

IDMA Course Material

– Some Notes on Mathematical Induction –

A Word of Advice

You really, really, really want to learn what mathematical induction is and how it works. Please do yourself a favour, and don't let yourself rest until you have grasped this. It is very hard to overemphasize how absolutely fundamental this will be for your future studies.

Preliminaries

Let us try to prove that for every $n \in \mathbb{N}$, we have that the formula

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6} \quad (1)$$

holds. Proving this gives a summation formula for the series of the first n square numbers.

The naive approach is to start from the beginning of the series. Explicit calculation gives us that

$$\begin{aligned} 1^2 &= 1 = \frac{6}{6} = \frac{1(1+1)(2+1)}{6}, \\ 1^2 + 2^2 &= 5 = \frac{30}{6} = \frac{2(2+1)(4+1)}{6}, \\ 1^2 + 2^2 + 3^2 &= 14 = \frac{84}{6} = \frac{3(3+1)(6+1)}{6}, \end{aligned}$$

It is tempting to continue in this manner, calculate a couple of statement more, and then just write

⋮
etc.

and say that the statement has been proven. This is, however, very much not a mathematical proof, and is actually borderline false as illustrated by the following example.

Example 1. Consider the following two sequences explicitly defined by

$$a_n = \left\lfloor \frac{2n}{\ln 2} \right\rfloor$$

and

$$b_n = \left\lceil \frac{2}{2^{1/n} - 1} \right\rceil.$$

Both of them start as

2, 5, 8, 11, 14, 17, 20, 23, 25, 28, 31, 34, 37, 40, 43, 46, 49, 51, 54, 57,
60, 63, 66, 69, 72, 75, 77, 80, 83, 86, 89, 92, 95, 98, 100, 103, 106,
109, 112, 115, 118, 121, 124, 126, 129, 132, 135, 138, 141, 144

It is actually easy to check on a computer that the two sequences have identical elements for the first couple of million values of n . It is therefore tempting to conclude that $a_n = b_n$ for all n but it turns out that

$$a_{777451915729368} = 2243252046704766 \neq 2243252046704767 = b_{777451915729368}$$

The above example illustrates that even if we used a computer to check that the formula (1) was true for all n between 1 and 100000, we would still not be able to answer the following two questions:

- (i) Is the formula valid for all $n \in \mathbb{N}$?
- (ii) Why is it true?

These two questions are highly connected and it is difficult to answer (i) without also answering (ii).

We will therefore try another approach. Let us define a sequence (a_n) by

$$a_n = \frac{n(n+1)(2n+1)}{6}$$

(note this is the right-hand side of the formula (1)) and try to prove the statement

$$a_n + (n+1)^2 = a_{n+1} \tag{2}$$

instead. This turns out to be relatively easy since by lifting the parenthesis, we get

$$\begin{aligned} a_n + (n+1)^2 &= \frac{1}{6} [n(n+1)(2n+1) + 6(n+1)^2] \\ &= \frac{1}{6} [2n^3 + 9n^2 + 13n + 6] \end{aligned}$$

and

$$\begin{aligned} a_{n+1} &= \frac{1}{6} ((n+1)(n+1+1)(2(n+1)+1)) \\ &= \frac{1}{6} [2n^3 + 9n^2 + 13n + 6] \end{aligned}$$

The formula (2) implies that **if** we know that (1) applies to some particular value of n then we can conclude that it also applies to $n+1$ since

$$\begin{aligned} 1^2 + \cdots + n^2 + (n+1)^2 &= a_n + (n+1)^2 && [\text{Formula (1) is valid for } n] \\ &= a_{n+1} && [\text{Formula (2)}] \\ &= \frac{[n+1]([n+1]+1)(2[n+1]+1)}{6} && [\text{Def. of } a_n] \end{aligned}$$

We have already shown that the formula is valid for $n = 3$. The above argument then shows that it is also valid for $n = 4$. This means that it is also valid for $n = 5$, which means that it is valid for $n = 6 \dots$ etc.

We are now getting close to answering questions (i) and (ii). But our argument still contains a sequence of dots and the statement “et cetera (etc.)”. Mathematical induction, which we will describe below, is the standard method to deal with “... etc.”.

Note how the statement (2) turned out to be the *key* for finding a proof. It is worth noticing how one can get the idea to construct such a statement by *analyzing* the problem. Because **if** (1) was true then

$$a_{n+1} = 1^2 + 2^2 + \cdots + n^2 + (n+1)^2 = a_n + (n+1)^2.$$

In more advanced proofs by induction, a significant part of the task is to find statements that can act as the key to a proof in the same way as (2).

Mathematical induction

Let us now formalize the above ideas as *the principle of mathematical induction*.

If a collection of statements $P(n)$, where n takes the values from $\{n_0, n_0 + 1, \dots\}$, satisfy that

$$P(n_0) \text{ is true} \quad (\text{S})$$

and that for all $n \geq n_0$, we have that

$$\text{If } P(n) \text{ is true, then } P(n + 1) \text{ is true} \quad (\text{T})$$

then $P(n)$ is true for all $n \geq n_0$.

