IDMA FUNDAMENTALS

Plans for today:
- Sets
- Sequences
- Sums
- Integers

Some of this already known, probably
Some of this might be new material

SETS

$\boxed{\text{SET}}$ collection of $\boxed{\text{ELEMENTS}}$ or $\boxed{\text{MEMBERS}}$

Ex $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Positive integers smaller than 10

Set of vertices in example graph in first lecture

Sets are $\boxed{\text{UNORDERED}}$ and $\boxed{\text{WITHOUT REPETITIONS}}$

$$\{1, 2, 3\} = \{2, 3, 1\} = \{3, 3, 1, 2, 1\}$$

What if we care about order and/or repetitions?
- Repetitions multi-set
- Order ordered set

$\boxed{\text{MEMBERSHIP}}$ in set $\boxed{x \in A}$

Ex For set A above

$$3 \in A$$
$$15 \notin A$$

Some important sets

$\mathbb{N} = \{0, 1, 2, 3, \dots\}$ non-negative integers

$\mathbb{Z} = \{\dots -2, -1, 0, 1, 2, \dots\}$ all integers

How to specify sets?

- list members
- description $\{ \text{element} \mid \text{description/condition} \}$

Ex $\mathbb{Z} = \{ a, -a \mid a \in \mathbb{N} \}$

$\mathbb{E} = \{ 2a \mid a \in \mathbb{Z} \}$ all even numbers

$\mathbb{N}^+ = \mathbb{Z}^+ = \{ a \in \mathbb{Z} \mid a > 0 \}$

$\mathbb{Q} = \{ a/b \mid a, b \in \mathbb{Z},\ b \neq 0 \}$ rational numbers

$\mathbb{R}$ all real numbers

A is a $\boxed{\text{SUBSET}}$ of B, denoted $\boxed{A \subseteq B}$ if for all $x \in A$ it holds that $x \in B$

Ex $B = \{2, 3, 4\}$; $C = \{1, 2, 3, 4, 5\}$ $D = \{4, 5, 6\}$

$B \subseteq C$ but $D \nsubseteq C$

When we are discussing sets, they are usually contained in some "universe" of objects that we are considering.

When this is clear from context, such a <u>universe</u> is denoted <u>$\mathcal{U}$</u>

$\boxed{\text{EMPTY SET } \emptyset}$ contains no element

A $\boxed{\text{FINITE}}$ set is a set that contains n elements for some $n \in \mathbb{N}$

$\boxed{\text{CARDINALITY} \quad \boxed{|A|}}$ = # elements

Sets that are not finite are called $\boxed{\text{INFINITE}}$

Sometimes write $|A| = \infty$

Ex $D = \{4,5,6\}$ finite $|D| = 3$

$|\mathbb{Z}| = \infty$; $\mathbb{Z}$ infinite

$\boxed{\text{POWER SET}}$ of A: set consisting of all subsets of A $\boxed{\mathcal{P}(A)}$ or $\boxed{2^A}$

Ex $\mathcal{P}(D) = \{\emptyset, \{4\}, \{5\}, \{6\}, \{4,5\}, \{4,6\}, \{5,6\}, \{4,5,6\}\}$

OPERATIONS ON SETS

$\boxed{\text{UNION}}$ $\boxed{A \cup B} = \{x \mid x \in A \text{ or } x \in B\}$

$\boxed{\text{INTERSECTION}}$ $\boxed{A \cap B} = \{x \mid x \in A \text{ and } x \in B\}$

$\boxed{\text{COMPLEMENT}}$ OF B WITH RESPECT TO A

$\boxed{A - B}$ or $\boxed{A \setminus B}$ $= \{x \mid x \in A \text{ and } x \notin B\}$

$\boxed{\text{COMPLEMENT}}$ OF B $\boxed{\overline{B}} = \mathcal{U} \setminus B = \{x \mid x \notin B\}$

assuming universe $\mathcal{U}$

Ex $B = \{2,3,4\}$ $D = \{4,5,6\}$

$B \cup D = \{2,3,4,5,6\}$

$B \cap D = \{4\}$

$B \setminus D = \{2,3\}$

F IV

Can take union and intersection of multiple sets

$\boxed{\bigcup_{i=1}^{n} A_i} = \{ x \mid \text{for some } A_i \text{ it holds that } x \in A_i \}$

$\boxed{\bigcap_{i=1}^{n} A_i} = \{ x \mid \text{for all } A_i \text{ it holds that } x \in A_i \}$

If $A \cap B = \emptyset$ then A and B are $\boxed{\text{DISJOINT}}$

Can also define more elaborate operations

$\boxed{\text{SYMMETRIC DIFFERENCE}}$ $\boxed{A \,\Delta\, B}$ or $\boxed{A \oplus B}$

