General Plan

In the first lecture of the week, we will wrap up the part of the course that deals with the important technique of **mathematical induction.**

Next up on our agenda is to talk more formally about **mathematical logic** and **proofs**. We will show how propositional logic can be used to structure and analyse mathematical arguments. Next, we will introduce predicate logic, which provides a convenient language for formalizing properties of objects (like "the integer n is a prime" or "the matrix A is invertible"). After this, we will move on to a discussion of proof techniques such as direct proofs, proofs by contraposition, and proof by contradiction. (There is obviously lots of fancy terminology flying around here—the important thing is not that you memorize all of it, but that you understand what we are doing.)

Note that mathematical logic is nothing magical—it is just a way to formalize well-known logical concept, and to describe in a clear, structured way different ways of reasoning that we already know to be valid. Indeed, in many cases common sense is sufficient to construct a valid mathematical argument. However, when mathematical statements become increasingly complicated it can be helpful to have a formal toolbox to keep track of the reasoning and to be sure that everything is correct. (As an illustration of this, already the proof we did of Theorem 4 on page 22 of KBR during the second week was not entirely trivial.)

We will also introduce some pieces of mathematical notation that allow us to make very precise statements in a brief, efficient way. For instance, we can describe that an asymptotically positive sequence (a_n) is $O(1)$ by writing

$$\exists c > 0 \; \exists k > 0 \; \forall n \geq k \;\; a_n \leq c \;.$$

In plain English, this can be read as *"there exist c greater than 0 and k greater than 0 such that for all n greater than or equal to k it holds that a_n is less than or equal to c."* This is still slightly informal, in that we assume that the reader understands from context that c is any positive real number whereas k and n are integers. If we wanted to be even more precise, then we could write

$$\exists c \in \mathbb{R}^+ \; \exists k \in \mathbb{Z}^+ \; \forall n \in \mathbb{Z}^+ (n \geq k \Rightarrow a_n \leq c) \;.$$

However, this second, more detailed version adds a lot of clutter, and sometimes it is preferable to be slightly more concise when the extra details are understood from context. There are no hard and fast rules here, but we will try during the lectures to learn by example what a suitable level of detail is. Also, when in doubt, it is never wrong to add extra details for clarity. At the end of the day, the reason we use mathematical notation is in order to communicate as clearly as possible.

As most other material in this course, our discussion of logic and proofs will involve fundamental topics that will reappear in later courses, and so you will do yourself great service by making sure to learn it inside out already now.

The course will cover KBR Sections 2.1–2.4 in detail. **Please make sure to read these sections before the lecture**, since much of it is better suited for self-study than for lecturing. Once we have covered all of it, then read all of it again and check that you have digested the material. Section 2.5 is not as important, and Section 2.6 we will not touch upon at all, but this is still useful reading.

Reading Instructions

- KBR Sections 2.1–2.4 in depth.

- KBR Section 2.5 is part of the course, but is not as important as Sections 2.1–2.4 and we will probably not be able to cover it in class.

- KBR Section 2.6 is not required to pass the course, but is recommended reading.

Exercises

Note that as for previous weeks, there are quite a few exercises suggested below. One question that tends to come up is how many of these exercises you are supposed to do. The answer is that this depends on how many exercises you yourself need to solve in order to digest the material.

If you feel that you understood what was covered in class and/or what you read in the textbook, and if the exercises seem straightforward, then it is fine to just do some of them (where I good idea would be to try to focus on the one that seem hardest to you). If some of the material seems harder, though, then it is a good idea to do more exercises until you really understand what is going on.

At the obvious risk of repeating myself, you will do yourself a great service by learning the material on this course well, because much (or most?) of what you will do in later courses will build on this material in one way or another.

