Heat Transfer

Analytical solutions for heat conduction

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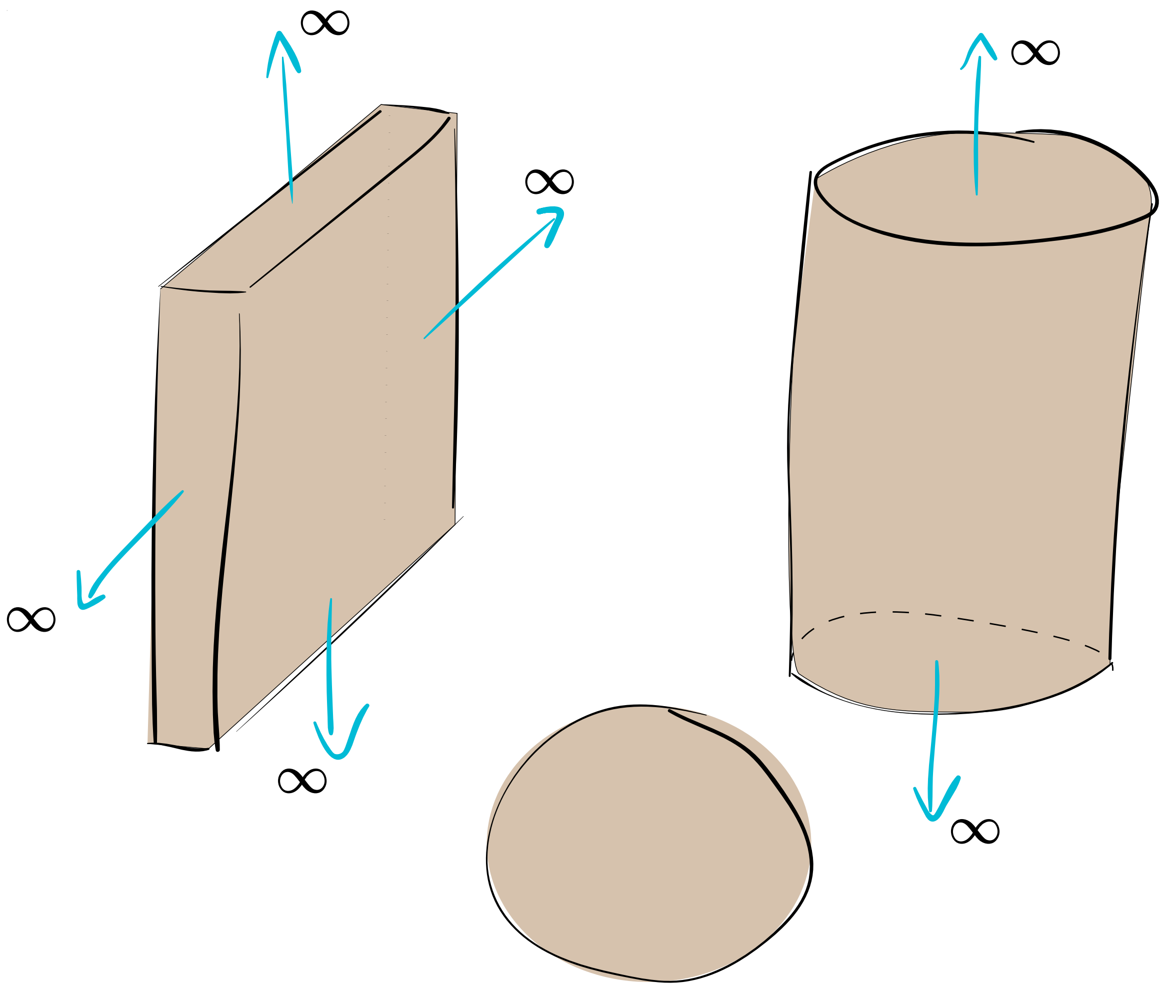
Why/why not analytical solutions?

Limitations

- Limited to *simplified geometries:*
  - Infinitely long plane wall
  - Infinitely long cylinder
  - Perfect sphere
- Limited to *simple conditions with thermal symmetry:*
  - Uniform and temperature-independent properties
  - Only convection on boundary with constant heat transfer coefficient, e.g. no heat generation inside, radiation on boundary etc
  - Uniform initial temperature



