Heat Transfer

Steady heat conduction using numerical methods

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Why look at *numerical methods for heat conduction*?

Why should I learn about numerical methods for heat conduction?

Learning about numerical methods for heat conduction is valuable for several reasons—both practical and conceptual:

✅ Real-world heat problems rarely have simple equations

Analytical solutions (like the ones from textbooks) only work for idealized cases—simple shapes, steady-state conditions, constant properties, etc.
In reality, heat conduction problems often involve:

- Complex geometries
- Time-dependent behavior (transient heat transfer)
- Non-uniform materials (like composites)
- Changing boundary conditions
- Phase changes (melting/freezing)

Numerical methods let you solve these realistic engineering problems.

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✅ It builds engineering intuition

Knowing the numerical foundations helps you:

- Understand stability, convergence, and error
- Choose correct grid sizes and time steps
- Spot unphysical results (like negative temperatures)
- Validate results instead of blindly trusting software

✅ It improves general problem-solving & programming skills

Heat conduction numerical methods teach:

- Finite difference / finite volume / finite element methods
- Linear algebra and matrix solvers
- Differential equations
- Algorithm design and coding (Python, MATLAB, C++, etc.)

These skills transfer to fluid mechanics, diffusion, vibrations, finance—even machine learning.

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✅ Industry relies on computational heat transfer

Industries where numerical heat conduction is essential:

Field	Why it matters
Aerospace	thermal protection systems, cooling electronics in aircraft/spacecraft
Automotive	engine and battery cooling, cabin systems
Electronics	chip thermal management
Energy systems	heat exchangers, nuclear reactor walls, geothermal systems
Manufacturing	welding, casting, 3D printing thermal analysis
Materials science	conduction in composites, metals, insulators

Tools like **ANSYS, COMSOL, Abaqus, OpenFOAM** use numerical methods internally. Understanding them helps you use these tools intelligently and avoid mistakes.

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Two approaches to solve heat conduction problems

Finite difference method

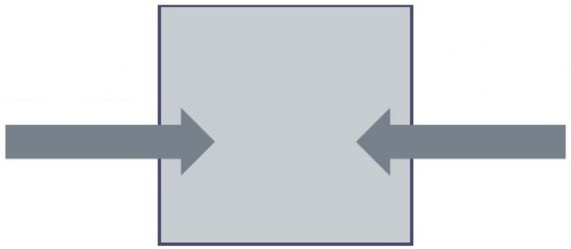
- Use equation for steady one-dimensional heat conduction as a starting point:

$$\frac{\mathrm{d}^2T}{\mathrm{d}x^2} + \frac{\dot{g}}{k} = 0$$

- .. now replace $\frac{\mathrm{d}^2T}{\mathrm{d}x^2}$ with *finite difference approximations* to form algebraic equations

Energy balance method

- Consider a small control volume element



- .. now write an *energy balance* on this volume element

$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = 0$$

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Finite difference method for 1D problems

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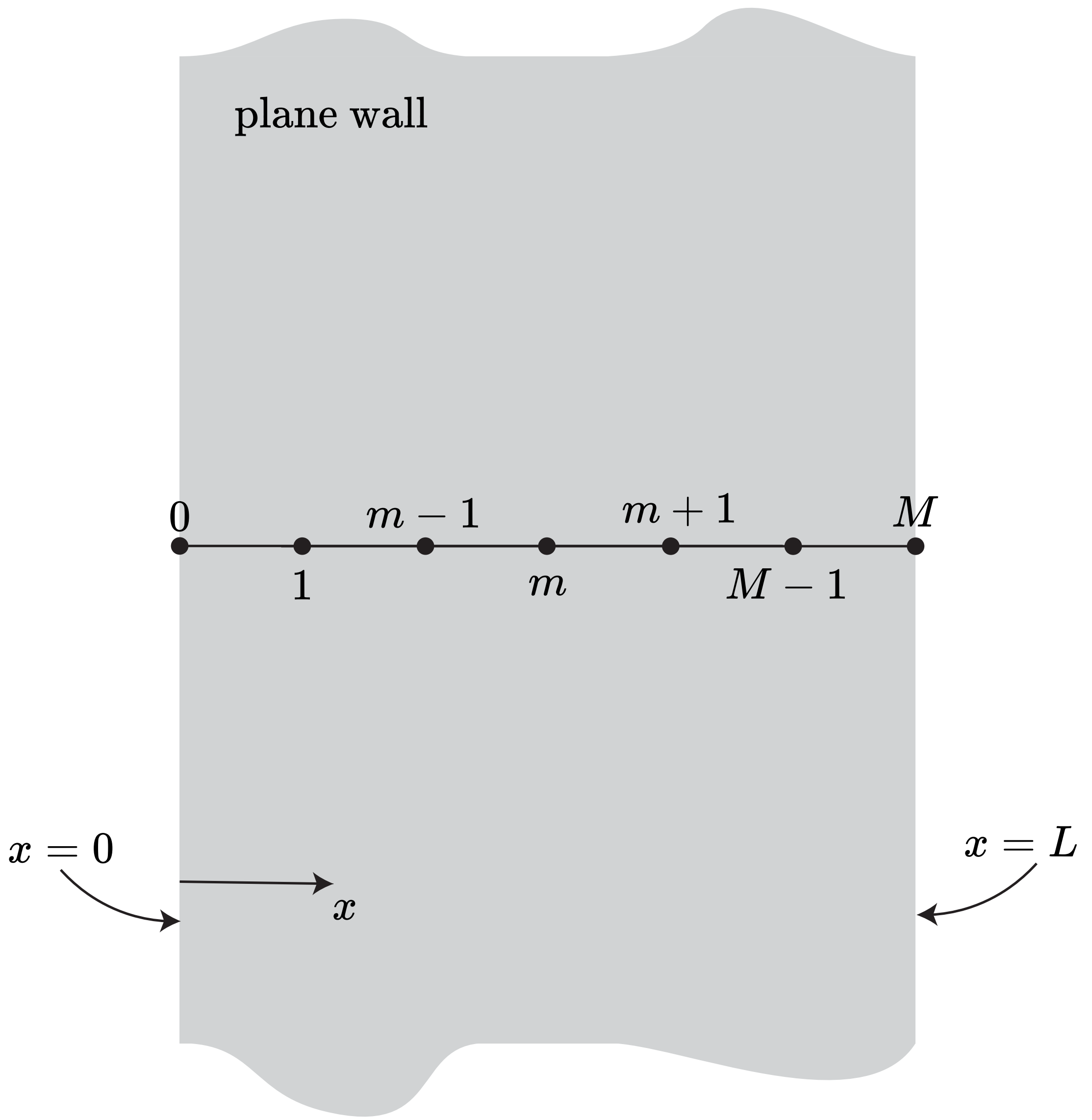
Notation for *1D finite difference* problems

A *1D plane wall* is defined as being infinitely long (i.e. heat transfer only occurs in one direction)

- *Position* is denoted by x
  - $x = 0$ at left side of wall
  - $x = L$ at right side of wall
- *Node number* is denoted by m
  - $m = 1$ at left side of wall
  - $m = M$ at right side of wall

- Notation:
  - T_5: Temperature at $m = 5$
  - ρ_5: Density at node $m = 5$
  - ...



