RELATIONS

CARTESIAN PRODUCT

$$A \times B = \{ (a,b) \mid a \in A,\ b \in B \}$$

ordered

Ex $A = \{a, b, c\}$ $\quad B = \{1, 2\}$

$$A \times B = \{ (a,1), (a,2), (b,1), (b,2), (c,1), (c,2) \}$$

$$A_1 \times A_2 \times \dots \times A_n = \{ (a_1, a_2, \dots, a_n) \mid a_i \in A_i \}$$

Ex Vectors in $\mathbb{R}^3$

OBSERVATION $|A_1 \times \dots \times A_n| = |A_1| \cdots |A_n|$
(Multiplication principle)

SUBSET $A' \subseteq A$ if for all $x \in A'$ it holds that $x \in A$

POWERSET $\mathcal{P}(A)$ or 2^A
Set of all subsets of A

$$2^A = \{ \emptyset, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\} \}$$

$$|2^A| = 2^{|A|}$$ (multiplication principle)

RELATION

Ex $A = \{\text{all people}\}$ $\quad B = \{\text{all bikes}\}$

Relation a owns bike b

$A = B =$ integers

Relation $a \leq b$

RELATION R from A to B

subset of $A \times B$

$(a,b) \in R$ is more often written $R(a,b)$ or aRb

$(a,b) \notin R$ written $a \not R b$

If $R \subseteq A \times A$, R is a relation on A

Ex

$A = \{2,3\}$ $B = \{1,2,3,4,5,6\}$

aRb if $a \mid b$

$R = \{(2,2), (2,4), (2,6), (3,3), (3,6)\}$

DOMAIN $Dom(R) = \{a \in A \mid \exists b \in B \; R(a,b)\}$

RANGE $Ran(R) = \{b \in B \mid \exists a \in A \; R(a,b)\}$

$Dom(R) = \{2,3\}$

$Ran(R) = \{2,3,4,6\}$

REPRESENTATION OF RELATION R

Matrix M_R $(m \times n)$ $m_{ij} = \begin{cases} 1 & \text{if } (a_i, b_j) \in R \\ 0 & \text{otherwise} \end{cases}$

$A = \{a_1, \ldots, a_m\}$

$B = \{b_1, \ldots, b_n\}$

Ex $A = \{1,2,3\}$ $B = \{r,s\}$ $R = \{(1,r), (2,s), (3,r)\}$

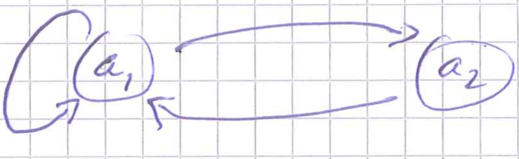
$$\begin{array}{c c} & \begin{array}{cc} r & s \end{array} \\ \begin{array}{c} 1 \\ 2 \\ 3 \end{array} & \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

Relation R on A: Directed graph (digraph)

vertices/nodes represent elements of a

directed edges $a_i \to a_j$ if $(a_i, a_j) \in R$

<u>Ex</u> $A = (a_1, a_2)$ $R = \{(a_1,a_1), (a_1,a_2), (a_2,a_1)\}$



Matrix
$$\begin{array}{c c} & \begin{array}{cc} a_1 & a_2 \end{array} \\ \begin{array}{c} a_1 \\ a_2 \end{array} & \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix} \end{array}$$

Same information – completely describes R

IN-DEGREE of a $|\{a' \in R \mid (a', a) \in R\}|$

OUT-DEGREE of a $|\{a' \in R \mid (a, a') \in R\}|$

Edges coming in / going out in digraph representation

R - RELATIVE SETS

R relation from A to B

R-relative set of a $\quad R(a) = \{ b \in B \mid aRb \}$

R-relative set of A_1

$$R(A_1) = \{ b \in B \mid \exists a \in A_1,\ aRb \}$$
$$= \bigcup_{a \in A_1} R(a)$$

<u>Ex</u> Consider again $A = \{2, 3\}$, $B = \{1, \dots, 6\}$
aRb if $a \mid b$

$$R(2) = \{2, 4, 6\}$$
$$R(3) = \{3, 6\}$$
$$R(\{2,3\}) = \{2, 3, 4, 6\}$$

<u>THM</u> ~~Let~~ R relation from A to B and $A_1, A_2 \subseteq A$.

