

Heat Transfer

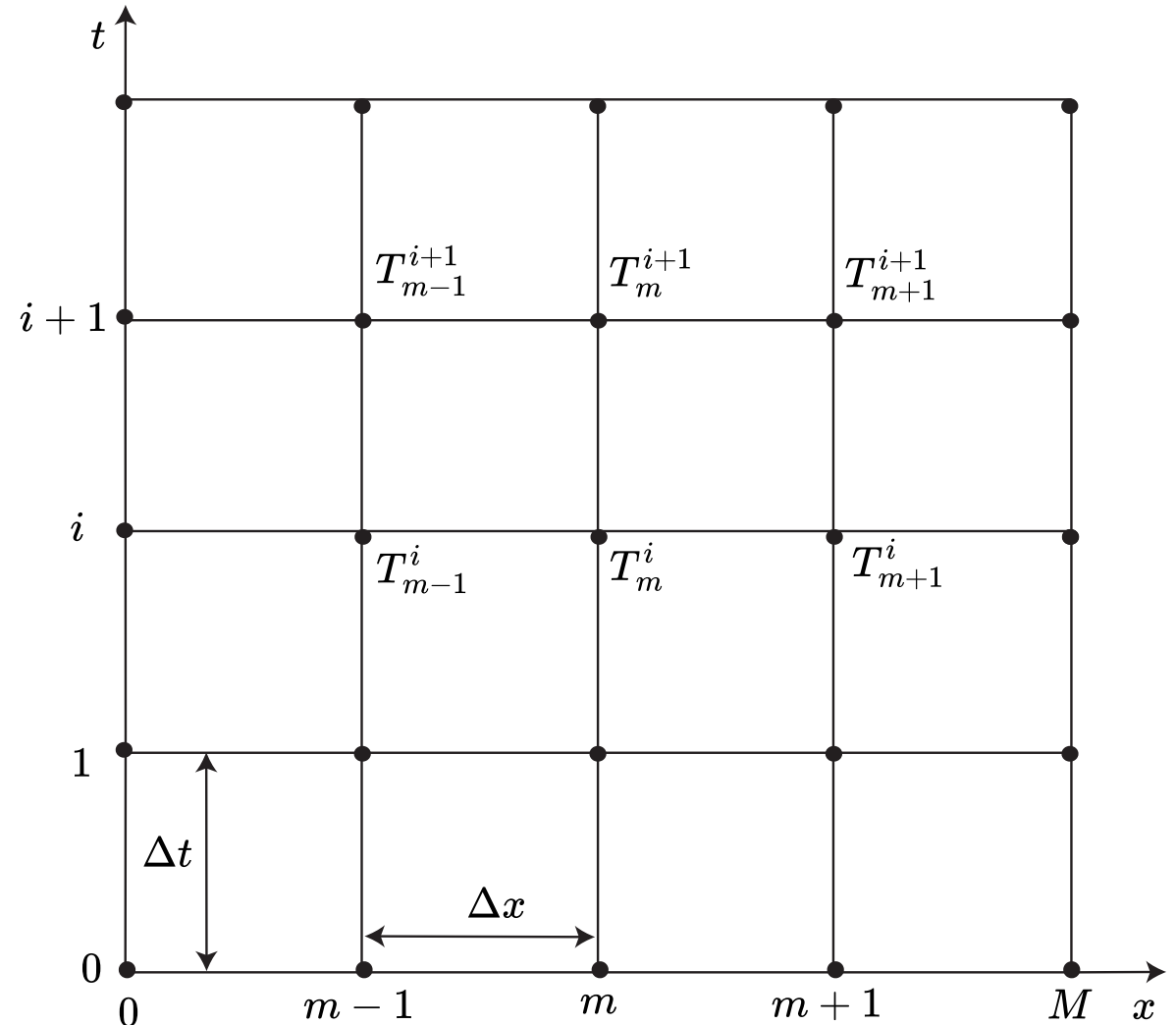
# Transient heat conduction using numerical methods

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# Notation for transient problems

