## Fluid Mechanics

# Analytical solutions for simple fluid flows

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# Complexities involved in Navier-Stokes equations

*No universal analytical solution* due to non-linearities ($u\partial u/\partial x$)

- Solution *approximated* using *numerical methods* (e.g. CFD)
- Solution found by *experiments*

Analytical solutions for *only a few problems*

- *Simple geometries*: Flow between two plates or in a tube
- *Simple flows*: Steady, laminar and fully-developed flows

*x-direction:*

$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho g_x$$

*y-direction:*

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho g_y$$

*z-direction:*

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho g_z$$

# Why look at these *simplified* problems?

For which applications is the flow between two plates important?

Flow between two plates—whether *pressure-driven* (plane Poiseuille flow) or *shear-driven* (Couette flow)—is a **canonical model** that underpins both engineering practice and natural phenomena. It's important in many applications, including:

### Engineering & Technology

- **Lubrication**: Thin films of oil in bearings, pistons, gears, or seals are modeled as flow between parallel plates.
- **Microfluidics**: Lab-on-a-chip devices use narrow rectangular channels, which are essentially parallel plate flows.
- **Heat exchangers**: Plate-type exchangers rely on fluid moving between closely spaced plates for efficient heat transfer.
- **Coating & printing**: Flow of paint, varnish, or ink in thin layers is approximated by plate flows.
- **Polymer processing**: Sheet extrusion and film casting involve molten polymer flow between plates.

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### Everyday Phenomena

- **Sticking of wet surfaces**: Two glass slides or a credit card and plastic sleeve resist separation due to viscous flow in the thin liquid film.
- **Windshield wipers**: The thin water film between blade and glass behaves like a parallel-plate channel.
- **Food preparation**: Spreading butter, cream, or sauces between slices involves thin-film flow.

### Biology & Medicine

- **Synovial joints**: Cartilage surfaces slide with synovial fluid in between, modeled as shear-driven thin-film flow.
- **Blood in microvessels**: Capillaries and arterioles often approximate parallel-plate geometries at small scales.
- **Dialysis and filtration**: Fluids move between thin membranes in channels resembling plate flows.

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### Fundamental & Educational Value

- Provides one of the **few exact solutions of Navier–Stokes**—great for teaching velocity profiles, shear stress, and pressure gradients.
- Serves as a **building block** for more complex flows (e.g., lubrication theory, Reynolds equation).
- Helps illustrate the importance of viscosity, low Reynolds number flows, and scaling laws.

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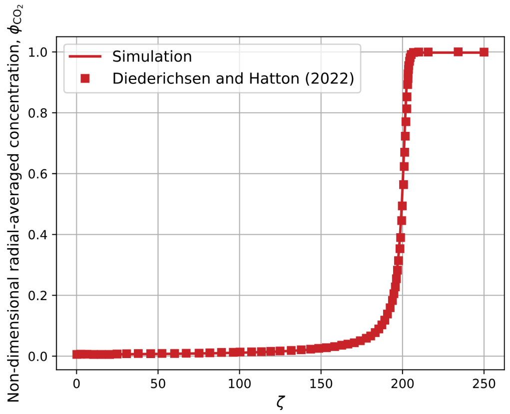
# Current research uses *Poiseuille flow* for verification of CFD

$CO_2$ diffuses into the fiber and is captured

- Really (!) long fibres ($l/D \approx 12000$)
- Steady, laminar, fully-developed flow is a *very good* assumption
- Prescribed boundary condition

