Heat Transfer

Transient heat conduction using numerical methods

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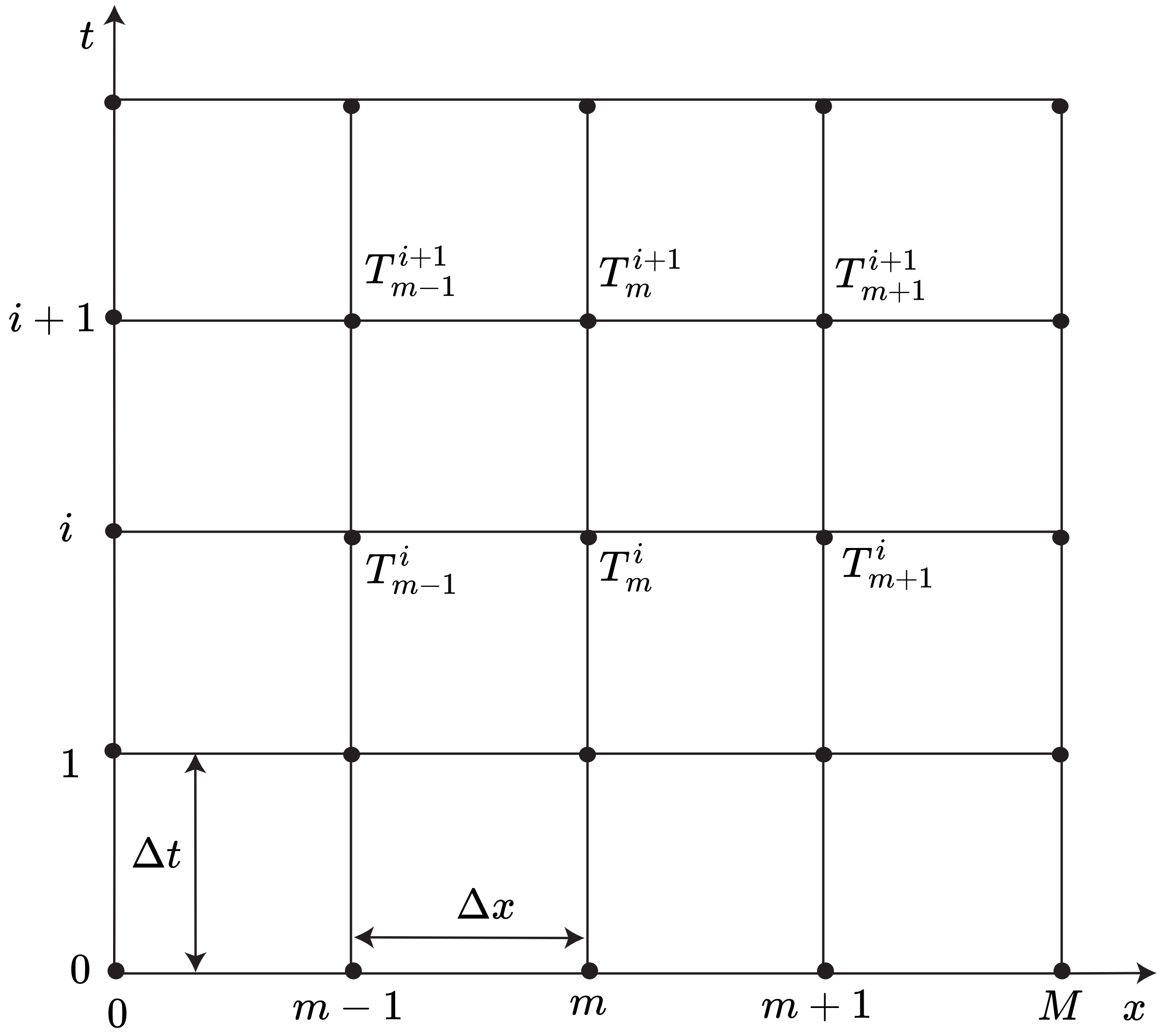
Notation for transient problems

- **On space axis** m, n, k denote node number in x, y, z-directions
  - Distance between nodes is Δx
- **On time axis** i denote node number in t-direction (time)
  - Time between nodes is Δt
- General notation:

$$T^i_{m,n,k}$$



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The energy balance equation for transient problems

We sum up heat transfer to/from volume element:

$$\begin{pmatrix}\text{Sum of heat}\\ \text{transfer rate}\\ \text{across surfaces}\\ \text{of volume element}\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of}\\ \text{the element}\end{pmatrix}$$

- For *transient* heat transfer (temperature changes with time):

$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t}$$

$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = m c_p \frac{\Delta T_{\text{element}}}{\Delta t}$$

$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = V_{\text{element}} \rho c_p \frac{\Delta T_{\text{element}}}{\Delta t}$$

