

## Fluid Mechanics

# Analytical solutions for simple fluid flows

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# Complexities involved in Navier-Stokes equations

*No universal analytical solution* due to non-linearities ( $u\partial u/\partial x$ )

- Solution *approximated* using *numerical methods* (e.g. CFD)
- Solution found by *experiments*

Analytical solutions for *only a few problems*

- *Simple geometries*: Flow between two plates or in a tube
- *Simple flows*: Steady, laminar and fully-developed flows

*x-direction:*

$$\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho g_x$$

*y-direction:*

$$\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \rho g_y$$

*z-direction:*

$$\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \rho g_z$$

# Why look at these *simplified* problems?

For which applications is the flow between two plates important?

Flow between two plates—whether *pressure-driven* (plane Poiseuille flow) or *shear-driven* (Couette flow)—is a **canonical model** that underpins both engineering practice and natural phenomena. It's important in many applications, including:

## Engineering & Technology

- **Lubrication:** Thin films of oil in bearings, pistons, gears, or seals are modeled as flow between parallel plates.
- **Microfluidics:** Lab-on-a-chip devices use narrow rectangular channels, which are essentially parallel plate flows.
- **Heat exchangers:** Plate-type exchangers rely on fluid moving between closely spaced plates for efficient heat transfer.
- **Coating & printing:** Flow of paint, varnish, or ink in thin layers is approximated by plate flows.
- **Polymer processing:** Sheet extrusion and film casting involve molten polymer flow between plates.

## Everyday Phenomena

- **Sticking of wet surfaces:** Two glass slides or a credit card and plastic sleeve resist separation due to viscous flow in the thin liquid film.
- **Windshield wipers:** The thin water film between blade and glass behaves like a parallel-plate channel.
- **Food preparation:** Spreading butter, cream, or sauces between slices involves thin-film flow.

## Biology & Medicine

- **Synovial joints:** Cartilage surfaces slide with synovial fluid in between, modeled as shear-driven thin-film flow.
- **Blood in microvessels:** Capillaries and arterioles often approximate parallel-plate geometries at small scales.
- **Dialysis and filtration:** Fluids move between thin membranes in channels resembling plate flows.

## Fundamental & Educational Value

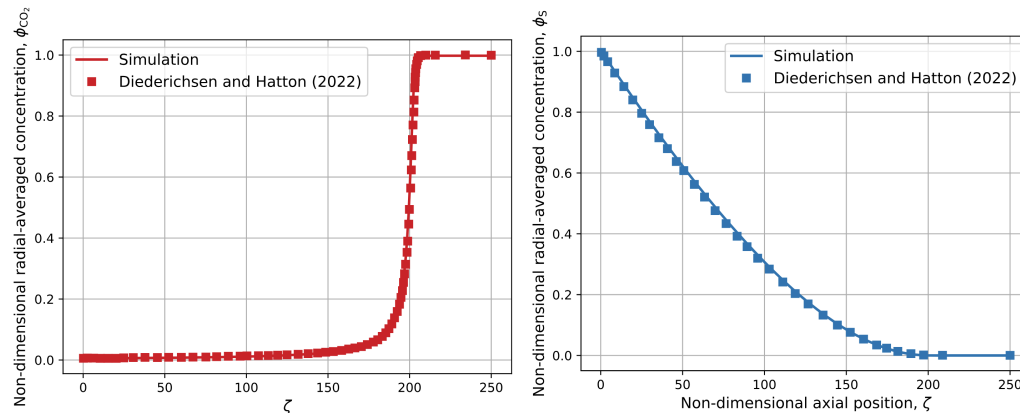
- Provides one of the **few exact solutions of Navier–Stokes**—great for teaching velocity profiles, shear stress, and pressure gradients.
- Serves as a **building block** for more complex flows (e.g., lubrication theory, Reynolds equation).
- Helps illustrate the importance of viscosity, low Reynolds number flows, and scaling laws.