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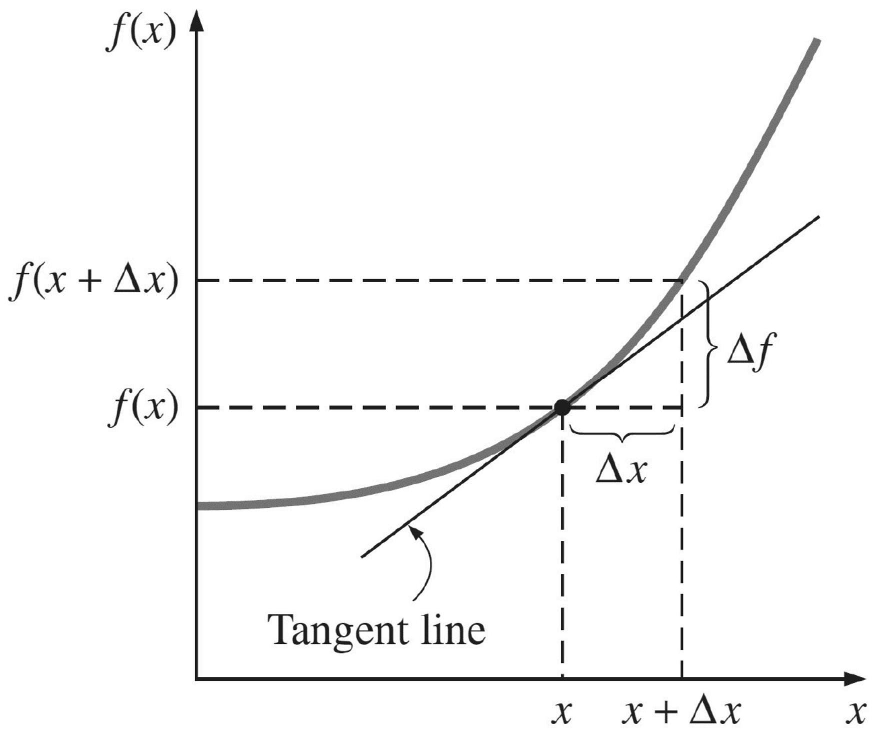
Recalling finite differences from old days 🙂

Consider a function f that depends on x. The first derivative is then *defined* as the slope of the tangent:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$$

- Leaving out the limit, we get an *approximate* solution:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} \approx \frac{f(x+\Delta x)-f(x)}{\Delta x}$$



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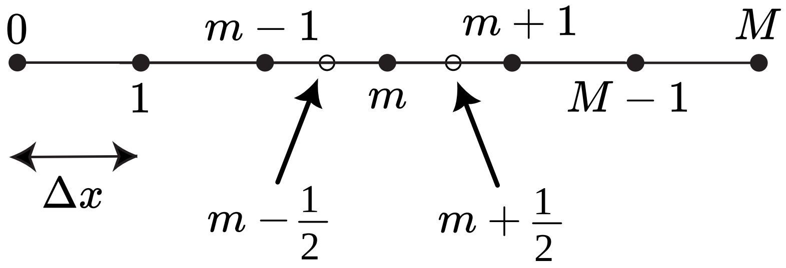
Applying finite differences to our 1D domain

Remember, we are looking for:

$$\frac{\partial^2 T}{\partial x^2}$$



We may approximate first derivative *at midpoints* as:

$$\left.\frac{\partial T}{\partial x}\right|_{x_{m-1/2}} \approx \frac{T_m - T_{m-1}}{\Delta x} \quad \text{and} \quad \left.\frac{\partial T}{\partial x}\right|_{x_{m+1/2}} \approx \frac{T_{m+1} - T_m}{\Delta x}$$

... now, we can approximate the second derivative using two first-order derivatives:

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{x_m} \approx \frac{\left.\frac{\partial T}{\partial x}\right|_{x_{m+1/2}} - \left.\frac{\partial T}{\partial x}\right|_{x_{m-1/2}}}{\Delta x} = \frac{\frac{T_{m+1}-T_m}{\Delta x} - \frac{T_m - T_{m-1}}{\Delta x}}{\Delta x}$$

... which reduces to:

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{x_m} \approx \frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2}$$

The equation for *steady one-dimensional heat conduction* now becomes:

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{g}}{k} = 0$$

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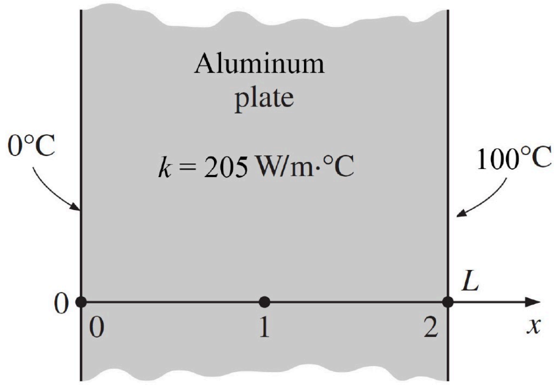
Exercise: Centre temperature in a plate by the finite difference method

Consider a aluminium plate of length $L = 0.5$ cm and thermal conductivity $k = 205$ W/(m·K) in which no heat is generated ($\dot{g} = 0$). One side of the plate is maintained at 0 °C by icy water while the other is maintained at 100 °C. Estimate the temperature in the centre at node 1.



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Energy balance method for 1D problems

