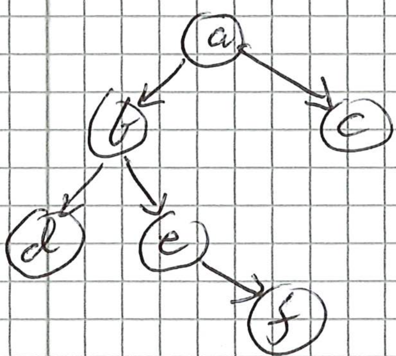
DIRECTED TREE: directed graph $T = (V, E)$ with

- special vertex v_0 (ROOT)
- for any $v \in V \setminus \{v_0\}$ there is a unique walk from v_0 to v
- there is no walk from v_0 to v_0

Can write (T, v_0) for clarity to specify clearly which vertex is the root

Ex $V = \{a, b, c, d, e, f\}$
$E = \{(a,b), (a,c), (b,d), (b,e), (e,f)\}$



Can visualize as tree (turned upside down, so that root is at the top

THEOREM (Properties of directed trees)
Let (T, v_0) be a directed tree. Then

1) T contains no cycles
2) v_0 is the only root
3) Indegree of v_0 is 0
Indegree of all other vertices is $\underline{1}$

Proof By contradiction

1) Suppose $\exists$ cycle $C = u_1 \to u_2 \to \dots \to u_k \to u_1$

By definition of directed tree, $\exists$ walk

$W: v_0 \to \dots \to u_1$

(in fact, has to be path! Why?)

But then W & C concatenated yields second path from v_0 to u_1. $\lightning$

2) Suppose that there exists second root u

Since u root, $\exists$ path $P_1: u \rightsquigarrow v_0$

Since v_0 root, $\exists$ path $P_2: v_0 \rightsquigarrow u$

But then concatenation of P_1 & P_2 yields cycle, contradicting 1) $\lightning$

3) Suppose indegree of v_0 is ≥ 1 and that $\exists$ edge (u, v_0)

Add edge (u, v_0) to walk $v_0 \rightsquigarrow u$ to get cycle $\lightning$

For $v \in V \setminus \{v_0\}$ suppose $\exists$ edges (u_1, v) and (u_2, v) for $u_1 \neq u_2$

By definition $\exists$ walks $P_1: v_0 \rightsquigarrow u_1$

$P_2: v_0 \rightsquigarrow u_2$

Concatenate with edges to get two different paths to v $\lightning$

THEOREM

If $T = (V, E)$ is a directed tree on $n = |V|$ vertices, then

$|E| = n - 1$

Proof By definition,

$$\# \text{ edges} = \sum_{v \in V} \text{indeg}(v)$$

Use previous theorem:

$$\text{indeg}(v_0) = 0$$

$$\text{indeg}(v) = 1 \quad \text{for } v \in V \setminus \{v_0\}$$

TREE TERMINOLOGY

LEAVES vertices with no outgoing edges

CHILDREN of v: the out-neighbours of v

PARENT of v: the in-neighbour of v

SIBLINGS of v: other children of ~~parents~~ parent of v

DESCENDANTS of v: all vertices reachable by path from v

LEVEL of v: length of path from root to v