KØBENHAVNS UNIVERSITET

Diskret Matematik og Formelle Sprog: Problem Set 3

Due: Monday March 6 at 12:59 CET .
Submission: Please submit your solutions via *Absalon* as a PDF file. State your name and e-mail address close to the top of the first page. Solutions should be written in LaTeX or some other math-aware typesetting system with reasonable margins on all sides (at least 2.5 cm). Please try to be precise and to the point in your solutions and refrain from vague statements. Make sure to explain your reasoning. *Write so that a fellow student of yours can read, understand, and verify your solutions.* In addition to what is stated below, the general rules for problem sets stated on *Absalon* always apply.
Collaboration: Discussions of ideas in groups of two to three people are allowed—and indeed, encouraged   but you should always write up your solutions completely on your own, from start to finish, and you should understand all aspects of them fully. It is not allowed to compose draft solutions together and then continue editing individually, or to share any text, formulas, or pseudocode. Also, no such material may be downloaded from or generated via the internet to be used in draft or final solutions. Submitted solutions will be checked for plagiarism.
Grading: A score of 120 points is guaranteed to be enough to pass this problem set.
Questions: Please do not hesitate to ask the instructor or TAs if any problem statement is unclear, but please make sure to send private messages—sometimes specific enough questions could give away the solution to your fellow students, and we want all of you to benefit from working on, and learning from, the problems. Good luck!

1 (50 p) Consider the relation S described by the directed graph D_S in Figure 1.

1a (10 p) Write down the matrix representation M_S of the relation S and describe briefly but clearly how you construct this matrix.

Solution: The matrix representation of the relation S is a 10×10 matrix where there is a 1 in position (i, j) if $(i, j) \in S$ and a 0 in this position otherwise. Looking at the directed graph representation D_S in Figure 1, this means that there should be a 1 in position (i, j) if and only if there is a directed edge from i to j in D_S. This yields the matrix

$$M_S = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(where we note, in particular, that from row 6 onwards there are only zeros in the matrix, since there are no outgoing edges from the vertices labelled $6, 7, \ldots, 10$ in D_S).

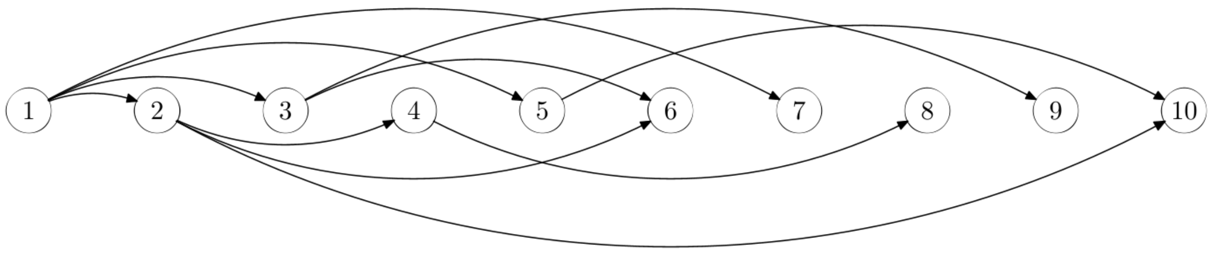

Figure 1: Directed graph D_S representing relation S in Problem 1.

1b (10 p) Let us write T to denote the transitive closure of the relation S. What is the matrix representation of T? Write it down and explain how you constructed it.

Solution: To obtain the transitive closure T of the relation S, we can proceed as follows:

1. Start by setting $T' = T = S$.
2. Go over all triples (i, j, k) such that $(i, j) \in T$ and $(j, k) \in T$, and add (i, k) to T' since the elements i and k should also be related by transitivity.
3. If new pairs were added in step 2, so that $T' \neq T$, then set $T = T'$ and go to step 2 again. Otherwise T is the transitive closure.

Referring to the directed graph representation D_S in Figure 1, a pair (i, k) should be in the transitive closure precisely when there is a path from i to k in D_S. Starting from each vertex $i = 1, 2, \ldots, 10$ and writing down which other vertices are reachable from i yields the matrix

$$M_T = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for the transitive closure T of S.

1c (10 p) Now let R be the reflexive closure of the relation T. What is the matrix representation of R? Write it down and explain how you constructed it.

Solution: To get the reflexive closure R of the relation T, we just need to add all pairs (i, i) so that the resulting relation is reflexive. For the matrix representation this corresponds to adding 1s on the diagonal, which yields the the matrix

$$M_R = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

for the reflexive closure R of T.

1d (20 p) Can you explain in words what the relation R is by describing how it can be interpreted? (In particular, is it similar to anything we have discussed during the course?)

