Fluid Mechanics
Analytical solutions for simple fluid flows

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Complexities involved in Navier-Stokes equations

No universal analytical solution due to non-linearities ($\partial u / \partial x$)

- Solution *approximated* using *numerical methods* (e.g. CFD)
- Solution found by *experiments*

Analytical solutions for *only a few problems*

- *Simple geometries*: Flow between two plates or in a tube
- *Simple flows*: Steady, laminar and fully-developed flows

x-direction:

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho g_x$$

y-direction:

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \rho g_y$$

z-direction:

$$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \rho g_z$$

Why look at these *simplified* problems?

For which applications is the flow between two plates important?

Flow between two plates—whether *pressure-driven* (plane Poiseuille flow) or *shear-driven* (Couette flow)—is a **canonical model** that underpins both engineering practice and natural phenomena. It's important in many applications, including:

Engineering & Technology

- **Lubrication**: Thin films of oil in bearings, pistons, gears, or seals are modeled as flow between parallel plates.
- **Microfluidics**: Lab-on-a-chip devices use narrow rectangular channels, which are essentially parallel plate flows.
- **Heat exchangers**: Plate-type exchangers rely on fluid moving between closely spaced plates for efficient heat transfer.
- **Coating & printing**: Flow of paint, varnish, or ink in thin layers is approximated by plate flows.
- **Polymer processing**: Sheet extrusion and film casting involve molten polymer flow between plates.

Everyday Phenomena

- **Sticking of wet surfaces**: Two glass slides or a credit card and plastic sleeve resist separation due to viscous flow in the thin liquid film.
- **Windshield wipers**: The thin water film between blade and glass behaves like a parallel-plate channel.
- **Food preparation**: Spreading butter, cream, or sauces between slices involves thin-film flow.

Biology & Medicine

- **Synovial joints**: Cartilage surfaces slide with synovial fluid in between, modeled as shear-driven thin-film flow.
- **Blood in microvessels**: Capillaries and arterioles often approximate parallel-plate geometries at small scales.
- **Dialysis and filtration**: Fluids move between thin membranes in channels resembling plate flows.

Fundamental & Educational Value

- Provides one of the **few exact solutions of Navier–Stokes**—great for teaching velocity profiles, shear stress, and pressure gradients.
- Serves as a **building block** for more complex flows (e.g., lubrication theory, Reynolds equation).
- Helps illustrate the importance of viscosity, low Reynolds number flows, and scaling laws.

Current research uses *Poiseuille flow* for verification of CFD

CO_2 diffuses into the fiber and is captured

- Really (!) long fibres ($l/D \approx 12000$)
- Steady, laminar, fully-developed flow is a *very good* assumption
- Prescribed boundary condition
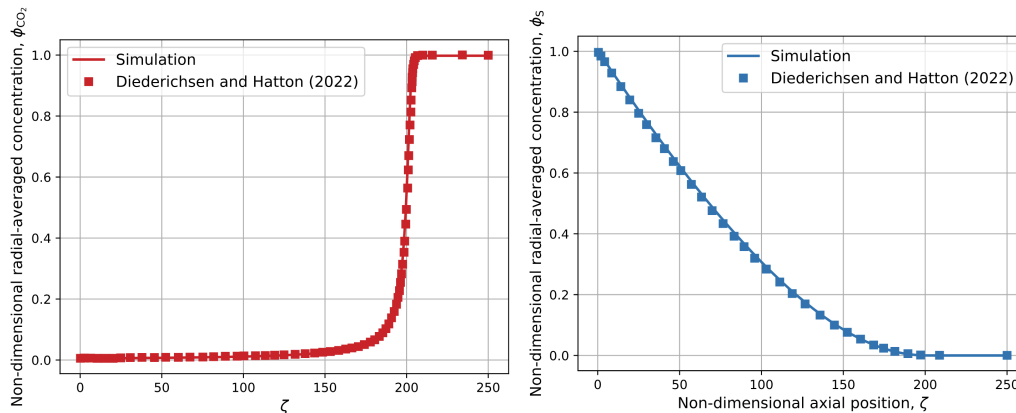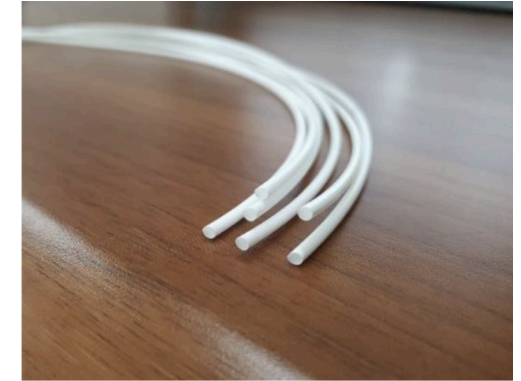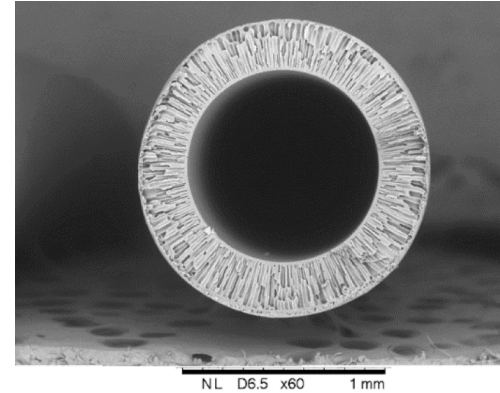
Results: Code verification results

