IDMA 2025: WEEK 4

COMBINATORICS, COUNTING & PROBABILITY

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Combinatorics

Counting: number of objects, outcomes, possibilities

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ORDER DOESN'T MATTER	MULTISET	SET

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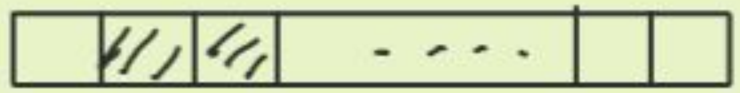

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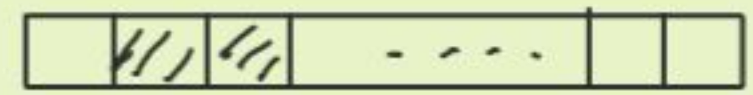
Size r multisets from $S=\{s_1,\ldots,s_n\}$ $\longleftrightarrow$ (1-1 correspondence)



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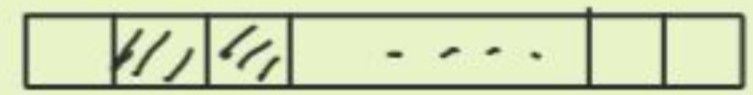

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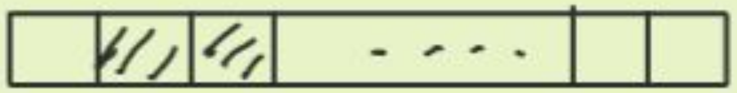
$$\left[\binom{n-1+r}{r} \text{ many} \right]$$

Combinatorial proof for Multisets

Size 6 multisets from $S = \{s_1, \dots, s_7\}$ $\xleftrightarrow[\text{correspondence}]{1-1}$ 

Row of 12 boxes with 6 coloured boxes & 6 uncoloured boxes

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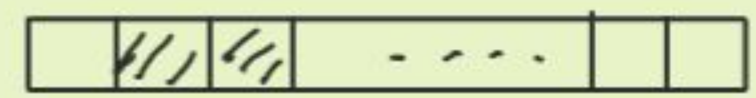
Size 6 multisets $\longleftrightarrow$ 

1–1

from $S = \{s_1, \ldots, s_7\}$ correspondence

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Correspondence

$\{s_2, s_2, s_3, s_5, s_5, s_7\} = A$

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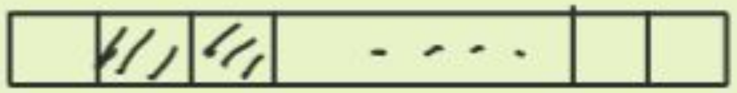
Size 6 multisets from $S = \{s_1, \ldots, s_7\}$ $\xleftrightarrow[\text{correspondence}]{1-1}$  Row of 12 boxes with 6 coloured boxes & 6 uncoloured boxes

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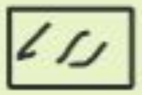
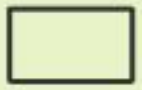
→  = A has another copy of current element

→  = skip to next element

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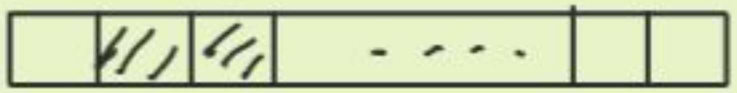

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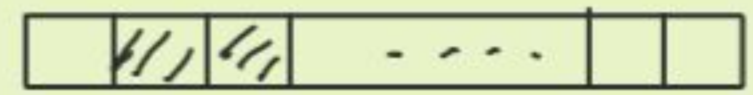
→ Can recover A from coloured boxes.

Combinatorial proof for Multisets

Size r multisets $\longleftrightarrow$ 

from $S=\{s_1,\ldots,s_n\}$

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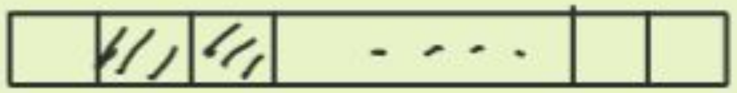

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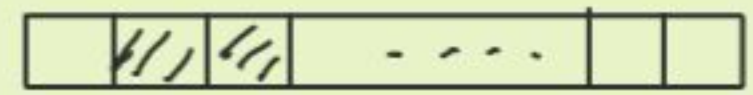
Size r multisets $\longleftrightarrow$ 

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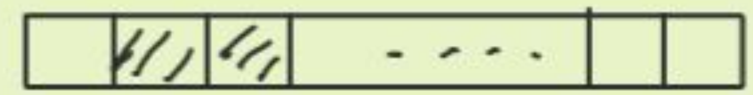
Size r multisets from $S = \{s_1, \ldots, s_n\}$ $\longleftrightarrow$ 

