IDMA 2025: WEEK 6

POSETS AND GRAPHS

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Algorithms & Complexity Section

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Last time

- Relation $R \subseteq A \times B$
- Relation on A : $R \subseteq A \times A$

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- Equivalence

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- Types of relations : Symmetric, Reflexive, Transitive
- Equivalence
- Partially Ordered Sets (Posets)

Partial Order

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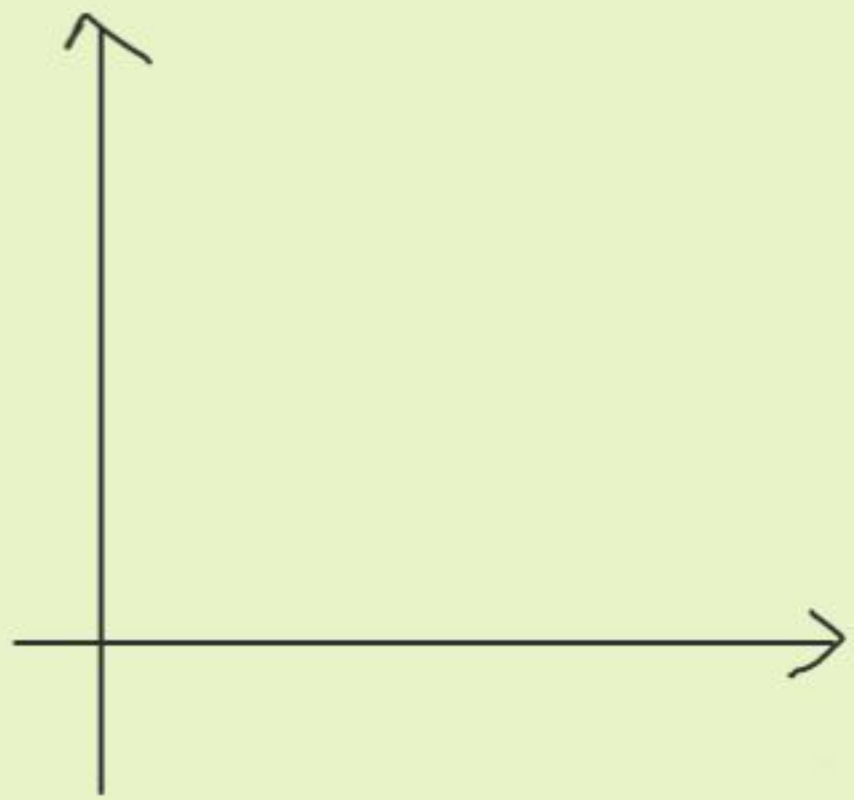

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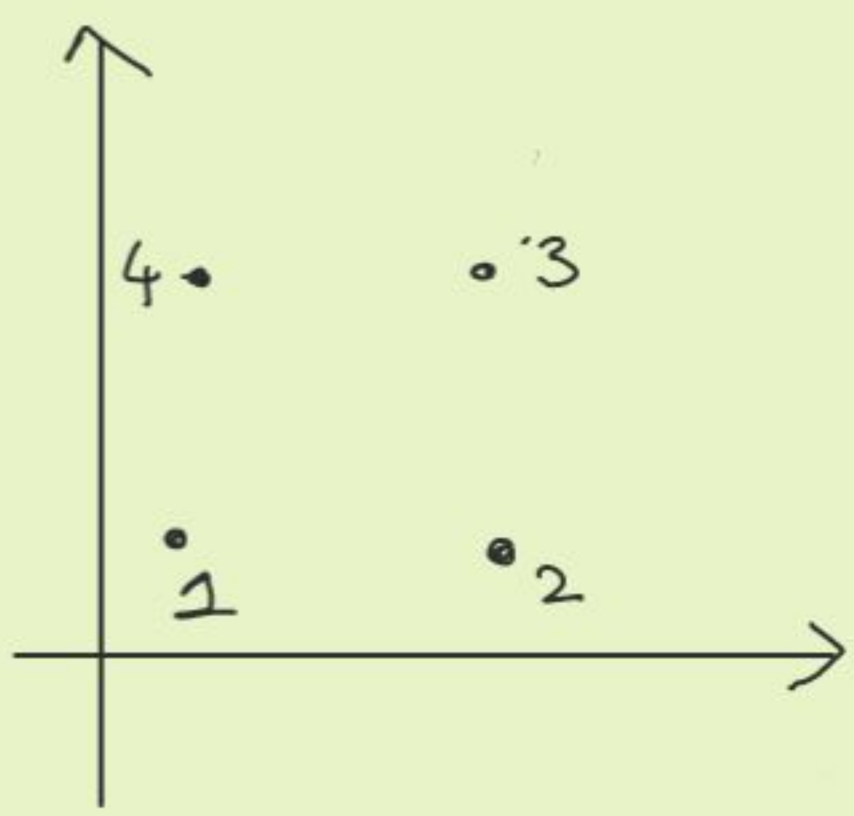
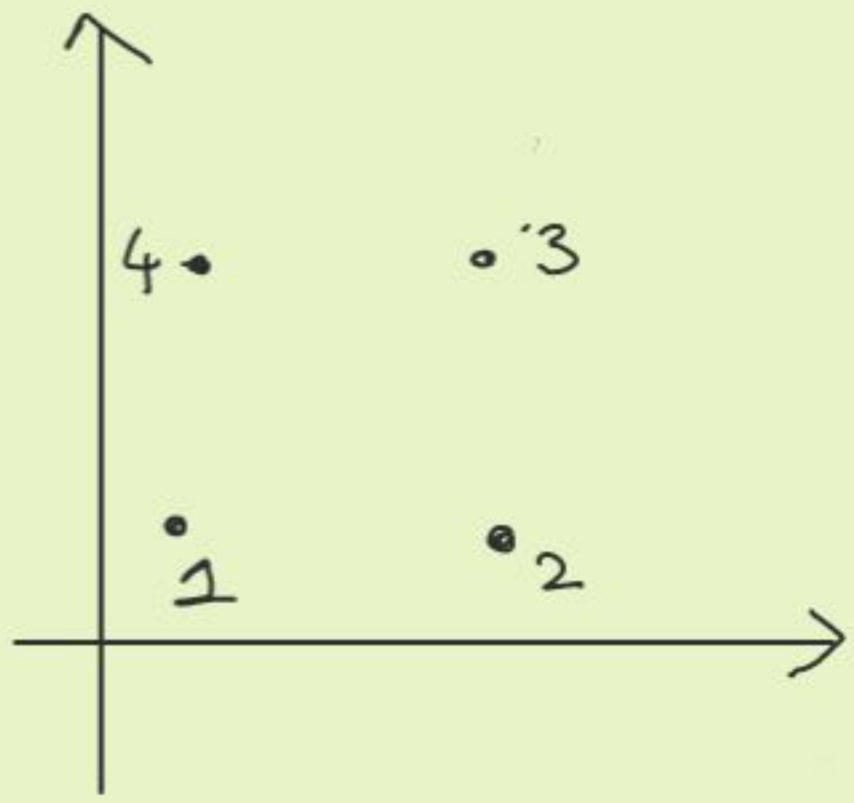


Partial Order

- (Total or Linear) Order: Relation R that allows us to compare all elements
- Partial Order: can compare some elements

- Reflexive: aRa for all $a \in A$
- Antisymmetric: aRb & $bRa \Rightarrow a = b$
- Transitive: aRb & $bRc \Rightarrow aRc$



Examples

$(\mathbb{N}, \leq)$ $(\mathbb{N}, |)$ $(\mathcal{P}(\{1,2,3\}), \subseteq)$ $(\{1,2,3\}, =)$

Reflexive?

aRa

Antisymmetric?

aRb & bRa

$\Rightarrow a=b$

Transitive?

aRb & bRc

$\Rightarrow aRc$

Total?

aRb OR bRa

Examples

$(\mathbb{N}, \leq)$ $(\mathbb{N}, |)$ $(\mathcal{P}(\{1,2,3\}), \subseteq)$ $(\{1,2,3\}, =)$

Reflexive? ✓

aRa

Antisymmetric? ✓

$aRb \;\&\; bRa$

$\Rightarrow a=b$

Transitive? ✓

$aRb \;\&\; bRc$

$\Rightarrow aRc$

Total? ✓

aRb OR bRa

Examples

	$(\mathbb{N}, \leq)$	$(\mathbb{N}, \mid)$	$(\mathcal{P}(\{1,2,3\}), \subseteq)$	$(\{1,2,3\}, =)$
Reflexive? aRa	✓	✓		
Antisymmetric? aRb & $bRa \Rightarrow a=b$	✓	✓		
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Total? aRb OR bRa	✓	✗		

Examples

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Total? aRb OR bRa	✓	✗	✗	✗

Product Posets

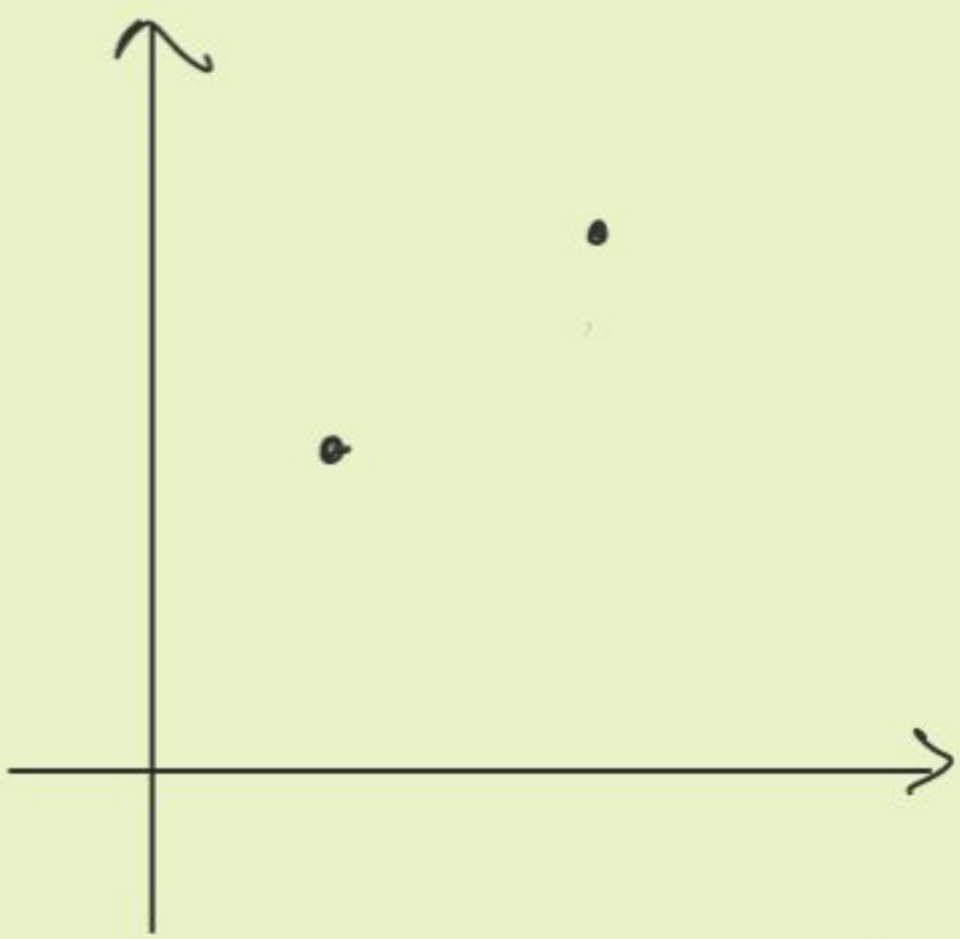
(A, R_1), (B, R_2) posets

$(A \times B, R_3)$

$(a_1, b_1)\, R_3\, (a_2, b_2)$ if

$a_1 R_1 a_2$ & $b_1 R_1 b_2$

This is a poset.



Representing finite Posets

$(\mathcal{P}(\{1,2,3\}), \subseteq)$

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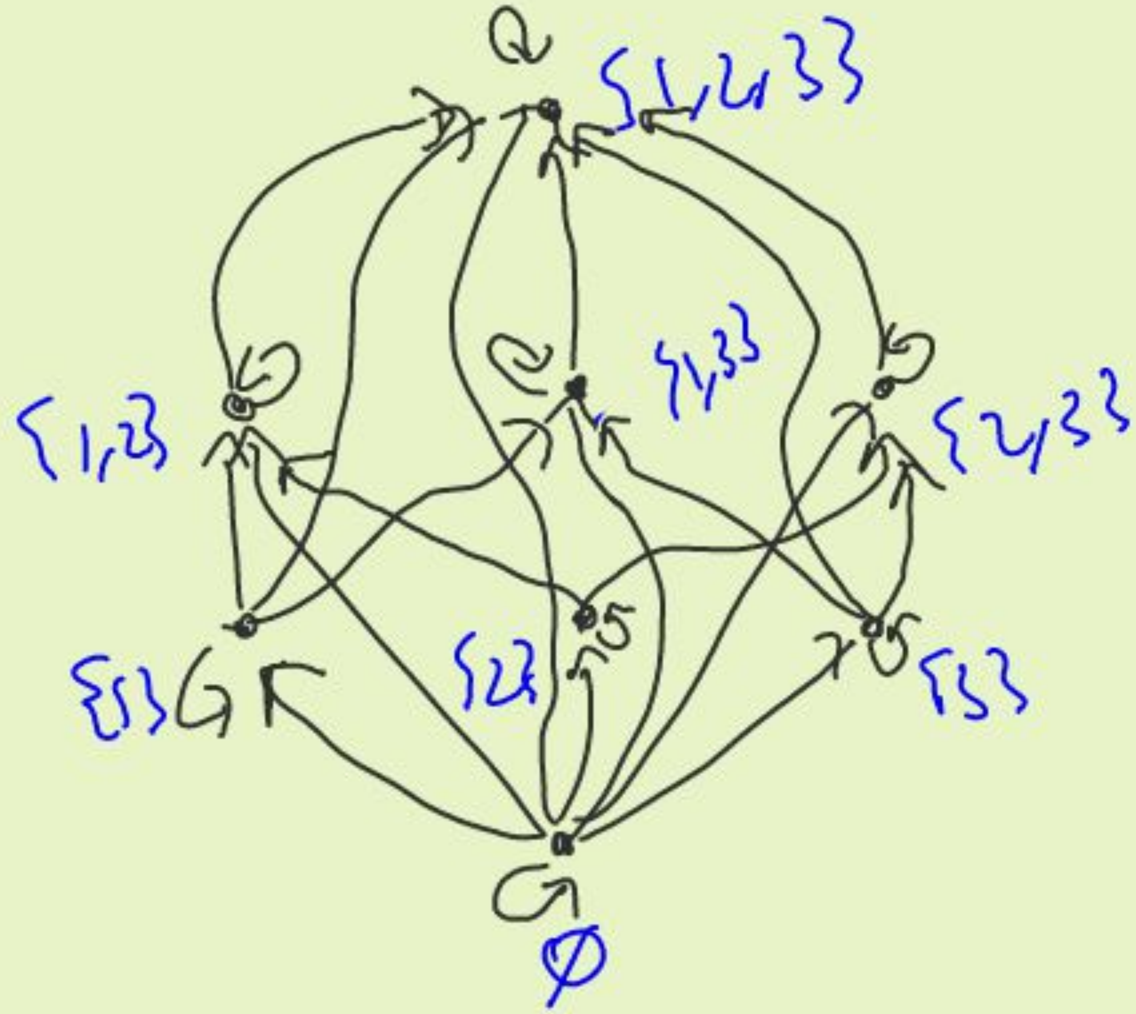

