Heat Transfer

Analytical solutions for heat conduction

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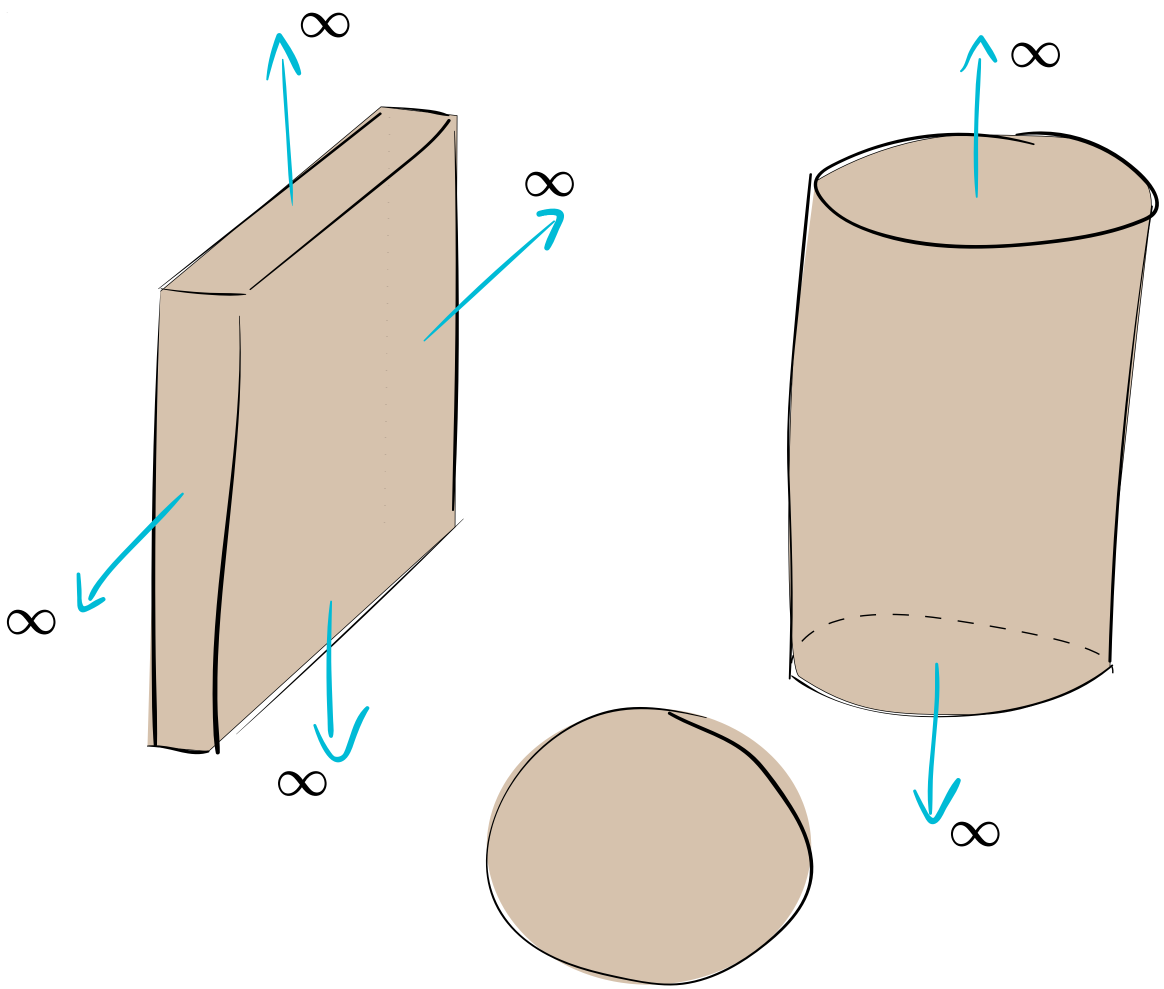
Why/why not analytical solutions?