- **On space axis**  $m, n, k$  denote node number in  $x, y, z$ -directions
  - Distance between nodes is  $\Delta x$
- **On time axis**  $i$  denote node number in  $t$ -direction (time)
  - Time between nodes is  $\Delta t$
- General notation:

$$T_{m,n,k}^i$$



# The energy balance equation for transient problems

We sum up heat transfer to/from volume element:

$$\left( \begin{array}{c} \text{Sum of heat} \\ \text{transfer rate} \\ \text{across surfaces} \\ \text{of volume element} \end{array} \right) + \left( \begin{array}{c} \text{Rate of heat} \\ \text{generation} \\ \text{inside the} \\ \text{element} \end{array} \right) = \left( \begin{array}{c} \text{Rate of change} \\ \text{of the energy} \\ \text{content of} \\ \text{the element} \end{array} \right)$$

- For *transient* heat transfer (temperature changes with time):

$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t}$$

$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = mc_p \frac{\Delta T_{\text{element}}}{\Delta t}$$

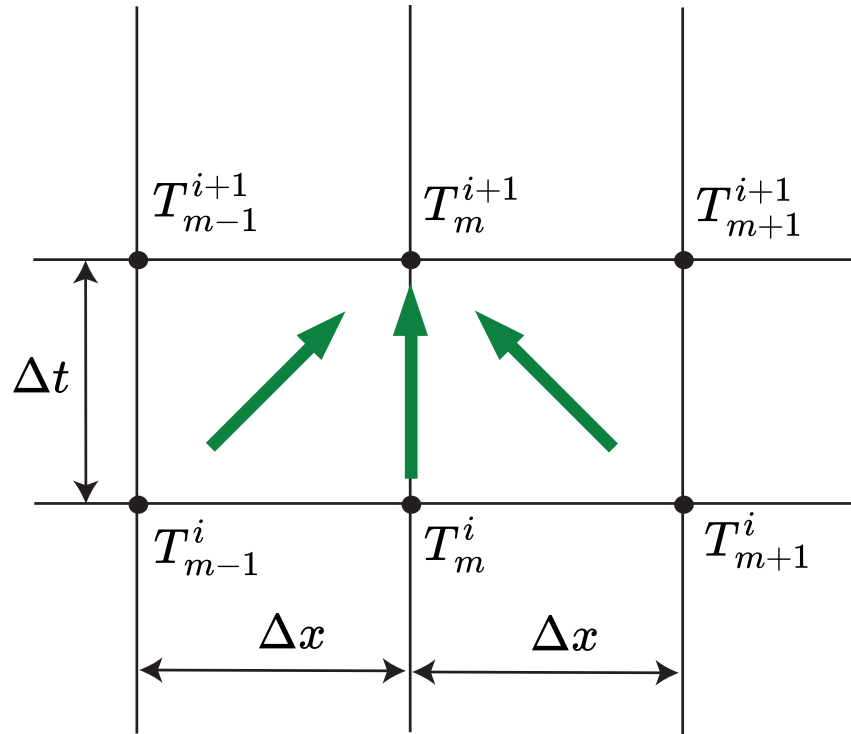
$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = V_{\text{element}} \rho c_p \frac{\Delta T_{\text{element}}}{\Delta t}$$

# Explicit and implicit solution methods

*Question is:* How do we advance from time  $t$  (timestep  $i$ ) to timestep  $t + \Delta t$  (timestep  $i + 1$ )?

