



## Introduktion til diskret matematik og algoritmer: Problem Set 3

**Due:** Wednesday March 11 at 12:59 CET.

**Submission:** Please submit your solutions via *Absalon* as a PDF file. State your name and e-mail address close to the top of the first page. Solutions should be written in L<sup>A</sup>T<sub>E</sub>X or some other math-aware typesetting system with reasonable margins on all sides (at least 2.5 cm). Please try to be precise and to the point in your solutions and refrain from vague statements. Never, ever just state the answer, but always make sure to explain your reasoning. *Write so that a fellow student of yours can read, understand, and verify your solutions.* In addition to what is stated below, the general rules for problem sets stated on *Absalon* always apply.

**Collaboration:** Discussions of ideas in groups of two to three people are allowed—and indeed, encouraged—but you should always write up your solutions completely on your own, from start to finish, and you should understand all aspects of them fully. It is not allowed to compose draft solutions together and then continue editing individually, or to share any text, formulas, or pseudocode. Also, no such material may be downloaded from or generated via the internet to be used in draft or final solutions. Submitted solutions will be checked for plagiarism.

**Grading:** A score of 120 points is guaranteed to be enough to pass this problem set.

**Questions:** Please do not hesitate to ask the instructor or TAs if any problem statement is unclear, but please make sure to send private messages—sometimes specific enough questions could give away the solution to your fellow students, and we want all of you to benefit from working on, and learning from, the problems. Good luck!

- 1 (90 p) Let  $A = \{1, 2, 3, 4\}$  and consider the following binary relations on  $A$ :

$$R = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$$

$$S = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$

$$T = \{(1, 1), (1, 4), (2, 2), (2, 3), (3, 2), (3, 3), (4, 1), (4, 4)\}$$

- 1a (60 p) For each of the relations above, determine whether it is

1. reflexive,
2. symmetric,
3. antisymmetric,
4. transitive.

Please make sure to explain, briefly but clearly, what these properties mean and why they are satisfied for a relation when they are. For any relation that fails to satisfy a property, make sure to provide a specific counterexample.

**Solution:** Recall that a relation  $R$  is:

- *reflexive* if for all  $x$  it holds that  $(x, x) \in R$ ;
- *symmetric* if whenever  $(x, y) \in R$  it also holds that  $(y, x) \in R$ ;
- *anti-symmetric* if  $(x, y) \in R$  and  $(y, x) \in R$  implies  $x = y$ ;

- *transitive* if whenever  $(x, y) \in R$  and  $(y, z) \in R$  it also holds that  $(x, z) \in R$ .

The relation  $R$  specified in the problem statement is anti-symmetric (simply since there is no pair  $(x, y)$  such that  $(x, y) \in R$  and  $(y, x) \in R$ ) and transitive (which can be verified by case analysis, or by observing that  $R$  is the greater-than relation). It is not reflexive since, e.g.,  $(1, 1) \notin R$ , and it is not symmetric since, e.g.,  $(2, 1) \in R$  but  $(1, 2) \notin R$ .

The relation  $S$  is the identity relation, which vacuously satisfies all the properties listed.

The relation  $T$  can be verified to be reflexive, symmetric, and transitive. It is not anti-symmetric since  $(2, 3) \in T$  and  $(3, 2) \in T$  but  $2 \neq 3$ .

- 1b** (30 p) Which of the relations above, if any, are equivalence relations or partial orders? Please make sure to justify your answers.

**Solution:** An *equivalence relation* is a relation that is reflexive, symmetric, and transitive. The relations  $S$  and  $T$  satisfy these conditions as argued above.

A *partial order* is a reflexive, anti-symmetric, and transitive relation. The identity relation  $S$  is formally speaking also a partial order, since it satisfies all the required properties (but it is of course a very boring partial order).

It might be worth pointing out that the relation  $R$  is *not* a partial order, since it is not reflexive. This is just a special case of the general fact that non-strict order relations define partial orders but strict order relations do not.

- 2** (80 p) Recall that the Fibonacci numbers are defined as

$$F_1 = 1$$

$$F_2 = 1$$

$$F_n = F_{n-1} + F_{n-2} \quad \text{for } n \geq 3.$$

Prove that consecutive Fibonacci numbers  $F_{n+1}$  and  $F_n$  are relatively prime, and show that for  $n \geq 2$  the Euclidean algorithm when run on  $F_{n+1}$  and  $F_n$  makes exactly  $n - 1$  function calls to determine that this is so (i.e., it reaches remainder 0 after exactly  $n - 1$  function calls).