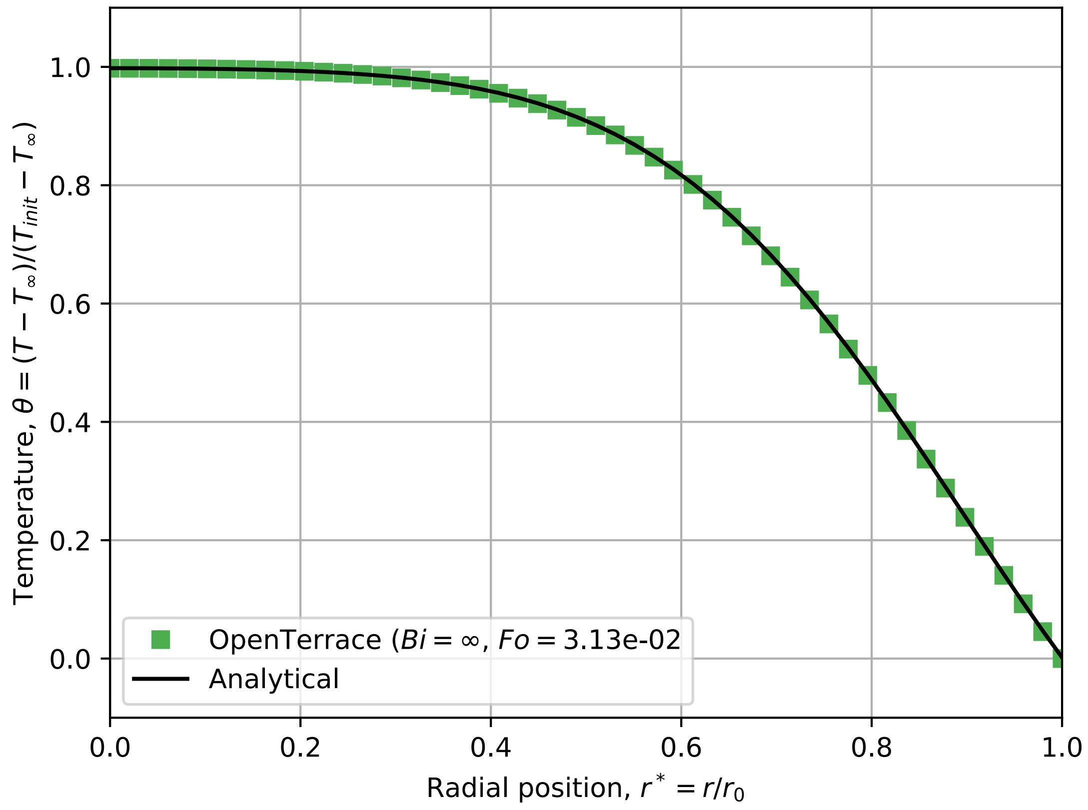
- Limitations
  - Limited to **simplified geometries:**
    - Infinitely long plane wall
    - Infinitely long cylinder
    - Perfect sphere
  - Limited to **simple conditions with thermal symmetry:**
    - Uniform and temperature-independent properties
    - Only convection on boundary with constant heat transfer coefficient, e.g. no heat generation inside, radiation on boundary etc
    - Uniform initial temperature
- Reasons to apply
  - **Fast to compute**, e.g. doing millions of evaluations
  - **Yield exact results**, e.g. excellent for verifying numerical codes



Verification studies

The OpenTerrace code is tested against known analytical solution in limiting cases. The following presents such code verification results. These are run automatically as part of the automated tests using GitHub actions.

Pure diffusion



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Pure advection

Starting point of analytical solutions

- Partial differential equation:

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$$

- Boundary conditions:

At $x = 0$:

$$\frac{\partial T(0,t)}{\partial x} = 0$$

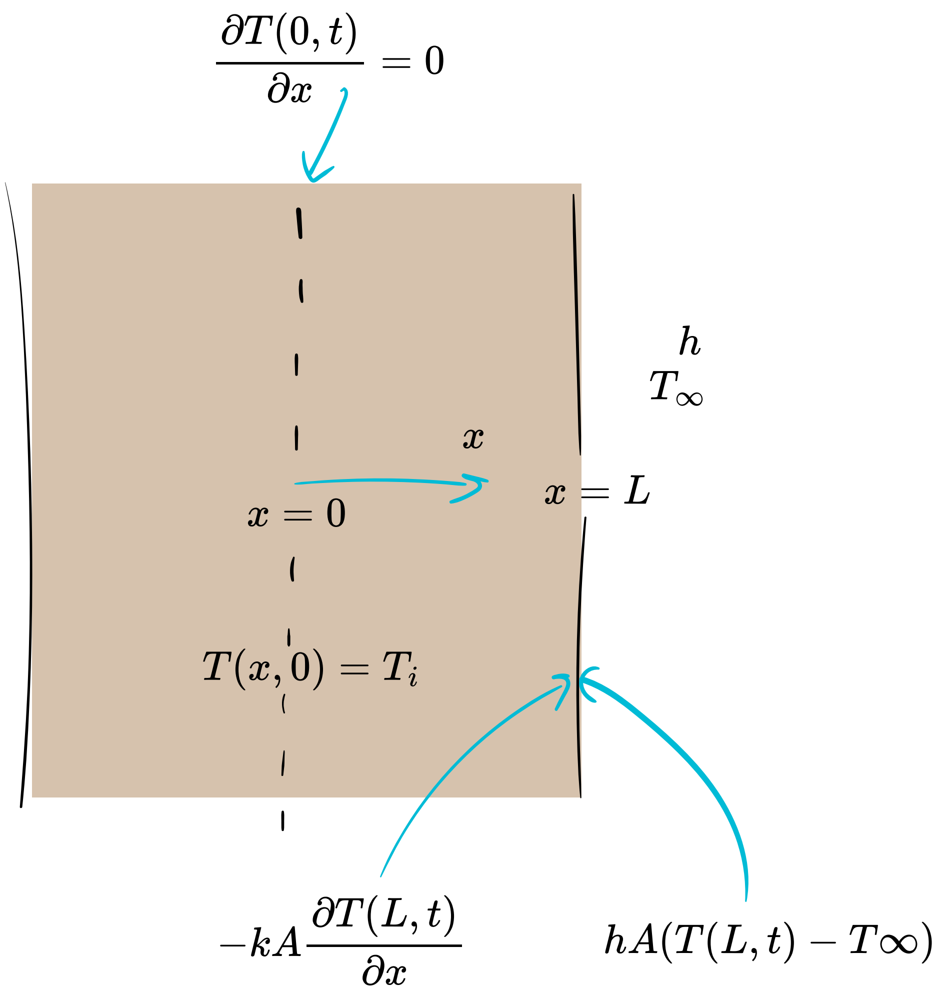
At $x = L$:

$$-kA\frac{\partial T(L,t)}{\partial x} = hA\left(T(L,t) - T_\infty\right)$$

- Initial condition:

At $t = 0$:

$$T(x,0) = T_i$$



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Introduction of non-dimensional variables

- Convert problem into non-dimensional form to reduce number of variables
  - Non-dimensional temperature:

$$\theta(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty}$$

  - Non-dimensional position:

$$X = \frac{x}{L}$$

  - Non-dimensional heat transfer coefficient (Biot number):

$$\mathrm{Bi} = \frac{hL}{k}$$

  - Non-dimensional time (Fourier number):

$$\tau = \frac{\alpha t}{L^2}$$

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Re-casting problem in non-dimensional form

- Starting from original partial differential equation:

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial T}$$

- Inserting definitions of non-dimensional numbers:
    - Inserting $X = x/L$:

$$\frac{\partial^2 T(X,t)}{\partial (X \cdot L)^2} = \frac{1}{\alpha}\frac{\partial T(X,t)}{\partial t}$$

$$\Leftrightarrow$$

$$\frac{\partial^2 T(X,t)}{\partial X^2} = \frac{L^2}{\alpha}\frac{\partial T(X,t)}{\partial t}$$

    - Inserting Fourier number $\tau = \alpha t/L^2$:

$$\frac{\partial^2 T(X,\tau)}{\partial (X \cdot L)^2} = \frac{\partial T(X,\tau)}{\partial \tau}$$

    - Inserting $\theta = \frac{T-T_\infty}{T_i-T_\infty}$:

$$\frac{\partial^2 \theta(X,\tau)}{\partial (X \cdot L)^2} = \frac{\partial \theta(X,\tau)}{\partial \tau}$$

