RELATIONS

CARTESIAN PRODUCT

$$A \times B = \{ (a,b) \mid a \in A,\ b \in B \}$$

ordered

Ex $A = \{a, b, c\}$ $\quad B = \{1, 2\}$

$$A \times B = \{ (a,1), (a,2), (b,1), (b,2), (c,1), (c,2) \}$$

$$A_1 \times A_2 \times \dots \times A_n = \{ (a_1, a_2, \dots, a_n) \mid a_i \in A_i \}$$

Ex Vectors in $\mathbb{R}^3$

OBSERVATION $|A_1 \times \dots \times A_n| = |A_1| \cdots |A_n|$
(Multiplication principle)

SUBSET $A' \subseteq A$ if for all $x \in A'$ it holds that $x \in A$

POWERSET $\mathcal{P}(A)$ or 2^A
Set of all subsets of A

$$2^A = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \}$$

$|2^A| = 2^{|A|}$ (multiplication principle)

RELATION

Ex $A = \{\text{all people}\}$ $\quad B = \{\text{all bikes}\}$
Relation a owns bike b

$A = B =$ integers
Relation $a \leq b$

RELATION R from A to B

subset of $A \times B$

$(a,b) \in R$ is more often written $R(a,b)$ or aRb

$(a,b) \notin R$ written $a \not R b$

If $R \subseteq A \times A$, R is a relation on A

Ex $A = \{2,3\}$ $B = \{1,2,3,4,5,6\}$

aRb if $a \mid b$

$$R = \{(2,2), (2,4), (2,6), (3,3), (3,6)\}$$

DOMAIN $Dom(R) = \{a \in A \mid \exists b \in B \; R(a,b)\}$

RANGE $Ran(R) = \{b \in B \mid \exists a \in A \; R(a,b)\}$

$Dom(R) = \{2,3\}$

$Ran(R) = \{2,3,4,6\}$

REPRESENTATION OF RELATION R

Matrix M_R $(m \times n)$ $m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{otherwise} \end{cases}$

$A = \{a_1, \ldots, a_m\}$

$B = \{b_1, \ldots, b_n\}$

Ex $A = \{1,2,3\}$ $B = \{r,s\}$ $R = \{(1,r), (2,s), (3,r)\}$

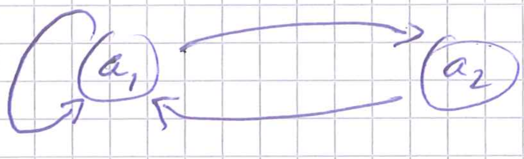
$$\begin{array}{cc} & \begin{array}{cc} r & s \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

Relation R on A: Directed graph (digraph)

vertices/nodes represent elements of A

directed edges $a_i \to a_j$ if $(a_i, a_j) \in R$

<u>Ex</u> $A = (a_1, a_2)$ $R = \{(a_1,a_1), (a_1,a_2), (a_2,a_1)\}$



Matrix
$$\begin{array}{c|cc} & a_1 & a_2 \\ \hline a_1 & 1 & 1 \\ a_2 & 1 & 0 \end{array}$$

Same information – completely describes R

IN-DEGREE of a $\quad |\{a' \in R \mid (a',a) \in R\}|$

OUT-DEGREE of a $\quad |\{a' \in R \mid (a,a') \in R\}|$

Edges coming in / going out in digraph representation

R - RELATIVE SETS

R relation from A to B

R-relative set of a $\qquad R(a) = \{ b \in B \mid aRb \}$

R-relative set of A_1

$$R(A_1) = \{ b \in B \mid \exists a \in A_1,\ aRb \}$$
$$= \bigcup_{a \in A_1} R(a)$$

__Ex__ Consider again $A = \{2, 3\}$, $B = \{1, \dots, 6\}$

aRb if $a \mid b$

$$R(2) = \{2, 4, 6\}$$
$$R(3) = \{3, 6\}$$
$$R(\{2,3\}) = \{2, 3, 4, 6\}$$

__THM__ Let R relation from A to B and $A_1, A_2 \subseteq A$.

1. If $A_1 \subseteq A_2$ then $R(A_1) \subseteq R(A_2)$
2. $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$
3. $R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$

Exercises:

- Read and understand proofs (Or better: Try yourself first!)
- Find example of when equality does __not__ hold in 3