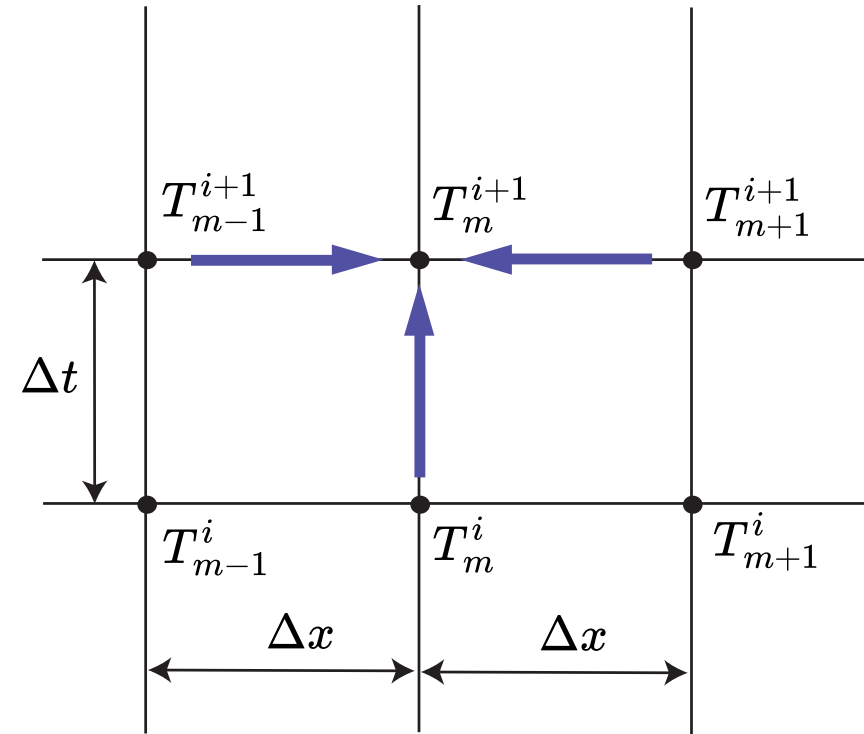
*Explicit* method

$$\sum_{\text{all surfaces}} \dot{Q}^i + \dot{G}_{\text{element}}^i = \frac{\rho V_{\text{element}} c_p (T_m^{i+1} - T_m^i)}{\Delta t}$$



*Implicit* method

$$\sum_{\text{all surfaces}} \dot{Q}^{i+1} + \dot{G}_{\text{element}}^{i+1} = \frac{\rho V_{\text{element}} c_p (T_m^{i+1} - T_m^i)}{\Delta t}$$



*Explicit:* Use data from current time step  $i$  (all known temperatures)

*Implicit:* Use data from next time step  $i + 1$  (unknown temperatures)

# Transient heat conduction using the *explicit method*

## Interior nodes using *explicit* solution method

Energy balance at interior node:

$$kA \frac{T_{m-1}^i - T_m^i}{\Delta x} + kA \frac{T_{m+1}^i - T_m^i}{\Delta x} + \dot{g}_m^i A \Delta x = \frac{\rho V c_p (T_m^{i+1} - T_m^i)}{\Delta t}$$

- Re-arranging and inserting thermal diffusivity  $\alpha = k/(\rho c_p)$ :

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{g}_m^i \Delta x^2}{k} = \frac{\Delta x^2}{\alpha \Delta t} (T_m^{i+1} - T_m^i)$$

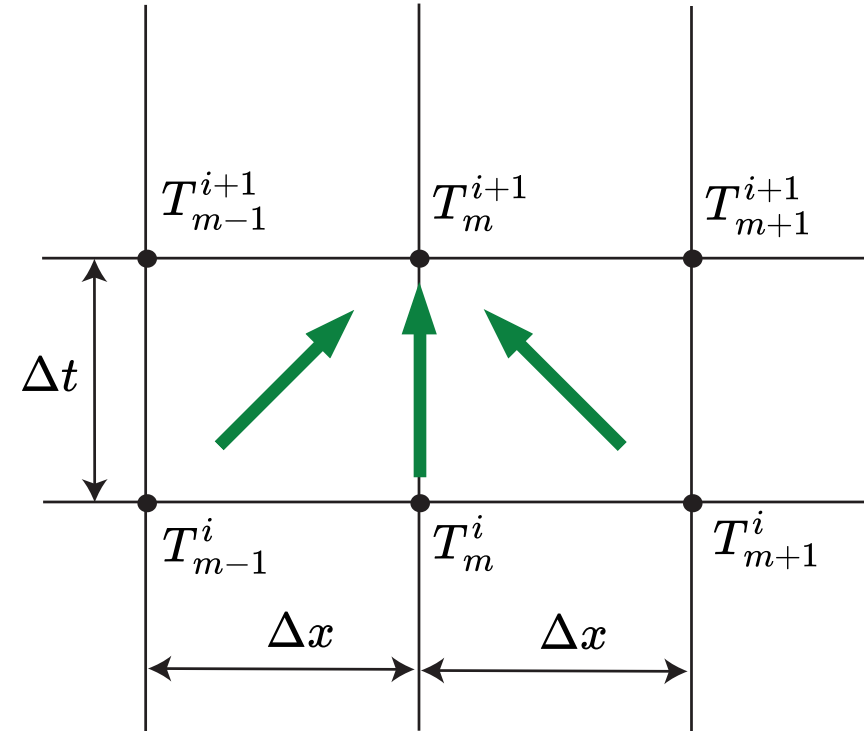
- Inserting mesh Fourier number  $\tau = \alpha \Delta t / \Delta x^2$ :