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Final formulation in non-dimensional form

- Partial differential equation:

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial T}$$

$$\Rightarrow$$

$$\frac{\partial^2 \theta(X,\tau)}{\partial X^2} = \frac{\partial T(X,\tau)}{\partial \tau}$$

- Boundary conditions:

At $x = 0$:

$$\frac{\partial T(0,t)}{\partial x} = 0$$

$$\Rightarrow$$

$$\frac{\partial \theta(0,\tau)}{\partial X} = 0$$

At $x = L$:

$$-k\frac{\partial T(L,t)}{\partial x} = h(T(L,t) - T_\infty)$$

A look at the solution for a plane wall

- Analytical solution:

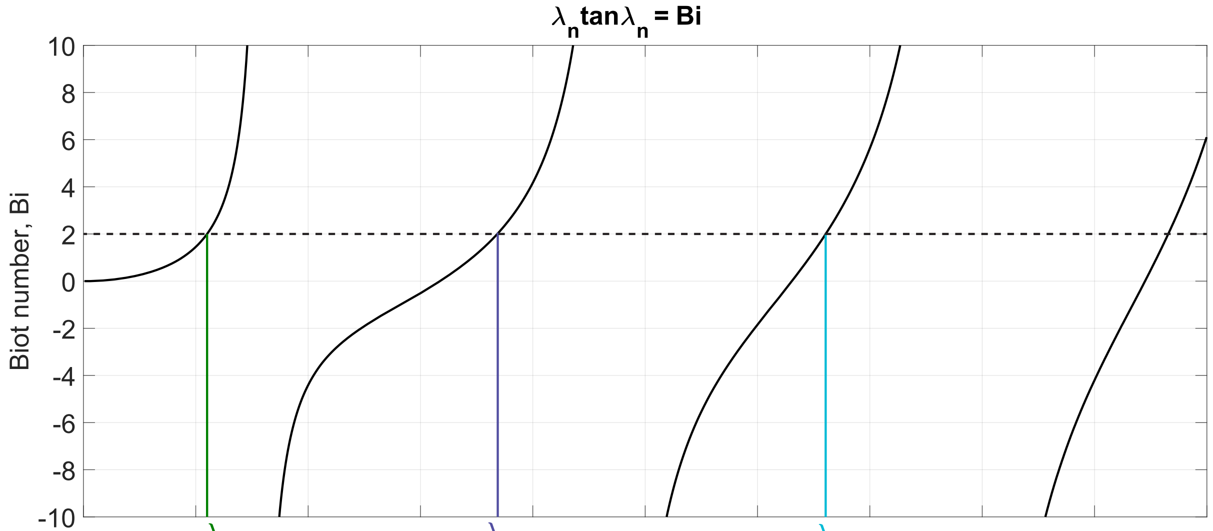
$$\theta = \sum_{n=1}^{\infty} \frac{4\sin\lambda_n}{2\lambda_n + \sin(2\lambda_n)} e^{-\lambda_n^2 \tau} \cos\left(\frac{\lambda_n x}{L}\right)$$

where λ_n are the roots of: $\lambda_n \tan\lambda_n = \mathrm{Bi}$

- Complicated because of:
  - The infinite sum ($n = 1, n = 2, n = 3$ ...)

$$\theta = \frac{4\sin\lambda_1}{2\lambda_1 + \sin(2\lambda_1)} e^{-\lambda_1^2 \tau} \cos\left(\frac{\lambda_1 x}{L}\right) + \frac{4\sin\lambda_2}{2\lambda_2 + \sin(2\lambda_2)} e^{-\lambda_2^2 \tau} \cos\left(\frac{\lambda_2 x}{L}\right) + \frac{4\sin\lambda_3}{2\lambda_3 + \sin(2\lambda_3)} e^{-\lambda_3^2 \tau} \cos\left(\frac{\lambda_3 x}{L}\right) + \ldots$$

  - Roots of trigonometric function ($\lambda_1, \lambda_2, \lambda_3$ ...) found here (e.g. for $\mathrm{Bi} = 2$):



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Example (centre temperature in insulation)

