

Fluid Mechanics

Analytical solutions for simple fluid flows

Lecturer: Jakob Hærvig

Slides by Jakob Hærvig (AAU Energy) and Jacob Andersen (AAU Build)

Complexities involved in Navier-Stokes equations

No universal analytical solution due to non-linearities ($u\partial u/\partial x$)

- Solution *approximated* using *numerical methods* (e.g. CFD)
- Solution found by *experiments*

Analytical solutions for *only a few problems*

- *Simple geometries*: Flow between two plates or in a tube
- *Simple flows*: Steady, laminar and fully-developed flows

x-direction:

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho g_x$$

y-direction:

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \rho g_y$$

z-direction:

$$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \rho g_z$$

Why look at these *simplified* problems?

For which applications is the flow between two plates important?

Flow between two plates—whether *pressure-driven* (plane Poiseuille flow) or *shear-driven* (Couette flow)—is a **canonical model** that underpins both engineering practice and natural phenomena. It's important in many applications, including:

Engineering & Technology

- **Lubrication:** Thin films of oil in bearings, pistons, gears, or seals are modeled as flow between parallel plates.
- **Microfluidics:** Lab-on-a-chip devices use narrow rectangular channels, which are essentially parallel plate flows.
- **Heat exchangers:** Plate-type exchangers rely on fluid moving between closely spaced plates for efficient heat transfer.
- **Coating & printing:** Flow of paint, varnish, or ink in thin layers is approximated by plate flows.
- **Polymer processing:** Sheet extrusion and film casting involve molten polymer flow between plates.

Everyday Phenomena

- **Sticking of wet surfaces:** Two glass slides or a credit card and plastic sleeve resist separation due to viscous flow in the thin liquid film.
- **Windshield wipers:** The thin water film between blade and glass behaves like a parallel-plate channel.
- **Food preparation:** Spreading butter, cream, or sauces between slices involves thin-film flow.

Biology & Medicine

- **Synovial joints:** Cartilage surfaces slide with synovial fluid in between, modeled as shear-driven thin-film flow.
- **Blood in microvessels:** Capillaries and arterioles often approximate parallel-plate geometries at small scales.
- **Dialysis and filtration:** Fluids move between thin membranes in channels resembling plate flows.

Fundamental & Educational Value

- Provides one of the **few exact solutions of Navier–Stokes**—great for teaching velocity profiles, shear stress, and pressure gradients.
- Serves as a **building block** for more complex flows (e.g., lubrication theory, Reynolds equation).
- Helps illustrate the importance of viscosity, low Reynolds number flows, and scaling laws.



Current research uses *Poiseuille flow* for verification of CFD

CO₂ diffuses into the fiber and is captured

- Really (!) long fibres ($l/D \approx 12000$)
- Steady, laminar, fully-developed flow is a *very good* assumption
- Prescribed boundary condition

Results: Code verification results

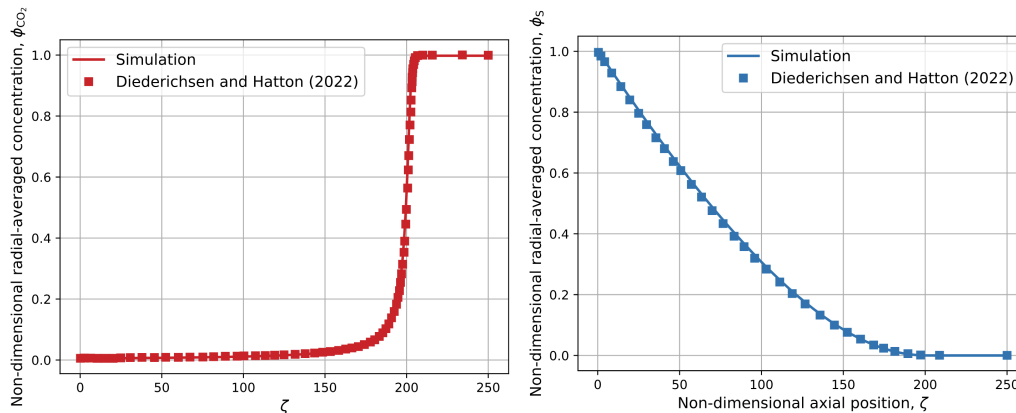
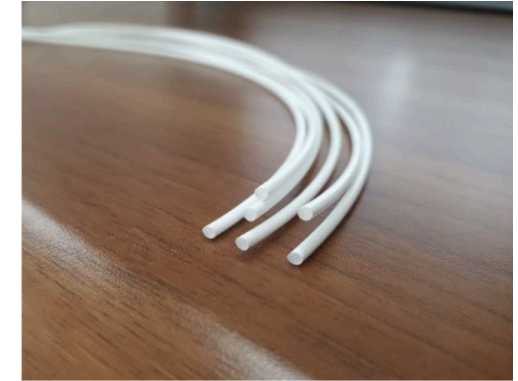
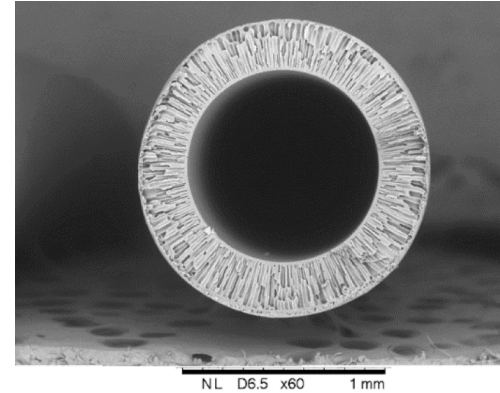


