KØBENHAVNS UNIVERSITET

Introduktion til diskret matematik og algoritmer: Problem Set 3

Due: Wednesday March 13 at 9:59 CET.
Submission: Please submit your solutions via *Absalon* as a PDF file. State your name and e-mail address close to the top of the first page. Solutions should be written in LaTeX or some other math-aware typesetting system with reasonable margins on all sides (at least 2.5 cm). Please try to be precise and to the point in your solutions and refrain from vague statements. Make sure to explain your reasoning. *Write so that a fellow student of yours can read, understand, and verify your solutions.* In addition to what is stated below, the general rules for problem sets stated on *Absalon* always apply.
Collaboration: Discussions of ideas in groups of two to three people are allowed—and indeed, encouraged—but you should always write up your solutions completely on your own, from start to finish, and you should understand all aspects of them fully. It is not allowed to compose draft solutions together and then continue editing individually, or to share any text, formulas, or pseudocode. Also, no such material may be downloaded from or generated via the internet to be used in draft or final solutions. Submitted solutions will be checked for plagiarism.
Grading: A score of 120 points is guaranteed to be enough to pass this problem set.
Questions: Please do not hesitate to ask the instructor or TAs if any problem statement is unclear, but please make sure to send private messages—sometimes specific enough questions could give away the solution to your fellow students, and we want all of you to benefit from working on, and learning from, the problems. Good luck!

1 (80 p) Decide for each of the propositional logic formulas below whether it is a tautology or a contradiction. If neither of these cases apply, then present a satisfying assignment for the formula. For this problem, it is sufficient (and necessary) for a full score to justify all your answers by presenting truth tables for all the subformulas, but you are also highly encouraged to *explain* why your answers are correct (and good explanations could compensate fully for minor slips in the truth tables).

(Note that logical not $\neg$ is assumed to bind harder than the binary connectives, but other than that all formulas are fully parenthesized for clarity. We write $\rightarrow$ for logical implication and $\leftrightarrow$ for equivalence.)

1a (20 p) $((p \rightarrow q) \rightarrow r) \rightarrow ((p \land q) \rightarrow r)$

Solution: For Problem 1a we will both present the truth table and explain why the answer is correct, but for the other formulas we will be a bit lazy in our solution sketches and skip the truth tables, spending more effort on providing good explanations (but note that students need to provide truth tables for all formulas).

The truth table for the formula in Problem 1a is as described below.

p	q	r	$p \to q$	$(p \to q) \to r$	$p \wedge q$	$(p \wedge q) \to r$	$((p \to q) \to r) \to ((p \wedge q) \to r)$
$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$
$\bot$	$\bot$	$\top$	$\top$	$\top$	$\bot$	$\top$	$\top$
$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\bot$	$\top$	$\top$
$\bot$	$\top$	$\top$	$\top$	$\top$	$\bot$	$\top$	$\top$
$\top$	$\bot$	$\bot$	$\bot$	$\top$	$\bot$	$\top$	$\top$
$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\top$
$\top$	$\top$	$\bot$	$\top$	$\bot$	$\top$	$\bot$	$\top$
$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$	$\top$

From the rightmost column in this table we can see that the formula is a tautology. Let us now explain why this is so.

An implication $A \to B$ is false if and only if $A = \top$ and $B = \bot$. With this in mind, for our formula to be falsified it is required that $((p \wedge q) \to r)$ evaluates to false and $(p \to q) \to r$ to true. For the first of these formulas to be false we must have $r = \bot$, $p = \top$, $q = \top$. If we insert these values into the second formula we get $(\top \to \top) \to \bot$, which means that it will also be false. Thus, there is no way to falsify the whole formula, meaning that it is a tautology.

1b (20 p) $((p \to q) \wedge (r \to s)) \leftrightarrow ((p \wedge r) \to (q \wedge s))$

Solution: This formula has a falsifying assignment $p = \top, r = \bot, q = \bot, s = \top$ and a satisfying assignment $p = \top, r = \top, q = \top, s = \top$. Hence, it is neither a tautology nor a contradiction.

