OPERATIONS ON RELATIONS

Relations are just sets, so can apply set operations

If R, S relations from A to B then $R \cup S$ and $R \cap S$ also relations

$a\,(R \cup S)\,b$ if aRb or aSb

$a\,(R \cap S)\,b$ if aRb and aSb

Ex Let $R, S \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ defined by

$aRb \iff 2 \mid ab$

$aSb \iff 3 \mid ab$

What is $R \cup S$? $R \cap S$?

COMPLEMENT or COMPLEMENTARY RELATION

$$\overline{R} = \{ (a,b) \in A \times B \mid (a,b) \notin R \}$$

INVERSE RELATION FROM B TO A

$$R^{-1} = \{ (b,a) \in B \times A \mid a,b \in R \}$$

Ex Consider $\geq$

Complementary relation $<$

Inverse relation $\leq$

Ex Consider divisibility $a \mid b$

Inverse relation a multiple of b

Complement: a Does not divide b

CLOSURE If $R \subseteq A \times A$ doesn't have some property P, try to add as few pairs (a,a') as possible to R so that property P holds

REFLEXIVE CLOSURE Smallest relation $S \subseteq A \times A$ such that
- $R \subseteq S$
- S reflexive

SMALLEST means: Whenever this holds for S', then $S \subseteq S'$

THM Reflexive closure of R is

$$R \cup \{(a,a) \mid a \in A\}$$

Proof think about it...

SYMMETRIC CLOSURE smallest relation $S \subseteq A \times A$ such that
- $R \subseteq S$
- S symmetric

THM Symmetric closure of R is

$$R \cup R^{-1}$$

$\boxed{\text{TRANSITIVE CLOSURE}}$ of R

Smallest $S \subseteq A \times A$ such that

$R \subseteq S$

S transitive

THM Transitive closure of R is R^{∞}

$\boxed{\text{Question}}$ Given undirected graph $G = (V, E)$, Define

$E(u,v)$ = there is an edge from u to v

① What is the transitive closure?

② Prove that the reflexive closure of the ~~transitive closure~~ is an equivalence relation

③ What are the equivalence classes of this relation?

THM

Let $P \in \{\text{symmetric}, \text{reflexive}\}$

If R and S are both P, then $R \cup S$ and $R \cap S$ are also P.

If R and S are transitive, then $R \cap S$ is transitive

$\boxed{\text{Question}}$ What about $R \cup S$ and transitivity?

COROLLARY

If R and S are equivalence relations, then so is $R \cap S$.

PARTIALLY ORDERED SETS (POSETS)

Relation R on A is a partial order if it is

- reflexive $\quad a \leq a$
- antisymmetric $\quad a \leq b \;\&\; b \leq a \Rightarrow a = b$
- transitive $\quad a \leq b \;\&\; b \leq c \Rightarrow a \leq c$

The pair (A, R) is called a $\boxed{\text{poset}}$

Often use $\leq$ or $\preceq$ (\preceq) instead of R

Write $a \prec b$ (\prec) as shorthand for "$a \preceq b$ and $a \neq b$"

Examples
- Less-than-or-equal $\leq$
- Divisibility $a \mid b$
- subset $A \subseteq B$ on power set

Non-example
- Less than $<$ on numbers (why?)

Two elements a, b are $\boxed{\text{comparable}}$ if $a \leq b$ or $b \leq a$

An order relation is a $\boxed{\text{total}}$ or $\boxed{\text{linear}}$ order if any two elements are comparable

Examples (continued)
- Less-than-or-equal is total
- Divisibility and subset are not.

PRODUCT PARTIAL ORDERS

Suppose $(A, \leq_1)$ and $(B, \leq_2)$ posets.
Can define new relation $\leq_3$ on $A \times B$ by

$$(a,b) \leq_3 (a',b') \text{ if } a \leq_1 a' \text{ and } b \leq_2 b'$$

THEOREM $(A \times B, \leq_3)$ as defined above is a poset.

Exercise: Prove this

Extends also to more than two posets

Example Coordinate-wise comparison of vectors

$$\vec{u} = (u_1, \ldots, u_n) \qquad u_i, v_i \in \mathbb{Z}/\mathbb{Q}/\mathbb{R}$$
$$\vec{v} = (v_1, \ldots, v_n)$$

