CVI

Alternative proof

$\#$ multisets $= \binom{n+r-1}{n-1}$

$n+r-1$ cells. Colour $n-1$ cells blue to show when we switch to next element

<u>From multiset to colouring</u>

Start with $s_i = s_1$

<u>Repeat</u> If c_i copies of s_i: skip c_i steps right
then colour next cell blue
update s_i to s_{i+1}

COULD BE ZERO

We colour $n-1$ cells
We skip total of r cells $\Big\}$ $n+r-1$

Different multisets $\Rightarrow$ different colourings
(look at lowest s_i for which there is difference)

Shows that all multisets give rise to colourings/subset choices. Now show that all colourings/subset choices give multisets

<u>From colouring to multiset</u>

Set $s_i := s_1$
~~for~~ position = leftmost cell up to rightmost
~~for~~ if position blue
$s_i := s_{i+1}$
else
add copy of s_i to set
[move position right]

CVII

Ex What is the number of possible different outcomes when rolling 3 (indistinguishable) dice?

$n = 6$
$r = 3$

$\#$ outcomes $= \#$ 3-element multisets

$$= \binom{6+3-1}{3} = \frac{8!}{3!\,5!} = \frac{8 \cdot 7 \cdot 6}{3 \cdot 2 \cdot 1} = 56$$

SUMMING UP what we did so far

n elements; choose r times

	WITH REPETITIONS	WITHOUT REPETITIONS
ORDERED	sequence n^r	permutation ${}_nP_r = \frac{n!}{(n-r)!}$
UNORDERED	multiset ${}_{n+r-1}C_r = \binom{n+r-1}{r}$	set ${}_nC_r = \binom{n}{r}$

BINOMIAL COEFFICIENT

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

"n choose r"

EXAMPLE COMBINATORIAL PROOF

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

How to prove?

Proof 1   Just calculate. Work on right-hand side

$$\binom{n-1}{k-1} + \binom{n-1}{k} = \frac{(n-1)!}{(k-1)!\,(n-k)!} + \frac{(n-1)!}{k!\,(n-k-1)!}$$

$$= \frac{k \cdot (n-1)!}{k \cdot (k-1)!\,(n-k)!} + \frac{(n-k)\,(n-1)!}{(n-k)\,k!\,(n-k-1)!}$$

$$= \frac{(k+n-k)\,(n-1)!}{k!\,(n-k)!} = \frac{n!}{k!\,(n-k)!}$$

$$= \binom{n}{k}$$

Proof 2 (combinatorial)

Right-hand side is talking about different ways of choosing subsets from set of $n-1$ elements...

Left-hand side talks about n elements.

Make case analysis for choosing k elements out of n

(a) nth element is in chosen set

(b) nth element is <u>not</u> in chosen set

In case (a), have to choose $k-1$ elements from remaining $n-1$ elements in $\binom{n-1}{k-1}$ ways

In case (b), have to choose all k elements from remaining $n-1$ elements in $\binom{n-1}{k}$ ways

These are all ways of choosing k elements out of n, so

$$\binom{n}{k} = \binom{n-1}{k-1} + \binom{n-1}{k}$$

PIGEONHOLE PRINCIPLE (skuffeprincippet)

If m pigeons are put into n pigeonholes for $m > n$, then at least one pigeonhole contains at least two pigeons

Ex Are there two people in Copenhagen with the same numbers of hair?

According to research, apparently $\leq 200,000$ hairs/person

Population of Copenhagen $> 600,000$

600,000 ↑ pigeons; 200,000 ↑ pigeonholes

So yes.

Ex For any 11 numbers between 1 and 20 one will be a multiple of another

Every $n \in \mathbb{Z}^+$ can be written $n = 2^k \cdot m$

$m \in \mathbb{Z}^+$ odd (the ODD PART)

Pigeonholes: odd parts 10 odd numbers between 1 and 20

Pigeons: The 11 chosen numbers

By PHP will be two numbers $2^{k_1} \cdot m$ $2^{k_2} \cdot m$ for same odd part. The larger one will be multiple of smaller one

