Introduktion til diskret matematik og algoritmer: Problem Set 3

Due: Wedneday March 12 at 12:59 CET.

Submission: Please submit your solutions via *Absalon* as a PDF file. State your name and e-mail address close to the top of the first page. Solutions should be written in LaTeX or some other math-aware typesetting system with reasonable margins on all sides (at least 2.5 cm). Please try to be precise and to the point in your solutions and refrain from vague statements. Never, ever just state the answer, but always make sure to explain your reasoning. *Write so that a fellow student of yours can read, understand, and verify your solutions.* In addition to what is stated below, the general rules for problem sets stated on *Absalon* always apply.

Collaboration: Discussions of ideas in groups of two to three people are allowed—and indeed, encouraged—but you should always write up your solutions completely on your own, from start to finish, and you should understand all aspects of them fully. It is not allowed to compose draft solutions together and then continue editing individually, or to share any text, formulas, or pseudocode. Also, no such material may be downloaded from or generated via the internet to be used in draft or final solutions. Submitted solutions will be checked for plagiarism.

Grading: A score of 120 points is guaranteed to be enough to pass this problem set.

Questions: Please do not hesitate to ask the instructor or TAs if any problem statement is unclear, but please make sure to send private messages—sometimes specific enough questions could give away the solution to your fellow students, and we want all of you to benefit from working on, and learning from, the problems. Good luck!

1 (40 p) Recall that a standard deck of cards has 52 cards partitioned into four *suits* (hearts, spades, clubs, and diamonds) with 13 ranks each (2–10 plus jack, queen, king, and ace). In this problem, we assume that you are dealt 5 cards from a perfectly shuffled deck of cards, and we wish to analyse the probability of getting flushes and/or straights. Note that there are two straights involving the ace: both ace–2–3–4–5 and 10–jack–queen–king–ace are valid straights.

1a What is the probability that you get a *flush*, i.e., 5 cards of the same suit but not all in sequence with respect to rank? (Because five cards of the same suit in sequential rank would be a *straight flush*.)

1b What is the probability that you get a *straight*, i.e., 5 cards of sequential rank but not all of the same suit? (Because if the latter condition also held, we would again have a straight flush.)

2 (40 p) Prove mathematically that among all numbers on the form $11\ldots100\ldots0$, i.e., numbers consisting of m ones followed by n zeros for some $m, n \in \mathbb{N}^+$ (sometimes notation like $1^m 0^n$ is used to describe text strings constructed in such a way), there is some number that is divisible by 2025.

Hint: Look at all numbers $1^m = 11\ldots1$ and consider what their remainders can be modulo 2025.

3 (40 p) Let $a \in \mathbb{R}^+$ be any positive real number. Show that for any integer $n \geq 2$ there is a rational number $\frac{c}{d}$, $c, d \in \mathbb{Z}$, $d \leq n$, that approximates a to within error $\frac{1}{dn}$, i.e., $\left| a - \frac{c}{d} \right| \leq \frac{1}{dn}$.

Hint: Consider the numbers $a, 2a, \ldots, n \cdot a$ and show that one of these numbers is at distance at most $1/n$ from some integer.

4 (70 p) In this problem we focus on relations. Suppose that $A = \{e_0, e_1, \ldots, e_5\}$ is a set of 6 elements and consider the relation R on A represented by the matrix

$$M_R = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}$$

(where element e_i corresponds to row and column $i + 1$).

4a Let us write S to denote the symmetric closure of the relation R. What is the matrix representation of S? Can you explain in words what the relation S is by describing how it can be interpreted?

4b Now let T be the transitive closure of the relation S. What is the matrix representation of T? Can you explain in words what the relation T is by describing how it can be interpreted?

4c Suppose that we instead let T' be the transitive closure of the relation R, and then let S' be the symmetric closure of T'. Are S' and T the same relation? If they are not the same, show some way in which they differ. If they are the same, is it true that S' and T constructed in this way from some relation R on a set A will always be the same? Please make sure to motivate your answers clearly.

5 (120 p) Recall that an undirected graph $G = (V, E)$ consists of a set of vertices V connected by edges E, where every edge is an unordered pair of vertices. If there is an edge (u, v) between two vertices u and v, then we say that u and v are the *endpoints* of the edge, and the two vertices are said to be *neighbours*. We say that a sequence of edges (v_1, v_2), (v_2, v_3), (v_3, v_4), ..., (v_{k-1}, v_k), in E is a *path* from v_1 to v_k.

In this problem, we wish to express properties of graphs in both natural language and predicate logic, and to translate between the two forms. We do this as follows:

- The universe is the set of vertices V of G.

