1)}$$

$$= q_0\left(\frac{1}{x_0} - \frac{\ln(x_0) - \ln(x_1)\frac{x_0-x_2}{x_1-x_2}}{x_0 - x_1} - \frac{\ln(x_0) - \ln(x_2)\frac{x_0-x_1}{x_2-x_1}}{x_0 - x_2}\right).$$

To determine the sign of the last line we consider the function $f : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0}$ where $f(x) = \ln(x)$. We let P be the unique degree-2 polynomial for which $P(x_0) = f(x_0)$, $P(x_1) = f(x_1)$, $P(x_2) = f(x_2)$ using the Lagrange basis polynomials:

$$P(x) = \ln(x_0)\frac{(x - x_1)(x - x_2)}{(x_0 - x_1)(x_0 - x_2)} + \ln(x_1)\frac{(x - x_0)(x - x_2)}{(x_1 - x_0)(x_1 - x_2)} + \ln(x_2)\frac{(x - x_0)(x - x_1)}{(x_2 - x_0)(x_2 - x_1)}.$$

We can now note that

$$\frac{dP}{dx}(x_0) = \frac{\ln(x_0) - \ln(x_1)\frac{x_0-x_2}{x_1-x_2}}{x_0 - x_1} - \frac{\ln(x_0) - \ln(x_2)\frac{x_0-x_1}{x_2-x_1}}{x_0 - x_2} \qquad \frac{df}{dx}(x_0) = \frac{1}{x_0},$$

so that

$$\frac{\partial}{\partial x_0}\left(\sum_{i=0}^{2}q_i x_i^p\right) = q_0\left(\frac{df}{dx}(x_0) - \frac{dP}{dx}(x_0)\right).$$

Analogously, for x_1 and x_2, we obtain

$$\frac{\partial}{\partial x_1}\left(\sum_{i=0}^{2}q_i x_i^p\right) = q_1\left(\frac{df}{dx}(x_1) - \frac{dP}{dx}(x_1)\right),$$

$$\frac{\partial}{\partial x_2}\left(\sum_{i=0}^{2}q_i x_i^p\right) = q_2\left(\frac{df}{dx}(x_2) - \frac{dP}{dx}(x_2)\right).$$

We now prove that f can intersect in at most 3 points with a degree-2 polynomial. If we let $P(x) = a_2 x^2 + a_1 x^1 + a_0$ for some a_0, a_1, a_2, we get that $\frac{d^2}{dx^2}(f - P) = -1/x^2 - 2a_2$. Since $1/x^2$ is monotone and non-negative, $\frac{d^2}{dx^2}(f - P)$ has at most one (single) root if a_2 is non-positive and no root otherwise. We conclude that $f - P$ has at most 3 roots.

We know x_0, x_1, x_2 are 3 distinct roots of $f - P$, so we know that $a_2 < 0$ (a_2 cannot be zero since the three points P is interpolating through are not collinear) and that all 3 roots are all single. Therefore, which one of f and P is greater 'switches' at each root. Since $a_2 < 0$, $P(x) < f(x)$ for $x > x_0$, so it follows that also $P(x) < f(x)$ for $x \in (x_2, x_1)$, while $P(x) > f(x)$ for $x < x_2$ and $x \in (x_1, x_0)$. Therefore

$$\frac{df}{dx}(x_0) - \frac{dP}{dx}(x_0) > 0, \qquad \frac{df}{dx}(x_1) - \frac{dP}{dx}(x_1) < 0, \qquad \frac{df}{dx}(x_2) - \frac{dP}{dx}(x_2) > 0;$$

the lemma for $p = 0$ follows. $\qquad\square$

We now apply Lemma 16 to solve for UMIN and UMAX. Note that in the lemma statement, the case $p = 0$ is the same as $p \neq 0$ when replacing the weighted p-mean by a weighted geometric mean.

Lemma 18. *Let $p < 1$ and $1 > \kappa_1 > \frac{\kappa_2 + u_{\min}}{1 + u_{\min}}$ and $\kappa_2 > \kappa_1^2 > u_{\min}^2$. Then, if $p \neq 0$,*

$$\text{UMIN}(\kappa_1, \kappa_2; p, u_{\min}) =$$

$$\left(\frac{(\kappa_1 - u_{\min})^2}{\kappa_2 - 2u_{\min}\kappa_1 + u_{\min}^2} \left(\frac{\kappa_2 - u_{\min}\kappa_1}{\kappa_1 - u_{\min}} \right)^p + \frac{\kappa_2 - \kappa_1^2}{\kappa_2 - 2u_{\min}\kappa_1 + u_{\min}^2} u_{\min}^p \right)^{\frac{1}{p}},$$

and

$$\text{UMAX}(\kappa_1, \kappa_2; p, u_{\min}) = \left(\frac{\kappa_2 - \kappa_1^2}{\kappa_2 - 2\kappa_1 + 1} + \frac{(1 - \kappa_1)^2}{\kappa_2 - 2\kappa_1 + 1} \left(\frac{\kappa_1 - \kappa_2}{1 - \kappa_1} \right)^p \right)^{\frac{1}{p}}.$$

If $p = 0$,

$$\text{UMAX}(\kappa_1, \kappa_2; 0, u_{\min}) = \left(\frac{\kappa_1 - \kappa_2}{1 - \kappa_1} \right)^{\left(\frac{(1 - \kappa_1)^2}{\kappa_2 - 2\kappa_1 + 1} \right)}$$

and

$$\text{UMIN}(\kappa_1, \kappa_2; 0, u_{\min}) = \left(\frac{\kappa_2 - u_{\min}\kappa_1}{\kappa_1 - u_{\min}} \right)^{\left(\frac{(\kappa_1 - u_{\min})^2}{\kappa_2 - 2u_{\min}\kappa_1 + u_{\min}^2} \right)} \cdot (u_{\min})^{\left(\frac{\kappa_2 - \kappa_1^2}{\kappa_2 - 2u_{\min}\kappa_1 + u_{\min}^2} \right)}.$$

Proof. W.l.o.g. we assume $x_0 > x_1 > x_2$. Since the weighted mean of x_0, x_1, x_2 is κ_1, we can furthermore assume w.l.o.g. that $x_0 > \kappa_1 > x_2$ (If $x_0 > \kappa_1$, then also $x_2 < \kappa_1$; if $x_0 = \kappa_1$, then $q_1 = q_2 = 0$, which is still captured by $x_0 > x_1 = \kappa_1 > x_2$ and $q_0 = q_2 = 0$; $x_0 < \kappa_1$ is not possible).

We now use this to rewrite the constraints $q_i \geq 0$ in terms of x_0, x_1, x_2. For q_0, we obtain

$$q_0 \geq 0 \Leftrightarrow \frac{1}{(x_0 - x_1)(x_0 - x_2)}(\kappa_2 - \kappa_1(x_1 + x_2) + x_1 x_2) \geq 0$$

$$\Leftrightarrow \kappa_2 - \kappa_1 x_2 + x_1(x_2 - \kappa_1) \geq 0$$

$$\Leftrightarrow x_1 \leq \frac{\kappa_2 - x_2 \kappa_1}{\kappa_1 - x_2}$$

$$\Leftrightarrow x_1 \leq \kappa_1 + \frac{\kappa_2 - \kappa_1^2}{\kappa_1 - x_2}. \tag{5}$$

Similarly, for q_1 and q_2, we obtain

$$q_1 \geq 0 \Leftrightarrow x_0 \geq \kappa_1 + \frac{\kappa_2 - \kappa_1^2}{\kappa_1 - x_2} \tag{6}$$

$$\Leftrightarrow x_2 \leq \kappa_1 - \frac{\kappa_2 - \kappa_1^2}{x_0 - \kappa_1}, \tag{7}$$

$$q_2 \geq 0 \Leftrightarrow x_1 \geq \kappa_1 - \frac{\kappa_2 - \kappa_1^2}{x_0 - \kappa_1}. \tag{8}$$

We now solve for UMIN and UMAX by finding the minimum and maximum of their defining optimization program from Lemma 14.

