KØBENHAVNS UNIVERSITET

Introduktion til diskret matematik og algoritmer: Problem Set 3

Due: Wednesday March 11 at 12:59 CET.
Submission: Please submit your solutions via *Absalon* as a PDF file. State your name and e-mail address close to the top of the first page. Solutions should be written in LaTeX or some other math-aware typesetting system with reasonable margins on all sides (at least 2.5 cm). Please try to be precise and to the point in your solutions and refrain from vague statements. Never, ever just state the answer, but always make sure to explain your reasoning. *Write so that a fellow student of yours can read, understand, and verify your solutions.* In addition to what is stated below, the general rules for problem sets stated on *Absalon* always apply.
Collaboration: Discussions of ideas in groups of two to three people are allowed—and indeed, encouraged—but you should always write up your solutions completely on your own, from start to finish, and you should understand all aspects of them fully. It is not allowed to compose draft solutions together and then continue editing individually, or to share any text, formulas, or pseudocode. Also, no such material may be downloaded from or generated via the internet to be used in draft or final solutions. Submitted solutions will be checked for plagiarism.
Grading: A score of 120 points is guaranteed to be enough to pass this problem set.
Questions: Please do not hesitate to ask the instructor or TAs if any problem statement is unclear, but please make sure to send private messages—sometimes specific enough questions could give away the solution to your fellow students, and we want all of you to benefit from working on, and learning from, the problems. Good luck!

1 (90 p) Let $A = \{1, 2, 3, 4\}$ and consider the following binary relations on A:

$$R = \{(2, 1), (3, 1), (3, 2), (4, 1), (4, 2), (4, 3)\}$$
$$S = \{(1, 1), (2, 2), (3, 3), (4, 4)\}$$
$$T = \{(1, 1), (1, 4), (2, 2), (2, 3), (3, 2), (3, 3), (4, 1), (4, 4)\}$$

1a For each of the relations above, determine whether it is

1. reflexive,
2. symmetric,
3. antisymmetric,
4. transitive.

Please make sure to explain, briefly but clearly, what these properties mean and why they are satisfied for a relation when they are. For any relation that fails to satisfy a property, make sure to provide a specific counterexample.

1b Which of the relations above, if any, are equivalence relations or partial orders? Please make sure to justify your answers.

2 (80 p) Recall that the Fibonacci numbers are defined as

$$
\begin{aligned}
F_1 &= 1 \\
F_2 &= 1 \\
F_n &= F_{n-1} + F_{n-2} && \text{for } n \geq 3.
\end{aligned}
$$

Prove that consecutive Fibonacci numbers F_{n+1} and F_n are relatively prime, and show that for $n \geq 2$ the Euclidean algorithm when run on F_{n+1} and F_n makes exactly $n-1$ function calls to determine that this is so (i.e., it reaches remainder 0 after exactly $n-1$ function calls).

3 (90 p) For a few years now the Copenhagen metropolitan area (including Lund) has had an unusually large number of researchers in computational complexity theory, and a team of such researchers have decided to submit a joint grant application to create the *Copenhagen Computational Complexity Centre* focusing on research in this scientific field. Since gender balance is a serious issue in computer science, a noteworthy aspect of the team of co-applicants is that the male professors Amir, Jakob, and Srikanth at the University of Copenhagen are balanced by the female professors Nutan and Paloma at the IT University of Copenhagen and Susanna at Lund University.

For the subproblems below, please make sure to answer not just with numbers but with more combinatorial-looking expressions, and to expand these expressions out to show that you understand the meaning of any notation used. Also make sure to explain how you reason to reach your answers.

3a Together with the application documents, the co-applicants are planning to enclose a group photo, and much thought has gone into how to choose the seating arrangement. All the researchers will be placed in a single row, but they have agreed that a great way to highlight the gender balance would be to make sure that male and female researchers alternate, so that every second person in the row is male or female, respectively. In how many different ways can the 6 researchers be arranged on the photo to satisfy this constraint?