Digraph

Representing finite Posets

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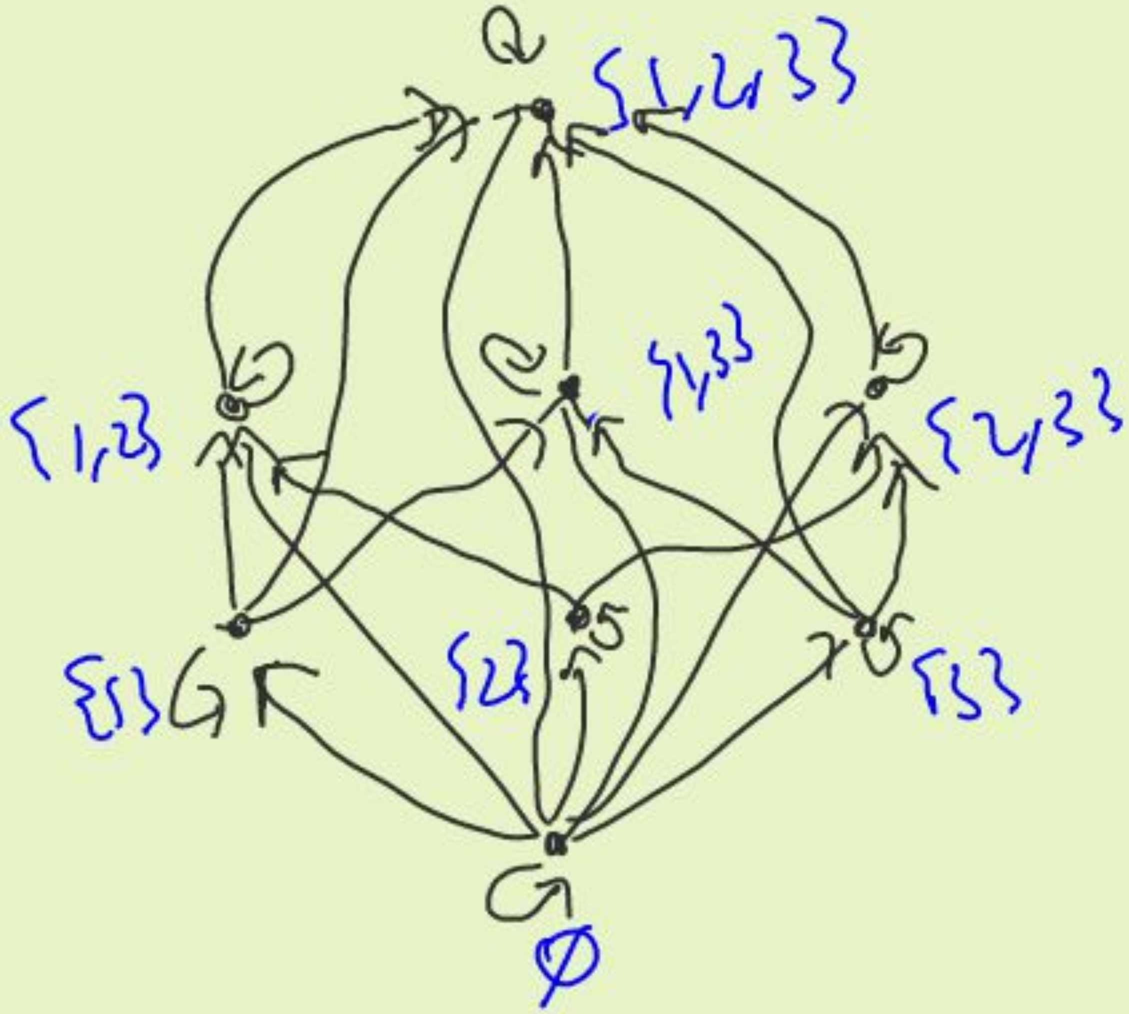

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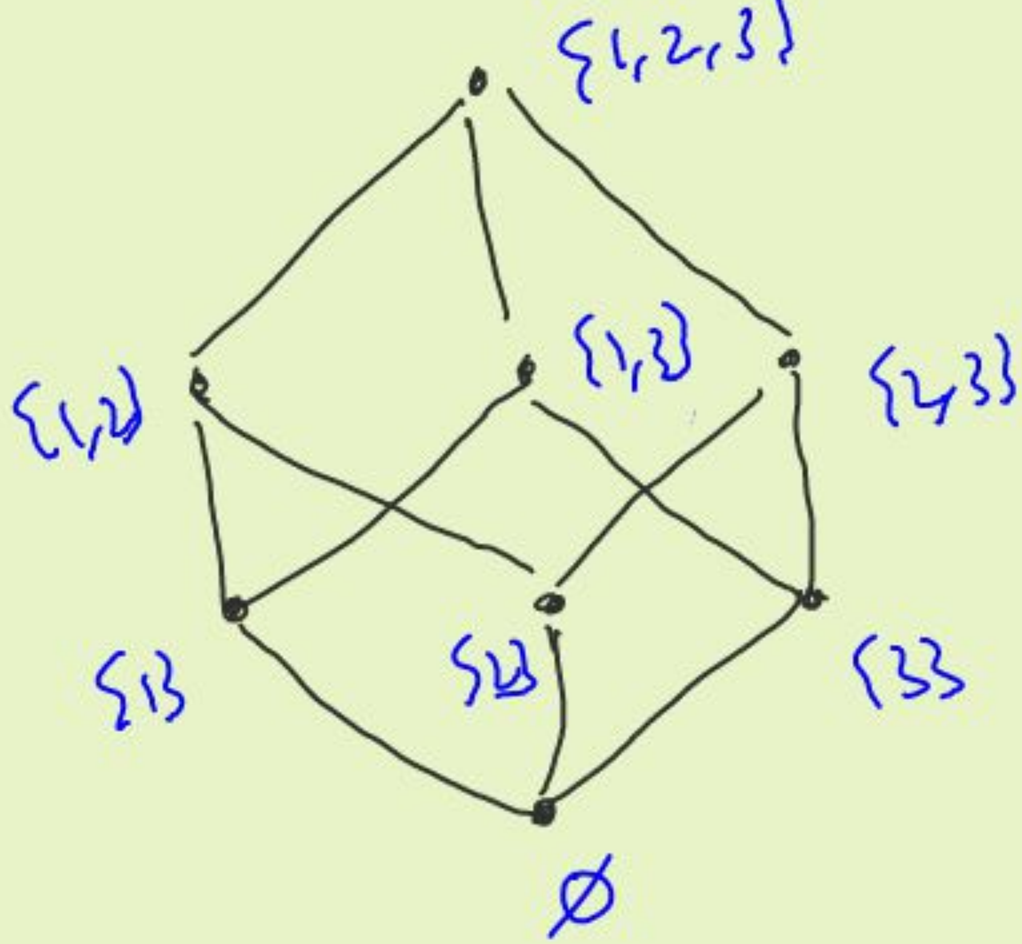
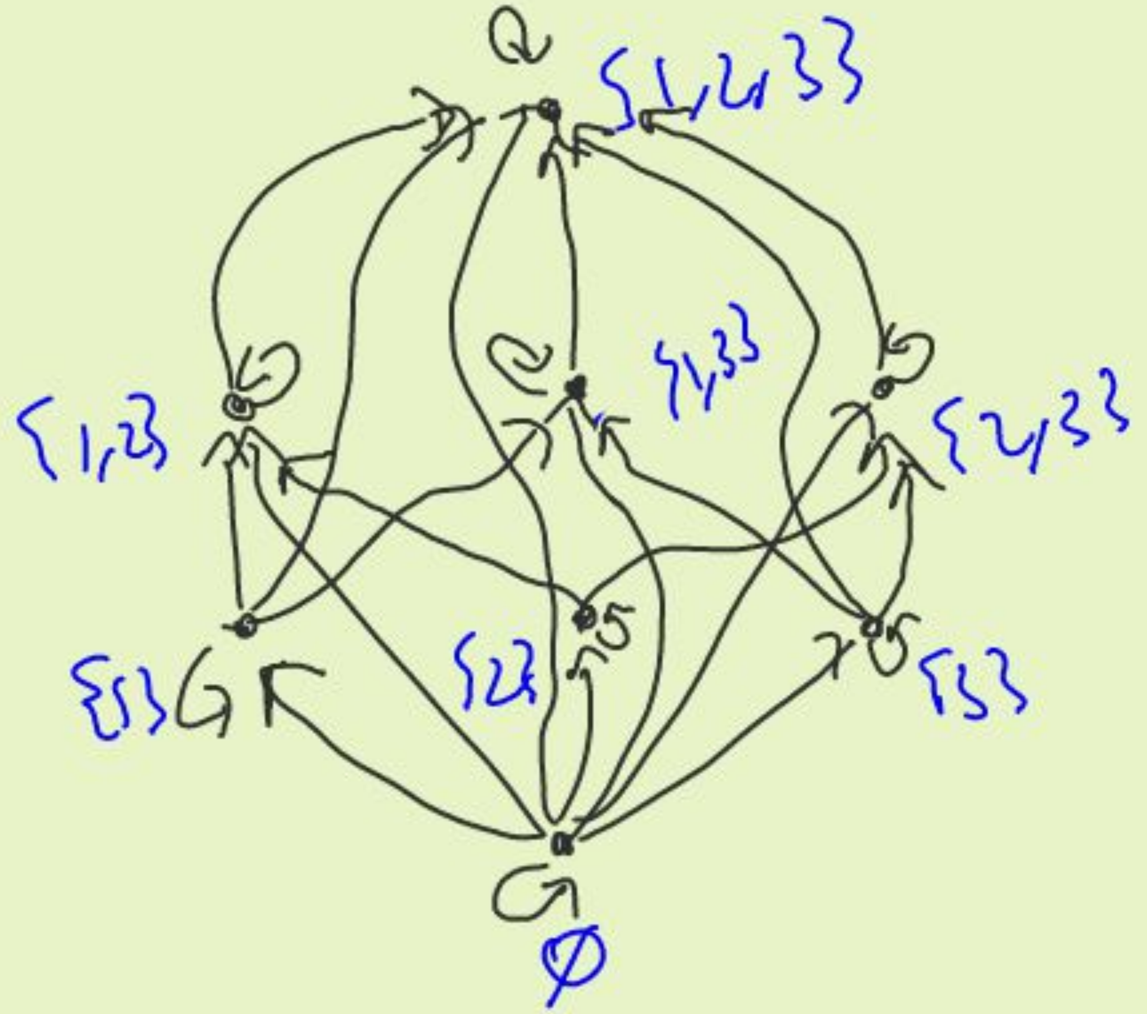


Hasse diagram

Representing finite Posets

$(\mathcal{P}(\{1,2,3\}), \subseteq)$

- No selfloops
- Remove arrows (edges go up)
- Remove edges implied by transitivity



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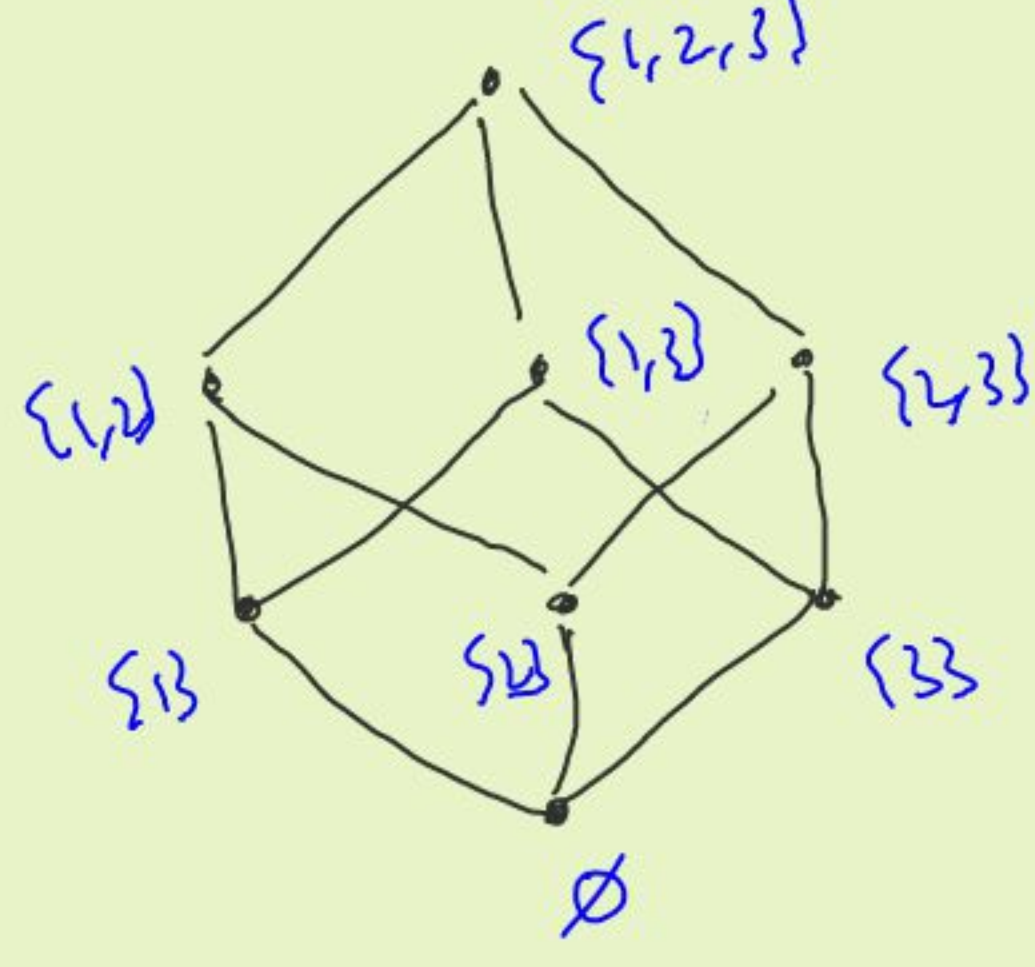
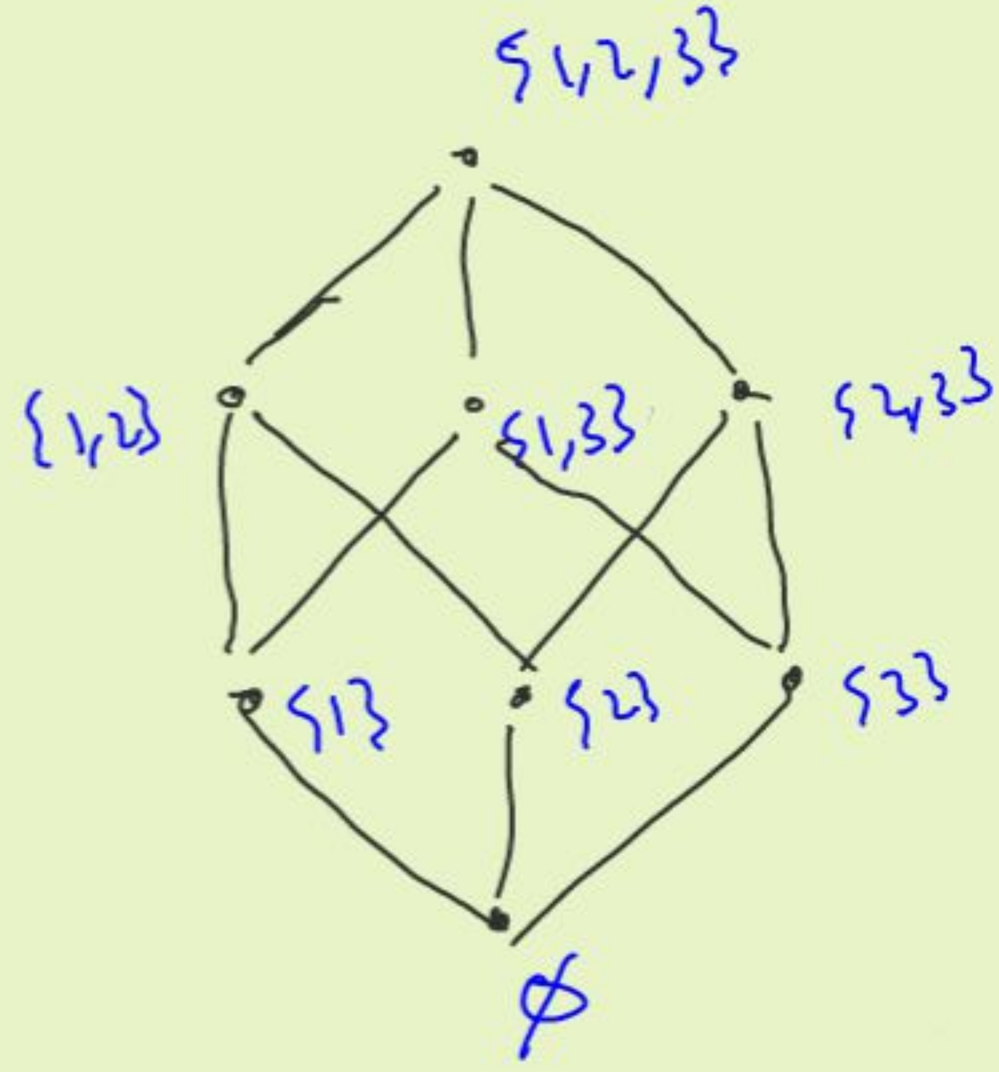