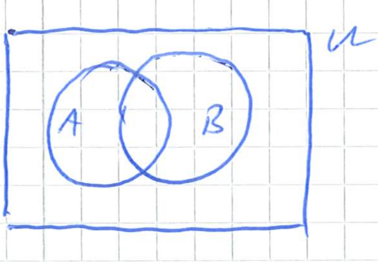
KBR, but non-standard

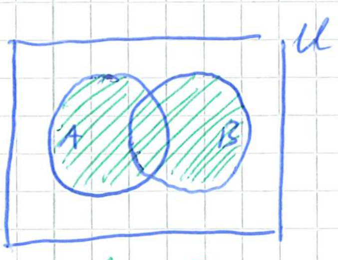
$$\left\{ x \;\middle|\; \text{or} \begin{array}{l} (x \in A \text{ and } x \notin B) \\ (x \notin A \text{ and } x \in B) \end{array} \right\}$$

So $A \,\Delta\, B = (A \cup B) \setminus (A \cap B)$

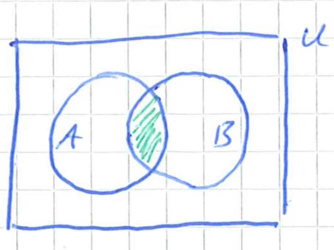
Ex $B \,\Delta\, D = \{2, 3, 5, 6\}$

$\boxed{\text{VENN DIAGRAMS}}$ can be used to illustrate relations between sets

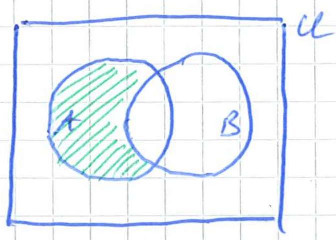




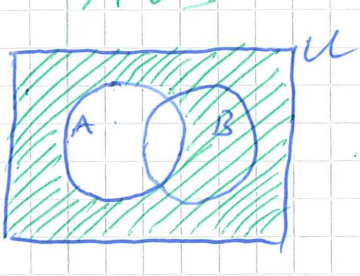
$A \cup B$



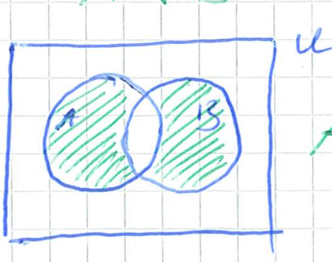
$A \cap B$



$A \setminus B$



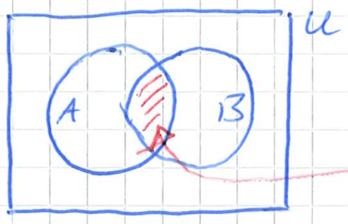
$\overline{A}$



$A \,\Delta\, B$

INCLUSION - EXCLUSION

How to count # elements in $A \cup B$?

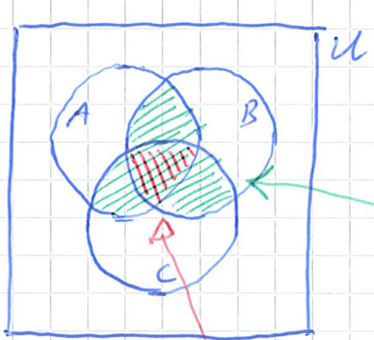


First attempt: $|A| + |B|$ Counting intersection twice!

Adjust

$$|A \cup B| = |A| + |B| - |A \cap B|$$

What about three sets?



$$|A \cup B \cup C| =$$

$$|A| + |B| + |C|$$
$$- |A \cap B| - |A \cap C| - |B \cap C|$$
$$+ |A \cap B \cap C|$$

Counting green area too much

Now removed $A \cap B \cap C$ one time too many

Can be generalized to more sets than 3

Important formula in combinatorics

SOME CLASSES OF FUNCTIONS

POLYNOMIALS $p(x) = a_n \cdot x^n + a_{n-1} x^{n-1} + \dots + a_1 x_1 + a_0$

$a_i \in \mathbb{R}$ **COEFFICIENTS**

(Assume $a_n \neq 0$; otherwise can be dropped)

DEGREE $\deg(p) = n$

Can also have x^r for $r \in \mathbb{R}$

Then require $x \geq 0$

EXPONENTIALS $f(y) = b^y$

$b > 0$ constant **BASE** $y \in \mathbb{R}$

LOGARITHM BASE b

$\log_b x$ is the number y s.t. $b^y = x$

Assume $x > 0$; focus on $b > 1$.

