# Steady heat conduction using numerical methods

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# Why look at *numerical methods for heat conduction*?

Learning about numerical methods for heat conduction is valuable for several reasons—both practical and conceptual:

## ✅ Real-world heat problems rarely have simple equations

Analytical solutions (like the ones from textbooks) only work for idealized cases—simple shapes, steady-state conditions, constant properties, etc.

In reality, heat conduction problems often involve:

- Complex geometries
- Time-dependent behavior (transient heat transfer)
- Non-uniform materials (like composites)
- Changing boundary conditions
- Phase changes (melting/freezing)

Numerical methods let you solve these realistic engineering problems.

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## ✅ It builds engineering intuition

Knowing the numerical foundations helps you:

- Understand stability, convergence, and error
- Choose correct grid sizes and time steps
- Spot unphysical results (like negative temperatures)
- Validate results instead of blindly trusting software

## ✅ It improves general problem-solving & programming skills

Heat conduction numerical methods teach:

- Finite difference / finite volume / finite element methods
- Linear algebra and matrix solvers
- Differential equations
- Algorithm design and coding (Python, MATLAB, C++, etc.)

These skills transfer to fluid mechanics, diffusion, vibrations, finance—even machine learning.

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## ✅ Industry relies on computational heat transfer

Industries where numerical heat conduction is essential:

| Field | Why it matters |
| --- | --- |
| Aerospace | thermal protection systems, cooling electronics in aircraft/spacecraft |
| Automotive | engine and battery cooling, cabin systems |
| Electronics | chip thermal management |
| Energy systems | heat exchangers, nuclear reactor walls, geothermal systems |
| Manufacturing | welding, casting, 3D printing thermal analysis |
| Materials science | conduction in composites, metals, insulators |

Tools like **ANSYS, COMSOL, Abaqus, OpenFOAM** use numerical methods internally. Understanding them helps you use these tools intelligently and avoid mistakes.

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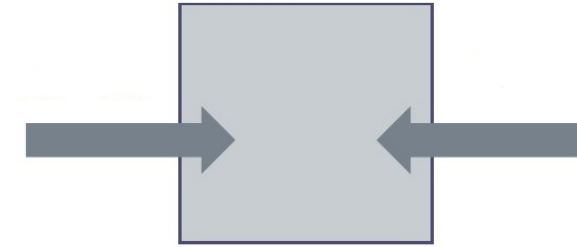# Two approaches to solve heat conduction problems

## Finite difference method

- Use equation for steady one-dimensional heat conduction as a starting point:

$$\frac{\mathrm{d}^2 T}{\mathrm{d}x^2} + \frac{\dot{g}}{k} = 0$$

- .. now replace $\frac{\mathrm{d}^2 T}{\mathrm{d}x^2}$ with *finite difference approximations* to form algebraic equations

## Energy balance method

- Consider a small control volume element



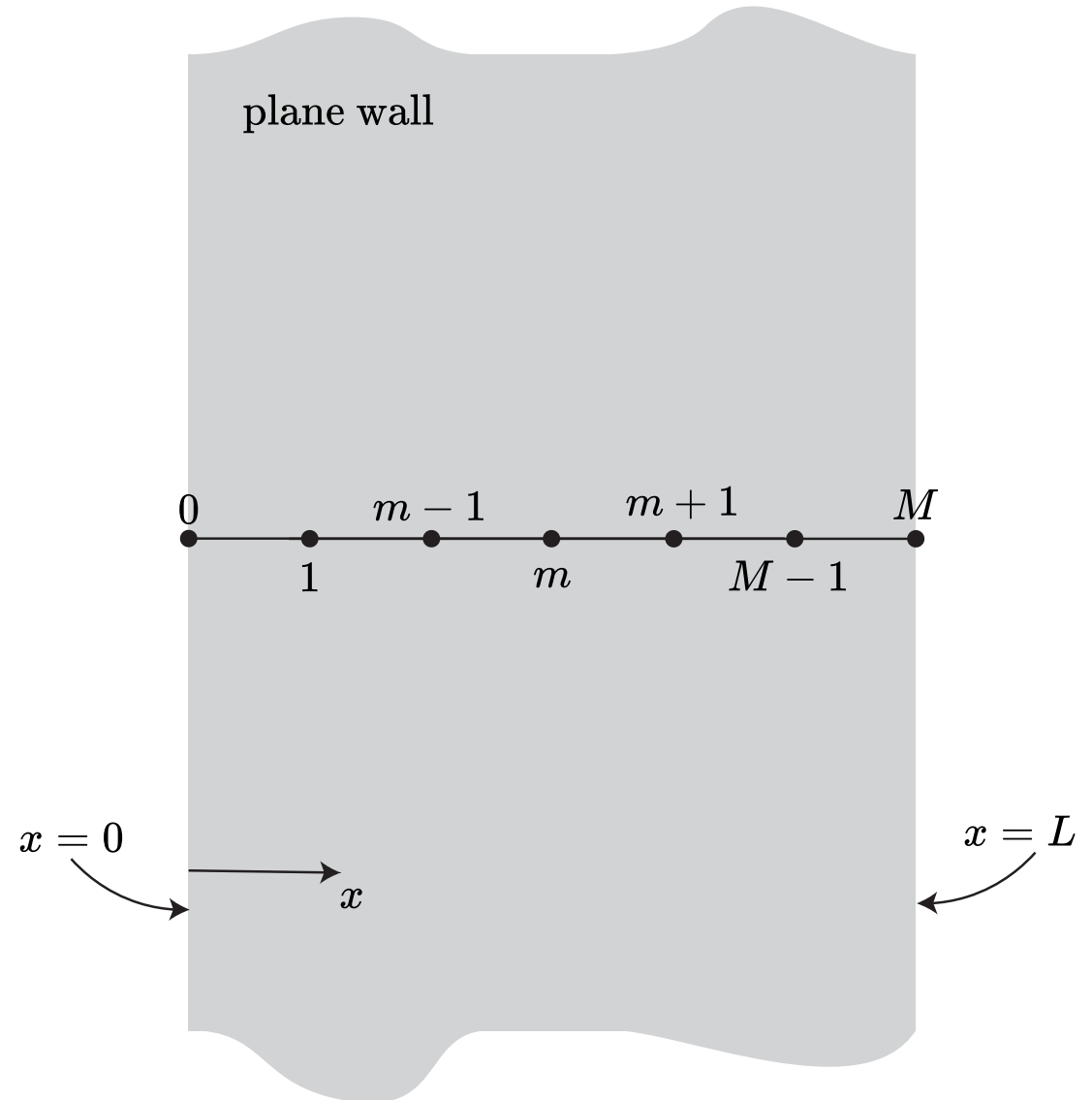- .. now write an *energy balance* on this volume element