Hasse diagram

Lattices (A, R) a poset



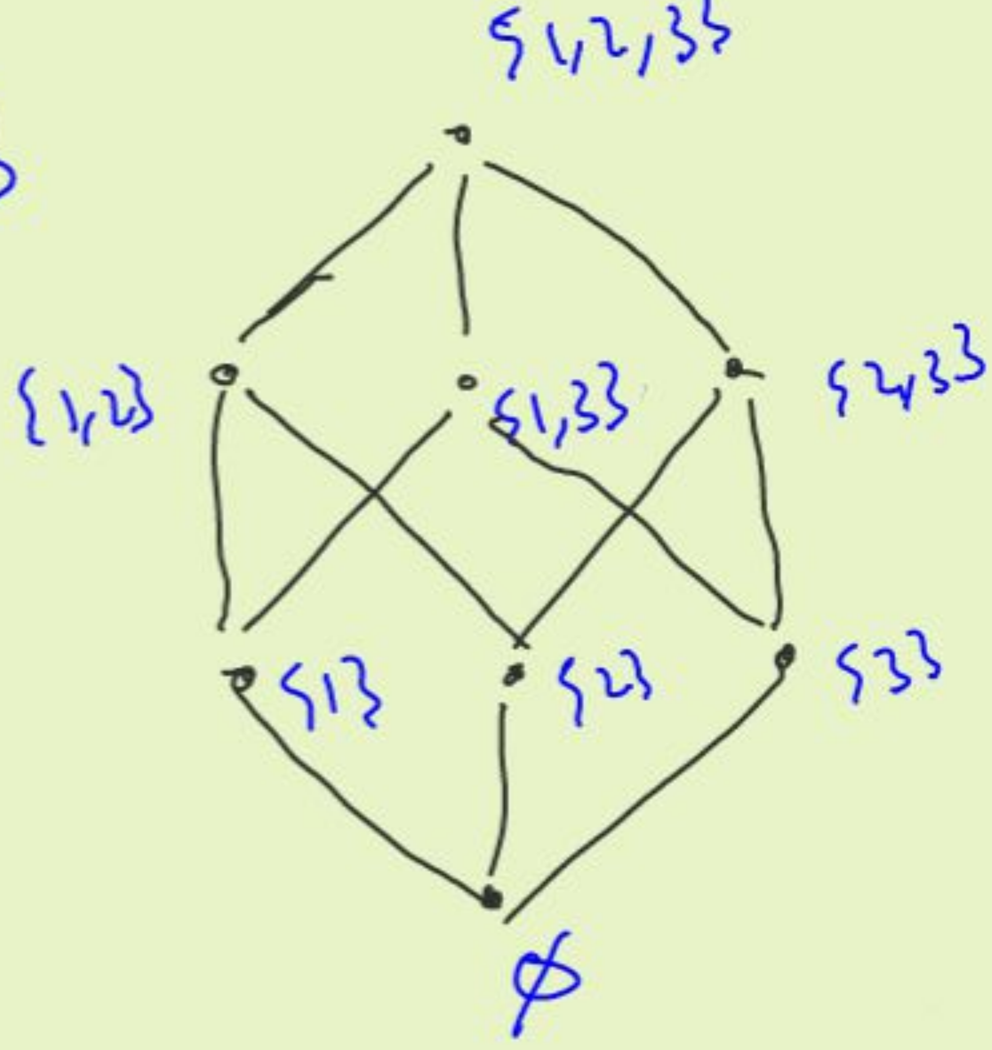
Lattices (A, R) a poset

Upper bound for $B \subseteq A$

$a \in A$ s.t. bRa for all $b \in B$.

Eg: $B = \{\emptyset, \{1\}, \{2\}\}$

Upper bounds: $\{1,2\}, \{1,2,3\}$



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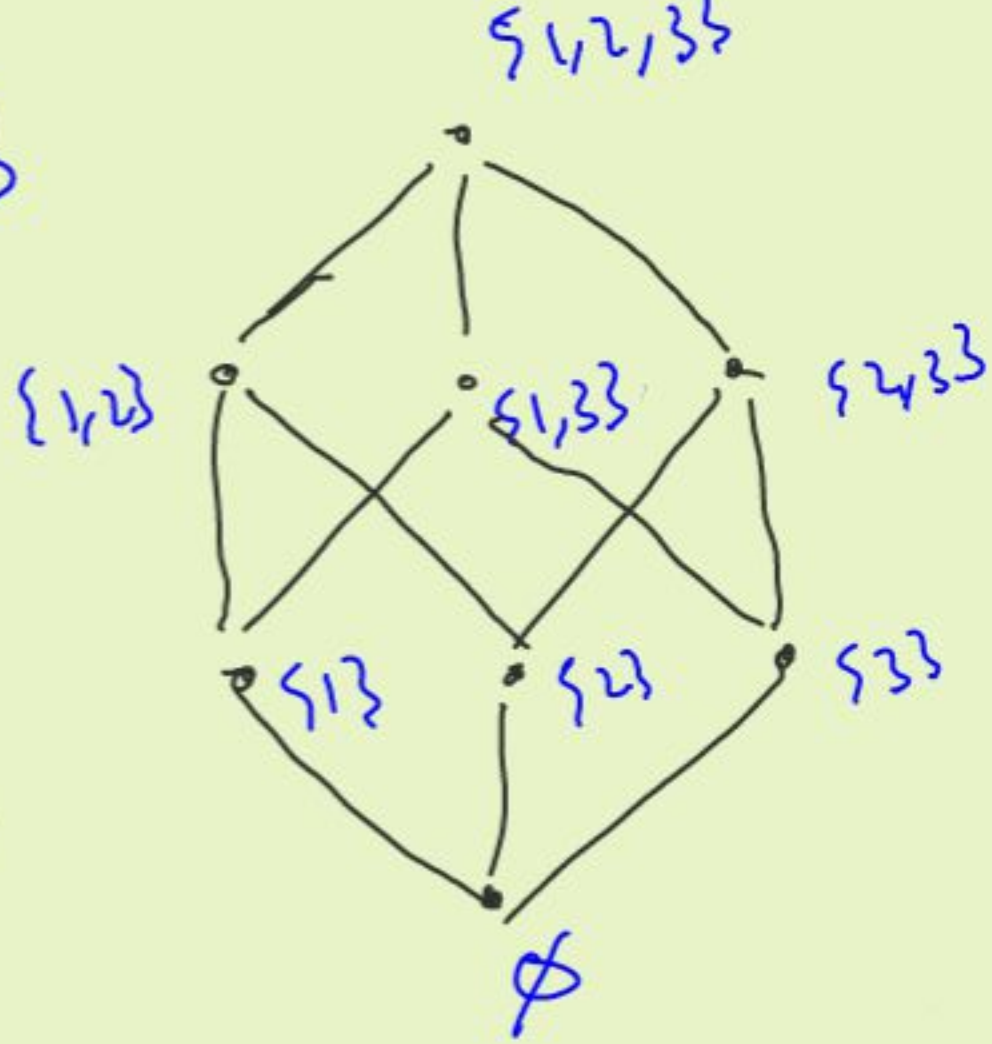

Least Upper bound (LUB)

for B : $a \in A$ s.t.

- Upper bound for B
- $a' \in A$ an upper bound for $B \Rightarrow aRa'$

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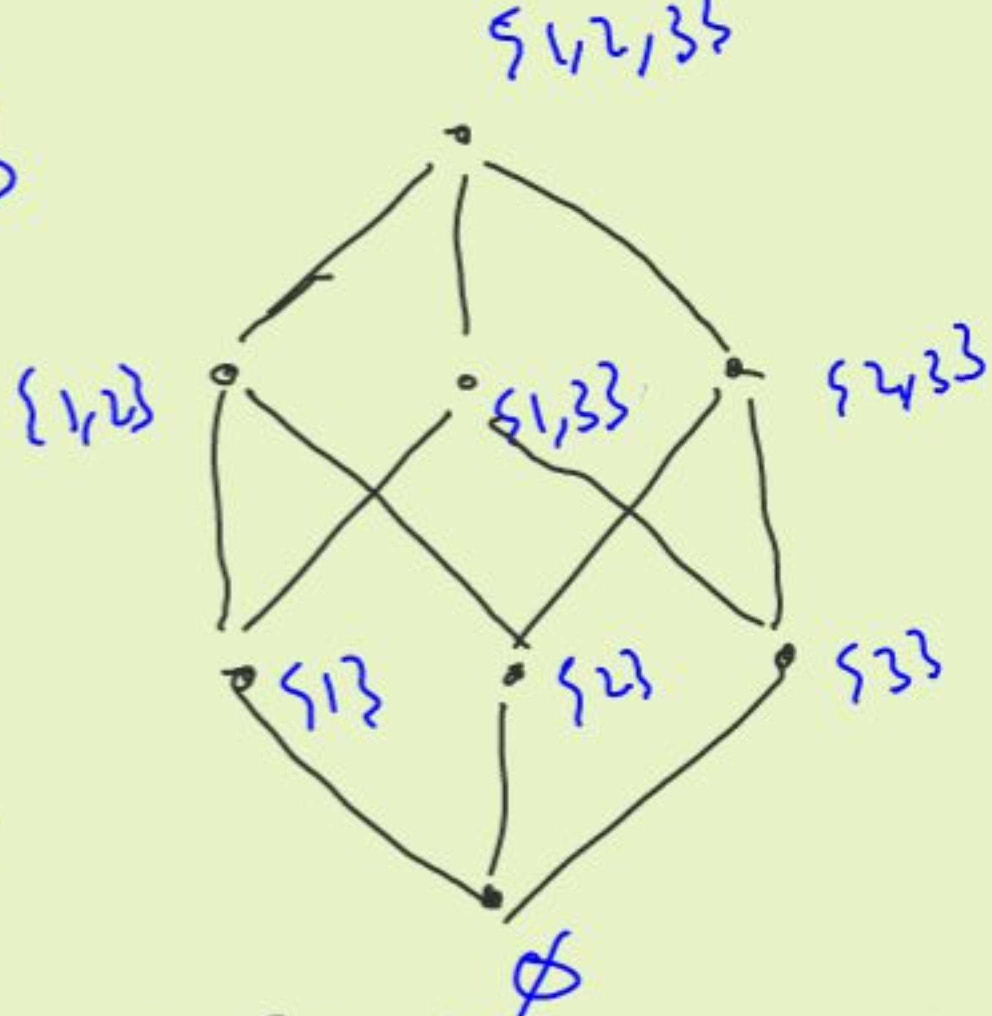

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Similar: Greatest lower bound (GLB)

Lattices (A, R) a poset

Examples

$(P(\{1,2,3\}), \subseteq)$ $(\mathbb{N}, \leq)$ $(\mathbb{N}, |)$ $(\{1,2,3\}, =)$

LUB

$(\{b_1, \ldots, b_r\})$

GLB

$(\{b_1, \ldots, b_r\})$

Lattices (A, R) a poset

Examples

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LUB $(\{b_1, \ldots, b_r\})$ $b_1 \cup \cdots \cup b_r$

GLB $(\{b_1, \ldots, b_r\})$ $b_1 \cap b_2 \cdots \cap b_r$

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Lattices (A, R) a poset

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LUB $(\{b_1, \ldots, b_r\})$	$b_1 \cup \cdots \cup b_r$	$\max\{b_1, \ldots, b_r\}$	$\mathrm{lcm}\{b_1, \ldots, b_r\}$	
GLB $(\{b_1, \ldots, b_r\})$	$b_1 \cap b_2 \cdots \cap b_r$	$\min\{b_1, \ldots, b_r\}$	$\gcd\{b_1, \ldots, b_r\}$	

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Obs: LUBs & GLBs unique (if they exist)

Lattice: Poset with LUBs & GLBs.

