RELATIONS RECAP

CARTESIAN PRODUCT $A \times B = \{(a,b) \mid a \in A, b \in B\}$

Ex $A = \{1,2,3\}$ $B = \{4,5,6\}$

$A \times B = \{(1,4),(1,5),(1,6),(2,4),(2,5),(2,6),(3,4),(3,5),(3,6)\}$

RELATION R from A to B

Subset of $A \times B$

If $A = B$: relation on A pretty common case

Ex

A = persons B = bikes R: a owns b

A = companies B = persons R: a employs b

$A = B$ = persons R: a is parent of b

Ways to represent relations

Running example $A = B = \{1,2,3,4\}$

$R(a,b)$ if $a < b$

Alternative notation $(a,b) \in R$

aRb

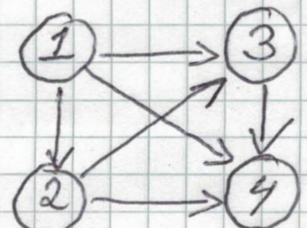
(1) SET R

$R = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$

(2) MATRIX M_R

$$\begin{array}{c|c} & \begin{array}{cccc} 1 & 2 & 3 & 4 \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \end{array} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

(3) DIGRAPH D_R For relation on A



R-RELATIVE SET $R(a) = \{b \in B \mid aRb\}$

$$R(A') = \{b \in B \mid \exists a \in A' \; aRb\}$$
$$= \bigcup_{a \in A'} R(a)$$

Ex

$A = \{6, 7, 8, 9, 10\}$ $\quad B = \{1, 2, 3, 4, 5\}$

aRb if a is a multiple of b

$R(6) = \{1, 2, 3\}$
$R(7) = \{1\}$
$R(8) = \{1, 2, 4\}$

$R(\{6, 7, 8\}) = \{1, 2, 3, 4\}$

DOMAIN $\quad \mathrm{Dom}(R) = \{a \in A \mid \exists b \; aRb\}$

RANGE $\quad \mathrm{Ran}(R) = \{b \in B \mid \exists a \; aRb\}$
$= R(A)$

For relation on A

IN-DEGREE of a $\quad |\{a' \in A \mid a'Ra\}|$

OUT-DEGREE of a $\quad |\{a' \in A \mid aRa'\}|$

incoming and outgoing edges, respectively, in digraph representation

Let us focus on relations on a set A

PATH IN DIGRAPH D_R

Path of length n in R

Sequence $a, x_1, x_2, \dots, x_{n-1}, b$ such that

$$a R x_1$$
$$x_{i-1} R x_i \qquad i \in [n-1]$$
$$x_{n-1} R b$$

Ex $a R b$ = edge in D_R = path of length 1

$a R x$, $x R b$ = path of length 2

Use this to define new relation

R^n: $a R^n b$ if $\exists$ path of length (exactly) n in R from a to b

Ex A = people in the world

$R(a,b)$ a knows b

Conjecture: SIX DEGREES OF SEPARATION

Everybody on earth is connected by chain of at most 6 people who are friends (pairwise) to anybody else

In math notation

$$\forall a \ \forall b \ \exists i \leq 6 \quad a R^i b$$

Given R, how can we find R^n?

<u>Idea 1</u> Draw digraph D_R
Find all paths of length n

<u>Idea 2</u> Use matrix representation M_R
Use Boolean matrix multiplication to compute

$$M_{R^n} = (M_R)^n_{\odot} =$$

$$= \underbrace{M_R \odot M_R \odot \cdots \odot M_R}_{\text{repeat matrix } n \text{ times}}$$

<u>Recall</u> For $n \times n$ Boolean matrices A, B
$C = A \odot B$ is matrix such that

$$c_{ij} = \begin{cases} 1 & \text{if } \exists k \text{ s.t. } a_{ik} = b_{kj} = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{i,n} \wedge b_{nj})$$

THEOREM For any $n \geq 2$ it holds that

$$M_{R^n} = (M_R)^n_{\odot}$$

When convenient,
Identify $A = \{a_1, a_2, \ldots, a_n\}$
with $\{1, 2, \ldots, n\}$
for notational simplicity

R VII

Proof By induction over n

BASE CASE $(n=2)$: $\exists$ path of length 2 between a_i and a_j precisely when for some $k \in [n]$

$$a_i \, R \, a_k \quad \text{and} \quad a_k \, R \, a_j$$

In matrix notation:

$$m_{ik} \wedge m_{kj} \quad \text{for some } k \in [n]$$

But this is entry c_{ij} for

$$C = M_R \odot M_R$$

INDUCTION STEP: Our $\boxed{\text{induction hypothesis}}$ is that R^n has matrix representation $(M_R)_{\odot}^{n}$.

