

Heat Transfer

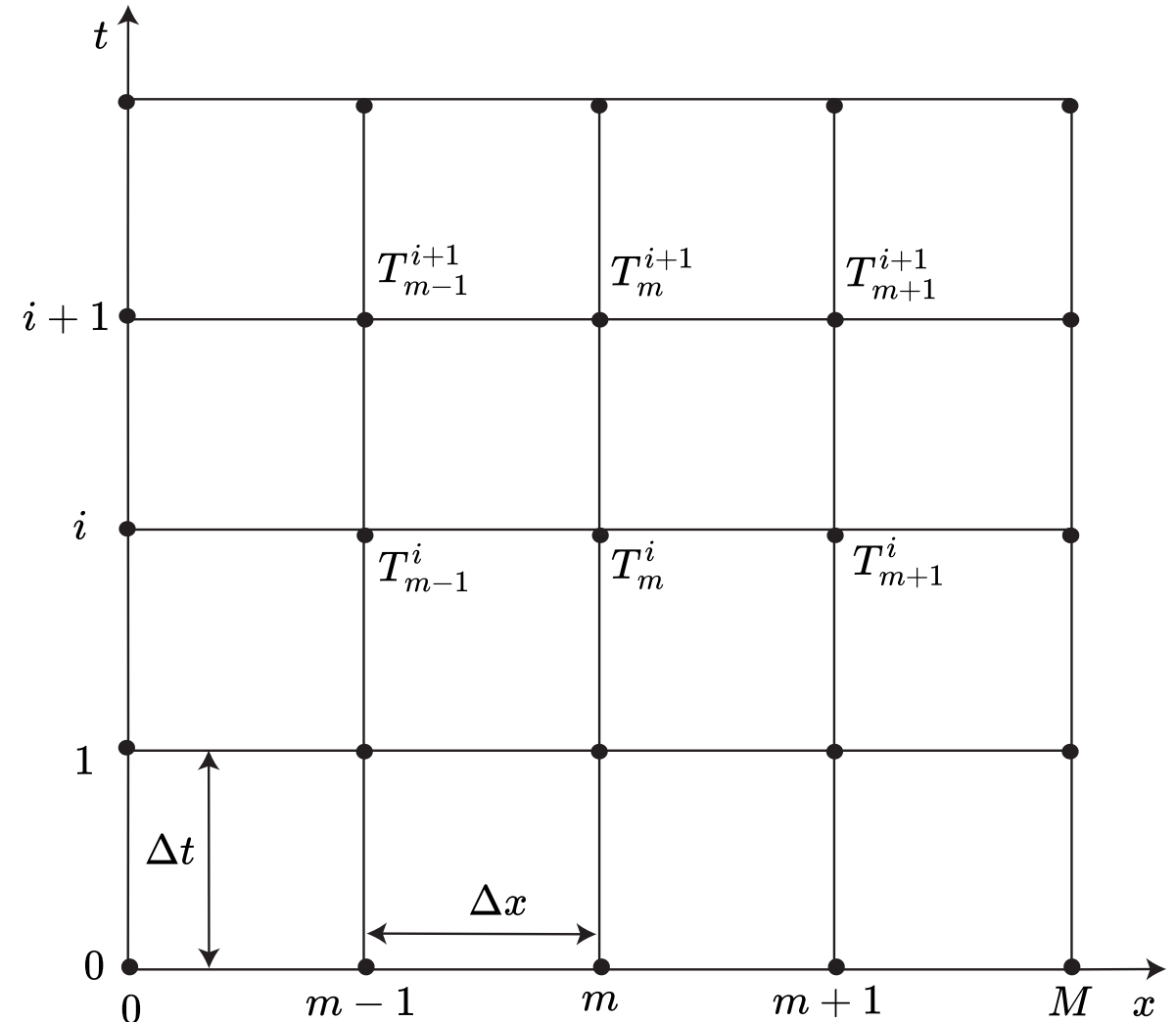
Transient heat conduction using numerical methods

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Notation for transient problems