Finding the minimum: We'll first assume $p \neq 0$. By Lemma 16, we know that increasing x_1 will (non-strictly) decrease the objective. The upper bounds on x_1 are Constraint (5) and $x_1 < x_0 \leq 1$. Thus, we know that there is a minimum either for $x_1 = \kappa_1 + \frac{\kappa_2 - \kappa_1^2}{\kappa_1 - x_2}$ (tight to the upper bound given by Constraint (5)) or for $x_0 \leq \kappa_1 + \frac{\kappa_2 - \kappa_1^2}{\kappa_1 - x_2}$ (the only case in which x_1 can't take this value since $x_1 < x_0$). By Constraint (6), we know $x_0 \geq \kappa_1 + \frac{\kappa_2 - \kappa_1^2}{\kappa_1 - x_2}$, so the second case becomes $x_0 = \kappa_1 + \frac{\kappa_2 - \kappa_1^2}{\kappa_1 - x_2}$.

We first consider the case $x_1 = \kappa_1 + \frac{\kappa_2 - \kappa_1^2}{\kappa_1 - x_2} = \frac{\kappa_2 - x_2 \kappa_1}{\kappa_1 - x_2}$. We get that

$$
\begin{aligned}
q_0 &= \frac{\kappa_2 - \kappa_1 \left(\frac{\kappa_2 - x_2 \kappa_1}{\kappa_1 - x_2} + x_2 \right) + \frac{\kappa_2 - x_2 \kappa_1}{\kappa_1 - x_2} x_2}{(x_0 - x_1)(x_0 - x_2)} \\
&= \frac{\kappa_2 - \left(\frac{\kappa_1 \kappa_2 - x_2 \kappa_1^2 + x_2 \kappa_1^2 - x_2^2 \kappa_1 - \kappa_2 x_2 + x_2^2 \kappa_1}{\kappa_1 - x_2} \right)}{(x_0 - x_1)(x_0 - x_2)} \\
&= \frac{\kappa_2 - \left(\frac{\kappa_2 (\kappa_1 - x_2)}{\kappa_1 - x_2} \right)}{(x_0 - x_1)(x_0 - x_2)} = 0,
\end{aligned}
$$

so the value of x_0 won't affect the objective. Furthermore,

$$
\begin{aligned}
q_1 &= \frac{\kappa_2 - \kappa_1 (x_0 + x_2) + x_0 x_2}{\left(\frac{\kappa_2 - x_2 \kappa_1}{\kappa_1 - x_2} - x_0 \right) \left(\frac{\kappa_2 - x_2 \kappa_1}{\kappa_1 - x_2} - x_2 \right)} \\
&= \frac{(\kappa_2 - \kappa_1 (x_0 + x_2) + x_0 x_2)(\kappa_1 - x_2)^2}{(\kappa_2 - x_2 \kappa_1 - x_0 \kappa_1 + x_0 x_2)(\kappa_2 - x_2 \kappa_1 - x_2 \kappa_1 + x_2^2)} \\
&= \frac{(\kappa_1 - x_2)^2}{\kappa_2 - 2 x_2 \kappa_1 + x_2^2},
\end{aligned}
$$

$$
q_2 = 1 - q_0 - q_1 = \frac{\kappa_2 - \kappa_1^2}{\kappa_2 - 2 x_2 \kappa_1 + x_2^2},
$$

so the objective becomes

$$
\left(\frac{(\kappa_1 - x_2)^2}{\kappa_2 - 2 x_2 \kappa_1 + x_2^2} \left(\frac{\kappa_2 - x_2 \kappa_1}{\kappa_1 - x_2} \right)^p + \frac{\kappa_2 - \kappa_1^2}{\kappa_2 - 2 x_2 \kappa_1 + x_2^2} x_2^p \right)^{1/p} \quad \text{if } p \neq 0 \tag{9}
$$

$$
\left(\frac{\kappa_2 - x_2 \kappa_1}{\kappa_1 - x_2} \right)^{\left(\frac{(\kappa_1 - x_2)^2}{\kappa_2 - 2 x_2 \kappa_1 + x_2^2} \right)} \cdot (x_2)^{\left(\frac{\kappa_2 - \kappa_1^2}{\kappa_2 - 2 x_2 \kappa_1 + x_2^2} \right)} \quad \text{if } p = 0. \tag{10}
$$

In both cases, the objective is a univariate function in x_2. We will show that it is monotonically increasing on the interval $[0, \kappa_1)$, which implies that the minimum is achieved at $x_2 = u_{\min}$, which implies $x_1 = \frac{\kappa_2 - u_{\min} \kappa_1}{\kappa_1 - u_{\min}}$. Since $\kappa_1 (1 + u_{\min}) > \kappa_2 + u_{\min}$, by assumption in the lemma statement, if follows that $x_1 < 1$. We know that x_0 doesn't affect the objective, so we can just set x_0 to be any value in $(x_1, 1]$. Plugging this in, we get

$$
\begin{aligned}
x_0 &\in (x_1, 1] & q_0 &= 0 \\
x_1 &= \frac{\kappa_2 - u_{\min} \kappa_1}{\kappa_1 - u_{\min}} & q_1 &= \frac{(\kappa_1 - u_{\min})^2}{\kappa_2 - 2 u_{\min} \kappa_1 + u_{\min}^2} \\
x_2 &= u_{\min} & q_2 &= \frac{\kappa_2 - \kappa_1^2}{\kappa_2 - 2 u_{\min} \kappa_1 + u_{\min}^2}.
\end{aligned} \tag{11}
$$

We'll now show for the case $p \neq 0$ that Equation (9) is monotonically increasing on $[0, \kappa_1)$. To do that, we first drop the p-th root and take the derivative of the expression without the root (with respect to x_2), to calculate its sign. We get

$$
\frac{(\kappa_2 - \kappa_1^2) \left(x_1^{p-1} ((p-2) \kappa_2 - 2(p-1) \kappa_1 x_2 + p x_2^2) + x_2^{p-1} (p \kappa_2 - 2(p-1) \kappa_1 x_2 + (p-2) x_2^2) \right)}{(\kappa_2 - 2 \kappa_1 x_2 + x_2^2)^2},
$$

where we write x_1 for $\frac{\kappa_2 - x_2\kappa_1}{\kappa_1 - x_2}$ for brevity. Since $\kappa_2 - \kappa_1^2 > 0$ and $\kappa_2 - 2\kappa_1 x_2 + x_2^2 = \kappa_2 - \kappa_1^2 + (\kappa_1 - x_2)^2 > 0$, we can drop those terms without changing the sign of the expression. We simplify further to get

$$x_1^{p-1}((p-2)\kappa_2 - 2(p-1)\kappa_1 x_2 + px_2^2) + x_2^{p-1}(p\kappa_2 - 2(p-1)\kappa_1 x_2 + (p-2)x_2^2)$$
$$= (x_1^{p-1} + x_2^{p-1})p(\kappa_2 - 2\kappa_1 x_2 + x_2^2) - 2x_1^{p-1}(\kappa_2 - \kappa_1 x_2) + 2x_2^{p-1}(\kappa_1 x_2 - x_2^2)$$
$$= (x_1^{p-1} + x_2^{p-1})p(x_1 - x_2)(\kappa_1 - x_2) - 2x_1^p(\kappa_1 - x_2) + 2x_2^p(\kappa_1 - x_2).$$

Since $\kappa_1 > x_2$ and $x_1 > x_2$, we can divide by $2(\kappa_1 - x_2)(x_1 - x_2)$ without changing the sign, to get:

$$\frac{px_1^{p-1} + px_2^{p-1}}{2} - \frac{x_1^p - x_2^p}{x_1 - x_2}.$$

If we let $f(x) = px^{p-1}$, it follows immediately from the Hermite–Hadamard inequality that the above expression is non-negative for $p \in (0, 1)$ and non-positive for $p < 0$.

By the chain rule, taking the p-th root switches the sign of the derivative if $p < 0$ and does not if $p \in (0, 1)$. Thus, we can conclude that the derivative of Equation (9) on $[0, \kappa_1)$ is non-negative for all $p < 0$ and $p \in (0, 1)$.