**Solution:** We prove that two consecutive Fibonacci numbers are relatively prime, i.e., that they have greatest common divisor 1, by using the Euclidean algorithm. While doing so, we count the number of function calls, i.e., the number of times the relation  $\text{gcd}(m, n) = \text{gcd}(n, m \bmod n)$  is applied before reaching the trivial base case where  $n$  divides  $m$  and the remainder is 0.

**Base case ( $n = 2$ ):** For  $F_3 = 2$  and  $F_2 = 1$ , we clearly have  $\text{gcd}(2, 1) = 1$ . If we run the Euclidean algorithm, we get that  $F_3 = 2 \cdot F_2 + 0$ , and so we reach remainder 0 after a single step.

**Induction step:** Suppose that for  $n - 1$  it holds that  $\text{gcd}(F_{n-1}, F_{n-2}) = 1$  and that the Euclidean algorithm reaches remainder 0 after  $(n - 1) - 1 = n - 2$  function calls.

As a first step when computing  $\text{gcd}(F_n, F_{n-1})$ , the Euclidean algorithm divides  $F_n$  by  $F_{n-1}$  to compute the remainder. This remainder is  $F_{n-2}$ , since by the definition of Fibonacci numbers we have that  $F_n = 1 \cdot F_{n-1} + F_{n-2}$ , and so the equalities

$$\text{gcd}(F_n, F_{n-1}) = \text{gcd}(F_{n-1}, F_n \bmod F_{n-1}) = \text{gcd}(F_{n-1}, F_{n-2}) \quad (1)$$

hold. By the induction hypothesis, the Euclidean algorithm computes  $\text{gcd}(F_{n-1}, F_{n-2}) = 1$  with  $n - 2$  additional recursive calls. Hence, we conclude that the Euclidean algorithm will determine that  $F_n$  and  $F_{n-1}$  are relatively prime after  $n - 1$  function calls.

The claim in the problem statement now follows by the induction principle.

- 3 (90 p) For a few years now the Copenhagen metropolitan area (including Lund) has had an unusually large number of researchers in computational complexity theory, and a team of such researchers have decided to submit a joint grant application to create the *Copenhagen Computational Complexity Centre* focusing on research in this scientific field. Since gender balance is a serious issue in computer science, a noteworthy aspect of the team of co-applicants is that the male professors Amir, Jakob, and Srikanth at the University of Copenhagen are balanced by the female professors Nutan and Paloma at the IT University of Copenhagen and Susanna at Lund University.

For the subproblems below, please make sure to answer not just with numbers but with more combinatorial-looking expressions, and to expand these expressions out to show that you understand the meaning of any notation used. Also make sure to explain how you reason to reach your answers.

- 3a (50 p) Together with the application documents, the co-applicants are planning to enclose a group photo, and much thought has gone into how to choose the seating arrangement. All the researchers will be placed in a single row, but they have agreed that a great way to highlight the gender balance would be to make sure that male and female researchers alternate, so that every second person in the row is male or female, respectively. In how many different ways can the 6 researchers be arranged on the photo to satisfy this constraint?

**Solution:** The seating arrangement is uniquely specified by determining whether the leftmost person on the photo is male or female, and then by specifying the internal order of the female and male researchers, respectively. This gives us:

- 2 choices for a male or female researcher at the leftmost position;
- $3! = 6$  ways of arranging the 3 female researchers from left to right;
- likewise  $3! = 6$  ways of arranging the 3 male researchers from left to right;

for a total of  $2 \cdot 3! \cdot 3! = 2 \cdot 6 \cdot 6 = 72$  different arrangements.

- 3b (40 p) Any serious research centre application these days should also identify a steering committee for the centre. After long deliberations, the co-applicants have decided that this committee should:

- consist of 4 persons all in all;
- include co-applicants representing all 3 partner institutions, i.e., the University of Copenhagen, the IT University of Copenhagen, and Lund University;
- have perfect gender balance, i.e., two male and two female members.

In how many different ways can the steering committee be composed?

**Solution:** Since Susanna is the only co-applicant from Lund University, she has to be on the steering committee.

This means that we need exactly one more female committee member, who will be from ITU, and this gives us 2 choices for either Nutan or Paloma.

Finally, we need two male members, who will both have to come from the University of Copenhagen. We can think of choosing either two persons among Amir, Jakob, and Srikanth in  $\binom{3}{2} = 3$  ways, or choosing one person to leave out in  $\binom{3}{1} = 3$  ways.

Summing up (or, rather, multiplying together), we see that the steering committee can be composed in  $1 \cdot 2 \cdot \binom{3}{2} = 1 \cdot 2 \cdot 3 = 6$  ways.