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→ For each object in $\mathcal{E}_1$, find an object in $\mathcal{E}_2$

→ For each object in $\mathcal{E}_2$, there is a unique corresponding object in $\mathcal{E}_1$

Another combinatorial identity

$$\binom{n}{k} = \binom{n-1}{k} + \binom{n-1}{k-1}$$

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$$\parallel$$

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$$\left.\begin{array}{l} B \cup \{s_n\}, \text{ if } |B| = k-1 \\ B \text{ ; if } |B| = k \end{array}\right\} \longleftarrow B$$

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Eg A Fair die

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In general, A = set of outcomes - Sample space

For each $a \in A$, $p(a)$ or $Pr[a]$ - "probability of a"

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$$\boxed{p(a) \geq 0, \quad \sum_{a \in A} p(a) = 1.}$$

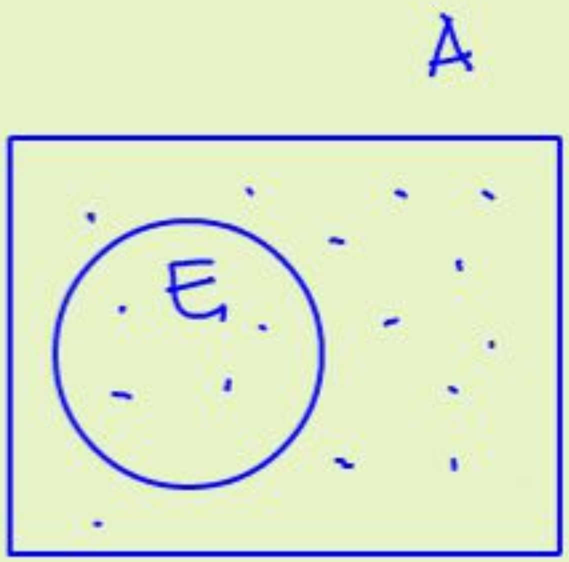
(p, A) - Probability Space

Events

(p, A): probability space

Event: subset of possible outcomes

$E \subseteq A$



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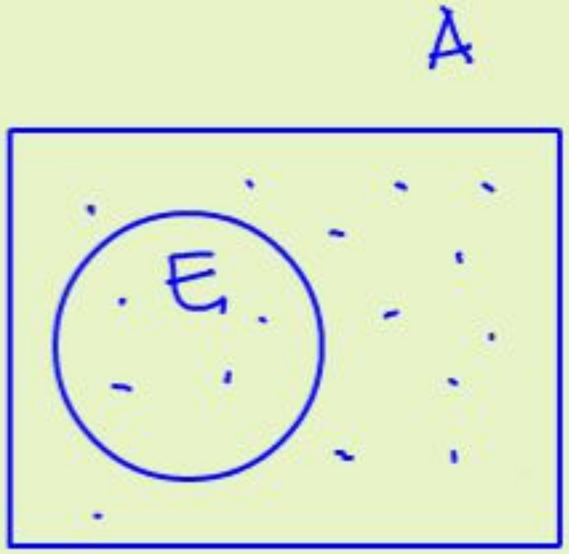

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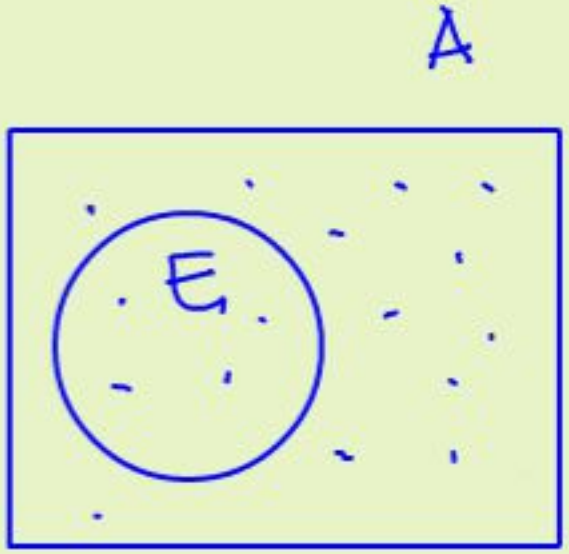