Graphs

Directed graph

$G = (V, E)$

V - vertices

E - edges, each edge
an ordered pair (u, v)

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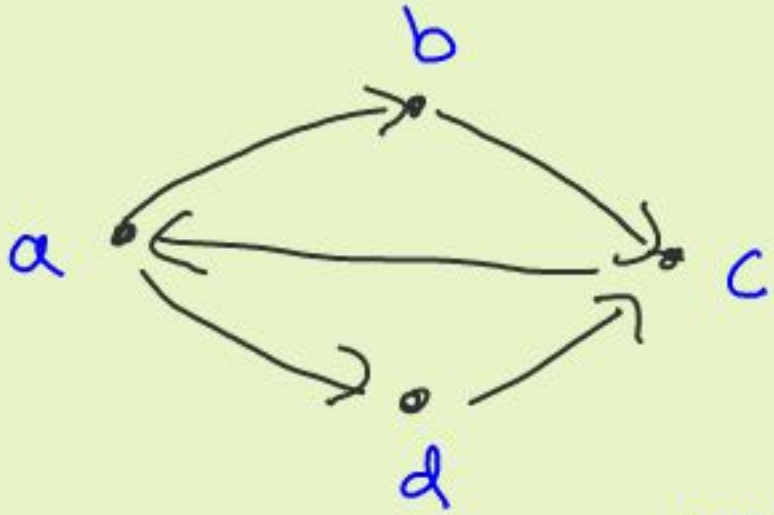

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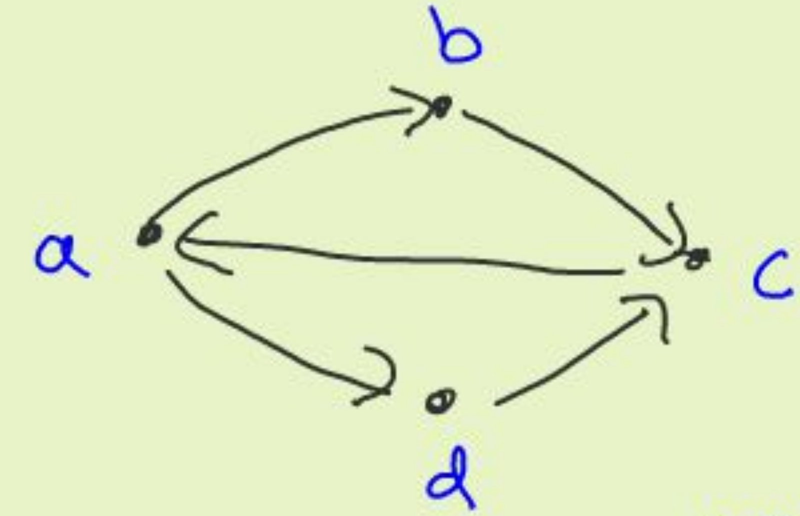

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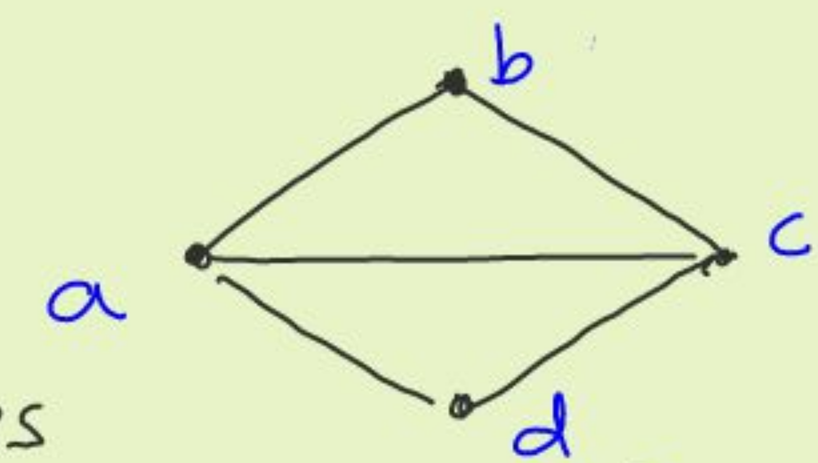
Undirected graph

E - each edge an "unordered pair", i.e. a set of two vertices



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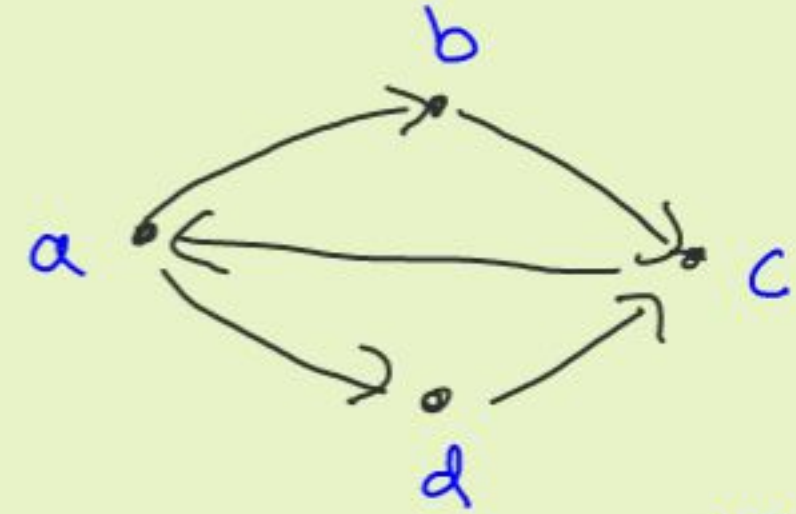

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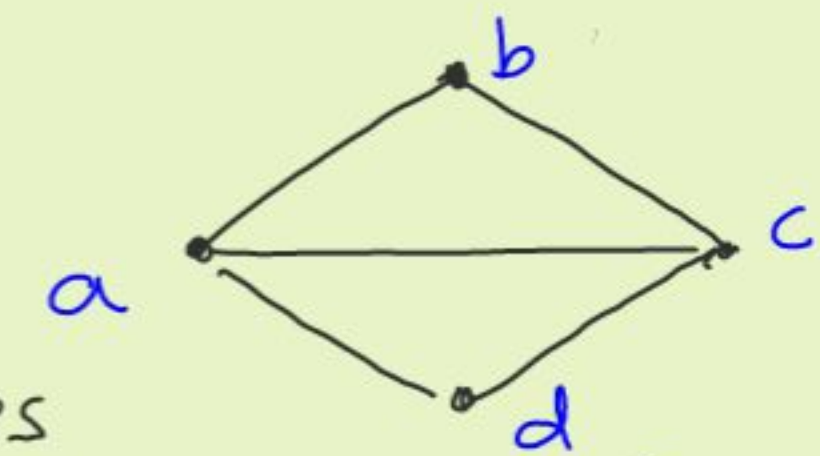

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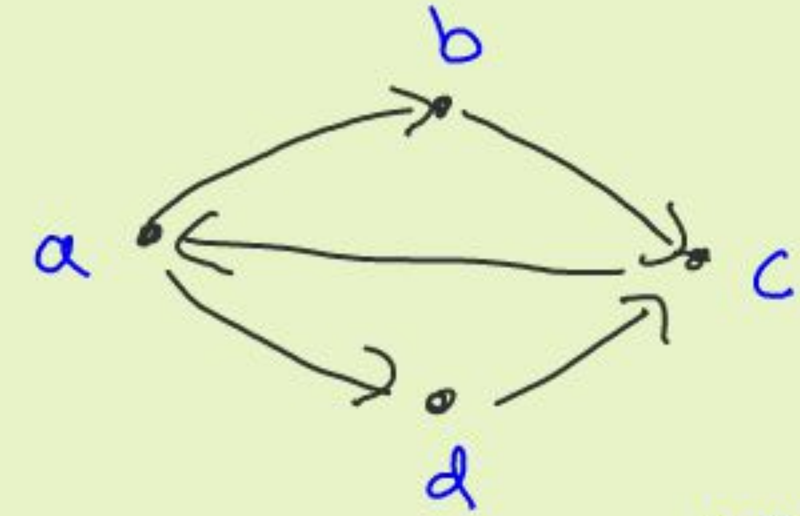
Simple graph - no self loops or parallel edges

Neighbours, degree



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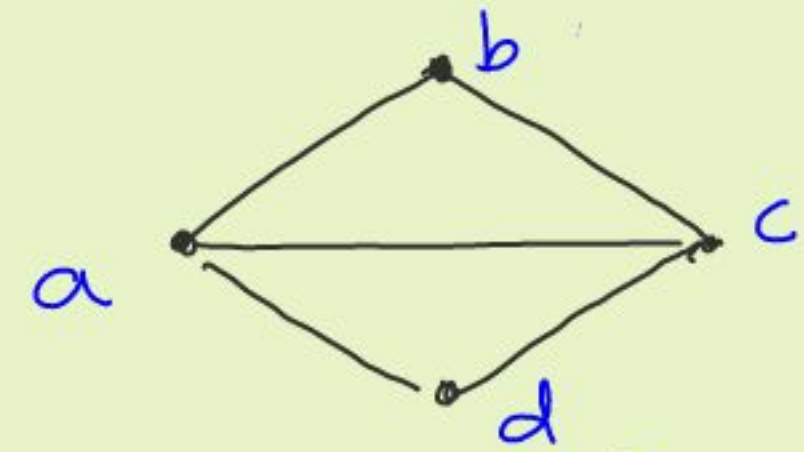

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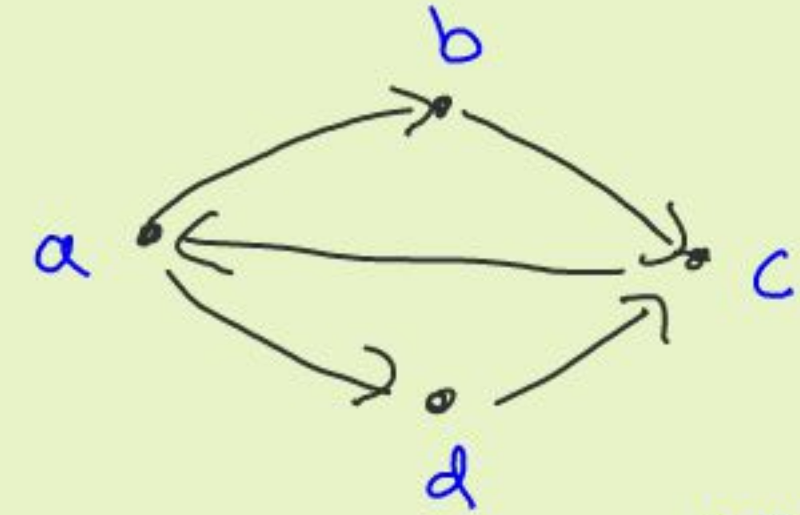

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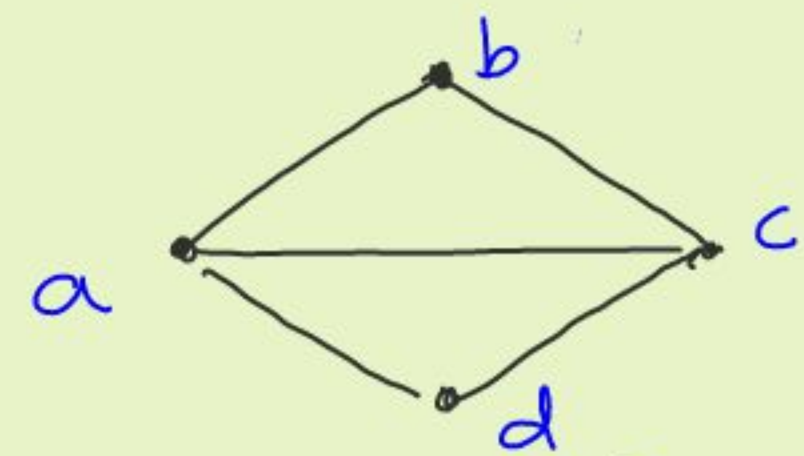

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Neighbours of $u \in V$

$N(u) = \{v \in V \mid \{u,v\} \in E\}$

Degree $\deg(u) = |N(u)|$

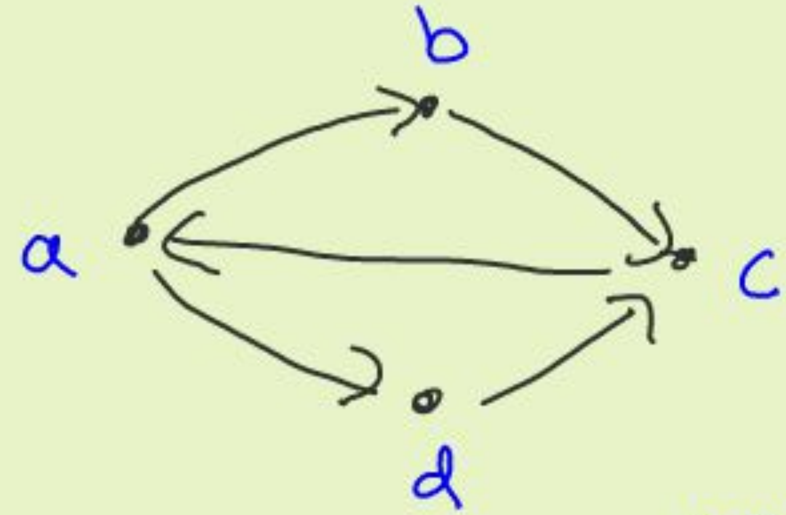
Neighbours, degree

Out-neighbours of $u \in V$

$$N_{out}(u) = \{v \mid (u,v) \in E\}$$

Out degree

$$\deg_{out}(u) = |N_{out}(u)|$$



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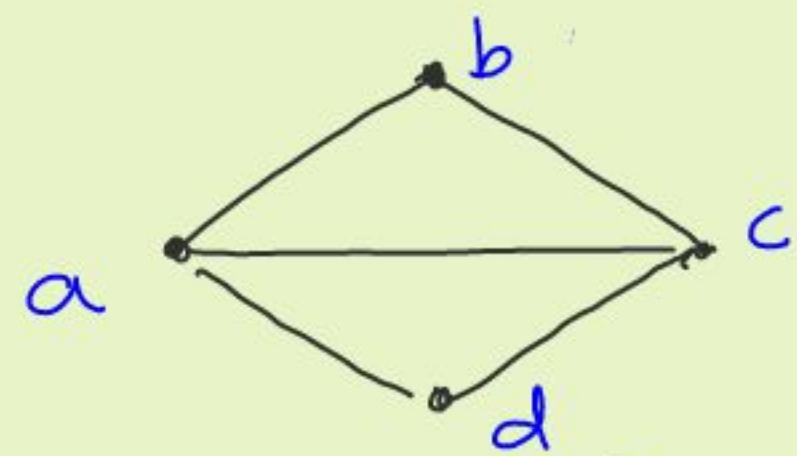

