Heat Transfer

Transient heat conduction using numerical methods

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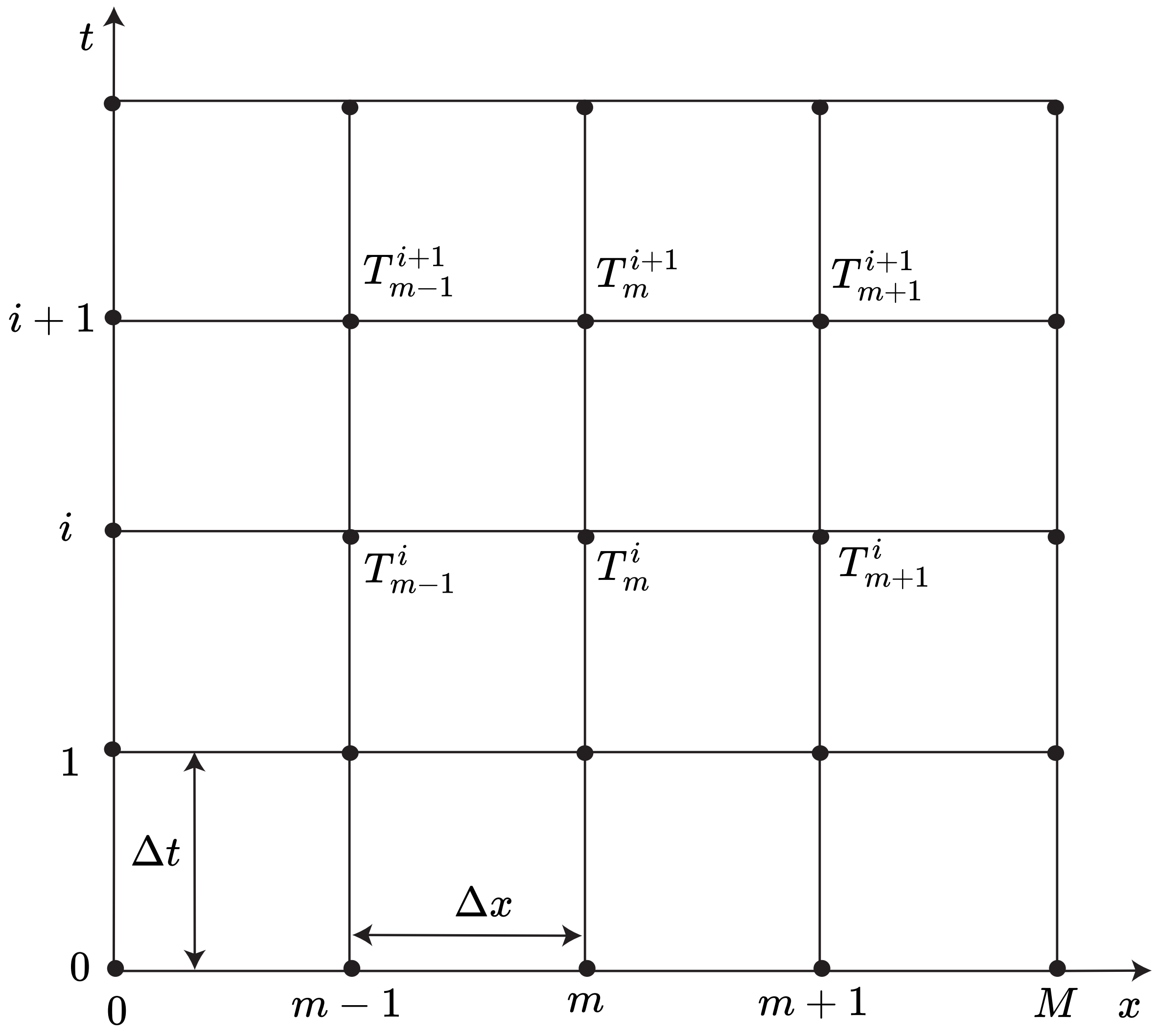
Notation for transient problems

- **On space axis** m, n, k denote node number in x, y, z-directions
  - Distance between nodes is Δx
- **On time axis** i denote node number in t-direction (time)
  - Time between nodes is Δt
- General notation:

$$T^i_{m,n,k}$$



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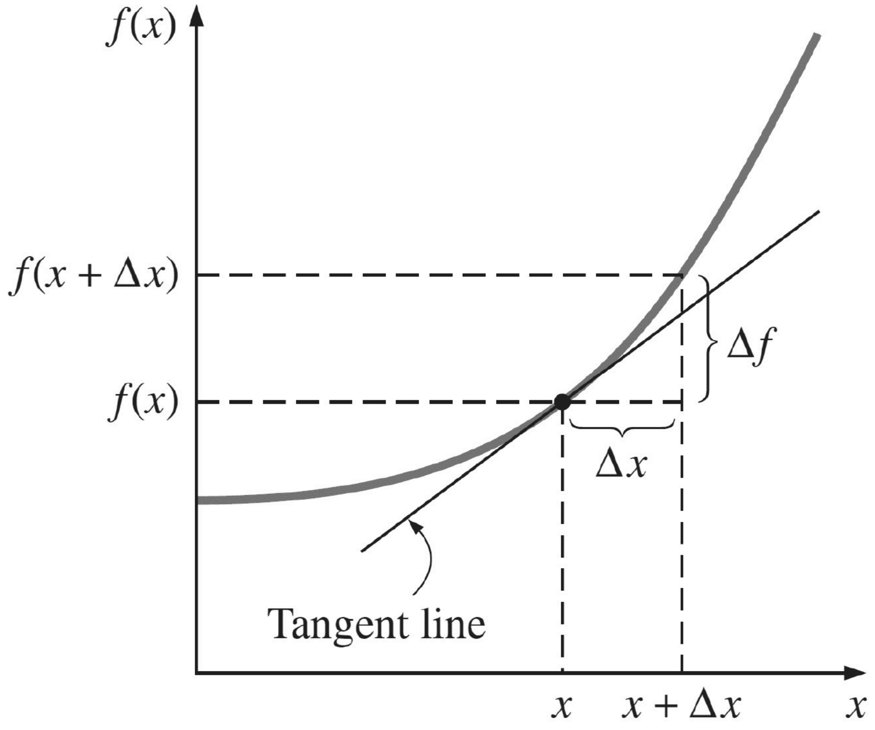
Recalling finite differences from old days 🙂

Consider a function f that depends on x. The first derivative is then *defined* as the slope of the tangent:

$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)-f(x)}{\Delta x}$

- Leaving out the limit, we get an *approximate* solution:

$\frac{\mathrm{d}f(x)}{\mathrm{d}x} \approx \frac{f(x+\Delta x)-f(x)}{\Delta x}$



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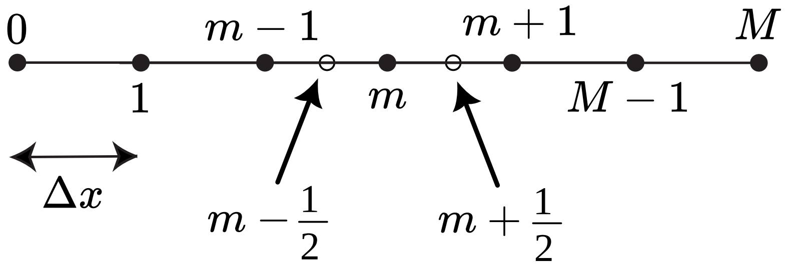
Applying finite differences to our 1D domain

Remember, we are looking for:

$$\frac{\partial^2 T}{\partial x^2}$$



We may approximate first derivative *at midpoints* as:

$$\left.\frac{\partial T}{\partial x}\right|_{x_{m-1/2}} \approx \frac{T_m - T_{m-1}}{\Delta x} \quad \text{and} \quad \left.\frac{\partial T}{\partial x}\right|_{x_{m+1/2}} \approx \frac{T_{m+1} - T_m}{\Delta x}$$

... now, we can approximate the second derivative using two first-order derivatives:

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{x_m} \approx \frac{\left.\frac{\partial T}{\partial x}\right|_{x_{m+1/2}} - \left.\frac{\partial T}{\partial x}\right|_{x_{m-1/2}}}{\Delta x} = \frac{\frac{T_{m+1}-T_m}{\Delta x} - \frac{T_m - T_{m-1}}{\Delta x}}{\Delta x}$$

... which reduces to:

$$\left.\frac{\partial^2 T}{\partial x^2}\right|_{x_m} \approx \frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2}$$

The equation for *steady one-dimensional heat conduction* now becomes:

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{g}}{k} = 0$$

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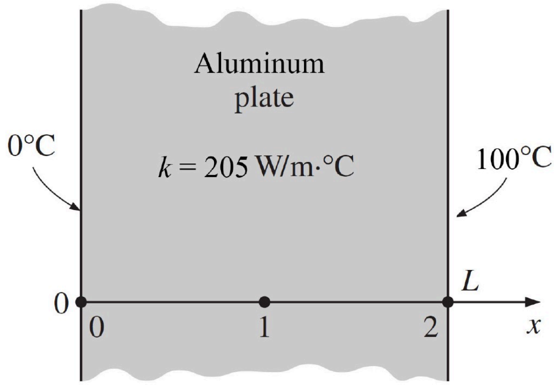
Exercise: Centre temperature in a plate by the finite difference method

Consider a aluminium plate of length $L = 0.5$ cm and thermal conductivity $k = 205$ W/(m·K) in which no heat is generated ($\dot{g} = 0$). One side of the plate is maintained at 0 °C by icy water while the other is maintained at 100 °C. Estimate the temperature in the centre at node 1.



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Energy balance method for 1D problems

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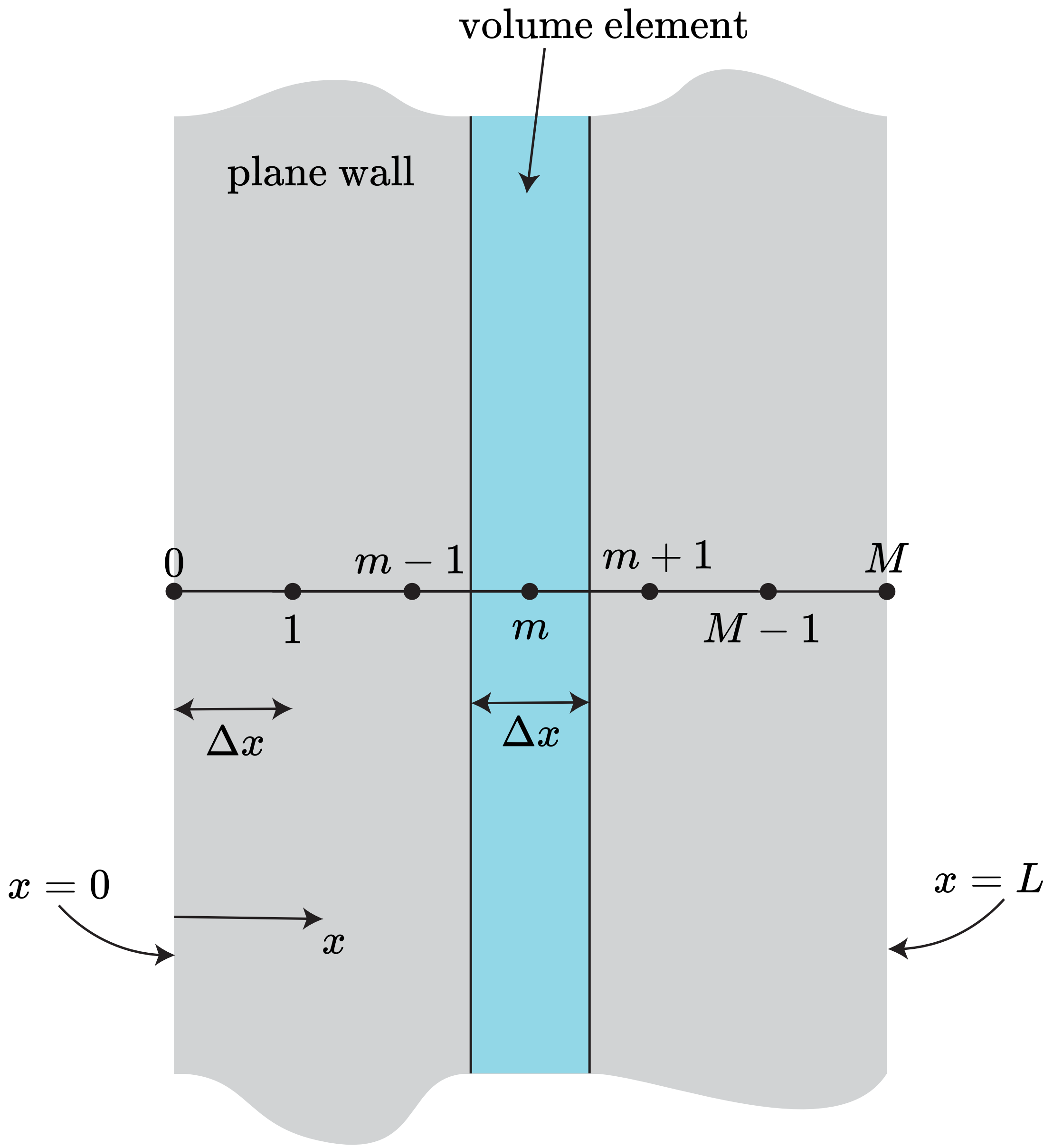
Notation for *1D energy balance* problems

A *1D plane wall* is defined as being infinitely long (i.e. heat transfer only occurs in one direction)

- *Position* is denoted by x
  - $x = 0$ at left side of wall
  - $x = L$ at right side of wall
- *Node number* is denoted by m
  - $m = 1$ at left side of wall
  - $m = M$ at right side of wall
- *Volume elements* with *width* Δx
  - Properties are assumed constant within each volume element
  - The finer the mesh (smaller Δx), the more accurate the solution

- Notation:
  - T_5: Temperature at $m = 5$
  - ρ_5: Density at node $m = 5$
  - ...