$$T_{m-1}^i - 2T_m^i + T_{m+1}^i + \frac{\dot{g}_m^i \Delta x^2}{k} = \frac{1}{\tau} (T_m^{i+1} - T_m^i)$$

- Isolating the only unknown  $T_m^{i+1}$ :

$$T_m^{i+1} = \tau (T_{m-1}^i + T_{m+1}^i) + (1 - 2\tau) T_m^i + \tau \frac{\dot{g}_m^i \Delta x^2}{k}$$

- 1 equation for each node with *only 1 unknown each* 😎



# Boundary nodes using *explicit* solution method

Energy balance for boundary node with convection:

$$hA(T_\infty - T_0^i) + kA \frac{T_1^i - T_0^i}{\Delta x} + \dot{g}_0 A \frac{\Delta x}{2} = \frac{\rho V c_p (T_0^{i+1} - T_0^i)}{\Delta t}$$

- Re-arranging and noting thermal diffusivity  $\alpha = k/(\rho c_p)$ :

$$\left( \frac{-2h\Delta x}{k} - 2 \right) T_0^i + 2T_1^i + \left( \frac{-2h\Delta x}{k} \right) T_\infty + \dot{g}_0 \frac{(\Delta x)^2}{k} = \frac{(\Delta x)^2}{\alpha \Delta t} (T_0^{i+1} - T_0^i)$$

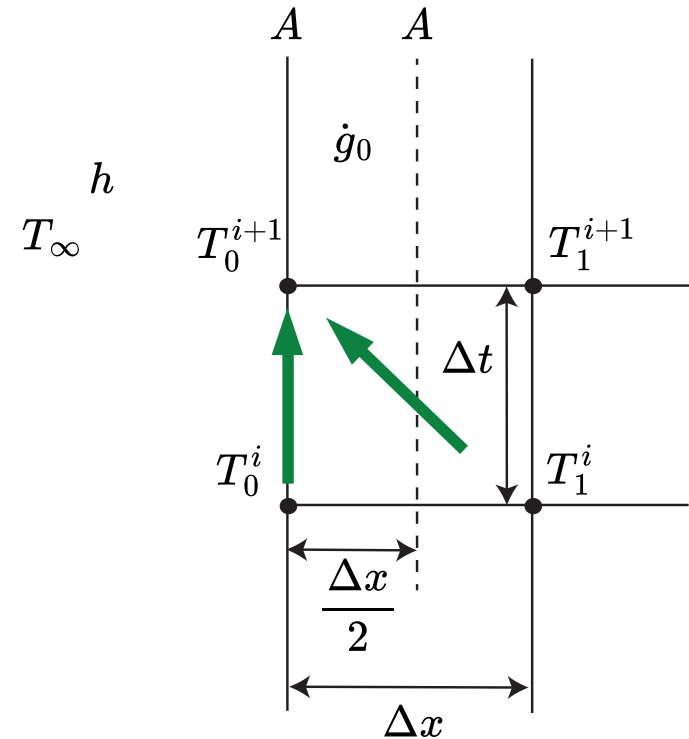
- Defining mesh Fourier number  $\tau = \alpha \Delta t / \Delta x^2$ :

$$\left( \frac{-2h\Delta x}{k} - 2 \right) T_0^i + 2T_1^i + \left( \frac{-2h\Delta x}{k} \right) T_\infty + \dot{g}_0 \frac{(\Delta x)^2}{k} = \frac{1}{\tau} (T_0^{i+1} - T_0^i)$$