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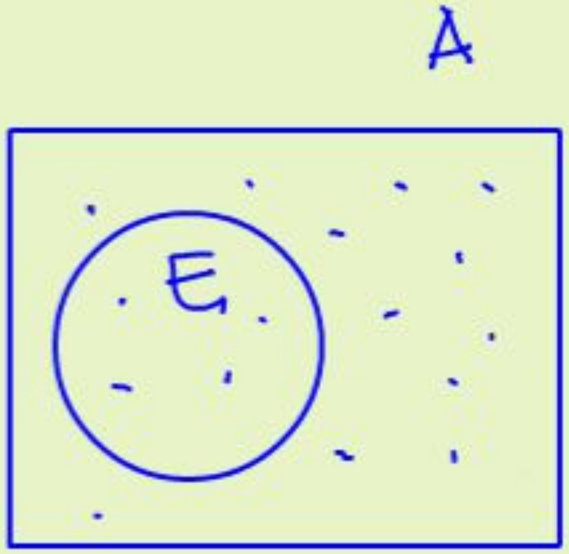
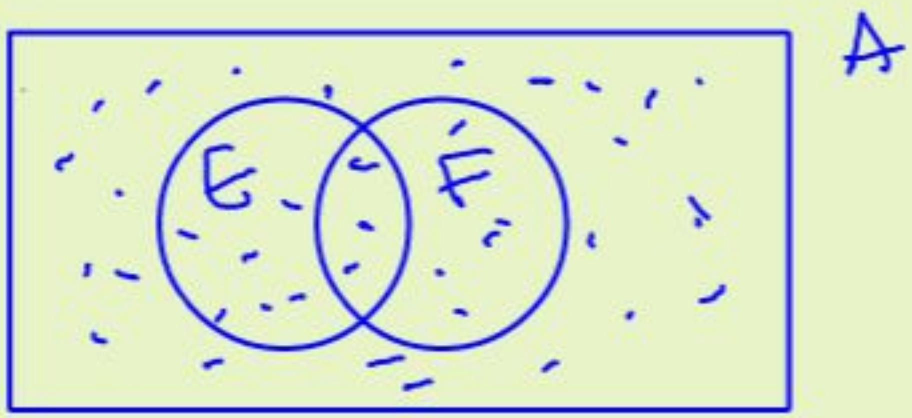
In general, $\Pr[E] = \sum_{a \in A} \Pr[a]$



Unions, Intersections, Complements



E, F - events

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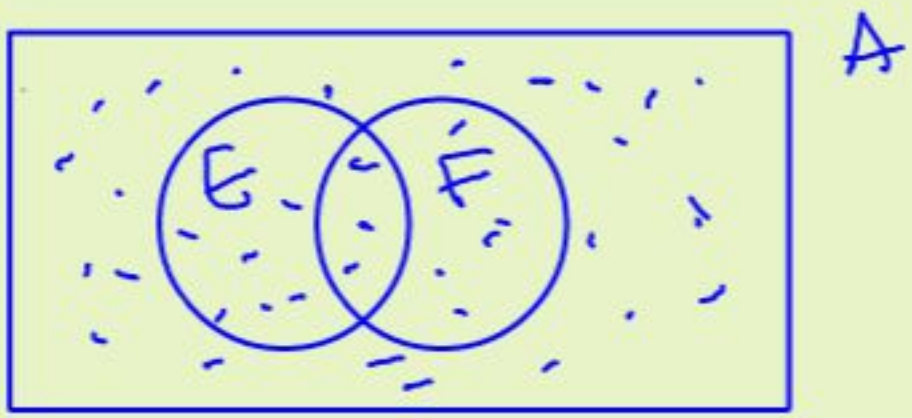

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Other events: $E \cup F$, $E \cap F$, $\bar{E}$ OR E^c

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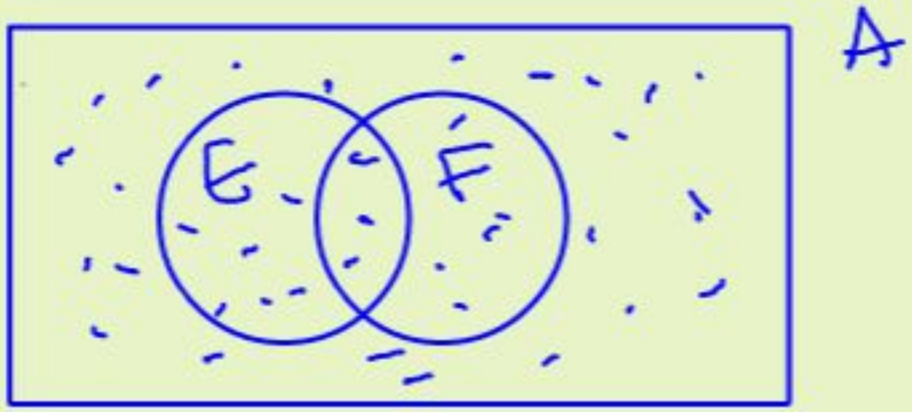