Let's next show for the case $p = 0$ that Equation (10) is also monotonically increasing on $[0, \kappa_1)$. The proof is very similar to the proof right above for Equation (9). First, we note that taking the logarithm of Equation (10) doesn't change the sign of its derivative, so we just need to show that the derivative of the logarithm is non-negative. This derivative is:

$$\frac{(\kappa_2 - \kappa_1^2)\left(2(\kappa_1 - x_2)(\ln(x_2) - \ln(x_1)) + \frac{(\kappa_2 - x_2^2)(\kappa_2 - 2x_2\kappa_1 + x_2^2)}{x(\kappa_2 - \kappa_1 x_2)}\right)}{(\kappa_2 - 2\kappa_1 x_2 + x_2^2)^2},$$

where again we write x_1 for $\frac{\kappa_2 - x_2\kappa_1}{\kappa_1 - x_2}$ for brevity. Since $\kappa_2 - \kappa_1^2 > 0$ and $\kappa_2 - 2\kappa_1 x_2 + x_2^2 = \kappa_2 - \kappa_1^2 + (\kappa_1 - x_2)^2 > 0$, we can drop those terms without changing the sign of the expression. We simplify further to get

$$2(\kappa_1 - x_2)(\ln(x_2) - \ln(x_1)) + \frac{(\kappa_2 - x_2^2)(\kappa_2 - 2x_2\kappa_1 + x_2^2)}{x(\kappa_2 - \kappa_1 x_2)}$$
$$= 2(\kappa_1 - x_2)(\ln(x_2) - \ln(x_1)) + \left(\frac{1}{x_2} + \frac{\kappa_1 - x_2}{\kappa_2 - x_2\kappa_1}\right)(\kappa_2 - 2x_2\kappa_1 + x_2^2)$$
$$= 2(\kappa_1 - x_2)(\ln(x_2) - \ln(x_1)) + \left(\frac{1}{x_2} + \frac{1}{x_1}\right)(x_1 - x_2)(\kappa_1 - x_2).$$

Since $\kappa_1 > x_2$ and $x_1 > x_2$, we can divide by $2(\kappa_1 - x_2)(x_1 - x_2)$ without changing the sign to get

$$\frac{\frac{1}{x_1} + \frac{1}{x_2}}{2} - \frac{\ln(x_1) - \ln(x_2)}{x_1 - x_2}.$$

If we let $f(x) = 1/x$, it follows immediately from the Hermite–Hadamard inequality that the above expression is non-negative. Thus, we can conclude that Equation (10) on $[0, \kappa_1)$ is non-negative.

It remains to consider the case $x_0 = \kappa_1 + \frac{\kappa_2 - \kappa_1^2}{\kappa_1 - x_2} = \frac{\kappa_2 - x_2\kappa_1}{\kappa_1 - x_2}$. We get analogous equations for q_0, q_1, q_2 as in the earlier case (with q_0 and q_1 switched, naturally), so we get the same objective as in Equations (9) and (10), which we know is minimized for $x_2 = u_{\min}$. Analogously to above, this implies $x_0 = \frac{\kappa_2 - u_{\min}\kappa_1}{\kappa_1 - u_{\min}}$. We know that $x_0 > u_{\min}$ since $\kappa_2 - u_{\min}\kappa_1 - u_{\min}(\kappa_1 - u_{\min}) = \kappa_2 - \kappa_1^2 + (\kappa_1 - u_{\min})^2 > 0$, so since x_1 doesn't affect the objective, we can just set it to be any value in (x_2, x_0). We get

$$
\begin{aligned}
x_0 &= \frac{\kappa_2 - u_{\min}\kappa_1}{\kappa_1 - u_{\min}} & q_0 &= \frac{(\kappa_1 - u_{\min})^2}{\kappa_2 - 2u_{\min}\kappa_1 + u_{\min}^2} \\
x_1 &\in (x_2, x_0) & q_1 &= 0 \\
x_2 &= u_{\min} & q_2 &= \frac{\kappa_2 - \kappa_1^2}{\kappa_2 - 2u_{\min}\kappa_1 + u_{\min}^2}.
\end{aligned}
\tag{12}
$$

We know that the objective is minimized for either the variable assignments from Solution (11) or from Solution (12). We note that in fact both assignments give the same objective value, thus, the global minimum. This value is exactly $\mathrm{UMIN}(\kappa_1, \kappa_2; p, u_{\min})$ in the lemma statement.

Finding the maximum: By Lemma 16, we know that decreasing x_1 will (non-strictly) increase the objective. The lower bounds on x_1 that we have are Constraint (8) and $x_1 > x_2 \geq u_{\min}$. Thus, we know that there is a maximum either for $x_1 = \kappa_1 - \frac{\kappa_2 - \kappa_1^2}{x_0 - \kappa_1}$ (tight to the lower bound given by Constraint (8)) or for $x_2 \geq \kappa_1 - \frac{\kappa_2 - \kappa_1^2}{x_0 - \kappa_1}$ (the only case in which x_1 can't take this value since $x_1 > x_2$). By Constraint (7), we know that $x_2 \leq \kappa_1 - \frac{\kappa_2 - \kappa_1^2}{x_0 - \kappa_1}$, so the second case becomes $x_2 = \kappa_1 - \frac{\kappa_2 - \kappa_1^2}{x_0 - \kappa_1}$.

We first consider the case $x_1 = \kappa_1 - \frac{\kappa_2 - \kappa_1^2}{x_0 - \kappa_1} = \frac{\kappa_1 x_0 - \kappa_2}{x_0 - \kappa_1}$. We get that

$$
\begin{aligned}
q_2 &= \frac{\kappa_2 - \kappa_1\left(x_0 + \frac{\kappa_1 x_0 - \kappa_2}{x_0 - \kappa_1}\right) + x_0 \frac{\kappa_1 x_0 - \kappa_2}{x_0 - \kappa_1}}{(x_2 - x_0)(x_2 - x_1)} \\
&= \frac{\kappa_2 - \left(\frac{\kappa_1 x_0^2 - \kappa_1^2 x_0 + \kappa_1^2 x_0 - \kappa_1 \kappa_2 - \kappa_1 x_0^2 + \kappa_2 x_0}{x_0 - \kappa_1}\right)}{(x_2 - x_0)(x_2 - x_1)} \\
&= \frac{\kappa_2 - \left(\frac{\kappa_2(x_0 - \kappa_1)}{x_0 - \kappa_1}\right)}{(x_2 - x_0)(x_2 - x_1)} = 0,
\end{aligned}
$$

so the value of x_2 won't affect the objective. Furthermore,

$$
\begin{aligned}
q_1 &= \frac{\kappa_2 - \kappa_1(x_0 + x_2) + x_0 x_2}{\left(\frac{\kappa_1 x_0 - \kappa_2}{x_0 - \kappa_1} - x_0\right)\left(\frac{\kappa_1 x_0 - \kappa_2}{x_0 - \kappa_1} - x_2\right)} \\
&= \frac{(\kappa_2 - \kappa_1(x_0 + x_2) + x_0 x_2)(x_0 - \kappa_1)^2}{(\kappa_1 x_0 - \kappa_2 - x_0^2 + \kappa_1 x_0)(\kappa_1 x_0 - \kappa_2 - x_0 x_2 + \kappa_1 x_2)} \\
&= \frac{(x_0 - \kappa_1)^2}{\kappa_2 - 2\kappa_1 x_0 + x_0^2}, \\
q_0 &= 1 - q_1 - q_2 = \frac{\kappa_2 - \kappa_1^2}{\kappa_2 - 2\kappa_1 x_0 + x_0^2},
\end{aligned}
$$

so the objective becomes

$$
\left(\frac{\kappa_2 - \kappa_1^2}{\kappa_2 - 2\kappa_1 x_0 + x_0^2} x_0^p + \frac{(x_0 - \kappa_1)^2}{\kappa_2 - 2\kappa_1 x_0 + x_0^2}\left(\frac{\kappa_1 x_0 - \kappa_2}{x_0 - \kappa_1}\right)^p\right)^{1/p} \quad \text{if } p \neq 0 \tag{13}
$$

$$
(x_0)^{\left(\frac{\kappa_2 - \kappa_1^2}{\kappa_2 - 2\kappa_1 x_0 + x_0^2}\right)} \cdot \left(\frac{\kappa_1 x_0 - \kappa_2}{x_0 - \kappa_1}\right)^{\left(\frac{(x_0 - \kappa_1)^2}{\kappa_2 - 2\kappa_1 x_0 + x_0^2}\right)} \quad \text{if } p = 0 \tag{14}
$$