- Isolating the only unknown  $T_0^{i+1}$ :

$$T_0^{i+1} = \left( 1 - 2\tau - 2\tau \frac{h\Delta x}{k} \right) T_0^i + (2\tau) T_1^i + 2\tau \frac{h\Delta x}{k} T_\infty + \frac{\tau \dot{g}_0 (\Delta x)^2}{k}$$

- 1 equation for each boundary node with *only 1 unknown* 😎

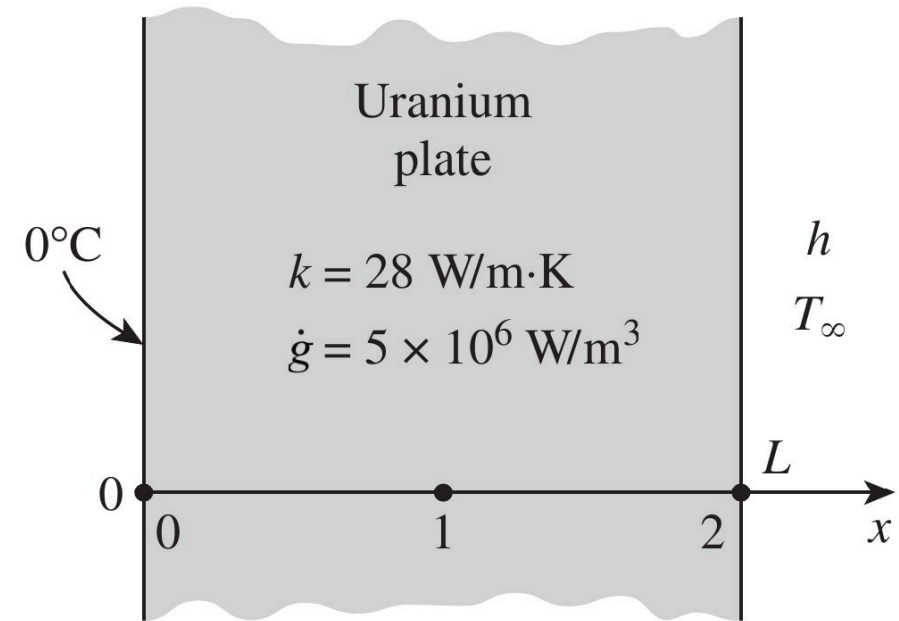


## Example: Transient heat conduction in a large plate using the explicit method

Consider a large uranium plate of thickness  $L = 4$  cm, thermal conductivity  $k = 28$  W/(m·K), and thermal diffusivity  $a = 12.5 \times 10^{-6}$  m<sup>2</sup>/s that is initially at a uniform temperature of 200°C. Heat is generated uniformly in the plate at a constant rate of  $\dot{g} = 5 \times 10^6$  W/m<sup>3</sup>.

At time  $t = 0$  s, one side of the plate is brought into contact with iced water and is maintained at 0°C at all times, while the other side is subjected to convection to an environment at  $T_\infty = 30^\circ\text{C}$  with a heat transfer coefficient of  $h = 45$  W/(m<sup>2</sup>·K).

*Estimate the exposed surface temperature of the plate 2.5 min* after start of cooling. Use three equally spaced nodes in the medium, two at the boundaries and one at the middle.