1. If $A_1 \subseteq A_2$ then $R(A_1) \subseteq R(A_2)$
2. $R(A_1 \cup A_2) = R(A_1) \cup R(A_2)$
3. $R(A_1 \cap A_2) \subseteq R(A_1) \cap R(A_2)$

Exercises:

- Read and understand proofs (Or better: Try yourself first!)
- Find example of when equality does <u>not</u> hold in 3

RELATIONS RECAP

CARTESIAN PRODUCT $A \times B = \{(a,b) \mid a \in A, b \in B\}$

Ex $A = \{1,2,3\}$ $B = \{4,5,6\}$

$A \times B = \{(1,4), (1,5), (1,6), (2,4), (2,5), (2,6), (3,4), (3,5), (3,6)\}$

RELATION R from A to B

Subset of $A \times B$

If $A = B$: relation on A pretty common case

Ex

A = persons B = bikes R: a owns b

A = companies B = persons R: a employs b

$A = B$ = persons R: a is parent of b

Ways to represent relations

Running example $A = B = \{1,2,3,4\}$

$R(a,b)$ if $a < b$

Alternative notation $(a,b) \in R$

aRb

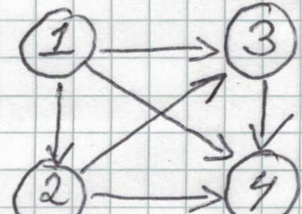
① SET R

$R = \{(1,2), (1,3), (1,4), (2,3), (2,4), (3,4)\}$

② MATRIX M_R

$$\begin{array}{c|c} & \begin{matrix} 1 & 2 & 3 & 4 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \end{array}$$

③ DIGRAPH D_R For relation on A



R-RELATIVE SET $R(a) = \{ b \in B \mid aRb \}$

$$R(A') = \{ b \in B \mid \exists a \in A' \ aRb \}$$
$$= \bigcup_{a \in A'} R(a)$$

Ex

$A = \{6, 7, 8, 9, 10\}$ $\quad B = \{1, 2, 3, 4, 5\}$

aRb if a is a multiple of b

$R(6) = \{1, 2, 3\}$
$R(7) = \{1\}$
$R(8) = \{1, 2, 4\}$

$R(\{6, 7, 8\}) = \{1, 2, 3, 4\}$

DOMAIN $\quad Dom(R) = \{ a \in A \mid \exists b \ aRb \}$
RANGE $\quad Ran(R) = \{ b \in B \mid \exists a \ aRb \}$
$= R(A)$

For relation on A

IN-DEGREE of a $\quad |\{ a' \in A \mid a' R a \}|$
OUT-DEGREE of a $\quad |\{ a' \in A \mid a R a' \}|$

incoming and outgoing edges, respectively, in digraph representation

Let us focus on relations on a set A

PATH IN DIGRAPH D_R

Path of length n in R

Sequence $a, x_1, x_2, \ldots, x_{n-1}, b$ such that

$$a R x_1$$
$$x_{i-1} R x_i \qquad i \in [n-1]$$
$$x_{n-1} R b$$

Ex $\quad a R b$ = edge in D_R = path of length 1

$a R x, x R b$ = path of length 2

Use this to define new relation

R^n: $\quad a R^n b$ if $\exists$ path of length (exactly) n in R from a to b

Ex $\quad A$ = people in the world

$R(a,b) \quad$ a knows b

Conjecture: SIX DEGREES OF SEPARATION

Everybody on earth is connected by chain of at most 6 people who are friends (pairwise) to anybody else

In math notation

$$\forall a \; \forall b \; \exists i \leq 6 \quad a R^i b$$

Given R, how can we find R^n? | R VI

<u>Idea 1</u> Draw digraph D_R

Find all paths of length n

<u>Idea 2</u> Use matrix representation M_R

Use Boolean matrix multiplication to compute

$$M_{R^n} = (M_R)^n_{\odot} =$$

$$= \underbrace{M_R \odot M_R \odot \dots \odot M_R}_{\text{repeat matrix } n \text{ times}}$$

<u>Recall</u> For $n \times n$ Boolean matrices A, B

$C = A \odot B$ is matrix such that

$$c_{ij} = \begin{cases} 1 & \text{if } \exists k \text{ s.t. } a_{ik} = b_{kj} = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$c_{ij} = (a_{i1} \wedge b_{1j}) \vee (a_{i2} \wedge b_{2j}) \vee \dots \vee (a_{i,n} \wedge b_{nj})$$