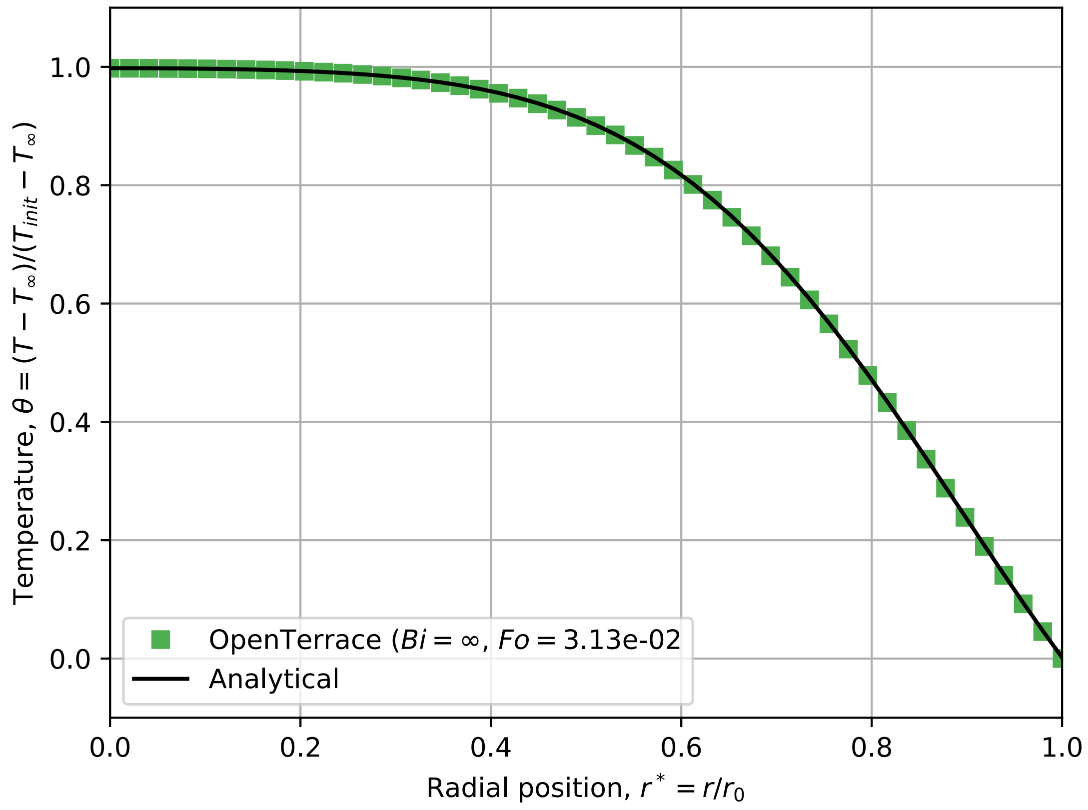
Reasons to apply

- Super *fast* to compute, e.g. doing millions of evaluations
- Yield *exact results*, e.g. excellent for verifying numerical codes

Verification studies

The OpenTerrace code is tested against known analytical solution in limiting cases. The following presents such code verification results. These are run automatically as part of the automated tests using GitHub actions.

Pure diffusion



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Pure advection

Starting point of analytical solutions

Partial differential equation:

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$$

- Boundary conditions:

  At $x = 0$:

$$\frac{\partial T(0,t)}{\partial x} = 0$$

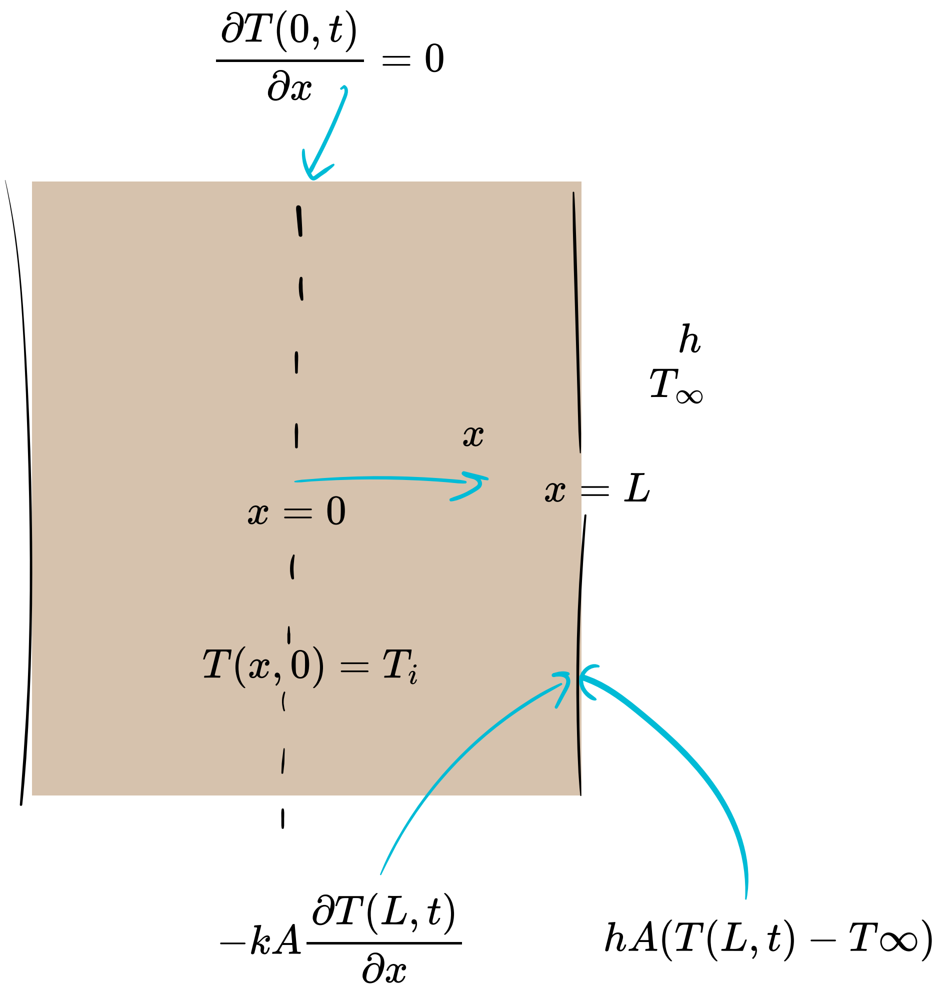
  At $x = L$:

$$-kA\frac{\partial T(L,t)}{\partial x} = hA\left(T(L,t) - T_\infty\right)$$

- Initial condition:

  At $t = 0$:

$$T(x,0) = T_i$$



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Introduction of non-dimensional variables

Convert problem into *non-dimensional* form to *reduce number of variables*

- Non-dimensional temperature:

$$\theta(x,t) = \frac{T(x,t) - T_\infty}{T_i - T_\infty}$$

- Non-dimensional position:

$$X = \frac{x}{L}$$

- Non-dimensional heat transfer coefficient (Biot number):

$$\mathrm{Bi} = \frac{hL}{k}$$

- Non-dimensional time (Fourier number):

$$\tau = \frac{\alpha t}{L^2}$$

Re-casting problem in non-dimensional form

Starting from original partial differential equation:

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$$

Then, inserting definitions of non-dimensional numbers:

- Inserting $X = x/L$:

$$\frac{\partial^2 T(X,t)}{\partial (X \cdot L)^2} = \frac{1}{\alpha}\frac{\partial T(X,t)}{\partial t} \Rightarrow \frac{\partial^2 T(X,t)}{\partial X^2} = \frac{L^2}{\alpha}\frac{\partial T(X,t)}{\partial t}$$

- Inserting Fourier number $\tau = \alpha t/L^2$:

$$\frac{\partial^2 T(X,\tau)}{\partial X^2} = \frac{\partial T(X,\tau)}{\partial \tau}$$

- Inserting non-dimensional temperature $\theta = \dfrac{T - T_\infty}{T_i - T_\infty}$:

$$\frac{\partial^2 \theta(X,\tau)}{\partial X^2} = \frac{\partial \theta(X,\tau)}{\partial \tau}$$

Equation now transformed to non-dimensional form

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Final formulation in non-dimensional form

Partial differential equation:

$$\frac{\partial^2 T(x,t)}{\partial x^2} = \frac{1}{\alpha}\frac{\partial T(x,t)}{\partial t}$$

Boundary conditions:

$$x = 0\text{:} \quad \frac{\partial T(0,t)}{\partial x} = 0$$

$$x = L\text{:} \quad -k\frac{\partial T(L,t)}{\partial x} = h(T(L,t) - T_\infty)$$

Initial conditions:

$$t = 0\text{:} \quad T(x,0) = T_i$$

Original problem:

- 9 variables: $T, x, L, t, k, \alpha, h, T_i, T_\infty$

Non-dimensional partial differential equation:

$$\frac{\partial^2 \theta(X,\tau)}{\partial X^2} = \frac{\partial T(X,\tau)}{\partial \tau}$$

Non-dimensional boundary conditions:

$$X = 0\text{:} \quad \frac{\partial \theta(0,\tau)}{\partial X} = 0$$

$$X = 1\text{:} \quad \frac{\partial \theta(1,\tau)}{\partial X} = -\mathrm{Bi}\theta(1,\tau)$$

Non-dimensional initial conditions:

$$\tau = 0\text{:} \quad \theta(X,0) = 1$$

Non-dimensional transformed problem:

- **Only 4 variables:** $\theta, X, \mathrm{Bi}, \tau$

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A look at the solution for a plane wall