Solution: The relation R is the divisibility relation restricted to the integers $\{1, 2, \ldots, 10\}$. In other words, we have that $(i, j) \in R$ if and only if $i \mid j$. This can be verified by going over all rows i in the matrix M_R and checking in each row that there is a 1 in column j if and only if $i \mid j$. The first row is all-1s, corresponding to that 1 divides all integers. In the second row, every other position is 1, corresponding to that every other number is even. In the third row, every third position is 1, since every third number is divisible by 3, et cetera.

2 (100 p) After the less than stellar performance by Jakob at the DIKU 50th anniversary outreach event *(see Problem Set 2 from DMFS 2022 for more information about this)*, only 12 brave children registered for a follow-up event that was arranged earlier this semester with the purpose of conveying the excitement of computer science to the younger generation. In view of this, the head of the Algorithms & Complexity Section Mikkel Thorup decided to take charge of the organization. In this problem we will study how Mikkel's follow-up event was arranged.

2a (40 p) Since Mikkel is an avid mushroom picker, he took the 12 children to the forest to collect mushrooms. Being a good host, he of course made sure that every child found at least one mushroom. When everybody returned to DIKU, it turned out that the children had collected exactly 77 mushrooms together. Mikkel explained to the children that this meant there had to be at least two kids who had collected the same number of mushrooms. Can you describe in detail how he could have proven such a claim?

Solution: Let us argue by contradiction. Suppose child number i picked m_i mushrooms, where we sort the children in increasing order with respect to the number of mushrooms picked, i.e., so that $m_1 \leq m_2 \leq \ldots \leq m_{12}$. Since all children found mushrooms we have $m_1 \geq 1$, and if no pair of children picked the same number of mushrooms, this means that $m_1 < m_2 < \ldots < m_{12}$. In particular, this implies for all i that $m_i \geq i$. Summing up, we get that the total number of mushrooms has to be

$$\sum_{i=1}^{12} m_i \;\geq\; \sum_{i=1}^{12} i \;=\; \frac{12 \cdot 13}{2} \;=\; 78 \;, \tag{1}$$

where for the next to last equality we can use the arithmetic sum formula $\sum_{i=1}^{n} i = n(n+1)/2$ that we have learned in class.

However, according to the problem statement the children only collected 77 mushrooms together. This contradicts the lower bound on the number of mushrooms that we just obtained. Hence, at least two of the children must have collected the same number of mushrooms.

2b (60 p) When all mushrooms had been cleaned, Mikkel taught the children what it means for two positive integers to be relatively prime. He then wrote the numbers $1, 2, 3, \ldots, 22$ on 22 sheets of paper and performed the following experiment:

- First, all the 22 sheets of papers were randomly shuffled.
- Then each of the 12 children randomly picked one sheet of paper.
- Finally, the children tried to identify a pair amongst themselves who held sheets of paper with relatively prime numbers.

This experiment was repeated several times, and every single time some pair of children found that they had drawn relatively prime numbers. Together with the children, Mikkel discussed whether this was just a weird coincidence or whether this always has to happen. What was the conclusion of this discussion? Please make sure to provide formal proofs backing up any claims you make.

Solution: This is not a coincidence, but always has to happen. This can be argued by using the pigeonhole principle. We consider our 12 "pigeons" to be the 12 numbers between 1 and 22 that are chosen. We create 11 "pigeonholes" by grouping consecutive numbers together into pairs $\{1, 2\}$, $\{3, 4\}$, $\ldots$, $\{19, 20\}$, $\{21, 22\}$. We now make two observations:

1. Two integers a and $a + 1$ are always relatively prime. Perhaps the easiest way to see this is that any divisor of two numbers a and b must also divide $b - a$, which, in particular, means the greatest common divisor of a and $b = a + 1$ must also divide $b - a = 1$. In other words, the greatest common divisor of a and $a + 1$ is 1, which is the definition of the two numbers being relatively prime.

2. Since we have 12 numbers but only 11 pairs, for at least some pair $\{a, a + 1\}$ both of the numbers in the pair are chosen.

This shows that at least one pair of children have to choose relatively prime numbers.

3 (50 p) In this problem we consider formulas in propositional logic. Decide for each of the formulas below whether it is tautological or not and then do the following:

- If the formula is a tautology, prove this by either (i) presenting a full truth table for all subformulas analogously to how we did it in class, or (ii) providing an explanation based on the rules and equivalences we have learned. You only need to to one of (i) or (ii), but you are free to do both if you like, and crisp and clear explanations can compensate for minor slips in the truth table.

- If the formula is *not* a tautology, present a falsifying assignment. Also, explain how you can change a single connective in the formula to turn it into a tautology, and try to provide a natural language description of what the tautology you obtain in this way encodes (i.e., not just mechanically replacing each connective by a word, but explaining what the underlying logical principle is).

