Lumped system assumption

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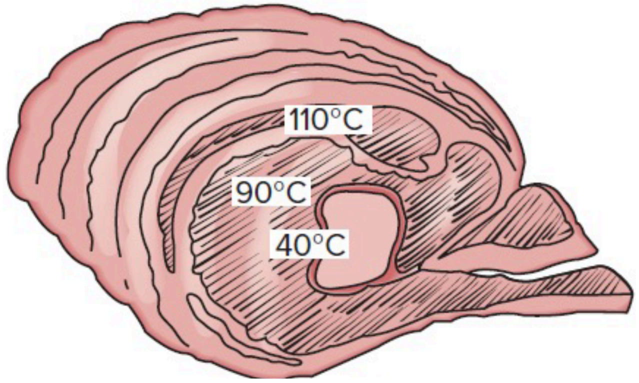
What is the lumped system assumption and why?

Most heat transfer problems are complex because temperature *varies with both space and time*

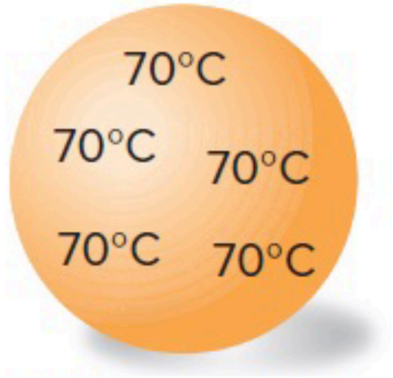
- In **lumped systems**, temperature *only varies with time (not space)*

$$T = f(x, y, z, t) \quad \rightarrow \quad T = f(t)$$

- Significantly *simplifies* the analysis!



A roast beef (not lumped system)



A copper ball (lumped system)

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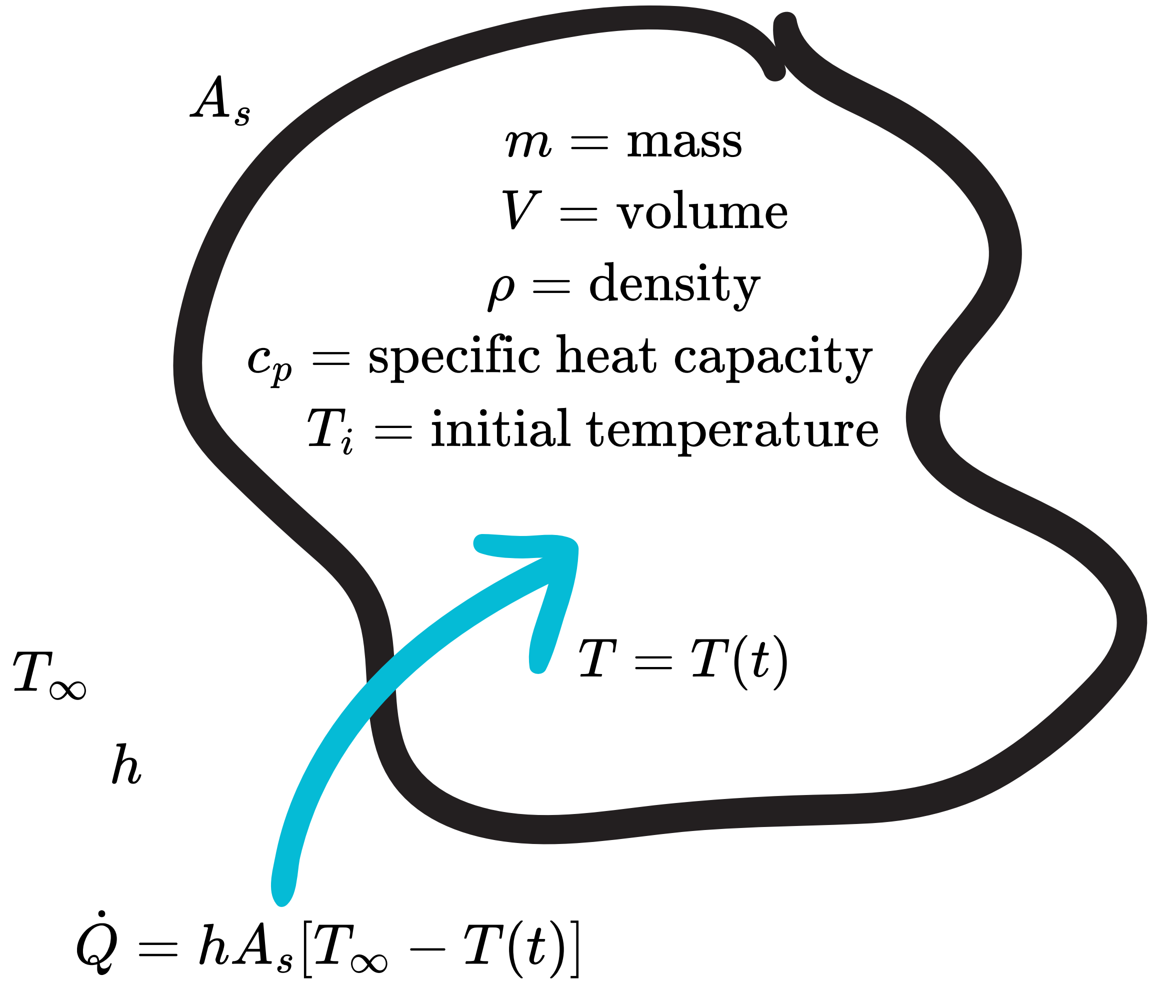
Starting point of lumped system analysis