$a_i \, R^{n+1} a_j$ if $\exists \, k \in [n]$ such that

$a_i \, R^n \, a_k$ and $a_k \, R \, a_j$ (by definition)

By induction hypothesis, $a_i \, R^n \, a_k$ iff

(i,k)-entry of $(M_R)_{\odot}^{n} = 1$

$a_k \, R \, a_j$ if $m_{kj} = 1$

This holds iff (i,j)-entry in $(M_R)_{\odot}^{n+1}$ is $=1$

This concludes the induction step.

The theorem follows by the induction principle ▨

CONNECTIVITY RELATION R^∞ on A

R VIII

$R^\infty(a,b)$ if $\exists n \in \mathbb{Z}^+$ s.t. $R^n(a,b)$

Equivalently,

$$R^\infty = R \cup R^2 \cup R^3 \cup \cdots$$
$$= \bigcup_{i=1}^{\infty} R^i$$

Note that this notation makes sense, because relations are sets

How large powers R^i of R do we need to consider?

Use pigeonhole principle

If $|A| < \infty$ (A finite set)

then in sequence

$a_1 R a_2$, $a_2 R a_3$, $a_3 R a_4$, ...

sooner or later will get repetition of element $a_i = a_j = y$ if length $\geq n$

$$a_1 R a_2 \cdots a_{i-1} R \overset{y}{a_i} \quad \underbrace{a_i R a_{i+1} \cdots a_{j-1} R \overset{y}{a_j}}_{\text{SKIP THIS PART IN SEQUENCE}} \quad a_j R a_{j+1}$$

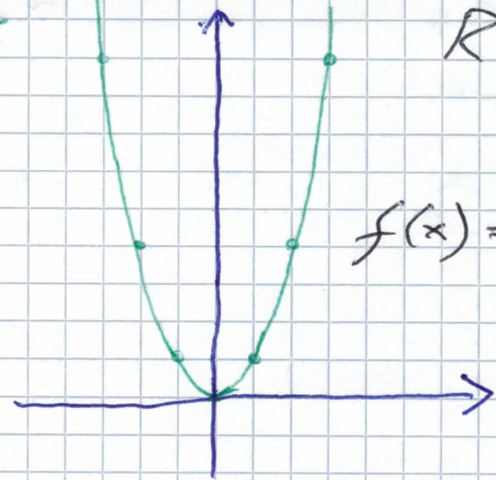
So $|A| - 1$ steps enough

FUNCTIONS

any sets

FUNCTION f FROM A TO B

Relation such that for every $a \in Dom(f)$ exists UNIQUE b for which $(a,b) \in f$



More common notation: $\boxed{f(a) = b}$

f is a function from A to B $\boxed{f : A \to B}$

If $f(a) = b$, also write $\boxed{a \mapsto b}$ (\mapsto)
when f understood from context

WARNING Normally required that f should be defined on all of A, i.e., $Dom(f) = A$
Otherwise PARTIAL FUNCTION

But KBR defines functions as partial functions — this is non-standard
We will try to be very clear when functions are partial or
TOTAL $(Dom(f) = A)$

Ex (a) $g = \{ (x, x^2) \mid x \in \mathbb{Z} \}$

$= \{(0,0), (1,1), (-1,1), (2,4), (-2,4), (3,9), \dots\}$

is a function illustrated above

(b) $h = \{ (x^2, x) \mid x \in \mathbb{Z} \}$

$= \{ (0,0), (1,1), (1,-1), (4,2), (4,-2), (9,3), \dots \}$

is NOT a function How do we make $\sqrt{}$ into a function?