1. More exercises on induction 2.4.10, 2.4.17, 2.4.22.

2. If you wish to check your basic understanding, solve KBR exercises 2.1.1, 2.1.2, 2.1.8.

3. If you wish to check your basic understanding, solve KBR exercises 2.1.15, 2.1.16, 2.1.18.

4. To check your basic understanding, solve KBR exercises 2.1.27, 2.1.28.

5. Check De-Morgan's laws by computing and comparing the truth tables of the left-hand-side and right-hand-side in each of the following

   (a) $\sim (p \vee q) \equiv (\sim p) \wedge (\sim q)$

   (b) $\sim (p \wedge q) \equiv (\sim p) \vee (\sim q)$

6. To check your basic understanding, solve KBR exercises 2.2.10, 2.2.11.

7. To get slightly more interesting challenges, solve KBR exercises 2.2.13, 2.2.15.

8. Let $\texttt{xor}$ (also known as *exclusive or*) be the logical connective with the following truth table:

P	Q	$P \texttt{ xor } Q$
T	T	F
T	F	T
F	T	T
F	F	F

   Find an equivalent expression for

   $$P \texttt{ xor } Q$$

   using only $\wedge$(and), $\vee$(or), and $\sim$(not). You can use P and Q any number of times and indicate the order of the operations using parentheses. Verify your answers by computing the truth table of your expression. (See Example 5 in KBR 2.1 for an example.)

9. Let $f, g : \mathbb{R}^+ \to \mathbb{R}$ be asymptotically positive functions. We can express the definition of "$f(x)$ is $O(g(x))$" using logical connectives and quantifiers as

$$\exists c > 0 \ \exists x_0 \in \mathbb{R}^+ \ \forall x \geq x_0 \ f(x) \leq cg(x) \qquad (1)$$

Recall that we defined "f is $o(g)$" as "for any constant $c > 0$ we can find $x_0 \in \mathbb{R}^+$ such that
$$f(x) < cg(x)$$
for all $x \geq x_0$."

(a) Express the above definition of "$f(x)$ is $o(g)$" using logical connectives and quantifiers.

(b) Write the negation of the proposition from the previous part and simplify it so that it does not contain the negation ($\sim$).
Hint: Theorem 3 from KBR 2.2 can be helpful here.

(c) Write a sentence in English that corresponds to your statement from the previous part.

10. If you wish to solve some more exercises to test basic understanding, solve KBR exercises 2.2.6, 2.2.21, 2.2.27.

11. To develop your mastery of the proof techniques that we learned in class, make sure to solve (and understand) a generous selection of KBR exercises 2.3.18, 2.3.23, 2.3.24, 2.3.27, 2.3.31, 2.3.34.

A Couple of Extra Challenging Exercises

(1) [*] Recall that we have discussed in class the *least number principle*, which can be formally stated as follows.

> If S is a non-empty set of non-negative integers, then there is a minimum element $x \in S$ (i.e., such that for all $y \in S$ it holds that $y \geq x$).

- Note that it is essential that we are talking about (a) integers that are (b) non-negative. Give examples showing that the least-number principle does not hold if if S is any non-empty set of integers, or if S is any set of non-negative rational numbers.

- Prove that the least number principle implies the induction principle. That is, we can prove a theorem saying that if the least number principle is true, then the induction principle (repeated below for your convenience) is also true.

> If a collection of statements $P(n)$, where n takes the values from a set of non-negative integers $\{n_0, n_0+1, n_0+2, \ldots\}$, satisfy that
>
> $$P(n_0) \text{ is true}$$
>
> and that for all $n \geq n_0$, we have that
>
> $$\text{If } P(n) \text{ is true, then } P(n+1) \text{ is true}$$
>
> then $P(n)$ is true for all $n \geq n_0$.

Hint: Let S be the set of all n for which $P(n)$ fails to hold.

- Prove that the induction principle implies the least number principle.

    Hint: Let S be a set of non-negative integers without a minimum element. Let $P(n)$ be the property that $m \notin S$ for all $m \leq n$ and use induction to prove that the set S is empty.

(2) [**] Find the mistake in the proof given in KBR Example 2.4.7 (page 73). *[This is a tricky one, since what is claimed in the example is definitely true, but the proof is taking a shortcut.]*