- The binary predicate $E(u, v)$ holds if and only if there is an edge between u and v in G.

- The unary predicate $S(v)$ is used to identify a subset of vertices $S = \{v \mid v \in V, S(v) \text{ is true}\}$ for which some property might or might not hold.

For example, we can write the natural language statement "S is a set containing exactly k distinct elements" as a formula

$$
\begin{aligned}
\mathsf{setsize}\,(S, k) \; := \quad & \exists u_1 \cdots \exists u_k \left(\bigwedge_{i=1}^{k-1} \bigwedge_{j=i+1}^{k} (u_i \neq u_j) \wedge \bigwedge_{i=1}^{k} S(u_i) \right) \\
& \wedge \; \neg \exists u_1 \cdots \exists u_{k+1} \left(\bigwedge_{i=1}^{k} \bigwedge_{j=i+1}^{k+1} (u_i \neq u_j) \wedge \bigwedge_{i=1}^{k} S(u_i) \right)
\end{aligned}
\tag{1}
$$

where $\bigwedge_{i=1}^{k} \phi_i$ is shorthand for $\phi_1 \wedge \phi_2 \wedge \cdots \wedge \phi_k$, and where we use standard notation $\neg$ for logical negation. In natural language, this formula can be read as: "There exist k elements u_1 to u_k such that (i) for every pair (u_i, u_j), $i \neq j$, the elements themselves are also distinct ($u_i \neq u_j$); and (ii) all the u_i for $i = 1, \ldots, k$ are members of S, but no such set of $k+1$ elements u_1 to u_{k+1} exists."

Below you find six graph properties defined in natural language and six graph properties written as predicate logic formulas. Most of the natural language definitions have equivalent predicate logic formulas, but not all.

Your task is to translate each predicate logic formula (a), ..., (f) into a natural language description, and argue which—if any—of the natural language definitions (1), ..., (6) it matches.

Natural Language Definitions:

(1) A *dominating set* of size k for a graph $G = (V, E)$ is a set S of k distinct vertices such that every vertex v in the graph either is in S or is a neighbour of a vertex in S.

(2) A *clique* S of size k in a graph $G = (V, E)$ is a set S of k distinct vertices such that all vertices in S are neighbours with each other.

(3) A *disconnected vertex set* of size k in a graph $G = (V, E)$ is a set S of k distinct vertices such that there are no edges from any $u \in S$ to any $v \in V \setminus S$. [Here $V \setminus S$ denotes set subtraction, so that $V \setminus S = \{v \mid v \in V \text{ and } v \notin S\}$.]

(4) A *vertex cover* of size k of a graph $G = (V, E)$ is a set S of k distinct vertices such that for every edge $(u, v) \in E$ it holds that at least one of the endpoints is in S.

(5) A graph $G = (V, E)$ is *bipartite*, with one of the parts in the bipartition having size k, if there a set S of k distinct vertices such that all edges in the graph go between S and $V \setminus S$.

(6) A *connected component* S of size k in a graph $G = (V, E)$ is a set S of k distinct vertices such that for every pair of distinct vertices u and v in S there is a path from u to v consisting of vertices in S.

Predicate Logic Formulas:

(a) $\mathsf{setsize}\,(S, k) \,\wedge\, \forall v \forall w \big(E(v, w) \rightarrow (S(v) \vee S(w))\big)$

(b) $\mathsf{setsize}\,(S, k) \,\wedge\, \forall v \big(S(v) \vee \exists w(S(w) \wedge E(v, w))\big)$

(c) $\mathsf{setsize}\,(S, k) \,\wedge\, \forall u \forall w \big((u \neq w \wedge S(u) \wedge S(w)) \rightarrow \exists v(E(u, v) \wedge E(v, w))\big)$

(d) $\mathsf{setsize}\,(S, k) \,\wedge\, \forall v \forall w \big(E(v, w) \rightarrow ((S(v) \wedge \neg S(w)) \vee (\neg S(v) \wedge S(w)))\big)$

(e) $\mathsf{setsize}\,(S, k) \,\wedge\, \forall v \big(S(v) \rightarrow \exists w(\neg S(w) \wedge E(v, w))\big)$

(f) $\mathsf{setsize}\,(S, k) \,\wedge\, \forall v \forall w \big((v \neq w \wedge S(v) \wedge S(w)) \rightarrow E(v, w)\big)$

For every predicate logic formula that matches a natural language description, explain clearly why there is a match. For formulas that do not match a description, write a natural language definition along the lines of (1), ..., (6) that describes the property that the formula encodes.