COMBINATORICS

Counting number of different objects, outcomes, possibilities

Ex

- Outcomes for gold, silver and bronze medals for 10 competitors
- Number of different ways 3 identical dice can come up
- Possible 5-card hands in poker and the probability of getting a good hand

TOPICS / CONCEPTS

Pigeonhole principle
Multiplication principle
Ordered / unordered choices with/without repetition
When order matters
- sequences (with repetitions)
- permutations (without repetitions)

When order doesn't matter
- subsets (without repetitions)
- multisets (with repetitions)

PIGEONHOLE PRINCIPLE (skuffeprincippet)

> If m pigeons fly into n pigeonholes for $m > n$, then at least one pigeonhole contains at least two pigeons

Ex Are there two persons in Copenhagen with exactly the same number of hair?

According to research, humans have $\leq 200\,000$ hair
Population of Copenhagen $> 600\,000$ persons

pigeons — pigeonholes

So YES, two persons in Copenhagen have exactly the same number of hair (but we don't know how to find them — this is a non-constructive, purely existential claim

EXTENDED PIGEONHOLE PRINCIPLE

> m pigeons, n pigeonholes $\Rightarrow$ at least one pigeonhole with at least $\lceil m/n \rceil$ pigeons

So at least 4 persons in Copenhagen have the same number of hair

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To apply the pigeonhole principle need to identify pigeons and pigeonholes, and show #pigeons $>$ #pigeonholes

III

Ex For any 11 numbers between 1 and 20, one will be a multiple of another

Every $n \in \mathbb{Z}^+$ can be written as
$n = 2^k \cdot m$ for
$m \in \mathbb{Z}^+$ odd (the ODD PART of n)

Pigeonholes: Odd parts 10 odd numbers between 1 and 20

Pigeons: the 11 chosen numbers.

By the pigeonhole principle two pigeons end up in the same hole
$\Longleftrightarrow$ two numbers $n_1 = 2^{k_1} \cdot m$ and $n_2 = 2^{k_2} \cdot m$ with the same odd part m.

Suppose without loss of generality (wlog) ~~that~~ $k_1 \leq k_2$

Then $n_2 = 2^{k_2} \cdot m = 2^{k_2 - k_1} \cdot 2^{k_1} \cdot m$
$= 2^{k_2 - k_1} \cdot n_1$

so $n_1 \mid n_2$ as claimed $\blacksquare$

Again, this is a (mostly) non-constructive result. It doesn't tell us too much about how to find the two numbers efficiently — only that two such numbers must exist

RECALL: $d \mid n$ if $\exists q \in \mathbb{Z}$ s.t. $n = q \cdot d$, and our q is $2^{k_2 - k_1}$

MULTIPLICATION PRINCIPLE

If n_1 ways of choosing T_1,
n_2 ways of choosing T_2,
and choices independent,
then $n_1 \cdot n_2$ ways of choosing T_1 and T_2 together

Ex A computer science researcher typically has
- jeans (blue or black)
- t-shirt (white, blue, or black)

How many possible outfits?

$2 \cdot 3 = 6$

GENERAL MULTIPLICATION PRINCIPLE

k independent choices / tasks
n_i choices for task i
Then $n_1 \cdot n_2 \cdots n_k = \prod_{i=1}^{k} n_i$ ways of performing all tasks together

Ex How many subsets are there of a set S of n elements?

$S = \{s_1, s_2, \ldots, s_n\}$

For each element $s_i \in S$, either include it in the subset or not: 2 choices per s_i
n elements $s_1, \ldots, s_n \Rightarrow$ $\boxed{2^n \text{ different choices} = \text{subsets}}$ (including the empty subset $\emptyset$ and S itself)

$\boxed{\text{SEQUENCE}}$ Ordered list of elements

Ex $(1, 1, 2, 3, 5, 8)$

$(2, 3, 5, 7, 11, 13)$

Order matters! $(1,2) \neq (2,1)$

$\boxed{\text{PERMUTATION}}$ Sequence without repetitions

$\boxed{\text{SET:}}$ Unordered collection of elements without repetitions

Note $\{2,4,8\} = \{8,4,2\} = \{2,4,8,4,2\}$

$\boxed{\text{MULTISET}}$ Unordered collection of elements, possibly with repetitions