Figure: Hollow fibre membrane



Bazhenov, S. D., Bilydukevich, A. V., & Volkov, A. V. (2018). Gas-Liquid Hollow Fiber Membrane Contactors for Different Applications. *Fibers*, 6(4), 76. <https://doi.org/10.3390/fib6040076>

Code: Inlet boundary condition in OpenFOAM

```
1
2 /*-----*-- C++ -*-----*/
3
4 //=====  
5 //      F ield      | OpenFOAM: The Open Source CFD Toolbox  
6 //      O peration  | Website:  https://openfoam.com  
7 //      A nd        | Version:  2506  
8 //      M anipulation |  
9 /*-----*--  
10 FoamFile  
11 {  
12     format      ascii;  
13     class       volVectorField;  
14     location    "0";  
15     object      U;  
16 }  
17 // * * * * *  
18 dimensions     [0 1 -1 0 0 0 0];  
19
```

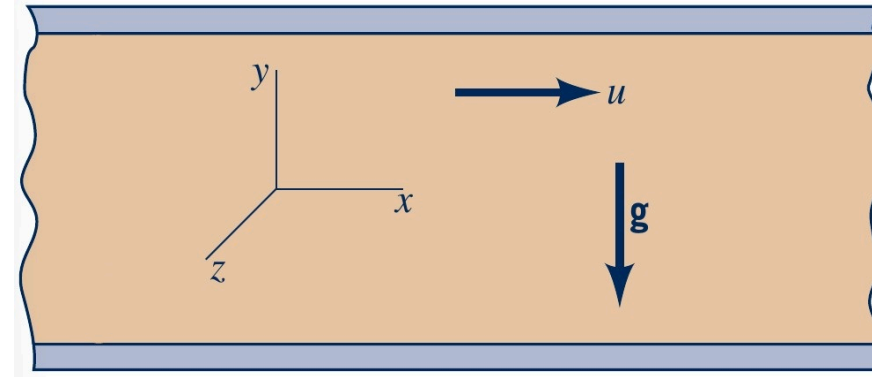
Flow between plates

Incompressible, steady, fully developed, laminar flow between two infinite plates

- Velocity only in x -direction:
 - *then*, $u = u(y), v = w = 0$
- Steady flow:
 - *then*, $\partial u / \partial t = \partial v / \partial t = \partial w / \partial t = 0$
- By continuity equation: $\partial u / \partial x + \partial v / \partial y + \partial w / \partial z = 0$
 - *then*, $\partial u / \partial x = 0$
- No gravity in x and z directions:
 - *then*, $g_x = g_z = 0$

Navier-Stokes equations *reduce* to:

$$x : 0 = -\frac{\partial p}{\partial x} + \mu \frac{\partial^2 u}{\partial y^2}, \quad y : 0 = -\frac{\partial p}{\partial y} + \rho g_y, \quad z : 0 = -\frac{\partial p}{\partial z}$$



x-direction:

$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho g_x$$

y-direction:

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \rho g_y$$

z-direction:

$$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \rho g_z$$

Flow between *two stationary* plates

Integrating twice to get $u(y)$:

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \frac{\partial p}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\mu} \frac{\partial p}{\partial x} y + C_1$$

$$u = \frac{1}{2\mu} \frac{\partial p}{\partial x} y^2 + C_1 y + C_2$$

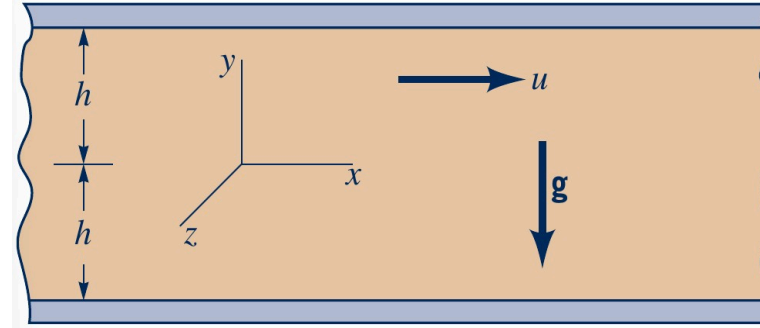
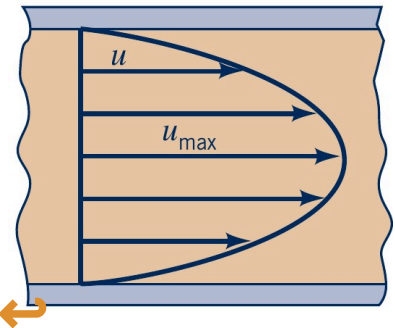
Using *boundary conditions*:

At $y = \pm h$ we have $u = 0$

$$C_1 = 0, C_2 = -\frac{1}{2\mu} \frac{\partial p}{\partial x} h^2$$

Velocity profile (by inserting C_1 and C_2):

$$u(y) = \frac{1}{2\mu} \frac{\partial p}{\partial x} (y^2 - h^2)$$



Volume flow rate (integrating velocity profile):

$$q = \int_{-h}^h u(y) dy = -\frac{2h^3}{3\mu} \left(\frac{\partial p}{\partial x} \right)$$

Average velocity:

$$V = \frac{q}{2h} = -\frac{h^2}{3\mu} \left(\frac{\partial p}{\partial x} \right)$$

Maximum velocity (centre of channel, $y = 0$):

$$u_{max} = u(y = 0) = -\frac{h^2}{2\mu} \left(\frac{\partial p}{\partial x} \right) = \frac{3}{2} V$$

Flow between *one stationary* and *one moving* plate (aka Couette flow)

Integrating twice to get $u(y)$:

$$\frac{\partial^2 u}{\partial y^2} = \frac{1}{\mu} \frac{\partial p}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{1}{\mu} \frac{\partial p}{\partial x} y + C_1$$

$$u = \frac{1}{2\mu} \frac{\partial p}{\partial x} y^2 + C_1 y + C_2$$

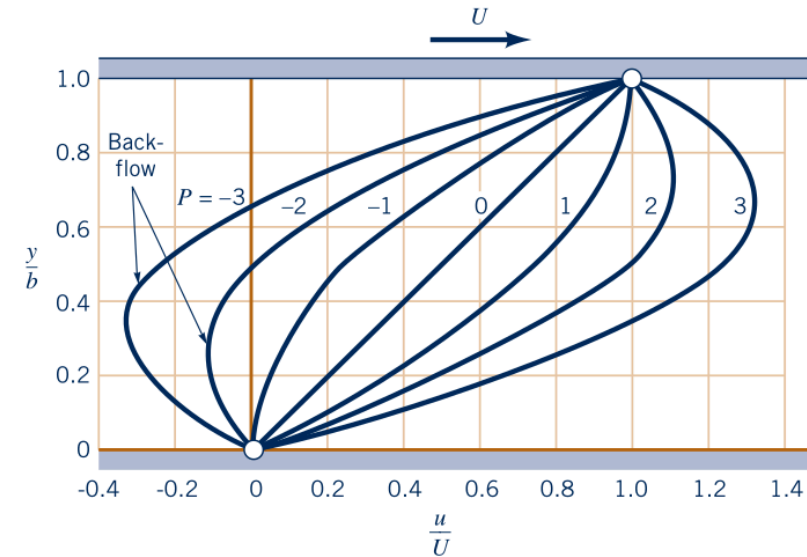
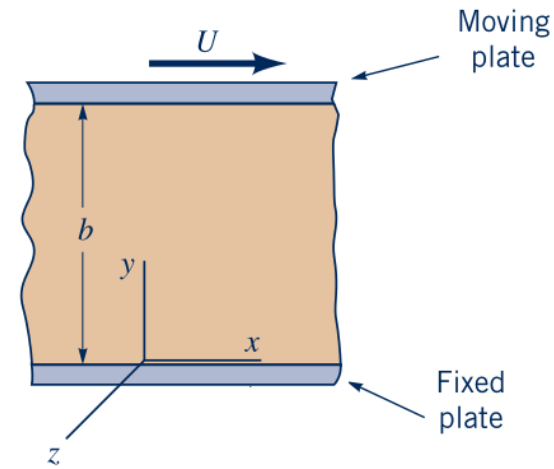
Using *boundary conditions*:

At $y = 0$ we have $u = 0$

At $y = b$ we have $u = U$

Finding C_1 and C_2 and inserting:

$$u(y) = U \frac{y}{b} + \frac{1}{2\mu} \frac{\partial p}{\partial x} (y^2 - by)$$



$$P = -\frac{b^2}{2\mu U} \left(\frac{\partial p}{\partial x} \right)$$

Flow in a *circular pipe* (aka Poiseuille flow)

Incompressible, steady, fully developed, laminar flow in a circular pipe

- Velocity only in z -direction:
 - *then*, $v_z = v_z(r)$, $v_\theta = v_r = 0$
- Steady flow:
 - *then*, $\partial u / \partial t = \partial v / \partial t = \partial w / \partial t = 0$
- By continuity equation: $\frac{1}{r} \frac{\partial(rv_r)}{\partial r} + \frac{1}{r} \frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$
 - *then*, $\partial v_z / \partial z = 0$
- No gravity in z direction:
 - *then*, $g_z = 0$

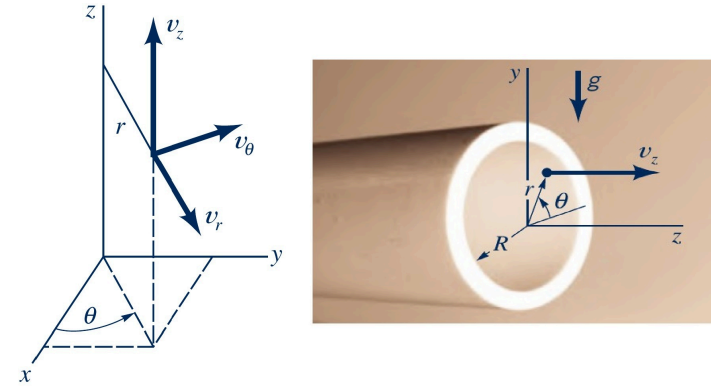
Navier-Stokes equations *reduce* to:

$$z : 0 = -\frac{\partial p}{\partial z} + \mu \frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right), \quad r : 0 = \rho g_r - \frac{\partial p}{\partial r}, \quad \theta : 0 = \rho g_\theta - \frac{1}{r} \frac{\partial p}{\partial \theta}$$

Integration, applying *boundary conditons* and *find* C_1 and C_2 :

$$v_z(r) = \frac{1}{4\mu} - \left(\frac{\partial p}{\partial z} \right) (r^2 - R^2)$$

- Velocity is a function of *viscosity* μ , *pressure gradient* $\partial p / \partial z$ and *pipe radius* R .



z-direction:

$$\rho \left(\frac{\partial v_z}{\partial t} + v_r \frac{\partial v_z}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_z}{\partial \theta} + v_z \frac{\partial v_z}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_z}{\partial r} \right) + \frac{1}{r^2} \frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2} \right) + \rho g_z$$

r-direction:

$$\rho \left(\frac{\partial v_r}{\partial t} + v_r \frac{\partial v_r}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z \frac{\partial v_r}{\partial z} \right) = -\frac{\partial p}{\partial r} + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_r}{\partial r} \right) - \frac{v_r}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_r}{\partial \theta^2} - \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_r}{\partial z^2} \right) + \rho g_r$$

theta-direction:

$$\rho \left(\frac{\partial v_\theta}{\partial t} + v_r \frac{\partial v_\theta}{\partial r} + \frac{v_\theta}{r} \frac{\partial v_\theta}{\partial \theta} - \frac{v_r v_\theta}{r} + v_z \frac{\partial v_\theta}{\partial z} \right) = -\frac{1}{r} \frac{\partial p}{\partial \theta} + \mu \left(\frac{1}{r} \frac{\partial}{\partial r} \left(r \frac{\partial v_\theta}{\partial r} \right) - \frac{v_\theta}{r^2} + \frac{1}{r^2} \frac{\partial^2 v_\theta}{\partial \theta^2} + \frac{2}{r^2} \frac{\partial v_r}{\partial \theta} + \frac{\partial^2 v_\theta}{\partial z^2} \right) + \rho g_\theta$$