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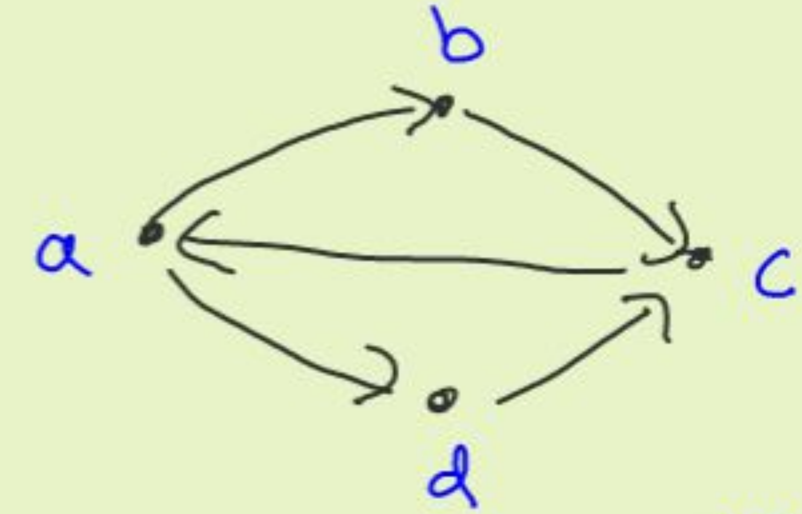

Out degree

$$\deg_{out}(u) = |N_{out}(u)|$$

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In-neighbours

$$N_{in}(u) = \{v \mid (v,u) \in E\}$$

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In-degree

$$\deg_{in}(u) = |N_{in}(u)|$$

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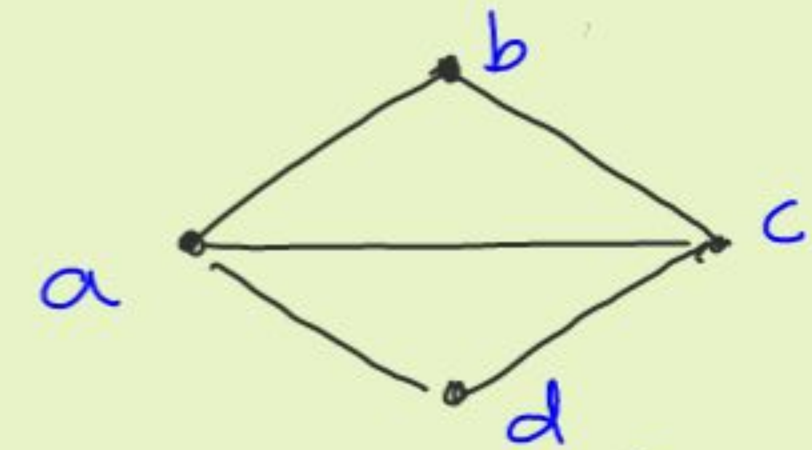

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$$\sum_{u \in V} \deg_{out}(u) = |E|$$

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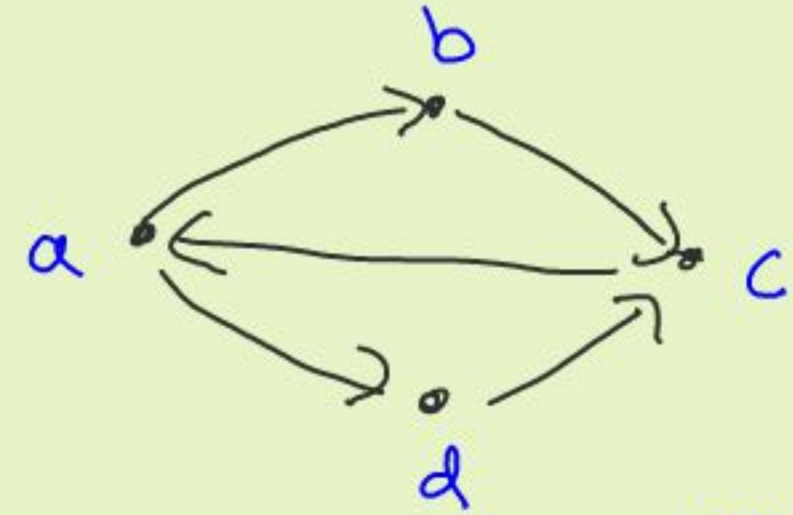

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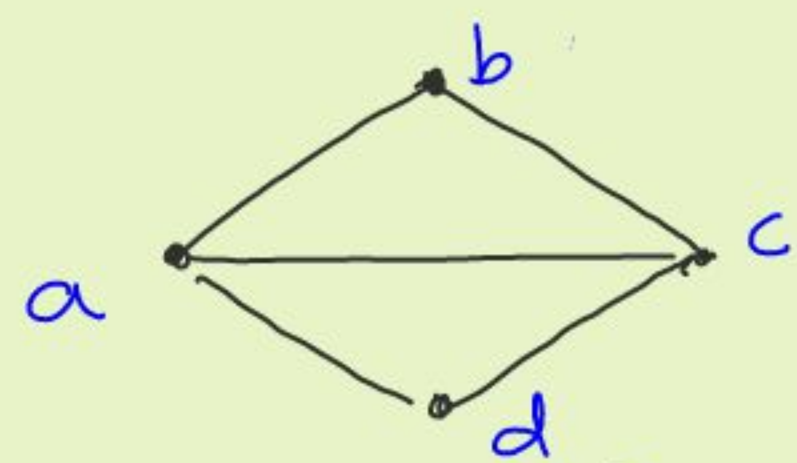

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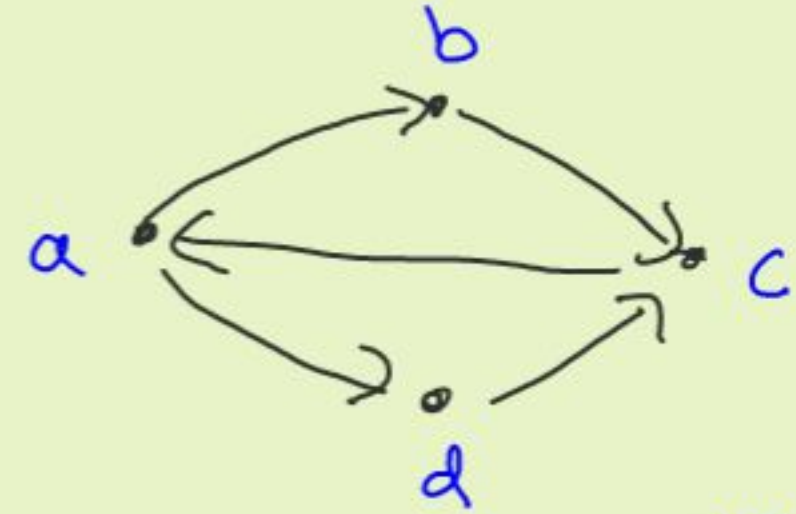

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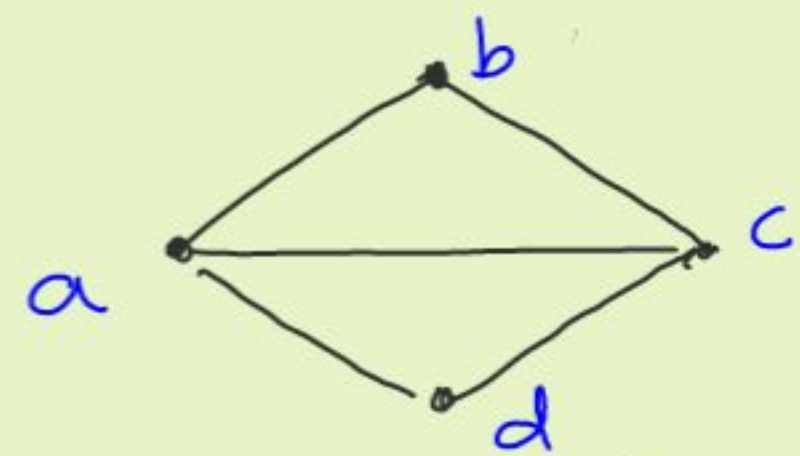
$$\frac{1}{2} \sum_{u \in V} \deg(u) = |E|$$



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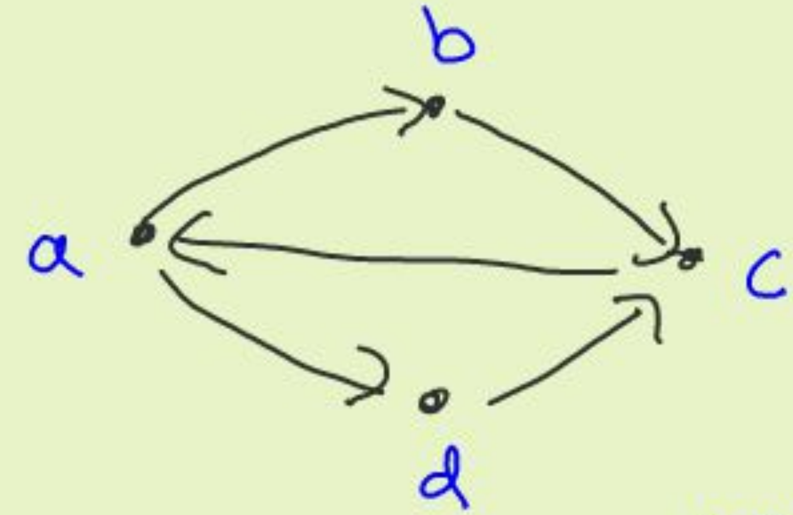
Paths & Simple Paths

Path - sequence of vertices

$(v_0, v_1, \dots, v_n)$ s.t.

$\forall i \in \{0, \dots, n-1\} : (v_i, v_{i+1}) \in E$



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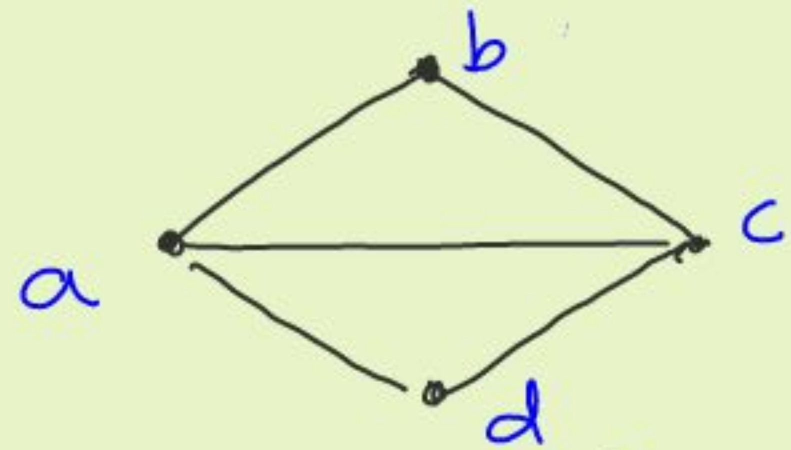

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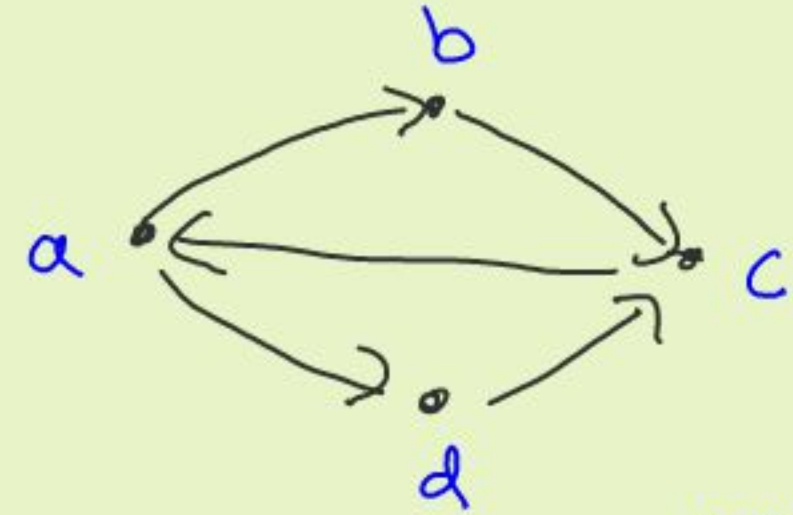
Simple Path - vertices are not repeated



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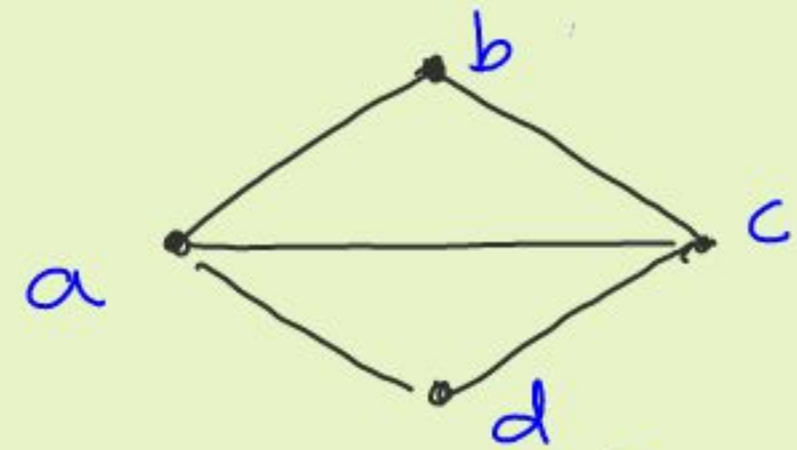

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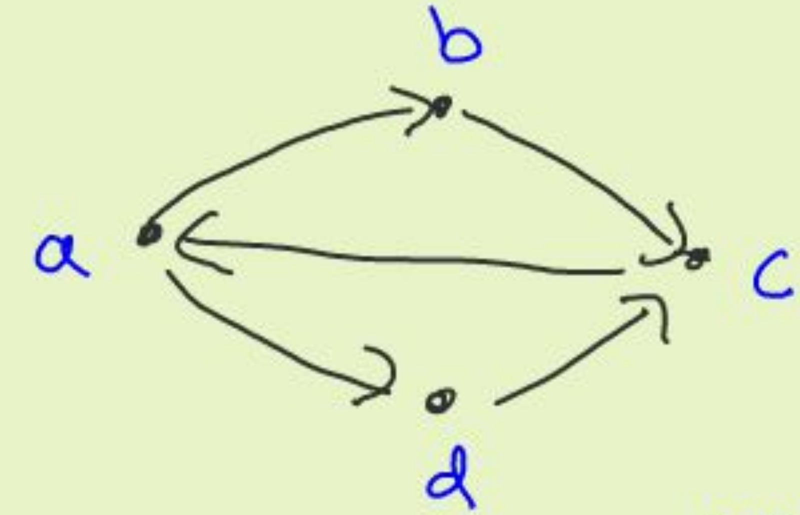
$\forall i \in \{0, \ldots, n-1\}: (v_i, v_{i+1}) \in E$