Note $[2,4,8] = [8,4,2] \neq [2,4,8,4,2]$

	WITH REPETITIONS	WITHOUT REPETITIONS
ORDERED	SEQUENCE	PERMUTATION
UNORDERED	MULTISET	SET

Suppose we have set of n elements
Choose r ordered / unordered and
with / without repetition

How many possibilities?

n elements, r choices

SEQUENCES

r choices
n alternatives for every choice
all outcomes yield different sequences

Multiplication principle:
n^r sequences

PERMUTATIONS

How many length-r sequences <u>without</u> repetitions (so $r \leq n$)

First choice: n alternatives
Second **choice**: $n-1$ alternatives
Third choice: $n-2$ alternatives
$\vdots$
r-th choice: $n-r+1$ alternatives

$$n(n-1)(n-2)\cdots(n-r+1) = {}_nP_r$$

in textbook notation $${}_nP_r = \frac{n!}{(n-r)!}$$

"number of permutations of n objects taken r at a time"

Ex With 10 competitors, how many possible outcomes are there for gold, silver, and bronze medal?

$${}_{10}P_3 = \frac{10!}{7!} = 10 \cdot 9 \cdot 8 = 720$$

How many permutations are there of a whole set S of n elements?

$${}_nP_n = n!$$

When we just say "permutation of S", we tend to mean permutation of all elements in S.

SET Like a permutation, except we don't care about the order

size-r permutations $\frac{n!}{(n-r)!}$

Except now we don't care how the r elements are ordered

Each subset shows up ordered in $r!$ different ways

So $\frac{n!}{(n-r)!\,r!} = {}_nC_r$ subsets

Much more common notation: $\boxed{\binom{n}{r} = \frac{n!}{(n-r)!\,r!}}$

BINOMIAL COEFFICIENT

"n-choose-r"

$$\binom{n}{r} = \frac{n!}{(n-r)!\,r!} = \frac{n!}{r!\,(n-r)!}$$

"number of combinations of n objects taken r at a time"

Why are these numbers called BINOMIAL COEFFICIENTS?

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

To get $x^i \cdot y^{n-i}$, choose x in i of the parentheses $(x+y)$ and choose y in the rest

Can be done in $\binom{n}{i}$ ways

<u>Note</u> that

$$\boxed{\binom{n}{r} = \binom{n}{n-r}}$$

ways to choose r out of n elements =

ways to leave $n-r$ out of n elements unchosen

<u>Ex</u> How many different 5-card hands can you get in poker?

$${}_{52}C_5 = \binom{52}{5} = \frac{52!}{5!\cdot 47!} = \frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5\cdot 4\cdot 3\cdot 2\cdot 1} =$$

$$= 2,598,960$$

Ex How many subsets are there of a set S of n elements?

subsets with r elements: $\binom{n}{r}$

r can range from 0 to n

So total number of subsets

$$\boxed{\sum_{r=0}^{n} \binom{n}{r} = 2^n}$$

[by the multiplication principle, as we argued before]

Combinatorial proof Prove an equality by counting the same collection of objects correctly in two different ways.

MULTISETS How many size-r multisets from a set of n elements?

THM If S is a set of n elements, then the number of multisets of size r with elements from S is

$$\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$$

Number of (standard) subsets of size r chosen from a set of $n+r-1$ elements

How to prove this?

Construct translation so that

- every size-r multiset is translated to a size-r subset of a set of $n-r+1$ elements
- different multisets are translated to different subsets
- All possible subsets are covered by translation

If so, the two different collections will have the same number of elements

Imagine we have

- $n+r-1$ cells on a row
- r cells coloured red to encode multiset

Then the red cells can we viewed as an r-subset

Example $\quad n = 7 \quad r = 6$

$n + r - 1 = 12$

Subset $\{2, 5, 6, 7, 9, 10\}$

How can we translate multisets to such subsets?

