General Plan

This week we will **move on to the more mathematical parts of the IDMA 2026 course.** We will start by drawing upon your knowledge about functions like

$$2^x \qquad x^{10} \qquad \log_{10} x$$

from your highschool curriculum (which you were already reminded about in the notes on Absalon last week about asymptotic analysis). You will, however, soon experience that the typical math tools within computer science are quite different from the ones you have dealt with before. Formally speaking, the mathematics you already know is mostly **continuous**, while in computer science we mostly care about **discrete** math.

This means that we start at the very beginning with many of the topics. For this, we have chosen the book

B. Kolby, R.C. Busby, and S.C. Ross
Discrete mathematical structures, 6th edition

which we will refer to as *KBR*. We will use a special edition of the book in the course as you have probably already noticed. In addition to this, some of the mathematical contents will come from the book

T. H. Cormen, C. E. Leiserson, R. L. Rivest, and C. Stein
Introduction to Algorithms, 4th edition

which we will refer to as *CLRS*.

*Note that problem set 1 is due on ~~Wednesday Feb 11~~ **Monday Feb 16 at 12:59 CET***.

Lectures

We will begin the week by talking about **sets, sequences and sums.** In particular, when analyzing algorithms it will not seldom be the case that the running time can be expressed as a sum, and so we will be interested in **summation formulas** when performing **asymptotic analysis** (which we already started getting acquainted with last week). The asymptotic notation

used in KBR differs from the one introduced in CLRS. We will favour the CLRS notation as it is the one commonly used in the context of computer science. The summation formulas are covered later on in the KBR book and could thus be difficult to read.

We will then switch to considering **fundamental properties of integers** related to division and discuss concepts such as divisor, multiple, prime, greatest common divisor (GCD), and least common multiple (LCM). In particular, we will go over Euclid's algorithm, one of the first algorithmic discoveries of mankind, which lets us calculate GCD of two integers efficiently. We will also discuss different ways to represent integers: first, in terms of their prime factorization and, second, using the base-b positional notation. Note that positional notation with $b = 10$ is our standard way of writing numbers while $b = 2$ is used to represent numbers on a computer. All of this is nicely covered in KBR Section 1.4, and quite likely you are already familiar with most of it.

Towards the end of the week, we will encounter for the first time *one of the most important topics in this whole course*, namely **mathematical induction**. This will be our preferred method to prove statements such as

$$\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$$

(which is, strictly speaking, a sequence of statements $P(n)$ parameterized by the positive integer n). During this and next week, we will discuss how to set up and execute a proof by induction, and we will see that this type of argument is well-suited for proving statements about recursive algorithms.

Since mathematical induction is such an important element of the course, wealso provide some brief notes on mathematical induction on Absalon that might be an easier read than KBR. The difference, however, is not that big, and so you should also read Section 2.4 in KBR. There might be some terminology in Section 2.4 that is not yet fully familiar—if so, please just ignore this for now, and we will get to it later (next week if everything goes according to plan).

Reading Instructions

- ***Recap:* CLRS Chapter 3:** We talked about asymptotics already last week, but stressed the intuition that this is all about *"focusing on the highest-order term and shaving away the constant factor in front."* This is a good intuition, and will help you solve most problems you will

encounter regarding asymptotic analysis, but of course we also need a firm, mathematical understanding of what it is we are doing. For this reason, do make sure to have read the discussion of asymptotic notation on pages 50–63. All of this material is needed, although little-o (o) and little-omega (ω) are less important that the other pieces of notation. Pages 63–68 discuss common functions and standard notation that you will want to become friends with. The more advanced material on pages 68–70 about function iteration, iterated logarithms, and Fibonacci sequences is not so important for now (but please note that for a basic course like this, pretty much anything you run into has a high chance/risk of appearing in later courses, so it might be a good time investment for the future to at least look at it anyway).

- **KBR Sections 1.1–1.3:** We will need all of this material on sets, sequences, and sums.

- **KBR Section 1.4** on integers. (Make sure you can follow the steps of the proof of Theorem 4, as this is a good example of a more advanced proof. In general, a good test for whether you have fully understood a proof is whether you can reproduce it—or at the main steps in it—without looking in the book.)

- **KBR Section 1.5** on matrices will be needed later in the course—why not read it right away?

- KBR Section 1.6 is **not** part of the required reading, but could be useful if you want to go deeper into the mathematics and think more abstractly about what is going on.

- **KBR Section 2.4** on mathematical induction. Again, it cannot be stressed enough how important the concept of mathematical induction is, so this is something that you should really study, and study hard (not just for this course).

- **Some Notes on Mathematical Induction:** There are also notes posted on Absalon which mostly overlap with the material above. It is recommended to read these notes to see a second exposition of the same material, and thus double your chances of getting a deep understanding of it. ;-)

Please note that all of the above is foundational material that we will use over and over again in the course and that you will also be assumed to know well in later courses. Therefore, it is to do good service to

yourself to read this carefully, and do enough exercises (see below) to be sure that you understand what is going on.

Suggested Exercises

How many exercises should you do? The short answer to this is that you should solve enough problems to be sure that you know the material. In general, we provide many exercises to work on, just to be sure that you will not run out of entertainment, but there is no need per se to solve absolutely all of these exercises. What you need to do, though, is to make sure that you work on enough exercises that you have fully digested the topics covered in the lectures and the textbooks.