THEOREM For any $n \geq 2$ it holds that

$$M_{R^n} = (M_R)^n_{\odot}$$

When convenient, identify $A = \{a_1, a_2, \dots, a_n\}$ with $\{1, 2, \dots, n\}$ for notational simplicity

Proof By induction over n

BASE CASE ($n=2$): $\exists$ path of length 2 between a_i and a_j precisely when for some $k \in [n]$

$$a_i \, R \, a_k \quad \text{and} \quad a_k \, R \, a_j$$

In matrix notation:

$$m_{ik} \wedge m_{kj} \quad \text{for some } k \in [n]$$

But this is entry c_{ij} for

$$C = M_R \odot M_R$$

INDUCTION STEP: Our $\boxed{\text{induction hypothesis}}$ is that R^n has matrix representation $(M_R)^n_\odot$.

$a_i \, R^{n+1} \, a_j$ if $\exists \, k \in [n]$ such that

$a_i \, R^n \, a_k$ and $a_k \, R \, a_j$ (by definition)

By induction hypothesis, $a_i \, R^n \, a_k$ iff

(i,k)-entry of $(M_R)^n_\odot = 1$

$a_k \, R \, a_j$ if $m_{kj} = 1$

This holds iff (i,j)-entry in $(M_R)^{n+1}_\odot$ is $= 1$

This concludes the induction step.

The theorem follows by the induction principle $\blacksquare$

CONNECTIVITY RELATION R^∞ on A

$R^\infty(a,b)$ if $\exists n \in \mathbb{Z}^+$ s.t. $R^n(a,b)$

Equivalently,

$$R^\infty = R \cup R^2 \cup R^3 \cup \cdots$$
$$= \bigcup_{i=1}^{\infty} R^i$$

Note that this notation makes sense, because relations are sets

How large powers R^i of R do we need to consider?

Use pigeonhole principle

If $|A| < \infty$ (A finite set)

then in sequence

$a_1 R a_2,\ a_2 R a_3,\ a_3 R a_4, \ldots$

sooner or later will get repetition of element $a_i = a_j = y$ if length $\geq n$

$$a_1 R a_2 \cdots a_{i-1} R \overset{y}{a_i} \ \underbrace{\boxed{a_i R a_{i+1} \cdots a_{j-1} R \overset{y}{a_j}}}_{\text{SKIP THIS PART IN SEQUENCE}} \ a_j R a_{j+1}$$

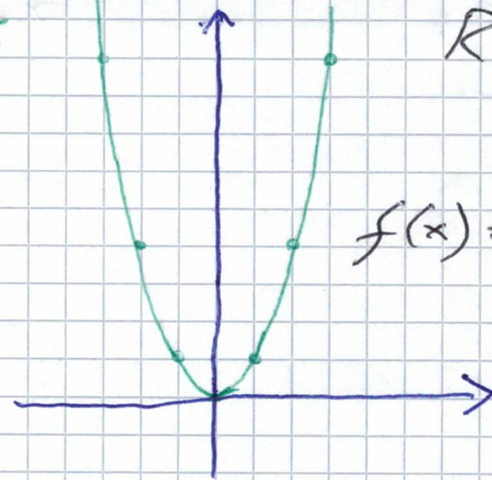
So $|A| - 1$ steps enough

FUNCTIONS

any sets

FUNCTION f FROM A TO B

Relation such that for every $a \in \text{Dom}(f)$ exists UNIQUE b for which $(a,b) \in f$



$f(x) = x^2$

More common notation: $\boxed{f(a) = b}$

f is a function from A to B $\boxed{f : A \to B}$

If $f(a) = b$, also write $\boxed{a \mapsto b}$ (\mapsto) when f understood from context

WARNING Normally required that f should be defined on all of A, i.e., $\text{Dom}(f) = A$. Otherwise PARTIAL FUNCTION

But KBR defines functions as partial functions — this is non-standard. We will try to be very clear when functions are partial or TOTAL ($\text{Dom}(f) = A$)

Ex (a) $g = \{ (x, x^2) \mid x \in \mathbb{Z} \}$

$= \{(0,0), (1,1), (-1,1), (2,4), (-2,4), (3,9), \ldots\}$

is a function illustrated above

(b) $h = \{ (x^2, x) \mid x \in \mathbb{Z} \}$

$= \{ (0,0), (1,1), (1,-1), (4,2), (4,-2), (9,3), \ldots \}$

is NOT a function How do we make $\sqrt{}$ into a function?