- **On space axis** m, n, k denote node number in x, y, z -directions
 - Distance between nodes is Δx
- **On time axis** i denote node number in t -direction (time)
 - Time between nodes is Δt
- General notation:

$$T_{m,n,k}^i$$



The energy balance equation for transient problems

We sum up heat transfer to/from volume element:

$$\left(\begin{array}{c} \text{Sum of heat} \\ \text{transfer rate} \\ \text{across surfaces} \\ \text{of volume element} \end{array} \right) + \left(\begin{array}{c} \text{Rate of heat} \\ \text{generation} \\ \text{inside the} \\ \text{element} \end{array} \right) = \left(\begin{array}{c} \text{Rate of change} \\ \text{of the energy} \\ \text{content of} \\ \text{the element} \end{array} \right)$$

- For *transient* heat transfer (temperature changes with time):

$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t}$$

$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = mc_p \frac{\Delta T_{\text{element}}}{\Delta t}$$

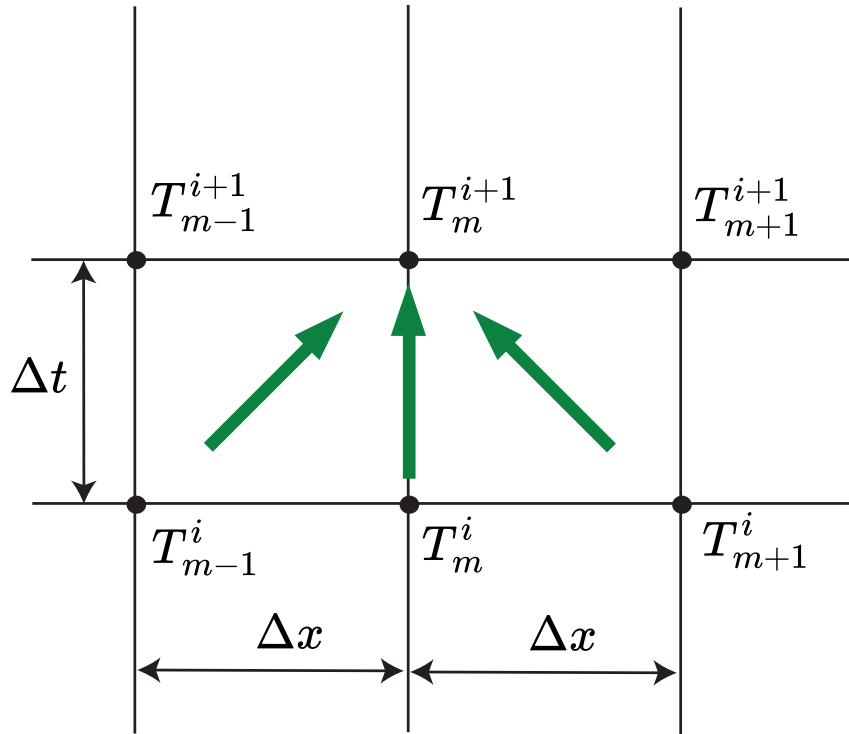
$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = V_{\text{element}} \rho c_p \frac{\Delta T_{\text{element}}}{\Delta t}$$

Explicit and implicit solution methods

Question is: How do we advance from time t (timestep i) to timestep $t + \Delta t$ (timestep $i + 1$)?