3b Any serious research centre application these days should also identify a steering committee for the centre. After long deliberations, the co-applicants have decided that this committee should:

- consist of 4 persons all in all;
- include co-applicants representing all 3 partner institutions, i.e, the University of Copenhagen, the IT University of Copenhagen, and Lund University;
- have perfect gender balance, i.e., two male and two female members.

In how many different ways can the steering committee be composed?

4 (150 p) We have learned in class about matrix multiplication, but there is also a way of multiplying matrices (or vectors) by just a number, which is called *scalar multiplication*. To multiply a matrix A by a number c, we multiply each entry $a_{i,j}$ in the matrix with the number c so that

$$c \cdot A = c \cdot \begin{pmatrix} a_{1,1} & a_{1,2} & \cdots & a_{1,n} \\ a_{2,1} & a_{2,2} & \cdots & a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m,1} & a_{m,2} & \cdots & a_{m,n} \end{pmatrix} = \begin{pmatrix} c \cdot a_{1,1} & c \cdot a_{1,2} & \cdots & c \cdot a_{1,n} \\ c \cdot a_{2,1} & c \cdot a_{2,2} & \cdots & c \cdot a_{2,n} \\ \vdots & \vdots & \ddots & \vdots \\ c \cdot a_{m,1} & c \cdot a_{m,2} & \cdots & c \cdot a_{m,n} \end{pmatrix}$$

is the result of the scalar multiplication.

An intriguing phenomenon that sometimes arises is that for some pairs of matrices and vectors matrix multiplication and scalar multiplication give the same result. As an example of this, we have

$$\begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \\ 2 \\ 2 \end{pmatrix} = 2 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} \tag{1}$$

and another example is

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ 3 \\ 3 \\ 3 \\ 3 \end{pmatrix} = 3 \cdot \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{pmatrix} . \tag{2}$$

When for a square matrix A there is a vector $\vec{x}$ (with not all entries equal to zero) and a number λ such that

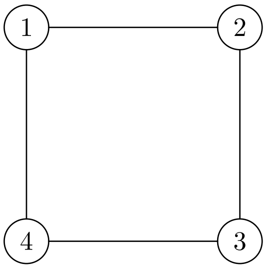
$$A \cdot \vec{x} = \lambda \cdot \vec{x} \ , \tag{3}$$

such a vector $\vec{x}$ is called an *eigenvector* of the matrix A with *eigenvalue* λ. We see that the matrix in (1) has the all-ones vector as eigenvector with eigenvalue 2, and the matrix in (2) also has the all-ones vector as eigenvector but with eigenvalue 3. In this problem, we want to develop our skills of matrix multiplication by studying such eigenvalues and eigenvectors, and also to establish some non-obvious connections between eigenvalues and -vectors on the one hand and graphs on the other hand.

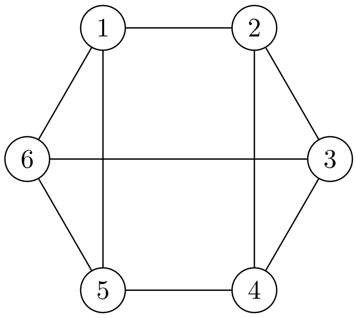
We say that an undirected, simple graph is *d-regular* if every vertex is incident to exactly d edges, or, in other words, has exactly d neighbours. For two illustrations of this, the graph in Figure 1a is 2-regular and the graph in Figure 1b is 3-regular. Now, a fun fact is that the 2-regular graph in Figure 1a has the adjacency matrix in (1), which has eigenvalue 2, and the 3-regular graph in Figure 1b has the adjacency matrix in (2) with eigenvalue 3. Your task is to show that this is not a coincidence, and to derive some other interesting connections between d-regular graphs and the eigenvalues and -vectors of their adjacency matrices.