In both cases, this is a univariate function in x_0. We know that $\kappa_1 x_0 = \sum_{i=0}^{2} q_i x_i x_0 \geq \sum_{i=0}^{2} q_i x_i^2 = \kappa_2$, so we have $x_0 \in [\frac{\kappa_2}{\kappa_1}, 1]$. In fact, note that if $p \leq 0$, the inequality is strict since $x_2 > 0$ and at least 2 of the q_i are positive (else $\kappa_1^2 = \kappa_2$), so we have $x_0 \in (\frac{\kappa_2}{\kappa_1}, 1]$ and Equations (13) and (14) are well-defined for all valid x_0 and p. We'll show that Equations (13) and (14) are monotonically increasing on the interval $(\frac{\kappa_2}{\kappa_1}, 1]$ ($[\frac{\kappa_2}{\kappa_1}, 1]$ for $p \in (0, 1)$). This implies that the maximum is achieved at $x_0 = 1$, which gives $x_1 = \frac{\kappa_1 - \kappa_2}{1 - \kappa_1}$. Since $\kappa_1(1 + u_{\min}) > \kappa_2 + u_{\min}$, by assumption in the lemma statement, it follows that $x_1 > u_{\min}$. We know that x_2 doesn't affect the objective, so we can just set x_2 to be any value in $[u_{\min}, x_1)$. Plugging this in, we get

$$
\begin{aligned}
x_0 &= 1 & q_0 &= \frac{\kappa_2 - \kappa_1^2}{\kappa_2 - 2\kappa_1 + 1} \\
x_1 &= \frac{\kappa_1 - \kappa_2}{1 - \kappa_1} & q_1 &= \frac{(1 - \kappa_1)^2}{\kappa_2 - 2\kappa_1 + 1} \\
x_2 &\in [u_{\min}, x_1] & q_2 &= 0.
\end{aligned} \tag{15}
$$

We now show that for $p \neq 0$, Equation (13) is monotonically increasing on $(\frac{\kappa_2}{\kappa_1}, 1]$ ($[\frac{\kappa_2}{\kappa_1}, 1]$ for $p \in (0,1)$). This proof is almost analogous to the minimization part. First, we note that Equation (13) is identical to Equation (9) with x_2 replaced by x_0. We can do the same algebraic steps, now noting that since $\kappa_1 < \kappa_2/\kappa_1 \leq x_0$ and $x_1 < x_0$, diving by $2(\kappa_1 - x_0)(x_1 - x_0)$ doesn't change the sign. We thus get, analogously to the minimization part, that the sign of the derivative of Equation (13) without the outer p-th root is the same as the sign of

$$\frac{p x_0^{p-1} + p x_1^{p-1}}{2} - \frac{x_0^p - x_1^p}{x_0 - x_1}.$$

Again, by the Hermite–Hadamard inequality with $f(x) = p x^{p-1}$, it follows immediately that the above expression is non-negative for $p \in (0,1)$ and non-positive for $p < 0$. By the chain rule, taking the p-th root switches the sign of the derivative if $p < 0$ and does not if $p \in (0,1)$. Thus, we can conclude that the derivative of Equation (13) on $(\frac{\kappa_2}{\kappa_1}, 1]$ ($[\frac{\kappa_2}{\kappa_1}, 1]$ for $p \in (0,1)$) is non-negative for all $p < 0$ and $p \in (0,1)$.

Let's next show for the case $p = 0$ that Equation (14) is monotonically increasing on $(\frac{\kappa_2}{\kappa_1}, 1]$. Also this proof is almost analogous to the minimization part. We note that Equation (14) is identical to Equation (10) with x_2 replaced by x_0. We can do the same algebraic steps, now noting that since $\kappa_1 < \kappa_2/\kappa_1 \leq x_0$ and $x_1 < x_0$, diving by $2(\kappa_1 - x_0)(x_1 - x_0)$ doesn't change the sign. We thus get, analogously to the minimization part, that the sign of the derivative of Equation (13) without the outer p-th root is the same as the sign of

$$\frac{\frac{1}{x_0} + \frac{1}{x_1}}{2} - \frac{\ln(x_0) - \ln(x_1)}{x_0 - x_1}.$$

If we let $f(x) = 1/x$, it follows immediately from the Hermite–Hadamard inequality that the above expression is non-negative. Thus, we can conclude that Equation (14) on $(\frac{\kappa_2}{\kappa_1}, 1]$ is non-negative.

It remains to consider the case $x_2 = \kappa_1 - \frac{\kappa_2 - \kappa_1^2}{x_0 - \kappa_1} = \frac{\kappa_1 x_0 - \kappa_2}{x_0 - \kappa_1}$. We get analogous equations for q_0, q_1, q_2 as in the earlier case (with q_1 and q_2 switched, naturally), so we get the same objective as in Equations (9) and (10), which we know is maximized for $x_0 = 1$. Analogously to above, this implies $x_2 = \frac{\kappa_1 - \kappa_2}{1 - \kappa_1}$. We know that $x_2 < 1$ since $(1 - \kappa_1) - (\kappa_1 - \kappa_2) = \kappa_2 - \kappa_1^2 + (1 - \kappa_1)^2 > 0$, so since x_1 doesn't affect the objective, we can just set it to be any value in (x_2, x_0). We get

$$
\begin{aligned}
x_0 &= 1 & q_0 &= \frac{\kappa_2 - \kappa_1^2}{\kappa_2 - 2\kappa_1 + 1} \\
x_1 &\in (x_2, x_0) & q_1 &= 0 \\
x_2 &= \frac{\kappa_1 - \kappa_2}{1 - \kappa_1} & q_2 &= \frac{(1 - \kappa_1)^2}{\kappa_2 - 2\kappa_1 + 1}.
\end{aligned}
\tag{16}
$$

We know that the objective is maximized for either the variable assignments from Solution (15) or from Solution (16). We note that in fact both assignments give the same objective value, thus, the global maximum. This value is exactly $\text{UMAX}(\kappa_1, \kappa_2; p, u_{\min})$ in the lemma statement. $\qquad\square$

We now have the tools to prove the main theorem.

Proof of Theorem 9. By Lemma 15, we know that any 2-representative panel $\mathcal{C}$ is ε-accurate with

$$
\begin{aligned}
\varepsilon = \max \quad & \text{UMAX}(\kappa_1, \kappa_2; p, u_{\min}) - \text{UMIN}(\kappa_1, \kappa_2; p, u_{\min}) \\
\text{s.t.} \quad & [\kappa_{i+j}]_{i,j=0}^1, \, [(1 + u_{\min})\kappa_{i+j+1} - \kappa_{i+j+2} - u_{\min}\kappa_{i+j}]_{i,j=0}^0 \succeq 0
\end{aligned}
$$

where $[M_{i,j}]_{i,j=0}^{d-1}$ is a $d \times d$ matrix with entry $M_{i,j}$ at position (i,j) and $M \succeq 0$ means that M is positive-semidefinite. The first matrix $[\kappa_{i+j}]_{i,j=0}^1$ thus is 2×2 and hence semi-definite if and only if $\kappa_0 \geq 0$, which is always true, and $\det([\kappa_{i+j}]_{i,j=0}^1) = \kappa_2 - \kappa_1^2 \geq 0$. The second matrix is 1×1 and thus positive semi-definite if and only if $(1 + u_{\min})\kappa_1 - \kappa_2 - u_{\min} \geq 0$. We get that the conditions are equivalent to $\kappa_2 \geq \kappa_1^2$ and $\kappa_1 \geq \frac{\kappa_2 + u_{\min}}{1 + u_{\min}}$.

We can combine the two inequalities to get $\kappa_1 \geq \frac{\kappa_1^2 + u_{\min}}{1 + u_{\min}}$ which implies $0 \geq (\kappa_1 - 1)(\kappa_1 - u_{\min})$, or equivalently $\kappa_1 \in [u_{\min}, 1]$. This gives us the constraints $\kappa_2 \geq \kappa_1^2 \geq u_{\min}^2$ and $1 \geq \kappa_1 \geq \frac{\kappa_2 + u_{\min}}{1 + u_{\min}}$ in the theorem statement.

We'll next show that if any of the 4 inequalities is not strict, then $\text{UMAX}(\kappa_1, \kappa_2; p, u_{\min}) - \text{UMIN}(\kappa_1, \kappa_2; p, u_{\min}) = 0$ (see Lemma 14 for the definition of UMAX and UMIN).