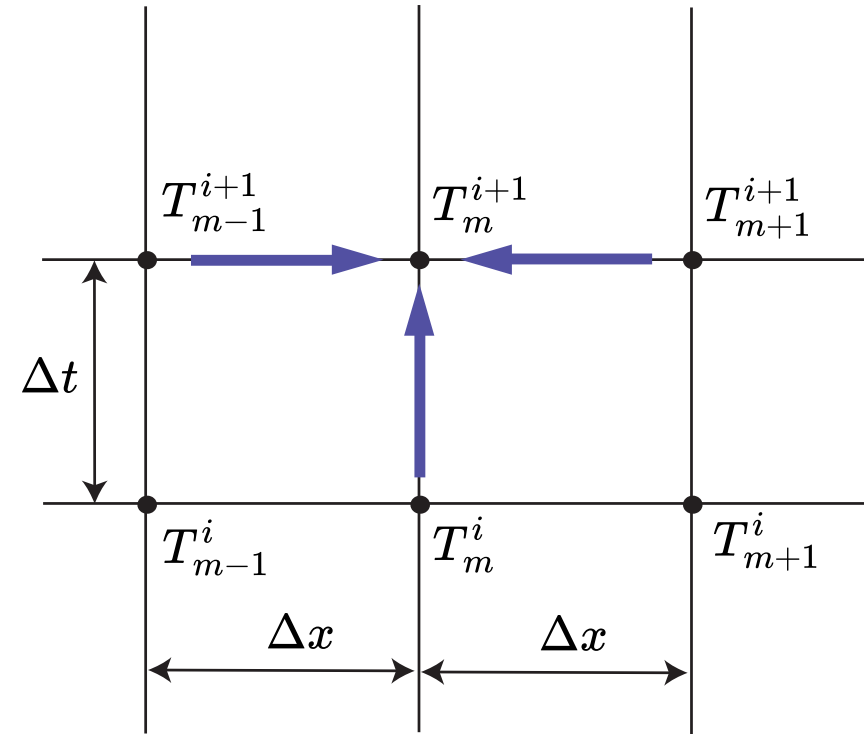
Explicit method

$$\sum_{\text{all surfaces}} \dot{Q}^i + \dot{G}_{\text{element}}^i = \frac{\rho V_{\text{element}} c_p (T_m^{i+1} - T_m^i)}{\Delta t}$$



Implicit method

$$\sum_{\text{all surfaces}} \dot{Q}^{i+1} + \dot{G}_{\text{element}}^{i+1} = \frac{\rho V_{\text{element}} c_p (T_m^{i+1} - T_m^i)}{\Delta t}$$



Explicit: Use data from current time step i (all known temperatures)

Implicit: Use data from next time step $i + 1$ (unknown temperatures)

Transient heat conduction using the *explicit method*

Interior nodes using explicit solution method

Energy balance at interior node:

$$kA \frac{T_{m-1}^i - T_m^i}{\Delta x} + kA \frac{T_{m+1}^i - T_m^i}{\Delta x} + \dot{q}_m^i A \Delta x = \frac{\rho V c_p (T_m^{i+1} - T_m^i)}{\Delta t}$$

- Re-arranging and inserting thermal diffusivity $\alpha = k/(\rho c_p)$:

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{q}_m^i \Delta x^2}{k} = \frac{\Delta x^2}{\alpha \Delta t} (T_m^{i+1} - T_m^i)$$

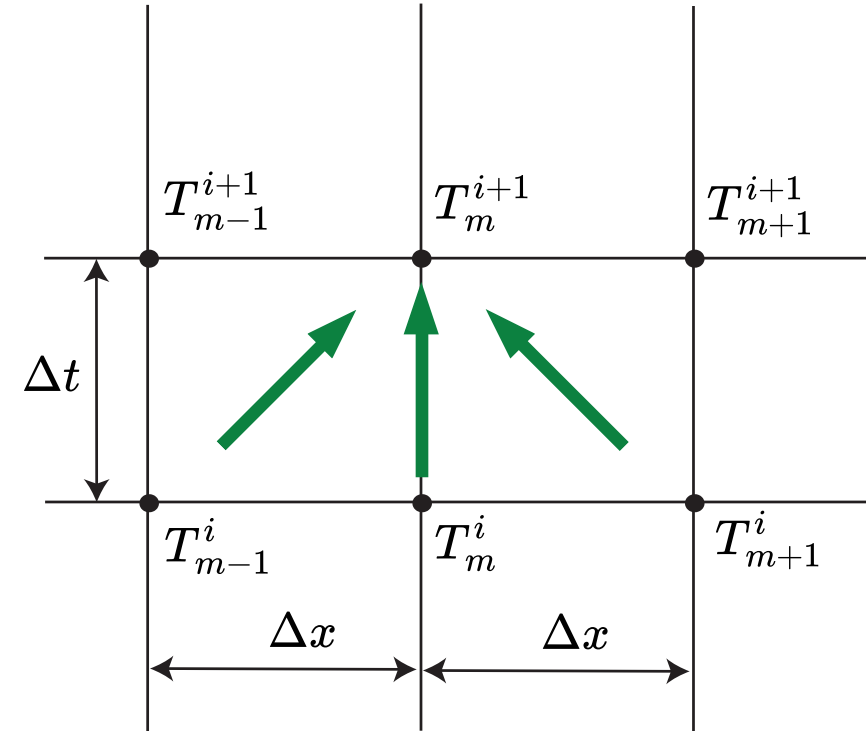
- Inserting mesh Fourier number $\tau = \alpha \Delta t / \Delta x^2$:

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{q}_m^i \Delta x^2}{k} = \frac{1}{\tau} (T_m^{i+1} - T_m^i)$$

- Isolating the only unknown T_m^{i+1} :

$$T_m^{i+1} = \tau (T_{m-1}^i + T_{m+1}^i) + (1 - 2\tau) T_m^i + \tau \frac{\dot{q}_m^i \Delta x^2}{k}$$

- 1 equation for each node with *only 1 unknown each* 😎



Boundary nodes using *explicit* solution method

Energy balance for boundary node with convection:

$$hA(T_\infty - T_0^i) + kA \frac{T_1^i - T_0^i}{\Delta x} + \dot{g}_0 A \frac{\Delta x}{2} = \frac{\rho V c_p (T_0^{i+1} - T_0^i)}{\Delta t}$$

- Re-arranging and noting thermal diffusivity $\alpha = k/(\rho c_p)$:

$$\left(\frac{-2h\Delta x}{k} - 2 \right) T_0^i + 2T_1^i + \left(\frac{-2h\Delta x}{k} \right) T_\infty + \dot{g}_0 \frac{(\Delta x)^2}{k} = \frac{(\Delta x)^2}{\alpha \Delta t} (T_0^{i+1} - T_0^i)$$

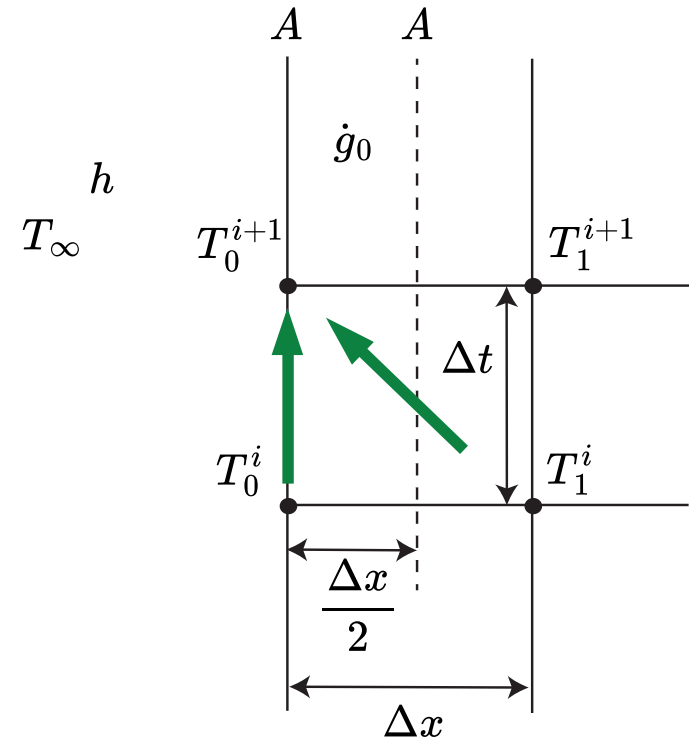
- Defining mesh Fourier number $\tau = \alpha \Delta t / \Delta x^2$:

$$\left(\frac{-2h\Delta x}{k} - 2 \right) T_0^i + 2T_1^i + \left(\frac{-2h\Delta x}{k} \right) T_\infty + \dot{g}_0 \frac{(\Delta x)^2}{k} = \frac{1}{\tau} (T_0^{i+1} - T_0^i)$$