1c (20 p) $((p \wedge \neg r) \vee (q \wedge \neg r)) \to ((p \vee q) \to r)$

Solution: This is again neither a tautology nor a contradiction. A satisfying assignment is $p, q, r = \top$ and a falsifying assignment is $p, q = \top$, $r = \bot$.

1d (20 p) $(p \to (q \vee r)) \vee (\neg(\neg p \vee q) \wedge \neg r)$

Solution: If we negate the formula $p \to (q \vee r)$ and rewrite it using the rules we have learned in class, then we get

$$\begin{aligned}
\neg(p \to (q \vee r)) &\iff \neg(\neg p \vee (q \vee r)) && [\text{since } A \to B \iff \neg A \vee B] \\
&\iff \neg((\neg p \vee q) \vee r) && [\text{since } A \vee (B \vee C) \iff (A \vee B) \vee C] \\
&\iff \neg(\neg p \vee q) \wedge \neg r && [\text{since } \neg(A \vee B) \iff \neg A \wedge \neg B]
\end{aligned}$$

and this final subformula exactly matches the second part of the formula we are interested in. This means that the whole formula is eqivalent to a formula of the form $A \vee \neg A$, which is a tautology.

2 (100 p) Recall that a graph $G = (V, E)$ consists of a set of vertices V connected by edges E, where every edge is a pair of vertices. If there is an edge (u, v) between two vertices u and v, then the two vertices are said to be *neighbours* and are both *incident* to the edge. We say that a sequence of edges (v_1, v_2), (v_2, v_3), (v_3, v_4), $\ldots$, (v_{k-1}, v_k), in E is a *path* from v_1 to v_k.

In this problem, we wish to express properties of graphs in both natural language and predicate logic, and to translate between the two forms. We do this as follows:

- The universe is the set of vertices V of G.
- The binary predicate $E(u, v)$ holds if and only if there is an edge from u to v in G.

- The unary predicate $S(v)$ is used to identify a subset of vertices $S = \{v \mid v \in V,\ S(v) \text{ is true}\}$ for which some property might or might not hold.

Below you find five graph properties written as predicate logic formulas and six graph properties defined in natural language. Most of the predicate logic formulas have corresponding natural language definitions, but not all.

Your task is to determine which of the predicate logic formulas (a), ..., (e) match which—if any—of the natural language definitions (1), ..., (6). Please make sure to motivate your answers clearly.

Predicate Logic Formulas:

(a) $\forall u \forall v \big(E(u,v) \rightarrow E(v,u)\big)$

(b) $\forall u \forall v \big(E(u,v) \rightarrow S(u) \vee S(v)\big)$

(c) $\forall u \forall v \exists w \big(S(w) \wedge (E(u,w) \vee E(v,w))\big)$

(d) $\forall u \forall v \big(E(u,v) \rightarrow ((S(u) \wedge \neg S(v)) \vee (\neg S(u) \wedge S(v)))\big)$

(e) $\forall u \forall v \big((u \neq v \wedge S(u) \wedge S(v)) \rightarrow E(u,v)\big)$

Natural Language Definitions:

(1) S is a *dominating set* in G, i.e., every vertex v in the graph either is in S or is a neighbour of a vertex in S.

(2) S is a *clique* in G, i.e., a set of vertices that are all neighbours with each other.

(3) The graph G is *undirected*.

(4) The graph G is *connected*.

(5) S is a *vertex cover* in G, i.e., for every edge at least one of the vertices incident to it is in S.

(6) The graph G is *bipartite* with bipartition $(S, V \setminus S)$, i.e., all edges go between S and $V \setminus S$.

Solution: Formula (a) matches the description (3) of the graph being undirected. The formulas says that whenever $E(u,v)$ holds, then $E(v,u)$ also holds. This is to say that the ordering of the vertices in an edge does not matter, but that "the edge exists in both direction"—i.e., that the graph is undirected.