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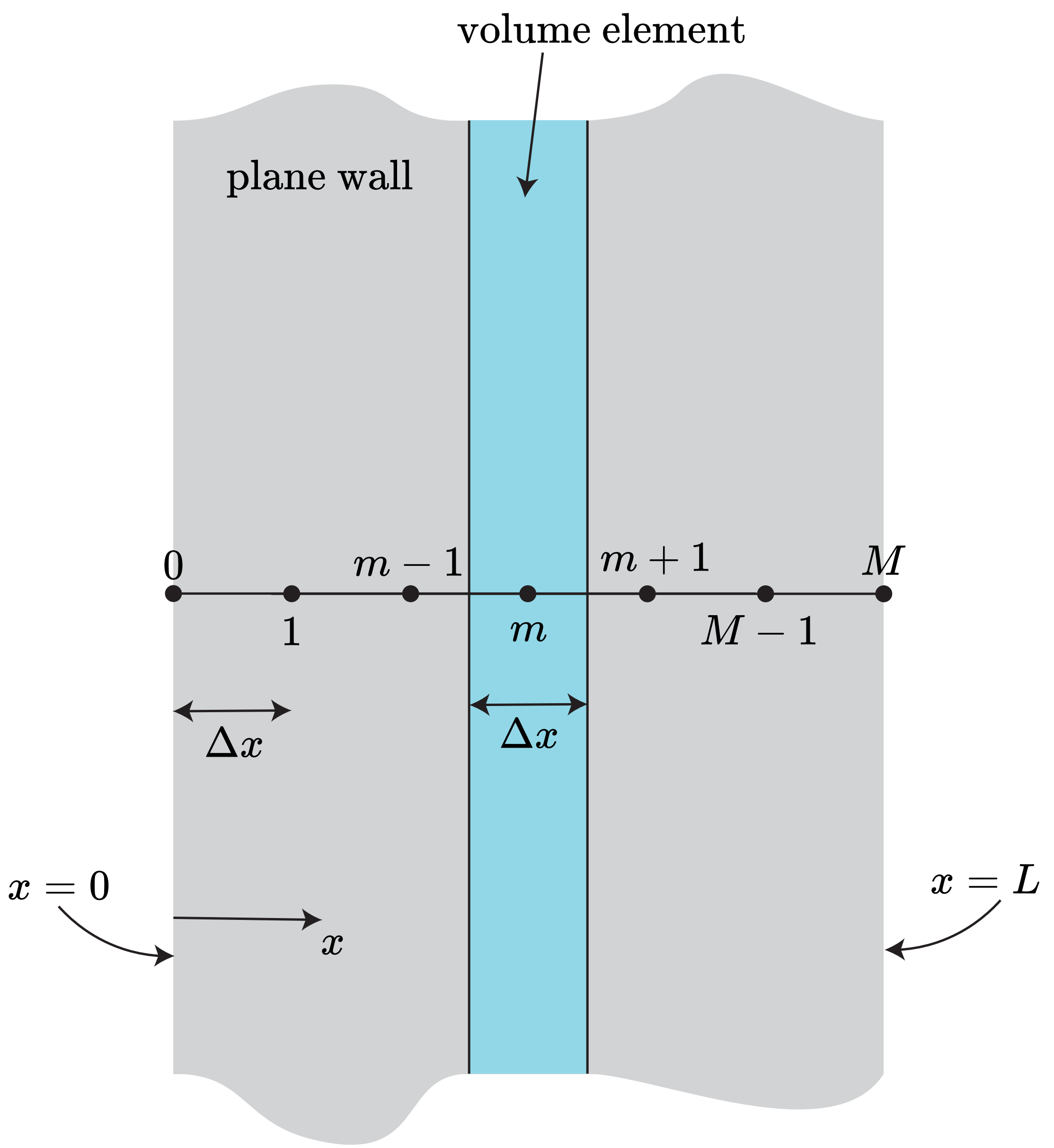
Notation for *1D energy balance* problems

A *1D plane wall* is defined as being infinitely long (i.e. heat transfer only occurs in one direction)

- *Position* is denoted by x
  - $x = 0$ at left side of wall
  - $x = L$ at right side of wall
- *Node number* is denoted by m
  - $m = 1$ at left side of wall
  - $m = M$ at right side of wall
- *Volume elements* with *width* Δx
  - Properties are assumed constant within each volume element
  - The finer the mesh (smaller Δx), the more accurate the solution
- Notation:
  - T_5: Temperature at $m = 5$
  - ρ_5: Density at node $m = 5$
  - ...



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The energy balance equation for interior nodes

We *sum up heat transfer to/from* volume element:

$$\left(\begin{array}{c}\text{Sum of heat}\\ \text{transfer rate}\\ \text{across surfaces}\\ \text{of volume element}\end{array}\right) + \left(\begin{array}{c}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{array}\right) = \left(\begin{array}{c}\text{Rate of change}\\ \text{of the energy}\\ \text{content of}\\ \text{the element}\end{array}\right)$$

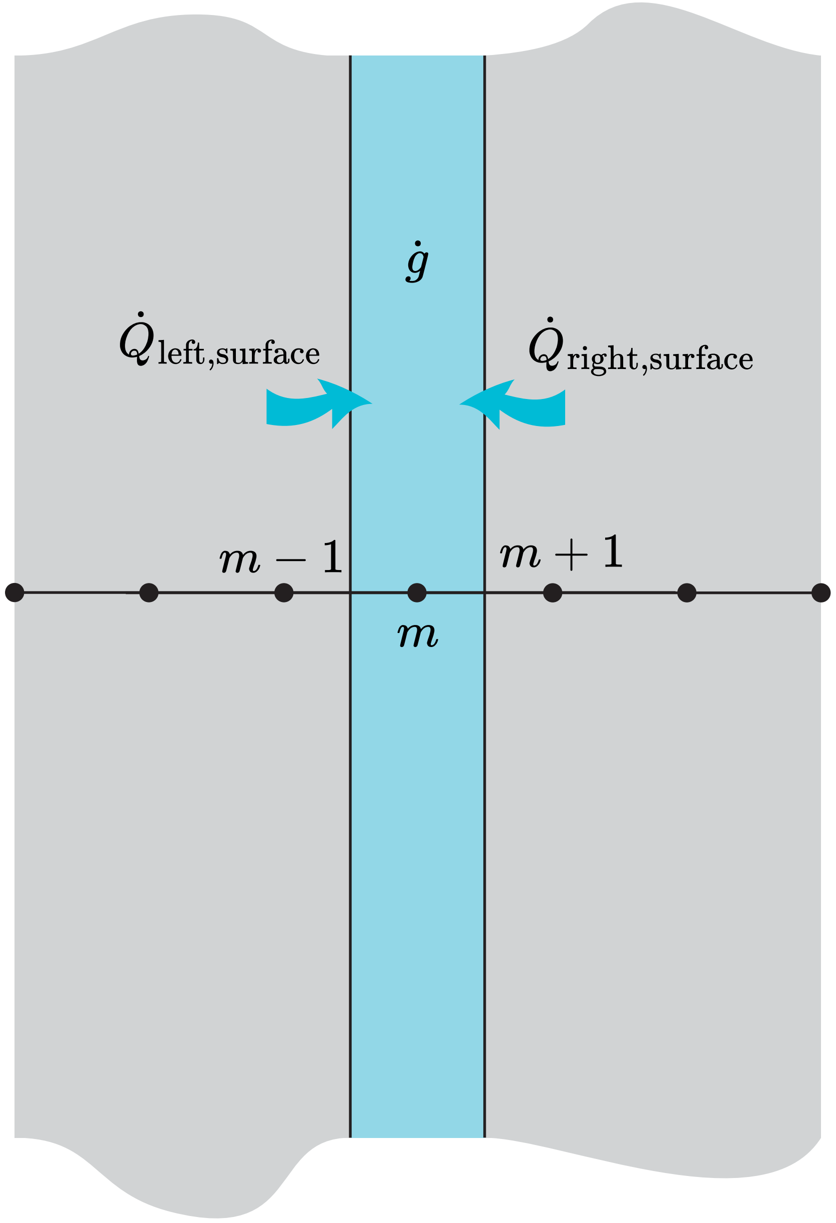
- For *steady* heat transfer (no change in time):

$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = 0$$

- $\dot{G}_{\text{element}}$ is the heat generation rate in the element (W):

$$\dot{G}_{\text{element}} = \dot{g} V_{\text{element}}$$

  - $\dot{g}$ is the volumetric heat generation rate in the element $(\mathrm{W/m^3})$