We refer to the statement (S) as *the base case* and the statement (T) as *the induction step*.

Let us complete the argument for (1) formally using this framework. We let $P(n)$ be the statement

$$P(n) : \quad \sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}$$

and we wish to show that $P(n)$ is true for all $n \geq n_0 = 1$.

base case: We need to show that $P(1)$ is true, i.e., that

$$\sum_{k=1}^1 k^2 = \frac{1(1+1)(2 \cdot 1 + 1)}{6}.$$

This follows trivially since both sides of the equality evaluate to 1.

Induction step: We need to show that if $P(n)$ is true then $P(n + 1)$ is also true. We can therefore assume that

$$\sum_{k=1}^n k^2 = \frac{n(n+1)(2n+1)}{6}.$$

We see that

$$\begin{aligned} \sum_{k=1}^{n+1} k^2 &= \sum_{k=1}^n k^2 + (n+1)^2 \\ &= \frac{n(n+1)(2n+1)}{6} + (n+1)^2 \\ &= \frac{1}{6} [n(n+1)(2n+1) + 6(n+1)^2] \\ &= \frac{1}{6} [2n^3 + 9n^2 + 13n + 6] \\ &= \frac{[n+1]([n+1]+1)(2[n+1]+1)}{6}, \end{aligned}$$

which means that $P(n + 1)$ is true.

Conclusion: Using the principle of mathematical induction we see that $P(n)$ is true for all $n \geq n_0$.

In some situations, it is necessary to use the *strong* principle of induction where the induction step does not only assume that $P(n)$ is true but that all $P(k)$, with $k \leq n$, are true. Formally:

If a collection of statements $P(n)$ indexed from the set $\{n_0, n_0+1, \dots\}$ satisfy that

$$P(n_0) \text{ is true} \quad (\text{S})$$

and that for all $n \geq n_0$, we have that

$$\text{If } P(n_0), \dots, P(n) \text{ are all true, then } P(n+1) \text{ is true} \quad (\text{T}')$$

then $P(n)$ is true for all $n \geq n_0$.

Depending on how the induction step is carried out in some cases as part of the base case it may be necessary to check extra statements in addition to $P(n_0)$. One case where this happens is if we need to make use of more than one of the previous statements $P(n_0), \dots, P(n)$ in order to complete the induction step.

We will conclude with an example of how to use the principle of strong induction. We define a sequence (b_n) recursively by setting $b_1 = 0$ and

$$b_n = b_{\lfloor n/2 \rfloor} + 1$$

for all $n > 1$. We want to show that

$$P(n) : \quad b_n = \lfloor \log_2(n) \rfloor.$$

Base case: We need to show that $P(1)$ is true, i.e,

$$b_1 = \lfloor \log_2(1) \rfloor,$$

which follows because both sides of the equality are 0.

Induction step: We use the principle of strong induction and thus, we have to show that if all statements $P(1), P(2), \dots, P(n)$ are true, then $P(n+1)$ is also true. Let us set $m = \lfloor (n+1)/2 \rfloor$. Then we have $b_{n+1} = b_m + 1$. We will use that $P(m)$ is true, which follows from our assumption since $m = \lfloor (n+1)/2 \rfloor \leq n$.

If n is odd then $n + 1$ is even and we have that $m = (n + 1)/2$ and

$$\begin{aligned}
 b_{n+1} &= b_m + 1 \\
 &= \lfloor \log_2(m) \rfloor + 1 && [\text{Here we use that } P(m) \text{ is true!}] \\
 &= \lfloor \log_2(m) + 1 \rfloor \\
 &= \lfloor \log_2(m) + \log_2(2) \rfloor \\
 &= \lfloor \log_2(2m) \rfloor \\
 &= \lfloor \log_2(n + 1) \rfloor.
 \end{aligned}$$

It follows that $P(n + 1)$ is true.

If n is even then $n + 1$ is odd and we have that $m = n/2$ and

$$\begin{aligned}
 b_{n+1} &= b_m + 1 \\
 &= \lfloor \log_2(m) \rfloor + 1 && [\text{Here we use that } P(m) \text{ is true!}] \\
 &= \lfloor \log_2(m) + 1 \rfloor \\
 &= \lfloor \log_2(m) + \log_2(2) \rfloor \\
 &= \lfloor \log_2(2m) \rfloor \\
 &= \lfloor \log_2(n) \rfloor.
 \end{aligned}$$

This is not yet the desired statement $P(n + 1)$. However, note that for all n between the even number 2^k and the odd number $2^{k+1} - 1$, we have that $\lfloor \log_2(n) \rfloor = k$. Hence, for all even $n > 0$, we have $\lfloor \log_2(n) \rfloor = \lfloor \log_2(n + 1) \rfloor$. So we conclude that $b_{n+1} = \lfloor \log_2(n + 1) \rfloor$ and thus $P(n + 1)$ is true in this case too.

Conclusion: Using the principle of strong induction we see that $P(n)$ is true for all $n \geq 1$.

Credits: This is a lightly edited version of notes prepared for the DMA 2019 course by Laura Mancinska, who was in turn inspired by notes written by Søren Eilers.