FROM MULTISET TO COLOURING

Let $S = \{s_1, s_2, \ldots, s_n\}$

Suppose we have a multiset M with r elements chosen from S (with repetitions)

For every $s_i \in S$, we can ask how many copies of s_i are in M (0 or more)

Here is how to colour our row of cells

MULTISET2COLOUR (M)

```
pos := 0
for i := 1 upto n
    pos := pos + 1          // skip one cell forward
    c := # copies of s_i in M
    for j := 1 upto c
        colour box in position pos red
        pos := pos + 1
```

Claim 1

The algorithm produces a valid colouring = size-r subset of set of $n+r-1$ elements

Proof

- Exactly r cells are coloured
- pos increases after every colouring — coloured cells are distinct
- pos is incremented $n+r$ times, but nothing is written after last increment, so $1 \leq pos \leq n+r-1$ for all coloured cells

Claim 2

Different multisets M and M' yield different colourings

Proof Suppose for $s_1, \ldots, s_{i-1}$ that M and M' have the same number of copies, but that s_i is the first element that appears different # times in M and M' (with more copies of s_i in M' than M, say)

Then translation of M' will have longer stretch of red cells for s_i than does M. And since position pos is always increased, this cannot be changed afterwards

Since all multisets yield different colourings, this shows that

$$\boxed{\# \text{multisets} \leq \binom{n+r-1}{r}}$$

But it could be that there are more colourings than multisets!

To show that this is not the case, we need to describe how to translate any row of coloured boxes (like the one example we saw before) to a multiset!

Let R be a row of $n+r-1$ boxes with r boxes coloured red

```
COLOUR2MULTISET(R)
  pos := 1              // position in row of boxes
  i := 1                // index of element in S
  while pos ≤ n+r-1
      if (cell in position pos red)
          add copy of s_i to multiset
      else
          i := i+1
      pos := pos+1
```

Claim 1

It always holds that $1 \le i \le n$, so s_i is always a valid element

Proof

There are r red cells (by assumption) so i is incremented $(n+r-1)-r = n-1$ times.

Claim 2

The algorithm outputs a multiset of r elements.

Proof

This is because there are r cells coloured red.

Claim 3

Different colourings yield different multisets.

Proof Exercise. (Argue in the same way as for the other algorithm.)