# The stability criterion for the *explicit* solution method

**Observation:** Explicit solution method may *diverge if time step size is too large!*

**Stability criterion:** When looking at node  $m$ , the *coefficient in front of  $T_m^i$  should be larger than zero*

- Equation for an *interior node* (derived here):

$$T_m^{i+1} = \tau(T_{m-1}^i + T_{m+1}^i) + (1 - 2\tau) T_m^i + \tau \frac{\dot{g}_m^i (\Delta x)^2}{k}$$

- Criterion becomes:*  $1 - 2\tau \geq 0 \rightarrow 1 - 2(\alpha \Delta t / \Delta x^2) \geq 0 \rightarrow \Delta t \leq \frac{(\Delta x)^2}{2\alpha}$

- Equation for a *boundary node with convection* (derived here):

$$T_0^{i+1} = \left(1 - 2\tau - 2\tau \frac{h\Delta x}{k}\right) T_0^i + (2\tau) T_1^i + 2\tau \frac{h\Delta x}{k} T_\infty + \frac{\tau \dot{g}_0^i (\Delta x)^2}{k}$$

- Criterion becomes:*  $\left(1 - 2\tau - 2\tau \frac{h\Delta x}{k}\right) \geq 0 \rightarrow \Delta t \leq \frac{(\Delta x)^2}{2\alpha(1 + \frac{h\Delta x}{k})}$

# Transient heat conduction using the *implicit method*

# Interior nodes using *implicit* solution method

Energy balance at interior node:

$$kA \frac{T_{m-1}^{i+1} - T_m^{i+1}}{\Delta x} + kA \frac{T_{m+1}^{i+1} - T_m^{i+1}}{\Delta x} + \dot{g}_m^{i+1} A \Delta x = \frac{\rho V c_p (T_m^{i+1} - T_m^i)}{\Delta t}$$

- Re-arranging:

$$T_{m-1}^{i+1} - 2T_m^{i+1} + T_{m+1}^{i+1} + \frac{\dot{g}_m^{i+1} \Delta x^2}{k} = \frac{\Delta x^2}{\alpha \Delta t} (T_m^{i+1} - T_m^i)$$

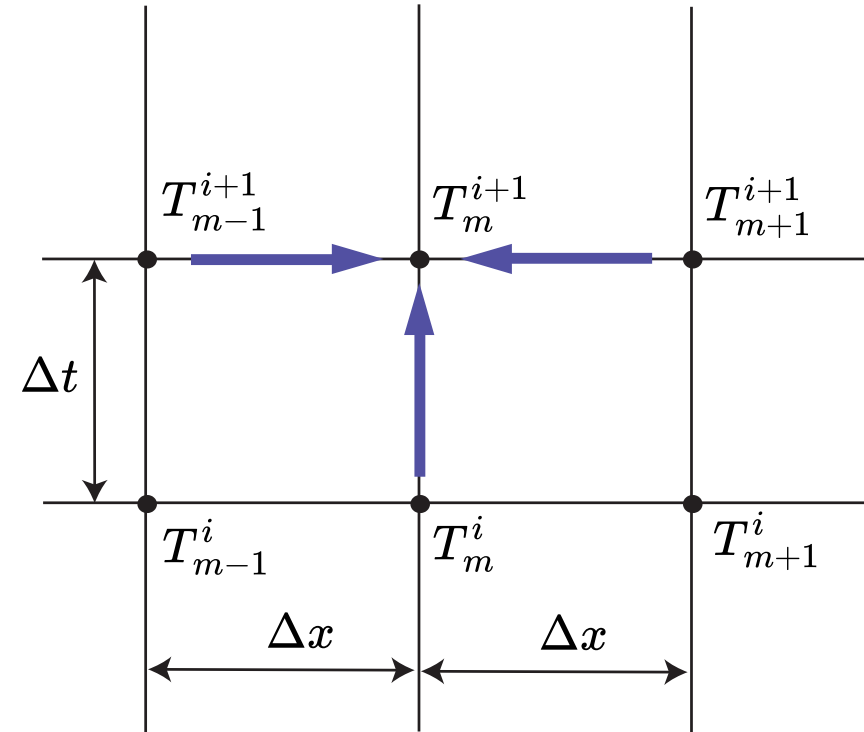
- Using  $\tau = \alpha \Delta t / \Delta x^2$ :

$$T_{m-1}^{i+1} - 2T_m^{i+1} + T_{m+1}^{i+1} + \frac{\dot{g}_m^{i+1} \Delta x^2}{k} = \frac{1}{\tau} (T_m^{i+1} - T_m^i)$$