4a (10 p) If $\vec{x}$ is an eigenvector of A corresponding to some eigenvalue λ, then so is $c \cdot \vec{x}$ for any $c \neq 0$. Explain why this is so.

Hint: This should be easy—just use the definitions above.



(a) Graph with adjacency matrix as in (1).



(b) Graph with adjacency matrix as in (2).

Figure 1: Two example regular graphs in Problem 4.

4b (20 p) Show that if G is a d-regular graph, then its adjacency matrix A_G always has d as an eigenvalue.

4c (20 p) Show that if G is a d-regular graph, then its adjacency matrix A_G can never have an eigenvalue λ such that $|\lambda| > d$.

Hint: Suppose that there is an eigenvector $\vec{x}$ with eigenvalue λ such that $|\lambda| > d$. Rescale the entries in $\vec{x}$ by some $c \neq 0$ so that the largest entry has value 1 and all other entries have absolute value at most 1. Consider the product $A_G \cdot \vec{x}$, focus on a largest entry in $\vec{x}$, and argue by contradiction.

4d (30 p) Show that if the d-regular graph G is connected, so that there is a path between any two vertices u and v in $V(G)$, then any eigenvector $\vec{x}$ of the adjacency matrix A_G with eigenvalue d must have all entries equal (i.e., $\vec{x}$ is the all-ones vector or some multiple of the all-ones vector).

Hint: Suppose $\vec{x}$ is an eigenvector of A_G with eigenvalue d in which not all entries are equal. Rescale the entries in $\vec{x}$ by some $c \neq 0$ so that the largest entry has value 1 and all other entries have absolute value at most 1, and consider the product $A_G \cdot \vec{x}$.

4e (30 p) Show that if the d-regular graph G is *not* connected, so that there exist two vertices u and v in $V(G)$ with no path between them, then there is in fact is an eigenvector $\vec{x}$ of the adjacency matrix A_G with eigenvalue d in which not all entries are equal.

Hint: Consider the different connected components of G and use them to define interesting vectors for which d is an eigenvalue.

4f (40 p) We say that an undirected graph $G = (V, E)$ is *bipartite* if there is a bipartition $V = V_1 \dot{\cup} V_2$ (where $\dot{\cup}$ denotes *disjoint union*, so that $V_1 \cup V_2 = V$ but $V_1 \cap V_2 = \emptyset$) such that any edge in G has one endpoint in V_1 and one endpoint in V_2, but there are no edges connecting vertices in V_1 to each other or vertices in V_2 to each other. (Just to give examples for this definition, it is not hard to verify that the graph in Figure 1a is bipartite but that the graph in Figure 1b is not.)

Show that if the d-regular graph G is bipartite and the adjacency matrix A_G has an eigenvector $\vec{x}$ with eigenvalue λ, then $-\lambda$ is also an eigenvalue for A_G.

Hint: Consider the bipartition $V = V_1 \dot{\cup} V_2$ and use it to modify the eigenvector $\vec{x}$ in some interesting way. (This connection between bipartiteness and negated eigenvalues is actually an if and only if—it holds that $-\lambda$ is also an eigenvalue only if G is bipartite—but you definitely do not need to prove this.)

Note that Problems 4d and 4e say that a d-regular graph G is connected if and only if the only eigenvectors of the adjacency matrix A_G with eigenvalue d are multiples of the all-ones vector, and Problem 4f says that G is bipartite only if the eigenvalues of A_G are symmetric with respect to 0. We are in fact only scratching the surface here, in that the eigenvalues of A_G can tell us much more about the properties of G. Perhaps the most important connection is that the second largest eigenvalue λ_2 in absolute value is a measure of how well-connected the graph is—if the gap between d and $|\lambda_2|$ is large, then G is an *expander graph* in which there are short paths between any two vertices. These and other highly nontrivial facts are further studied in *spectral graph theory.*