PROPERTIES OF LOGARITHMS

For $b, c > 0$, $b, c \neq 1$

$x, x_1, x_2 > 0$

$r \in \mathbb{R}$

$$\boxed{\begin{aligned} \log_b x^r &= r \log_b x \\ \log_b (x_1 \cdot x_2) &= \log_b x_1 + \log_b x_2 \\ \log_b x &= \frac{\log_c x}{\log_c b} \end{aligned}}$$

Verify directly from definition

ASYMPTOTICS

Let $k, a, b, c \in \mathbb{R}$

$k > 0$ $\quad b > 0$

$a > 1$ $\quad c > 1$

Then

"constant < log" $\quad$ k is $o(\log_a x)$

"log < polynomial" $\quad$ $\log_a x$ is $o(x^b)$

"polynomial < exponential" $\quad$ x^b is $o(c^x)$

Logarithms scale the same regardless of base

For polynomials:
Larger exponent $\Rightarrow$ asymptotically faster growth

For exponentials:
Larger base $\Rightarrow$ asymptotically faster growth

$\boxed{\text{SEQUENCE}}$ ordered list of numbers

Numbers called ELEMENTS or TERMS

Ex

$1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots \quad (a_n)$

$1, 2, 6, 24, 120, 720, \ldots \quad (b_n)$

$0, 3, 8, 15, 24, 35, 48, \ldots \quad (c_n)$

NOTATION

(a_n) — whole sequence

a_n — nth term

n usually ranges from 0 or 1 up to infinity

How to specify sequence?

(a) $\boxed{\text{EXPLICIT}}$ $\quad a_n = f(n)$

Ex $\quad c_n = n^2 - 1$

(b) $\boxed{\text{RECURSIVE}}$

- Specify finite number of initial elements $a_1, \ldots, a_k$
- Define a_n for $n > k$ in terms of previous elements

Ex

$a_1 = a_2 = 1$

$a_n = a_{n-1} + a_{n-2}$ FIBONACCI SEQUENCE

$b_1 = 1$

$b_n = n \cdot b_{n-1}$ $\quad n!$ n - FACTORIAL

F IX

Recursive definitions often appear when analysing running time of recursive programs

Ex $\boxed{\text{BINARY SEARCH}}$

T_n : time for list of size n

$T_1 = k_1$ some constant k_1

$T_n \leq k_2 + T_{\lfloor n/2 \rfloor}$ some constant k_2

$\boxed{\text{MERGE SORT}}$

T_n : time for list of size n

$T_1 = k_1$

$T_n \leq k_2 \cdot n + 2 \cdot T_{\lceil n/2 \rceil}$

Can identify sequence (a_n) (with) function $f : \mathbb{N} \to \mathbb{R}$ defined by

$$f(n) = a_n$$

Can use asymptotic notation also for sequences

BINARY SEARCH $T_n = O(\log n)$

MERGE SORT $T_n = O(n \log n)$

SERIES

F X

Summation symbol $\displaystyle\sum_{i=1}^{n} a_i$ — UPPER LIMIT (n), SUMMATION INDEX (i), LOWER LIMIT (1)

"Sum of a_i from $i=1$ to n"

Meaning: $a_1 + a_2 + \ldots + a_{n-1} + a_n$

Given sequence (a_n) $\quad a_1, a_2, a_3, \ldots$

A SERIES is a sequence (S_k) whose kth term is given by

$$\underline{S_k} = a_1 + a_2 + \ldots + a_k = \boxed{\sum_{i=1}^{k} a_i}$$

Equivalently, can write

$$\begin{cases} S_k = S_{k-1} + a_k \\ S_1 = a_1 \end{cases}$$

Series also show up in analysis of algorithm running times

<u>Ex</u> For INSERTION SORT we had (essentially)

$$c \cdot \sum_{i=1}^{n} i = \Theta(n^2)$$

USEFUL IDENTITIES

$$\sum_{i=1}^{n} (a_i + b_i) = \sum_{i=1}^{n} a_i + \sum_{i=1}^{n} b_i$$

$$\sum_{i=1}^{n} c \cdot a_i = c \cdot \sum_{i=1}^{n} a_i$$

$$\sum_{i=1}^{n} a_i = \sum_{i=1}^{m} a_i + \sum_{i=m+1}^{n} a_i \qquad [m \leq n]$$

SOME USEFUL FACTS

$$\boxed{\log(n!) = \Theta(n \log n)}$$

[see notes on Absalon]

$$\boxed{\sum_{i=1}^{n} \frac{1}{i} = \Theta(\log n)}$$

Exercise Prove this by

- comparing $\sum_{i=1}^{n} \frac{1}{i}$ and $\int_{i=1}^{n} \frac{1}{x}\, dx$
- using that $\frac{d \ln(x)}{dx} = \frac{1}{x}$