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The energy balance equation for interior nodes

We *sum up heat transfer to/from* volume element:

$$\begin{pmatrix}\text{Sum of heat}\\ \text{transfer rate}\\ \text{across surfaces}\\ \text{of volume element}\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of}\\ \text{the element}\end{pmatrix}$$

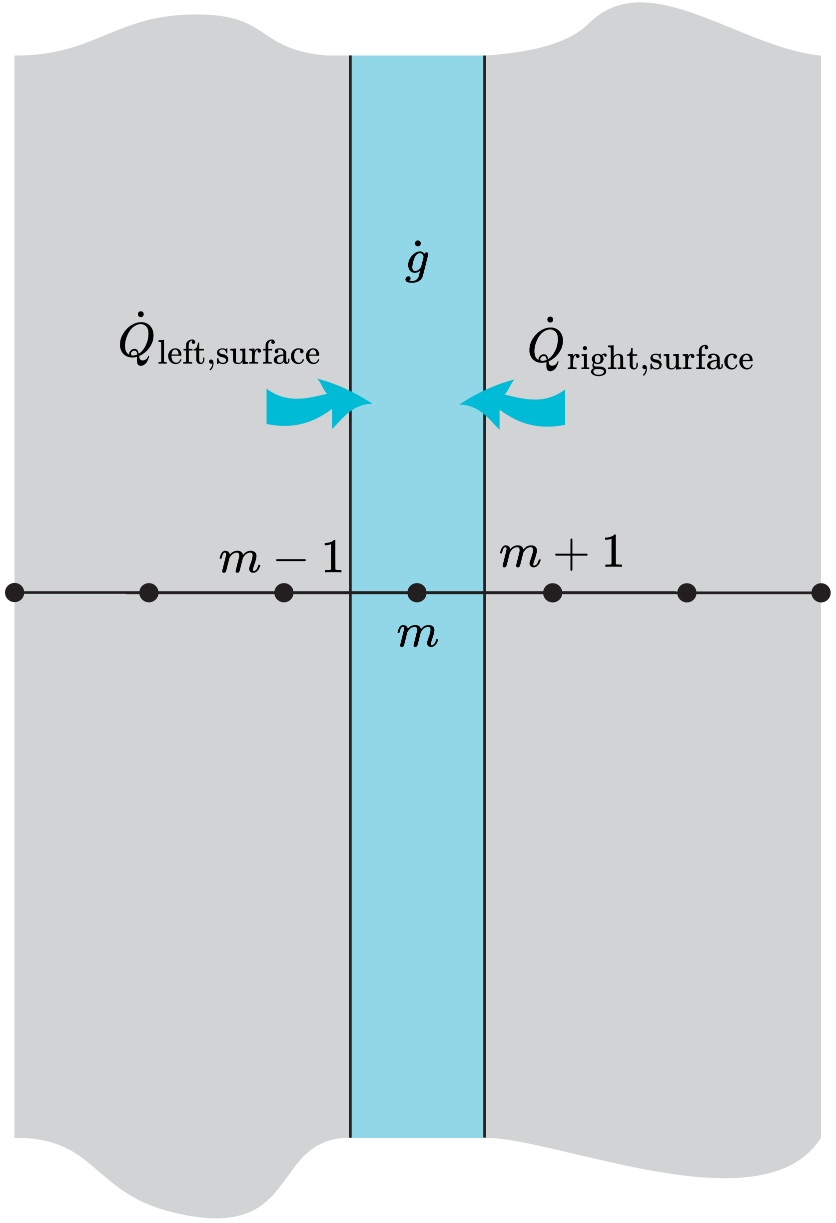
- For *steady* heat transfer (no change in time):

$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = 0$$

- $\dot{G}_{\text{element}}$ is the heat generation rate in the element (W):

$$\dot{G}_{\text{element}} = \dot{g} V_{\text{element}}$$

  - $\dot{g}$ is the volumetric heat generation rate in the element $(\mathrm{W/m^3})$



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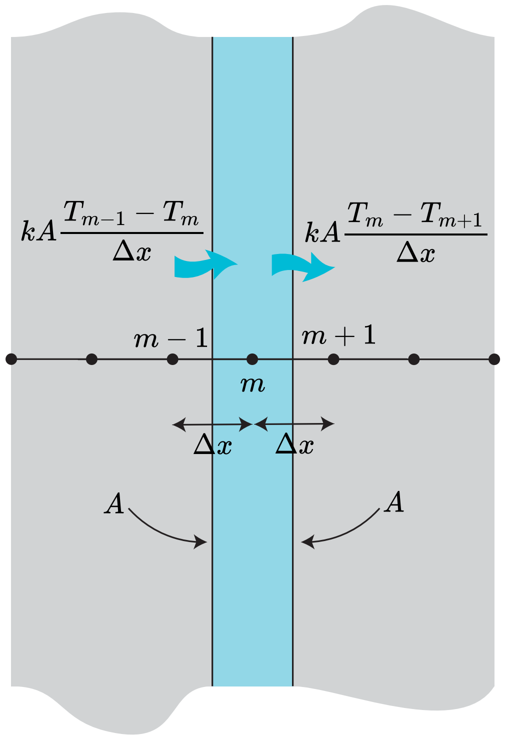
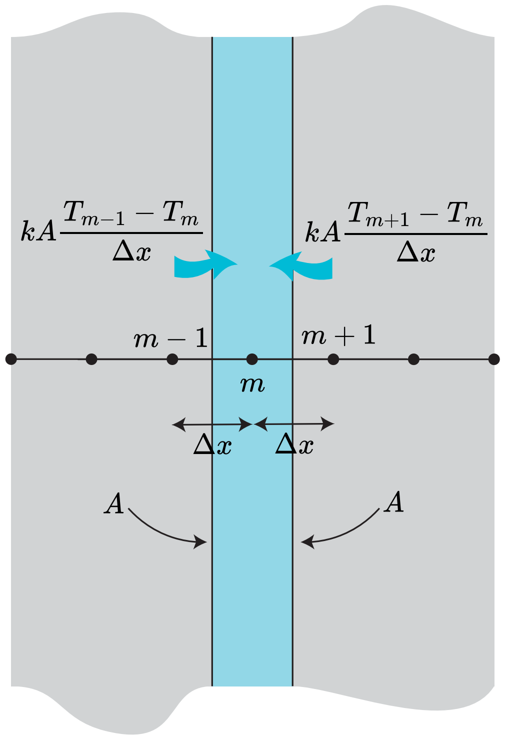
Convention for heat transfer direction

Assumed direction of heat transfer is *irrelevant as long as we are consistent*





Energy balance on control volume:

$$kA\frac{T_{m-1}-T_m}{\Delta x} - kA\frac{T_m-T_{m+1}}{\Delta x} + \dot{g}_m A\Delta x = 0$$

Energy balance on control volume:

$$kA\frac{T_{m-1}-T_m}{\Delta x} + kA\frac{T_{m+1}-T_m}{\Delta x} + \dot{g}_m A\Delta x = 0$$

Conclusion: In both cases, it reduces to *the same expression*:

$$T_{m-1} - 2T_m + T_{m+1} + \frac{\dot{g}_m\Delta x^2}{kA} = 0$$

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The four boundary condition types

We have to specify **boundary conditions** on each non-internal node.