As already mentioned, you are also encouraged to work on the problem set problems from previous years that are posted on Absalon, but it is probably a good idea to do some of the exercises below first to make sure that you have understood the course material.

Exercises from KBR on Integers and Induction

At the end of the KBR book you can find solution to all odd-numbered exercises. It goes without saying, though, that it is advisable not to look at the solution until after you have solved the exercise or if you are completely stuck. Many of the early KBR exercises are testing basic understanding and so should hopefully be fairly quick to solve—if not, then do make use of the fact that the TAs are there to help you!

(1) KBR exercises 1.2.1, 1.3.7, and 1.3.10

(2) KBR exercises 1.4.10, 1.4.13, 1.4.24, and 1.4.35.

(3) Solve KBR exercises 1.4.43 and 1.4.44, and

   (a) Argue that the running time of the algorithm given in KBR for finding the base b expansion of $n \in \mathbb{Z}$ is $O(\log n)$.

   (b) Why is it clear that the runtime of this algorithm must be $\Theta(\log n)$?

(4) KBR exercises 2.4.3, 2.4.6, and 2.4.16.

(5) If you read about matrices, then solve (some selection of) KBR exercises 1.5.5, 1.5.9, 1.5.12, 1.5.16, and 1.5.21 to verify your basic understanding of the relevant concepts (we will return to this later in the course).

Some More Exercises on Asymptotic Analysis and Such

Unless explicitly stated otherwise, you are free to use theorems from the notes on Absalon without a proof, as well as make use of any previously solved exercises.

(1) KBR exercises 5.3.1, 5.3.5, and 5.3.9.

(2) Let $f, g : \mathbb{R}^+ \to \mathbb{R}$ be asymptotically positive functions. Prove the following:

- **(Θ-relation is symmetric)** $f(x)$ is $\Theta(g(x))$ if and only if $g(x)$ is $\Theta(f(x))$.
- If $f(x)$ is $o(g(x))$ then $g(x)$ is *not* $\Theta(f(x))$.

(3) Assume that $a, b, c > 0$. Use the properties of the logarithms from last week's notes on functions and asymptotic analysis to show that

- $(\log_a b)(\log_c d) = (\log_a d)(\log_c b)$.
- $\log_b a = (\log_a b)^{-1}$

where, in each equation above, logarithm bases are not 1.

(4) Use summation formulas to calculate

$$\sum_{k=1}^{100}(k^2 + k) + \sum_{k=0}^{10} 5^k$$

Some Additional Exercises (Possibly for Later)

Here are some extra exercises that might be useful if you want to repeat this material when preparing for the exam.

(1) KBR exercises 1.2.14 and 1.3.38.

(2) KBR exercises 1.4.11–12, 1.4.16, and 1.4.25.

(3) KBR exercise 1.4.41.

(4) KBR exercises 2.4.8, 2.4.10, 2.4.27, and 2.4.29.

(5) KBR exercises 5.3.7 and 5.3.10.

(6) Use the definition of big-O to prove or disprove the following

- 2^{x+1} is $O(2^x)$
- 2^{2x} is $O(2^x)$

(7) Let $f, g : \mathbb{R}^+ \to \mathbb{R}$ be asymptotically positive functions. Consider the following two statements:

   (a) $g(x)$ is $O(f(x))$ and $f(x)$ is $O(g(x))$.

   (b) There exist constants $c_1, c_2 > 0$ and x_0 such that $c_1 g(x) \leq f(x) \leq c_2 g(x)$ for all $x \geq x_0$.

Show that (a) holds if and only if (b) holds thus establishing the equivalence of the two definition of big-Θ from last week's notes on functions and asymptotic analysis.

Remember that proving an "if and only if" statement involves showing two directions.

(8) Use the rules $(B1)$–$(B6)$, $(L1)$–$(L6)$, $(M1)$–$(M4)$ from last week's notes on functions and asymptotic analysis to show the following.

- $10x^{10}$ is $O(2^x)$
- $10x^{10}$ is $o(2^x)$
- $x^2 - x$ is $O(3x^2 + 2x)$
- $3x^2 + 2x$ is $O(x^2 - x)$.
- $x^2 - x$ is $\Theta(3x^2 + 2x)$
- x^3 is $o(x2^x)$
- $\frac{\log_2 x}{x^2}$ is $O(1)$

(9) Let $f, g : \mathbb{R}^+ \to \mathbb{R}$ be asymptotically positive functions. Prove rule $(L1)$ from the notes on functions and asymptotic analysis. In your proof you can use the rest of the rules and theorems from the notes as well as the following rule:

(L1') If $f(x)$ is $o(g(x))$ then $g(x) \pm f(x)$ is $\Theta(g(x))$.

(10) Would rule $(M2)$ hold if we dropped the requirement that the constant c is positive? Justify your answer.

(11) Let $f(n) = 2^{\log_3(n)}$ and $g(n) = n$. Find a such that $f(n) = n^a$. Then determine whether $f(n)$ is $O(g(n))$ and whether $g(n)$ is $O(f(n))$.