Consider a body of arbitrary shape with

- At time $t = 0$ the body is exposed to convective heat transfer from the outside:
  - Surrounding temperature T_∞
  - Heat transfer coefficient h
  - Surface area A_s



Derivation of the lumped system equation

Energy transferred to body during $\mathrm{d}t$:

$$\dot{Q} = hA_s(T_\infty - T)\mathrm{d}t = mc_p\mathrm{d}T = \rho V c_p \mathrm{d}T$$

Because T_∞ is constant, we may expand $\mathrm{d}T$:

$$\dot{Q} = hA_s(T_\infty - T)\mathrm{d}t = \rho V c_p \mathrm{d}(T - T_\infty)$$

..Re-arranging:

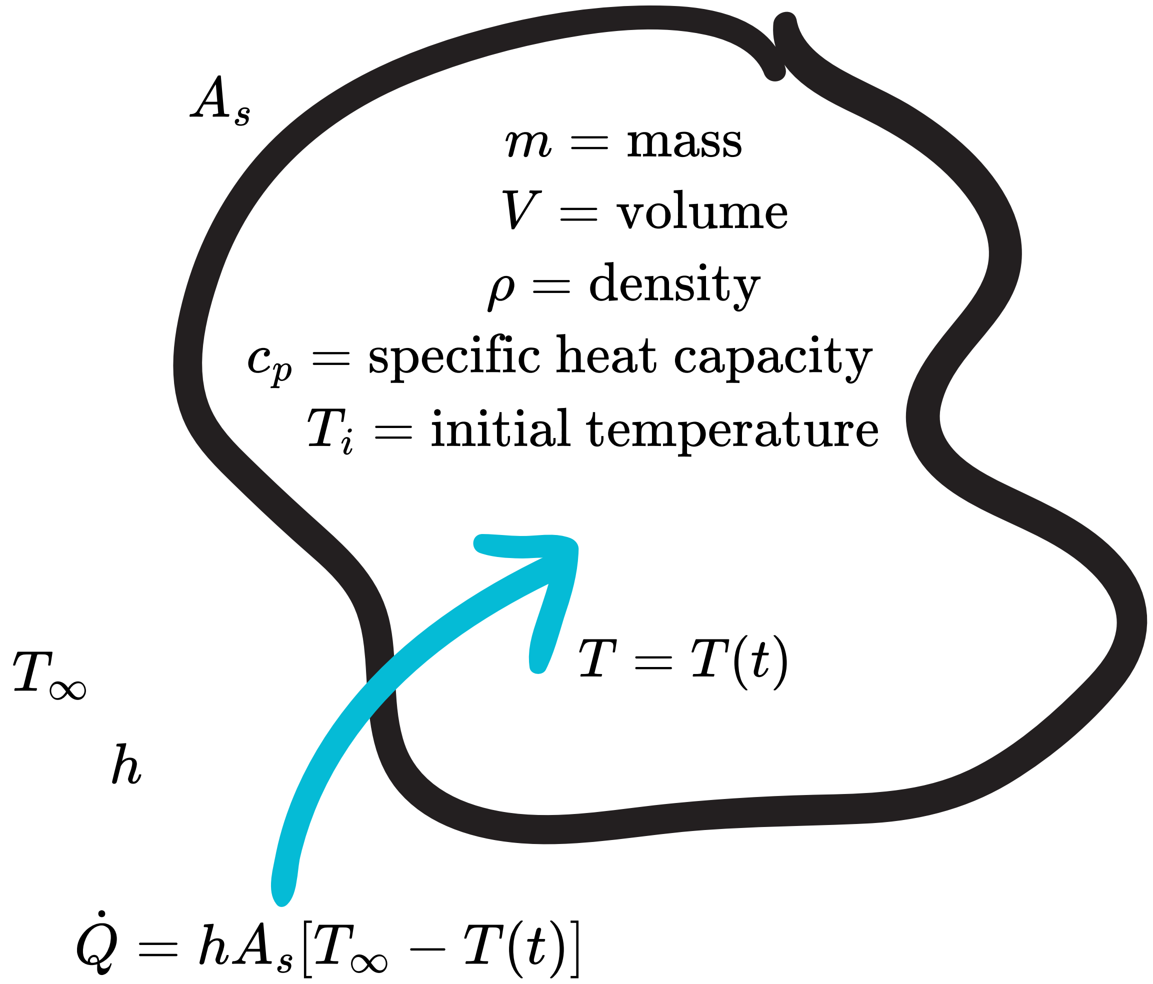
$$\frac{\mathrm{d}(T - T_\infty)}{T - T_\infty} = -\frac{hA_s}{\rho V c_p}\mathrm{d}t$$