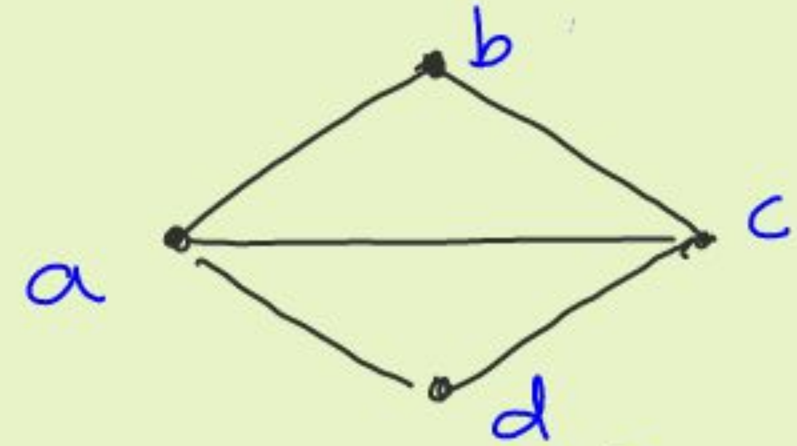
$V = \{a, b, c, d\}$

$E = \{(a,b), (b,c), (c,a), (a,d), (d,c)\}$

Simple Path - vertices are not repeated

(a, b, c, d) - Simple path

(a, b, c, a, d) - Path, not simple



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Paths & Simple Paths

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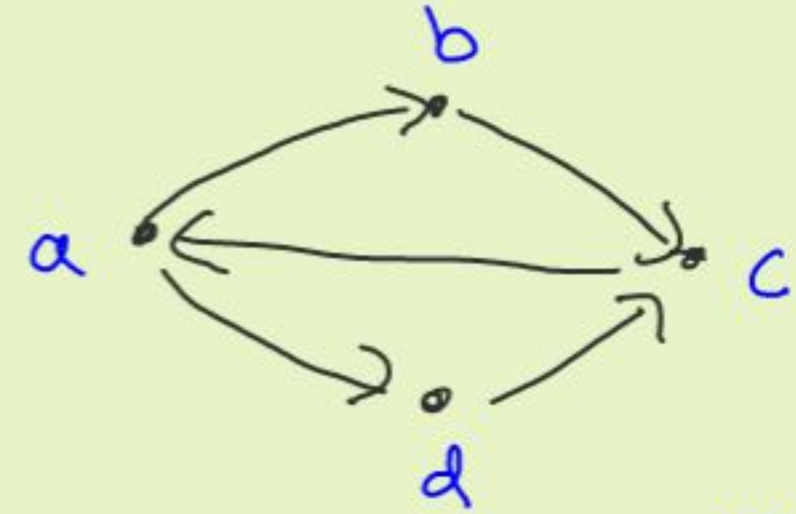

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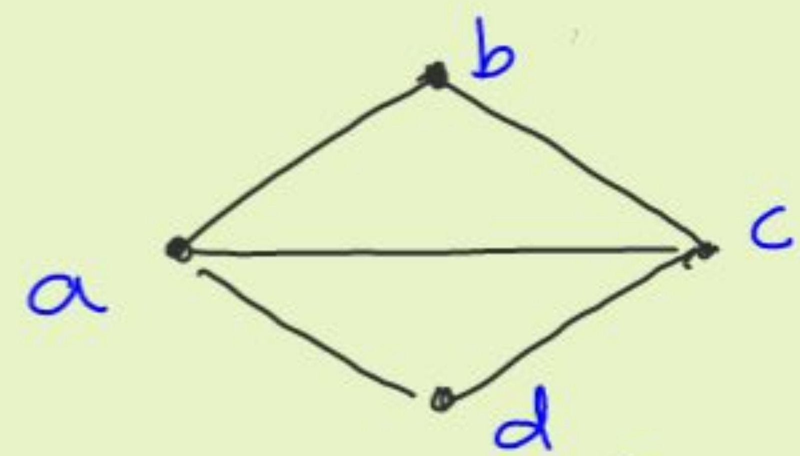
Simple Path - vertices are not repeated

(a, b, c, d) - Simple path

(a, b, c, a, d) - Path, not simple

Cycle - $v_0 = v_n$ & no repeated edges.



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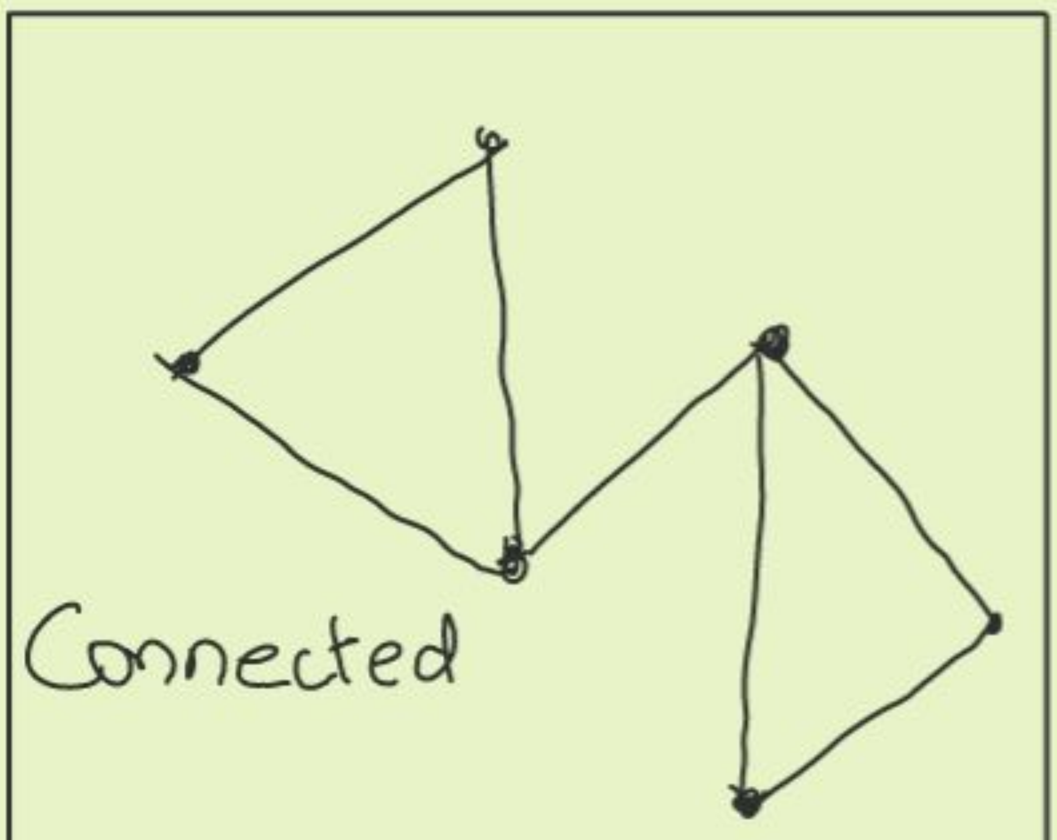
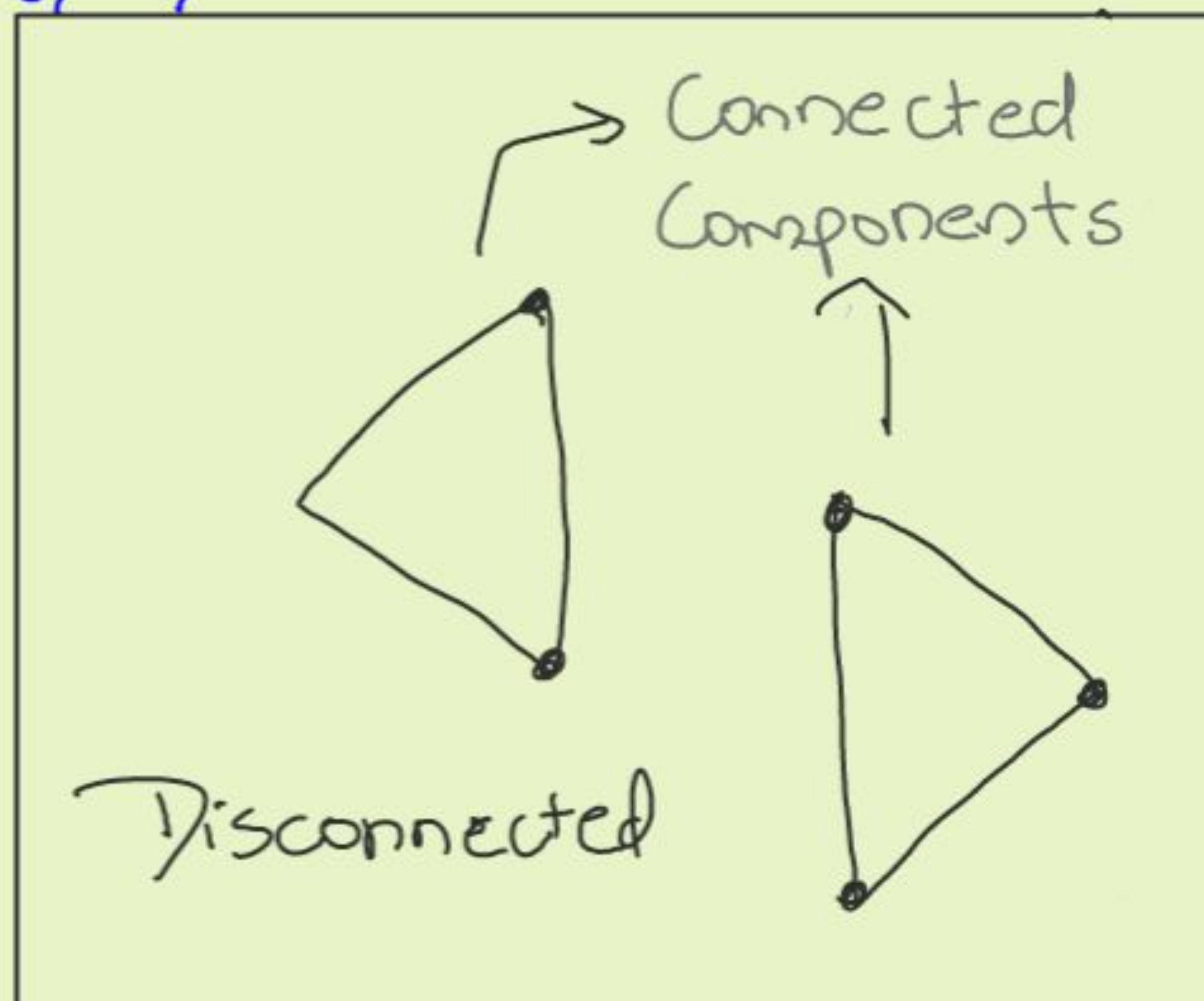
Connectedness

$G = (V, E)$ Undirected graph

G is connected if

$\forall u, v \in V : \exists$ path $(v_0, \ldots, v_n)$ $v_0 = u$ & $v_n = v$





Undirected Tree

$G = (V, E)$ undirected tree if it is connected & contains no cycles (acyclic).

Undirected Tree

$G=(V,E)$ undirected tree if it is connected & contains no cycles (acyclic).

Theorem G finite graph is a tree if & only if:

1. G connected & acyclic
2. G connected & $|E|=|V|-1$
3. G acyclic & $|E|=|V|-1$
4. $\forall u \neq v \in V$: there is a unique simple path between u & v

Proof of Theorem (1 $\Leftrightarrow$ 4)

1 $G=(V,E)$ connected & acyclic

4 $\forall u \neq v$: there is a unique simple path b/w u, v

Proof of Theorem (1 $\Longleftrightarrow$ 4)

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4 $\Rightarrow$ 1 or $\neg$ 1 $\Rightarrow$ $\neg$ 4

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4 $\Rightarrow$ 1 or $\neg$ 1 $\Rightarrow$ $\neg$ 4

Assume $\neg$ 1. Then either

→ G disconnected

or

→ G has a cycle

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$\rightarrow$ G disconnected $\Rightarrow$ $\exists\, u \neq v$: no path b/w u, v

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→ G disconnected $\Rightarrow$ $\exists u \neq v$: no path b/w u, v

or

→ G has a cycle $\Rightarrow$ $\exists u \neq v$: two distinct simple paths b/w u & v

Proof of Theorem (① $\Leftrightarrow$ ④)

① $G=(V,E)$ connected & acyclic

④ $\forall u \neq v$: there is a unique simple path b/w u, v

④ $\Rightarrow$ ① or $\neg$① $\Rightarrow$ $\neg$④

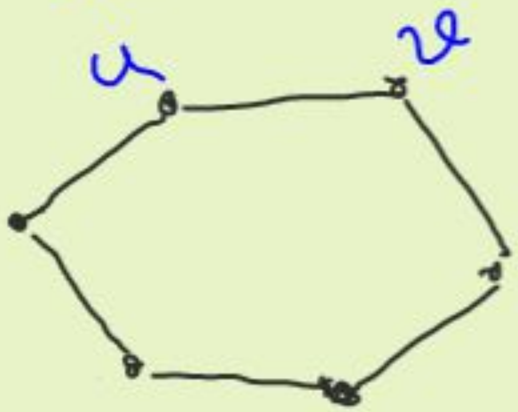
Assume $\neg$①. Then either

$\rightarrow$ G disconnected $\Rightarrow$ $\exists\, u \neq v$: no path b/w u, v

or

$\rightarrow$ G has a cycle $\Rightarrow$ $\exists\, u \neq v$: two distinct simple paths b/w u & v

because



$\rightarrow$ Shortest cycle in G does not repeat vertices

Proof of Theorem (1 $\Leftrightarrow$ 4)

1 $G=(V,E)$ connected & acyclic

4 $\forall u \neq v$: there is a unique simple path b/w u, v

1 $\Rightarrow$ 4. or $\neg$ 4 $\Rightarrow$ $\neg$ 1

Assume $\neg$ 4.

Proof of Theorem (1 $\Leftrightarrow$ 4)

1 $G=(V,E)$ connected & acyclic

4 $\forall u \neq v$: there is a unique simple path b/w u, v

$1 \Rightarrow 4$. or $\neg 4 \Rightarrow \neg 1$

Assume $\neg 4$. Then either

$\to$ $\exists u \neq v$: no simple path b/w u, v

$\to$ $\exists u \neq v$: two simple paths b/w u, v

Proof of Theorem (1 $\Leftrightarrow$ 4)

1 $G=(V,E)$ connected & acyclic

4 $\forall u\neq v$: there is a unique simple path b/w u,v

(Shortest paths are simple.)

1 $\Rightarrow$ 4. or $\neg$ 4 $\Rightarrow$ $\neg$ 1

Assume $\neg$ 4. Then either

$\rightarrow$ $\exists u\neq v$: no simple path b/w u,v $\Rightarrow$

$\exists u\neq v$: no path b/w u,v $\Rightarrow$ G disconneted

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$\rightarrow$ $\exists u \neq v$: no simple path b/w u, v $\Rightarrow$ $\exists u \neq v$: no path b/w u, v $\Rightarrow$ G disconneted

$\rightarrow$ $\exists u \neq v$: two simple paths b/w u, v $\Rightarrow$ G has a cycle.

Proof of Theorem (① ⟺ ④)

① $G = (V, E)$ connected & acyclic

④ $\forall u \neq v$: there is a unique simple path b/w u, v

(Shortest paths are simple.)

① ⇒ ④ or ¬④ ⇒ ¬①

Assume ¬④. Then either

→ $\exists u \neq v$: no simple path b/w u, v ⇒
$\exists u \neq v$: no path b/w u, v ⇒ G disconneted

→ $\exists u \neq v$: two simple paths b/w u, v ⇒ G has a cycle.