EXTENDED PIGEONHOLE PRINCIPLE

m pigeons, n holes $\Rightarrow$ $\lceil m/n \rceil$ pigeons in one hole

So 4 people in Copenhagen have the same #hair

SOME PROBABILITY THEORY

$\boxed{\text{(DISCRETE) SAMPLE SPACE}}$: set of outcomes

Rolling a dice: $A = \{1, 2, 3, 4, 5, 6\}$

Probability function $p : A \to \mathbb{R}$ ← also denoted $Pr[a]$

$$p(a) \geq 0 \quad \forall a \in A$$

$$\sum_{a \in A} p(a) = 1$$

(p, A) is a $\boxed{\text{PROBABILITY SPACE}}$

$\boxed{\text{EVENT}}$ $E \subseteq A$ subset of sample space

Rolling an even number $E = \{2, 4, 6\}$

$$p(E) = \sum_{a \in E} p(a)$$

When all events equally likely, probabilities reduce to counting

$$p(E) = \sum_{a \in E} p(a) = \sum_{a \in E} \frac{1}{|A|} = \frac{|E|}{|A|}$$

$$\frac{\text{number of outcomes in } E}{\text{total \# outcomes}}$$

Ex Probability of drawing 3 kings from deck of cards when drawing 3 cards

All outcomes equally likely

A = all possible 3-card hands $|A| = {}_{52}C_3 = \binom{52}{3}$

E = only kings $|E| = 4$ (why?)

$$p(E) = \frac{4}{\binom{52}{3}} = \frac{4}{22100} = \frac{1}{5525}$$

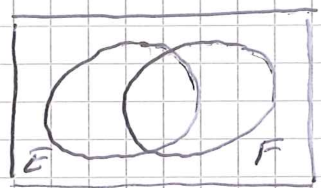
Set operations for events

$\boxed{\text{UNION}}$ $E \cup F = \{a \in A \mid a \in E \text{ or } a \in F\}$

$\boxed{\text{INTERSECTION}}$ $E \cap F = \{a \in A \mid a \in E \text{ and } a \in F\}$

$\boxed{\text{COMPLEMENT}}$ $\overline{E}$ (E^c) $= \{a \in A \mid a \notin E\}$

A



$$p(\overline{E}) = 1 - p(E)$$

$$p(E \cup F) = p(E) + p(F) - p(E \cap F)$$

Ex Probability of rolling $\overbrace{\text{odd number}}^{E}$ or $\overbrace{\text{prime}}^{F}$ (with fair die)

$E = \{1, 3, 5\}$ $\quad p(E) = 3/6 = 1/2$

$F = \{2, 3, 5\}$ $\quad p(F) = 3/6 = 1/2$

$E \cup F = \{1, 2, 3, 5\}$ $\quad p(E \cup F) = 4/6 = 2/3$

$E \cap F = \{3, 5\}$ $\quad p(E \cap F) = 2/6 = 1/3$

$$p(E \cup F) = p(E) + p(F) - p(E \cap F)$$

$$2/3 = 1/2 + 1/2 - 1/3$$

Probabilities of poker hands

Deck of 52 cards ; 4 suits

spades ♠ hearts ♡ clubs ♣ diamonds ♢

13 ranks : 2 – 10, J, Q, K (jack, queen, king)

How many different 5-card hands ?

$$ {}_{52}C_5 = \binom{52}{5} = \frac{52!}{5!\,47!} = 2{,}598{,}960 $$

HANDS

Straight flush : 5 cards of sequential rank, all of same suit

Four of a kind : 4 cards of same rank

Full house : 3 cards of one rank, 2 cards of other rank

Flush : 5 cards of same suit (not in sequence)

Straight : 5 cards of sequential rank (not all same suit)

Three of a kind : 3 cards of one rank 2 cards of other ranks

Two pairs 2 cards of one rank 2 cards of other rank (and one of 3rd rank)

A pair 2 cards of one rank, 3 cards of other ranks

Pr IV

$\boxed{\text{FOUR OF A KIND}}$ How many hands?

Choose rank in 13 ways

Remaining card (kicker) in 48 ways

hands $13 \cdot 48 = 624$

Probability of getting four of a kind when being dealt 5 cards:

$$\frac{624}{\binom{52}{5}} \approx 0{,}00024$$

$\boxed{\text{TWO PAIRS}}$ How many hands

First rank 13 ways

suits $\binom{4}{2}$ ways

Second rank 12 choices remaining

suits again $\binom{4}{2}$ ways

Last card of some other rank

$11 \cdot 4$ such cards

$$13 \cdot \binom{4}{2} \cdot 12 \cdot \binom{4}{2} \cdot 11 \cdot 4 = 123{,}552$$

Probability of getting two pairs

$$\frac{123{,}552}{\binom{52}{5}} \approx 0{,}0475$$