- Isolating the only unknown T_0^{i+1} :

$$T_0^{i+1} = \left(1 - 2\tau - 2\tau \frac{h\Delta x}{k} \right) T_0^i + (2\tau) T_1^i + 2\tau \frac{h\Delta x}{k} T_\infty + \frac{\tau \dot{g}_0 (\Delta x)^2}{k}$$

- 1 equation for each boundary node with *only 1 unknown* 😎

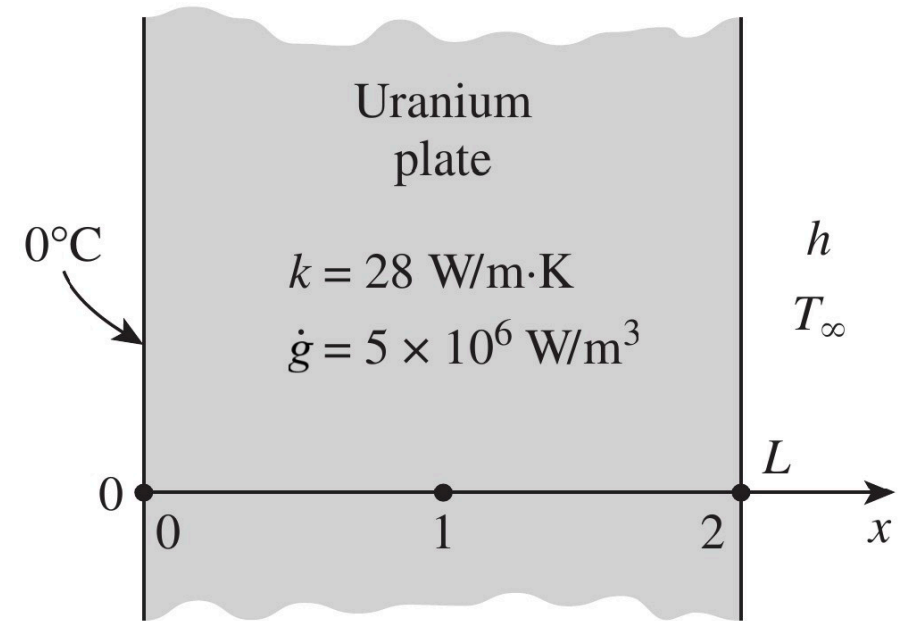


Example: Transient heat conduction in a large plate using the explicit method

Consider a large uranium plate of thickness $L = 4$ cm, thermal conductivity $k = 28$ W/(m·K), and thermal diffusivity $a = 12.5 \times 10^{-6}$ m²/s that is initially at a uniform temperature of 200°C. Heat is generated uniformly in the plate at a constant rate of $\dot{g} = 5 \times 10^6$ W/m³.

At time $t = 0$ s, one side of the plate is brought into contact with iced water and is maintained at 0°C at all times, while the other side is subjected to convection to an environment at $T_\infty = 30^\circ\text{C}$ with a heat transfer coefficient of $h = 45$ W/(m²·K).

Estimate the exposed surface temperature of the plate 2.5 min after start of cooling. Use three equally spaced nodes in the medium, two at the boundaries and one at the middle.