Figure: Hollow fibre membrane

Bazhenov, S. D., Bildyukevich, A. V., & Volkov, A. V. (2018). Gas-Liquid Hollow Fiber Membrane Contactors for Different Applications. Fibers, 6(4), 76. https://doi.org/10.3390/fib6040076

Code: Inlet boundary condition in OpenFOAM

```
1
2  /*--------------------------------*- C++ -*----------------------------------*
3     =========                 |
4     \\      /  F ield         | OpenFOAM: The Open Source CFD Toolbox
5      \\    /   O peration     | Website:  https://openfoam.com
6       \\  /    A nd           | Version:  2506
7        \\/     M anipulation  |
8  \*---------------------------------------------------------------------------*
9  FoamFile
10 {
11     format      ascii;
12     class       volVectorField;
13     location    "0";
14     object      U;
15 }
16 // * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * //
17
18 dimensions      [0 1 -1 0 0 0 0];
19
```
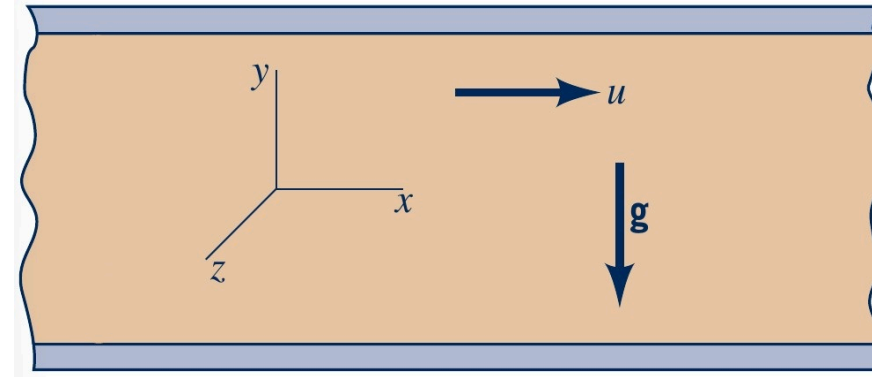
Flow between plates

Steady, *fully developed*, *laminar flow* between two *infinite plates*

- Velocity only in x-direction:
  - *then,* $u = u(y), v = w = 0$
- Steady flow:
  - *then,* $\partial u/\partial t = \partial v/\partial t = \partial w/\partial t = 0$
- By continuity equation: $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$
  - *then,* $\partial u/\partial x = 0$
- No gravity in x and z directions:
  - *then,* $g_x = g_z = 0$

Navier-Stokes equations *reduce* to:

$$x : 0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}, \quad y : 0 = -\frac{\partial p}{\partial y} + \rho g_y, \quad z : 0 = -\frac{\partial p}{\partial z}$$

x-direction:

$$\rho \left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho g_x$$

y-direction:

$$\rho \left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \rho g_y$$

z-direction:

$$\rho \left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \rho g_z$$

Flow between *two stationary* plates

Integrating twice to get $u(y)$:

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\mu}\frac{\partial p}{\partial x}y + C_1$$

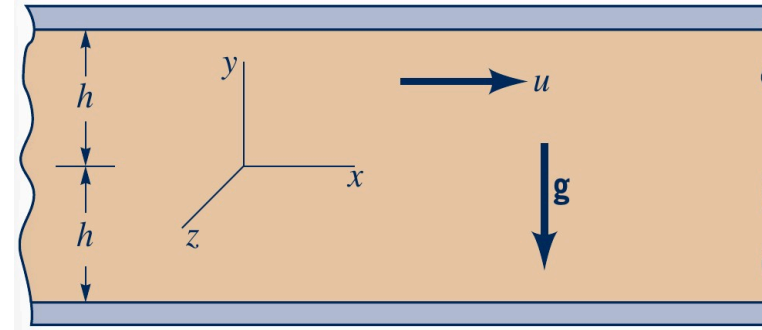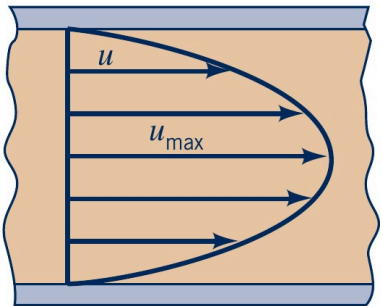$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + C_1 y + C_2$$