*Results*: Code verification results



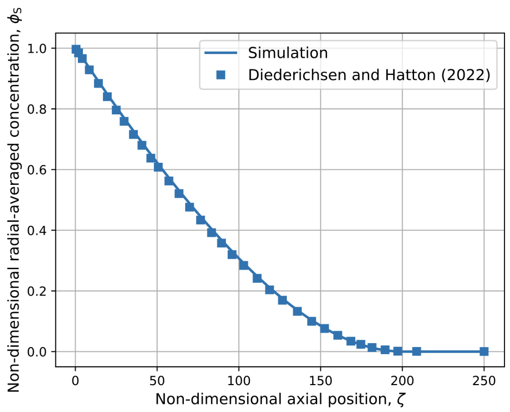
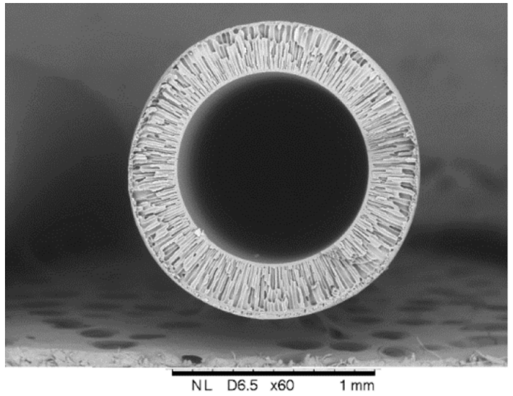
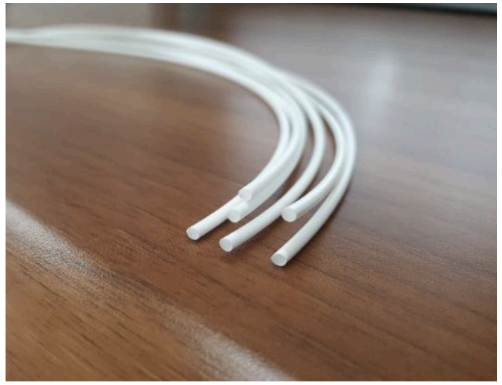




*Figure*: Hollow fibre membrane



Bazhenov, S. D., Bildyukevich, A. V., & Volkov, A. V. (2018). Gas-Liquid Hollow Fiber Membrane Contactors for Different Applications. Fibers, 6(4), 76. https://doi.org/10.3390/fib6040076

*Code*: Inlet boundary condition in OpenFOAM

```

/*--------------------------------*- C++ -*----------------------------------*\
  =========                 |
  \\      /  F ield         | OpenFOAM: The Open Source CFD Toolbox
   \\    /   O peration     | Website:  https://openfoam.com
    \\  /    A nd           | Version:  2506
     \\/     M anipulation  |
\*---------------------------------------------------------------------------*/
FoamFile
{
    format      ascii;
    class       volVectorField;
    location    "0";
    object      U;
}
// * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * * //

dimensions      [0 1 -1 0 0 0 0];

```

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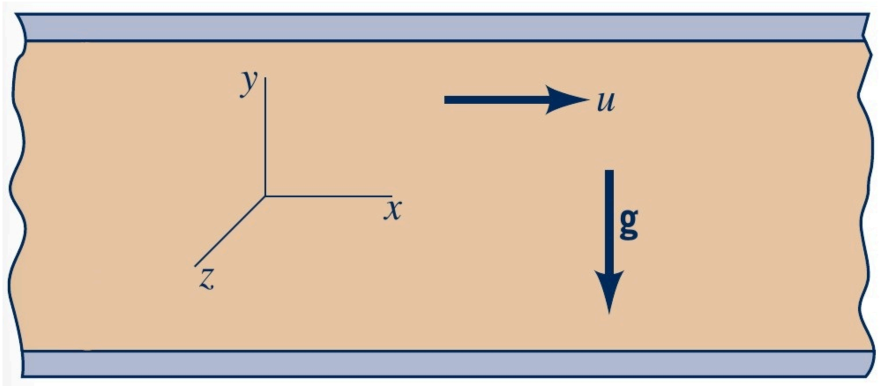
# Flow between plates

*Incompressible*, *steady*, *fully developed*, *laminar flow* between two *infinite plates*




- Velocity only in $x$-direction:
  - *then,* $u = u(y), v = w = 0$
- Steady flow:
  - *then,* $\partial u/\partial t = \partial v/\partial t = \partial w/\partial t = 0$
- By continuity equation: $\partial u/\partial x + \partial v/\partial y + \partial w/\partial z = 0$
  - *then,* $\partial u/\partial x = 0$
- No gravity in $x$ and $z$ directions:
  - *then,* $g_x = g_z = 0$

Navier-Stokes equations *reduce* to:

$$x : 0 = -\frac{\partial p}{\partial x} + \mu\frac{\partial^2 u}{\partial y^2}, \quad y : 0 = -\frac{\partial p}{\partial y} + \rho g_y, \quad z : 0 = -\frac{\partial p}{\partial z}$$

**$x$-direction:**

$$\rho\left(\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} + w\frac{\partial u}{\partial z}\right) = -\frac{\partial p}{\partial x} + \mu\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}\right) + \rho g_x$$

