KØBENHAVNS UNIVERSITET

Introduktion til diskret matematik og algoritmer: Problem Set 3

Due: Wednesday March 13 at 9:59 CET.
Submission: Please submit your solutions via *Absalon* as a PDF file. State your name and e-mail address close to the top of the first page. Solutions should be written in LaTeX or some other math-aware typesetting system with reasonable margins on all sides (at least 2.5 cm). Please try to be precise and to the point in your solutions and refrain from vague statements. Make sure to explain your reasoning. *Write so that a fellow student of yours can read, understand, and verify your solutions.* In addition to what is stated below, the general rules for problem sets stated on *Absalon* always apply.
Collaboration: Discussions of ideas in groups of two to three people are allowed—and indeed, encouraged—but you should always write up your solutions completely on your own, from start to finish, and you should understand all aspects of them fully. It is not allowed to compose draft solutions together and then continue editing individually, or to share any text, formulas, or pseudocode. Also, no such material may be downloaded from or generated via the internet to be used in draft or final solutions. Submitted solutions will be checked for plagiarism.
Grading: A score of 120 points is guaranteed to be enough to pass this problem set.
Questions: Please do not hesitate to ask the instructor or TAs if any problem statement is unclear, but please make sure to send private messages—sometimes specific enough questions could give away the solution to your fellow students, and we want all of you to benefit from working on, and learning from, the problems. Good luck!

1 (80 p) Decide for each of the propositional logic formulas below whether it is a tautology or a contradiction. If neither of these cases apply, then present a satisfying assignment for the formula. For this problem, it is sufficient (and necessary) for a full score to justify all your answers by presenting truth tables for all the subformulas, but you are also highly encouraged to *explain* why your answers are correct (and good explanations could compensate fully for minor slips in the truth tables).

(Note that logical not $\neg$ is assumed to bind harder than the binary connectives, but other than that all formulas are fully parenthesized for clarity. We write $\to$ for logical implication and $\leftrightarrow$ for equivalence.)

1a $((p \to q) \to r) \to ((p \land q) \to r)$

1b $((p \to q) \land (r \to s)) \leftrightarrow ((p \land r) \to (q \land s))$

1c $((p \land \neg r) \lor (q \land \neg r)) \to ((p \lor q) \to r)$

1d $(p \to (q \lor r)) \lor (\neg(\neg p \lor q) \land \neg r)$

2 (100 p) Recall that a graph $G = (V, E)$ consists of a set of vertices V connected by edges E, where every edge is a pair of vertices. If there is an edge (u, v) between two vertices u and v, then the two vertices are said to be *neighbours* and are both *incident* to the edge. We say that a sequence of edges (v_1, v_2), (v_2, v_3), (v_3, v_4), $\ldots$, (v_{k-1}, v_k), in E is a *path* from v_1 to v_k.

In this problem, we wish to express properties of graphs in both natural language and predicate logic, and to translate between the two forms. We do this as follows:

- The universe is the set of vertices V of G.
- The binary predicate $E(u, v)$ holds if and only if there is an edge from u to v in G.
- The unary predicate $S(v)$ is used to identify a subset of vertices $S = \{v \mid v \in V,\ S(v) \text{ is true}\}$ for which some property might or might not hold.

Below you find five graph properties written as predicate logic formulas and six graph properties defined in natural language. Most of the predicate logic formulas have corresponding natural language definitions, but not all.

Your task is to determine which of the predicate logic formulas (a), $\ldots$, (e) match which—if any—of the natural language definitions (1), $\ldots$, (6). Please make sure to motivate your answers clearly.

Predicate Logic Formulas:

(a) $\forall u \forall v \big(E(u, v) \rightarrow E(v, u)\big)$

(b) $\forall u \forall v \big(E(u, v) \rightarrow S(u) \lor S(v)\big)$

(c) $\forall u \forall v \exists w \big(S(w) \land (E(u, w) \lor E(v, w))\big)$

(d) $\forall u \forall v \big(E(u, v) \rightarrow ((S(u) \land \neg S(v)) \lor (\neg S(u) \land S(v)))\big)$

(e) $\forall u \forall v \big((u \neq v \land S(u) \land S(v)) \rightarrow E(u, v)\big)$

Natural Language Definitions:

(1) S is a *dominating set* in G, i.e., every vertex v in the graph either is in S or is a neighbour of a vertex in S.

(2) S is a *clique* in G, i.e., a set of vertices that are all neighbours with each other.

(3) The graph G is *undirected.*

(4) The graph G is *connected.*

(5) S is a *vertex cover* in G, i.e., for every edge at least one of the vertices incident to it is in S.

(6) The graph G is *bipartite* with bipartition $(S, V \setminus S)$, i.e., all edges go between S and $V \setminus S$.

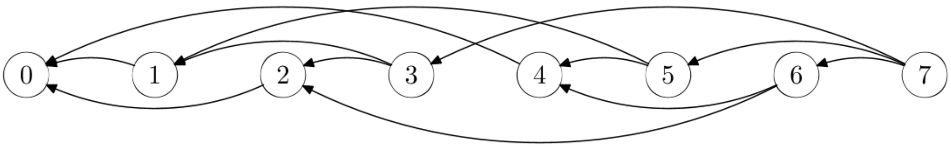

Figure 1: Directed graph D_S representing relation S in Problem 4.

3 (60 p) When Jakob is socializing in the evenings after conferences in the United States, he tries to convince his colleagues that poker should really be played with a *republican deck of cards*, i.e., with cards of all ranks 2–10 plus the aces, but without kings, queens, or jacks.

3a Suppose that you are dealt 5 cards from a perfectly shuffled republican deck of cards. What is the probability of getting a *full house*, i.e., three of a kind (three cards of the same rank) with a pair (two other cards of the same rank)? Explain clearly how you obtain the expression in your answer.

3b It is a sad fact that so far Jakob has had quite limited success in convincing colleagues that a republican deck of cards is the true American way of playing poker. Jakob believes this is because the colleagues have all been brainwashed by the big casinos in Las Vegas, and that this in turn is because with a republican deck of cards the probability for the players of getting a strong hand (like a full house) would be higher than with a standard deck of cards, making it more likely that casinos might lose money.

Ignoring the wider ramifications of the worldwide conspiracy that Jakob alleges to have discovered, is the factual claim true that the probability of getting, e.g., a full house is higher with a republican deck of cards than with a standard deck of cards when being dealt 5 cards from a shuffled deck? (Note that you should not have to do very precise calculations to be able to determine whether this claim is true or not, and a clear intuitive explanation of which answer should be the correct one can give generous partial credits.)

4 (60 p) Consider the relation S described by the directed graph D_S in Figure 1.

4a Write down the matrix representation M_S of the relation S and describe briefly but clearly how you constructed this matrix.

4b Let us write I to denote the inverse of the relation S. What is the matrix representation of I? Write it down and explain how you constructed it.

4c Now let T be the transitive closure of the relation I. What is the matrix representation of T? Write it down and explain how you constructed it.

4d Finally, let R be the reflexive closure of the relation T. Can you explain in words what the relation R is by describing how it can be interpreted? (In particular, is it similar to anything we have discussed during the course? Be as specific in your reply as you can.)