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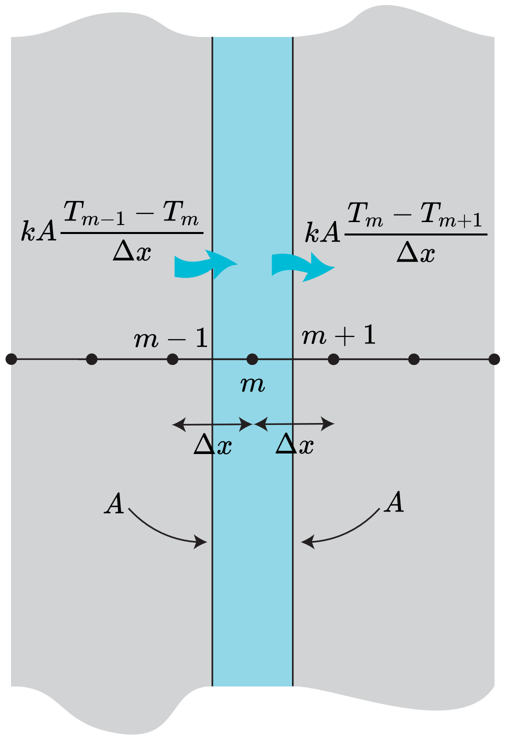
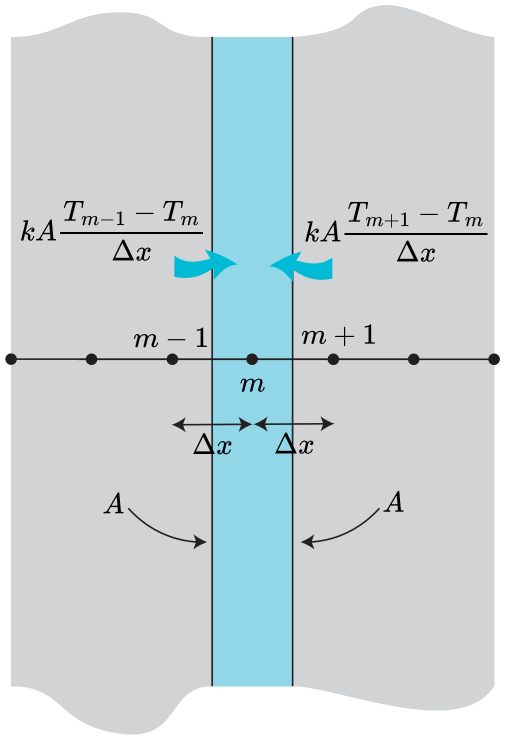
Convention for heat transfer direction

Assumed direction of heat transfer is *irrelevant as long as we are consistent*





Energy balance on control volume:

$$kA\frac{T_{m-1}-T_m}{\Delta x} - kA\frac{T_m-T_{m+1}}{\Delta x} + \dot{g}_m A\Delta x = 0$$

Energy balance on control volume:

$$kA\frac{T_{m-1}-T_m}{\Delta x} + kA\frac{T_{m+1}-T_m}{\Delta x} + \dot{g}_m A\Delta x = 0$$

Conclusion: In both cases, it reduces to *the same expression*:

$$T_{m-1} - 2T_m + T_{m+1} + \frac{\dot{g}_m \Delta x^2}{kA} = 0$$

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The four boundary condition types

We have to specify **boundary conditions** on each non-internal node.

- *Four common* boundary condition types:

1. *Constant temperature*

$T_0 = c$

2. *Constant heat flux*

$$\dot{Q}_{\text{left,surface}} = \dot{q}A = c$$

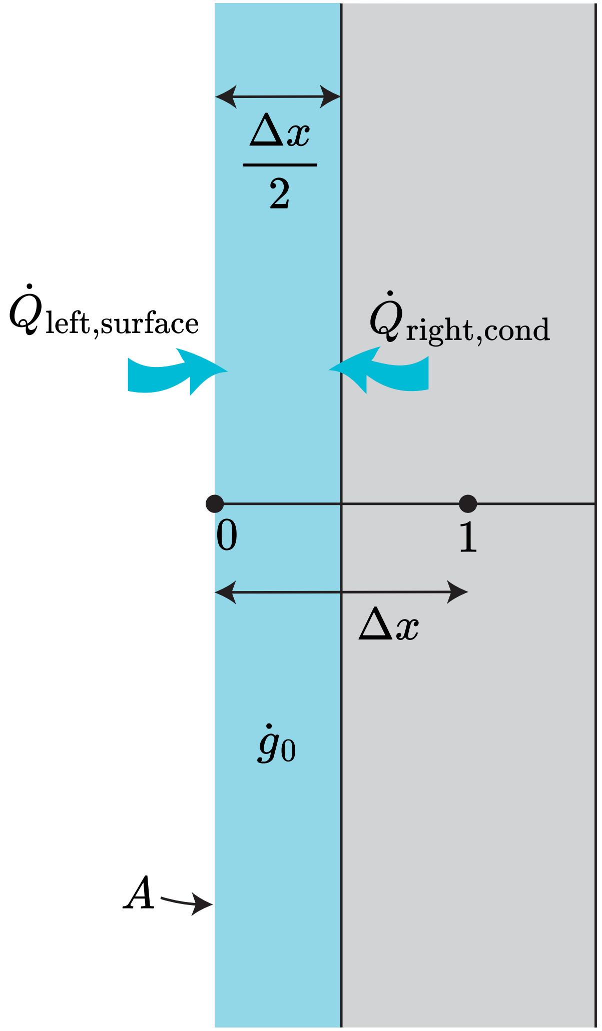
3. *Convection*

$$\dot{Q}_{\text{left,surface}} = hA\left(T_\infty - T_0\right) = c$$

4. *Radiation*

$$\dot{Q}_{\text{left,surface}} = \epsilon\sigma A\left(T_{\text{surr}}^4 - T_0^4\right) = c$$

- .. or any combination of type 2, 3 and 4



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Incorporating the different boundary conditions

Type 1 (constant temperature):

- We simply *set the temperature* of the node, e.g. $T_0 = c$

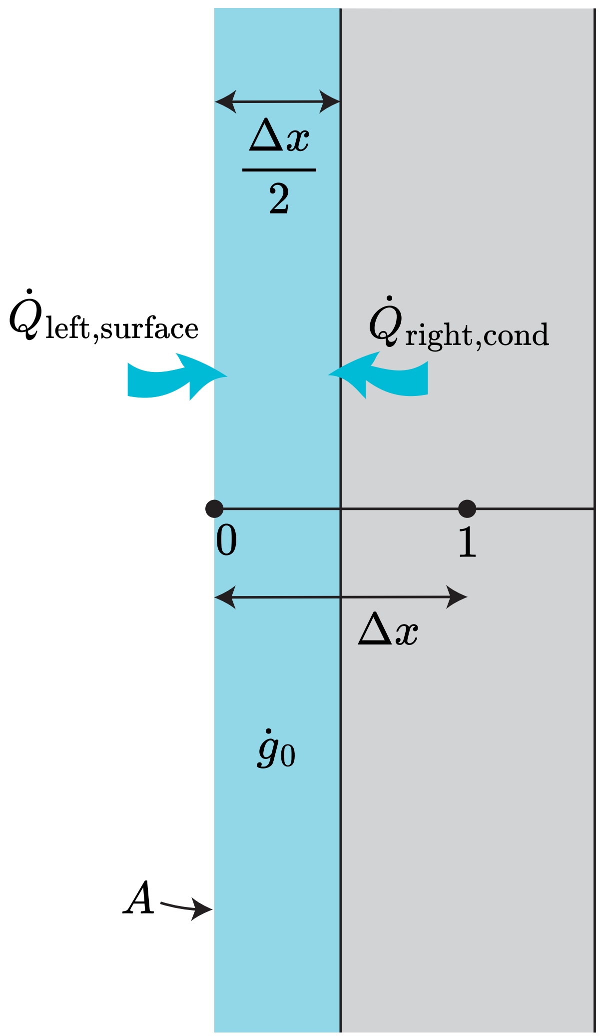
Type 2-4:

- We apply an *energy balance* on the volume element:

$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = 0,$$

$$\dot{Q}_{\text{left,surface}} + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0 \left(\frac{\Delta x}{2} A\right) = 0.$$

- Expression for $\dot{Q}_{\text{left,surface}}$ depends on boundary condition type type



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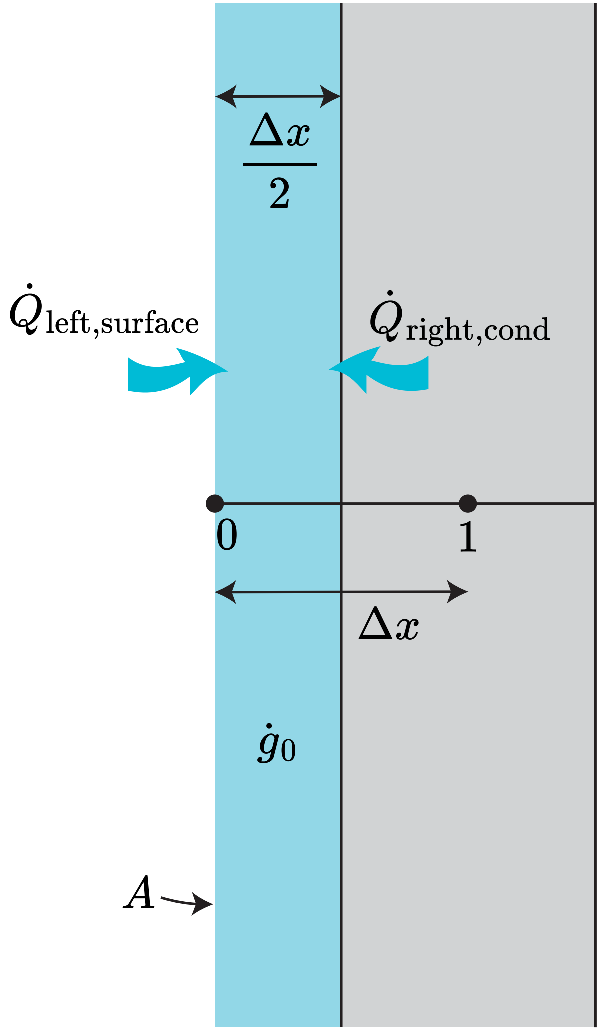
Incorporating mixed boundary condition types

All BC types (**heat flux**, convection and radiation) can easily be added to form mixed types:

$$\dot{Q}_{\text{left,surface}} = \dot{q}_0 A \; + \; hA\left(T_\infty - T_0\right) \; + \; \epsilon\sigma A\left(T_{\text{surr}}^4 - T_0^4\right)$$



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Mirror concept for insulated boundaries (e.g zero heat flux)

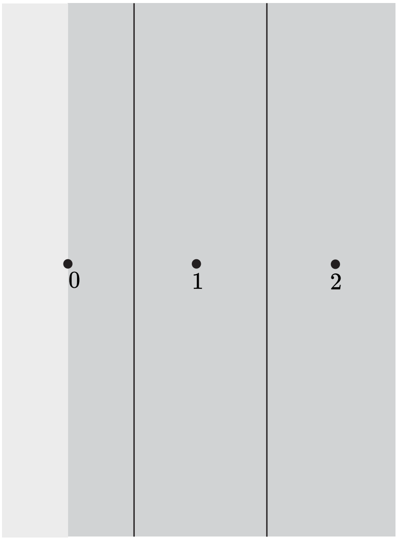
Insulated boundary (zero heat flux) can be modelled using *mirror nodes*:

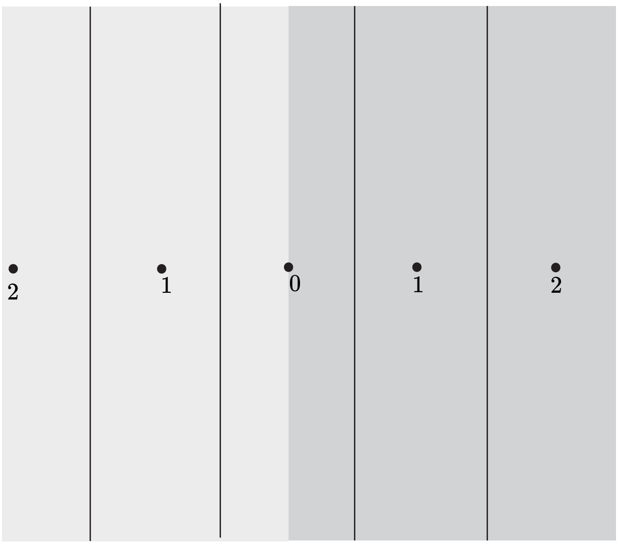
- For interior nodes:

$$\frac{T_{m+1} - 2T_m + T_{m-1}}{\Delta x^2} + \frac{\dot{g}}{k} = 0$$

- For insulated boundaries:

$$\frac{T_1 - 2T_0 + T_1}{\Delta x^2} + \frac{\dot{g}}{k} = 0$$





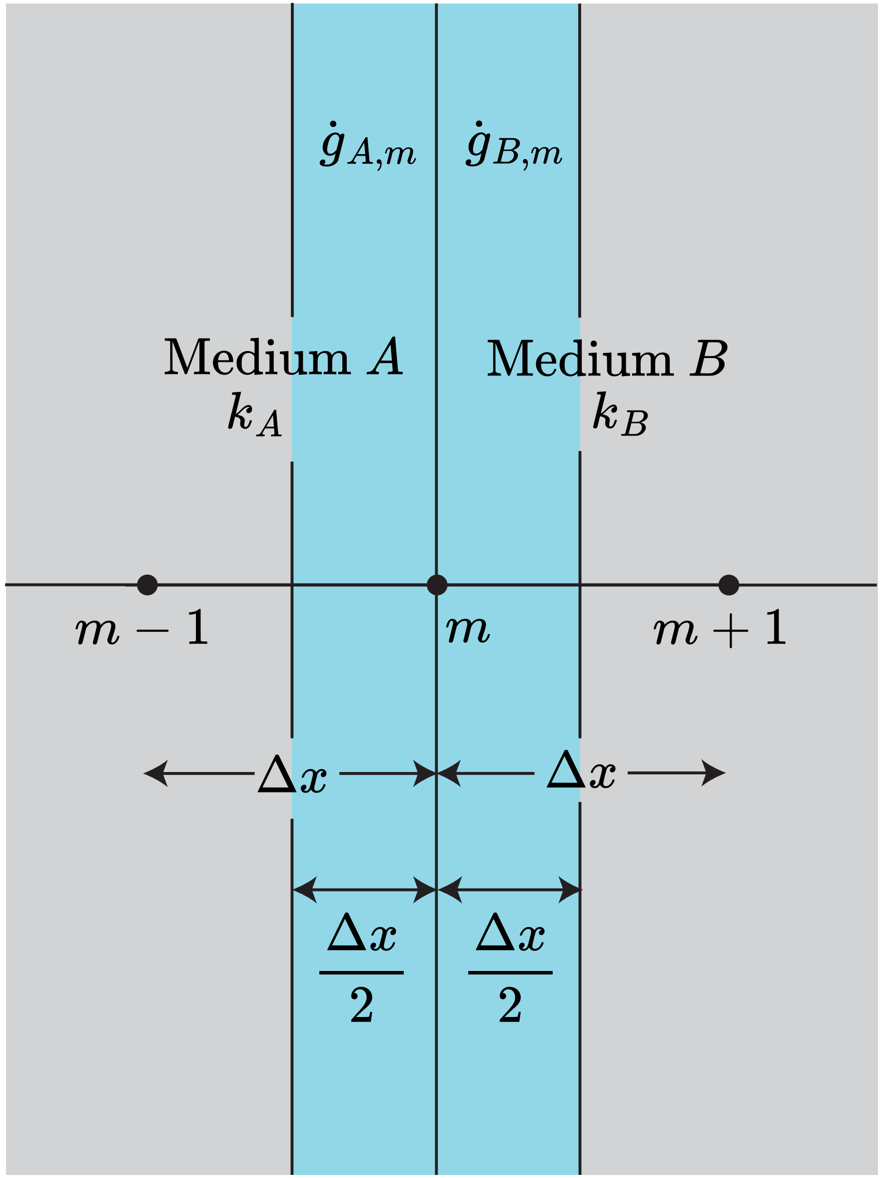
Interfaces between different materials

Starting with the general energy balance:

$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = 0,$$

- Assuming *perfect contact (no air gaps)* at the interface, the energy balance becomes:

$$k_1 A \frac{T_{m-1} - T_m}{\Delta x_1} + k_2 A \frac{T_{m+1} - T_m}{\Delta x_2} + \dot{g}_A A \Delta \frac{x}{2} + \dot{g}_B A \Delta \frac{x}{2} = 0$$



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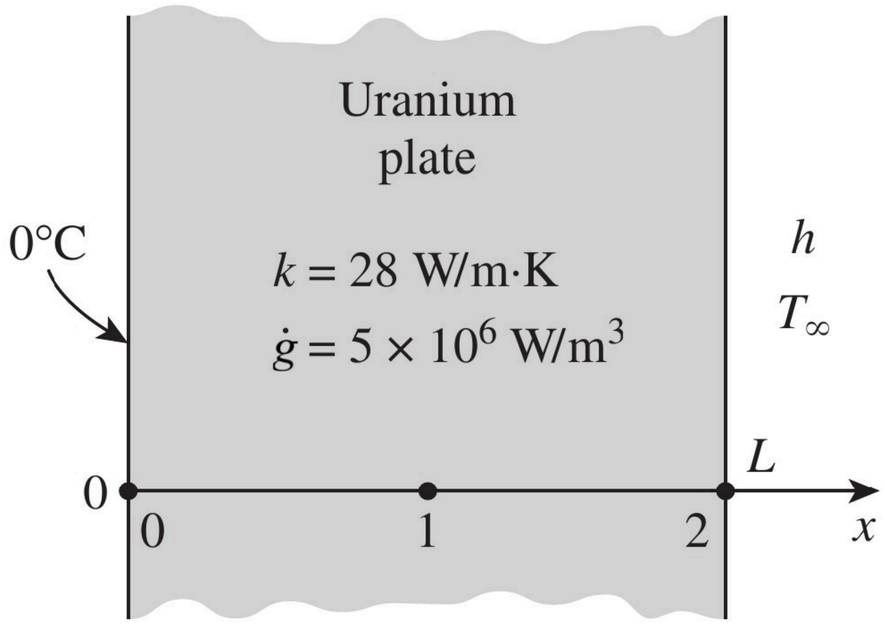
Exercise: Centre temperature in a plate by the energy balance method

Consider a aluminium plate of length $L = 4$ cm and thermal conductivity $k = 28$ W/(m·K) in which heat is generated at a rate $\dot{g} = 5 \cdot 10^6$ W/m³. The left side is maintained at 0°C and the right side is exposed to convection with $h = 45$ W/(m²·K) and ambient temperature of $T_\infty = 30$°C. Find T_2.



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Energy balance method for 2D (and 3D) problems

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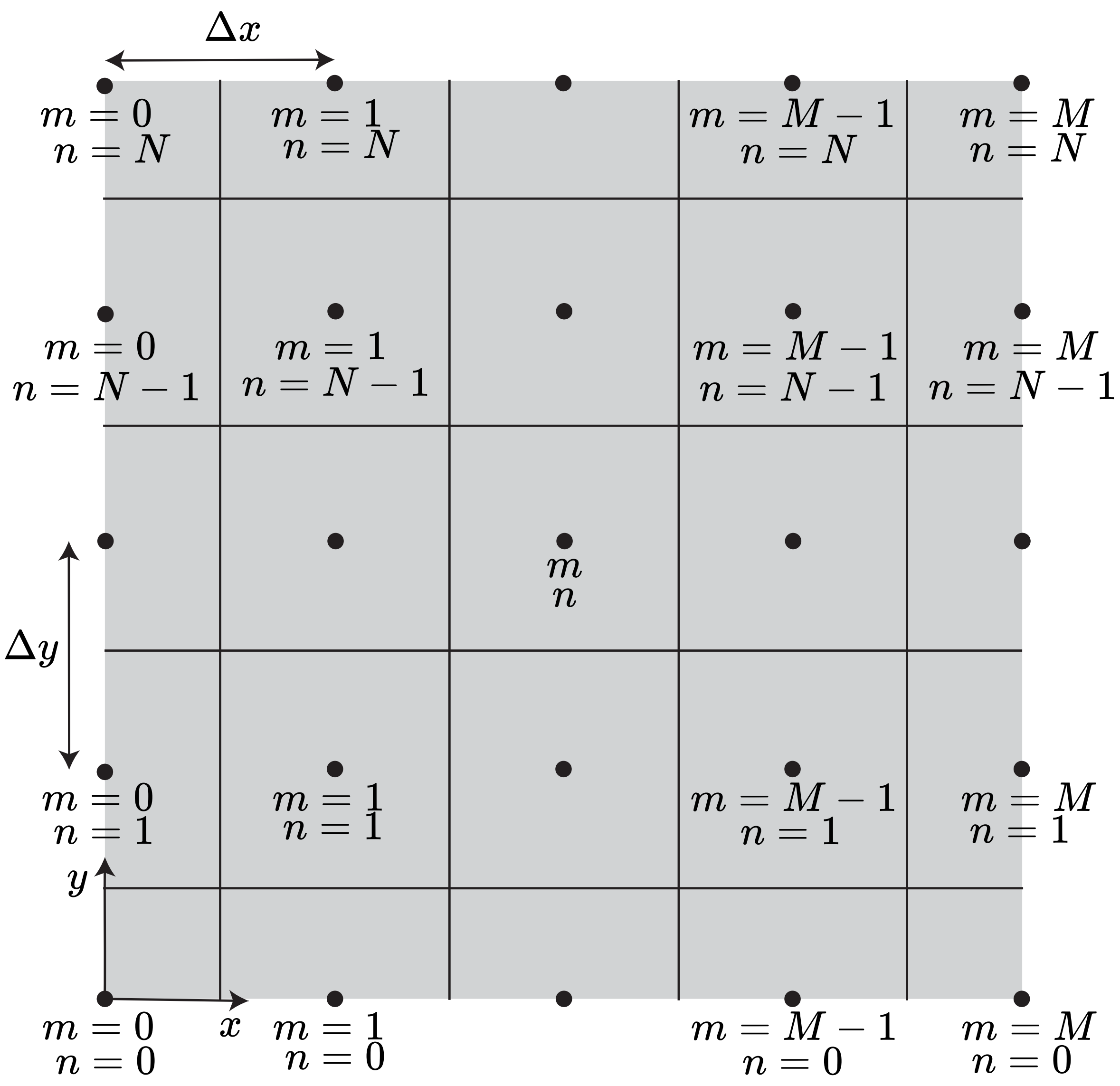
Notation for *2D energy balance* problems