Details on flow in *circular pipes* (aka Poiseuille flow)

- Volume flow rate (integrate velocity):

$$Q = 2\pi \int_0^R v_z(r) r dr = -\frac{\pi R^4}{8\mu} \left(\frac{\partial p}{\partial z} \right) = \frac{\pi R^4 \Delta p}{8\mu l}$$

- Flow rate is a function of *viscosity* μ , *pressure gradient* $\partial p / \partial z$ and *pipe radius* R .

- Average velocity (divide by area):

$$V = \frac{Q}{A} = \frac{-\frac{\pi R^4}{8\mu} \left(\frac{\partial p}{\partial z} \right)}{\pi R^2} = -\frac{R^2}{8\mu} \left(\frac{\partial p}{\partial z} \right) = \frac{R^2 \Delta p}{8\mu l}$$

- Mean velocity is a function of *viscosity* μ , *pressure gradient* $\partial p / \partial z$ and *pipe radius* R .

- Maximum velocity ($r = 0$):

$$v_z(r = 0) = \frac{1}{4\mu} - \left(\frac{\partial p}{\partial z} \right) (0^2 - R^2) = -\frac{R^2}{4\mu} \left(\frac{\partial p}{\partial z} \right) = \frac{R^2 \Delta p}{4\mu l}$$

- Mean velocity is a function of *viscosity* μ , *pressure gradient* $\partial p / \partial z$ and *pipe radius* R .

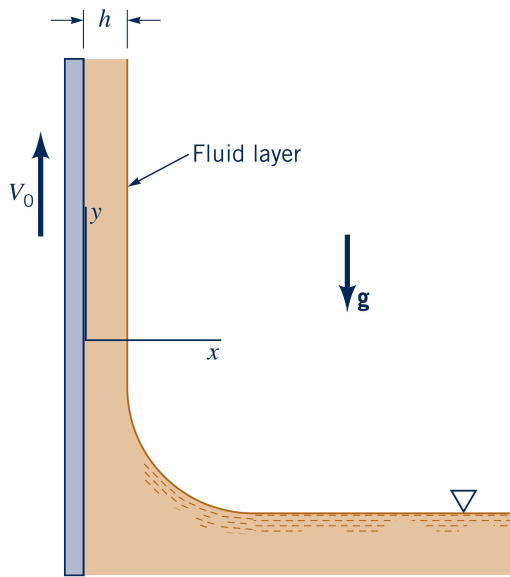
- Ratio of centerline to mean velocity ($v_z(r = 0) / V$):

$$\frac{v_z(r = 0)}{V} = \frac{\frac{R^2 \Delta p}{4\mu l}}{\frac{R^2 \Delta p}{8\mu l}} = 2$$

- Centerline velocity is exactly *twice* the mean velocity.

Exercise: Upward moving fluid film

A wide moving belt passes through a container of a viscous liquid. The belt moves vertically upward with a constant velocity, V_0 , as illustrated in the figure. Because of viscous forces the belt picks up a film of fluid of thickness h . Gravity tends to make the fluid drain down the belt. Assume that the flow is laminar, steady, and fully developed.



$$\rho \left(\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + w \frac{\partial u}{\partial z} \right) = -\frac{\partial p}{\partial x} + \mu \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right) + \rho g_x$$

$$\rho \left(\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + w \frac{\partial v}{\partial z} \right) = -\frac{\partial p}{\partial y} + \mu \left(\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} + \frac{\partial^2 v}{\partial z^2} \right) + \rho g_y$$

$$\rho \left(\frac{\partial w}{\partial t} + u \frac{\partial w}{\partial x} + v \frac{\partial w}{\partial y} + w \frac{\partial w}{\partial z} \right) = -\frac{\partial p}{\partial z} + \mu \left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} + \frac{\partial^2 w}{\partial z^2} \right) + \rho g_z$$