3a $(p \to (q \land r)) \to ((q \lor \neg p) \land (r \lor \neg p))$

Solution: This formula is a tautology. To show this, let us work on the second half of the formula, i.e., the conclusion of the outermost implication. Using commutativity and distributivity, we know that

$$(q \lor \neg p) \land (r \lor \neg p) \;\equiv\; (\neg p \lor q) \land (\neg p \lor r) \;\equiv\; \neg p \lor (q \land r) \ . \tag{2a}$$

For the implication connective we know that $x \to y \equiv \neg x \lor y$, and applying this to (2a) we get

$$\neg p \lor (q \land r) \;\equiv\; p \to (q \land r) \ . \tag{2b}$$

But this last formula in (2b) is the premise of the outermost implication, and any formula that is logically equivalent to a formula on the form $x \to x$ is certainly a tautology.

We can also construct a truth table for $(p \to (q \land r)) \to ((q \lor \neg p) \land (r \lor \neg p))$ as follows:

p	q	r	$(p \to$	$(q \land r))$	$\to$	$((q \lor \neg p)$	$\land$	$(r \lor \neg p))$
$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\top$	$\top$	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\bot$
$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\top$
$\top$	$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\top$	$\bot$	$\bot$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

We see that in the middle column corresponding to the outermost implication we only have $\top$, so the formula always evaluates to true and hence is a tautology.

3b $((p \land q) \to r) \to ((r \lor \neg p) \land (r \lor \neg q))$

Solution: This formula is *not* a tautology—if we set $p = \top$ and $q = r = \bot$, then the premise $((p \land q) \to r)$ of the outermost implication is true (since $p \land q$ is false), but the conclusion $(r \lor \neg p) \land (r \lor \neg q)$ of the outermost implication is false (since $r \lor \neg p$ is false).

To analyse the structure of this formula, let us work on the conclusion of the outermost implication. Using the distributivity, commutativity, and De Morgan rules we can rewrite the conclusion as

$$(r \lor \neg p) \land (r \lor \neg q) \;\equiv\; r \lor (\neg p \land \neg q) \;\equiv\; (\neg p \land \neg q) \lor r \;\equiv\; \neg(p \lor q) \lor r \ , \tag{3a}$$

and by applying the equivalence $x \to y \equiv \neg x \lor y$ for the implication connective we can rewrite (3a) further as

$$\neg(p \lor q) \lor r \;\equiv\; (p \lor q) \to r \ . \tag{3b}$$

From this we see that if we we change the AND in the premise of the formula $((p \land q) \to r) \to ((r \lor \neg p) \land (r \lor \neg q))$ in Problem 3b to an OR, we get the formula

$$((p \lor q) \to r) \to ((r \lor \neg p) \land (r \lor \neg q)) \tag{4}$$

which is clearly a tautology, since in view of the rewriting in (3a) and (3b) this is equivalent to a formula on the form $x \to x$. We can also argue directly that the formula (4) is a tautology by observing that what the formula says is that if the implication $(p \lor q) \to r$ holds and r is false, it has to hold that both p and q are false. This is certainly a true statement, since an implication is not true if the premise is true and the conclusion is false.

Another possible solution is to change the AND in the conclusion of the formula in Problem 3b to an OR to get

$$((p \wedge q) \rightarrow r) \rightarrow ((r \vee \neg p) \vee (r \vee \neg q)) \ , \tag{5}$$

and this formula can also be argued to be a tautology in a similar way.

(Note that $\rightarrow$ denotes logical implication and $\neg$ denotes logical negation. Negation is assumed to bind harder than the binary connectives, but other than that all formulas are fully parenthesized for clarity.)

4 (60 p) Suppose that we have an 8×8 square grid with Othello (or Reversi) markers on all cells. Markers have one black side and one white side. The marker in the upper right corner in the grid has the black side up, and all other cells have the white side of their markers up.

Consider now a game where in one single move we can choose one row or column, and then flip all the markers in that row or column, so that all markers with the black side up instead have the white side up, and vice versa.

Is it possible to find a sequence of moves that leads to a configuration where all markers on the grid have the black side up? If your answer is yes, explain what such a sequence of moves looks like. If your answer is no, then give a proof that no sequence of moves can lead to an all-black grid.

Solution: No, it is not possible to find a sequence of moves leading to only black markers on the grid.

To see this, consider what happens when we flip all markers in a row or column. Suppose that before the flip there are s black markers. Then after the flip $8-s$ white markers have turned black, and s black markers have turned white. The total increase (or decrease) of black markers is thus $(8-s)-s = 8-2s$, which is an even number. That is, the number of black markers on the grid always changes by an even number after each move. Since we are starting with an odd number of black markers, we get the invariant that the total number of black markers on the grid will always be odd. But if so, it can never be the case that all 64 positions on the grid are black.