KØBENHAVNS UNIVERSITET

Introduktion til diskret matematik og algoritmer: Exam April 2, 2025

Submission: Please write your solutions with ample margins on all sides, and make sure your handwriting is legible. *Start your solution of every new problem on a new page. Please mark every page with your name, exam number or something else that uniquely identifies your exam,* so that it is easy to see for every page which exam submission it is part of.
Exposition: Please try to be precise in your solutions and refrain from vague statements. Never, ever just state an answer, but always make sure to *explain why* the answer is what it is. Provide clear references to any facts in the course literature used. *Write your solutions in such a way that a fellow student of yours could read, understand, and verify your solutions.*
Collaboration: All problems should be solved individually. No communication or collaboration is allowed, and solutions will be checked for plagiarism.
Reference material: Textbooks and handwritten notes (including lecture notes) are allowed. Other typewritten material, including (but not limited to) problem sets or previous exams with solutions, is not allowed. Please note that you deviate from definitions and algorithm descriptions in the course material at your own risk! If, however, slight variations of algorithms were presented in the textbooks and/or in class, such minor details do not matter.
Grading: A score of 220 points is guaranteed to be enough to pass the exam.
About the problems: Note that the problem are of quite different types, and are *not sorted in increasing order of difficulty. Please read through the whole exam first,* before you start working on any single problem, so that you can plan which order of dealing with the problems makes most sense to you. Note that this is a fairly large exam, and you can get a top grade without solving all problems. Also, partial answers to problems can sometimes give substantial amounts of points. ***Good luck!***

1 (50 p) This problem is about different representations of integers.

1a Write the binary number $(110)_2$ in decimal notation.

1b Write the decimal number 110 in binary notation.

1c Write the octal number $(2025)_8$ in decimal notation.

2 (50 p) Use the algorithm we have learned for determining $d = \gcd(m, n)$ for the numbers below, showing details of all function calls made, and then express d as a linear combination of m and n.

2a $m = 38$ and $n = 14$.

2b $m = 117$ and $n = 69$.

3 (60 p) Provide formal proofs of the following claims using proof techniques that we have learned during the course.

3a Define $a_1 = 1$ and $a_n = 2a_{n-1} + 1$ for $n \geq 2$. Prove that for all positive integers $n \in \mathbb{N}^+$ it holds that $a_n = 2^n - 1$.

3b Prove that for all non-negative integers $n \in \mathbb{N}$ it holds that $3 \mid 4^n + 5$.

4 (90 p) Let $A = \{1, 2, 3, 4\}$ and consider the following binary relations on A:

$$
\begin{aligned}
R &= \{(2,1), (3,1), (3,2), (4,1), (4,2), (4,3)\} \\
S &= \{(1,1), (2,2), (3,3), (4,4)\} \\
T &= \{(1,1), (1,4), (2,2), (2,3), (3,2), (3,3), (4,1), (4,4)\}
\end{aligned}
$$

4a For each of the relations above, determine whether it is

1. reflexive,
2. symmetric,
3. antisymmetric,
4. transitive.

Please make sure to explain, briefly but clearly, what these properties mean and why they are satisfied for a relation when they are. For any relation that fails to satisfy a property, make sure to provide a specific counterexample.

4b Which of the relations above, if any, are equivalence relations or partial orders? Please make sure to justify your answers.

5 (60 p) For each of the propositional logic formulas below, determine whether it is a tautology or not. If the formula is not a tautology, show how to add a single connective to make it into a tautology. Please make sure to justify your answers (e.g., by presenting truth tables, or by using rules for rewriting logic formulas that we have learned in class).

5a $\neg\big((p \to q) \lor r\big) \to \big((\neg q \land \neg r) \land p\big)$

5b $\big((p \land q) \to r\big) \leftrightarrow \big((q \lor r) \lor \neg p\big)$

6 (80 p) Recall that the Fibonacci numbers are defined as

$$
\begin{aligned}
F_1 &= 1 \\
F_2 &= 1 \\
F_n &= F_{n-1} + F_{n-2} && \text{for } n \geq 3.
\end{aligned}
$$

Prove that consecutive Fibonacci numbers F_{n+1} and F_n are relatively prime, and show that for $n \geq 2$ the Euclidean algorithm when run on F_{n+1} and F_n makes exactly $n-1$ function calls to determine that this is so (i.e., it reaches remainder 0 after exactly $n-1$ function calls).

7 (90 p) For a few years now the Copenhagen metropolitan area (including Lund) has had an unusually large number of researchers in computational complexity theory, and a team of such researchers have decided to submit a joint grant application to create the *Copenhagen Computational Complexity Centre* focusing on research in this scientific field. Since gender balance is a serious issue in computer science, a noteworthy aspect of the team of co-applicants is that the male professors Amir, Jakob, and Srikanth at the University of Copenhagen are balanced by the female professors Nutan and Paloma at the IT University of Copenhagen and Susanna at Lund University.

For the subproblems below, please make sure to answer not just with numbers but with more combinatorial-looking expressions, and to expand these expressions out to show that you understand the meaning of any notation used. Also make sure to explain how you reason to reach your answers.

