DMFS: COMBINATORICS

C I

COMBINATORICS: Counting different objects/possibilities

Ex 1) Outcomes for gold, silver, bronze medals for 10 competitors
2) ~~Sum~~ Number of ~~three~~ ways dice can come up
3) Possible 5-card hands

- Multiplication principle
- Ordered / unordered; with/without repetition
- When order matters
  - sequences (with repetitions)
  - permutations (without repetitions)
- When order doesn't matter
  - subsets (without repetitions)
  - multisets (with repetitions)

MULTIPLICATION PRINCIPLE

If n_1 ways of performing task T_1
" n_2 — " — T_2
then $n_1 \cdot n_2$ ways of performing both tasks together

Ex CS researcher outfits
Blue or black jeans
white, black, or blue t-shirt
How many outfits?

GENERAL MULTIPLICATION PRINCIPLE

k tasks, n_i ways for task i
Then $\prod_{i=1}^{k} n_i = n_1 \cdots n_k$ ways of performing all tasks together

How many subsets of a set S of n elements ? C II

For each element $s_i \in S$, either include or not
2 choices.
All in all 2^n different choices = subsets
(including empty set)

SEQUENCE ordered collection / list of elements
Ex $(1,2)$; $(2,2,2)$; $(2,1)$; $(1,1,2,3,5,8)$
Note $(1,2) \neq (2,1)$
Sequences without repetitions: PERMUTATION

SET unordered collection of elements without repetitions
$\{1,2\}$; $\{2,1,3\}$; $\{2,1\}$; $\{1,2,3,5,8\}$
Note $\{1,2\} = \{2,1\}$

MULTISET unordered collection of elements, possibly with repetitions
$[1,2]$; $[2,1,2]$, $[2,1]$, $[1,1,2,3,5,8]$
Note $[1,2] = [2,1] \neq [2,1,2]$

	WITH REPETITIONS	WITHOUT REPETITIONS
ORDERED	SEQUENCE	PERMUTATION
UNORDERED	MULTISET	SET

How many different sequences of length r choosing elements from set of size S ?

r choices; n alternatives every time: n^r sequences
by multiplication principle

How many length-r sequences <u>without repetitions</u>?

$$n\cdot(n-1)\cdots(n-r+1) = {}_nP_r$$

"number of permutations of n objects taken r at a time"

$$_nP_r = \frac{n!}{(n-r)!}$$

10 competitions, # gold-silver-bronze outcomes?

$10\cdot 9\cdot 8 = 720$

How many permutations of whole set S?

$$_nP_n = n!$$

"permutation of S" tends to mean permutation of all of S

What if we don't care about order?

size-r subsets of S

$\frac{n!}{(n-r)!}$ possibilities when order matters

Each subset can be chosen in $r!$ ways

So $\frac{n!}{r!(n-r)!} = {}_nC_r$ subsets

More common notation $\binom{n}{r}$ "n-choose-r"

BINOMIAL COEFFICIENT

"number of combinations of n objects taken r at a time"

$$(x+y)^n = \sum_{i=0}^{n} \binom{n}{i} x^i y^{n-i}$$

Note $\binom{n}{r} = \binom{n}{n-r}$

ways to choose r elements =
ways to leaven $n-r$ elements unchosen

How many different 5-card hands?

$${}_{52}C_5 = \binom{52}{5} = \frac{52!}{5!\,47!} = \frac{52\cdot 51\cdot 50\cdot 49\cdot 48}{5\cdot 4\cdot 3\cdot 2\cdot 1}$$

$$= 2,598,960$$

Combinations with repeats:
S set with n elements
How many multisets of r elements?

If S is a set of n elements, then there are $\binom{n+r-1}{r} = \binom{n+r-1}{n-1}$ multisets of size r with elements from S.

<u>Proof</u>

Need to show how to count so that
- each multiset counted once
- no multiset counted more than once
- nothing except multisets counted

Imagine we have $n+r-1$ cells on a row
Colour r cells red to encode multiset

Assume $|S| = n = 7$ and $r = 5$. Every multiset of 5 elements from S should result in some colouring of 5 out of $7+5-1 = 11$ boxes as above

FROM MULTISET TO COLOURING

Let $S = \{s_1, s_2, \ldots, s_n\}$

MULTISET 2 COLOUR

(e will take values $s_1, s_2, \ldots, s_n$ in that order)

Note that we can have c==0

```
pos := 1
for e in S
    Let c := #copies of e in multiset
    for i := 1 upto c
        colour box in position pos red
        pos := pos + 1
    pos := pos + 1    // skip a cell
```

ACTUALLY NOT QUITE RIGHT: MORE CAREFUL ANALYSIS NEEDED

SEE NEXT PAGE

~~Algorithm will colour r cells red~~
~~Index pos will loop over $1, 2, \ldots, n+r-1$~~
~~and final increment pos := pos + 1~~
~~will set pos = $n+r$ (just to the right of boxes)~~

Different multisets will give different colourings
For two multisets $M \neq M'$, look at first s_i that appears different # times in M and M'
There colourings will diverge.

Detour

CV + ε/2

More careful analysis of

MULTISET 2 COLOUR:

By construction, if multiset has r elements, then algorithm will colour r cells red

Great!

But we need to check that it never writes out of bounds, i.e., in position $pos \geq n+r$

Note that at end of algorithm we have $pos = n+r+1$, so there is ~~reason~~ potentially reason to worry

Consider when $e = s_n$ ← LAST ELEMENT at top of for loop. Then

$$\boxed{pos = 1 + (n-1) + (\#\text{elements in multiset except } s_n)}$$

Also note that pos was incremented after colouring, so all coloured cells have indices

$$\leq (n-1) + (\#\text{elements in multiset except } s_n)$$

$$\leq n - 1 + r$$ OK so far!