$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = 0$$

$\leftarrow$

# Finite difference method for 1D problems

# Notation for *1D finite difference* problems

A *1D plane wall* is defined as being infinitely long (i.e. heat transfer only occurs in one direction)

- *Position* is denoted by $x$
  - $x = 0$ at left side of wall
  - $x = L$ at right side of wall
- *Node number* is denoted by $m$
  - $m = 1$ at left side of wall
  - $m = M$ at right side of wall

- Notation:
  - $T_5$: Temperature at $m = 5$
  - $\rho_5$: Density at node $m = 5$
  - ...

plane wall

$$0 \qquad m-1 \qquad m+1 \qquad M$$
$$\bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet\!-\!\!-\!\bullet$$
$$1 \qquad\qquad m \qquad M-1$$
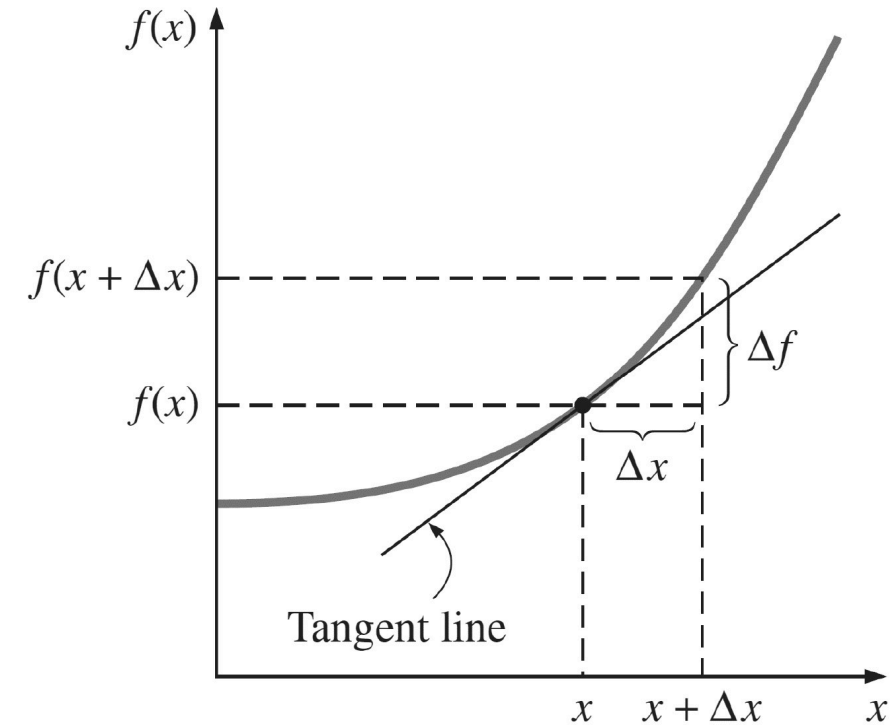
$x = 0$

$x$

$x = L$

←

# Recalling finite differences from old days 🙂

Consider a function $f$ that depends on $x$. The first derivative is then *defined* as the slope of the tangent:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

- Leaving out the limit, we get an *approximate* solution:

$$\frac{\mathrm{d}f(x)}{\mathrm{d}x} \approx \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

# Applying finite differences to our 1D domain

Remember, we are looking for:

$$\frac{\partial^2 T}{\partial x^2}$$

We may approximate first derivative *at midpoints* as:

$$\frac{\partial T}{\partial x}\Big|_{x_{m-1/2}} \approx \frac{T_m - T_{m-1}}{\Delta x} \quad \text{and} \quad \frac{\partial T}{\partial x}\Big|_{x_{m+1/2}} \approx \frac{T_{m+1} - T_m}{\Delta x}$$

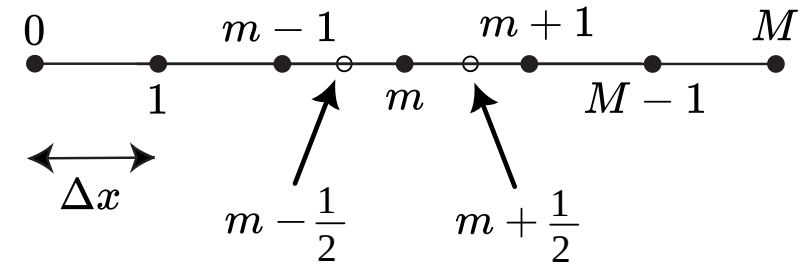... now, we can approximate the second derivative using two first-order derivatives:

$$\frac{\partial^2 T}{\partial x^2}\Big|_{x_m} \approx \frac{\frac{\partial T}{\partial x}\big|_{x_{m+1/2}} - \frac{\partial T}{\partial x}\big|_{x_{m-1/2}}}{\Delta x} = \frac{\frac{T_{m+1}-T_m}{\Delta x} - \frac{T_m - T_{m-1}}{\Delta x}}{\Delta x}$$

... which reduces to:

$$\frac{\partial^2 T}{\partial x^2}\Big|_{x_m} \approx \frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2}$$
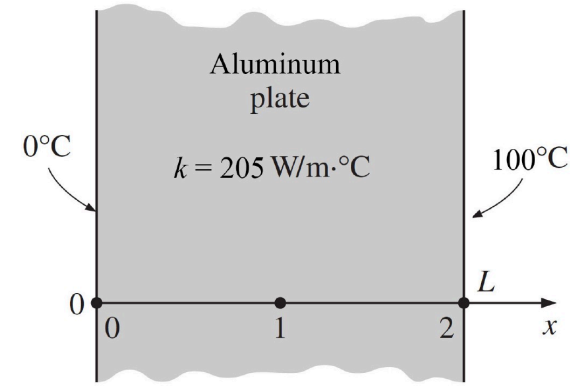
The equation for *steady one-dimensional heat conduction* now becomes:

$$\frac{T_{m-1} - 2T_m + T_{m+1}}{\Delta x^2} + \frac{\dot{g}}{k} = 0$$

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# Exercise: Centre temperature in a plate by the finite difference method