First, if $\kappa_1^2 = \kappa_2$, we get that $\sqrt{\sum_{i=0}^{2} q_i x_i^2} = \sum_{i=0}^{2} q_i x_i$ which by the arithmetic-quadratic inequality only holds if $x_0 = x_1 = x_2$ (unless $q_i = 0$ for some i, in which case the corresponding x_i can take any value). Thus, $\text{UMAX}(\kappa_1, \kappa_2; p, u_{\min}) = \text{UMIN}(\kappa_1, \kappa_2; p, u_{\min}) = \kappa_1$.

Next, if $\kappa_1 = \frac{\kappa_2 + u_{\min}}{1 + u_{\min}}$, we get that

$$
\begin{aligned}
0 &= \kappa_1(1 + u_{\min}) - \kappa_2 - u_{\min} \\
&= \sum_{i=0}^{2} q_i(x_i + x_i u_{\min} - x_i^2) - u_{\min} \\
&= \sum_{i=0}^{2} q_i(x_i + x_i u_{\min} - x_i^2 - u_{\min}) + \sum_{i=0}^{2} q_i u_{\min} - u_{\min} \\
&= \sum_{i=0}^{2} q_i(1 - x_i)(x_i - u_{\min}).
\end{aligned}
$$

Since $q_i(1 - x_i)(x_i - u_{\min}) \geq 0$ for $x_i \in [u_{\min}, 1]$, it follows that $x_i = u_{\min}$ or $x_i = 1$ for all $i = 0, 1, 2$. To obtain first moment κ_1, the weights (q_i) need to be $\frac{1 - \kappa_1}{1 - u_{\min}}$ for $u_{\min}$ and $\frac{\kappa_1 - u_{\min}}{1 - u_{\min}}$ for 1, which also gives second moment $\kappa_2 = \kappa_1(1 + u_{\min}) - u_{\min}$ as desired. It follows that $\text{UMAX}(\kappa_1, \kappa_2; p, u_{\min}) = \text{UMIN}(\kappa_1, \kappa_2; p, u_{\min}) = \left(\frac{1 - \kappa_1}{1 - u_{\min}} u_{\min}^p + \frac{\kappa_1 - u_{\min}}{1 - u_{\min}} \right)^{1/p}$.

Lastly, both $\kappa_1 = 1$ and $\kappa_1 = u_{\min}$ imply $\kappa_1^2 = \kappa_2$, so the first case applies.

Since there always exists a point at which all constraints are strictly followed, and $\text{UMAX}(\kappa_1, \kappa_2; p, u_{\min}) - \text{UMIN}(\kappa_1, \kappa_2; p, u_{\min}) \geq 0$ at any such point, as follows immediately from their definition, we can restrict the constraints to being strict. This allows us to plug in the equations for UMAX and UMIN from Lemma 18 to obtain the theorem statement. □

A.4 Proof of Theorem 8

The proof of Theorem 8 consists of two parts: showing that every 1-representative panel is ε-accurate for the ε in the statement of the theorem, and then showing that this ε cannot be made smaller. The proof of the first part follows as a special case of our general analysis of m-representative panels in Appendix A.3, in particular it can be found in Appendix A.3.3. Here, we show the second part, i.e., that this bound is tight. We show that for any $\varepsilon' < \varepsilon$ (and $p < 1$, $u_{\min} \in [0, 1)$), for ε as specified in the theorem statement, there exists a 1-representative panel that is not ε'-accurate.

We'll first show Lemma 19, which will be useful for this. Intuitively, it says that if $u_{\min}$ and κ_1 are integers divided by $|\mathcal{F}|$, we can have two panels in which each feature value appears in the same share of individuals, and so that the two panels have the highest and lowest welfare possible for panels with first moment κ_1 and individual utilities in $[u_{\min}, 1]$, as specified by UMAX and UMIN in Lemma 17.

Note that this lemma assumes that either $p < 1$ and $u_{\min} \in [0, 1)$ or $p \in (0, 1)$ and $u_{\min} = 0$, but excludes the case $p \leq 0$ and $u_{\min} = 0$. This is because in this setting the best possible upper bound is the (trivial) upper bound of 1, as stated in Theorem 8, for the same reasons why the lower bound in Theorem 11 holds: A single person with utility 0 makes the population utility 0 regardless of the other individuals' utilities. We thus treat this case separately.

Lemma 19. *Let $p < 1$ and $u_{\min} \in [0, 1)$, or $p \in (0, 1)$ and $u_{\min} = 0$. Let ℓ be an integer, and $\kappa_1, u_{\min} \in \mathbb{Q}$ such that $1 > \kappa_1 > u_{\min}$ and $\ell u_{\min}, \ell \kappa_1 \in \mathbb{Z}$. Then, there exist two panels $\mathcal{C} = \{(r_1, y_1), \ldots, (r_T, y_T)\}, \mathcal{C}' = \{(r_1', y_1), \ldots, (r_T', y_T)\}$ with individuals of types in $\mathcal{I} = \{y_1, \ldots, y_T\}$ with $|\mathcal{F}| = \ell$ features and an alternative a s.t. $u_i(a) \in [u_{\min}, 1]$ for all $y_i \in \mathcal{I}$, so*

that each feature value appears in the same share in $\mathcal{C}$ and $\mathcal{C}'$, i.e. $\sum_{y_i \in \mathcal{I}} r_i \mathbf{y_i} = \sum_{y_i \in \mathcal{I}} r'_i \mathbf{y_i}$ and

$$u_{\mathcal{C},p}(a) = \mathrm{UMIN}(\kappa_1; p, u_{\min}) = \begin{cases} \left(\frac{\kappa_1 - u_{\min}}{1 - u_{\min}} + \frac{1 - \kappa_1}{1 - u_{\min}} u_{\min}{}^p \right)^{1/p} & \text{if } p \neq 0 \\ (u_{\min})^{\frac{1 - \kappa_1}{1 - u_{\min}}} & \text{if } p = 0 \end{cases},$$

and $u_{\mathcal{C}',p}(a) = \mathrm{UMAX}(\kappa_1; p, u_{\min}) = \kappa_1$.

Proof. Let $\mathcal{F} = \{f_1, ..., f_\ell\}$ and $V_f = \{0, 1\}$ for all $f \in \mathcal{F}$. We let y_1 be the type that has value 1 for all features, i.e. $y_1(f, i) = i$ for all $f \in \mathcal{F}, i = 0, 1$. We let

$$Y_{\kappa_1} = \{y : |\{f \in \mathcal{F} : y(f, 1) = 1\}| = \ell \kappa_1\} \quad Y_{u_{\min}} = \{y : |\{f \in \mathcal{F} : y(f, 1) = 1\}| = \ell u_{\min}\}$$

be the set of all possible types that have value 1 for exactly $\ell \kappa_1$ and $\ell u_{\min}$ of their features, respectively. We let $\mathcal{I} = \{y_1\} \cup Y_{\kappa_1} \cup Y_{u_{\min}}$.

Let's now define the panels. We let $\mathcal{C}$ be the panel consisting of $(\ell \kappa_1 - \ell u_{\min}) \binom{\ell}{\ell u_{\min}} \in \mathbb{Z}$ individuals of type y_1 and of $(\ell - \ell \kappa_1) \in \mathbb{Z}$ individuals of each of the $\binom{\ell}{\ell u_{\min}}$ types in $Y_{u_{\min}}$. We let $\mathcal{C}'$ be the panel consisting of 1 individual for each type in Y_{κ_1}.

We need to confirm that for every feature $f \in \mathcal{F}$, each value $v \in V_f$ appears in the same share of $\mathcal{C}$ and $\mathcal{C}'$. Since there are only two values, 0 and 1, for each feature, it suffices to show that value 1 appears in the same share of individuals in $\mathcal{C}$ and $\mathcal{C}'$ for any feature.

Let's fix a feature $f \in \mathcal{F}$. First, we can see that a fraction of $\binom{\ell-1}{\ell \kappa_1 - 1} / \binom{\ell}{\ell \kappa_1} = \kappa_1$ of the individuals in $\mathcal{C}'$ have value 1 for feature f. Also for $\mathcal{C}$ we get that a fraction of

$$\frac{(\ell \kappa_1 - \ell u_{\min}) \binom{\ell}{\ell u_{\min}} + (\ell - \ell \kappa_1) \binom{\ell-1}{\ell u_{\min} - 1}}{(\ell \kappa_1 - \ell u_{\min}) \binom{\ell}{\ell u_{\min}} + (\ell - \ell \kappa_1) \binom{\ell}{\ell u_{\min}}} = \frac{(\kappa_1 - u_{\min}) + (1 - \kappa_1) u_{\min}}{(\kappa_1 - u_{\min}) + (1 - \kappa_1)} = \kappa_1$$

of the individuals in $\mathcal{C}'$ have value 1 for feature f.