# Current research uses *Poiseuille flow* for verification of CFD

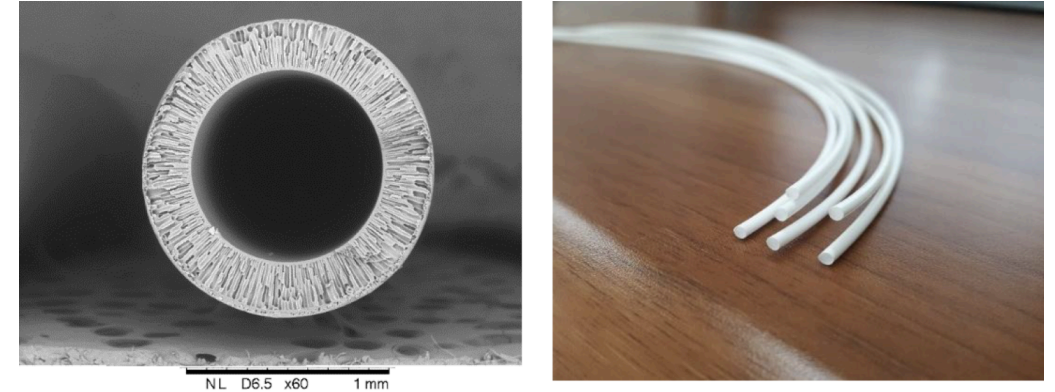
CO<sub>2</sub> diffuses into the fiber and is captured

- Really (!) long fibres ( $l/D \approx 12000$ )
- Steady, laminar, fully-developed flow is a *very good* assumption
- Prescribed boundary condition

*Results*: Code verification results



*Figure*: Hollow fibre membrane



Bazhenov, S. D., Bilydukevich, A. V., & Volkov, A. V. (2018). Gas-Liquid Hollow Fiber Membrane Contactors for Different Applications. *Fibers*, 6(4), 76. <https://doi.org/10.3390/fib6040076>

*Code*: Inlet boundary condition in OpenFOAM

```
1  1
2  /*-----*-- C++ -----*--
3  =====
4  \ \ / / F i e l d           | OpenFOAM: The Open Source CFD Toolbox
5  \ \ / / O peration         | Website: https://openfoam.com
6  \ \ / / A nd               | Version: 2506
7  \ \ / / M anipulation      |
8  /*-----*--
9  FoamFile
10 {
11     format      ascii;
12     class       volVectorField;
13     location    "0";
14     object      U;
15 }
16 // * * * * *
17
18 dimensions     [0 1 -1 0 0 0 0];
19
```

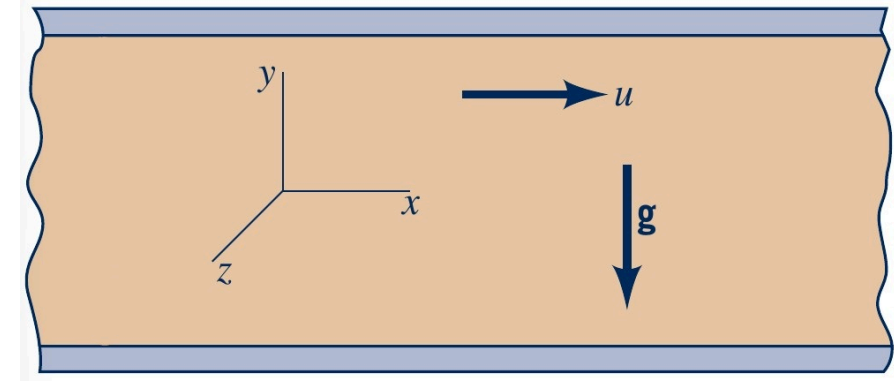
# Flow between plates

*Incompressible, steady, fully developed, laminar flow* between two infinite plates

- Velocity only in  $x$ -direction:
  - *then*,  $u = u(y)$ ,  $v = w = 0$
- Steady flow:
  - *then*,  $\partial u / \partial t = \partial v / \partial t = \partial w / \partial t = 0$
- By continuity equation:  $\partial u / \partial x + \partial v / \partial y + \partial w / \partial z = 0$ 
  - *then*,  $\partial u / \partial x = 0$
- No gravity in  $x$  and  $z$  directions:
  - *then*,  $g_x = g_z = 0$