IMPORTANT TERMINOLOGY FOR FUNCTIONS

$f : A \to B$ is

- $\boxed{\text{EVERYWHERE DEFINED}}$ or $\boxed{\text{TOTAL}}$ if $Dom(f) = A$
- $\boxed{\text{SURJECTIVE}}$ or $\boxed{\text{ONTO}}$ if $Ran(f) = B$
- $\boxed{\text{INJECTIVE}}$ or $\boxed{\text{ONE-TO-ONE}}$ if for $x_1 \neq x_2$ it holds that $f(x_1) \neq f(x_2)$
- $\boxed{\text{BIJECTIVE}}$ if f is total, surjective, and injective.
  If so, f also called $\boxed{\text{BIJECTION}}$

Ex (a) $f : \mathbb{R} \to \mathbb{R}$ defined by $x \mapsto x^2$
- total
- not surjective
- not injective

(b) $g : \mathbb{R} \to \mathbb{R}_{\geq 0}$ defined by $x \mapsto x^2$
- total
- surjective
- not injective

(c) $h : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined by $x \mapsto x^2$
- total
- surjective
- injective

so bijection

IMPORTANT TERMINOLOGY FOR RELATIONS

Relation R on A is

- $\boxed{\text{REFLEXIVE}}$ if $\forall a \ (a,a) \in R$
- $\boxed{\text{IRREFLEXIVE}}$ if $\forall a \ (a,a) \notin R$

Ex
$=$ and $\leq$ are reflexive
$<$ is irreflexive

- $\boxed{\text{SYMMETRIC}}$ if $\forall a,b \ (a,b) \in R \Rightarrow (b,a) \in R$
- $\boxed{\text{ASYMMETRIC}}$ if $\forall a,b \ (a,b) \in R \Rightarrow (b,a) \notin R$
- $\boxed{\text{ANTISYMMETRIC}}$ if $\forall a,b \ (a,b) \in R$ and $(b,a) \in R \Rightarrow a = b$

WARNING Distinguish between asymmetric and antisymmetric!
Asymmetric relations are vacuously antisymmetric

Ex
- Divisibility $\mid$ is antisymmetric
- Strictly less than $<$ is asymmetric
- "u and v are neighbours in an undirected graph" is symmetric

Can a relation be both symmetric and asymmetric?

Well, yes, empty relation $R = \emptyset$ can... Other than that NO!

Can a nontrivial relation be both symmetric and antisymmetric?

Yes, equality satisfies this.

Exercise   Prove that this is only such relation.

Can we read off any of these properties from matrix representation M_R?

REFLEXIVE : Diagonal all 1s

IRREFLEXIVE : Diagonal all 0s

SYMMETRIC : Matrix symmetric, i.e., $(M_R)^T = M_R$

Relation R on A is

- $\boxed{\text{TRANSITIVE}}$ if $\forall a, b, c \in A$
  $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$

Ex - Strictly less than $<$ and divisibility $|$ are transitive

- Neighbour relation in graph is NOT transitive in general

OBSERVATION

R is transitive if and only if $R^2 \subseteq R$

Proof ($\Rightarrow$) Suppose $(a,c) \in R^2$. By definition $\exists b$ such that $(a,b) \in R$ and $(b,c) \in R$. Since R is transitive by assumption, we have $(a,c) \in R$. This shows that $R^2 \subseteq R$

($\Leftarrow$) Suppose $R^2 \subseteq R$. Consider a, b, c such that $(a,b) \in R$ and $(b,c) \in R$. We need to show that $(a,c) \in R$. But $(a,b) \in R$ and $(b,c) \in R$ means $(a,c) \in R^2$ by definition, and since $R^2 \subseteq R$ by assumption we have $(a,c) \in R$, which shows that R is transitive ▨

Relation R on A is an **EQUIVALENCE RELATION** if it is

- reflexive
- symmetric
- transitive

INTUITION: An equivalence relation tells us which elements in A are "essentially the same" or, well, equivalent

<u>Ex</u> A classical equivalence relation is $\boxed{\text{CONGRUENCE CLASSES MODULO}}$ some $n \in \mathbb{Z}^+$

$(a,b) \in R$ if $(a \bmod n) = (b \bmod n)$

i.e., if a and b yield same remainder when divided by n

Say, $n = 5$. Then

- 2, 7, 12, 17 all equivalent
- 3, 8, 13, 18 all equivalent
- 2 and 3 are NOT equivalent

A $\boxed{\text{PARTITION}}$ $\mathcal{P}$ of a set A is a collection of subsets $P_1, P_2, \ldots, P_k$ such that

1. $\forall i \in [k] \quad P_i \neq \emptyset$
2. $\forall i,j \in [k],\ i \neq j \Rightarrow P_i \cap P_j = \emptyset$
3. $\bigcup_{i=1}^{k} P_i = A$

(Actually, partitions need not be finite if A is infinite. This definition works for infinite partitions also.)