Double subscripts (m,n) to denote two dimensions (triple for three dimensions)

- m used for x-direction
- n used for y-direction
- Temperature at node $(m = 1, n = 0)$ is written $T_{1,0}$
- Distances between nodes are Δx and Δy
- Form volume elements around each node
- Convert from node number to actual coordinates by:
  - $x = m\Delta x$
  - $y = n\Delta y$



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The energy balance equation for *interior nodes*

We *sum up heat transfer to/from* volume element:

$$\left(\begin{array}{c}\text{Sum of heat}\\ \text{transfer rate}\\ \text{across surfaces}\\ \text{of volume element}\end{array}\right) + \left(\begin{array}{c}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{array}\right) = \left(\begin{array}{c}\text{Rate of change}\\ \text{of the energy}\\ \text{content of}\\ \text{the element}\end{array}\right)$$

- For *steady* heat transfer (no changes in time):

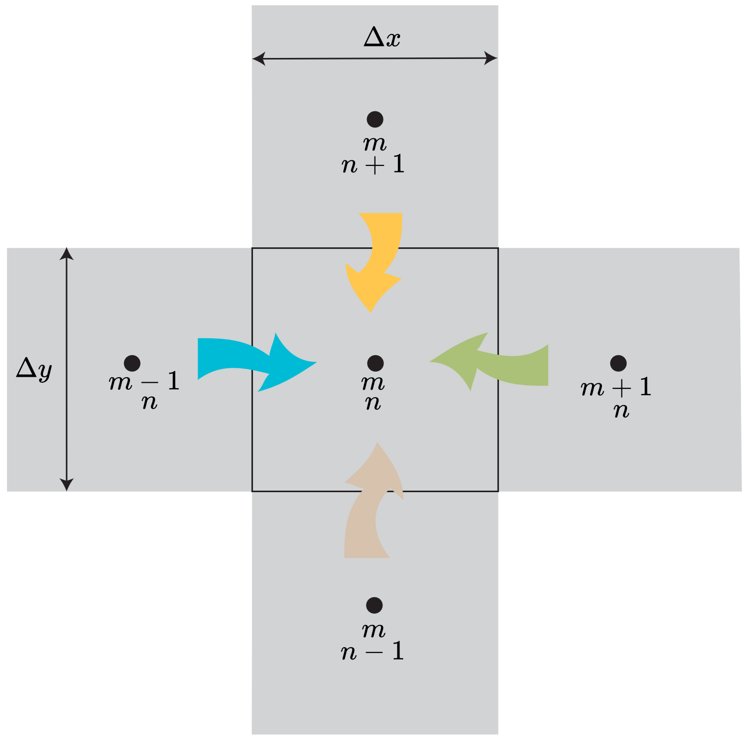
$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = 0$$

- Sum up heat transfer to/from volume element:

$$\dot{Q}_{\text{left,cond}} + \dot{Q}_{\text{right,cond}} + \dot{Q}_{\text{bottom,cond}} + \dot{Q}_{\text{top,cond}} + \dot{G}_{\text{element}} = 0$$

  - .. or in terms of node numbers:

$$kA\frac{T_{m-1,n} - T_{m,n}}{\Delta x} + kA\frac{T_{m+1,n} - T_{m,n}}{\Delta x} + kA\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + kA\frac{T_{m,n+1} - T_{m,n}}{\Delta y} + \dot{g}_{m,n}(\Delta x \Delta y) = 0$$



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The energy balance equation for *boundary nodes*

We *sum up heat transfer to/from* volume element:

$$\left(\begin{array}{c}\text{Sum of heat}\\ \text{transfer rate}\\ \text{across surfaces}\\ \text{of volume element}\end{array}\right) + \left(\begin{array}{c}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{array}\right) = \left(\begin{array}{c}\text{Rate of change}\\ \text{of the energy}\\ \text{content of}\\ \text{the element}\end{array}\right)$$

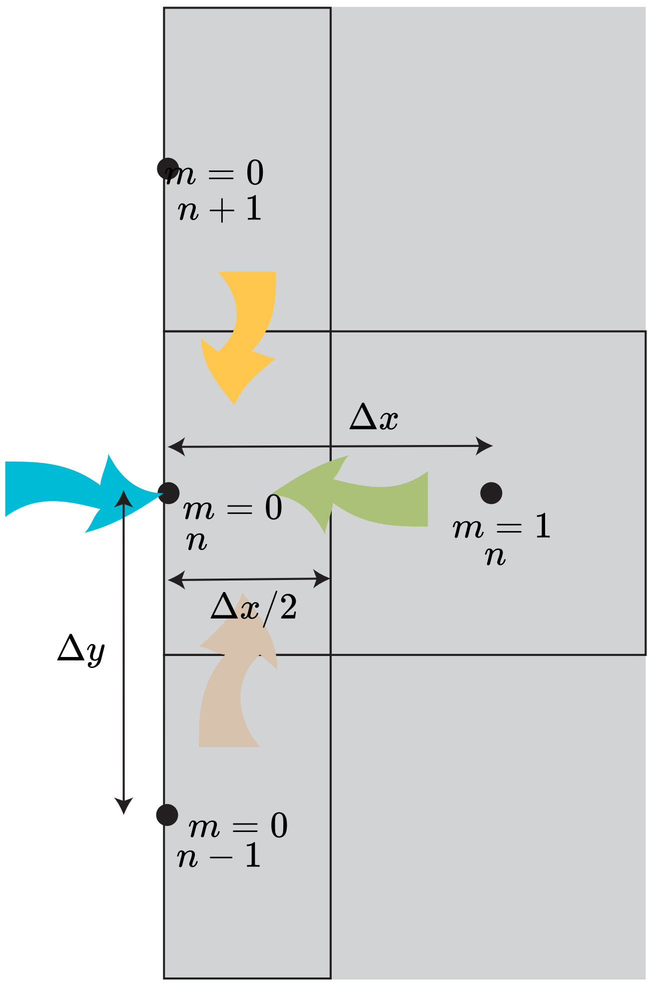
- For a volume element on the left boundary:

$$\dot{Q}_{\text{left,surface}} + \dot{Q}_{\text{right,cond}} + \dot{Q}_{\text{bottom,cond}} + \dot{Q}_{\text{top,cond}} + \dot{G}_{\text{element}} = 0$$