- *Four common* boundary condition types:

1. *Constant temperature*

$$T_0 = c$$

2. *Constant heat flux*

$$\dot{Q}_{\text{left,surface}} = \dot{q}A = c$$

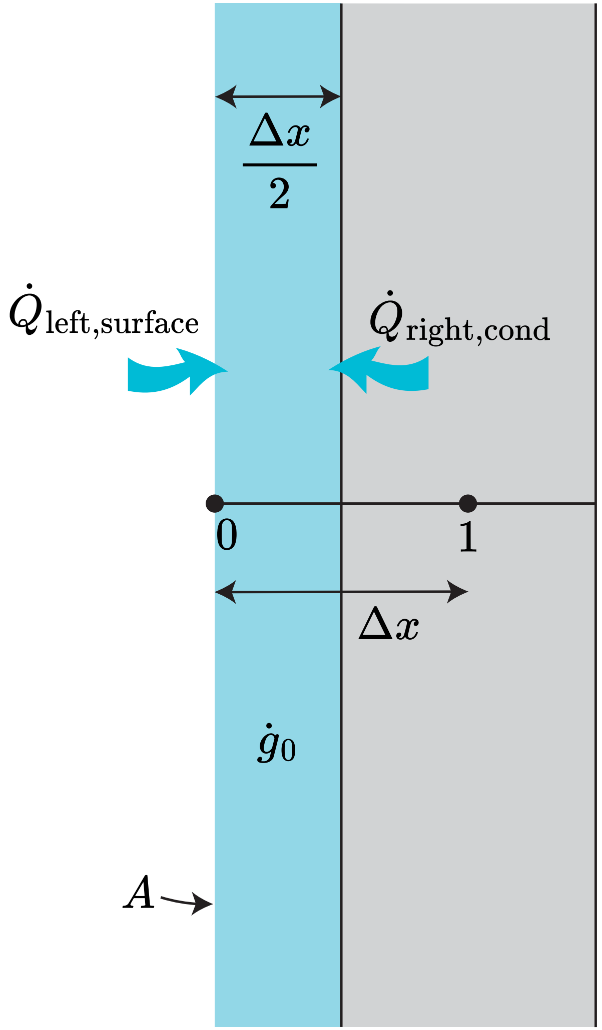
3. *Convection*

$$\dot{Q}_{\text{left,surface}} = hA\left(T_\infty - T_0\right) = c$$

4. *Radiation*

$$\dot{Q}_{\text{left,surface}} = \epsilon\sigma A\left(T_{\text{surr}}^4 - T_0^4\right) = c$$

- .. or any combination of type 2, 3 and 4



Incorporating the different boundary conditions

Type 1 (constant temperature):

- We simply *set the temperature* of the node, e.g. $T_0 = c$

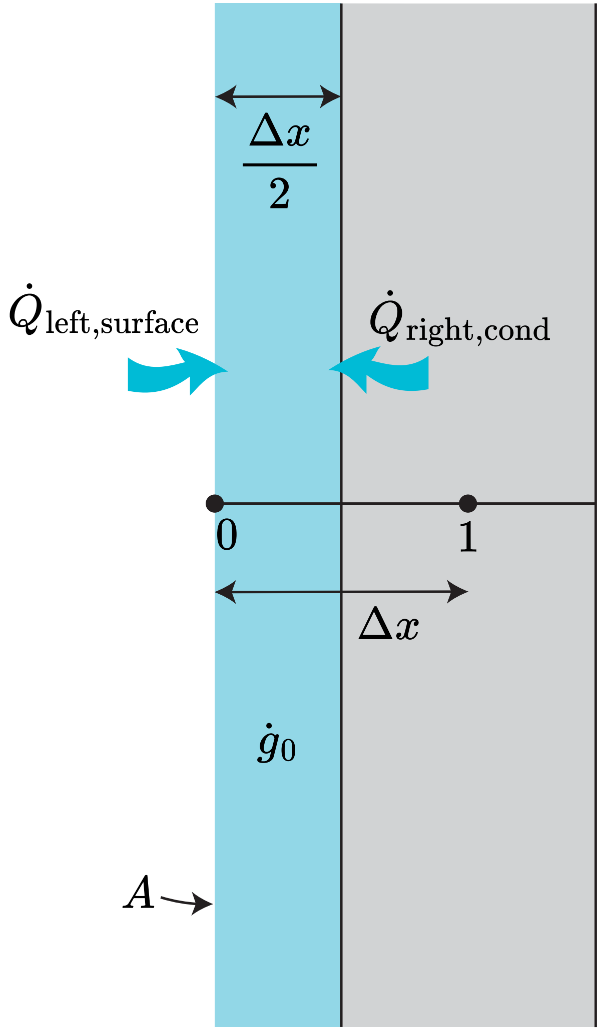
Type 2-4:

- We apply an *energy balance* on the volume element:

$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = 0,$$

$$\dot{Q}_{\text{left,surface}} + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0\left(\frac{\Delta x}{2}A\right) = 0.$$

- Expression for $\dot{Q}_{\text{left,surface}}$ depends on boundary condition type