(A "collection of sets" is just a set consisting of sets, but we often call a set of sets a "collection")

Ex Let

$$A = \{1, 2, 3, 4, 5, 6, 7\}$$

$$P_1 = \{1, 4, 7\}$$
$$P_2 = \{2, 5\}$$
$$P_3 = \{3, 6\}$$
$$P_4 = \{2, 3, 5, 7\}$$

- P_1, P_2, P_3, P_4 do NOT form a partition
- P_1, P_2 do NOT form a partition
- P_1, P_2, P_3 do form a partition

Partitions give rise to equivalence relations and vice versa

THEOREM (From equivalence relations to partitions)

If R is an equivalence relation on A, then the collection $\mathcal{P}$ of all DISTINCT R-relative sets $R(a)$, $a \in A$, is a partition of A.

Proof Let $\mathcal{P} = \{P_i\}$ be all distinct R-relative sets. We need to prove ① – ③ above.

① $P_i \neq \emptyset$ since every $R(a)$ contains a by reflexivity

③ <u>By contradiction</u>

Suppose $\bigcup_i P_i \neq A$

Means $A \setminus \bigcup_i P_i \neq \emptyset$, so

$\exists\, b \in A \setminus \bigcup_i P_i$

But then we can add $R(b)$ to $\mathcal{P}$
Clearly distinct from all other P_i,
since $b \notin P_i$ but $b \in R(b)$
However $\mathcal{P}$ already contains all
distinct R-relative sets. $\lightning$

Finally, we need to prove
② If $P_i \neq P_j$, then $P_i \cap P_j = \emptyset$

Let us do <u>proof by contraposition</u>
If $P_i \cap P_j \neq \emptyset$, then $P_i = P_j$

Let $P_i = R(a)$
$P_j = R(c)$

Suppose $b \in R(a) \cap R(c)$
Want to prove $R(a) = R(c)$

That is:

(i) $R(a) \subseteq R(c)$ For every $d \in R(a)$ we have $d \in R(c)$

(ii) $R(c) \subseteq R(a)$ For every $d \in R(c)$ we have $d \in R(a)$

How to prove this? Not much we can do...
We only know that R equivalence relation.
But this makes the task EASY! All we can do is
to play with the definitions — this has got to work!

$b \in R(a) \cap R(c)$

$\left.\begin{array}{l}(a,b) \in R \\ (c,b) \in R\end{array}\right\}$ by definition

$(b,c) \in R$ by symmetry

$(a,c) \in R$ by transitivity

$(c,a) \in R$ by symmetry

Suppose $d \in R(a)$ Then

$(a,d) \in R$ by definition

$(d,a) \in R$ by symmetry

$(d,c) \in R$ by transitivity

$(c,d) \in R$ by symmetry

So $d \in R(c)$ and $R(a) \subseteq R(c)$

In EXACTLY THE SAME WAY, just swapping "a" and "c" in the six lines above, we prove $R(c) \subseteq R(a)$, and hence $R(a) = R(c)$ or $P_i = P_j$ as claimed

Hence, $\mathcal{P}$ is a partition.
The theorem follows ▨

THEOREM

If $\mathcal{P} = \{P_i\}$ is a partition of the set A, then the relation

$$R = \{ (a,b) \mid \exists i \text{ such that } a, b \in P_i \}$$

is an equivalence relation

Proof R as defined above is definitely some kind of relation. We need to establish that R is reflexive, symmetric, and transitive.

REFLEXIVITY $(a,a) \in R$

Every a is in some P_i, since $\mathcal{P}$ partition
Since a and a are both in P_i :-)
$(a,a) \in R$ — CHECK! ✓

SYMMETRY $(a,b) \in R \Rightarrow (b,a) \in R$

~~Also~~ clear by construction. The definition of R doesn't care about the order
If $(a,b) \in R$, then $\exists i$ such that $b \in P_i$ and $a \in P_i$, so $(b,a) \in R$ — CHECK! ✓

TRANSITIVITY $(a,b) \in R$ and $(b,c) \in R \Rightarrow (a,c) \in R$

$(a,b) \in R \Rightarrow \exists i \quad a, b \in P_i$
$(b,c) \in R \Rightarrow \exists j \quad b, c \in P_j$