- .. or in terms of node numbers:

$$\dot{Q}_{\text{left surface}} + k\Delta y \frac{T_{1,n} - T_{0,n}}{\Delta x} + k\frac{\Delta x}{2}\frac{T_{0,n-1} - T_{0,n}}{\Delta y} + k\frac{\Delta x}{2}\frac{T_{0,n+1} - T_{0,n}}{\Delta y} + \dot{g}_{0,n}\left(\frac{\Delta x}{2}\Delta y\right) = 0$$

- The expression for $\dot{Q}_{\text{left surface}}$ depends on the type of boundary condition



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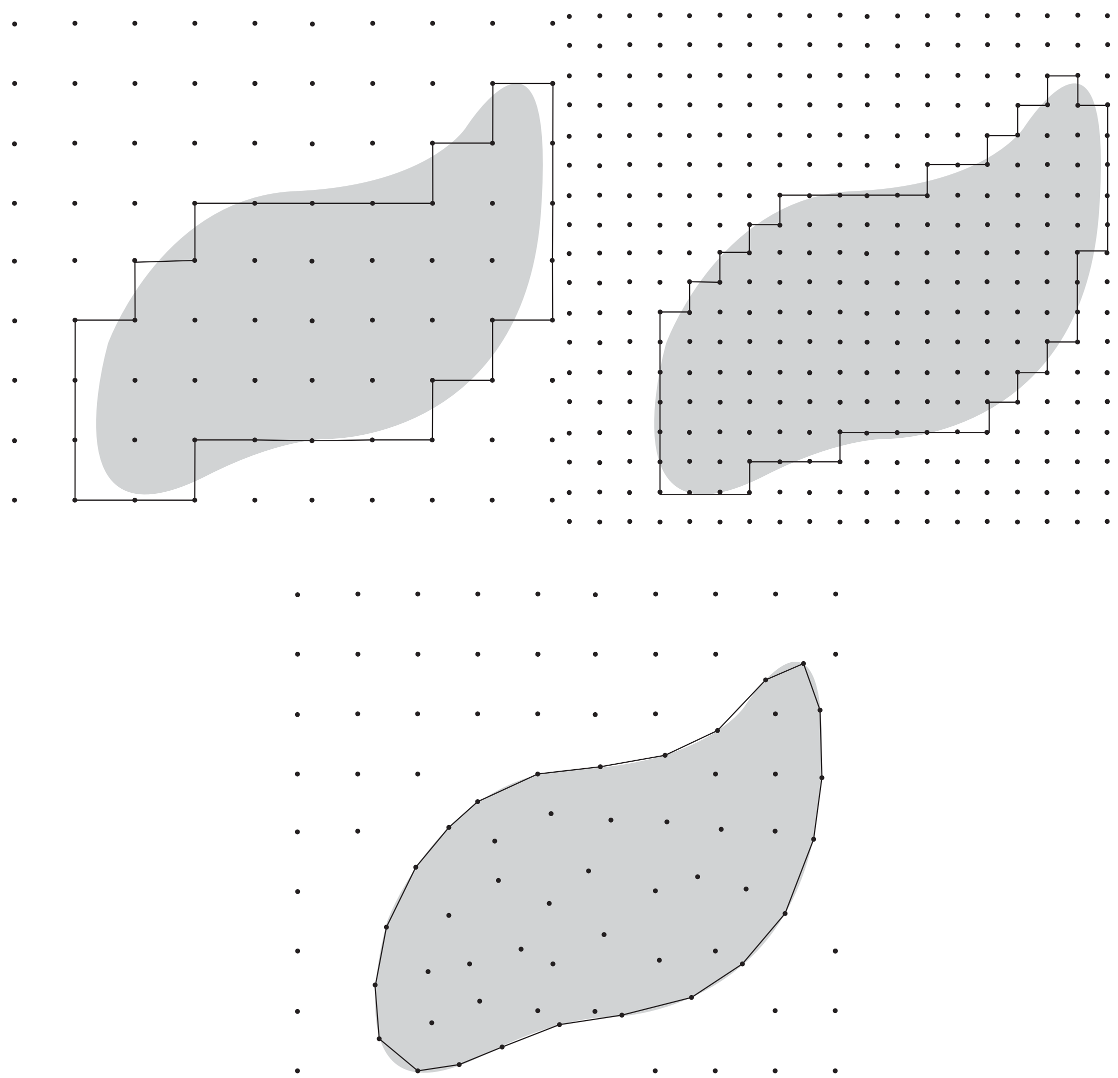
Handling *irregular* boundaries

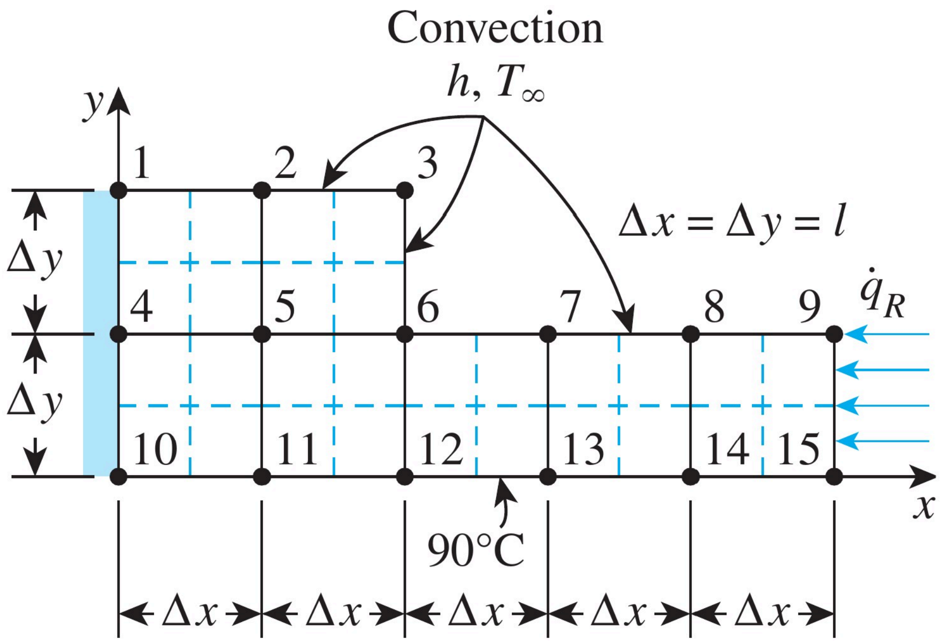
Different approaches to irregular boundaries:

- *Baseline*: Approximate the shape using regular grid (stair-step approach)
- Use *smaller* volume elements near the boundary to better approximate the shape
- Use *irregular* boundary volume elements that conform to the shape



Exercise: Steady Two-Dimensional Heat Conduction

Determine the temperature at the 15 nodes. Thermal conductivity is $k = 15$ W/m·K, and heat is generated in the body at a rate of $\dot{q} = 2 \times 10^6$ W/m³. The left surface of the body is insulated, and the bottom surface is maintained at a uniform temperature of $90°\mathrm{C}$. The entire top surface is subjected to convection to ambient air at $T_\infty = 25°\mathrm{C}$ with a convection coefficient of $h = 80$ W/m²·K, and the right surface is subjected to heat flux at a uniform rate of $\dot{q}_R = 5000$ W/m². The nodal network of the problem consists of 15 equally spaced nodes with $\Delta x = \Delta y = 1.2$ cm.



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