IMPORTANT TERMINOLOGY FOR FUNCTIONS

$f: A \to B$ is

- $\boxed{\text{EVERYWHERE DEFINED}}$ or $\boxed{\text{TOTAL}}$ if $\mathrm{Dom}(f) = A$
- $\boxed{\text{SURJECTIVE}}$ or $\boxed{\text{ONTO}}$ if $\mathrm{Ran}(f) = B$
- $\boxed{\text{INJECTIVE}}$ or $\boxed{\text{ONE-TO-ONE}}$ if for $x_1 \neq x_2$ it holds that $f(x_1) \neq f(x_2)$
- $\boxed{\text{BIJECTIVE}}$ if f is total, surjective, and injective.
  If so, f also called $\boxed{\text{BIJECTION}}$

__Ex__ (a) $f: \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x^2$
- total
- not surjective
- not injective

(b) $g: \mathbb{R} \to \mathbb{R}_{\geq 0}$ defined by $x \mapsto x^2$
- total
- surjective
- not injective

(c) $h: \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $x \mapsto x^2$
- total
- surjective
- injective

so bijection

IMPORTANT TERMINOLOGY FOR RELATIONS

Relation R on A is

- $\boxed{\text{REFLEXIVE}}$ if $\forall a \ (a,a) \in R$
- $\boxed{\text{IRREFLEXIVE}}$ if $\forall a \ (a,a) \notin R$

Ex

$=$ and $\leq$ are reflexive

$<$ is irreflexive

- $\boxed{\text{SYMMETRIC}}$ if $\forall a,b \ (a,b) \in R \Rightarrow (b,a) \in R$
- $\boxed{\text{ASYMMETRIC}}$ if $\forall a,b \ (a,b) \in R \Rightarrow (b,a) \notin R$
- $\boxed{\text{ANTISYMMETRIC}}$ if $\forall a,b \ (a,b) \in R$ and $(b,a) \in R \Rightarrow a = b$

WARNING Distinguish between asymmetric and antisymmetric!

Asymmetric relations are vacuously antisymmetric

Ex – Divisibility $|$ is antisymmetric

– Strictly less than $<$ is asymmetric

– "u and v are neighbours in an undirected graph" is symmetric

Can a relation be both symmetric and asymmetric?

Well, yes, empty relation $R = \emptyset$ can... Other than that **NO!**

Can a nontrivial relation be both symmetric and antisymmetric?

Yes, equality satisfies this.

Exercise Prove that this is only such relation.

Can we read off any of these properties from matrix representation M_R?

REFLEXIVE : Diagonal all 1s

IRREFLEXIVE : Diagonal all 0s

SYMMETRIC : Matrix symmetric, i.e., $(M_R)^T = M_R$

Relation R on A is

- TRANSITIVE if $\forall a, b, c \in A$ $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$

Ex - Strictly less than $<$ and divisibility $|$ are transitive
- Neighbour relation in graph is NOT transitive in general

OBSERVATION

R is transitive if and only if $R^2 \subseteq R$

Proof ($\Rightarrow$) Suppose $(a,c) \in R^2$. By definition $\exists b$ such that $(a,b) \in R$ and $(b,c) \in R$. Since R is transitive by assumption, we have $(a,c) \in R$. This shows that $R^2 \subseteq R$

($\Leftarrow$) Suppose $R^2 \subseteq R$. Consider a, b, c such that $(a,b) \in R$ and $(b,c) \in R$. We need to show that $(a,c) \in R$. But $(a,b) \in R$ and $(b,c) \in R$ means $(a,c) \in R^2$ by definition, and since $R^2 \subseteq R$ by assumption we have $(a,c) \in R$, which shows that R is transitive ▨

Relation R on A is an **EQUIVALENCE RELATION** if it is

- reflexive
- symmetric
- transitive

INTUITION: An equivalence relation tells us which elements in A are "essentially the same" or, well, equivalent

Ex A classical equivalence relation is **CONGRUENCE CLASSES MODULO** some $n \in \mathbb{Z}^+$

$(a, b) \in R$ if $(a \bmod n) = (b \bmod n)$

i.e., if a and b yield same remainder when divided by n

Say, $n = 5$. Then

- 2, 7, 12, 17 all equivalent
- 3, 8, 13, 18 all equivalent
- 2 and 3 are NOT equivalent

A **PARTITION** $\mathcal{P}$ of a set A is a collection of subsets $P_1, P_2, \ldots, P_k$ such that

① $\forall i \in [k] \quad P_i \neq \emptyset$

② $\forall i, j \in [k], \; i \neq j \Rightarrow P_i \cap P_j = \emptyset$

③ $\bigcup_{i=1}^{k} P_i = A$

(Actually, partitions need not be finite if A is infinite. This definition works for infinite partitions also.)