Formula (b) matches the description (5) of vertex cover. The formula says that if there is an edge between two vertices u and v, then at least one of these vertices is in S (i.e., the set defined by the unary predicate $S(\cdot)$). This is the definition of S being a vertex cover.

Formula (c) says that for any pair of vertices u and v, there is a vertex w that is in the set S and that is a neighbour of either u or v. This does not match any of the natural language descriptions.

Formula (d) matches the description (6) of bipartiteness. It says that if there is an edge between u and v, then either u is in S and v is not, or the other way round. That is, exactly one of the vertices is any edge is in S, which shows that $(S, V \setminus S)$ is indeed a bipartition of the graph.

Formula (e) matches the description (2) of clique. It says that if two distinct vertices u and v are both in S, then there has to be an edge between them. This is exactly the condition for vertices in a clique.

3 (60 p) When Jakob is socializing in the evenings after conferences in the United States, he tries to convince his colleagues that poker should really be played with a *republican deck of cards*, i.e., with cards of all ranks 2–10 plus the aces, but without kings, queens, or jacks.

3a Suppose that you are dealt 5 cards from a perfectly shuffled republican deck of cards. What is the probability of getting a *full house*, i.e., three of a kind (three cards of the same rank) with a pair (two other cards of the same rank)? Explain clearly how you obtain the expression in your answer.

Solution: In a "republican deck of cards" we have 10 ranks and 4 cards in each rank, for a total of 40 cards in the deck. The total number of hands of 5 cards is therefore $\binom{40}{5}$. All of these hands are equally likely (assuming a perfectly shuffled deck of cards).

In order to determine the probability of getting a full house, we therefore only need to count all (distinct) ways of getting a full house, and then divide by the total number of hands, to get the probability. In order to get a full house, we first choose the first rank in one of 10 ways, and then we have to draw 3 cards of this rank, which can be done in $\binom{4}{3}$ ways. The second rank can be chosen in 9 ways, and two cards of that rank can be drawn in $\binom{4}{2}$ ways. Hence, the total number of full houses is $10 \cdot \binom{4}{3} \cdot 9 \cdot \binom{4}{2}$, and the probability of getting a full house when 5 cards are drawn randomly is

$$\frac{10 \cdot \binom{4}{3} \cdot 9 \cdot \binom{4}{2}}{\binom{40}{5}} .$$

3b It is a sad fact that so far Jakob has had quite limited success in convincing colleagues that a republican deck of cards is the true American way of playing poker. Jakob believes this is because the colleagues have all been brainwashed by the big casinos in Las Vegas, and that this in turn is because with a republican deck of cards the probability for the players of getting a strong hand (like a full house) would be higher than with a standard deck of cards, making it more likely that casinos might lose money.

Ignoring the wider ramifications of the worldwide conspiracy that Jakob alleges to have discovered, is the factual claim true that the probability of getting, e.g., a full house is higher with a republican deck of cards than with a standard deck of cards when being dealt 5 cards from a shuffled deck? (Note that you should not have to do very precise calculations to be able to determine whether this claim is true or not, and a clear intuitive explanation of which answer should be the correct one can give generous partial credits.)

Solution: Jakob actually happens to be right this time (not necessarily about the worldwide conspiracy, but about the probability). In a standard deck of cards, the probability of getting a full house is

$$\frac{13 \cdot \binom{4}{3} \cdot 12 \cdot \binom{4}{2}}{\binom{52}{5}} .$$

If we compare this with the probability above for a "republican deck of cards", then we see that the denominator grows much more when going from a republican deck to a full deck than does the numerator. Phrased differently, the number of ways of getting full houses is certainly larger with a full, standard deck of cards, but the number of getting all kinds of crazy 5-card hands increases even more with a standard deck of cards. (This can of course be verified also by performing the calculations comparing the two probabilities.)