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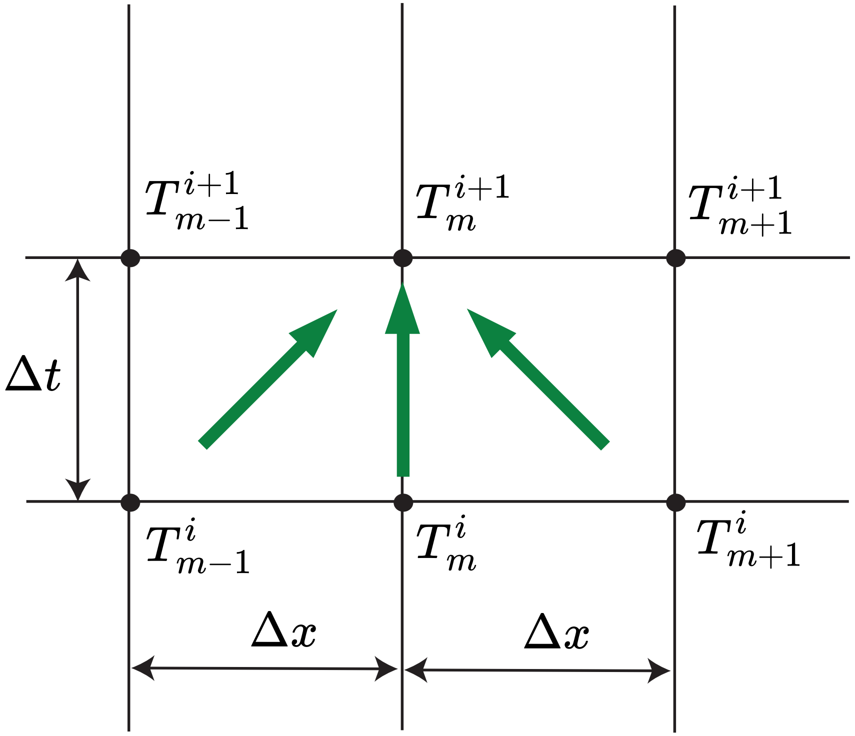
Explicit and implicit solution methods

Question is: How do we advance from time t (timestep i) to timestep $t + \Delta t$ (timestep $i + 1$)?

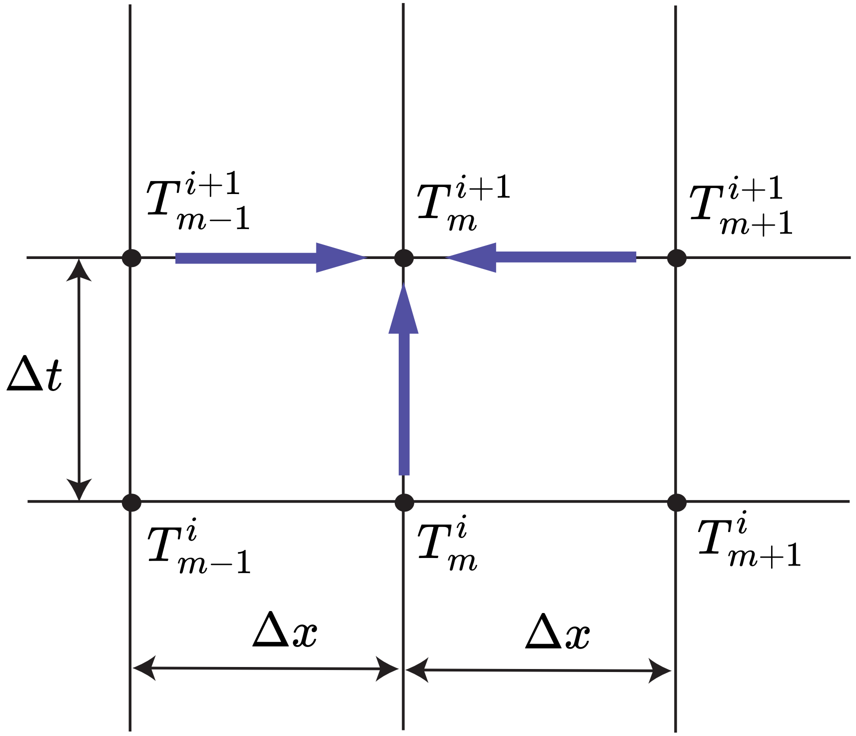
Explicit method

$$\sum_{\text{all surfaces}} Q^i + \dot{G}^i_{\text{element}} = \frac{\rho V_{\text{element}} c_p (T_m^{i+1} - T_m^i)}{\Delta t}$$



Implicit method

$$\sum_{\text{all surfaces}} Q^{i+1} + \dot{G}^{i+1}_{\text{element}} = \frac{\rho V_{\text{element}} c_p (T_m^{i+1} - T_m^i)}{\Delta t}$$



Explicit: Use data from current time step i (all known temperatures)

Implicit: Use data from next time step $i + 1$ (unknown temperatures)