Find closest such u, v & consider shortest & second shortest path. Concatenate them.

Proof of Theorem (1 $\Leftrightarrow$ 2)

1 $G=(V,E)$ connected & acyclic

2 G connected & $|E| = |V| - 1$

Proof of Theorem (1 $\Leftrightarrow$ 2)

1. $G=(V,E)$ connected & acyclic

2. G connected & $|E| = |V| - 1$

Induction on $n = |V|$.

Base case: $n=1$. $|E| = 0 = n-1$.

Proof of Theorem (① ⟺ ②)

① $G=(V,E)$ connected & acyclic

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Induction on $n=|V|$.

Base case: $n=1$. $|E|=0=n-1$.

Induction: Assume for n. Say $|V|=n+1$.

Proof of Theorem (① $\Leftrightarrow$ ②)

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① $\Rightarrow$ $\exists v\in V : \deg(v)=1$ $\longrightarrow$ If every vertex has degree ≥ 2, G has a cycle.

If any vertex has degree 0, G is disconnected.

Proof of Theorem (① ⟺ ②)

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② G connected & $|E| = |V|-1$

Induction on $n=|V|$.

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① $\Rightarrow \exists v \in V : deg(v) = 1$ ⟶ If every vertex has degree ≥ 2, G has a cycle. If any vertex has degree 0, G is disconnected.

Consider $G' = (V \setminus \{v\}, E \setminus \{\{u,v\}\})$ where u is the only nbr. of v.

G' is connected & acyclic.

$\Rightarrow |E|-1 = n-1$

$\Rightarrow |E| = n$

Proof of Theorem ((1) $\Leftrightarrow$ (2))

(1) $G=(V,E)$ connected & acyclic

(2) G connected & $|E| = |V|-1$

Induction on $n=|V|$.

Base case: $n=1$. $|E|=0=n-1$.

Induction: Assume for n. Say $|V|=n+1$.

(2) $\Rightarrow$ $\exists v\in V : \deg(v)=1$

Proof of Theorem (1 $\Leftrightarrow$ 2)

1 $G=(V,E)$ connected & acyclic

2 G connected & $|E| = |V|-1$

Induction on $n=|V|$.

Base case: $n=1$. $|E|=0=n-1$.

Induction: Assume for n. Say $|V|=n+1$.

2 $\Rightarrow \exists v \in V : \deg(v)=1 \longrightarrow \sum_{u \in V} \deg(u) = 2|E| = 2n.$

&

If any vertex has degree 0, G is disconnected.

Proof of Theorem (① $\Leftarrow\Rightarrow$ ②)

① $G=(V,E)$ connected & acyclic

② G connected & $|E|=|V|-1$

Induction on $n=|V|$.

Base case: $n=1$. $|E|=0=n-1$.

Induction: Assume for n. Say $|V|=n+1$.

② $\Rightarrow$ $\exists v\in V : \deg(v)=1$ $\longrightarrow$ $\sum_{u\in V} \deg(u) = 2|E| = 2n$.

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If any vertex has degree 0, G is disconnected.

Consider $G' = (V\setminus\{v\}, E\setminus\{\{u,v\}\})$

where u is the only nbr. of v.

G' is connected & $|E\setminus\{\{u,v\}\}|=n-1$

$\Rightarrow$ G' is connected & acyclic

$\Rightarrow$ So is G.

Spanning Trees

$G = (V, E)$

Subgraph G' of G: obtained by removing some subset of edges and/or vertices.

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Spanning tree: Subgraph $G' = (\underset{\text{same vertex set}}{V}, E')$ that is a **tree**.

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$G=(V,E)$

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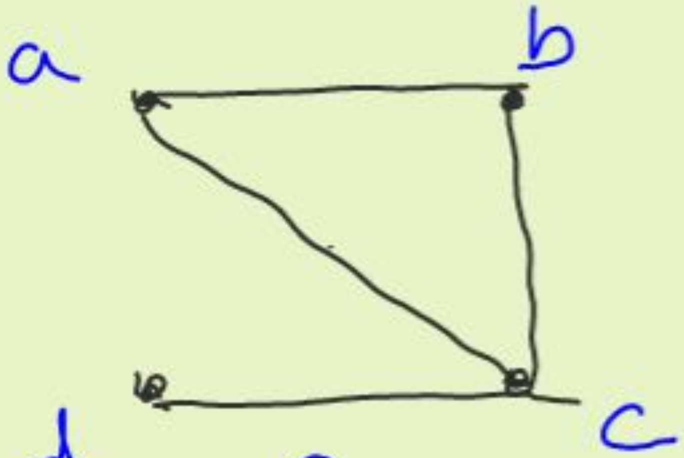
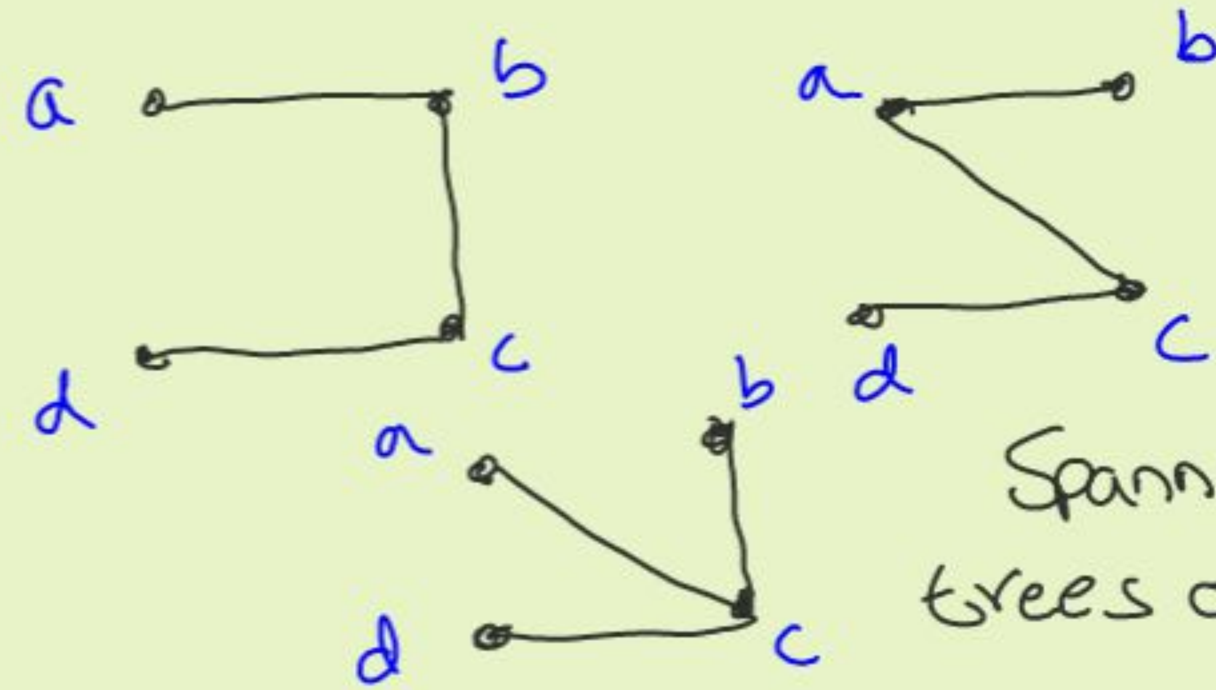
Spanning tree: Subgraph $G'=(V,E')$ that is a tree.

same vertex set



G



Spanning trees of G

Directed Tree $T = (V, E)$ directed graph

- → Special vertex v_0 (root)
- → For $v \in V \setminus \{v_0\}$, unique walk from v_0 to v.
- → No walk from v_0 to v_0.

Directed Tree

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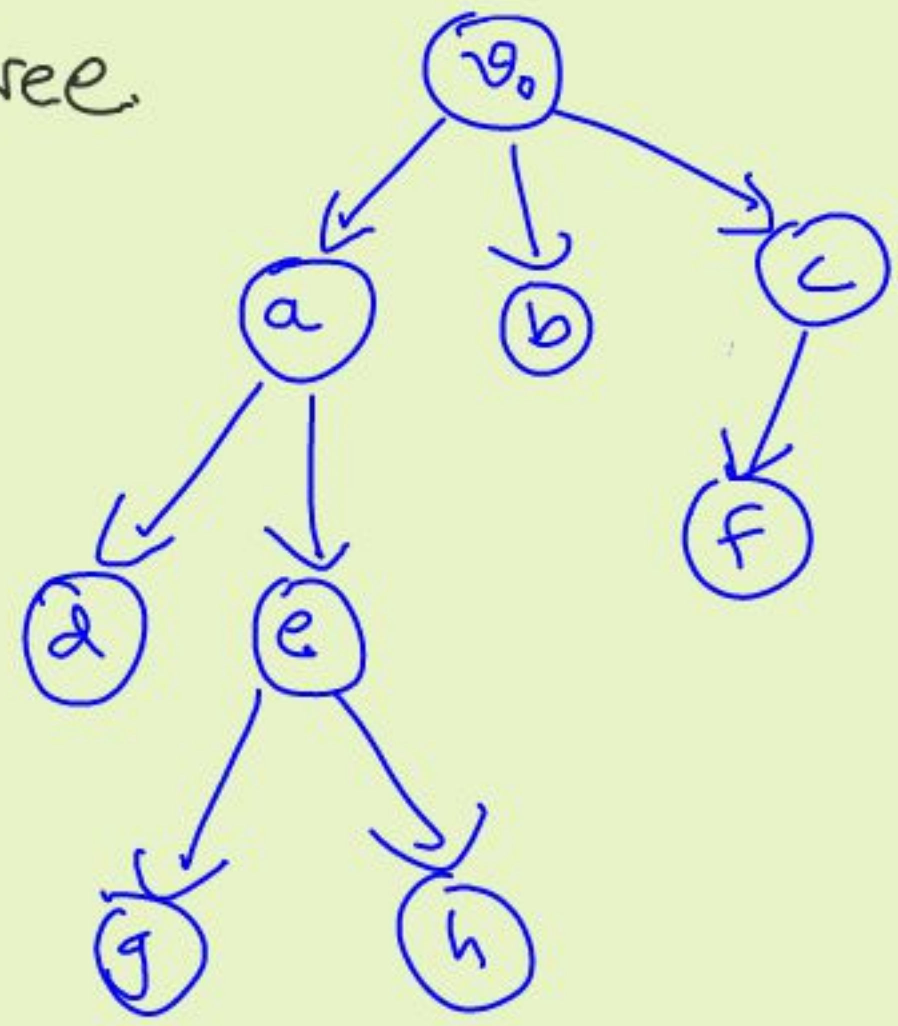
Thm: (T, v_0) a directed tree.



① T contains no cycles

② v_0 is the unique root.

③ $\deg_{in}(v_0) = 0$

$\deg_{in}(v) = 1 \quad \forall v \in V \setminus \{v_0\}$

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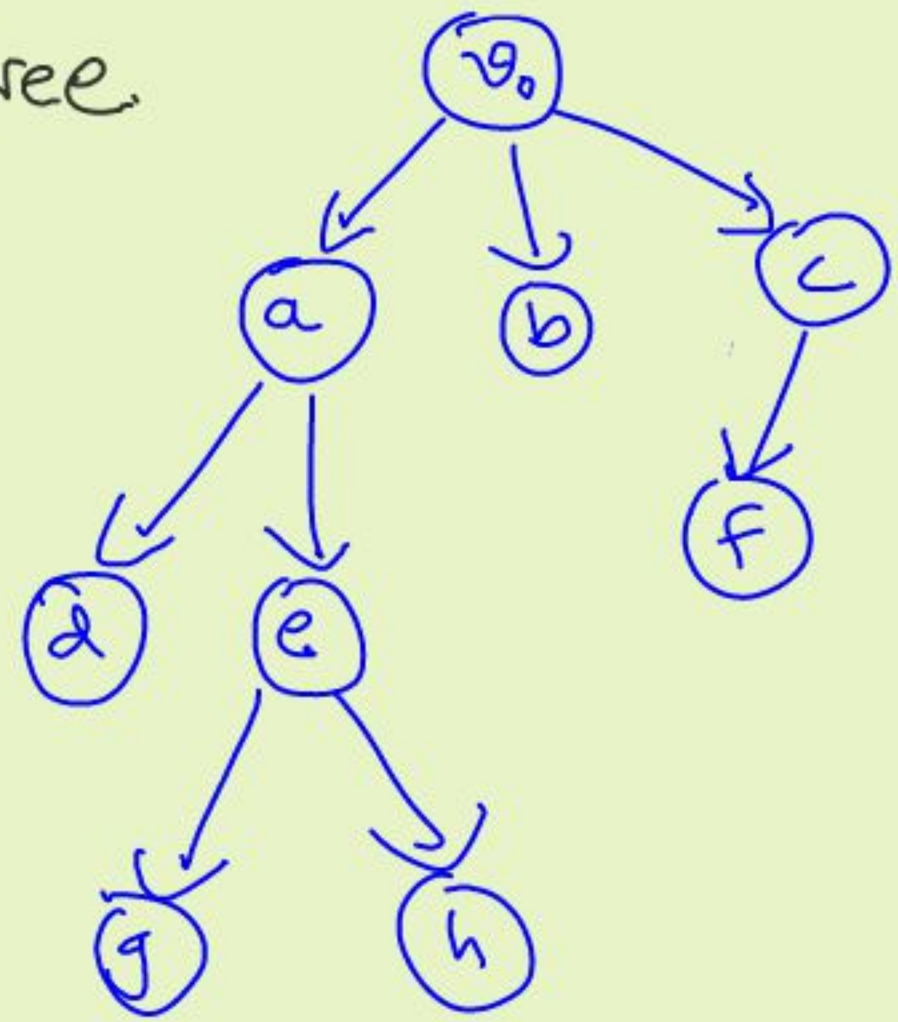

Thm: (T, v_0) a directed tree.

① T contains no cycles

② v_0 is the unique root.