Consider a aluminium plate of length $L = 0.5$ cm and thermal conductivity $k = 205$ W/(m·K) in which no heat is generated ($\dot{g} = 0$). One side of the plate is maintained at 0 °C by icy water while the other is maintained at 100 °C. Estimate the temperature in the centre at node 1.
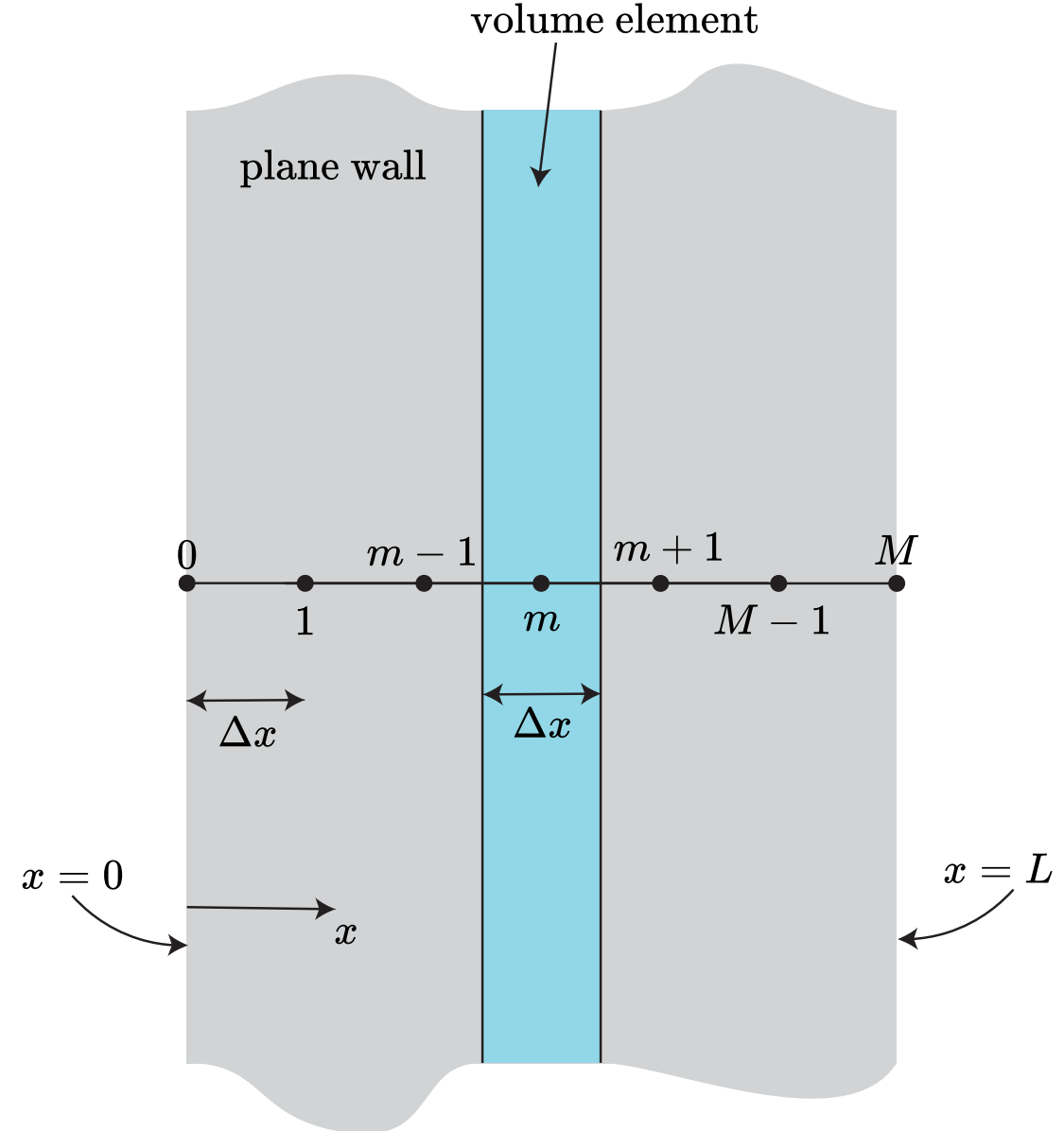
# Energy balance method for 1D problems

# Notation for *1D energy balance* problems

A *1D plane wall* is defined as being infinitely long (i.e. heat transfer only occurs in one direction)

- *Position* is denoted by $x$
  - $x = 0$ at left side of wall
  - $x = L$ at right side of wall
- *Node number* is denoted by $m$
  - $m = 1$ at left side of wall
  - $m = M$ at right side of wall
- *Volume elements* with *width $\Delta x$*
  - Properties are assumed constant within each volume element
  - The finer the mesh (smaller $\Delta x$), the more accurate the solution

- Notation:
  - $T_5$: Temperature at $m = 5$
  - $\rho_5$: Density at node $m = 5$
  - ...

$\leftarrow$

# The energy balance equation for interior nodes

We *sum up heat transfer to/from* volume element:

$$
\begin{pmatrix} \text{Sum of heat} \\ \text{transfer rate} \\ \text{across surfaces} \\ \text{of volume element} \end{pmatrix} + \begin{pmatrix} \text{Rate of heat} \\ \text{generation} \\ \text{inside the} \\ \text{element} \end{pmatrix} = \begin{pmatrix} \text{Rate of change} \\ \text{of the energy} \\ \text{content of} \\ \text{the element} \end{pmatrix}
$$

- For *steady* heat transfer (no change in time):

$$
\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = 0
$$

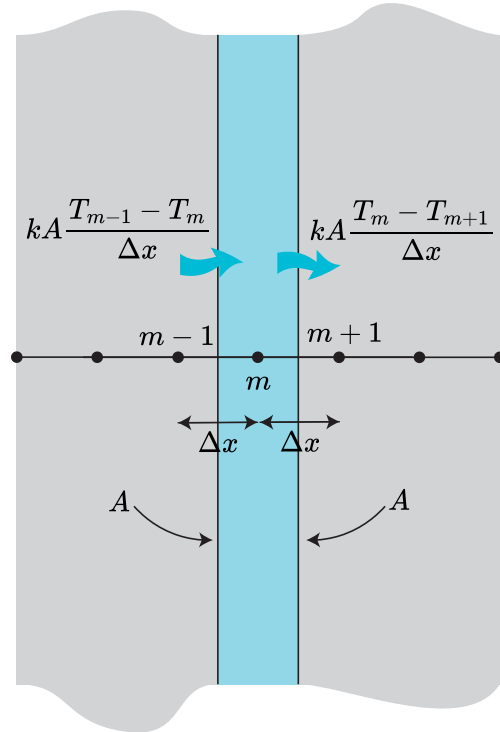- $\dot{G}_{\text{element}}$ is the heat generation rate in the element (W):

$$
\dot{G}_{\text{element}} = \dot{g} V_{\text{element}}
$$

  - $\dot{g}$ is the volumetric heat generation rate in the element ($\text{W/m}^3$)