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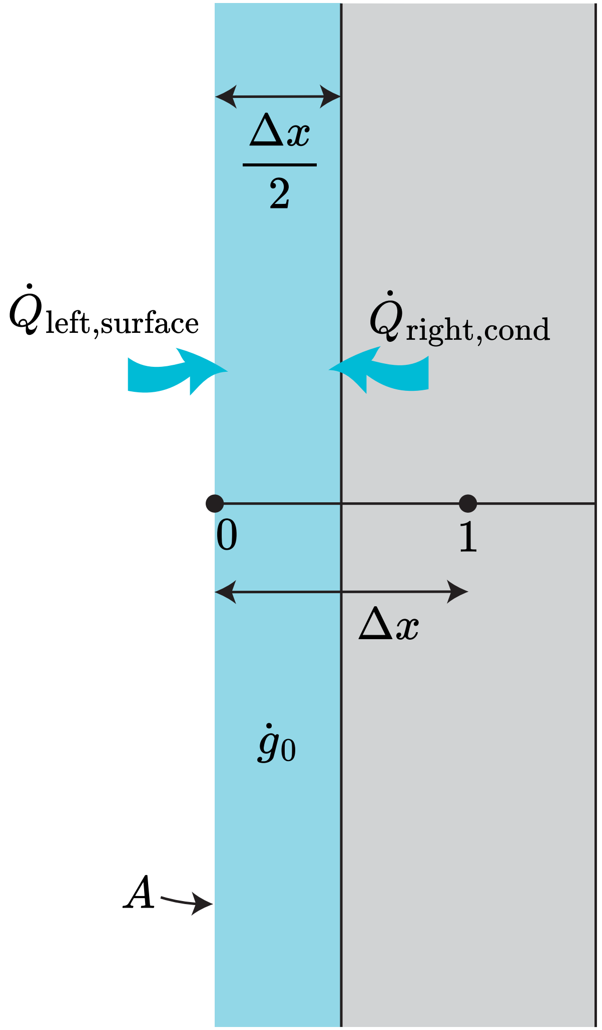
Incorporating mixed boundary condition types

All BC types (**heat flux**, **convection** and **radiation**) can easily be added to form mixed types:

$$\dot{Q}_{\text{left,surface}} = \dot{q}_0 A \;+\; hA\left(T_\infty - T_0\right) \;+\; \epsilon\sigma A\left(T_{\text{surr}}^4 - T_0^4\right)$$



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Mirror concept for insulated boundaries (e.g zero heat flux)

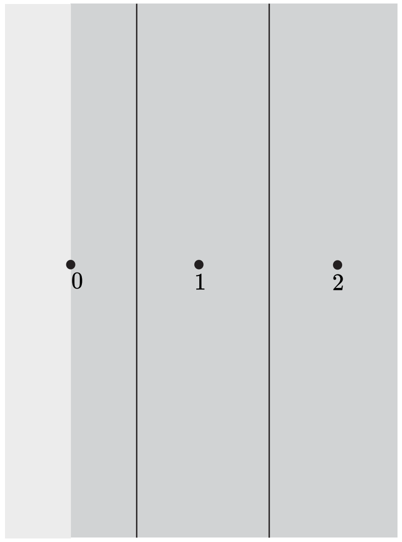
Insulated boundary (zero heat flux) can be modelled using *mirror nodes*:

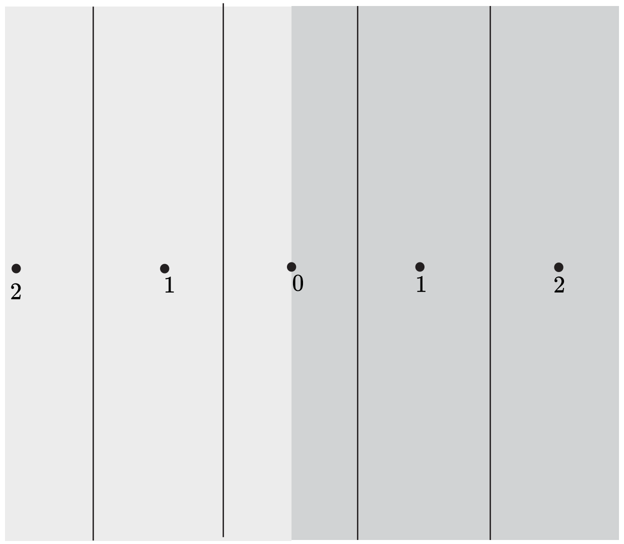
- For interior nodes:

$$\frac{T_{m+1} - 2T_m + T_{m-1}}{\Delta x^2} + \frac{\dot{g}}{k} = 0$$

- For insulated boundaries:

$$\frac{T_1 - 2T_0 + T_1}{\Delta x^2} + \frac{\dot{g}}{k} = 0$$





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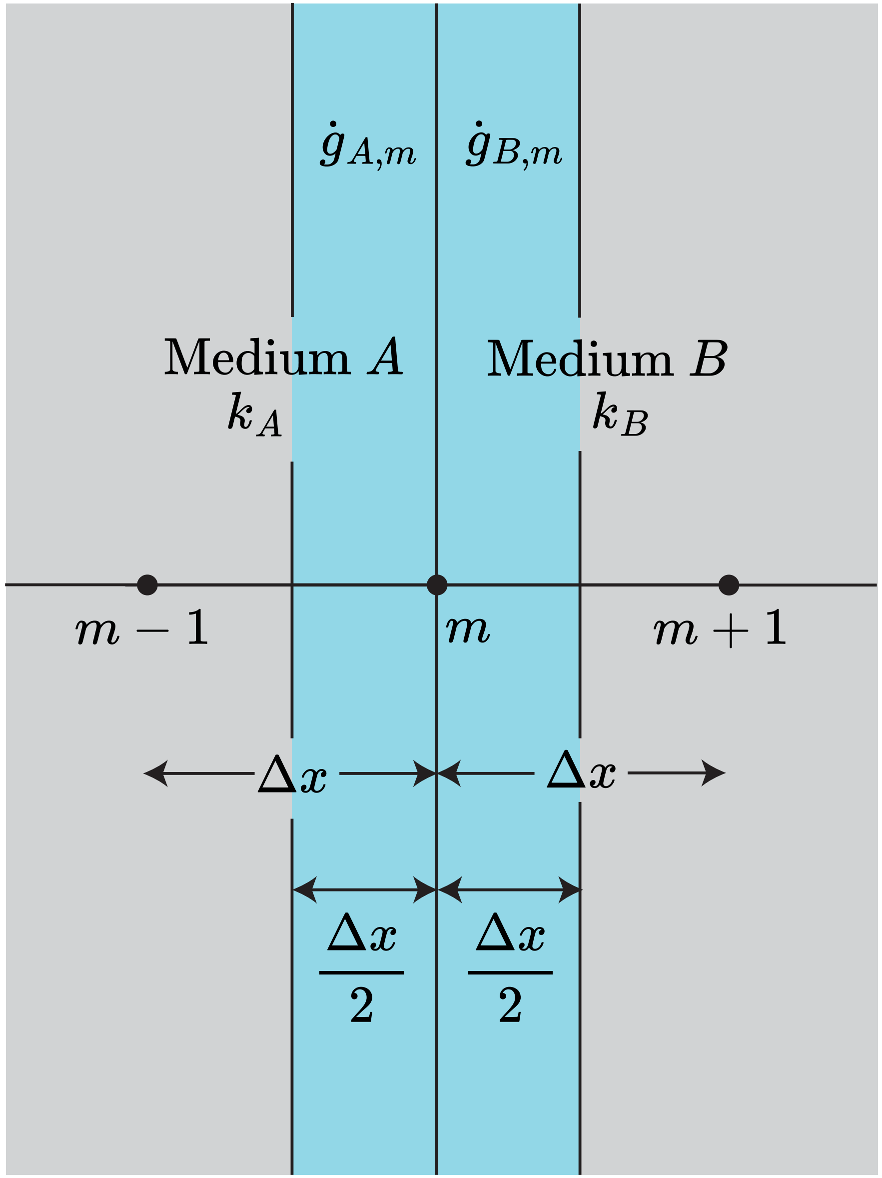
Interfaces between different materials