We let a be an alternative such that $a(f, i) = i/\ell$ for all $f \in \mathcal{F}$ and $i = 0, 1$; i.e., an individual's utility is the number of features for which they have value 1, divided by the total number of features. Thus, all individuals in $\mathcal{C}'$ get utility κ_1 from a, so $u_{\mathcal{C}',p}(a) = \kappa_1$. Furthermore, the individuals of type y_1 get utility 1 and the individuals of a type in $Y_{u_{\min}}$ get utility $u_{\min}$. Note that thus all individuals $y \in \mathcal{I}$ get a utility in $[u_{\min}, 1]$ from a, as needed. The share of individuals with utility 1 in $\mathcal{C}$ is

$$\frac{(\ell \kappa_1 - \ell u_{\min}) \binom{\ell}{\ell u_{\min}}}{(\ell \kappa_1 - \ell u_{\min}) \binom{\ell}{\ell u_{\min}} + (\ell - \ell \kappa_1) \binom{\ell}{\ell u_{\min}}} = \frac{\kappa_1 - u_{\min}}{1 - u_{\min}}$$

and the share of with utility $u_{\min}$ in $\mathcal{C}$ is $1 - \frac{\kappa_1 - u_{\min}}{1 - u_{\min}} = \frac{1 - \kappa_1}{1 - u_{\min}}$. Applying the definition of panel welfare, we find that $u_{\mathcal{C},p}(a)$ is as specified in the lemma. □

Proof of Theorem 8, lower bound. Given any $u_{\min} \in [0, 1)$ and $p < 1$, we'll show that we can construct a population $\mathcal{P}$, an alternative a so that the utility of every individual in the population for a is in $[u_{\min}, 1]$, and a 1-representative panel $\mathcal{C}$ in $\mathcal{P}$, such that $|u_{\mathcal{C},p}(a) - u_{\mathcal{P},p}(a)|$ gets as close to ε as desired (in particular, greater than any fixed $\varepsilon' < \varepsilon$) if the size of the population and the number of features is sufficiently large. This implies that there are m-representative committees $\mathcal{C}$ that are not ε'-accurate for any $\varepsilon' < \varepsilon$. To make it clear that ε depends on p and $u_{\min}$, we'll write

$$\varepsilon(p, u_{\min}) = \begin{cases} \frac{1-p}{p} \left(p \cdot \frac{1 - u_{\min}}{1 - u_{\min}{}^p} \right)^{\frac{1}{1-p}} + \frac{u_{\min} - u_{\min}{}^p}{1 - u_{\min}{}^p} & \text{for } p > 0 \vee (p < 0 \wedge u_{\min} > 0) \\ 1 + \frac{1 - u_{\min}}{\ln (1/u_{\min})} \left(\ln \left(\frac{1 - u_{\min}}{\ln (1/u_{\min})} \right) - 1 \right) & \text{for } p = 0 \wedge u_{\min} > 0 \\ 1 & \text{for } p \leq 0 \wedge u_{\min} = 0 \end{cases}$$

for ε as defined in the theorem statement.

Let's first consider the case that $p \leq 0$, $u_{\min} = 0$, where we want to show that for any $\varepsilon' < 1 = \varepsilon(p, u_{\min})$ there exists a 1-representative panel that is not ε'-accurate. By Theorem 11, we know that

for any $|\mathcal{F}|$, there exists a 1-representative panel that is not ε-accurate for any $\varepsilon < (|\mathcal{F}| - 1)/|\mathcal{F}|$. If we set $|\mathcal{F}|$ large enough in terms of ε', for example $|\mathcal{F}| > 1 + \lceil 1/(1 - \varepsilon') \rceil$, we get that

$$\frac{|\mathcal{F}| - 1}{|\mathcal{F}|} = \frac{\lceil 1/(1 - \varepsilon') \rceil}{1 + \lceil 1/(1 - \varepsilon') \rceil} > \varepsilon',$$

so there exists a 1-representative panel that is not ε'-accurate.

We now turn to the case $p < 1$ and $u_{\min} \in (0, 1)$ or $p \in (0, 1)$ and $u_{\min} = 0$, noting that these cases cover all of "$p > 0$", "$p < 0 \wedge u_{\min} > 0$," and "$p = 0 \wedge u_{\min} > 0$," as needed. We let

$$\kappa_1 = \begin{cases} \frac{1 - u_{\min}}{1 - u_{\min}{}^p} \left(p \cdot \frac{1 - u_{\min}}{1 - u_{\min}{}^p} \right)^{\frac{p}{1-p}} + \frac{u_{\min} - u_{\min}{}^p}{1 - u_{\min}{}^p} & \text{if } p \neq 0 \\ 1 - (1 - u_{\min}) \frac{\ln\left(\frac{1}{1 - u_{\min}} \right) + \ln\ln\left(\frac{1}{u_{\min}} \right)}{\ln\left(\frac{1}{u_{\min}} \right)} & \text{if } p = 0 \end{cases},$$

to be the value in $(u_{\min}, 1)$ maximizing the difference between $\textsc{Umin}(\kappa_1; p, u_{\min})$ and $\textsc{Umax}(\kappa_1; p, u_{\min})$, as found in Equations (3) and (4), so that

$$\textsc{Umax}(\kappa_1; p, u_{\min}) - \textsc{Umin}(\kappa_1; p, u_{\min}) = \varepsilon(p, u_{\min})$$

as shown in the proof of the first part of this theorem.

We now consider a fixed $|\mathcal{F}| = \ell$, and let $\hat{u}_{\min} = \lceil \ell u_{\min} \rceil / \ell$ and $\hat{\kappa}_1 = \lceil \ell \kappa_1 \rceil / \ell$ be the smallest rational numbers with ℓ as denominator that are greater than $u_{\min}$ and κ_1, respectively. We assume ℓ is large enough so that $1 > \hat{\kappa}_1 > \hat{u}_{\min}$. Then, Lemma 19 tells us that for any $p < 1, u_{\min} \in (0, 1)$ or $p \in (0, 1), u_{\min} = 0$, there exist two panels $\mathcal{C} = \{(r_1, y_1), \ldots, (r_T, y_T)\}$, $\mathcal{C}' = \{(r'_1, y_1), \ldots, (r'_T, y_T)\}$ with ℓ features, so that each feature value appears in the same frequency in $\mathcal{C}$ and $\mathcal{C}'$, and that there exists an alternative a that has the maximum and minimum social welfare possible for a panel with first moment $\hat{\kappa}_1$ and minimum population utility $\hat{u}_{\min}$, i.e., $\textsc{Umax}(\hat{\kappa}_1; p, \hat{u}_{\min})$ and $\textsc{Umin}(\hat{\kappa}_1; p, \hat{u}_{\min})$, respectively.

Now, note that as $\ell \to \infty$, we get that $\hat{\kappa}_1 \to \kappa_1$ and $\hat{u}_{\min} \to u_{\min}$. Since $\textsc{Umax}(\kappa_1; p, u_{\min})$ and $\textsc{Umin}(\kappa_1; p, u_{\min})$ are continuous for $p < 1, u_{\min} \in (0, 1)$ or $p \in (0, 1), u_{\min} = 0$, and $\kappa_1 \in [u_{\min}, 1]$, we get that as $\ell \to \infty$,

$$\textsc{Umax}(\hat{\kappa}_1; p, \hat{u}_{\min}) \to \textsc{Umax}(\kappa_1; p, u_{\min}), \qquad \textsc{Umin}(\hat{\kappa}_1; p, \hat{u}_{\min}) \to \textsc{Umin}(\kappa_1; p, u_{\min}),$$

and therefore also

$$|u_{\mathcal{C}', p}(a) - u_{\mathcal{C}, p}(a)| = \textsc{Umax}(\hat{\kappa}_1; p, \hat{u}_{\min}) - \textsc{Umin}(\hat{\kappa}_1; p, \hat{u}_{\min}) \to \varepsilon(p, u_{\min}).$$

We now construct a population $\mathcal{P}$ that consists of the individuals in $\mathcal{C}$ once and n' times each individual in $\mathcal{C}'$. Since each feature value appears in the same share of $\mathcal{C}$ and $\mathcal{C}'$, it also appears in the same share of $\mathcal{P}$, so $\mathcal{C}$ and $\mathcal{C}'$ are 1-representative panels in $\mathcal{P}$. If we let $n' \to \infty$, we see that

$$u_{\mathcal{P}, p}(a) = \sum_{y_i \in \mathcal{I}} \frac{r_i + n' r'_i}{n' + 1} (u_i(a))^p \to \sum_{y_i \in \mathcal{I}} r'_i (u_i(a))^p = u_{\mathcal{C}', p}(a),$$

so $|u_{\mathcal{P}, p}(a) - u_{\mathcal{C}, p}(a)| \to |u_{\mathcal{C}', p}(a) - u_{\mathcal{C}, p}(a)|$ as $n' \to \infty$.