Now, intergrate from $t = 0$ where $T = T_i$ to t at which $T = T(t)$:

$$\ln\frac{T(t) - T_\infty}{T_i - T_\infty} = -\frac{hA_s}{\rho V c_p}t$$

Exponential on both sides gives **lumped system equation**:

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = \mathrm{e}^{-\frac{hA_s}{\rho V c_p}t}$$



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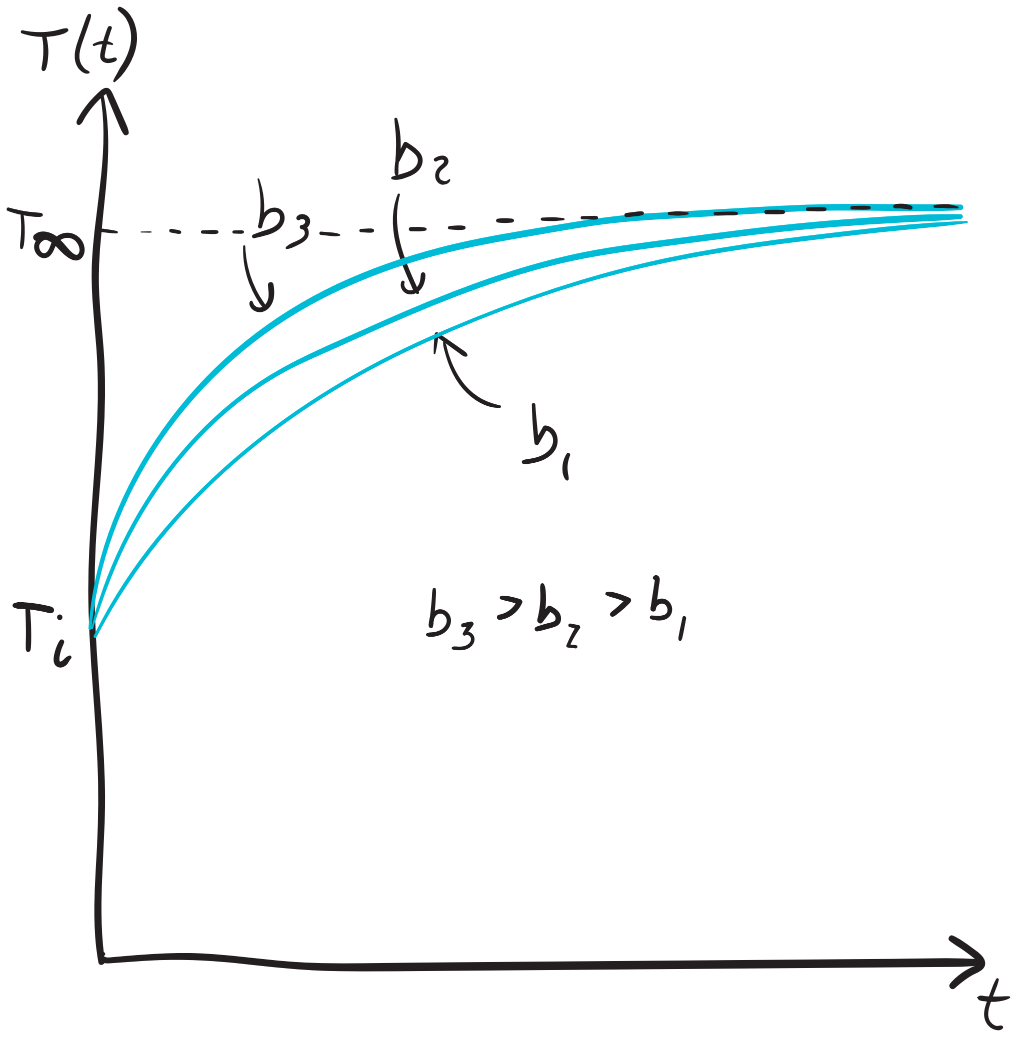
The *time constant* for lumped systems