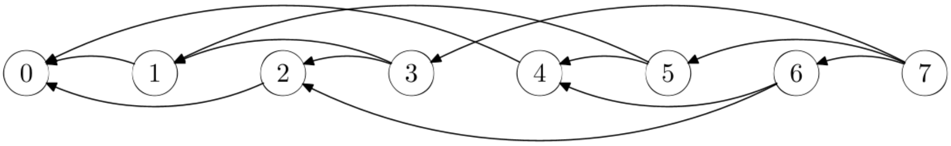

Figure 1: Directed graph D_S representing relation S in Problem 4.

4 (60 p) Consider the relation S described by the directed graph D_S in Figure 1.

4a (10 p) Write down the matrix representation M_S of the relation S and describe briefly but clearly how you constructed this matrix.

Solution: The matrix representation of the relation S is an 8×8 matrix (with indexing starting from row and column 0, just to be consistent with the names of the elements in the relation) where there is a 1 in position (i, j) if $(i, j) \in S$ and a 0 in this position otherwise. Looking at the directed graph representation D_S in Figure 1, this means that there should be a 1 in position (i, j) if and only if there is a directed edge from i to j in D_S. This yields the matrix

$$M_S = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \end{pmatrix}$$

(where we note, in particular, that row 0 is an all-zero row in the matrix, since there are no outgoing edges from the vertex labelled 0 in D_S).

4b (10 p) Let us write I to denote the inverse of the relation S. What is the matrix representation of I? Write it down and explain how you constructed it.

Solution: For the inverse relation I of S, we have that $(i, j) \in I$ if and only if $(j, i) \in S$. This means that we could generate the matrix for I by first reversing all edges in Figure 1 and then proceeding as in our solution to Problem 4a. However, a faster way of getting the same result is to use that the matrix representation of the inverse of a relation is the transpose of the matrix representing the original relation. Therefore, we have that

$$M_I = \begin{pmatrix} 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

is the matrix representation of the inverse relation I.

4c (10 p) Now let T be the transitive closure of the relation I. What is the matrix representation of T? Write it down and explain how you constructed it.

Solution: To obtain the transitive closure T of the relation I, we can proceed as follows:

1. Start by setting $T' = T = I$.
2. Go over all triples (i, j, k) such that $(i, j) \in I$ and $(j, k) \in I$, and add (i, k) to T' since the elements i and k should also be related by transitivity.
3. If new pairs were added in step 2, so that $T' \neq T$, then set $T = T'$ and go to step 2 again. Otherwise T is the transitive closure.

Looking at the directed graph representation D_S in Figure 1, but reversing the edges so that we get the graph for I, a pair (i, k) should be in the transitive closure precisely when there is a path from i to k in the graph. Starting from each element $i = 0, 1, \ldots, 7$ and writing down which other vertices are reachable from i yields the matrix

$$M_T = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

for the transitive closure T of I.

4d (20 p) Finally, let R be the reflexive closure of the relation T. Can you explain in words what the relation R is by describing how it can be interpreted? (In particular, is it similar to anything we have discussed during the course? Be as specific in your reply as you can.)

Solution: Although the problem statement does not ask for it, let us write down the matrix representation

$$M_R = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

of the reflexive closure R of T. This is the subset relation on the universe of three elements. If we write all numbers in binary, each number can be viewed as an indicator vector for a subset, so that, e.g.:

- $0 = (000)_2$ corresponds to $\emptyset$,
- $1 = (001)_2$ corresponds to $\{1\}$,

- $2 = (010)_2$ corresponds to $\{2\}$,
- $3 = (011)_2$ corresponds to $\{1, 2\}$,
- $4 = (100)_2$ corresponds to $\{3\}$,
- et cetera all the way up to $7 = (111)_2$ corresponding to $\{1, 2, 3\}$.

With this interpretation, we can verify that the relation T in Problem 4c is the strict subset relation, and taking the reflexive closure gives the relation where every subset is also related to itself.