- **Problem:** Find the centre temperature in a 10 cm thick layer of insulation after 30 minutes. Initially its temperature is 10°C and at time $t = 0$ s it's suddenly exposed to an ambient temperature of 25°C on one side.
- **Properties:**
  - Plate half thickness, $L = 0.05$ m
  - Initial temperature, $T_i = 10^\circ$C
  - Ambient temperature, $T_\infty = 25^\circ$C
  - Heat transfer coefficient, $h = 20$ W/(m^{2}K)
  - Thermal conductivity, $k = 0.05$ W/(m K)
  - Thermal diffusivity, $\alpha = 3 \cdot 10^{-7}$ m^2/s
- **Solution:** It's a plane wall, so analytical solution is:

$$\theta = \sum_{n=1}^{\infty} \frac{4\sin\lambda_n}{2\lambda_n + \sin(2\lambda_n)} e^{-\lambda_n^2 \tau} \cos\left(\frac{\lambda_n x}{L}\right)$$

where λ_n are the roots of: $\lambda_n \tan\lambda_n = \mathrm{Bi}$

- Let's calculate up to $n = 3$ and because $x = 0$ in the centre, we get:

$$\theta = \frac{4\sin\lambda_1}{2\lambda_1 + \sin(2\lambda_1)} e^{-\lambda_1^2 \tau} + \frac{4\sin\lambda_2}{2\lambda_2 + \sin(2\lambda_2)} e^{-\lambda_2^2 \tau} + \frac{4\sin\lambda_3}{2\lambda_3 + \sin(2\lambda_3)} e^{-\lambda_3^2 \tau}$$

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- Calculate τ:

Example (centre temperature in insulation)

- Find λ_1, λ_2 and λ_3 as the first 3 roots in $\lambda_n \tan(\lambda_n) = \mathrm{Bi} = h\ L/k$

```
clc, clear all, close all

    h = 20;
    k = 0.05;
    L = 0.05;
    Bi = h*L/k;
    fcn=@(lambda)lambda.*tan(lambda)-Bi;

    fplot(fcn,[0,10])
    xlabel('\lambda_n')
    ylabel('Biot number, Bi')
    grid on
    ylim([-5 5])

    for n = 1:3
      lambda(n) = fzero(fcn,(n-1)*pi+[0 pi/2])
    end
```