Want to prove $P_i = P_j$ — then $a, c \in P_i$ so $(a,c) \in R$
But $b \in P_i \cap P_j$, and $\mathcal{P}$ is partition, so by (2) we indeed have $P_i = P_j$ — CHECK! ✓ $\blacksquare$

OPERATIONS ON RELATIONS

Relations are just sets, so can apply set operations

If R, S relations from A to B then $R \cup S$ and $R \cap S$ also relations

$a \, (R \cup S) \, b$ if aRb or aSb

$a \, (R \cap S) \, b$ if aRb and aSb

__Ex__ Let $R, S \subseteq \mathbb{Z}^+ \times \mathbb{Z}^+$ defined by

$aRb \iff 2 \mid ab$

$aSb \iff 3 \mid ab$

What is $R \cup S$? $R \cap S$?

$\boxed{\text{COMPLEMENT}}$ or COMPLEMENTARY RELATION

$$\overline{R} = \{ (a,b) \in A \times B \mid (a,b) \notin R \}$$

$\boxed{\text{INVERSE RELATION}}$ FROM B TO A

$$R^{-1} = \{ (b,a) \in B \times A \mid a,b \in R \}$$

__Ex__ Consider $\geq$

Complementary relation $<$

Inverse relation $\leq$

__Ex__ Consider divisibility $a \mid b$

Inverse relation a multiple of b

Complement: a Does not divide b

CLOSURE If $R \subseteq A \times A$ doesn't have some property P, try to add as few pairs (a, a') as possible to R so that property P holds

REFLEXIVE CLOSURE Smallest relation $S \subseteq A \times A$ such that
- $R \subseteq S$
- S reflexive

SMALLEST means: Whenever this holds for S', then $S \subseteq S'$

THM Reflexive closure of R is

$$R \cup \{(a,a) \mid a \in A\}$$

Proof think about it...

SYMMETRIC CLOSURE smallest relation $S \subseteq A \times A$ such that
- $R \subseteq S$
- S symmetric

THM Symmetric closure of R is

$$R \cup R^{-1}$$

$\boxed{\text{TRANSITIVE CLOSURE}}$ of R

Smallest $S \subseteq A \times A$ such that

$R \subseteq S$

S transitive

THM Transitive closure of R is R^{∞}

$\boxed{\text{Question}}$ Given undirected graph $G = (V, E)$, Define

$E(u,v)$ = there is an edge from u to v

① What is the transitive closure?

② Prove that the reflexive closure of the ~~transitive closure~~ is an equivalence relation

③ What are the equivalence classes of this relation?

THM

Let $P \in \{\text{symmetric, reflexive}\}$

If R and S are both P, then $R \cup S$ and $R \cap S$ are also P.

If R and S are transitive, then $R \cap S$ is transitive

Question What about $R \cup S$ and transitivity?

COROLLARY

If R and S are equivalence relations, then so is $R \cap S$.

PARTIALLY ORDERED SETS (POSETS)

Relation R on A is a partial order if it is

- reflexive $a \leq a$
- antisymmetric $a \leq b$ & $b \leq a \Rightarrow a = b$
- transitive $a \leq b$ & $b \leq c \Rightarrow a \leq c$

The pair (A, R) is called a poset

Often use $\leq$ or $\preceq$ (\preceq) instead of R

Write $a \prec b$ (\prec) as shorthand for "$a \preceq b$ and $a \neq b$"

Examples
- Less-than-or-equal $\leq$
- Divisibility $a \mid b$
- subset $A \subseteq B$ on power set

Non-example
- Less than $<$ on numbers (why?)

Two elements a, b are comparable if $a \leq b$ or $b \leq a$

An order relation is a total or linear order if any two elements are comparable

Examples (continued)
- Less-than-or-equal is total
- Divisibility and subset are not.

PRODUCT PARTIAL ORDERS

Suppose $(A, \leq_1)$ and $(B, \leq_2)$ posets.
Can define new relation $\leq_3$ on $A \times B$ by
$$(a,b) \leq_3 (a',b') \text{ if } a \leq_1 a' \text{ and } b \leq_2 b'$$

THEOREM $(A \times B, \leq_3)$ as defined above is a poset.