Starting with the general energy balance:

$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = 0,$$

- Assuming *perfect contact (no air gaps)* at the interface, the energy balance becomes:

$$k_1 A \frac{T_{m-1} - T_m}{\Delta x_1} + k_2 A \frac{T_{m+1} - T_m}{\Delta x_2} + \dot{g}_A A \Delta \frac{x}{2} + \dot{g}_B A \Delta \frac{x}{2} = 0$$



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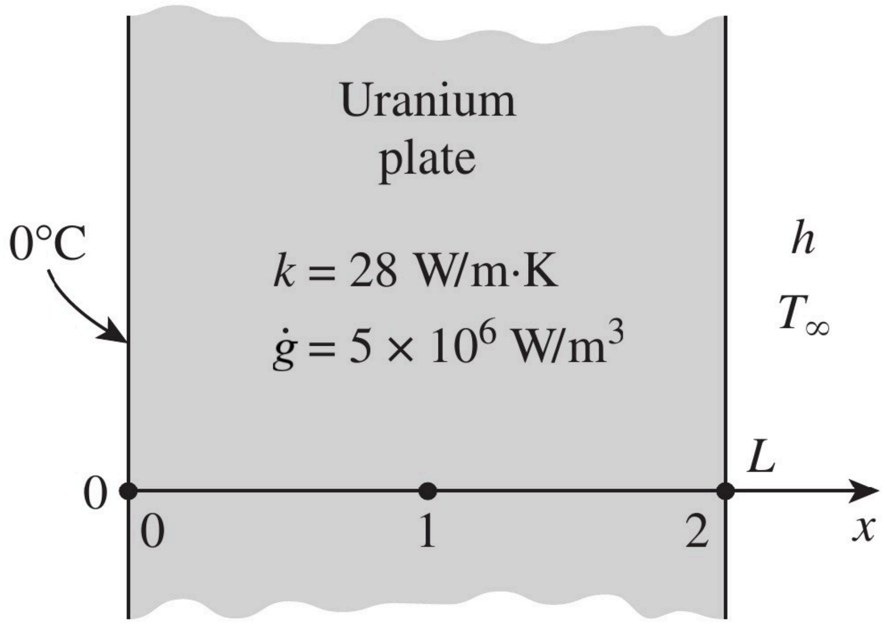
Exercise: Centre temperature in a plate by the energy balance method

Consider a aluminium plate of length $L = 4$ cm and thermal conductivity $k = 28$ W/(m·K) in which heat is generated at a rate $\dot{g} = 5 \cdot 10^6$ W/m³. The left side is maintained at 0°C and the right side is exposed to convection with $h = 45$ W/(m²·K) and ambient temperature of $T_\infty = 30$°C. Find T_2.



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Energy balance method for 2D (and 3D) problems

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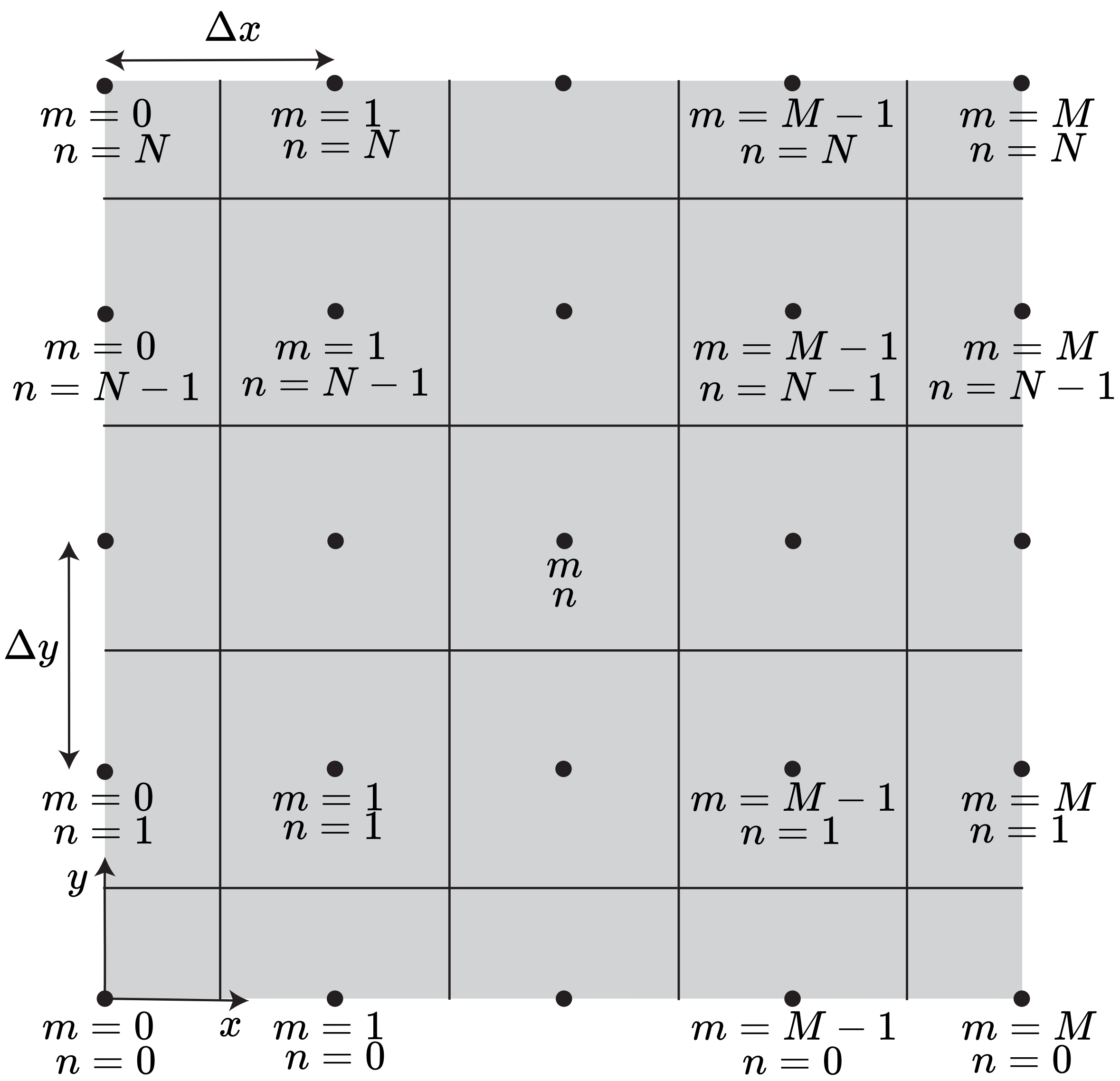
Notation for *2D energy balance* problems