$\dot{g}$

$\dot{Q}_{\text{left,surface}}$     $\dot{Q}_{\text{right,surface}}$

$m-1$     $m+1$

$m$

←

# Convention for heat transfer direction

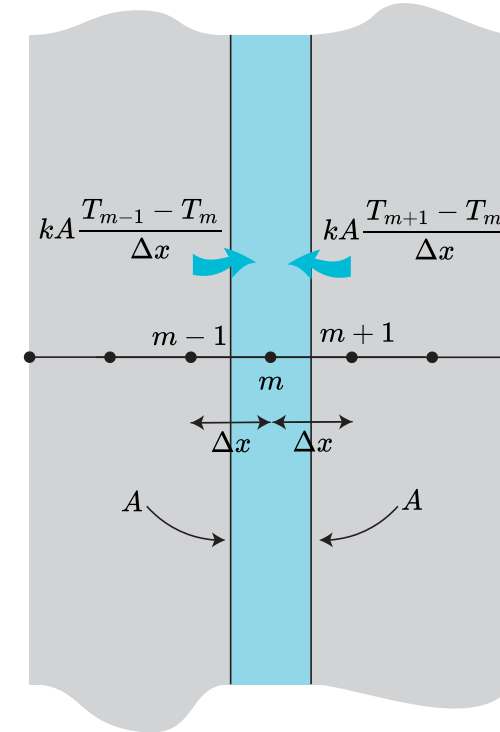Assumed direction of heat transfer is *irrelevant as long as we are consistent*



Energy balance on control volume:

$$kA\frac{T_{m-1} - T_m}{\Delta x} - kA\frac{T_m - T_{m+1}}{\Delta x} + \dot{g}_m A\Delta x = 0$$

**Conclusion:** In both cases, it reduces to *the same expression*:

$$T_{m-1} - 2T_m + T_{m+1} + \frac{\dot{g}_m \Delta x^2}{kA} = 0$$

Energy balance on control volume:

$$kA\frac{T_{m-1} - T_m}{\Delta x} + kA\frac{T_{m+1} - T_m}{\Delta x} + \dot{g}_m A\Delta x = 0$$

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# The four boundary condition types

We have to specify **boundary conditions** on each non-internal node.

- *Four common* boundary condition types:

1. *Constant temperature*

$$T_0 = c$$

2. *Constant heat flux*
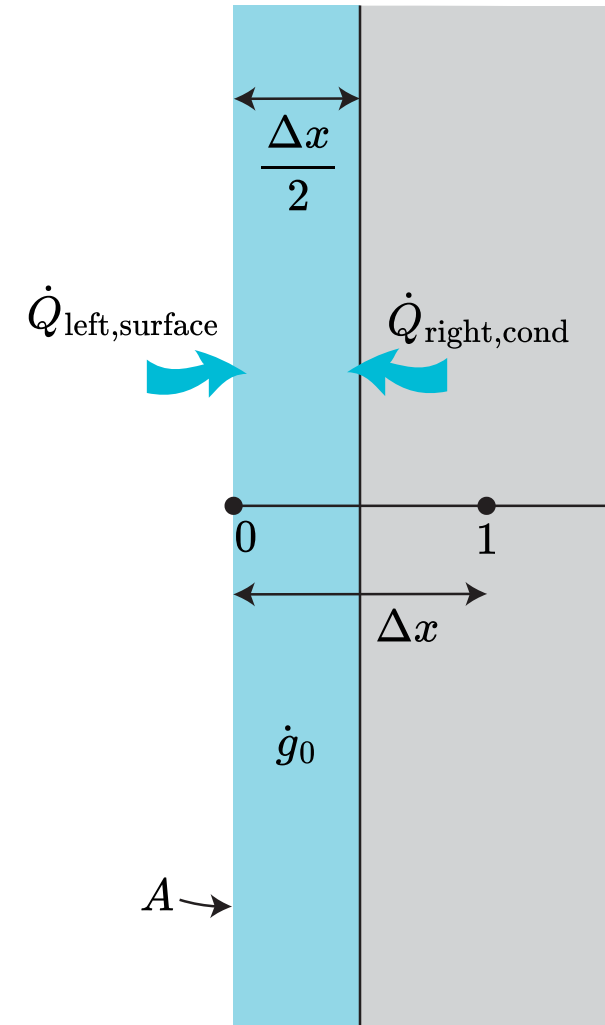
$$\dot{Q}_{\text{left,surface}} = \dot{q}A = c$$

3. *Convection*

$$\dot{Q}_{\text{left,surface}} = hA\left(T_\infty - T_0\right) = c$$

4. *Radiation*

$$\dot{Q}_{\text{left,surface}} = \epsilon\sigma A\left(T_{\text{surr}}^4 - T_0^4\right) = c$$

- .. or any combination of type 2, 3 and 4

$\leftarrow$

# Incorporating the different boundary conditions

**Type 1** (constant temperature):

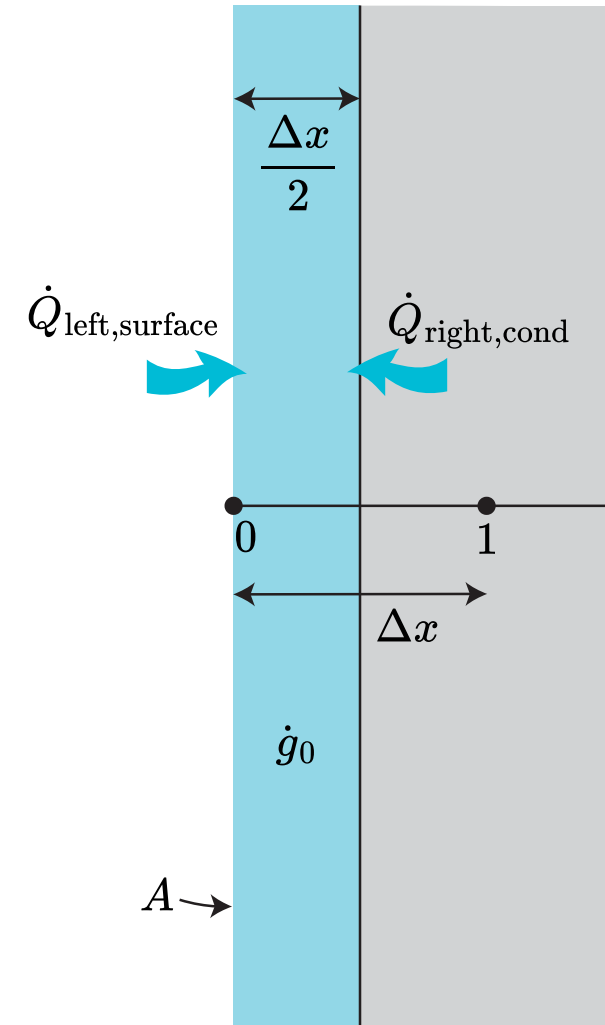- We simply *set the temperature* of the node, e.g. $T_0 = c$