Exercise: Prove this

Extends also to more than two posets

Example Coordinate-wise comparison of vectors

$\vec{u} = (u_1, \dots, u_n)$

$\vec{v} = (v_1, \dots, v_n)$ $u_i, v_i \in \mathbb{Z}/\mathbb{Q}/\mathbb{R}$

$\vec{u} \leq \vec{v}$ if $\forall i \in [n]$ $u_i \leq v_i$

Note Product of totally ordered posets is not totally ordered

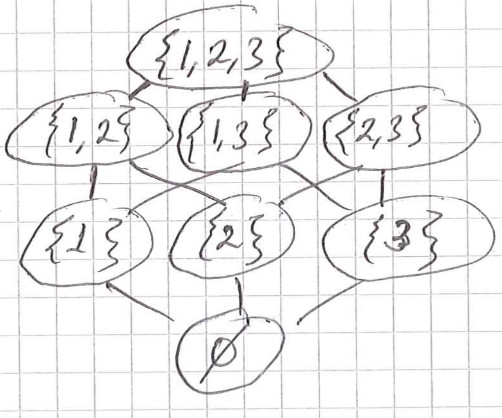
Exercise: Show this! (What assumptions do you need)

HASSE DIAGRAM

- Start with digraph representation
- Remove all loops
- Remove all edges implied by transitivity
- Make edges point upwards and remove arrow heads

Example

Powerset of $\{1,2,3\}$ with $\subseteq$



$z \in A$ is a

- $\boxed{\text{maximal}}$ element if there is no $a \in A$ s.t. $z < a$
- $\boxed{\text{minimal}}$ element if no $a \in A$ s.t. $a < z$

NOTE distinguish maximal – maximum
minimal – minimum

THEOREM Any finite, nonempty poset has a maximal and a minimal element.

Exercises

1) Does $(\mathbb{Z}, \leq)$ have any maximal or minimal elements, and if so which?
2) Ditto for $(\mathbb{Z}^+, \leq)$
3) Ditto for $(\mathcal{P}(\mathbb{Z}), \subseteq)$

We say that $z \in A$ is

- the $\boxed{\text{greatest / maximum}}$ element of A if $\forall a \in A \;\; a \leq z$
- the $\boxed{\text{least/ smallest}}$/ $\boxed{\text{minimum}}$ element of A if $\forall a \;\; z \leq a$

THEOREM A poset has at most one greatest element and at most one least element.

Let $(A, \leq)$ poset and $B \subseteq A$. Then $a \in A$ is

- an **upper bound** for B if $b \leq a$ for all $b \in B$
- a **lower bound** for B if $\forall b \in B$ $a \leq b$

We say that $a \in A$ is

- the **least upper bound (LUB)** for B if a is an upper bound for B and $a \leq a'$ for any upper bound a' for B
- the **greatest lower bound (GLB)** for B if a is a lower bound and $a' \leq a$ for any other lower bound

Alternative terminology

LUB: supremum, join

GLB: infimum, meet

THEOREM For poset $(A, \leq)$ any $B \subseteq A$ has at most one LUB and one GLB.

Note In general, LUB & GLB might not exist

Exercise: Construct example!

DEFINITION A **lattice** is a poset $(A, \leq)$ in which every pair $a, b \in A$ has a LUB and a GLB.

Example $(\mathbb{Z}, \mid)$ ← divisibility

What is LUB for a, b? LCM(a,b)

— " — GLB — " — ? GCD(a,b)

THEOREM If $(A, \leq)$ is a lattice, then any $B \subseteq A$ has a GLB and a LUB

Proof Exercise.

WARNING "Lattice" is a popular word in math and has several different meanings.

DIRECTED GRAPH (DIGRAPH) $G = (V, E)$ [or $D = (V, A)$]

- Vertices V
- Edges $E \subseteq V \times V$ ordered pairs

Given relation R on A, can represent as digraph D_R — different ways of looking at same mathematical object

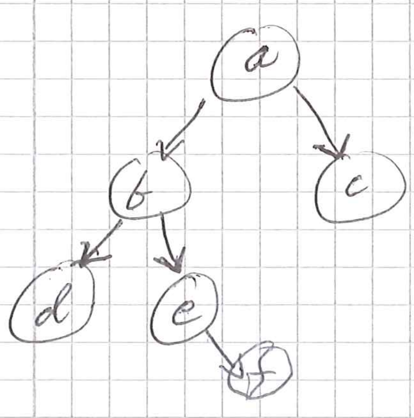
DIRECTED TREE digraph T with

- special vertex v_0 (ROOT)
- for any $v \in V(T) \setminus \{v_0\}$ there is a unique walk from v_0 to v
- no walk from v_0 to v_0