③ $\deg_{in}(v_0) = 0$

$\deg_{in}(v) = 1 \quad \forall v \in V \setminus \{v_0\}$

Thm: $|E| = \sum_{v \in V} \deg_{in}(v) = |V| - 1.$

Directed Tree $T = (V, E)$ directed graph

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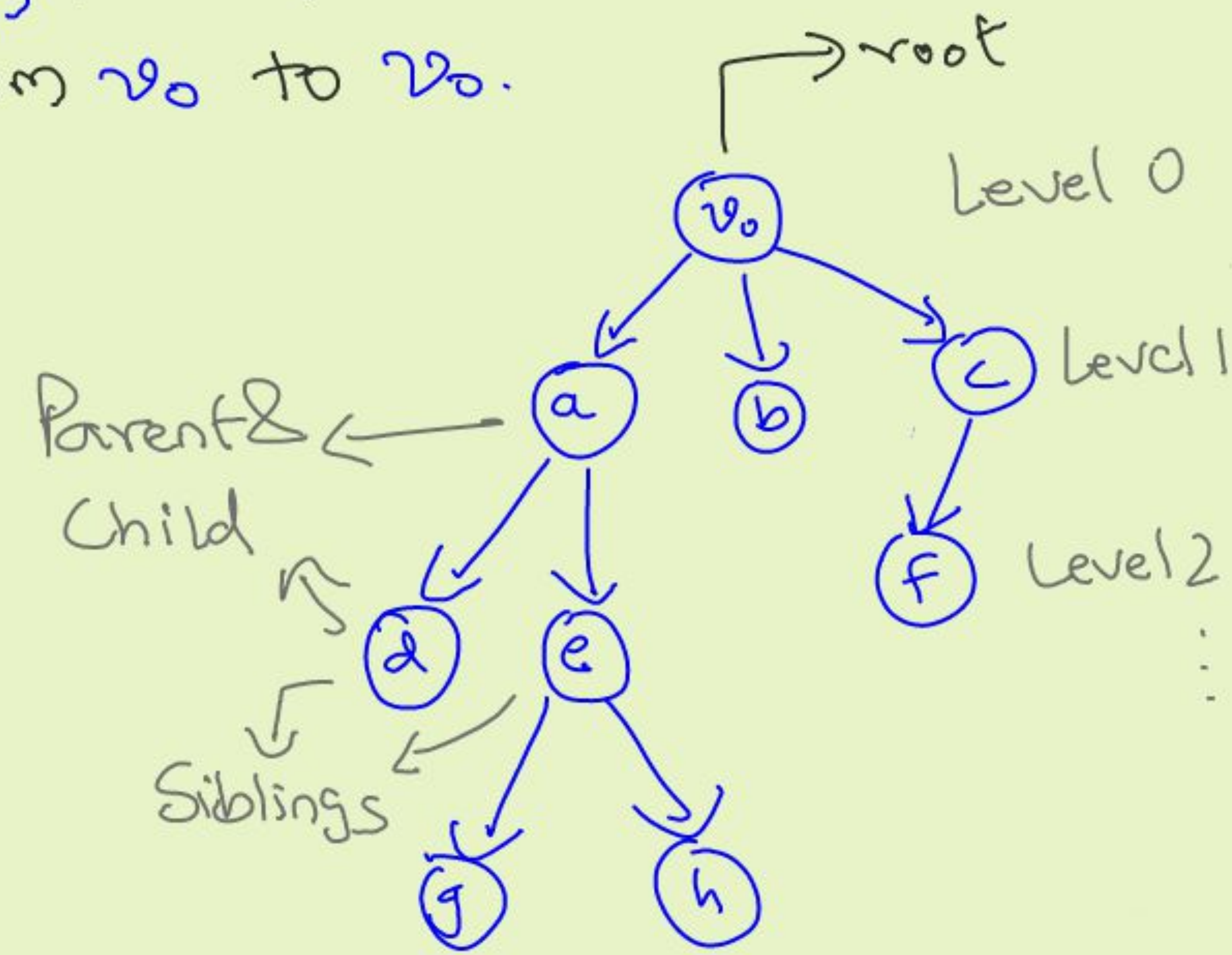

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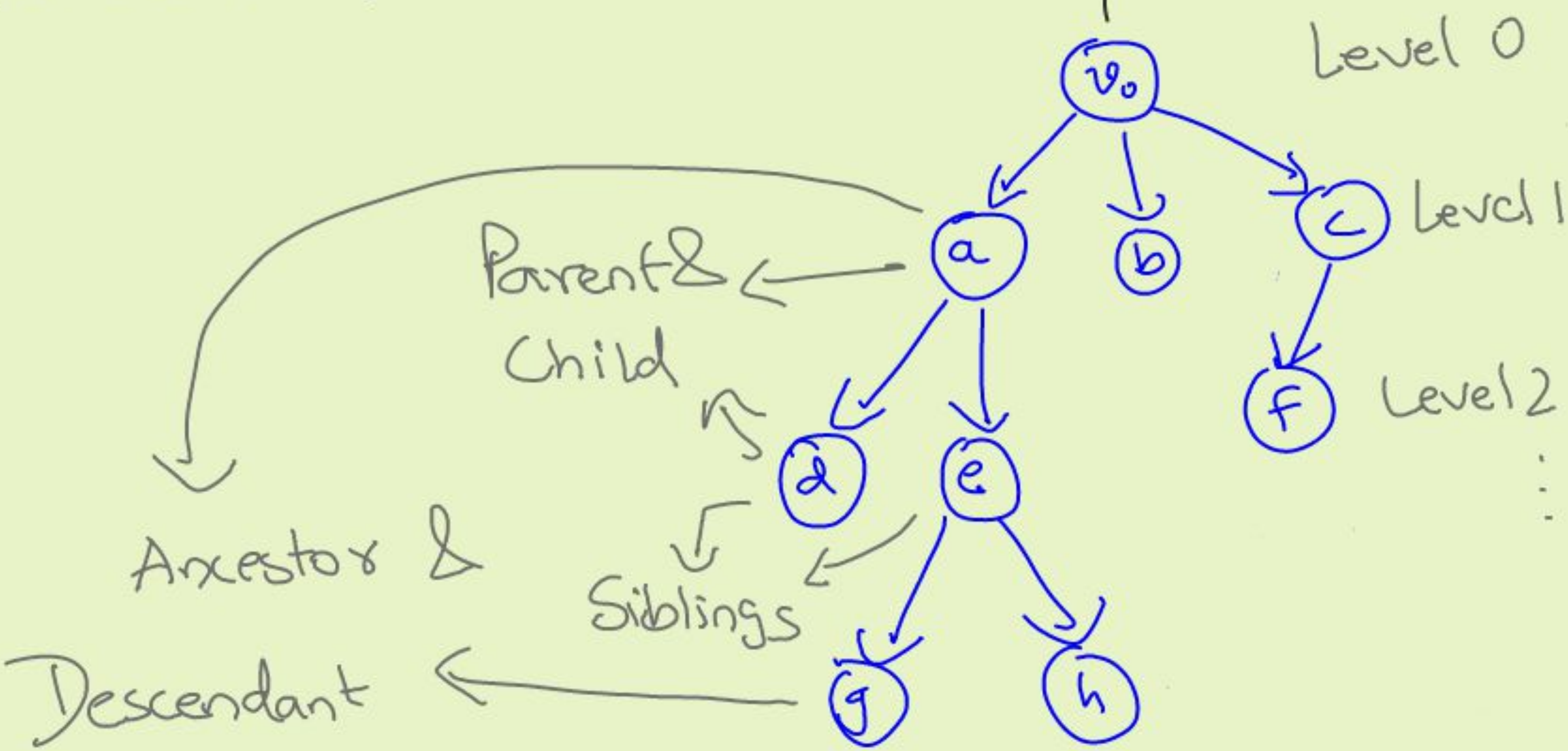

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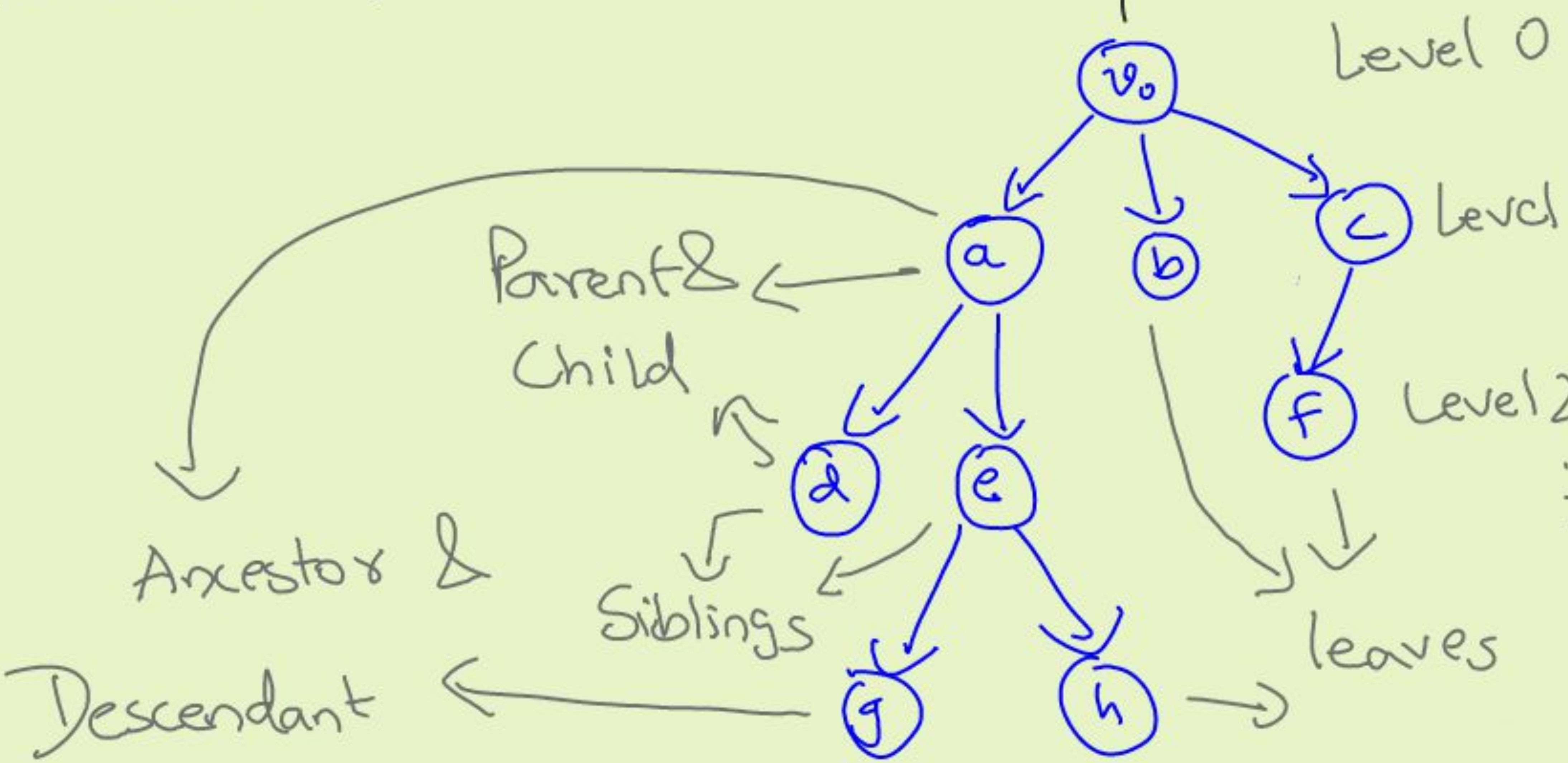

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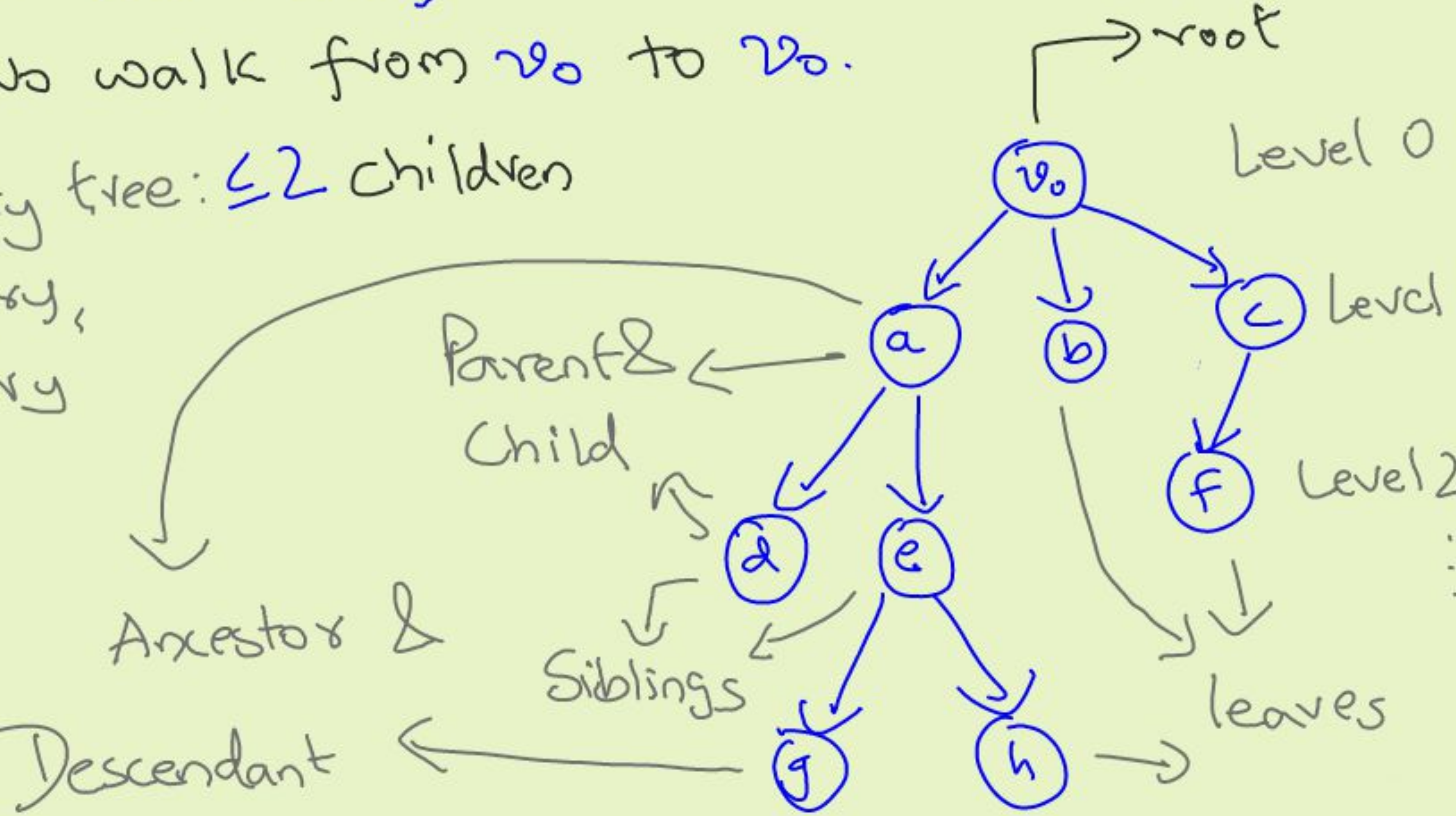
Binary tree: ≤ 2 children

Ternary,

n-ary



Outline

$\to$ Recap of relations

- $\to$ Digraph rep
- $\to$ Reflexive $\quad \to$ Antisymmetric
- $\to$ Symmetric $\quad \to$ Transitive

$\to$ Posets

$\to$ Examples & non-examples

$\to$ Product posets

$\to$ Posets via Hasse diagrams

$\to$ Lattices

$\to$ Examples

<u>Graphs</u>

→ Defn of a graph
→ Nbrs, adjacency, degree
  → Paths & Walks

→ Trees, directed
→ Props + pf
→ Terminology
→ Undirected trees
→ Props
→ Spanning trees