**Type 2-4**:

- We apply an *energy balance* on the volume element:

$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = 0,$$

$$\dot{Q}_{\text{left,surface}} + kA\frac{T_1 - T_0}{\Delta x} + \dot{g}_0 \left(\frac{\Delta x}{2}A\right) = 0.$$

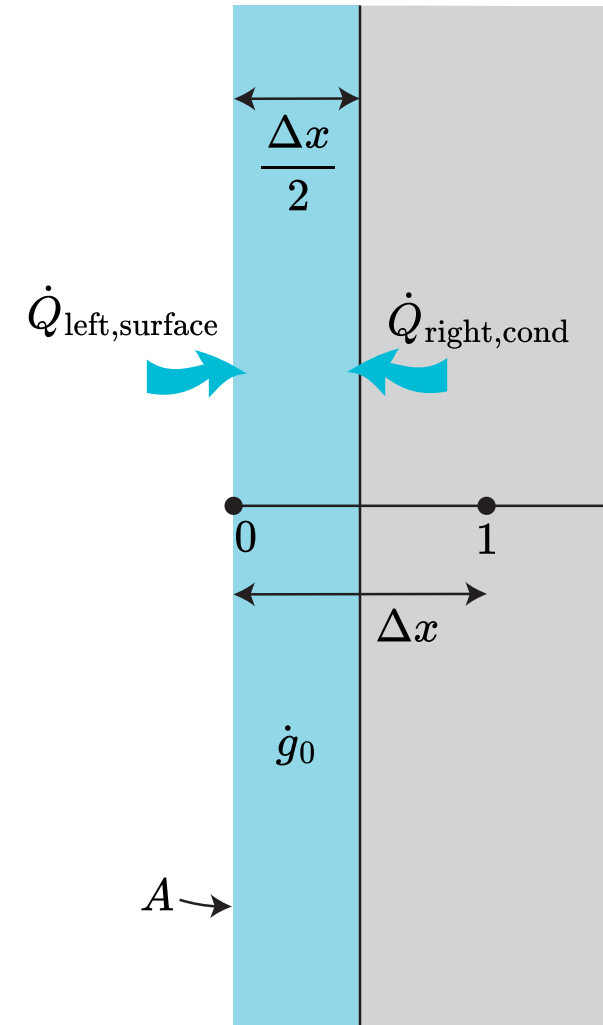- Expression for $\dot{Q}_{\text{left,surface}}$ depends on boundary condition type
type

# Incorporating mixed boundary condition types

All BC types (heat flux, convection and radiation) can easily be added
to form mixed types:

$$\dot{Q}_{\text{left,surface}} = \dot{q}_0 A \; + \; hA\left(T_\infty - T_0\right) \; + \; \epsilon\sigma A\left(T_{\text{surr}}^4 - T_0^4\right)$$

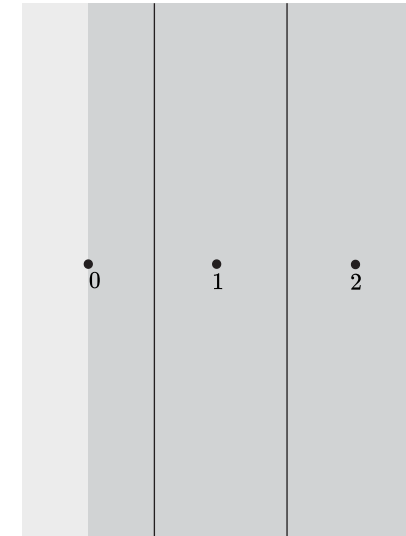# Mirror concept for insulated boundaries (e.g zero heat flux)

Insulated boundary (zero heat flux) can be modelled using *mirror nodes*:

- For interior nodes:

$$\frac{T_{m+1} - 2T_m + T_{m-1}}{\Delta x^2} + \frac{\dot{g}}{k} = 0$$

- For insulated boundaries:

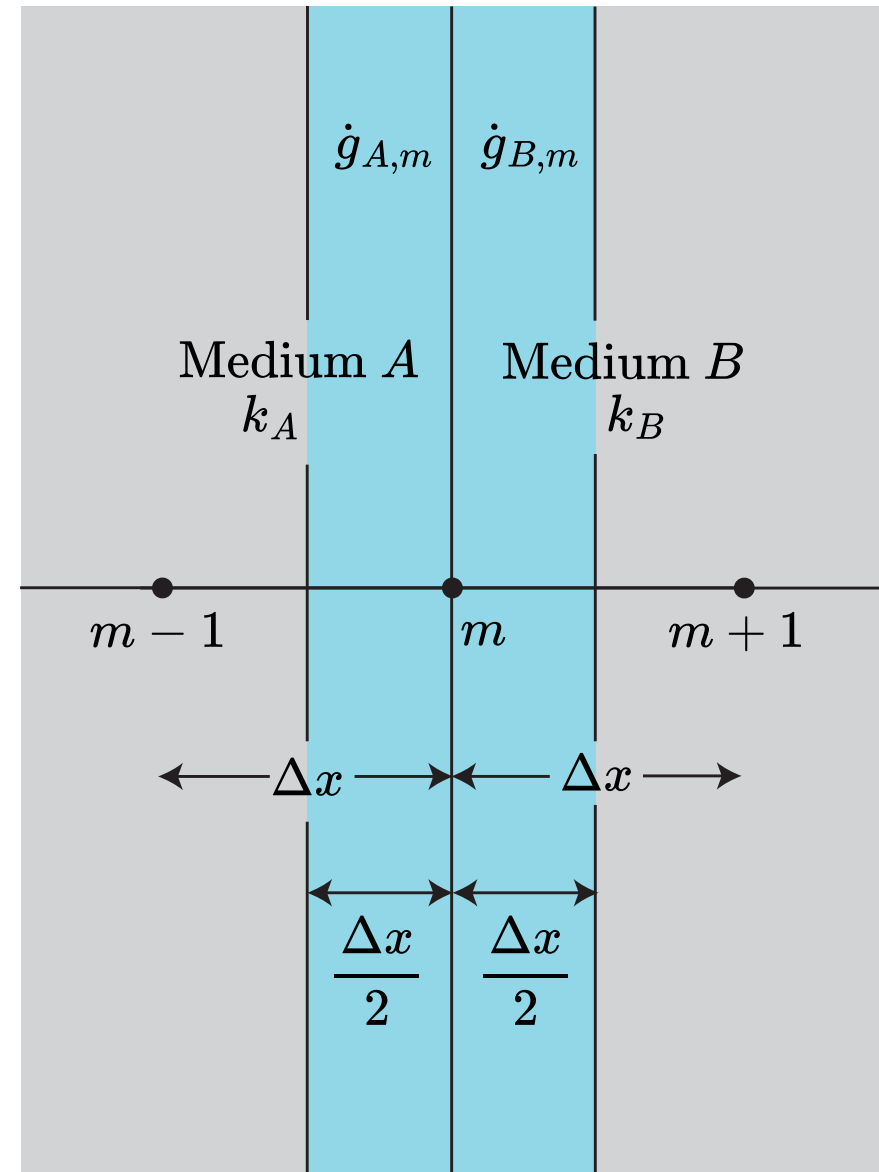$$\frac{T_1 - 2T_0 + T_1}{\Delta x^2} + \frac{\dot{g}}{k} = 0$$