Double subscripts (m,n) to denote two dimensions (triple for three dimensions)

- m used for x-direction
- n used for y-direction
- Temperature at node $(m = 1, n = 0)$ is written $T_{1,0}$
- Distances between nodes are Δx and Δy
- Form volume elements around each node
- Convert from node number to actual coordinates by:
  - $x = m\Delta x$
  - $y = n\Delta y$



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The energy balance equation for *interior nodes*

We *sum up heat transfer to/from* volume element:

$$\left(\begin{array}{c}\text{Sum of heat}\\ \text{transfer rate}\\ \text{across surfaces}\\ \text{of volume element}\end{array}\right) + \left(\begin{array}{c}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{array}\right) = \left(\begin{array}{c}\text{Rate of change}\\ \text{of the energy}\\ \text{content of}\\ \text{the element}\end{array}\right)$$

- For *steady* heat transfer (no changes in time):

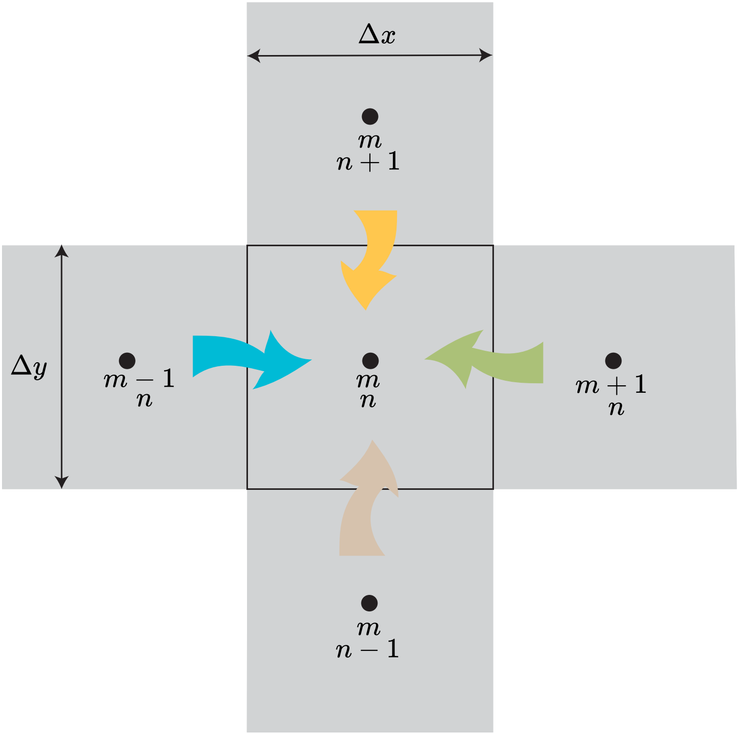
$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = 0$$

- Sum up heat transfer to/from volume element:

$$\dot{Q}_{\text{left,cond}} + \dot{Q}_{\text{right,cond}} + \dot{Q}_{\text{bottom,cond}} + \dot{Q}_{\text{top,cond}} + \dot{G}_{\text{element}} = 0$$

  - .. or in terms of node numbers:

$$kA\frac{T_{m-1,n} - T_{m,n}}{\Delta x} + kA\frac{T_{m+1,n} - T_{m,n}}{\Delta x} + kA\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + kA\frac{T_{m,n+1} - T_{m,n}}{\Delta y} + \dot{g}_{m,n}(\Delta x \Delta y) = 0$$



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The energy balance equation for *boundary nodes*

We *sum up heat transfer to/from* volume element:

$$\begin{pmatrix}\text{Sum of heat}\\ \text{transfer rate}\\ \text{across surfaces}\\ \text{of volume element}\end{pmatrix} + \begin{pmatrix}\text{Rate of heat}\\ \text{generation}\\ \text{inside the}\\ \text{element}\end{pmatrix} = \begin{pmatrix}\text{Rate of change}\\ \text{of the energy}\\ \text{content of}\\ \text{the element}\end{pmatrix}$$

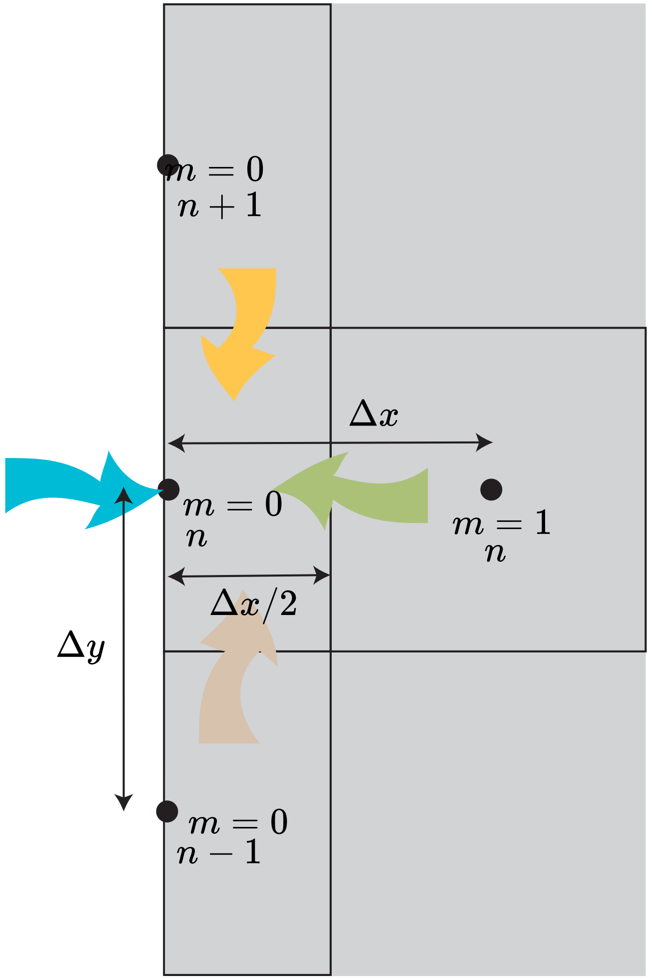
- For a volume element on the left boundary:

$$\dot{Q}_{\text{left,surface}} + \dot{Q}_{\text{right,cond}} + \dot{Q}_{\text{bottom,cond}} + \dot{Q}_{\text{top,cond}} + \dot{G}_{\text{element}} = 0$$

- .. or in terms of node numbers:

$$\dot{Q}_{\text{left surface}} + k\Delta y\frac{T_{1,n} - T_{0,n}}{\Delta x} + k\frac{\Delta x}{2}\frac{T_{0,n-1} - T_{0,n}}{\Delta y} + k\frac{\Delta x}{2}\frac{T_{0,n+1} - T_{0,n}}{\Delta y} + \dot{g}_{0,n}\left(\frac{\Delta x}{2}\Delta y\right) = 0$$