EXPLICIT EXPRESSIONS FOR COMMON SERIES

$$1+2+3+4+\ldots+(n-1)+n$$

$$\boxed{\sum_{i=1}^{n} i = \frac{n(n+1)}{2}} = \Theta(n^2)$$

$$1+4+9+16+\ldots+(n-1)^2+n^2$$

$$\boxed{\sum_{i=1}^{n} i^2 = \frac{n(n+1)(2n+1)}{6}} = \Theta(n^3)$$

$$c+c^2+c^3+c^4+\ldots+c^{n-1}+c^n$$

$$\boxed{\sum_{i=1}^{n} c^i = \frac{c^{n+1}-c}{c-1}} = \Theta(c^n)$$

$c \neq 1$ for $c > 1$

F XIII

How can we prove

$$\sum_{i=1}^{n} i = \frac{n(n+1)}{2} \quad ?$$

Check some values

n	$\sum_{i=1}^{n} i$
1	$1 = 1 \cdot 2/2$
2	$1+2 = 3 = 2 \cdot 3/2$
3	$1+2+3 = 6 = 3 \cdot 4/2$
4	$1+2+3+4 = 10 = 4 \cdot 5/2$

Not a proof... Check some more

5

$$1+2+3+4+5 =$$

$$\boxed{(1+2+3+4)} + 5 =$$

$$\boxed{\frac{4 \cdot 5}{2}} + 5 =$$

$$\frac{4 \cdot 5 + 2 \cdot 5}{2} =$$

$$\frac{5(4+2)}{2} = \frac{5 \cdot 6}{2}$$

6

$$\boxed{1+2+3+4+5} + 6 =$$

$$\boxed{\frac{5 \cdot 6}{2}} + 6 =$$

$$\frac{5 \cdot 6 + 2 \cdot 6}{2} =$$

$$\frac{6 \cdot (5+2)}{2} = \frac{6 \cdot 7}{2}$$

Suppose we checked up to n
For $n+1$ we get

$$\boxed{1+2+\dots+(n-1)+n}+(n+1) =$$

$$\boxed{\frac{n(n+1)}{2}} + (n+1) =$$

$$\frac{n(n+1) + 2(n+1)}{2} =$$

THE EXPRESSION
WE WERE
LOOKING FOR

$$\frac{(n+2)(n+1)}{2} = \frac{(n+1)(n+2)}{2}$$

$\boxed{\text{Is this a proof?}}$

What about $c + c^2 + c^3 + c^4 + \dots$

n	$\sum_{i=1}^{n} c^i$
1	$c = \frac{c^2-c}{c-1} = c\left(\frac{c-1}{c-1}\right)$
2	$c+c^2 = \frac{(c^2+c)(c-1)}{c-1} = \frac{c^3-c}{c-1}$
3	$c+c^2+c^3 = \frac{(c^3+c^2+c)(c-1)}{c-1} = \frac{c^4-c}{c-1}$

F XV

(4)

$$\boxed{c + c^2 + c^3} + c^4 =$$

$$\boxed{\frac{c^4 - c}{c - 1}} + c^4 =$$

$$\frac{c^4 - c + (c-1)\,c^4}{c - 1} =$$

$$\frac{\cancel{c^4} - c + c^5 - \cancel{c^4}}{c-1} = \frac{c^5 - c}{c - 1}$$

(5)

$$\boxed{c + c^2 + c^3 + c^4} + c^5 =$$

$$\boxed{\frac{c^5 - c}{c - 1}} + c^5 =$$

$$\frac{c^5 - c + (c-1)\,c^5}{c - 1} =$$

$$\frac{\cancel{c^5} - c + c^6 - \cancel{c^5}}{c-1} = \frac{c^6 - c}{c - 1}$$

Suppose we checked correctness up to c^n. For c^{n+1} we get

$$\boxed{c + c^2 + \dots + c^{n-1} + c^n} + c^{n+1} =$$

$$\boxed{\frac{c^{n+1} - c}{c-1}} + c^{n+1} =$$

$$\frac{c^{n+1} - c + (c-1)\,c^{n+1}}{c-1} =$$

$$\frac{\cancel{c^{n+1}} - c + c^{n+2} - \cancel{c^{n+1}}}{c-1} =$$

$$= \frac{c^{n+2} - c}{c-1}$$

Is this a proof?

We have shown that

(a) formula correct for small values of n

(b) if formula correct for n, then correct for $n+1$

MATHEMATICAL INDUCTION — more about this next time!