Interior nodes using *explicit* solution method

Energy balance at interior node:

$$kA\frac{T_{m-1}^i - T_m^i}{\Delta x} + kA\frac{T_{m+1}^i - T_m^i}{\Delta x} + \dot{g}_{\mathrm{m}}^i A\Delta x = \frac{\rho V c_p (T_m^{i+1} - T_m^i)}{\Delta t}$$

- Re-arranging and inserting thermal diffusivity $\alpha = k/(\rho c_p)$:

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{g}_m^i \Delta x^2}{k} = \frac{\Delta x^2}{\alpha \Delta t}\left(T_m^{i+1} - T_m^i\right)$$

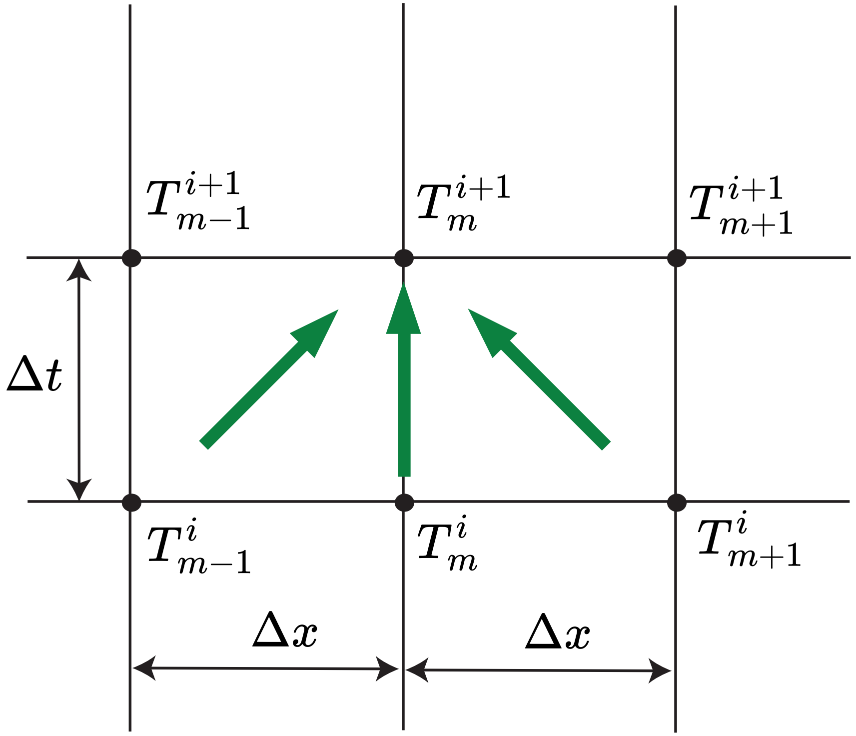
- Inserting mesh Fourier number $\tau = \alpha\Delta t/\Delta x^2$:

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{g}_m^i \Delta x^2}{k} = \frac{1}{\tau}\left(T_m^{i+1} - T_m^i\right)$$

- Isolating the only unknown T_m^{i+1}:

$$T_m^{i+1} = \tau(T_{m-1}^i + T_{m+1}^i) + (1 - 2\tau)T_m^i + \tau\frac{\dot{g}_m^i \Delta x^2}{k}$$

- 1 equation for each node with *only 1 unknown each* 🤩



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Boundary nodes using explicit solution method

- Energy balance for boundary node with convection:

$$hA(T_\infty - T_0^i) + kA\frac{T_1^i - T_0^i}{\Delta x} + \dot{g}_0^i A\frac{\Delta x}{2} = \frac{\rho V c_p (T_0^{i+1} - T_0^i)}{\Delta t}$$

- Re-arranging and noting thermal diffusivity $\alpha = k/(\rho c_p)$:

$$\left(\frac{-2h\Delta x}{k} - 2\right) T_0^i + 2T_1^i + \left(\frac{-2h\Delta x}{k}\right) T_\infty + \dot{g}_0^i \frac{(\Delta x)^2}{k} = \frac{(\Delta x)^2}{\alpha \Delta t}\left(T_0^{i+1} - T_0^i\right)$$

- Defining mesh Fourier number $\tau = \alpha\Delta t/\Delta x^2$:

$$\left(\frac{-2h\Delta x}{k} - 2\right) T_0^i + 2T_1^i + \left(\frac{-2h\Delta x}{k}\right) T_\infty + \dot{g}_0^i \frac{(\Delta x)^2}{k} = \frac{1}{\tau}\left(T_0^{i+1} - T_0^i\right)$$

- Isolating the only unknown T_m^{i+1}:

$$T_0^{i+1} = \left(1 - 2\tau - 2\tau\frac{h\Delta x}{k}\right) T_0^i + (2\tau)T_1^i + 2\tau\frac{h\Delta x}{k}T_\infty + \frac{\tau \dot{g}_0^i (\Delta x)^2}{k}$$

- 1 equation for each boundary node with 1 unknown 😁