# Interfaces between different materials

Starting with the general energy balance:

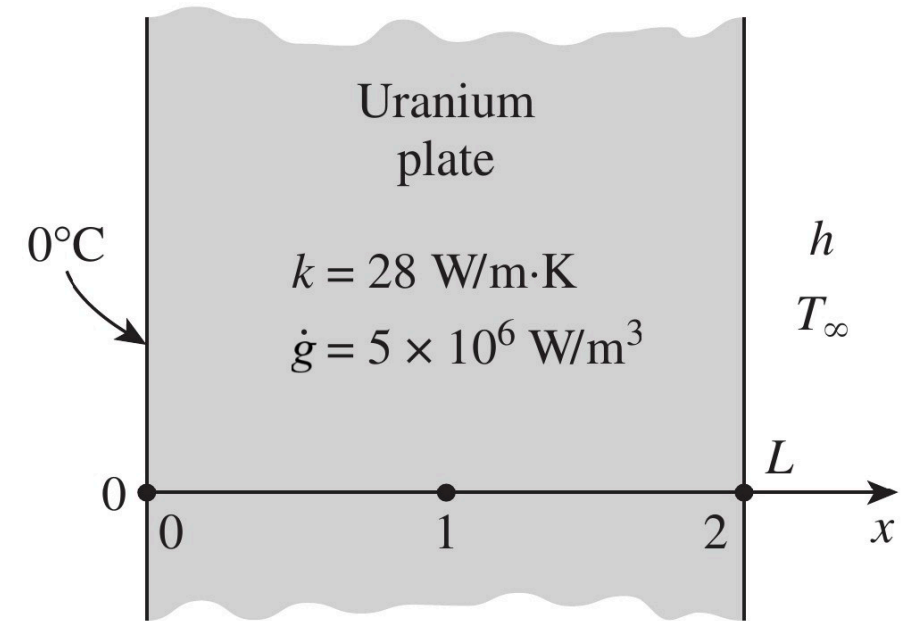$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = 0,$$

- Assuming *perfect contact (no air gaps)* at the interface, the energy balance becomes:

$$k_1 A \frac{T_{m-1} - T_m}{\Delta x_1} + k_2 A \frac{T_{m+1} - T_m}{\Delta x_2} + \dot{g}_A A \Delta \frac{x}{2} + \dot{g}_B A \Delta \frac{x}{2} = 0$$

# Exercise: Centre temperature in a plate by the energy balance method

Consider a aluminium plate of length $L = 4$ cm and thermal conductivity $k = 28$ W/(m·K) in which heat is generated at a rate $\dot{g} = 5 \cdot 10^6$ W/m³. The left side is maintained at 0°C and the right side is exposed to convection with $h = 45$ W/(m²·K) and ambient temperature of $T_\infty = 30$°C. Find $T_2$.



