SOME BASICS ABOUT MATRICES

MATRIX Rectangular array of elements (for us: numbers)

$$A = \begin{array}{cl}
\begin{pmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2n} \\ a_{31} & a_{32} & a_{33} & \cdots & a_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \cdots & a_{mn} \end{pmatrix} & \begin{array}{l} \text{ELEMENT} \\ \\ \text{3rd ROW} \\ \\ \\ \end{array} \\
\quad \text{2nd COLUMN} &
\end{array}$$

$m \times n$ ("m by n") matrix SIZE or DIMENSION

Unless otherwise stated, matrix A has m rows and n columns

Understood from context :-)

Notation

$$A = (a_{ij})$$

$m = n$ SQUARE MATRIX

a_{ii}, $i \in [n]$ MAIN DIAGONAL

a_{ij}, $i \neq j$ OFF-DIAGONAL ELEMENTS

DIAGONAL MATRIX

All off-diagonal elements 0

$$\begin{pmatrix} 3 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{pmatrix}$$

IDENTITY MATRIX I_n (or just I)

Diagonal matrix with diagonal entries 1

ZERO MATRIX $0_{m \times n}$ all entries 0 (or just 0)

VECTOR matrix with $m = 1$ or $n = 1$

$m = 1$ ROW VECTOR

$n = 1$ COLUMN VECTOR

MATRIX ADDITION

If A, B are matrices with same dimensions $m \times n$, can add them

$C = A + B$ $m \times n$ matrix such that

$$c_{ij} = a_{ij} + b_{ij}$$

How to compute this?

```
MATRIX-ADD (A, B, C)
    for i := 1 upto m
        for j := 1 upto n
            c_ij := a_ij + b_ij
```

Let us focus on square matrices for simplicity (pretty much the only kind of matrices we will care about in this course)

That is, $\boxed{m = n}$

What is the time complexity of MATRIX-ADD? $\Theta(n^2)$

Is it possible to do better?

No. Output has n^2 numbers. Takes $\Omega(n^2)$ time to output these numbers

Addition of matrices works just like addition of numbers

$A + B = B + A$ COMMUTATIVITY

$(A+B)+C = A+(B+C)$ ASSOCIATIVITY

$A + 0 = A$ ZERO MATRIX is ADDITIVE IDENTITY

MATRIX MULTIPLICATION

A $m \times p$ matrix

B $p \times n$ matrix

MUST MATCH!

$C = A \cdot B$ is $m \times n$ matrix such that

$$\boxed{c_{ij} = \sum_{k=1}^{p} a_{ik} \cdot b_{kj}}$$

Dot product of ith row in A with jth column in B

NOT COMMUTATIVE in general! Can have $AB \neq BA$ even if both AB & BA defined

Ex
$$\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -2 & 5 \end{pmatrix}$$

$$\begin{pmatrix} -1 & 1 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 2 \\ 2 & 3 \end{pmatrix}$$

If A, B, C are matrices of COMPATIBLE DIMENSIONS, then

$$(AB)C = A(BC) \qquad \text{ASSOCIATIVITY}$$

$$\left.\begin{aligned} A(B+C) &= AB + AC \\ (A+B)C &= AC + BC \end{aligned}\right\} \text{DISTRIBUTIVITY}$$

For $m \times n$ matrix A

$$A \cdot I_n = I_m \cdot A = A$$

IDENTITY MATRIX is MULTIPLICATIVE IDENTITY

How to compute matrix product $C = AB$?

A $m \times p$ matrix

B $p \times n$ matrix

```
MATRIX-MUL (A, B, C)
  for i := 1 upto m
  |   for j := 1 upto n
  |   |   c_ij := 0
  |   |   for k := 1 upto p
  |   |   L   c_ij := c_ij + a_ik · b_kj
  L   L
```

Focus again on square matrices, $m = n = p$

Time complexity of MATRIX-MUL? $\Theta(n^3)$

Is it possible to do better?

Hmmm... Need to compute all products $a_{ik} \cdot b_{kj}$ for $i, j, k \in [n]$

That is n^3 numbers... SO MAYBE AGAIN NO?

But it is possible to do much better!

- Strassen $O(n^{2.807})$ $\boxed{1969}$
- Coppersmith–Winograd $O(n^{2.375})$ $\boxed{1990}$

Further improvements in lower-order digits in last few years by

- Stothers $\boxed{2010}$ $O(n^{2.374})$
- Vassilevska–Williams $\boxed{2011}$ $O(n^{2.3728642})$
- Le Gall $\boxed{2014}$ $O(n^{2.3728639})$

Lots of research into this over the years...

Matrix multiplication used as subroutine in many other algorithms

Notation: MATRIX MULTIPLICATION EXPONENT ω (lower-case omega)

Currently $\omega \approx 2.37$

Improvement in $\omega \Rightarrow$ automatic improvements in lots of algorithms

Somewhat widely believed that

$$\boxed{\omega = 2}$$

should be possible

BIG open research problem

In practice, matrices can get very large – important to speed up computations.

Even $O(n^2)$ might be too slow

A (square) matrix A is SPARSE if
\# non-zero entries $\ll n^2$
(or $o(n^2)$, as we can write using asymptotic notation)

Lots of research into efficient algorithms for sparse matrices

Can also consider
- APPROXIMATION ALGORITHMS: not exactly correct answer, but close
- RANDOMIZED ALGORITHMS: use randomness to try to find answer faster (but might fail with small probability)
- Combinations of the two...