**$y$-direction:**

$$\rho\left(\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial x} + v\frac{\partial v}{\partial y} + w\frac{\partial v}{\partial z}\right) = -\frac{\partial p}{\partial y} + \mu\left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2}\right) + \rho g_y$$

**$z$-direction:**

$$\rho\left(\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial x} + v\frac{\partial w}{\partial y} + w\frac{\partial w}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2}\right) + \rho g_z$$

# Flow between *two stationary* plates

Integrating twice to get $u(y)$:

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\mu}\frac{\partial p}{\partial x}y + C_1$$

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + C_1 y + C_2$$

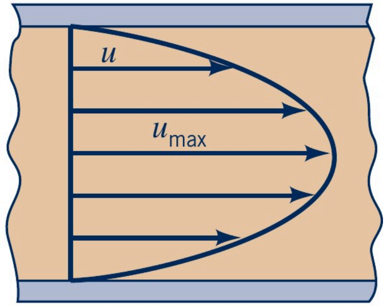
Using *boundary conditions*:

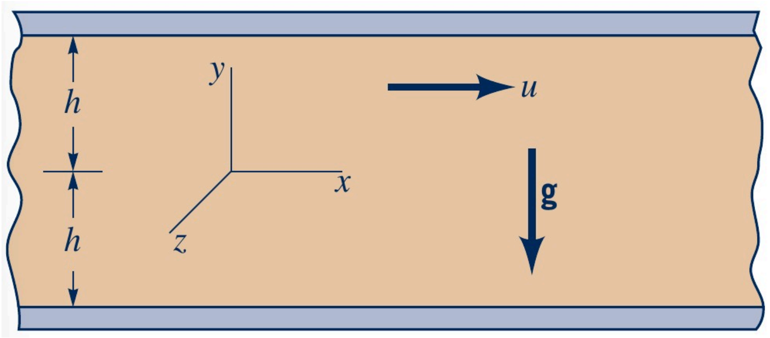
At $y = \pm h$ we have $u = 0$

$$C_1 = 0, C_2 = -\frac{1}{2\mu}\frac{\partial p}{\partial x}h^2$$

*Velocity profile* (by inserting $C_1$ and $C_2$):

$$u(y) = \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - h^2)$$




*Volume flow rate* (integrating velocity profile):

$$q = \int_{-h}^{h} u(y) = -\frac{2h^3}{3\mu}\left(\frac{\partial p}{\partial x}\right)$$

*Average velocity*:

$$V = \frac{q}{2h} = -\frac{h^2}{3\mu}\left(\frac{\partial p}{\partial x}\right)$$

*Maximum velocity* (centre of channel, $y = 0$):

$$u_{max} = u(y=0) = -\frac{h^2}{2\mu}\left(\frac{\partial p}{\partial x}\right) = \frac{3}{2}V$$

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# Flow between *one stationary* and *one moving* plate (aka Couette flow)

Integrating twice to get $u(y)$:

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu}\frac{\partial p}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\mu}\frac{\partial p}{\partial x}y + C_1$$

$$u = \frac{1}{2\mu}\frac{\partial p}{\partial x}y^2 + C_1 y + C_2$$

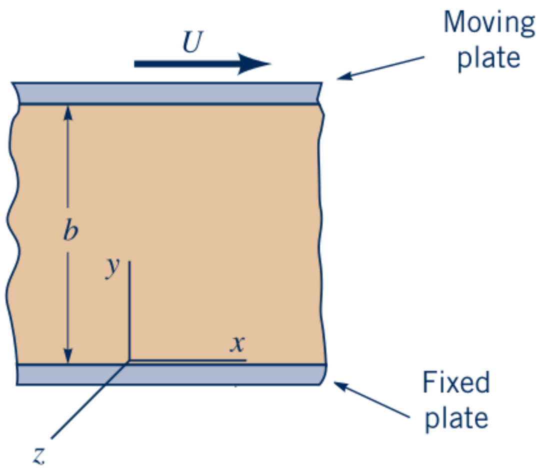
Using *boundary conditions*:

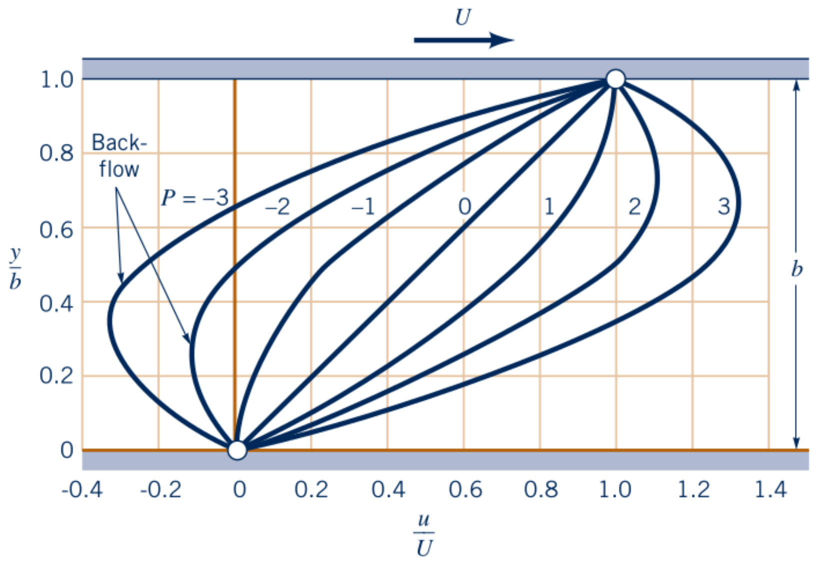
At $y = 0$ we have $u = 0$

At $y = b$ we have $u = U$

Finding $C_1$ and $C_2$ and inserting:

$$u(y) = U\frac{y}{b} + \frac{1}{2\mu}\frac{\partial p}{\partial x}(y^2 - by)$$




$$P = -\frac{b^2}{2\mu U}\left(\frac{\partial p}{\partial x}\right)$$

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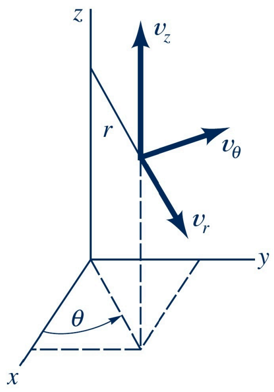
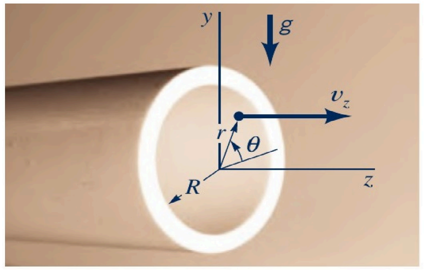
# Flow in a *circular pipe* (aka Poiseuille flow)

*Incompressible*, *steady*, *fully developed*, *laminar flow* in a circular pipe




- Velocity only in $z$-direction:
  - *then,* $v_z = v_z(r), v_\theta = v_r = 0$
- Steady flow:
  - *then,* $\partial u/\partial t = \partial v/\partial t = \partial w/\partial t = 0$
- By continuity equation: $\frac{1}{r}\frac{\partial(rv_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$
  - *then,* $\partial v_z/\partial z = 0$
- No gravity in $z$ direction:
  - *then,* $g_z = 0$

Navier-Stokes equations *reduce* to:

$$z : 0 = -\frac{\partial p}{\partial z} + \mu\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right), \quad r : 0 = \rho g_r - \frac{\partial p}{\partial r}, \quad \theta : 0 = \rho g_\theta - \frac{1}{r}\frac{\partial p}{\partial \theta}$$

*Integration*, applying *boundary condtions* and *find $C_1$* and $C_2$:

$$v_z(r) = \frac{1}{4\mu}\left(\frac{\partial p}{\partial z}\right)(r^2 - R^2)$$

- Velocity is a function of *viscosity* $\mu$, *pressure gradient* $\partial p/\partial z$ and *pipe radius* $R$.