$\vec{u} \leq \vec{v}$ if $\forall i \in [n]$ $u_i \leq v_i$

Note Product of totally ordered posets is not ~~totally ordered~~

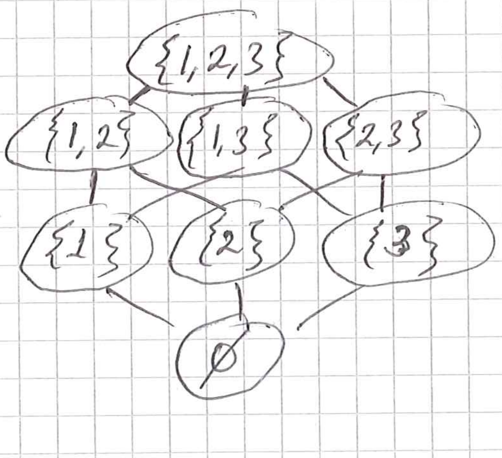
Exercise: Show this! (What assumptions do you need)

HASSE DIAGRAM

- Start with digraph representation
- Remove all loops
- Remove all edges implied by transitivity
- Make edges point upwards and remove arrow heads

Example

Powerset of $\{1,2,3\}$ with $\subseteq$



$z \in A$ is a

- $\boxed{\text{maximal}}$ element if there is no $a \in A$ s.t. $z < a$
- $\boxed{\text{minimal}}$ element if no $a \in A$ s.t. $a < z$

NOTE distinguish maximal – maximum
minimal – minimum

THEOREM Any finite, nonempty poset has a maximal and a minimal element.

Exercises

1) Does $(\mathbb{Z}, \leq)$ have any maximal or minimal elements, and if so which?
2) Ditto for $(\mathbb{Z}^+, \leq)$
3) Ditto for $(\mathcal{P}(\mathbb{Z}), \subseteq)$

We say that $z \in A$ is :

- the $\boxed{\text{greatest / maximum}}$ element of A if $\forall a \in A \quad a \leq z$
- the $\boxed{\text{least/ smallest}}$ / $\boxed{\text{minimum}}$ element of A if $\forall a \quad z \leq a$

THEOREM A poset has at most one greatest element and at most one least element.

Let $(A, \leq)$ poset and $B \subseteq A$. Then $a \in A$ is

- an upper bound for B if $b \leq a$ for all $b \in B$
- a lower bound for B if $\forall b \in B \; a \leq b$

We say that $a \in A$ is

- the least upper bound (LUB) for B if a is an upper bound for B and $a \leq a'$ for any upper bound a' for B
- the greatest lower bound (GLB) for B if a is a lower bound and $a' \leq a$ for any other lower bound

Alternative terminology

LUB: supremum, join

GLB: infimum, meet

THEOREM For poset $(A, \leq)$ any $B \subseteq A$ has at most one LUB and one GLB.

Note In general, LUB & GLB might not exist

Exercise: Construct example!

DEFINITION A lattice is a poset $(A, \leq)$ in which every pair $a, b \in A$ has a LUB and a GLB.

Example $(\mathbb{Z}, |)$ — divisibility

What is LUB for a, b? LCM(a,b)

— " — GLB — " — ? GCD(a,b)

THEOREM If $(A, \leq)$ is a lattice, then any $B \subseteq A$ has a GLB and a LUB

Proof Exercise.

WARNING "Lattice" is a popular word in math and has several different meanings.

$\boxed{\text{DIRECTED GRAPH (DIGRAPH)}}$ $G = (V, E)$ [or $D = (V, A)$]

- Vertices V
- Edges $E \subseteq V \times V$ ordered pairs

Given relation R on A, can represent as digraph D_R — different ways of looking at same mathematical object

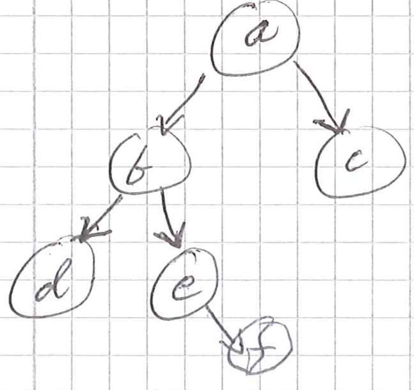
$\boxed{\text{DIRECTED TREE}}$ digraph T with

- special vertex v_0 (ROOT)
- for any $v \in V(T) \setminus \{v_0\}$ there is a unique walk from v_0 to v
- no walk from v_0 to v_0

Can write (T, v_0) for clarity

Ex $A = \{a, b, c, d, e, f\}$

$R = \{(a,b), (a,c), (b,d), (b,e), (e,f)\}$



Can visualize as tree (usually turned upside down so root is at the top)

THEOREM (Properties of directed trees)

Let (T, v_0) be a directed tree. Then

1) T contains no cycles
2) v_0 is the only root
3) Indegree of v_0 is 0 and indegree of all other vertices is 1

Proof By contradiction

1) Suppose $\exists$ cycle $C = u_1 \to u_2 \to \dots \to u_k \to u_1$

By def $\exists$ walk (in fact, has to be path! why?)