(A "collection of sets" is just a set consisting of sets, but we often call a set of sets a "collection")

Ex Let

$$A = \{1, 2, 3, 4, 5, 6, 7\}$$

$$P_1 = \{1, 4, 7\}$$
$$P_2 = \{2, 5\}$$
$$P_3 = \{3, 6\}$$
$$P_4 = \{2, 3, 5, 7\}$$

- P_1, P_2, P_3, P_4 do NOT form a partition
- P_1, P_2 do NOT form a partition
- P_1, P_2, P_3 do form a partition

Partitions give rise to equivalence relations and vice versa

THEOREM (From equivalence relations to partitions)

If R is an equivalence relation on A, then the collection $\mathcal{P}$ of all DISTINCT R-relative sets $R(a)$, $a \in A$, is a partition of A.

Proof Let $\mathcal{P} = \{P_i\}$ be all distinct R-relative sets. We need to prove 1 – 3 above.

1 $P_i \neq \emptyset$ since every $R(a)$ contains a by reflexivity

③ <u>By contradiction</u>

Suppose $\bigcup_i P_i \neq A$

Means $A \setminus \bigcup_i P_i \neq \emptyset$, so

$\exists\, b \in A \setminus \bigcup_i P_i$

But then we can add $R(b)$ to $\mathcal{P}$

Clearly distinct from all other P_i, since $b \notin P_i$ but $b \in R(b)$

However $\mathcal{P}$ already contains all distinct R-relative sets. $\lightning$

Finally, we need to prove

② If $P_i \neq P_j$, then $P_i \cap P_j = \emptyset$

Let us do <u>proof by contraposition</u>

If $P_i \cap P_j \neq \emptyset$, then $P_i = P_j$

Let $P_i = R(a)$

$P_j = R(c)$

Suppose $b \in R(a) \cap R(c)$

Want to prove $R(a) = R(c)$

That is:

(i) $R(a) \subseteq R(c)$ For every $d \in R(a)$ we have $d \in R(c)$

(ii) $R(c) \subseteq R(a)$ For every $d \in R(c)$ we have $d \in R(a)$

How to prove this? Not much we can do...
We only know that R equivalence relation.
But this makes the task EASY! All we can do is to play with the definitions — this has got to work!

$b \in R(a) \cap R(c)$

$(a,b) \in R$ } by definition
$(c,b) \in R$ }

$(b,c) \in R$ by symmetry

$(a,c) \in R$ by transitivity

$(c,a) \in R$ by symmetry

Suppose $d \in R(a)$ Then

$(a,d) \in R$ by definition

$(d,a) \in R$ by symmetry

$(d,c) \in R$ by transitivity

$(c,d) \in R$ by symmetry

So $d \in R(c)$ and $R(a) \subseteq R(c)$

In EXACTLY THE SAME WAY, just swapping "a" and "c" in the six lines above, we prove $R(c) \subseteq R(a)$, and hence $R(a) = R(c)$ or $P_i = P_j$ as claimed

Hence, P is a partition.
The theorem follows $\blacksquare$

THEOREM

If $\mathcal{P} = \{P_i\}$ is a partition of the set A, then the relation

$$R = \{ (a,b) \mid \exists i \text{ such that } a, b \in P_i \}$$

is an equivalence relation

Proof R as defined above is definitely some kind of relation. We need to establish that R is reflexive, symmetric, and transitive.

REFLEXIVITY $(a,a) \in R$

Every a is in some P_i, since $\mathcal{P}$ partition
Since a and a are both in P_i :-)
$(a,a) \in R$ — CHECK! ✓

SYMMETRY $(a,b) \in R \Rightarrow (b,a) \in R$

~~Also~~ clear by construction. The definition of R doesn't care about the order

If $(a,b) \in R$, then $\exists i$ such that $b \in P_i$ and $a \in P_i$, so $(b,a) \in R$ — CHECK! ✓

TRANSITIVITY $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$

$(a,b) \in R \Rightarrow \exists i \quad a, b \in P_i$
$(b,c) \in R \Rightarrow \exists j \quad b, c \in P_j$

Want to prove $P_i = P_j$ — then $a, c \in P_i$ so $(a,c) \in R$

But $b \in P_i \cap P_j$, and $\mathcal{P}$ is partition, so by (2) we indeed have $P_i = P_j$ — CHECK! ✓

$\blacksquare$