- 4 (150 p) We have learned in class about matrix multiplication, but there is also a way of multiplying matrices (or vectors) by just a number, which is called *scalar multiplication*. To multiply a matrix  $A$  by a number  $c$ , we multiply each entry  $a_{i,j}$  in the matrix with the number  $c$  so that

$$c \cdot A = c \cdot \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} = \begin{pmatrix} c \cdot a_{1,1} & c \cdot a_{1,2} & \cdots & c \cdot a_{1,n} \\ c \cdot a_{2,1} & c \cdot a_{2,2} & \cdots & c \cdot a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c \cdot a_{m,1} & c \cdot a_{m,2} & \cdots & c \cdot a_{m,n} \end{pmatrix}$$

is the result of the scalar multiplication.

An intriguing phenomenon that sometimes arises is that for some pairs of matrices and vectors matrix multiplication and scalar multiplication give the same result. As an example of this, we have

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad (2)$$

and another example is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \quad (3)$$

When for a square matrix  $A$  there is a vector  $\vec{x}$  (with not all entries equal to zero) and a number  $\lambda$  such that

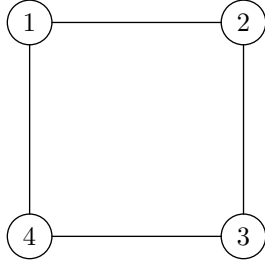
$$A \cdot \vec{x} = \lambda \cdot \vec{x} \quad , \quad (4)$$

such a vector  $\vec{x}$  is called an *eigenvector* of the matrix  $A$  with *eigenvalue*  $\lambda$ . We see that the matrix in (2) has the all-ones vector as eigenvector with eigenvalue 2, and the matrix in (3) also has the all-ones vector as eigenvector but with eigenvalue 3. In this problem, we want to develop our skills of matrix multiplication by studying such eigenvalues and eigenvectors, and also to establish some non-obvious connections between eigenvalues and -vectors on the one hand and graphs on the other hand.

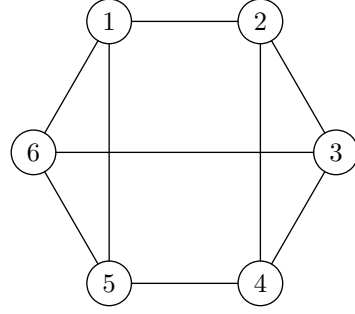
We say that an undirected, simple graph is *d-regular* if every vertex is incident to exactly  $d$  edges, or, in other words, has exactly  $d$  neighbours. For two illustrations of this, the graph in Figure 1a is 2-regular and the graph in Figure 1b is 3-regular. Now, a fun fact is that the 2-regular graph in Figure 1a has the adjacency matrix in (2), which has eigenvalue 2, and the 3-regular graph in Figure 1b has the adjacency matrix in (3) with eigenvalue 3. Your task is to show that this is not a coincidence, and to derive some other interesting connections between  $d$ -regular graphs and the eigenvalues and -vectors of their adjacency matrices.

- 4a (10 p) If  $\vec{x}$  is an eigenvector of  $A$  corresponding to some eigenvalue  $\lambda$ , then so is  $c \cdot \vec{x}$  for any  $c \neq 0$ . Explain why this is so.

*Hint:* This should be easy—just use the definitions above.



(a) Graph with adjacency matrix as in (2).



(b) Graph with adjacency matrix as in (3).

Figure 1: Two example regular graphs in Problem 4.

**Solution:** Suppose that  $A$  is an  $n \times n$  matrix, just to fix the dimension. By definition, coordinate  $i$  in the vector  $A\vec{x}$  is  $\sum_{j=1}^n a_{i,j}x_j$ . If  $\lambda$  is an eigenvalue with eigenvector  $\vec{x}$ , it holds that  $\sum_{j=1}^n a_{i,j}x_j = \lambda x_i$ . But then for the vector  $c\vec{x}$  we get that coordinate  $i$  in the vector  $A(c\vec{x})$  is  $\sum_{j=1}^n a_{i,j}(cx_j) = c \sum_{j=1}^n a_{i,j}x_j = c \cdot (\lambda x_i) = \lambda(c \cdot x_i)$ , so clearly  $c \cdot \vec{x}$  is also an eigenvector. Just explaining briefly that  $A(c\vec{x}) = c(A\vec{x}) = c \cdot (\lambda \cdot \vec{x}) = \lambda(c \cdot \vec{x})$  is also fine.

**4b** (20 p) Show that if  $G$  is a  $d$ -regular graph, then its adjacency matrix  $A_G$  always has  $d$  as an eigenvalue.