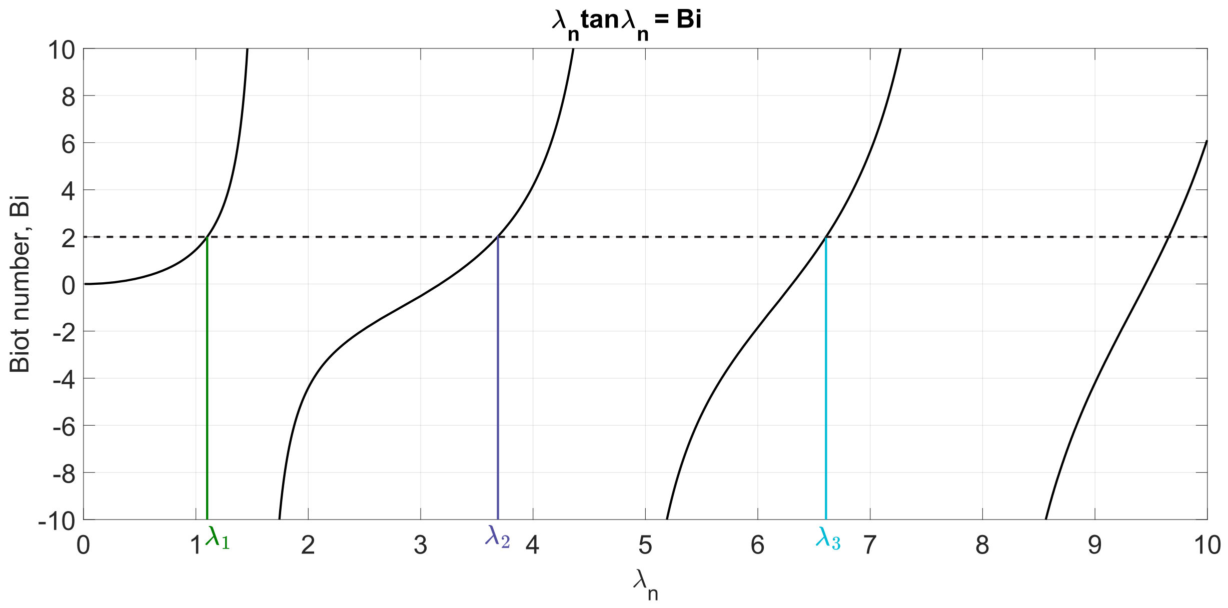
Analytical solution:

$$\theta = \sum_{n=1}^{\infty} \frac{4\sin\lambda_n}{2\lambda_n + \sin(2\lambda_n)} e^{-\lambda_n^2 \tau} \cos\left(\frac{\lambda_n x}{L}\right)$$

where λ_n are the roots of: $\lambda_n \tan\lambda_n = \mathrm{Bi}$

which we can expand as:

$$\theta = \frac{4\sin\lambda_1}{2\lambda_1 + \sin(2\lambda_1)} e^{-\lambda_1^2 \tau} \cos\left(\frac{\lambda_1 x}{L}\right) + \frac{4\sin\lambda_2}{2\lambda_2 + \sin(2\lambda_2)} e^{-\lambda_2^2 \tau} \cos\left(\frac{\lambda_2 x}{L}\right) + \frac{4\sin\lambda_3}{2\lambda_3 + \sin(2\lambda_3)} e^{-\lambda_3^2 \tau} \cos\left(\frac{\lambda_3 x}{L}\right) + \ldots$$



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Example: centre temperature in insulation

Find the centre temperature in a 10 cm thick layer of insulation after 30 minutes. Initially, its temperature is 10°C and at time $t = 0$ s it's suddenly exposed to an ambient temperature of 25°C on one side. Assume its properties are:

- Plate half thickness, $L = 0.05$ m
- Initial temperature, $T_i = 10^\circ$C
- Ambient temperature, $T_\infty = 25^\circ$C
- Heat transfer coefficient, $h = 20$ W/(m^{2}K)
- Thermal conductivity, $k = 0.05$ W/(m K)
- Thermal diffusivity, $\alpha = 3 \cdot 10^{-7}$ m^2/s

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One-term approximations

- Assume all terms for $n \geq 2$ **to be zero**, e.g. $\theta_2 = 0$, $\theta_3 = 0$ ...

$$\theta = \frac{4\sin\lambda_1}{2\lambda_1 + \sin(2\lambda_1)} e^{-\lambda_1^2\tau} \cos\left(\frac{\lambda_1 x}{L}\right) + \cancel{\frac{4\sin\lambda_2}{2\lambda_2 + \sin(2\lambda_2)} e^{-\lambda_2^2\tau} \cos\left(\frac{\lambda_2 x}{L}\right) + \frac{4\sin\lambda_3}{2\lambda_3 + \sin(2\lambda_3)} e^{-\lambda_3^2\tau} \cos\left(\frac{\lambda_3 x}{L}\right) + \ldots}$$

- Typically less than 2% error if $\tau = \alpha t / L_c^2 > 0.2$