If we have seen all of multiset, then

$pos = n+r$

but no further colouring will be made.

Otherwise, c copies of s_n will colour positions $n+r-c, n+r-c+1, \ldots, n+r-2, n+r-1$

so we are OK. GOOD, LET'S GO BACK TO THE PROOF...

Since all multisets yield different colourings, this proves that

CV + ε

$$\boxed{\#\text{multisets} \leq \binom{n+r-1}{r}}$$

But it could be that there are <u>more colourings</u> that multisets! To show this is not so, describe how to translate any set of coloured boxes (like on the previous page) to a multiset!

FROM COLOURING TO MULTISET

$S = \{s_1, s_2, \ldots, s_n\}$

COLOUR2MULTISET

```
pos := 1
i := 1      // index of element in S
while pos < n + r
    if cell in position pos is red
        output copy of s_i to multiset
    else
        i := i + 1
    pos := pos + 1
```

There are r red cells (by assumption) so i will be incremented $(n+r-1) - r = n-1$ times, i.e., loops over $1, 2, \ldots, n$, SO s_i ALWAYS VALID ELEMENT

For same reason, OUTPUTS EXACTLY r ELEMENTS

So we get valid multiset. Can also see that different colourings yield different multisets, so $\boxed{\#\text{multisets} \geq \binom{n+r-1}{r}}$

WHY?

ILLUSTRATION OF PROOF

CV 1/2

S set of n elements, $|S| = n$

Choose multiset of r elements

(order doesn't matter, but count copies)

Claim: $\boxed{\binom{n+r-1}{r} \text{ such sets}}$

= Given row with $n+r-1$ cells, how many ways to colour r cells red

Present "translations"

(a) from multiset to colouring

(b) from colouring to multiset

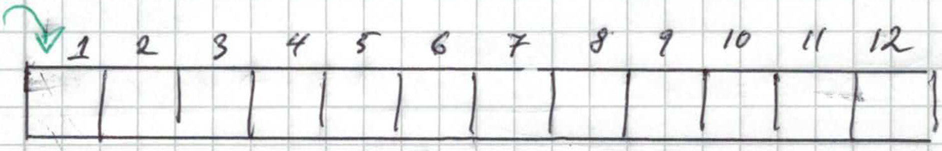
"COMBINATORIAL PROOF"

Example $n = 7$, $r = 6$

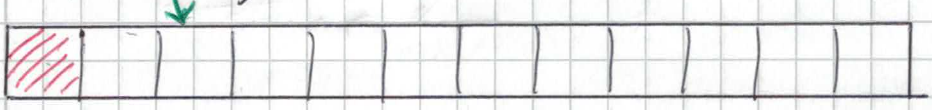
Consider multiset $[1, 3, 3, 3, 4, 4]$

$n + r - 1 = 12$ cells

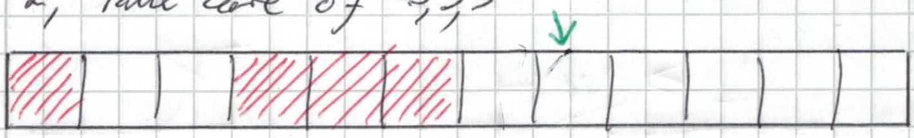
START HERE



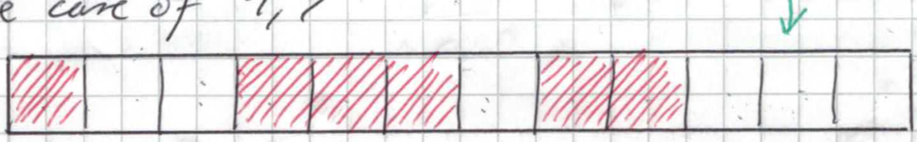
Take care of 1



No 2, take care of 3,3,3



Take care of 4, 4



No 5, no 6, so done!

$\underbrace{}_{\text{no } 7}$

For 5: skip from position 11 to 12

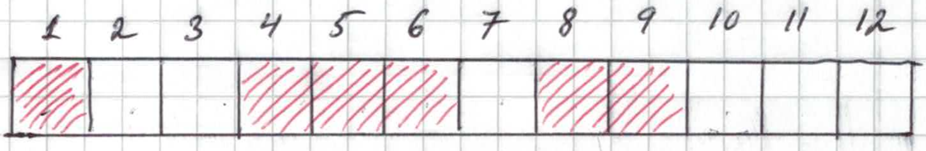
For 6: skip from position 12 right

For 7: Do nothing — last element in universe

How to translate back? C V 2/3

We have $n = 7$, $r = 6$

Consider



Translate back to multiset

(1) Current position 1
1 red, then blank $\Rightarrow$ One copy of 1 in multiset

(2) Current position 3
Blank $\Rightarrow$ No 2 in multiset

(3) Current position 4
3 red, then blank $\Rightarrow$ Three copies of 3 in multiset

(4) Current position 8
2 red, then blank $\Rightarrow$ Two copies of 4 in multiset

(5) Current position 11
Blank $\Rightarrow$ No 5 in multiset

(6) Current position 12
Blank $\Rightarrow$ No 6 in multiset

(7) Now (current position 13) out of bounds $\Rightarrow$ No 7 in multiset

Makes sense, since we have already chosen our 6 elements

Constructed multiset: $[1, 3, 3, 3, 4, 4]$