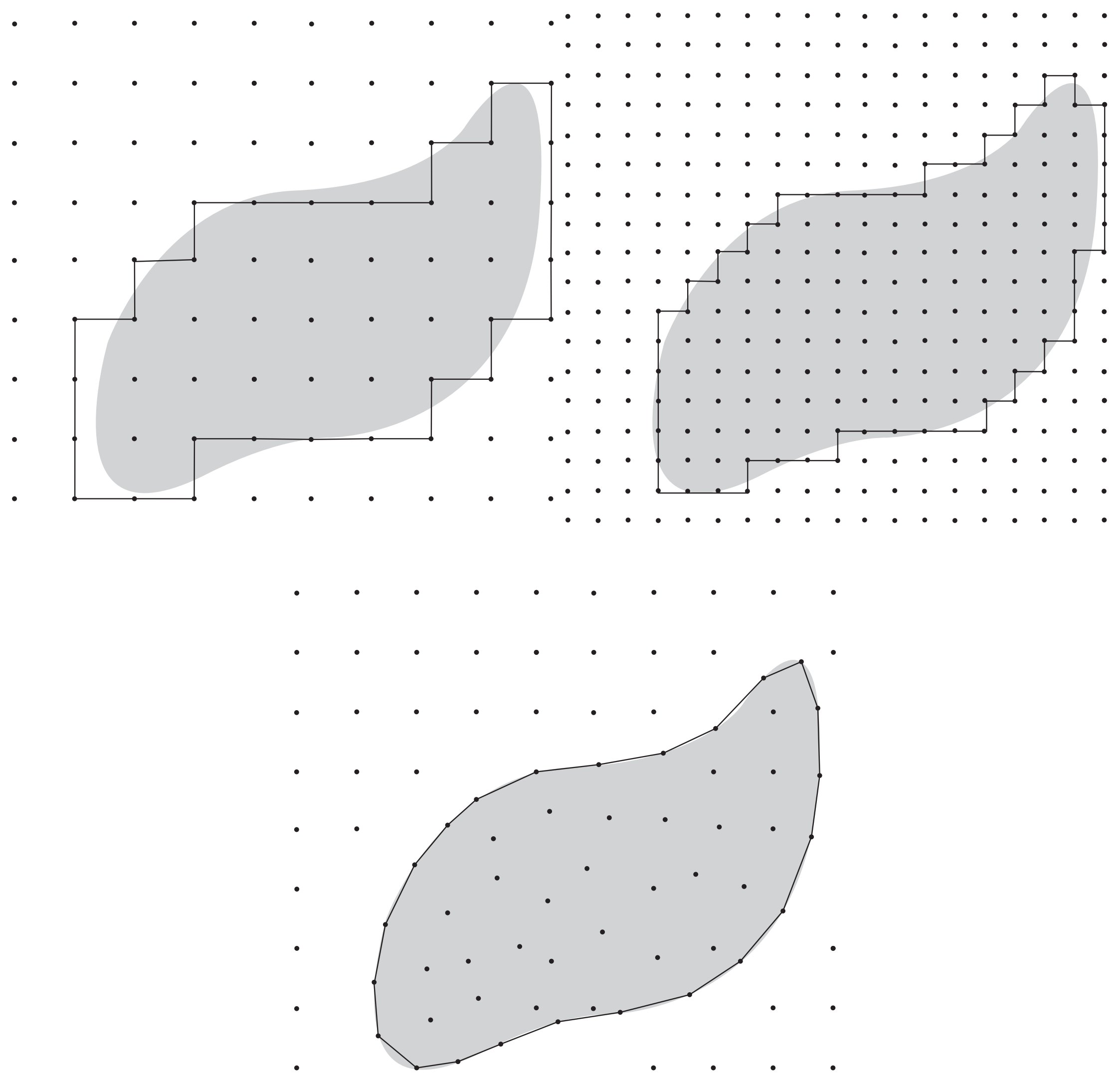
- The expression for $\dot{Q}_{\text{left surface}}$ depends on the type of boundary condition



Handling *irregular* boundaries

Different approaches to irregular boundaries:

- *Baseline*: Approximate the shape using regular grid (stair-step approach)
- Use *smaller* volume elements near the boundary to better approximate the shape
- Use *irregular* boundary volume elements that conform to the shape



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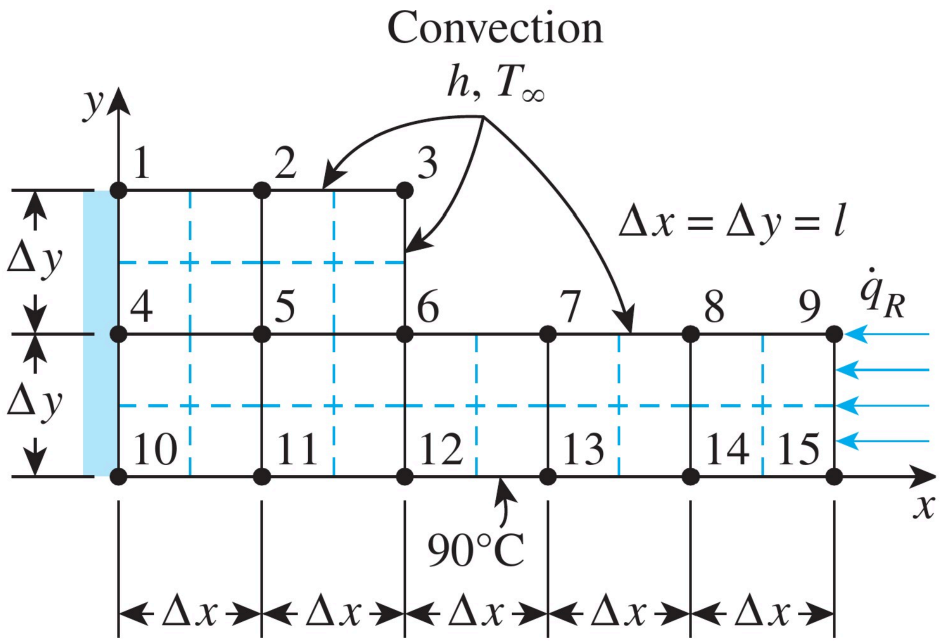
Exercise: Steady Two-Dimensional Heat Conduction

Determine the temperature at the 15 nodes. Thermal conductivity is $k = 15$ W/m·K, and heat is generated in the body at a rate of $\dot{q} = 2 \times 10^6$ W/m³. The left surface of the body is insulated, and the bottom surface is maintained at a uniform temperature of $90°\text{C}$. The entire top surface is subjected to convection to ambient air at $T_\infty = 25°\text{C}$ with a convection coefficient of $h = 80$ W/m²·K, and the right surface is subjected to heat flux at a uniform rate of $\dot{q}_R = 5000$ W/m². The nodal network of the problem consists of 15 equally spaced nodes with $\Delta x = \Delta y = 1.2$ cm.



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