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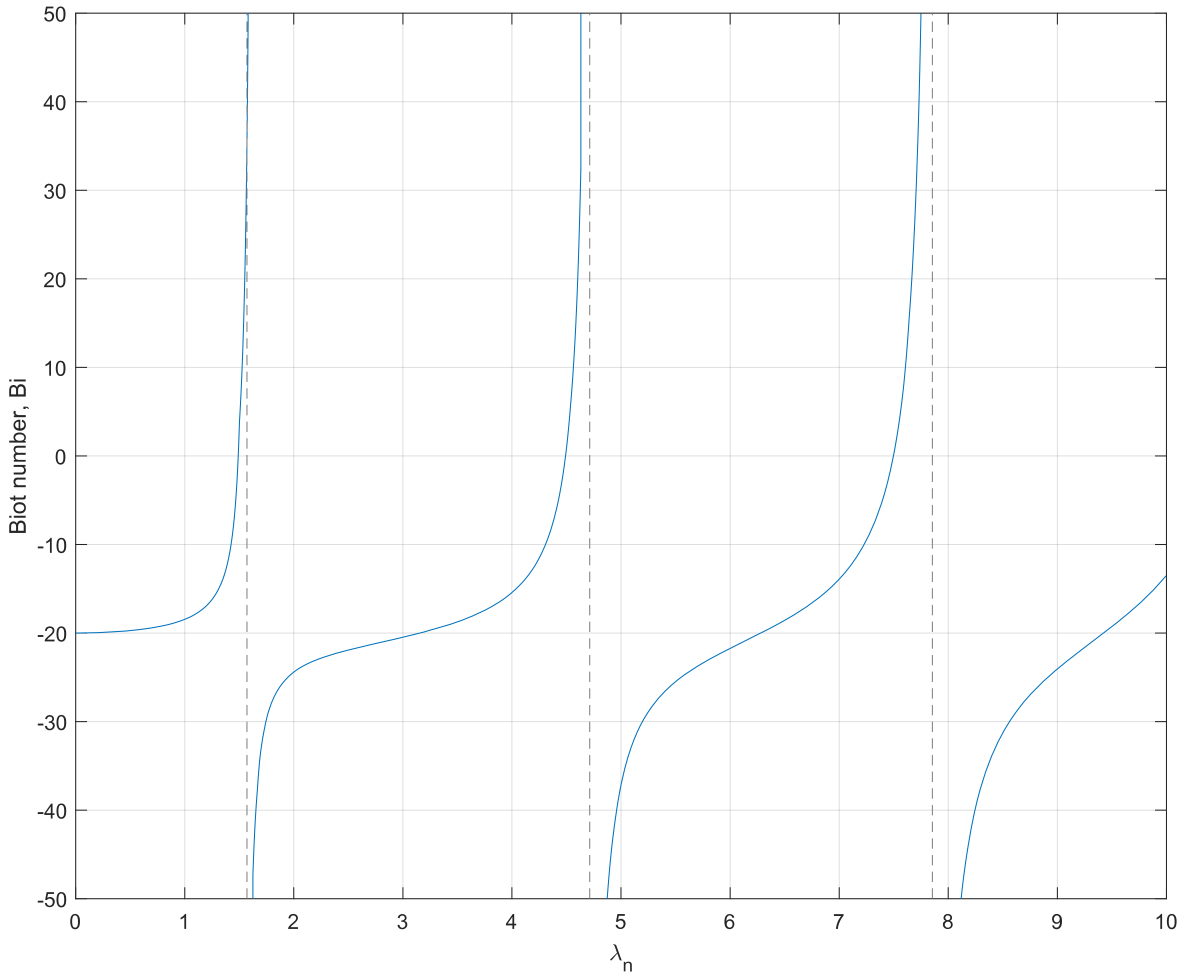
lambda = [1.4961    4.4915    7.4954]
```



Example (centre temperature in insulation)

- Now we got λ_1, λ_2 and λ_3 in:

$$\theta = \frac{4\sin\lambda_1}{2\lambda_1 + \sin(2\lambda_1)} e^{-\lambda_1^2\tau} + \frac{4\sin\lambda_2}{2\lambda_2 + \sin(2\lambda_2)} e^{-\lambda_2^2\tau} + \frac{4\sin\lambda_3}{2\lambda_3 + \sin(2\lambda_3)} e^{-\lambda_3^2\tau}$$

- Now considering each term:

$$\theta_1 = \frac{4\sin\lambda_1}{2\lambda_1 + \sin(2\lambda_1)} e^{-\lambda_1^2\tau} = 0.7831$$

$$\theta_2 = \frac{4\sin\lambda_2}{2\lambda_2 + \sin(2\lambda_2)} e^{-\lambda_2^2\tau} = -0.0053$$

$$\theta_3 = \frac{4\sin\lambda_3}{2\lambda_3 + \sin(2\lambda_3)} e^{-\lambda_3^2\tau} = 1.2848 \cdot 10^{-6}$$

- θ_2 and θ_3 are **really small** compared to θ_1, e.g. $\theta \approx \theta_1$:

$$T(0, 1800) \approx \theta_1(T_i - T_\infty) + T_\infty = 13.25^\circ\mathrm{C}$$

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One-term approximations

- Assume all terms for $n \geq 2$ **to be zero**, e.g. $\theta_2 = 0$, $\theta_3 = 0$ ...

$$\theta = \frac{4\sin\lambda_1}{2\lambda_1 + \sin(2\lambda_1)} e^{-\lambda_1^2\tau} \cos\left(\frac{\lambda_1 x}{L}\right) + \cancel{\frac{4\sin\lambda_2}{2\lambda_2 + \sin(2\lambda_2)} e^{-\lambda_2^2\tau} \cos\left(\frac{\lambda_2 x}{L}\right) + \frac{4\sin\lambda_3}{2\lambda_3 + \sin(2\lambda_3)} e^{-\lambda_3^2\tau} \cos\left(\frac{\lambda_3 x}{L}\right)} + \ldots$$

- Typically less than 2% error if $\tau = \alpha t / L_c^2 > 0.2$