Putting it all together, we get that as $\ell, n' \to \infty$, $|u_{\mathcal{P}, p}(a) - u_{\mathcal{C}, p}(a)| \to \varepsilon(p, u_{\min})$ for the 1-representative panel $\mathcal{C}$ in $\mathcal{P}$. In particular, this implies that for any $\delta > 0$, for all ℓ and n' large enough, $|\varepsilon(p, u_{\min}) - |u_{\mathcal{P}, p}(a) - u_{\mathcal{C}, p}(a)|| < \delta$. If we set $\delta = \varepsilon(p, u_{\min}) - \varepsilon' > 0$ (and notice that always $|u_{\mathcal{P}, p}(a) - u_{\mathcal{C}, p}(a)| \leq \varepsilon(p, u_{\min})$ as shown in the first, upper bound, part of the proof), it follows that for any $\varepsilon' < \varepsilon(p, u_{\min})$, for all ℓ and n' large enough, $|u_{\mathcal{P}, p}(a) - u_{\mathcal{C}, p}(a)| \in (\varepsilon', \varepsilon(p, u_{\min})]$, and in particular, $|u_{\mathcal{P}, p}(a) - u_{\mathcal{C}, p}(a)| > \varepsilon'$. Thus, $\mathcal{C}$ is a 1-representative panel that is not ε' accurate. □

A.5 Proof of Theorem 11

To keep the math in the proof concise, we use the (well-known) extension of binomial coefficients to allow for negative integers in the top part. We then apply the Chu-Vandermonde identity for generalized binomial coefficients multiple times throughout the proof.

Definition 20 ([15, Definition 3.1.]). Let $k \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}$. We define the generalized binomial coefficient as $\binom{n}{k} = \frac{n(n-1)\ldots(n-k+1)}{k!}$.

There is a nice identity for binomial coefficients with a negative top part, which will be useful for the proof.

Lemma 21 ([15, Proposition 3.16.]). *Let $k \in \mathbb{Z}_{\geq 0}$ and $n \in \mathbb{Z}$. Then*

$$\binom{n}{k} = (-1)^k \binom{k-n-1}{k}.$$

A key part of our proof is the so called Chu-Vandermon identity, first stated by mathematician Chu Shih-chieh already in 1303.

Theorem 22 ([15, Theorem 3.29.]). *Let $k \in \mathbb{Z}_{\geq 0}$, $n, m \in \mathbb{Z}$. Then*

$$\binom{n+m}{k} = \sum_{i=0}^{k} \binom{n}{i}\binom{m}{k-i}.$$

We refer to [15] for contemporary proofs of Lemma 21 and Theorem 22.

Proof of Theorem 11. For this proof, we'll construct two panels, $\mathcal{C}$ and $\mathcal{C}'$, of equal size that are m-representative with respect to each other, i.e., for every $\bar{m} \leq m$, every $\bar{m}$-tuple of feature values appears in the same number of individuals in $\mathcal{C}$ and $\mathcal{C}'$. We'll construct an alternative a that gives one person in $\mathcal{C}$ utility 0, while every person in panel $\mathcal{C}'$ has utility at least $(|\mathcal{F}| - m)/|\mathcal{F}|$. We then let $\mathcal{P}$ be the population consisting of the individuals in $\mathcal{C}$ and $\mathcal{C}'$, so $\mathcal{C}'$ is a m-representative panel in $\mathcal{P}$ with welfare at least $(|\mathcal{F}| - m)/|\mathcal{F}|$ for a, while to overall population has welfare 0.

First, note that if $|\mathcal{F}| \leq m$, the bound in the theorem follows trivially. Thus, let $\mathcal{F} = \{f_1, \ldots, f_\ell\}$ for $\ell = |\mathcal{F}| > m$. Let $V_f = \{0, 1\}$ for all $f \in \mathcal{F}$. Let y_ℓ be the type that has value 0 for all features, i.e. $y_\ell(f, i) = 1 - i$ for all $f \in \mathcal{F}, i = 0, 1$. For $j \in \{0, \ldots, m\}$, we let $Y_j = \{y : |f \in \mathcal{F} : y(f, 0) = 1| = j\}$ be the set of all types that have value 0 for exactly j features.

We define $\mathcal{C}$ to be the panel consisting of one individual of type y_ℓ and $\binom{\ell-j-1}{m-j}$ individuals of each type $y \in Y_j$, for $j \in \{0, \ldots, m\}, j \not\equiv m \mod 2$. We define $\mathcal{C}'$ to be the panel consisting of $\binom{\ell-j-1}{m-j}$ individuals of each type $y \in Y_j$, for $j \in \{0, \ldots, m\}, j \equiv m \mod 2$. We'll first prove that the number of individuals in $\mathcal{C}$, k, and in $\mathcal{C}'$, denoted k', is indeed the same. Let's first calculate k and k':

$$k = 1 + \sum_{\substack{j \in \{0,\ldots,m\} \\ j \not\equiv m \mod 2}} \binom{\ell-j-1}{m-j}|Y_j| = 1 - \sum_{\substack{j \in \{0,\ldots,m\} \\ j \not\equiv m \mod 2}} \binom{m-\ell}{m-j}\binom{\ell}{j},$$

$$k' = \sum_{\substack{j \in \{0,\ldots,m\} \\ j \equiv m \mod 2}} \binom{\ell-j-1}{m-j}|Y_j| = \sum_{\substack{j \in \{0,\ldots,m\} \\ j \equiv m \mod 2}} \binom{m-\ell}{m-j}\binom{\ell}{j},$$

where the two right equalities follow from Lemma 21 and the fact that $|Y_j| = \binom{\ell}{j}$. We now note that by Theorem 22,

$$\sum_{j=0}^{m} \binom{m-\ell}{m-j}\binom{\ell}{j} = \binom{m}{m} = 1,$$

which immediately implies $k = k'$.

Next, we'll argue that $\mathcal{C}$ and $\mathcal{C}'$ are m-representative in respect to each other, i.e., any $\bar{m}$-tuple for feature values for $\bar{m} \in [m]$ appears in $\mathcal{C}$ with the same frequency as $\mathcal{C}'$. Since $k = k'$, this means it appears in the same number of individuals in $\mathcal{C}$ as in $\mathcal{C}'$.

Let $F \subset \mathcal{F}, |F| = \bar{m}$, be any set of $\bar{m} \leq m$ features in $\mathcal{F}$ and let $\mathcal{V} = (v_f)_{f \in F} \in \bigtimes_{f \in F} V_f$ be any assignment of values to these features. Let $k_\mathcal{V}$ and $k'_\mathcal{V}$ be the number of individuals in $\mathcal{C}$ and $\mathcal{C}'$, respectively, with feature values $\mathcal{V}$. We thus want to show that $k_\mathcal{V} = k'_\mathcal{V}$ for all $\mathcal{V}$.

For any $\mathcal{V}$, let $i = |v_f \in \mathcal{V} : v_f = 0|$ be the number of zeros in this assignment. For $j = \{0, \ldots, m\}$, the number of types in Y_j that have feature values $\mathcal{V}$ is $\binom{\ell-\bar{m}}{j-i}$ if $j \geq i$, since there are $\ell - \bar{m}$ feature values not defined by $\mathcal{V}$, out of which $j - i$ are zeros. If $j < i$, we know that no type in Y_j has feature

values $\mathcal{V}$, since types in Y_j have value 0 for less features than required by $\mathcal{V}$. Type y_0 has feature values $\mathcal{V}$ if and only if $i = \bar{m}$ since it is all zeros..