Introducing $b = hA_s/(\rho V c_p)$, we obtain:

$$\frac{T(t) - T_\infty}{T_i - T_\infty} = \mathrm{e}^{-\frac{hA_s}{\rho V c_p} t} = \mathrm{e}^{-bt}$$

The *time constant* $(1/b)$ describes the rate at which the system approaches the surrounding temperature T_∞

- *Large* b: Temperature approaches T_∞ quickly
- *Small* b: Temperature approaches T_∞ slowly



Validity of lumped system assumption

We define a *Biot number*

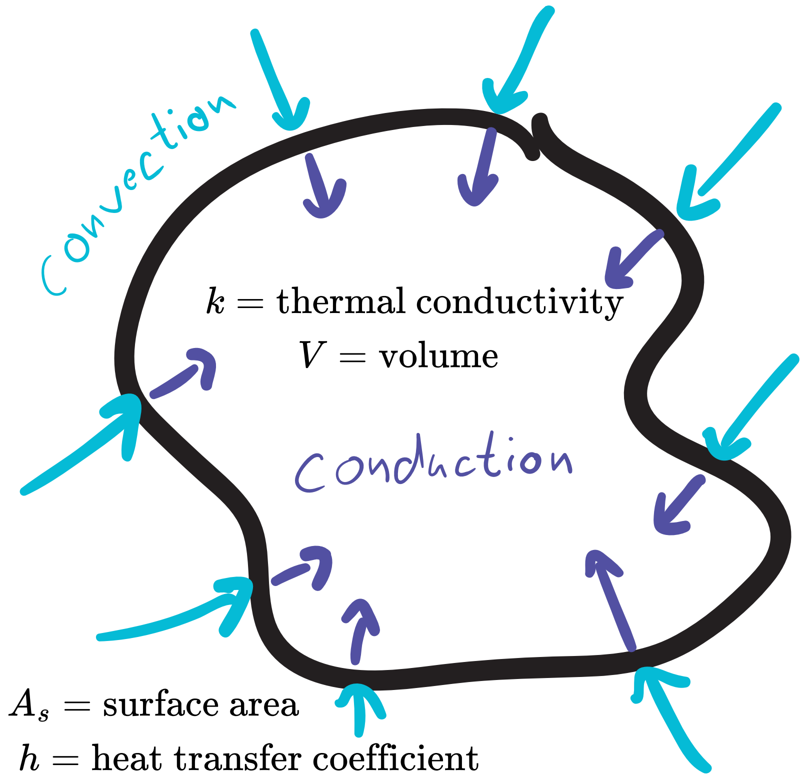
$$\mathrm{Bi} = hL_c/k$$

- with the characteristic length $L_c = V/A_s$

If $\mathrm{Bi} = 0$: lumped system assumption is **exact**

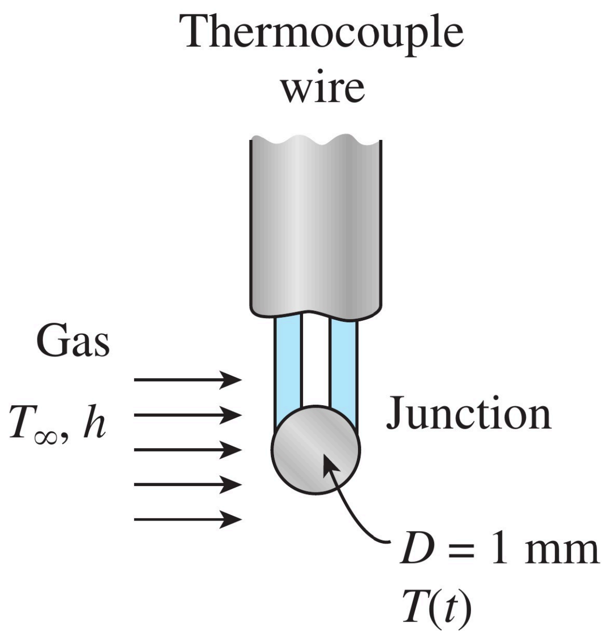
If $\mathrm{Bi} \leq 0.1$: lumped system assumption is **resonable accurate**

If $\mathrm{Bi} > 0.1$: lumped system assumption is **inaccurate**



Example: Fast temperature measurements with thermocouples

The temperature of a gas stream is to be measured by a thermocouple. The thermocouple junction can be approximated as a 1 mm diameter sphere. The junction properties are $k = 35$ W/m·K, $\rho = 8500$ kg/m^3 and $c_p = 320$ J/kg·K.
How quickly can the thermocouple respond to changes in temperature when the convective heat transfer coefficient between the junction and the gas is $h = 210$ W/m^2·K?



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