Since all colourings yield different multisets, it follows that

$$\boxed{\#\text{multisets} \geq \binom{n+r-1}{r}}$$

ILLUSTRATION OF MULTISET2COLOUR

$M = [2, 4, 4, 4, 5, 5]$ $\quad n = 7$, $r = 6$

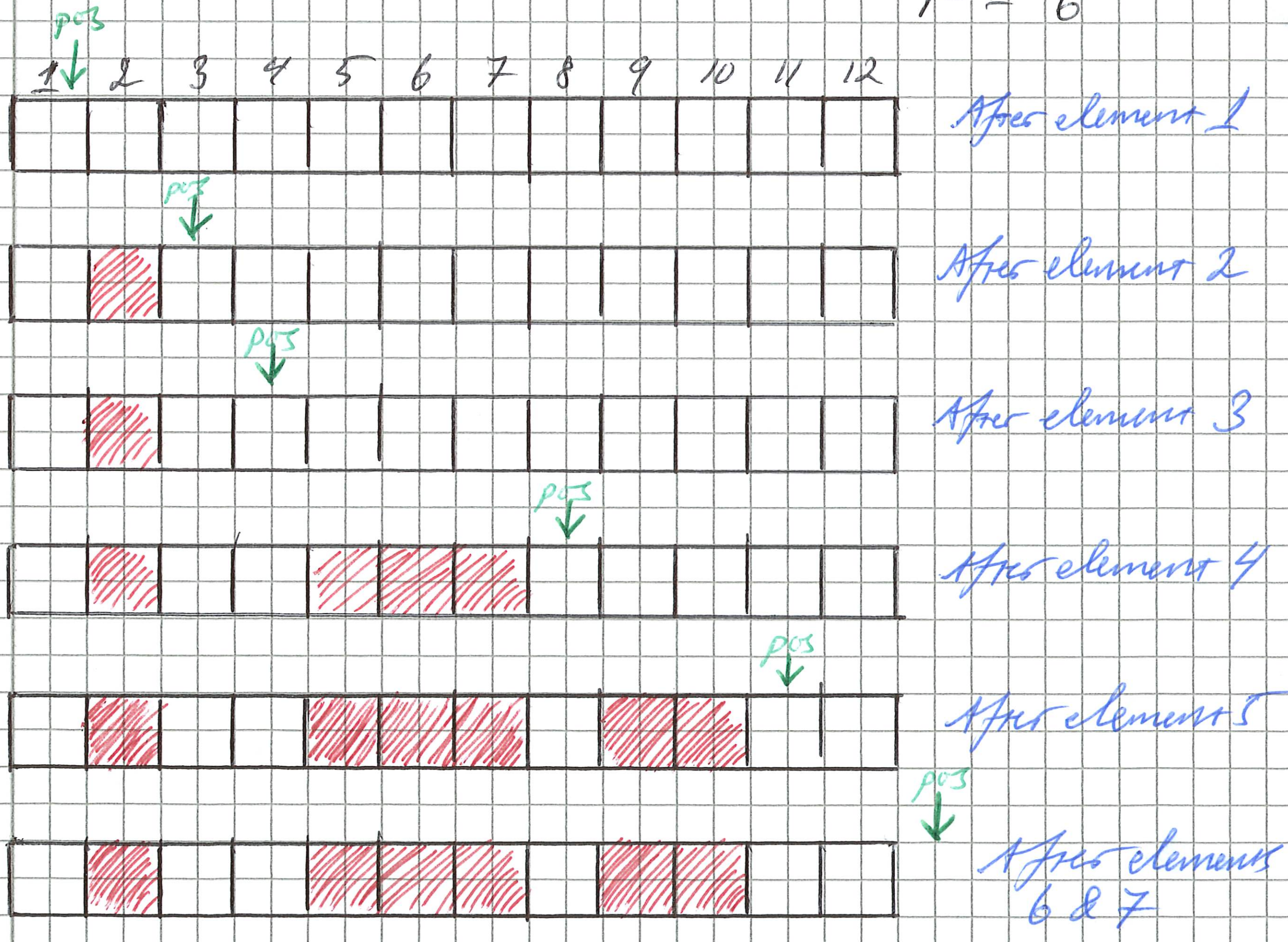

ILLUSTRATION OF COLOUR 2 MULTISET

Recall $n = 7$, $r = 6$, $n + r - 1 = 12$

VALUE OF i CIRCLED

(1) Current position 1 $\Rightarrow$ No copy of 1 in multiset
Blank

(2) Current position 2 $\Rightarrow$ One copy of 2 in multiset
1 red, then blank

(3) Current position 4 $\Rightarrow$ No copy of 3 in multiset
Blank

(4) Current position 5 $\Rightarrow$ Three copies of 4 in multiset
3 red, then blank

(5) Current position 9 $\Rightarrow$ Two copies of 5 in multiset
2 red, then blank

(6) Current position 12 $\Rightarrow$ No copy of 6 in multiset
Blank

(7) Now current position 13 $\Rightarrow$ No copy of 7 in multiset
is out of bounds

[we have already chosen 6 elements, so it makes sense that we already know that adding 7 is not an option]

Constructed multiset: [2, 4, 4, 4, 5, 5]

Translation back worked as it should – phew!

Ex What is the number of possible different outcomes when rolling 3 (indistinguishable) dice?

$n = 6 \qquad r = 3$

\# outcomes = # 3-element multisets =

$$= \binom{6+3-1}{3} = \frac{8!}{3!\,5!} =$$

$$= \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$$

SUMMING UP our counting so far

n elements, choose r times

	WITH REPETITIONS	WITHOUT REPETITIONS
ORDERED	SEQUENCE n^r	PERMUTATION ${}_nP_r = \frac{n!}{(n-r)!}$
UNORDERED	MULTISET ${}_{n+r-1}C_r = \binom{n+r-1}{r}$	SET ${}_nC_r = \binom{n}{r} = \frac{n!}{r!\,(n-r)!}$

BINOMIAL COEFFICIENT

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$ "n choose r"

\# ways to choose r elements out of n without repetitions when order does not matter