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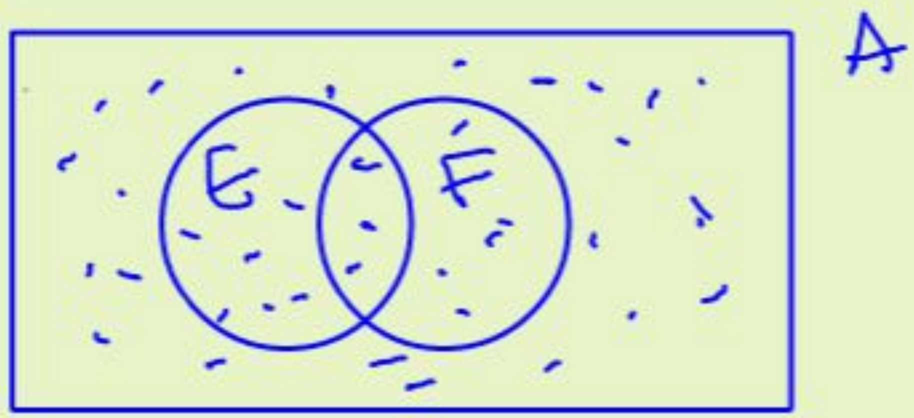

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$$Pr[E \cup F] = \sum_{a \in E \cup F} p(a) = Pr[E] + Pr[F] - Pr[E \cap F]$$

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We can compute the probability of E by counting to compute the size of E.

Poker Hands

Standard deck of 52 cards

4 suits: Spades Clubs Hearts Diamonds

♠ ♣ ♥ ♦

13 cards per suit: A, 2-10, J, Q, K (ranks)

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$A = \{\text{Set of 5 cards}\}$, $|A| = \binom{52}{5} = \dfrac{52!}{5!\,47!}$

$\downarrow$

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$= 2.598.960$

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↓

Poker Hands

$p(a) = \frac{1}{\binom{52}{5}}$

Event Two Pairs

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Two Pairs = $\{$Hands with 2 cards of one rank, 2 cards of another rank & a fifth card of a 3rd rank$\}$

$\subseteq A$

Event Two Pairs

Two Pairs = {Hands with 2 cards of one rank, 2 cards of another rank & a fifth card of a 3rd rank}

$\subseteq A$

3 spades, 3 diamonds, 7 diamonds, 7 clubs, 2 hearts ✓

Event Two Pairs

Two Pairs = {Hands with 2 cards of one rank, 2 cards of another rank & a fifth card of a 3rd rank}

$\subseteq A$

3 spades, 3 diamonds, 7 diamonds, 7 clubs, 2 hearts

✓

3 spades, 3 diamonds, 7 diamonds, 7 clubs, 7 hearts

✗

Event Two Pairs

Two Pairs = {Hands with 2 cards of one rank, 2 cards of another rank & a fifth card of a 3rd rank}

$\subseteq A$

$$\Pr[\text{Two Pairs}] = \frac{|\text{Two Pairs}|}{|A|}$$

Event Two Pairs

Two Pairs = {Hands with 2 cards of one rank, 2 cards of another rank & a fifth card of a 3rd rank}

$\subseteq A$

$$\Pr[\text{Two Pairs}] = \frac{|\text{Two Pairs}|}{|A|} = \frac{}{\binom{52}{5}}$$

Event Two Pairs

Two Pairs = { Hands with 2 cards of one rank, 2 cards of another rank & a fifth card of a 3rd rank }

$\subseteq A$

$$\Pr[\text{Two Pairs}] = \frac{|\text{Two Pairs}|}{|A|} = \frac{}{\binom{52}{5}}$$

Number of two pairs $= \binom{13}{2} \cdot 11 \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 4$

Event Two Pairs

Two Pairs = { Hands with 2 cards of one rank, 2 cards of another rank & a fifth card of a 3rd rank }

$\subseteq A$

$$\Pr[\text{Two Pairs}] = \frac{|\text{Two Pairs}|}{|A|} = \frac{}{\binom{52}{5}}$$

Number of two pairs $= \binom{13}{2} \cdot 11 \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 4$

Choose ranks appearing twice

Choose rank appearing once

Suits of first rank

Suits of second rank

Suit of last rank

Event Two Pairs

Two Pairs = { Hands with 2 cards of one rank, 2 cards of another rank & a fifth card of a 3rd rank }

$\subseteq A$

$$\Pr[\text{Two Pairs}] = \frac{|\text{Two Pairs}|}{|A|} = \frac{123.552}{\binom{52}{5}}$$

Number of two pairs $= \binom{13}{2} \cdot 11 \cdot \binom{4}{2} \cdot \binom{4}{2} \cdot 4$

Choose ranks appearing twice

Choose rank appearing once

Suits of first rank

Suits of second rank

Suit of last rank