**Solution:** For this and the following problems, an important first observation is that if we identify the vertices  $v_1, v_2, \dots, v_n$  of  $G$  with the integers  $1, 2, \dots, n$  (as in Figure 1), then the  $i$ th entry of  $A_G\vec{x}$  is

$$(A_G\vec{x})_i = \sum_{j=1}^n a_{i,j}x_j = \sum_{j \in N(i)} x_j, \quad (5)$$

where as usual  $N(\cdot)$  denotes the set of neighbours of a vertex. This is so since  $a_{i,j}$  is 1 when vertices  $i$  and  $j$  are neighbours and is 0 otherwise.

If we follow the examples in the problem statement and let  $\vec{x} = \mathbf{1}$  be the all-ones vector, then we see from (5) that  $(A_G\mathbf{1})_i = \sum_{j \in N(i)} 1$  will simply count the number of neighbours of vertex  $i$ . Since all vertices have  $d$  neighbours in a  $d$ -regular graph, we have  $(A_G\mathbf{1})_i = d = d \cdot 1$ , and so it follows that  $\mathbf{1}$  is an eigenvector with eigenvalue  $d$ .

**4c** (20 p) Show that if  $G$  is a  $d$ -regular graph, then its adjacency matrix  $A_G$  can never have an eigenvalue  $\lambda$  such that  $|\lambda| > d$ .

*Hint:* Suppose that there is an eigenvector  $\vec{x}$  with eigenvalue  $\lambda$  such that  $|\lambda| > d$ . Rescale the entries in  $\vec{x}$  by some  $c \neq 0$  so that the largest entry has value 1 and all other entries have absolute value at most 1. Consider the product  $A_G \cdot \vec{x}$ , focus on a largest entry in  $\vec{x}$ , and argue by contradiction.

**Solution:** Follow the hint and rescale the vector  $\vec{x}$ , so that  $x_i = 1$  for the largest entry  $i$  and  $|x_j| \leq 1$  for all  $j$ . Then we get

$$|(A_G\vec{x})_i| = \left| \sum_{j \in N(i)} x_j \right| \leq \sum_{j \in N(i)} |x_j| \leq \sum_{j \in N(i)} 1 \leq d < |\lambda|, \quad (6)$$

and so  $|\lambda|$  is just too large to possibly be the absolute value of an eigenvalue of  $A_G$ .

- 4d** (30 p) Show that if the  $d$ -regular graph  $G$  is connected, so that there is a path between any two vertices  $u$  and  $v$  in  $V(G)$ , then any eigenvector  $\vec{x}$  of the adjacency matrix  $A_G$  with eigenvalue  $d$  must have all entries equal (i.e.,  $\vec{x}$  is the all-ones vector or some multiple of the all-ones vector).

*Hint:* Suppose  $\vec{x}$  is an eigenvector of  $A_G$  with eigenvalue  $d$  in which not all entries are equal. Rescale the entries in  $\vec{x}$  by some  $c \neq 0$  so that the largest entry has value 1 and all other entries have absolute value at most 1, and consider the product  $A_G \cdot \vec{x}$ .

**Solution:** Again we rescale the vector  $\vec{x}$  so that the largest entry has value 1 and so that  $|x_j| \leq 1$  holds for all  $j$ . Suppose that  $i^*$  is a coordinate such that  $x_{i^*} = 1$  but there is a neighbour  $j^*$  of  $i^*$  such that  $x_{j^*} < 1$ . (If there are no such  $i^*$  and  $j^*$ , then all entries in the vector are the same—note that we are using here the assumption that the graph is connected.) We now have

$$(A_G \vec{x})_{i^*} = \sum_{j \in N(i^*)} x_j < \sum_{j \in N(i^*)} 1 = d, \quad (7)$$

where we get a strict inequality since  $x_j \leq 1$  holds for all  $j$  and for the particular neighbour  $j^*$  we have strict inequality  $x_{j^*} < 1$ . This shows that  $\vec{x}$  is not an eigenvector of  $A_G$  with eigenvalue  $d$ .

- 4e** (30 p) Show that if the  $d$ -regular graph  $G$  is *not* connected, so that there exist two vertices  $u$  and  $v$  in  $V(G)$  with no path between them, then there is in fact an eigenvector  $\vec{x}$  of the adjacency matrix  $A_G$  with eigenvalue  $d$  in which not all entries are equal.

*Hint:* Consider the different connected components of  $G$  and use them to define interesting vectors for which  $d$  is an eigenvalue.