The stability criterion for the *explicit* solution method

Observation: Explicit solution method may *diverge if time step size is too large!*

Stability criterion: When looking at node m , the *coefficient in front of T_m^i should be larger than zero*

- Equation for an *interior node* (derived here):

$$T_m^{i+1} = \tau(T_{m-1}^i + T_{m+1}^i) + (1 - 2\tau) T_m^i + \tau \frac{\dot{g}_m^i (\Delta x)^2}{k}$$

- Criterion becomes:* $1 - 2\tau \geq 0 \rightarrow 1 - 2(\alpha \Delta t / \Delta x^2) \geq 0 \rightarrow \Delta t \leq \frac{(\Delta x)^2}{2\alpha}$

- Equation for a *boundary node with convection* (derived here):

$$T_0^{i+1} = \left(1 - 2\tau - 2\tau \frac{h\Delta x}{k}\right) T_0^i + (2\tau) T_1^i + 2\tau \frac{h\Delta x}{k} T_\infty + \frac{\tau \dot{g}_0^i (\Delta x)^2}{k}$$

- Criterion becomes:* $\left(1 - 2\tau - 2\tau \frac{h\Delta x}{k}\right) \geq 0 \rightarrow \Delta t \leq \frac{(\Delta x)^2}{2\alpha(1 + \frac{h\Delta x}{k})}$

Transient heat conduction using the *implicit method*

Interior nodes using *implicit* solution method

Energy balance at interior node:

$$kA \frac{T_{m-1}^{i+1} - T_m^{i+1}}{\Delta x} + kA \frac{T_{m+1}^{i+1} - T_m^{i+1}}{\Delta x} + \dot{g}_m^{i+1} A \Delta x = \frac{\rho V c_p (T_m^{i+1} - T_m^i)}{\Delta t}$$

- Re-arranging:

$$T_{m-1}^{i+1} - 2T_m^{i+1} + T_{m+1}^{i+1} + \frac{\dot{g}_m^{i+1} \Delta x^2}{k} = \frac{\Delta x^2}{\alpha \Delta t} (T_m^{i+1} - T_m^i)$$

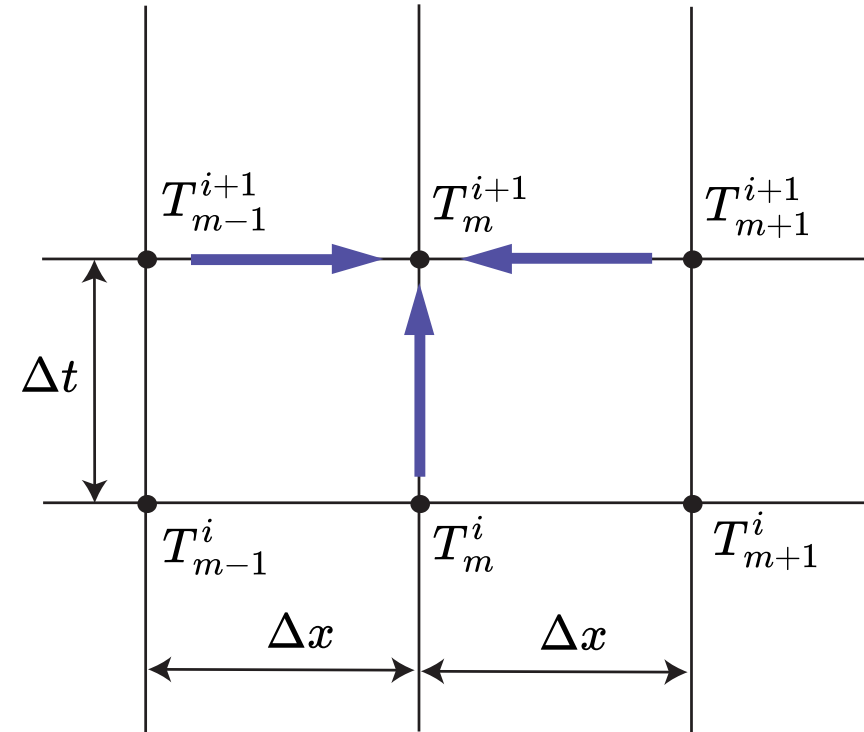
- Using $\tau = \alpha \Delta t / \Delta x^2$:

$$T_{m-1}^{i+1} - 2T_m^{i+1} + T_{m+1}^{i+1} + \frac{\dot{g}_m^{i+1} \Delta x^2}{k} = \frac{1}{\tau} (T_m^{i+1} - T_m^i)$$