7a Together with the application documents, the co-applicants are planning to enclose a group photo, and much thought has gone into how to choose the seating arrangement. All the researchers will be placed in a single row, but they have agreed that a great way to highlight the gender balance would be to make sure that male and female researchers alternate, so that every second person in the row is male or female, respectively. In how many different ways can the 6 researchers be arranged on the photo to satisfy this constraint?

7b Any serious research centre application these days should also identify a steering committee for the centre. After long deliberations, the co-applicants have decided that this committee should:

- consist of 4 persons all in all;
- include co-applicants representing all 3 partner institutions, i.e, the University of Copenhagen, the IT University of Copenhagen, and Lund University;
- have perfect gender balance, i.e., two male and two female members.

In how many different ways can the steering committee be composed?

8 (90 p) When Jakob has international visitors, he needs to give them travel directions from Kastrup Airport to the Department of Computer Science (DIKU) at Universitetsparken. Jakob is aware of the following relevant public transport options in Copenhagen with travel times as stated (in either direction, and with time for switching transport mode included):

- Between Kastrup Airport and Kongens Nytorv by metro: 13 minutes.
- Between Kastrup Airport and København H by train: 20 minutes.
- Between København H and Nørreport by train: 3 minutes.
- Beween København H and Kongens Nytorv by metro: 4 minutes.
- Between Kongens Nytorv and Vibenshus Runddel by metro: 10 minutes.
- Between Vibenshus Runddel and Universitetsparken by foot: 9 minutes.
- Between Nørreport and Universitetsparken by bus: 8 minutes.

What is not so clear to Jakob is how he should use this information to find as fast a route as possible between the airport and DIKU to suggest to his visitors.

8a Help Jakob by modelling this problem as a graph. Explain what the vertices and edges represent and what other information you need to add to the graph. Make sure to show concretely what graph you obtain for Jakob's problem above.

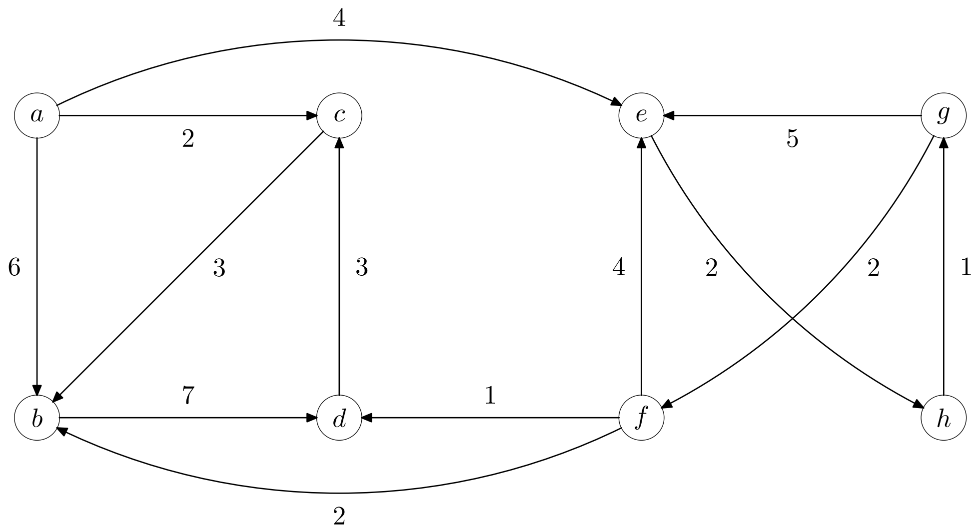

Figure 1: Graph for Problem 9

8b Propose a suitable graph algorithm to solve Jakob's problem. Explain what this algorithm is and why it is the right choice for this problem. Make a dry-run of the algorithm and explain the relevant steps in the execution (similarly to what has been done in class and in the lecture notes).

If your algorithm uses any auxiliary data structures, then explain in detail for the first two vertices processed how these data structures change. For the rest of the algorithm execution, just report what the relevant outputs of the data structures are without going into any details.

What travel directions for Jakob's visitors does your algorithm produce?

9 (120 p) Consider the graph in Figure 1 and the following ordered sequences listing the vertices in this graph:

1. a, c, e, b, h, g, f, d.
2. a, b, c, e, d, h, g, f.
3. a, c, b, e, h, g, f, d.
4. a, b, d, c, e, h, g, f.

For each of the sequences above, determine whether it can be the result of:

(a) a breadth-first search with vertices listed in order of visits;
(b) a depth-first search with vertices listed in order of discovery;
(c) a shortest-path computation with vertices listed in the order they are removed from the priority queue.

Each sequence above is the output of at most one of the algorithms, but since there are four sequences there could be a sequence that cannot be produced by any of the algorithms.

Partial credit is given for matching algorithms and vertex sequences correctly. For full credit, you need to give an overview of how and why the algorithms process the vertices in the given order (such as explaining the order of recursive calls, edges processed, or similar), or why none of the proposed algorithms could yield the sequence in question, but you do not have to provide detailed information about any auxiliary data structures used in the algorithms.