**Solution:** Fix some connected component  $C$  and define a vector  $\vec{x}$  such that  $x_i = 1$  for  $i \in C$  and  $x_i = 0$  for  $i \notin C$ . We know from (5) that  $(A_G \vec{x})_i = \sum_{j \in N(i)} x_j$ . Since for every vertex  $i$  it holds that  $i$  and  $N(i)$  lie in the same connected component, we get for all  $i' \notin C$  that  $(A_G \vec{x})_{i'} = \sum_{j \in N(i')} x_j = \sum_{j \in N(i')} 0 = 0 = d \cdot x_{i'}$ . For vertices  $i'' \in C$  we get that  $C$  must contain all the  $d$  neighbours, and so in this case it holds that  $(A_G \vec{x})_{i''} = \sum_{j \in N(i'')} x_j = \sum_{j \in N(i'')} 1 = d = d \cdot x_{i''}$ . Hence,  $\vec{x}$  is an eigenvector with eigenvalue  $d$ .

- 4f** (40 p) We say that an undirected graph  $G = (V, E)$  is *bipartite* if there is a bipartition  $V = V_1 \dot{\cup} V_2$  (where  $\dot{\cup}$  denotes *disjoint union*, so that  $V_1 \cup V_2 = V$  but  $V_1 \cap V_2 = \emptyset$ ) such that any edge in  $G$  has one endpoint in  $V_1$  and one endpoint in  $V_2$ , but there are no edges connecting vertices in  $V_1$  to each other or vertices in  $V_2$  to each other. (Just to give examples for this definition, it is not hard to verify that the graph in Figure 1a is bipartite but that the graph in Figure 1b is not.)

Show that if the  $d$ -regular graph  $G$  is bipartite and the adjacency matrix  $A_G$  has an eigenvector  $\vec{x}$  with eigenvalue  $\lambda$ , then  $-\lambda$  is also an eigenvalue for  $A_G$ .

*Hint:* Consider the bipartition  $V = V_1 \dot{\cup} V_2$  and use it to modify the eigenvector  $\vec{x}$  in some interesting way. (This connection between bipartiteness and negated eigenvalues is actually an if and only if—it holds that  $-\lambda$  is also an eigenvalue only if  $G$  is bipartite—but you definitely do not need to prove this.)

**Solution:** Suppose that  $G$  is a bipartite  $d$ -regular graph with bipartition  $V = V_1 \dot{\cup} V_2$ , and that  $\vec{x}$  is an eigenvector with eigenvalue  $\lambda$ . Define the vector  $\vec{y}$  by

$$y_i = \begin{cases} x_i & \text{if } i \in V_1, \\ -x_i & \text{if } i \in V_2. \end{cases} \quad (8)$$

We are again going to use that  $(A_G \vec{y})_i = \sum_{j \in N(i)} y_j$  according to (5).

Consider first a vertex  $i \in V_1$ . Note that for all its neighbours  $j \in N(i)$  it holds that  $j \in V_2$  because of bipartiteness, and so  $y_j = -x_j$  by our construction of  $\vec{y}$  in (8). This means that

$$(A_G \vec{y})_i = \sum_{j \in N(i)} y_j = \sum_{j \in N(i)} -x_j = -\lambda x_i = (-\lambda) \cdot y_i \quad (9)$$

(where we used that  $\sum_{j \in N(i)} x_j = \lambda x_i$  by assumption, since  $\vec{x}$  is an eigenvector with eigenvalue  $\lambda$ ).

In exactly the same way, for  $i \in V_2$  we have that  $y_i = -x_i$  but that the neighbours  $j \in N(i)$  have vector entries  $y_j = x_j$  since  $j \in V_1$ . The calculation

$$(A_G \vec{y})_i = \sum_{j \in N(i)} y_j = \sum_{j \in N(i)} x_j = \lambda x_i = (-\lambda) \cdot y_i \quad (10)$$

completes the proof that  $\vec{y}$  is an eigenvector with eigenvalue  $-\lambda$ .

Note that Problems 4d and 4e say that a  $d$ -regular graph  $G$  is connected if and only if the only eigenvectors of the adjacency matrix  $A_G$  with eigenvalue  $d$  are multiples of the all-ones vector, and Problem 4f says that  $G$  is bipartite only if the eigenvalues of  $A_G$  are symmetric with respect to 0. We are in fact only scratching the surface here, in that the eigenvalues of  $A_G$  can tell us much more about the properties of  $G$ . Perhaps the most important connection is that the second largest eigenvalue  $\lambda_2$  in absolute value is a measure of how well-connected the graph is—if the gap between  $d$  and  $|\lambda_2|$  is large, then  $G$  is an *expander graph* in which there are short paths between any two vertices. These and other highly nontrivial facts are further studied in *spectral graph theory*.