$W: v_0 \to \dots \to u_1$

But then W and C concatenated yield second path violates uniqueness.

2) Suppose 2nd root u

Since u root, $\exists$ path $u \rightsquigarrow v_0$

Since v_0 root, — " — $v_0 \rightsquigarrow u$

Yields cycle, contradicting 1)

3) Suppose indegree of $v_0 \geq 1$ and $\exists$ edge(u, v_0)

Concatenate with $v_0 \rightsquigarrow u$ to get cycle

for $v \in V(T) \setminus \{v_0\}$ suppose $\exists$ edges (u_1, v), (u_2, v) $u_1 \neq u_2$

Then $v_0 \rightsquigarrow u_1, v$ and $v_0 \rightsquigarrow u_2, v$ yields two different paths, violating uniqueness

THEOREM If $T_1 = (V, E)$ tree on $n = |V|$ vertices, then $|E| = n - 1$

Proof
$$\# \text{ edges} = \sum_{v \in V} \text{indeg}(v)$$
Use previous theorem

Directed trees as relations

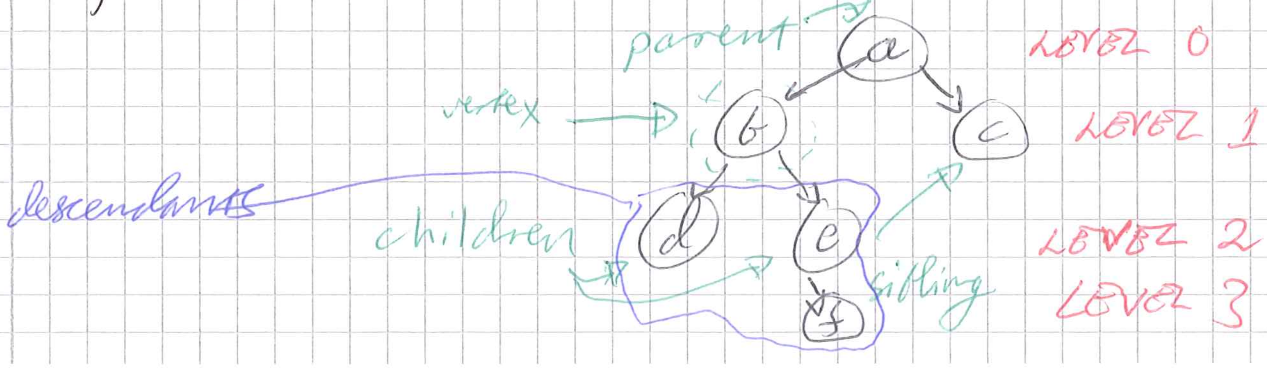
Given directed tree T, corresponding relation R_T is

- irreflexive
- asymmetric
- antitransitive $(a R_T b \ \& \ b R_T c \Rightarrow a \not R_T c)$

$\boxed{\text{TREE TERMINOLOGY}}$

LEAVES: vertices with no outgoing edges
CHILDREN of v: the out-neighbours of v
PARENT of v: the in-neighbour of v
SIBLINGS of v: other children of parent of v
DESCENDANTS of v: all vertices reachable by path from v

LEVEL of vertex: Root v_0 at level 0
For $v \neq v_0$ length of path from v_0
HEIGHT of tree: max level in it



PS VIII

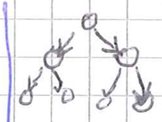
A tree is an $\boxed{n\text{-TREE}}$ (for $n \in \mathbb{Z}^+$)
if every vertex has at most n children

$n = 2$ $\boxed{\text{BINARY TREE}}$ (or $\boxed{n\text{-ARY TREE}}$)

$n = 3$ $\boxed{\text{TERNARY TREE}}$

An n-tree is $\boxed{\text{COMPLETE}}$ if all vertices except for leaves have n children

The $\boxed{\text{COMPLETE } n\text{-TREE OF HEIGHT } h}$ is a complete n-tree of height h that has all leaves at level h



Complete binary tree of height 2

Exercise Suppose T the complete n-tree of height h

- How many vertices are there at level k for $k = 0, 1, \dots, h$?
- Determine $|V|$.

$\boxed{\text{SUBTREE}}$ rooted at b $T(b)$

Take b and all descendants $D(b)$ of b

Remove all vertices not in $\{b\} \cup D(b)$

and all edges not in $(\{b\} \cup D(b)) \times (\{b\} \cup D(b))$

Then $\boxed{T(b) \text{ is a tree with root } b}$

Exercise Prove this!