Navier-Stokes equations *reduce* to:

$$x : 0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}, \quad y : 0 = -\frac{\partial p}{\partial y} + \rho g_y, \quad z : 0 = -\frac{\partial p}{\partial z}$$



*x*-direction:

$$\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho g_x$$

*y*-direction:

$$\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \rho g_y$$

*z*-direction:

$$\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \rho g_z$$

# Flow between *two stationary* plates

Integrating twice to get  $u(y)$ :

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \frac{\partial p}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\mu} \frac{\partial p}{\partial x} y + C_1$$

$$u = \frac{1}{2\mu} \frac{\partial p}{\partial x} y^2 + C_1 y + C_2$$

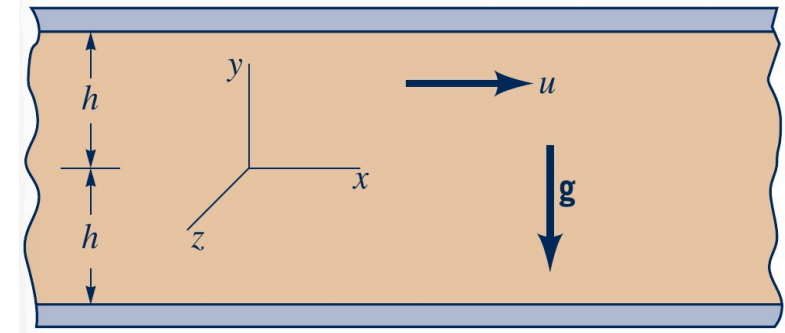
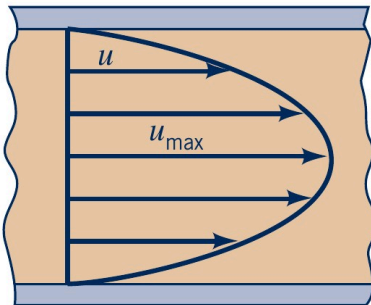
Using *boundary conditions*:

At  $y = \pm h$  we have  $u = 0$

$$C_1 = 0, C_2 = -\frac{1}{2\mu} \frac{\partial p}{\partial x} h^2$$

*Velocity profile* (by inserting  $C_1$  and  $C_2$ ):

$$u(y) = \frac{1}{2\mu} \frac{\partial p}{\partial x} (y^2 - h^2)$$



*Volume flow rate* (integrating velocity profile):

$$q = \int_{-h}^h u(y) dy = -\frac{2h^3}{3\mu} \left( \frac{\partial p}{\partial x} \right)$$

*Average velocity*:

$$V = \frac{q}{2h} = -\frac{h^2}{3\mu} \left( \frac{\partial p}{\partial x} \right)$$

*Maximum velocity* (centre of channel,  $y = 0$ ):

$$u_{\max} = u(y = 0) = -\frac{h^2}{2\mu} \left( \frac{\partial p}{\partial x} \right) = \frac{3}{2} V$$

# Flow between *one stationary* and *one moving* plate (aka Couette flow)

Integrating twice to get  $u(y)$ :

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \frac{\partial p}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\mu} \frac{\partial p}{\partial x} y + C_1$$

