PARENTHESIS THEOREM (THM 20.7)

In depth-first search of (directed or undirected) G, for any two vertices u, v exactly one of the following 3 conditions hold

(1) Intervals [u.dtime, u.ftime] and [v.dtime, v.ftime] are entirely disjoint, and neither u nor v i descendant of the other in depth-first forest

(2) [u.dtime, u.ftime] ⊆ [v.dtime, v.ftime] and u is descendant of v

(3) [v.dtime, v.ftime] ⊆ [u.dtime, u.ftime] and v is descendant of u

Proof By case analysis

Suppose $\boxed{u.dtime < v.dtime}$

Subcase (a): $\boxed{v.dtime < u.ftime}$

Then v descendant of u.

⟹ Recursive call for v finishes before recursive call for u

⟹ v.ftim < u.ftime so [v.dtime, v.ftime] ⊆ [u.dtime, u.ftime]

Subcase (b): $\boxed{v.dtime > u.ftime}$, so

u.dtime < u.ftime < v.dtime < v.ftime

Neither vertex was discovered while other was grey, so neither is descendant of other

Other case is $\boxed{u.dtime > v.dtime}$
Run same proof but with u & v exchanged $\boxed{\equiv}$

COROLLARY (COR 20.8)

v is a proper descendant of u in depth first forest for (directed or undirected) G
IF AND ONLY IF
u.dtime < v.dtime < v.ftime < u.ftime

Proof Immediate from Parenthesis Theorem

WHITE-PATH THEOREM (THM 20.7)

In depth-first forest for (directed or undirected) G it holds that w is descendant of v
IF AND ONLY IF
at time $v.d_{time}$ there is a path in G from v to w consisting only of white vertices

Intuition The only way w can become descendant of v is by chain of DFS-visit calls to white vertices along path to w

Formal proof

($\Rightarrow$) If $w = v$, then vacuously true.
Suppose w any proper descendant of v
By corollary of Parenthesis Theorem, all such w have $w.d_{time} > v.d_{time}$, i.e. whole subtree of descendants is white

($\Leftarrow$) Suppose $v \underset{\text{path}}{\overset{\text{white}}{\rightsquigarrow}} w$

Argue by induction over length of path

Base case $w = v$ trivial

Induction hypothesis

For white path of length $\leq k$ $v \rightsquigarrow w$ it holds that w descendant of v

Induction step

Consider path of length $k+1$

$v \rightsquigarrow w' \to w$

By IH, w' descendant of v

By corollary $\boxed{w'.ftime \leq v.ftime}$

w must be discovered after v, since $\boxed{v.dtime < w.dtime}$

w is definitely discovered before w' is finished. Hence

$v.dtime < w.dtime \leq w'.ftime \leq v.ftime$

But then by Parenthesis Theorem

$v.dtime < w.dtime < w.ftime < v.ftime$

and w is also descendant of v

CLASSIFICATION OF DIRECTED EDGES IN G after DFS on G

1) $\boxed{\text{TREE EDGES}}$ (u,v) edges in the depth-first forest

2) $\boxed{\text{BACK EDGES}}$ (u,v) connecting vertex u back to ancestor in depth-first tree

3) $\boxed{\text{FORWARD EDGES}}$ (u,v) non-tree edge connecting u to descendant v in depth-first tree

4) $\boxed{\text{CROSS EDGES}}$ (u,v) all other edges in G

Type of edge (u,v) determined when out-neighbour v inspected in call DFS-VISIT(G, u):

- If v WHITE, then (u,v) tree edge
- If v GRAY, then (u,v) back edge
- If v BLACK, then (u,v) forward edge or cross edge

We will care most about tree edges and back edges. See CLRS pages 567–570 for more about all of this