0°C

Uranium plate

$k = 28$ W/m·K

$\dot{g} = 5 \times 10^6$ W/m³

$h$

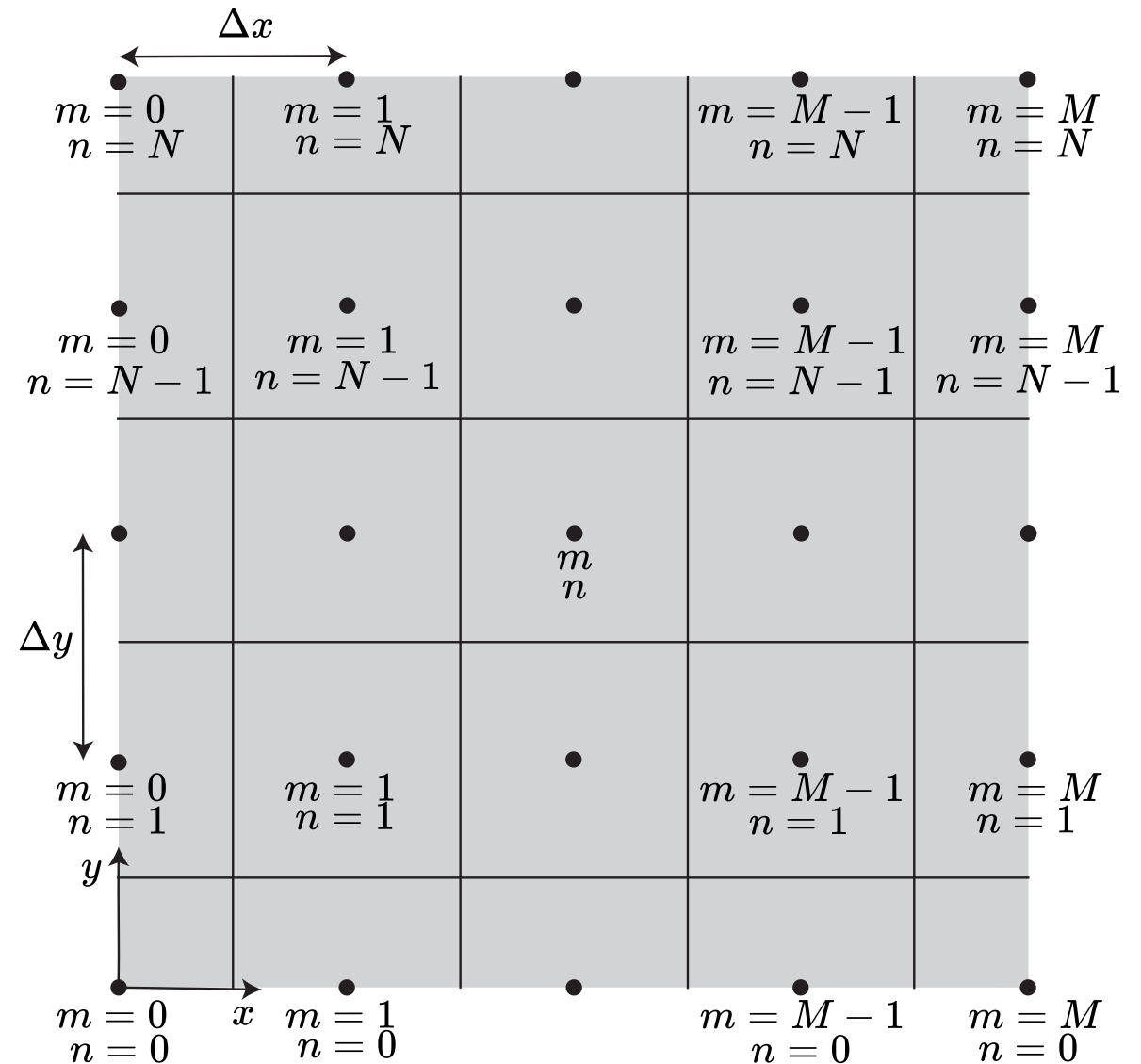$T_\infty$

$L$

0    1    2    $x$

# Energy balance method for 2D (and 3D) problems

# Notation for *2D energy balance* problems

*Double subscripts (m,n)* to denote two dimensions (triple for three dimensions)

- $m$ used for $x$-direction
- $n$ used for $y$-direction
- Temperature at node ($m = 1, n = 0$) is written $T_{1,0}$
- Distances between nodes are $\Delta x$ and $\Delta y$
- Form volume elements around each node
- Convert from node number to actual coordinates by:
  - $x = m\Delta x$
  - $y = n\Delta y$

# The energy balance equation for *interior nodes*

We *sum up heat transfer to/from* volume element:

$$\begin{pmatrix} \text{Sum of heat} \\ \text{transfer rate} \\ \text{across surfaces} \\ \text{of volume element} \end{pmatrix} + \begin{pmatrix} \text{Rate of heat} \\ \text{generation} \\ \text{inside the} \\ \text{element} \end{pmatrix} = \begin{pmatrix} \text{Rate of change} \\ \text{of the energy} \\ \text{content of} \\ \text{the element} \end{pmatrix}$$

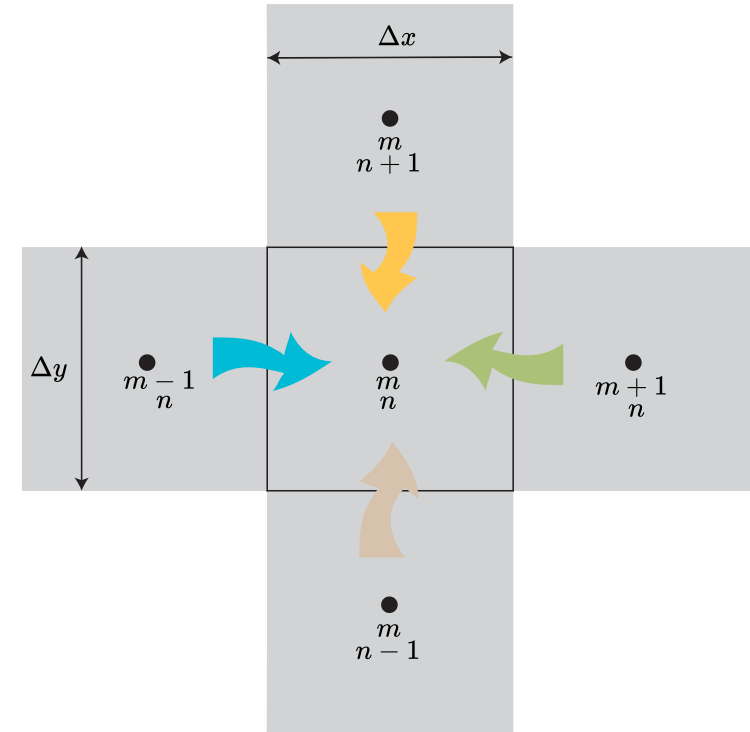- For *steady* heat transfer (no changes in time):

$$\sum_{\text{all surfaces}} \dot{Q} + \dot{G}_{\text{element}} = \frac{\Delta E_{\text{element}}}{\Delta t} = 0$$

- Sum up heat transfer to/from volume element:

$$\dot{Q}_{\text{left,cond}} + \dot{Q}_{\text{right,cond}} + \dot{Q}_{\text{bottom,cond}} + \dot{Q}_{\text{top,cond}} + \dot{G}_{\text{element}} = 0$$

- .. or in terms of node numbers:

$$kA\frac{T_{m-1,n} - T_{m,n}}{\Delta x} + kA\frac{T_{m+1,n} - T_{m,n}}{\Delta x}$$
$$+ kA\frac{T_{m,n-1} - T_{m,n}}{\Delta y} + kA\frac{T_{m,n+1} - T_{m,n}}{\Delta y} + \dot{g}_{m,n}(\Delta x \Delta y) = 0$$