- Separating the 3 unknowns  $T_{m-1}^{i+1}$ ,  $T_m^{i+1}$  and  $T_{m+1}^{i+1}$ :

$$\tau T_{m-1}^{i+1} - (1 + 2\tau) T_m^{i+1} + \tau T_{m+1}^{i+1} + \tau \frac{\dot{g}_m^{i+1} \Delta x^2}{k} + T_m^i = 0$$

- Can't solve each equation individually *because each equation has three unknowns*,  $T_{m-1}^{i+1}$ ,  $T_m^{i+1}$ , and  $T_{m+1}^{i+1}$  😞



## Boundary nodes using *implicit* solution method

Energy balance at boundary node with convection:

$$hA(T_\infty - T_0^{i+1}) + kA \frac{T_1^{i+1} - T_0^{i+1}}{\Delta x} + \dot{g}_m^{i+1} A \frac{\Delta x}{2} = \frac{\rho V c_p (T_0^{i+1} - T_0^i)}{\Delta t}$$

- Re-arranging:

$$\left(\frac{-2h\Delta x}{k} - 2\right) T_0^{i+1} + 2T_1^{i+1} + \left(\frac{2h\Delta x}{k}\right) T_\infty + \dot{g}_0^i \frac{(\Delta x)^2}{k} = \frac{(\Delta x)^2}{\alpha \Delta t} (T_0^{i+1} - T_0^i)$$

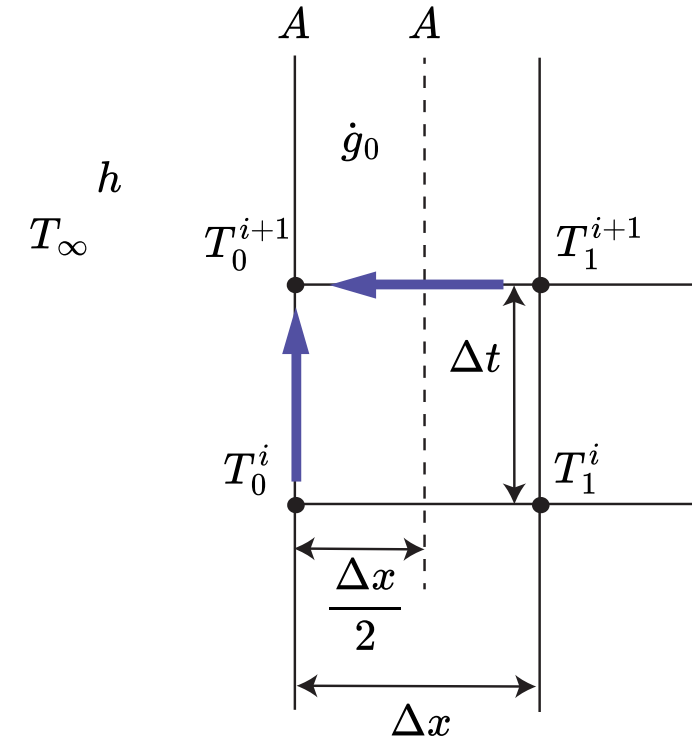
- Defining  $\tau = \alpha \Delta t / \Delta x^2$ :

$$\left(\frac{-2h\Delta x}{k} - 2\right) T_0^{i+1} + 2T_1^{i+1} + \left(\frac{2h\Delta x}{k}\right) T_\infty + \dot{g}_0^i \frac{(\Delta x)^2}{k} = \frac{1}{\tau} (T_0^{i+1} - T_0^i)$$

- Separating the 2 unknowns  $T_0^{i+1}$  and  $T_1^{i+1}$ :

$$\left(-1 - 2\tau - 2\tau \frac{h\Delta x}{k}\right) T_0^{i+1} + (2\tau) T_1^{i+1} + 2\tau \frac{h\Delta x}{k} T_\infty + \tau \frac{\dot{g}_0^{i+1} (\Delta x)^2}{k} + T_0^i = 0$$

- Can't solve each equation individually *because each equation has two unknowns*,  $T_0^{i+1}$  and  $T_1^{i+1}$  😞



## Example: Transient heat conduction in a large plate using the implicit method

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