Diskret Matematik og Formelle Sprog: Problem Set 3

Due: Monday March 6 at 12:59 CET .
Submission: Please submit your solutions via *Absalon* as a PDF file. State your name and e-mail address close to the top of the first page. Solutions should be written in LaTeX or some other math-aware typesetting system with reasonable margins on all sides (at least 2.5 cm). Please try to be precise and to the point in your solutions and refrain from vague statements. Make sure to explain your reasoning. *Write so that a fellow student of yours can read, understand, and verify your solutions.* In addition to what is stated below, the general rules for problem sets stated on *Absalon* always apply.
Collaboration: Discussions of ideas in groups of two to three people are allowed—and indeed, encouraged—but you should always write up your solutions completely on your own, from start to finish, and you should understand all aspects of them fully. It is not allowed to compose draft solutions together and then continue editing individually, or to share any text, formulas, or pseudocode. Also, no such material may be downloaded from or generated via the internet to be used in draft or final solutions. Submitted solutions will be checked for plagiarism.
Grading: A score of 120 points is guaranteed to be enough to pass this problem set.
Questions: Please do not hesitate to ask the instructor or TAs if any problem statement is unclear, but please make sure to send private messages   sometimes specific enough questions could give away the solution to your fellow students, and we want all of you to benefit from working on, and learning from, the problems. Good luck!

1 (50 p) Consider the relation S described by the directed graph D_S in Figure 1.

1a (10 p) Write down the matrix representation M_S of the relation S and describe briefly but clearly how you construct this matrix.

1b (10 p) Let us write T to denote the transitive closure of the relation S. What is the matrix representation of T? Write it down and explain how you constructed it.

1c (10 p) Now let R be the reflexive closure of the relation T. What is the matrix representation of R? Write it down and explain how you constructed it.

1d (20 p) Can you explain in words what the relation R is by describing how it can be interpreted? (In particular, is it similar to anything we have discussed during the course?)

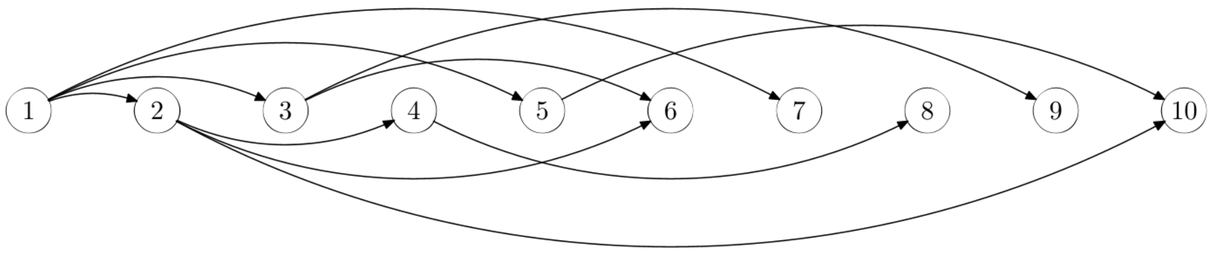

Figure 1: Directed graph D_S representing relation S in Problem 1.

2 (100 p) After the less than stellar performance by Jakob at the DIKU 50th anniversary outreach event *(see Problem Set 2 from DMFS 2022 for more information about this)*, only 12 brave children registered for a follow-up event that was arranged earlier this semester with the purpose of conveying the excitement of computer science to the younger generation. In view of this, the head of the Algorithms & Complexity Section Mikkel Thorup decided to take charge of the organization. In this problem we will study how Mikkel's follow-up event was arranged.

2a (40 p) Since Mikkel is an avid mushroom picker, he took the 12 children to the forest to collect mushrooms. Being a good host, he of course made sure that every child found at least one mushroom. When everybody returned to DIKU, it turned out that the children had collected exactly 77 mushrooms together. Mikkel explained to the children that this meant there had to be at least two kids who had collected the same number of mushrooms. Can you describe in detail how he could have proven such a claim?

2b (60 p) When all mushrooms had been cleaned, Mikkel taught the children what it means for two positive integers to be relatively prime. He then wrote the numbers $1, 2, 3, \ldots, 22$ on 22 sheets of paper and performed the following experiment:

- First, all the 22 sheets of papers were randomly shuffled.
- Then each of the 12 children randomly picked one sheet of paper.
- Finally, the children tried to identify a pair amongst themselves who held sheets of paper with relatively prime numbers.

This experiment was repeated several times, and every single time some pair of children found that they had drawn relatively prime numbers. Together with the children, Mikkel discussed whether this was just a weird coincidence or whether this always has to happen. What was the conclusion of this discussion? Please make sure to provide formal proofs backing up any claims you make.

3 (50 p) In this problem we consider formulas in propositional logic. Decide for each of the formulas below whether it is tautological or not and then do the following:

- If the formula is a tautology, prove this by either (i) presenting a full truth table for all subformulas analogously to how we did it in class, or (ii) providing an explanation based on the rules and equivalences we have learned. You only need to to one of (i) or (ii), but you are free to do both if you like, and crisp and clear explanations can compensate for minor slips in the truth table.
- If the formula is *not* a tautology, present a falsifying assignment. Also, explain how you can change a single connective in the formula to turn it into a tautology, and try to provide a natural language description of what the tautology you obtain in this way encodes (i.e., not just mechanically replacing each connective by a word, but explaining what the underlying logical principle is).

3a $(p \rightarrow (q \wedge r)) \rightarrow ((q \vee \neg p) \wedge (r \vee \neg p))$

3b $((p \wedge q) \rightarrow r) \rightarrow ((r \vee \neg p) \wedge (r \vee \neg q))$

(Note that $\rightarrow$ denotes logical implication and $\neg$ denotes logical negation. Negation is assumed to bind harder than the binary connectives, but other than that all formulas are fully parenthesized for clarity.)

4 (60 p) Suppose that we have an 8×8 square grid with Othello (or Reversi) markers on all cells. Markers have one black side and one white side. The marker in the upper right corner in the grid has the black side up, and all other cells have the white side of their markers up.

Consider now a game where in one single move we can choose one row or column, and then flip all the markers in that row or column, so that all markers with the black side up instead have the white side up, and vice versa.

Is it possible to find a sequence of moves that leads to a configuration where all markers on the grid have the black side up? If your answer is yes, explain what such a sequence of moves looks like. If your answer is no, then give a proof that no sequence of moves can lead to an all-black grid.