KØBENHAVNS UNIVERSITET

Diskret Matematik og Formelle Sprog: Problem Set 3

Due: Monday March 8 at 23:59 CET.
Submission: Please submit your solutions via *Absalon* as PDF file. State your name and e-mail address close to the top of the first page. Solutions should be written in LaTeX or some other math-aware typesetting system with reasonable margins on all sides (at least 2.5 cm). Please try to be precise and to the point in your solutions and refrain from vague statements. *Write so that a fellow student of yours can read, understand, and verify your solutions.* In addition to what is stated below, the general rules in the course information always apply.
Collaboration: Discussions of ideas in groups of two to three people are allowed—and indeed, encouraged—but you should always write up your solutions completely on your own, from start to finish, and you should understand all aspects of them fully. It is not allowed to compose draft solutions together and then continue editing individually, or to share any text, formulas, or pseudo-code. Also, no such material may be downloaded from the internet and/or used verbatim. Submitted solutions will be checked for plagiarism.
Grading: A score of 140 points is guaranteed to be enough to pass this problem set.
Questions: Please do not hesitate to ask the instructor or TAs if any problem statement is unclear, but please make sure to send private messages sometimes specific enough questions could give away the solution to your fellow students, and we want all of you to benefit from working on and learning on the problems. Good luck!

1 (40 p) For positive integers $a_1, a_2, \ldots, a_k$, define $\gcd(a_1, a_2, \ldots, a_k)$ to be the largest positive integer d such that d divides every a_i and any positive integer c that divides every a_i also has to divide d. Is it true that there are integers m_i, not necessarily positive, such that $d = \sum_{i=1}^{k} m_i a_i$? Prove this or give a simple counter-example.

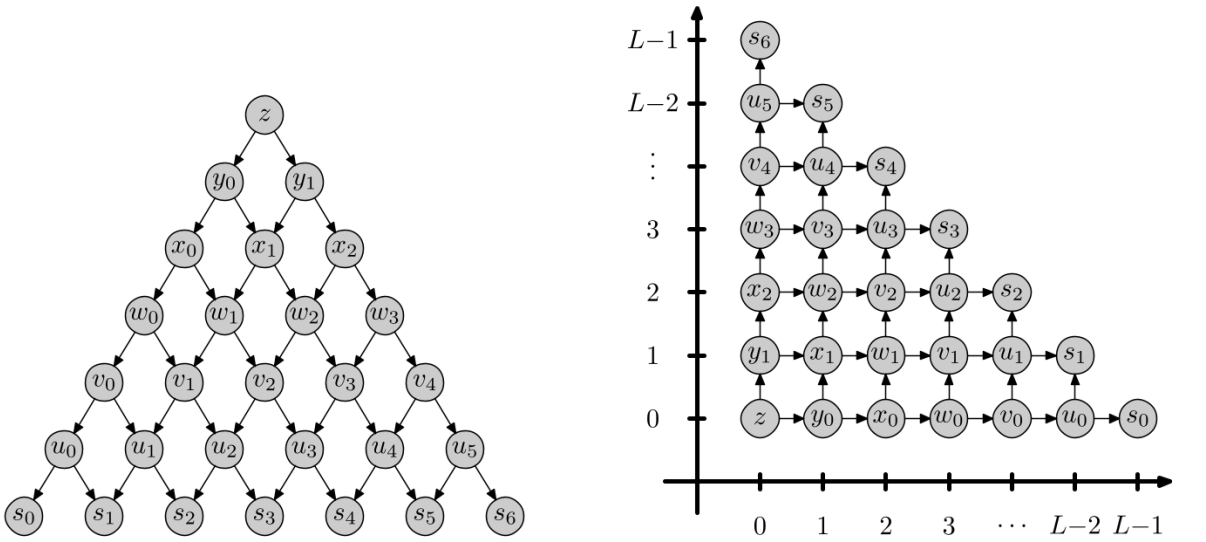
2 (80 p) Decide for each of the propositional logic formulas below whether it is a tautology or a contradiction. If neither of these cases apply, then present a satisfying assignment for the formula. It is sufficient (and necessary) for a full score to justify all your answers by presenting truth tables for all the subformulas analogously to how we did it in class, but you are also encourage to *explain* why your answers are correct (and good explanations could compensate fully for minor slips in the truth tables).

(Note that logical not $\neg$ is assumed to bind harder than the binary connectives, but other than that all formulas are fully parenthesized for clarity. We write $\rightarrow$ for logical implication and $\leftrightarrow$ for equivalence.)

2a (20 p) $((p \rightarrow q) \rightarrow r) \rightarrow ((p \wedge q) \rightarrow r)$

2b (20 p) $((p \rightarrow q) \wedge (r \rightarrow s)) \leftrightarrow ((p \wedge r) \rightarrow (q \wedge s))$

2c (20 p) $((p \wedge \neg r) \vee (q \wedge \neg r)) \rightarrow ((p \vee q) \rightarrow r)$



(a) Lattice graph with 7 layers.

(b) Graph as connected points in 2D integer lattice.

Figure 1: Example graph for Problem 3.

2d (20 p) $(p \to (q \vee r)) \vee (\neg(\neg p \vee q) \wedge \neg r)$

3 (100 p) Consider a directed graph that consists of L layers numbered from 0 up to $L-1$, with $i+1$ vertices in every layer i numbered from 0 to i, and with outgoing edges from vertex j in layer i to vertices j and $j+1$ in layer $i+1$. See Figure 1a for an illustration of this graph with 7 layers. We can agree to call this graph a *lattice graph* (because it can be obtained from a fragment of the integer lattice in 2 dimensions as shown in Figure 1b, but this is actually completely irrelevant to this problem).

Suppose we start in the unique source vertex on level 0 and walk along edges in the graph, flipping a fair coin at every vertex to decide whether to go left or right, until we reach one of the sinks in the last layer. For instance, in Figure 1a the walk "left–left–right–left–right–right" would visit vertices z, y_0, x_0, w_1, v_1, u_2, and end in s_3.

3a (40 p) For every vertex s_i in the lattice graph in Figure 1a, calculate the probability that such a walk ends in vertex s_i. Which vertex is the walk most likely to end up in?

3b (60 p) For a lattice graph with L layers, where $L \geq 2$ is a positive integer, calculate the probability that a walk as described above ends in vertex j in layer $L-1$ for $j = 0, 1, \ldots, L-1$.

Hint: Use the fact that all walks are equally likely to turn this into a counting problem.

4 (60 p) In this problem we focus on relations. In particular, suppose that $A = \{e_1, e_2, \ldots, e_9, e_{10}\}$ is a set of 10 elements and consider the relation R on A represented by the matrix

$$M_R = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(where element e_i corresponds to row and column i).

4a (20 p) Let us write T to denote the transitive closure of the relation R. What is the matrix representation of T in this case? Can you describe in words what the relation T is?

4b (20 p) Suppose that we create a new relation S_1 from R by first taking the reflexive closure and then the symmetric closure. What does the matrix representation of S_1 look like? Can you describe in words what the relation S_1 is?

4c (20 p) Suppose instead that we first take the symmetric closure and then the reflexive closure to derive another relation S_2 from R. Are S_1 and S_2 different relations, or are they the same relation in the end? If the latter case holds, is it true for any relation R on a set A that relations S_1 and S_2 derived in this way will be the same, or can they sometimes be different? Give a proof or present a simple counter-example.