$$u = \frac{1}{2\mu} \frac{\partial p}{\partial x} y^2 + C_1 y + C_2$$

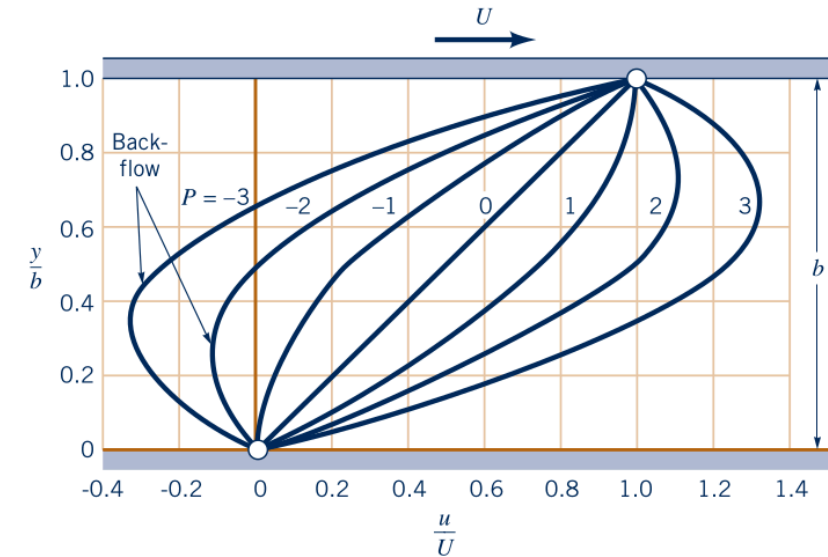
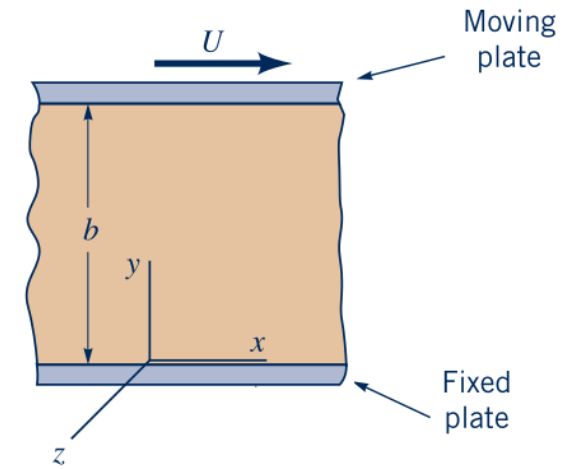
Using *boundary conditions*:

At  $y = 0$  we have  $u = 0$

At  $y = b$  we have  $u = U$

Finding  $C_1$  and  $C_2$  and inserting:

$$u(y) = U \frac{y}{b} + \frac{1}{2\mu} \frac{\partial p}{\partial x} (y^2 - by)$$



$$P = -\frac{b^2}{2\mu U} \left( \frac{\partial p}{\partial x} \right)$$

# Flow in a *circular pipe* (aka Poiseuille flow)

*Incompressible, steady, fully developed, laminar flow* in a circular pipe

- Velocity only in  $z$ -direction:
  - *then*,  $v_z = v_z(r)$ ,  $v_\theta = v_r = 0$
- Steady flow:
  - *then*,  $\partial u / \partial t = \partial v / \partial t = \partial w / \partial t = 0$
- By continuity equation:  $\frac{1}{r} \frac{\partial(rv_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$ 
  - *then*,  $\partial v_z / \partial z = 0$
- No gravity in  $z$  direction:
  - *then*,  $g_z = 0$

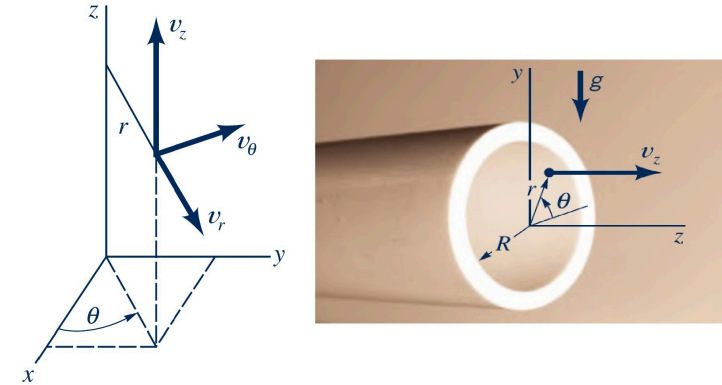
Navier-Stokes equations *reduce* to:

$$z : 0 = -\frac{\partial p}{\partial z} + \mu \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right), \quad r : 0 = \rho g_r - \frac{\partial p}{\partial r}, \quad \theta : 0 = \rho g_\theta - \frac{1}{r} \frac{\partial p}{\partial \theta}$$

*Integration*, applying *boundary conditions* and *find*  $C_1$  and  $C_2$ :

$$v_z(r) = \frac{1}{4\mu} - \left( \frac{\partial p}{\partial z} \right) (r^2 - R^2)$$

- Velocity is a function of *viscosity*  $\mu$ , *pressure gradient*  $\partial p / \partial z$  and *pipe radius*  $R$ .