Areas of very active research

You might touch on some of this in later courses.

(Last two pages just nice-to-know
Won't be on exam)

BOOLEAN MATRIX A

All $a_{ij} \in \{0, 1\}$

Identify $\quad 1 \Leftrightarrow$ True $\quad$ Standard in

$\quad\quad\quad\quad 0 \Leftrightarrow$ False $\quad$ computer science

For $a, b \in \{0, 1\}$

$$a \vee b = \begin{cases} 0 & \text{if } a = b = 0 \\ 1 & \text{otherwise} \end{cases}$$

$$a \wedge b = \begin{cases} 1 & \text{if } a = b = 1 \\ 0 & \text{otherwise} \end{cases}$$

Define

$$\boxed{\begin{aligned} A \vee B &= (a_{ij} \vee b_{ij}) \\ A \wedge B &= (a_{ij} \wedge b_{ij}) \end{aligned}}$$

Ex Let $A = \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$

Then

$$A \vee B = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} \quad A \wedge B = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$$

BOOLEAN PRODUCT $A \odot B$

A $m \times p$ matrix

B $p \times n$ matrix

$\boxed{C = A \odot B}$ is $m \times n$ matrix such that

$$\boxed{c_{ij} = \begin{cases} 1 & \text{if } a_{ik} = b_{kj} = 1 \text{ for some } k \in [p] \\ 0 & \text{otherwise} \end{cases}}$$

Compare:

$$A \cdot B = \left(\sum_{k=1}^{p} a_{ik} \cdot b_{kj} \right)$$

$$A \odot B = \Big((a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \cdots \vee (a_{ip} \wedge b_{pj}) \Big)$$

$$= \left(\bigvee_{k=1}^{p} (a_{ik} \wedge b_{kj}) \right)$$

<u>Example</u> $A \odot B = C$

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \odot \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$$

$c_{1,1}$: $a_{11} = b_{11} = 1$ so 1

$c_{1,2}$: $a_{11} = 1, b_{21} = 0$ but $a_{12} = b_{22} = 1$, so 1

$c_{2,1}$: $a_{21} = 0$; $b_{12} = 0$ so 0

$c_{2,2}$: $a_{21} = 0$ but $a_{22} = b_{22} = 1$ so 1

<u>Exercise</u> Show that the identity matrix is the multiplicative identity also for Boolean multiplication

GRAPHS $G = (V, E)$

V VERTICES

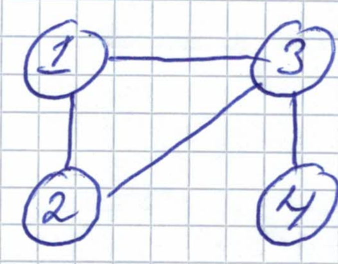
E EDGES between pairs of vertices

UNDIRECTED or DIRECTED

Focus on undirected here

Example

$G =$



Can represent graph by ADJACENCY MATRIX

Graph G over $|V| = n$ vertices

$n \times n$ adjacency matrix A_G

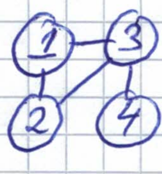
$$a_{ij} = \begin{cases} 1 & \text{if edge from } i \text{ to } j \text{; i.e., } (i,j) \in E \\ 0 & \text{otherwise} \end{cases}$$

$$A_G = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

MATRIX TRANSPOSE A^T

$A^T = (a_{ji})$ Exchange rows and columns

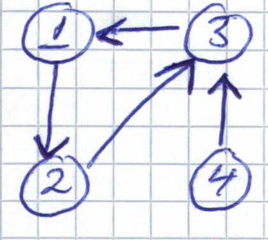
Matrix is SYMMETRIC if $A^T = A$

$G =$ 

$$A_G = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$\boxed{\text{Graph } G \text{ is undirected} \iff A_G \text{ symmetric}}$$

Let us quickly compare with a directed graph

$H =$ 

$$A_H = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

A_H is <u>not</u> symmetric

Back to our undirected graph G

Consider

$$(A_G)^2 = A_G \cdot A_G = \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 1 \\ 1 & 1 & 3 & 0 \\ 1 & 1 & 0 & 1 \end{pmatrix}$$

What is the meaning of this?

$(A_G^2)_{ii}$ = DEGREE of vertex i

= #edges INCIDENT to vertex i

$(A_G^2)_{ij}$ = # ways to take "two hops" from vertex i to vertex j

$$\left(A_G^2\right)_{ij} = \sum_{k=1}^{n} \underbrace{a_{ik} \cdot a_{kj}}_{\substack{1 \text{ if possible to hop} \\ i \to k \to j}}$$

$\boxed{\text{PATH}}$ in graph: Sequence of vertices $(v_0, v_1, \ldots, v_k)$ such that for all $i \in [k]$ $(v_{i-1}, v_i) \in E$

LENGTH k

$\boxed{\text{SIMPLE PATH}}$: all vertices v_i distinct

Sometimes only simple paths are called paths and (non-simple) paths are called $\boxed{\text{walks}}$ — that is what I am used to

We have just argued:

$\left(A_G^2\right)_{ij}$ = # paths from i to j of length exactly 2

Exercise: Prove by induction that

$$\boxed{\left(A_G^k\right)_{ij} = \#\text{paths from } i \text{ to } j \text{ of length exactly } k}$$

Exercise:
What is the meaning of the (i,j)-entry of

$$\underbrace{A_G \odot A_G \odot \ldots \odot A_G}_{A_G \; k \text{ times}} \;?$$