Using *boundary conditions*:

At $y = \pm h$ we have $u = 0$

$$C_1 = 0, C_2 = -\frac{1}{2\mu}\frac{\partial p}{\partial x}h^2$$

Velocity profile (by inserting C_1 and C_2):

$$u(y) = \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - h^2)$$

Volume flow rate (integrating velocity profile):

$$q = \int_{-h}^{h} u(y) = -\frac{2h^3}{3\mu}\left(\frac{\partial p}{\partial x}\right)$$

Average velocity:

$$V = \frac{q}{2h} = -\frac{h^2}{3\mu}\left(\frac{\partial p}{\partial x}\right)$$

Maximum velocity (centre of channel, $y = 0$):

$$u_{max} = u(y = 0) = -\frac{h^2}{2\mu}\left(\frac{\partial p}{\partial x}\right) = \frac{3}{2}V$$

Flow between *one stationary* and *one moving* plate (aka Couette flow)

Integrating twice to get $u(y)$:

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\mu}\frac{\partial p}{\partial x}y + C_1$$

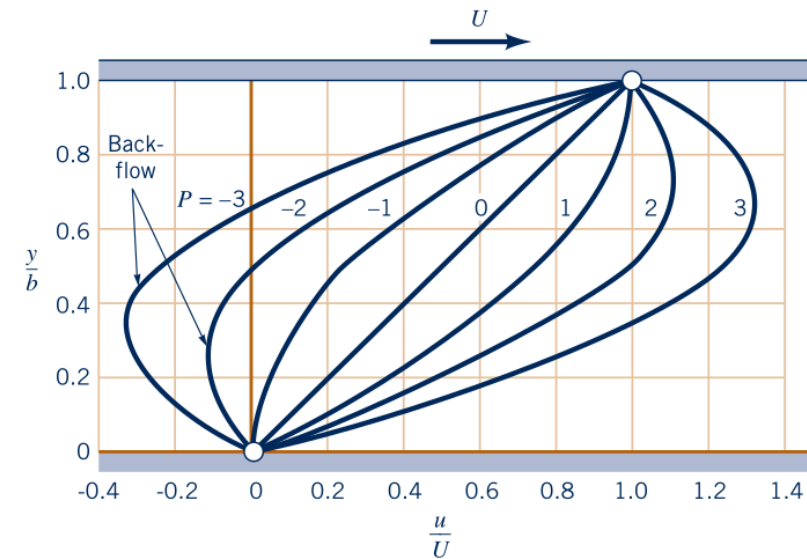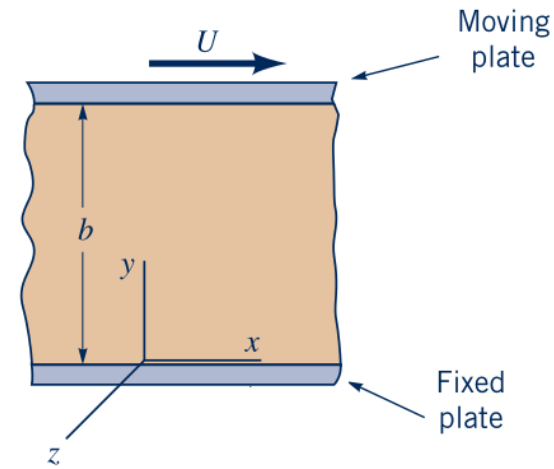$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + C_1 y + C_2$$

Using *boundary conditions*:

At $y = 0$ we have $u = 0$

At $y = b$ we have $u = U$

Finding C_1 and C_2 and inserting:

$$u(y) = U\frac{y}{b} + \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - by)$$

$$P = -\frac{b^2}{2\mu U}\left(\frac{\partial p}{\partial x}\right)$$

Flow in a *circular pipe* (aka Poiseuille flow)