←

# The energy balance equation for *boundary nodes*

We *sum up heat transfer to/from* volume element:

$$
\begin{pmatrix}
\text{Sum of heat} \\
\text{transfer rate} \\
\text{across surfaces} \\
\text{of volume element}
\end{pmatrix}
+
\begin{pmatrix}
\text{Rate of heat} \\
\text{generation} \\
\text{inside the} \\
\text{element}
\end{pmatrix}
=
\begin{pmatrix}
\text{Rate of change} \\
\text{of the energy} \\
\text{content of} \\
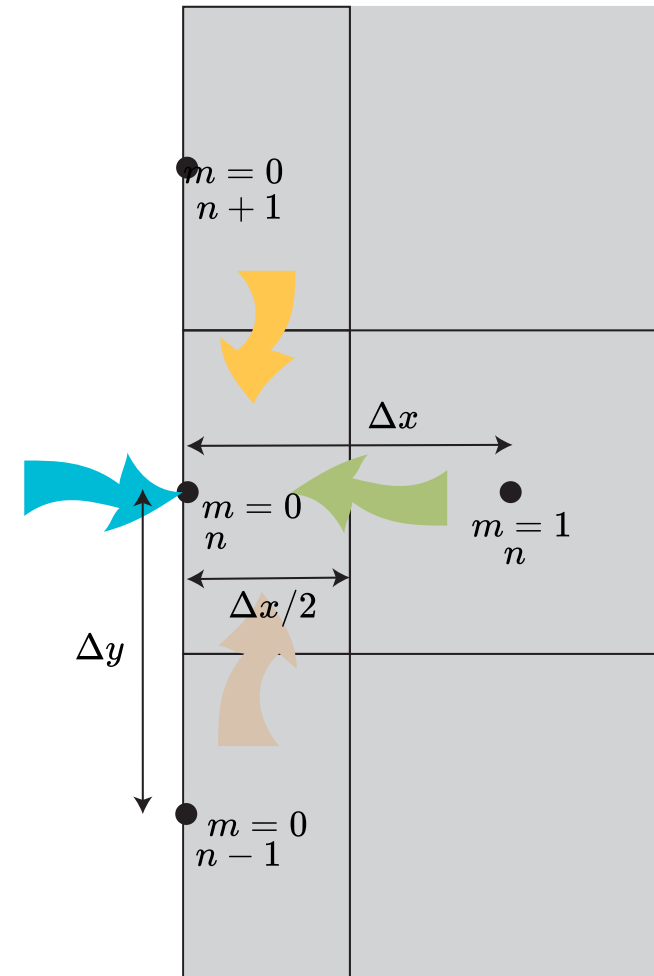\text{the element}
\end{pmatrix}
$$

- For a volume element on the left boundary:

$$
\dot{Q}_{\text{left,surface}} + \dot{Q}_{\text{right,cond}} + \dot{Q}_{\text{bottom,cond}} + \dot{Q}_{\text{top,cond}} + \dot{G}_{\text{element}} = 0
$$

- .. or in terms of node numbers:

$$
\dot{Q}_{\text{left surface}} + k\Delta y \frac{T_{1,n} - T_{0,n}}{\Delta x} + k\frac{\Delta x}{2}\frac{T_{0,n-1} - T_{0,n}}{\Delta y}
$$
$$
+ k\frac{\Delta x}{2}\frac{T_{0,n+1} - T_{0,n}}{\Delta y} + \dot{g}_{0,n}\left(\frac{\Delta x}{2}\Delta y\right) = 0
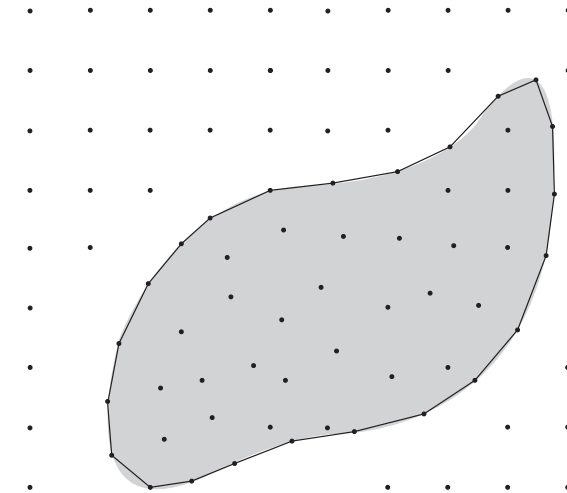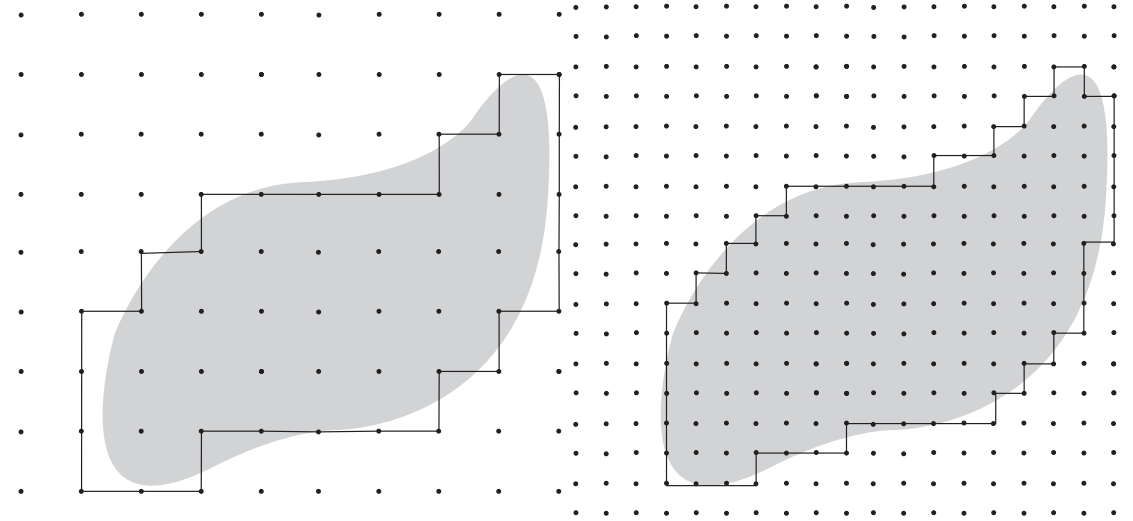$$

- The expression for $\dot{Q}_{\text{left surface}}$ depends on the type of boundary condition



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# Handling *irregular* boundaries

Different approaches to irregular boundaries:

- *Baseline*: Approximate the shape using regular grid (stair-step approach)
- Use *smaller* volume elements near the boundary to better approximate the shape
- Use *irregular* boundary volume elements that conform to the shape

# Exercise: Steady Two-Dimensional Heat Conduction

Determine the temperature at the 15 nodes. Thermal conductivity is $k = 15$ W/m·K, and heat is generated in the body at a rate of $\dot{q} = 2 \times 10^6$ W/m³. The left surface of the body is insulated, and the bottom surface is maintained at a uniform temperature of $90°$C. The entire top surface is subjected to convection to ambient air at $T_\infty = 25°$C with a convection coefficient of $h = 80$ W/m²·K, and the right surface is subjected to heat flux at a uniform rate of $\dot{q}_R = 5000$ W/m². The nodal network of the problem consists of 15 equally spaced nodes with $\Delta x = \Delta y = 1.2$ cm.