KØBENHAVNS UNIVERSITET

Introduktion til diskret matematik og algoritmer: Problem Set 2

Due: Wednesday February 28 at 9:59 CET.
Submission: Please submit your solutions via *Absalon* as a PDF file. State your name and e-mail address close to the top of the first page. Solutions should be written in LaTeX or some other math-aware typesetting system with reasonable margins on all sides (at least 2.5 cm). Please try to be precise and to the point in your solutions and refrain from vague statements. Make sure to explain your reasoning. *Write so that a fellow student of yours can read, understand, and verify your solutions.* In addition to what is stated below, the general rules for problem sets stated on *Absalon* always apply.
Collaboration: Discussions of ideas in groups of two to three people are allowed—and indeed, encouraged—but you should always write up your solutions completely on your own, from start to finish, and you should understand all aspects of them fully. It is not allowed to compose draft solutions together and then continue editing individually, or to share any text, formulas, or pseudocode. Also, no such material may be downloaded from or generated via the internet to be used in draft or final solutions. Submitted solutions will be checked for plagiarism.
Grading: A score of 120 points is guaranteed to be enough to pass this problem set.
Questions: Please do not hesitate to ask the instructor or TAs if any problem statement is unclear, but please make sure to send private messages—sometimes specific enough questions could give away the solution to your fellow students, and we want all of you to benefit from working on, and learning from, the problems. Good luck!

1 (100 p) Suppose that we are given an array $A = [5, 6, 4, 7, 3, 8, 2, 9, 1, 10]$ to be sorted in increasing order.

1a Run insertion sort by hand on this array (as the lecturer has been doing in class), and describe in detail in every step of the algorithm how the elements in the array are moved.

1b Run merge sort by hand on this array (as the lecturer has been doing in class), and show in detail in every step of the algorithm what recursive calls are made and how the results from these recursive calls are combined.

1c Suppose that we are given another array B that is already sorted in increasing order. How fast do the insertion sort and merge sort algorithms run in this case? Is any of them asymptotically faster than the other as the size of the array B grows?

1d Suppose that we are given a third array C that is sorted in *decreasing* order, so that it needs to be reversed to be sorted in the order that we prefer, namely increasing. How fast do the insertion sort and merge sort algorithms run in this case? Is any of them asymptotically faster than the other as the size of the array C grows?

2 (60 p) Post-pandemic life in academia has meant noticeable changes for both Jakob and his family, in ways both good and bad.

2a In the autumn of 2022, Jakob attended his first non-virtual international conference in a very long time (since February 2020, to be precise), and the conference had more than 100 participants. A colleague at the conference pointed out to Jakob that this meant that either there were participants from more than 10 different countries, or else more than 10 people from some particular country attended the conference. Jakob, who had not studied the list of participants that closely, was amazed at this claim. Can you explain to Jakob, without even having looked at the list of participants, why the claim has to be true?

2b While Jakob was away, he suggested to his children that they should entertain themselves at home with the following game: Write down the numbers 1 to 20 on a sheet of paper. Erase any two distinct numbers a and b and replace them by the number $a + b - 1$. Now do the same again with all numbers currently on the sheet, i.e., pick any two members a' and b' of the multi-set $\big(\{1, 2, \ldots, 20\} \setminus \{a, b\}\big) \cup \{a + b - 1\}$ of all numbers between 1 and 20 except a and b plus the new number $a + b - 1$, and replace the two chosen numbers by their sum minus one $a' + b' - 1$. Repeat this procedure until there is only one single number left on the sheet of paper (i.e., until the multi-set has size 1). The children found this game a little bit repetitive, however. Can you describe the range of possible outcomes for this game? For a full score, provide proofs of any claims you make.

3 (80 p) Provide formal proofs of the following claims using proof techniques that we have learned during the course.

3a For all $s \in \mathbb{N}$ and all $k > 0$ it holds that $1 + sk \leq (1 + k)^s$.

3b For the sequence (T_i) defined by

$$T_i = \begin{cases} 0 & \text{for } i = 1, \\ 1 & \text{for } i = 2, \\ T_{i-1} + T_{i-2} & \text{for } i \geq 3, \end{cases}$$

it holds for all $i \geq 2$ that

$$1 \leq \frac{T_{i+1}}{T_i} \leq 2 \ .$$

3c For the sequence (a_n) defined by

$$a_n = \begin{cases} 5 & \text{if } n = 1, \\ 13 & \text{if } n = 2, \\ 5a_{n-1} - 6a_{n-2} & \text{if } n > 2, \end{cases}$$

it holds for all positive integers n that $a_n = 2^n + 3^n$.