Can write (T, v_0) for clarity

Ex $A = \{a, b, c, d, e, f\}$

$R = \{(a,b), (a,c), (b,d), (b,e), (e,f)\}$



Can visualize as tree (usually turned upside down so root is at the top)

THEOREM (Properties of directed trees)

Let (T, v_0) be a directed tree. Then

1) T contains no cycles
2) v_0 is the only root
3) Indegree of v_0 is 0 and indegree of all other vertices is 1

<u>Proof</u> By contradiction

1) Suppose $\exists$ cycle $C = u_1 \to u_2 \to \dots \to u_k \to u_1$

By def $\exists$ walk (in fact, has to be path! why?)

$W : v_0 \to \dots \to u_1$

But then W and C concatenated yield second path violates uniqueness.

2) Suppose 2nd root u

Since u root, $\exists$ path $u \rightsquigarrow v_0$

Since v_0 root, —"— $v_0 \rightsquigarrow u$

Yields cycle, contradicting 1)

3) Suppose indegree of $v_0 \geq 1$ and $\exists$ edge (u, v_0)

Concentenate with $v_0 \rightsquigarrow u$ to get cycle

for $v \in V(T) \setminus \{v_0\}$ suppose $\exists$ edges (u_1, v), (u_2, v) $u_1 \neq u_2$

Then $v_0 \rightsquigarrow u_1, v$ and $v_0 \rightsquigarrow u_2, v$ yields two different paths, violating uniqueness

THEOREM If $T_1 = (V, E)$ tree on $n = |V|$ vertices, then $|E| = n - 1$

Proof

$$\# \text{ edges} = \sum_{v \in V} \text{indeg}(v)$$

Use previous theorem

Directed trees as relations

Given directed tree T, corresponding relation R_T is

- irreflexive
- asymmetric
- antitransitive $(a R_T b \;\&\; b R_T c \Rightarrow a \not R_T c)$

TREE TERMINOLOGY

LEAVES: vertices with no outgoing edges

CHILDREN of v: the out-neighbours of v

PARENT of v: the in-neighbour of v

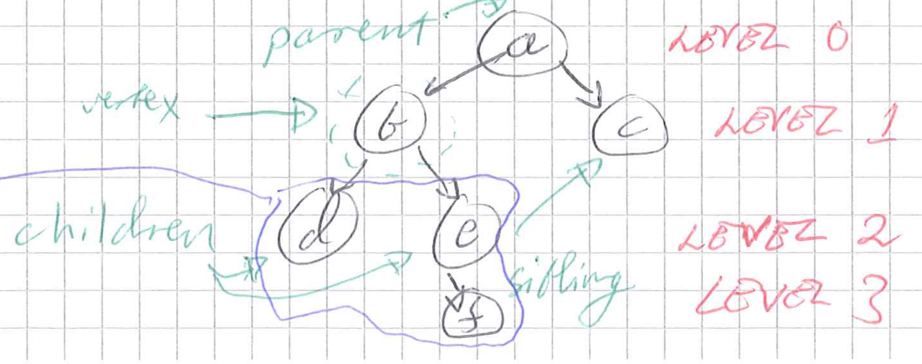
SIBLINGS of v: other children of parent of v

DESCENDANTS of v: all vertices reachable by path from v

LEVEL of vertex: Root v_0 at level 0

For $v \neq v_0$ length of path from v_0

HEIGHT of tree: max level in it

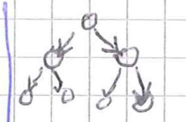


A tree is an **n-TREE** (for $n \in \mathbb{Z}^+$) (or **n-ARY TREE**)
if every vertex has at most n children

$n = 2$ **BINARY TREE**

$n = 3$ **TERNARY TREE**

An n-tree is **COMPLETE** if all vertices except for leaves have n children



Complete binary tree of height 2

The **COMPLETE n-TREE OF HEIGHT h** is a complete n-tree of height h that has all leaves at level h

Exercise: Suppose T the complete n-tree of height h

- How many vertices are there at level k for $k = 0, 1, \ldots, h$?
- Determine $|V|$.

SUBTREE rooted at b $\quad T(b)$

Take b and all descendants $D(b)$ of b

Remove all vertices not in $\{b\} \cup D(b)$

and all edges not in $(\{b\} \cup D(b)) \times (\{b\} \cup D(b))$

Then $\boxed{T(b) \text{ is a tree with root } b}$

Exercise: Prove this!