- Separating the 3 unknowns T_{m-1}^{i+1} , T_m^{i+1} and T_{m+1}^{i+1} :

$$\tau T_{m-1}^{i+1} - (1 + 2\tau) T_m^{i+1} + \tau T_{m+1}^{i+1} + \tau \frac{\dot{g}_m^{i+1} \Delta x^2}{k} + T_m^i = 0$$

- Can't solve each equation individually *because each equation has three unknowns*, T_{m-1}^{i+1} , T_m^{i+1} , and T_{m+1}^{i+1} 😞



Boundary nodes using *implicit* solution method

Energy balance at boundary node with convection:

$$hA(T_\infty - T_0^{i+1}) + kA \frac{T_1^{i+1} - T_0^{i+1}}{\Delta x} + \dot{g}_m^{i+1} A \frac{\Delta x}{2} = \frac{\rho V c_p (T_0^{i+1} - T_0^i)}{\Delta t}$$

- Re-arranging:

$$\left(\frac{-2h\Delta x}{k} - 2 \right) T_0^{i+1} + 2T_1^{i+1} + \left(\frac{2h\Delta x}{k} \right) T_\infty + \dot{g}_0^i \frac{(\Delta x)^2}{k} = \frac{(\Delta x)^2}{\alpha \Delta t} (T_0^{i+1} - T_0^i)$$

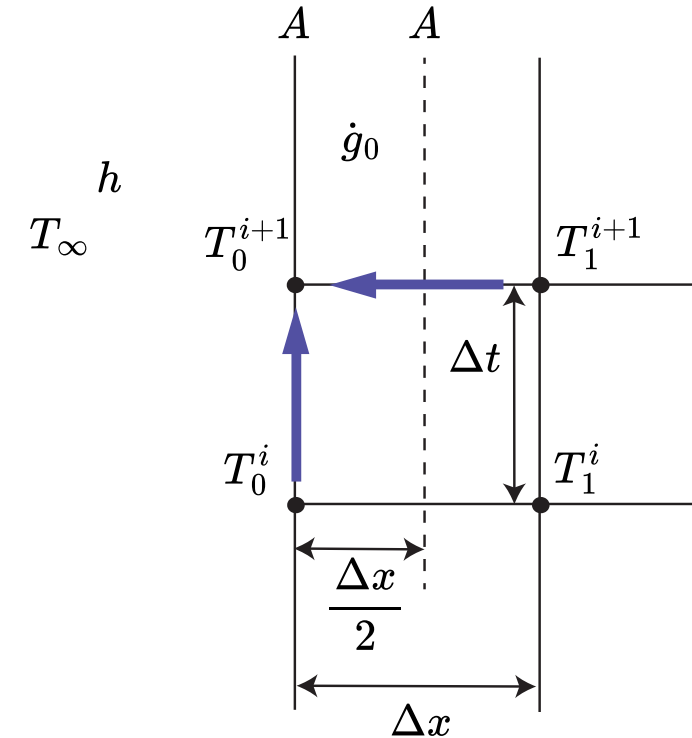
- Defining $\tau = \alpha \Delta t / \Delta x^2$:

$$\left(\frac{-2h\Delta x}{k} - 2 \right) T_0^{i+1} + 2T_1^{i+1} + \left(\frac{2h\Delta x}{k} \right) T_\infty + \dot{g}_0^i \frac{(\Delta x)^2}{k} = \frac{1}{\tau} (T_0^{i+1} - T_0^i)$$

- Separating the 2 unknowns T_0^{i+1} and T_1^{i+1} :

$$\left(-1 - 2\tau - 2\tau \frac{h\Delta x}{k} \right) T_0^{i+1} + (2\tau) T_1^{i+1} + 2\tau \frac{h\Delta x}{k} T_\infty + \tau \frac{\dot{g}_0^{i+1} (\Delta x)^2}{k} + T_0^i = 0$$

- Can't solve each equation individually *because each equation has two unknowns*, T_0^{i+1} and T_1^{i+1} 😞



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