**$z$-direction:**

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right) + \rho g_z$$

**$r$-direction:**

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_r}{\partial r}\right) - \frac{v_r}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2}\right) + \rho g_r$$

**$\theta$-direction:**

$$\rho\left(\frac{\partial v_\theta}{\partial t} + v_r\frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_\theta}{\partial \theta} - \frac{v_r v_\theta}{r} + v_z\frac{\partial v_\theta}{\partial z}\right) = -\frac{1}{r}\frac{\partial p}{\partial \theta} + \mu\left(\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_\theta}{\partial r}\right) - \frac{v_\theta}{r^2} + \frac{1}{r^2}\frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2}\frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2}\right) + \rho g_\theta$$

# Details on flow in *circular pipes* (aka Poiseuille flow)

- Volume flow rate (integrate velocity):

$$Q = 2\pi \int_0^R v_z(r) r dr = -\frac{\pi R^4}{8\mu}\left(\frac{\partial p}{\partial z}\right) = \frac{\pi R^4 \Delta p}{8\mu l}$$

  - Flow rate is a function of *viscosity* $\mu$, *pressure gradient* $\partial p/\partial z$ and *pipe radius* $R$.

- Average velocity (divide by area):

$$V = \frac{Q}{A} = \frac{-\frac{\pi R^4}{8\mu}\left(\frac{\partial p}{\partial z}\right)}{\pi R^2} = -\frac{R^2}{8\mu}\left(\frac{\partial p}{\partial z}\right) = \frac{R^2 \Delta p}{8\mu l}$$

  - Mean velocity is a function of *viscosity* $\mu$, *pressure gradient* $\partial p/\partial z$ and *pipe radius* $R$.

- Maximum velocity ($r = 0$):

$$v_z(r=0) = \frac{1}{4\mu} - \left(\frac{\partial p}{\partial z}\right)(0^2 - R^2) = -\frac{R^2}{4\mu}\left(\frac{\partial p}{\partial z}\right) = \frac{R^2 \Delta p}{4\mu l}$$

  - Mean velocity is a function of *viscosity* $\mu$, *pressure gradient* $\partial p/\partial z$ and *pipe radius* $R$.

- Ratio of centerline to mean velocity ($v_z(r=0)/V$):

$$\frac{v_z(r=0)}{V} = \frac{\frac{R^2 \Delta p}{4\mu l}}{\frac{R^2 \Delta p}{8\mu l}} = 2$$

  - Centerline velocity is exactly *twice* the mean velocity.

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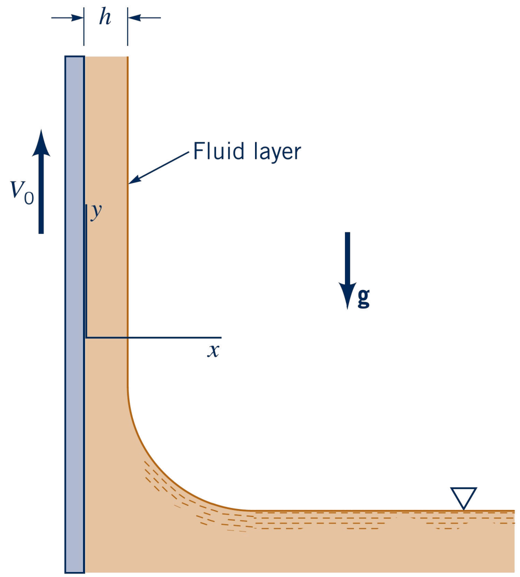
# Exercise: Upward moving fluid film

A wide moving belt passes through a container of a viscous liquid. The belt moves vertically upward with a constant velocity, $V_0$, as illustrated in the figure. Because of viscous forces the belt picks up a film of fluid of thickness $h$. Gravity tends to make the fluid drain down the belt. Assume that the flow is laminar, steady, and fully developed.




$$\rho\left(\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+v\frac{\partial u}{\partial y}+w\frac{\partial u}{\partial z}\right)=-\frac{\partial p}{\partial x}+\mu\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 u}{\partial z^2}\right)+\rho g_x$$

$$\rho\left(\frac{\partial v}{\partial t}+u\frac{\partial v}{\partial x}+v\frac{\partial v}{\partial y}+w\frac{\partial v}{\partial z}\right)=-\frac{\partial p}{\partial y}+\mu\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}+\frac{\partial^2 v}{\partial z^2}\right)+\rho g_y$$

$$\rho\left(\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+v\frac{\partial w}{\partial y}+w\frac{\partial w}{\partial z}\right)=-\frac{\partial p}{\partial z}+\mu\left(\frac{\partial^2 w}{\partial x^2}+\frac{\partial^2 w}{\partial y^2}+\frac{\partial^2 w}{\partial z^2}\right)+\rho g_z$$

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