Steady, *fully developed*, *laminar flow* in a circular pipe

- Velocity only in z-direction:
  - *then,* $v_z = v_z(r), v_\theta = v_r = 0$
- Steady flow:
  - *then,* $\partial u/\partial t = \partial v/\partial t = \partial w/\partial t = 0$
- By continuity equation: $\dfrac{1}{r}\dfrac{\partial(rv_r)}{\partial r} + \dfrac{1}{r}\dfrac{\partial v_\theta}{\partial \theta} + \dfrac{\partial v_z}{\partial z} = 0$
  - *then,* $\partial v_z/\partial z = 0$
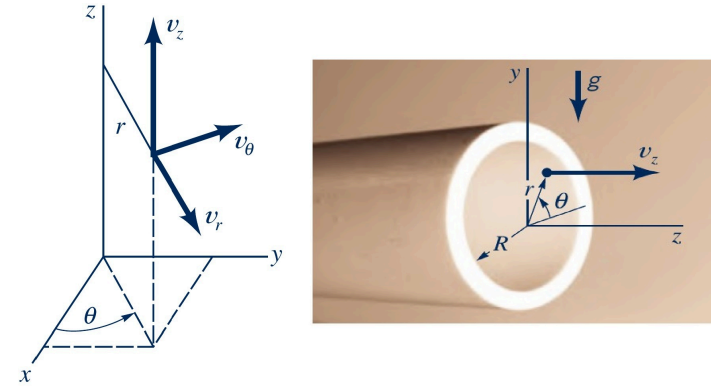- No gravity in z direction:
  - *then,* $g_z 0$

Navier-Stokes equations *reduce* to:

$$z: 0 = -\frac{\partial p}{\partial z} + \mu \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right), \quad r: 0 = \rho g_r - \frac{\partial p}{\partial r}, \quad \theta: 0 = \rho g_\theta - \frac{\partial p}{\partial \theta}$$

Integration, applying *boundary condtions* and *find C_1* and *C_2*:

$$v_z(r) = \frac{1}{4\mu} - \left(\frac{\partial p}{\partial z}\right)\left(r^2 - R^2\right)$$

- Velocity is a function of *viscosity μ*, *pressure gradient $\partial p/\partial z$* and *pipe radius R*.

z-direction:

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z$$

r-direction:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_r$$

θ-direction:

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} - \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right) + \rho g_\theta$$

Details on flow in *circular pipes* (aka Poiseuille flow)

- Volume flow rate (integrate velocity):

$$Q = 2\pi \int_0^R v_z(r)r\,dr = -\frac{\pi R^4}{8\mu}\left(\frac{\partial p}{\partial z}\right) = \frac{\pi R^4 \Delta p}{8\mu l}$$

  ○ Flow rate is a function of *viscosity* μ, *pressure gradient* $\partial p/\partial z$ and *pipe radius* R.

- Average velocity (divide by area):

$$V = \frac{Q}{A} = \frac{-\dfrac{\pi R^4}{8\mu}\left(\dfrac{\partial p}{\partial z}\right)}{\pi R^2} = -\frac{R^2}{8\mu}\left(\frac{\partial p}{\partial z}\right) = \frac{R^2 \Delta p}{8\mu l}$$

  ○ Mean velocity is a function of *viscosity* μ, *pressure gradient* $\partial p/\partial z$ and *pipe radius* R.

- Maximum velocity ($r = 0$):

$$v_z(r=0) = \frac{1}{4\mu} - \left(\frac{\partial p}{\partial z}\right)(0^2 - R^2) = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial z}\right) = \frac{R^2 \Delta p}{4\mu l}$$

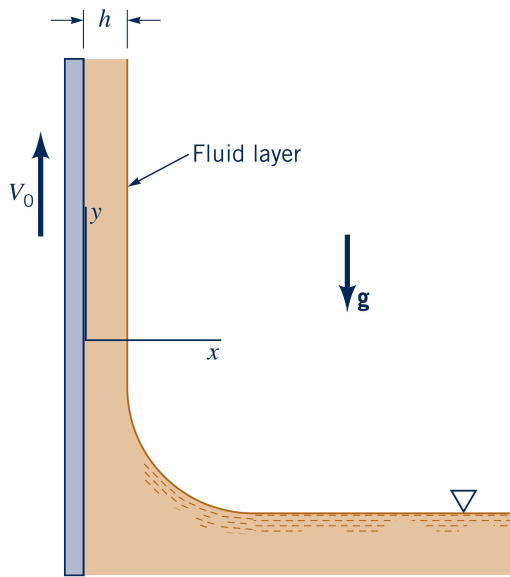 ○ Mean velocity is a function of *viscosity* μ, *pressure gradient* $\partial p/\partial z$ and *pipe radius* R.

- Ratio of centerline to mean velocity ($v_z(r=0)/V$):

$$\frac{v_z(r=0)}{V} = \frac{\dfrac{R^2 \Delta p}{4\mu l}}{\dfrac{R^2 \Delta p}{8\mu l}} = 2$$

  ○ Centerline velocity is exactly *twice* the mean velocity.

Exercise: Upward moving fluid film

A wide moving belt passes through a container of a viscous liquid. The belt moves vertically upward with a constant velocity, V_0, as illustrated in the figure. Because of viscous forces the belt picks up a film of fluid of thickness h. Gravity tends to make the fluid drain down the belt. Assume that the flow is laminar, steady, and fully developed.

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho g_x$$

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \rho g_y$$

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \rho g_z$$