*z*-direction:

$$\rho \left( \frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z$$

*r*-direction:

$$\rho \left( \frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial p}{\partial r} + \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_r}{\partial r} \right) - \frac{v_r}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right) + \rho g_r$$

*θ*-direction:

$$\rho \left( \frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} - \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left( \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial v_\theta}{\partial r} \right) - \frac{v_\theta}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right) + \rho g_\theta$$



# Details on flow in *circular pipes* (aka Poiseuille flow)

- Volume flow rate (integrate velocity):

$$Q = 2\pi \int_0^R v_z(r) r dr = -\frac{\pi R^4}{8\mu} \left( \frac{\partial p}{\partial z} \right) = \frac{\pi R^4 \Delta p}{8\mu l}$$

- Flow rate is a function of *viscosity*  $\mu$ , *pressure gradient*  $\partial p / \partial z$  and *pipe radius*  $R$ .

- Average velocity (divide by area):

$$V = \frac{Q}{A} = \frac{-\frac{\pi R^4}{8\mu} \left( \frac{\partial p}{\partial z} \right)}{\pi R^2} = -\frac{R^2}{8\mu} \left( \frac{\partial p}{\partial z} \right) = \frac{R^2 \Delta p}{8\mu l}$$

- Mean velocity is a function of *viscosity*  $\mu$ , *pressure gradient*  $\partial p / \partial z$  and *pipe radius*  $R$ .

- Maximum velocity ( $r = 0$ ):

$$v_z(r = 0) = \frac{1}{4\mu} - \left( \frac{\partial p}{\partial z} \right) (0^2 - R^2) = -\frac{R^2}{4\mu} \left( \frac{\partial p}{\partial z} \right) = \frac{R^2 \Delta p}{4\mu l}$$

- Mean velocity is a function of *viscosity*  $\mu$ , *pressure gradient*  $\partial p / \partial z$  and *pipe radius*  $R$ .

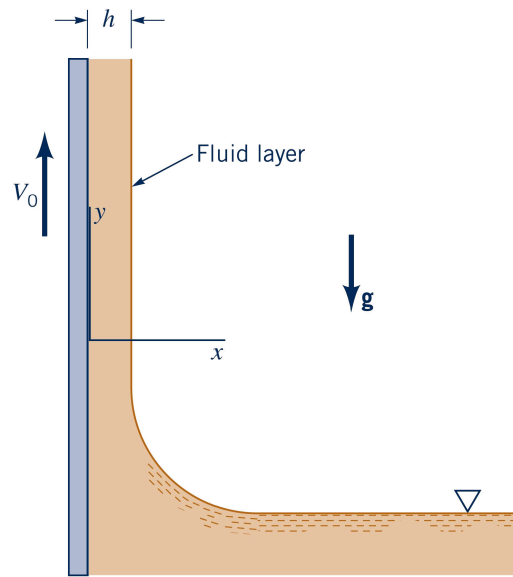
- Ratio of centerline to mean velocity ( $v_z(r = 0) / V$ ):

$$\frac{v_z(r = 0)}{V} = \frac{\frac{R^2 \Delta p}{4\mu l}}{\frac{R^2 \Delta p}{8\mu l}} = 2$$

- Centerline velocity is exactly *twice* the mean velocity.

## Exercise: Upward moving fluid film

A wide moving belt passes through a container of a viscous liquid. The belt moves vertically upward with a constant velocity,  $V_0$ , as illustrated in the figure. Because of viscous forces the belt picks up a film of fluid of thickness  $h$ . Gravity tends to make the fluid drain down the belt. Assume that the flow is laminar, steady, and fully developed.



$$\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho g_x$$

$$\rho \left( \frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \rho g_y$$

$$\rho \left( \frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left( \frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \rho g_z$$