We get that

$$k_{\mathcal{V}} = \mathbb{1}[i = \bar{m}] + \sum_{\substack{j \in \{i,\ldots,m\} \\ j \not\equiv m \mod 2}} \binom{\ell - j - 1}{m - j}\binom{\ell - \bar{m}}{j - i} = \mathbb{1}[i = \bar{m}] - \sum_{\substack{j \in \{i,\ldots,m\} \\ j \not\equiv m \mod 2}} \binom{m - \ell}{m - j}\binom{\ell - \bar{m}}{j - i},$$

$$k'_{\mathcal{V}} = \sum_{\substack{j \in \{i,\ldots,m\} \\ j \equiv m \mod 2}} \binom{\ell - j - 1}{m - j}\binom{\ell - \bar{m}}{j - i} = \sum_{\substack{j \in \{i,\ldots,m\} \\ j \equiv m \mod 2}} \binom{m - \ell}{m - j}\binom{\ell - \bar{m}}{j - i},$$

where again the two right equalities follow from Lemma 21. We now note that

$$\sum_{j=i}^{m} \binom{m - \ell}{m - j}\binom{\ell - \bar{m}}{j - i} = \sum_{j=0}^{m-i} \binom{m - \ell}{m - i - j}\binom{\ell - \bar{m}}{j} = \binom{m - \bar{m}}{m - i},$$

where the last equality follows from Theorem 22. Now, note that since $i \leq \bar{m}$, the bottom part of $\binom{m - \bar{m}}{m - i}$ is at least its top part, so by Definition 20, we get

$$\binom{m - \bar{m}}{m - i} = \begin{cases} 0 & \text{if } i < \bar{m} \\ 1 & \text{if } i = \bar{m} \end{cases} = \mathbb{1}[i = \bar{m}].$$

This immediately implies $k_{\mathcal{V}} = k'_{\mathcal{V}}$.

We let $\mathcal{P}$ be the population consisting of all individuals in $\mathcal{C}$ and $\mathcal{C}'$. Since for any $\bar{m} \leq m$, any $\bar{m}$-tuple of feature values appears in the same share of individuals in $\mathcal{C}$ and in $\mathcal{C}'$, we know they also appear in the same share in $\mathcal{P}$. Thus, $\mathcal{C}$ and $\mathcal{C}'$ are m-representative panels in $\mathcal{P}$.

We now define alternative a such that $a(f, i) = i/\ell$ for all $f \in \mathcal{F}$, $i = 0, 1$. That is, an individual gets benefit $1/\ell$ for each feature value that is 1. Thus, individual y_ℓ has utility 0, and any individual $y \in Y_j$ for $j \in \{0, \ldots, m\}$ has utility $(\ell - j)/\ell$. In particular, note that the utility from a for any individual in $\mathcal{P}$ is in $[0, 1]$, as required.

The p-mean, for $p \leq 0$, of a list of numbers including 0 is 0, so the welfare of $\mathcal{P}$ is $u_{\mathcal{P},p}(a) = 0$. However, all individuals in $\mathcal{C}'$ have at least utility $(\ell - m)/\ell$. Since for any p, the p-mean of a list of numbers is at least as big as the smallest number in the list, $u_{\mathcal{C}',p}(a) \geq \frac{\ell - m}{\ell}$. We conclude that for this alternative a and m-representative panel $\mathcal{C}'$ in $\mathcal{P}$,

$$|u_{\mathcal{C}',p}(a) - u_{\mathcal{P},p}(a)| \geq \frac{\ell - m}{\ell}.$$

The theorem statement follows. □

A.6 Additional details on empirical experiments

A.6.1 European nation-wide panel

For this panel, 30,000 letters were mailed to invite individuals to participate. 1,727 individuals volunteered to be in the candidate pool; the desired panel size was 110. Seven features were being considered when selecting the panel, which we list in Table 2.

We obtained the underlying population data from the 2016 European Social Survey [21]. The dataset contains 1,959 respondents with post-stratification weights to account for sampling and participation bias. We dropped 44 people from the dataset: 40 because they had missing data and 4 because they were younger than 16, the minimum age set for the assembly. Furthermore, since the European Social Survey only has the fields 'Male' and 'Female' for gender, we removed the 12 individuals with gender 'Other' from the pool of candidates, reducing the size to 1,715.

One of the seven features considered was "Concern about climate change". Only 4 individuals in the pool of candidates had the value "Not at all concerned", making it impossible to even form a 1-representative committee of the desired size with this candidate pool. We thus excluded this feature from our analysis.

Table 2: The features considered in selecting panel EUR1

Features	Values
Gender	Male, Female, ~~Other~~
Age	16–29, 30–44, 45–59, 60+
Place of residence within the country	12 distinct regions
Educational attainment	3 levels
~~Concern about climate change~~	~~Very concerned, Fairly concerned, Not very concerned, Not at all concerned, Other~~
Ethnicity	White, Non-white
Residential density	Urban, Rural

We note that due to its limited sample size of 1,959, the European Social Survey doesn't contain a person for all possible tuples of m feature values, for $m \geq 3$. However, this is not concerning for our positive results on the existence of m-representative panels, since these are tuples of feature values that appear in a sufficiently small share of the population so that an m-representative panel would satisfy the corresponding quota by containing 0 or 1 individuals with that tuple of feature values. In our experiments, for tuples of feature values that didn't appear in the European Social Survey, we enforced that 0 people on the panel have this tuple of feature values, tightening the constraints on the existence of m-representative panels.

A.6.2 Australian state-wide panels 1 & 2

These two panels were being conducted in series with overlap in the pool of candidates and individuals on the panels. Initially, 25,000 letters were mailed to invite individuals to participate, with additional letters being sent to groups with expected low participation rates. 1,070 individuals volunteered to be in the first candidate pool, from which a panel of 50 candidates was selected. For the second panel, additional individuals were recruited for the candidate pool, then including 1,145 individuals (some of which were on the first panel). A panel of 328 individuals was selected. Four features were being considered when selecting the panels, which we list in Table 3.

Table 3: The features considered in selecting panels AUS1 and AUS2

Features	Values
Sex	Male, Female
Age	18–24, 25–34, 35–44, 45–54, 55–64, 65+
Residential density	Metropolitan, Regional
~~Residential status~~	~~Owner/Occupier, Tenant~~

We obtained the underlying population data from the Australian Bureau of Statistics' 2021 Census data [1]. Up to small deviations due to differential privacy noise being applied to the dataset, we obtained the exact population share of each feature value tuple including "Sex", "Age", and "Residential Density". However, the publicly available data from the Australian Census separates between individuals and dwellings, so that no data on the intersection of the "Residential status" feature with the features "Age" and "Sex" is available. We therefore dropped the residential status feature in our analysis.

A.6.3 Australian state-wide panel 3

For this panel, individuals could self-register interest in participating. 3,500 individuals did so, and were emailed an invitation to participate. 518 of those individuals ended up volunteering to be in the candidate pool; the desired panel size was 45. Four features were being considered when selecting the panel, which we list in Table 4.

We obtained the underlying population data from the Australian Bureau of Statistics' 2021 Census data [1]. Since the Australian Census only has the fields 'Male' and 'Female' for sex, we removed

Table 4: The features considered in selecting panel AUS3

Features	Values
Gender	Male, Female, ~~Non-binary~~, ~~I would prefer not to say~~
Age	18–24, 25–34, 35–44, 45–54, 55–64, 65–74, 75+
Place of residence within the state	3 distinct regions
Ability to attend all panel sessions	Yes, No, Unsure, Unsure-1

the 9 individuals with gender 'Non-binary' or 'I would prefer not to say' from the pool of candidates, reducing the size to 509. Additionally, the feature 'Ability to attend all panel sessions' cannot be obtained for the underlying population, by itself or in intersection with other features. Thus, we only consider this feature individually, ensuring that at most $1/45$ of the individuals on the panel have each of the values 'No', 'Unsure', or 'Unsure-1', as was the requirement for the actually selected panel. For each feature value tuple including "Gender" (without "Other"), "Age", and "Place of residence", we obtained the exact population share, up to small deviations due to differential privacy noise being applied to the dataset.