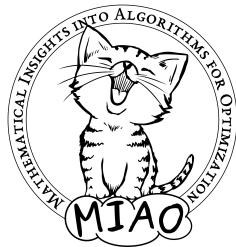


Complexity Theory for Real-World Computation

Jakob Nordström

University of Copenhagen and Lund University

Universidade Federal de Minas Gerais
Belo Horizonte, Brazil
September 19, 2025

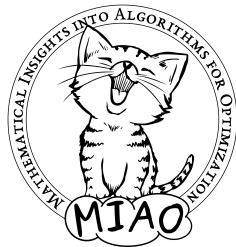


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Three Simple Problems...

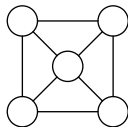
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Does the graph $G = (V, E)$ have a **colouring** with k colours such that all neighbours have distinct colours?

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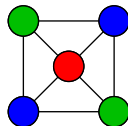


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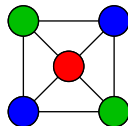


3-colouring? Yes

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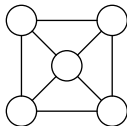
3-colouring? Yes, but no 2-colouring

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CLIQUE

Is there a **clique** in the graph $G = (V, E)$ with k vertices that are all pairwise connected by edges in E ?

Three Simple Problems...

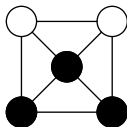


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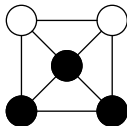


3-clique? Yes

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Three Simple Problems...



3-clique? Yes, but no 4-clique

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Given propositional logic formula, is there a **satisfying assignment**?

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- Variables should be set to **true** or **false**
- Constraint $(x \vee \neg y \vee z)$: means x or z should be true or y false
- \wedge means all constraints should hold simultaneously
- Is there a truth value assignment satisfying all constraints?

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COLOURING: frequency allocation for mobile base stations

CLIQUE: bioinformatics, computational chemistry

SAT: easily models these and many other problems

...with Huge Practical Implications

- Some more examples of problems that can be encoded as propositional logic formulas:
 - computer hardware verification
 - computer software testing
 - artificial intelligence
 - operations research
 - cryptography
 - bioinformatics
 - et cetera...
- Leads to **humongous formulas** (100,000s or even 1,000,000s of variables)
- Can we use computers to solve these problems efficiently?

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How can we understand real-world algorithms for NP-hard problems?

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How can we understand real-world algorithms for NP-hard problems?

This talk: Use proof complexity (not only conceivable answer)

Algorithmic View of Proof Complexity

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Question 2: Separate talk — lots of recent exciting progress; mostly negative (worst-case) results, e.g., [AM20, GKMP20, dRGN⁺21]

Applications of Proof Complexity

Three applied reasons for proof complexity:

- 1 Understand real-world applied algorithmic paradigms [**this talk**]
- 2 Get ideas for algorithmic improvements (e.g., [EN18, EN20, LBD⁺20, DGD⁺21, DGN21, KBBN22])
- 3 Enhance algorithms to write machine-verifiable certificates of correctness (e.g., [EGMN20, GMN20, GMM⁺20, GN21, GMN22, GMNO22, BBN⁺23, BGMN23, MM23, BBN⁺24, DMM⁺24, GMM⁺24, HOGN24, IOT⁺24, MMN24, DHN⁺25, KLM⁺25, MM25])

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Or just view this as a convenient excuse to study nice computational complexity problems for their own sake... ☺

Outline

- 1 Conflict-Driven Clause Learning and Resolution
 - The SATISFIABILITY Problem in Different Shapes
 - Conflict-Driven Clause Learning (CDCL)
 - Resolution Proof System
- 2 Algebraic and Semi-algebraic Methods
 - Nullstellensatz
 - Gröbner Bases and Polynomial Calculus
 - Cutting Planes and Pseudo-Boolean Solving
- 3 Some Proof Systems We Won't Have Time for
 - Sherali-Adams and Sums of Squares
 - Stabbing Planes
 - Extended Resolution

Formal Description of SAT Problem

- **Variable** x : takes value **true** ($= 1$) or **false** ($= 0$)
- **Literal** ℓ : variable x or its negation \bar{x} (write \bar{x} instead of $\neg x$)
- **Clause** $C = \ell_1 \vee \dots \vee \ell_k$: disjunction of literals
(Consider as sets, so no repetitions and order irrelevant)
- **Conjunctive normal form (CNF) formula** $F = C_1 \wedge \dots \wedge C_m$:
conjunction of clauses

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The SATISFIABILITY (or just SAT) Problem

Given a CNF formula F , is it satisfiable?

Here is our example formula again:

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The Same Problem in Three Different Shapes

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$$(1 - x)(1 - z) = 0$$

$$(1 - y)z = 0$$

$$(1 - x)y(1 - u) = 0$$

$$yu = 0$$

$$(1 - u)(1 - v) = 0$$

$$xv = 0$$

$$u(1 - w) = 0$$

$$xuw = 0$$

For **true** = 1 and **false** = 0, is there a $\{0, 1\}$ -valued solution?

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$$z - yz = 0 \qquad y + (1 - z) \geq 1$$

$$y - xy - yu + xyu = 0 \qquad x + (1 - y) + u \geq 1$$

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$$x - y + u \geq 0$$

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$$-y - u \geq -1$$

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$$-u + w \geq 0$$

$$xuw = 0$$

$$-x - u - w \geq -2$$

For **true** = 1 and **false** = 0, is there a $\{0, 1\}$ -valued solution?

State-of-the-Art SAT Solving in One Slide

High-level description of modern **conflict-driven clause learning (CDCL)** SAT solving (as pioneered in [BS97, MS99, MMZ⁺01]):

- Try to build satisfying assignment for formula (**branching** or **decision heuristic** crucial)
- When partial assignment violates formula, **compute explanation for conflict** and **add to formula** as new clause (**clause learning**)
- Every once in a while, **restart** from beginning (but save computed info)

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$

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Free choice to assign value to variable

Notation $p \stackrel{d}{=} 0$

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Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \bar{u}$ forces $u = 0$

Notation $u \stackrel{p \vee \bar{u}}{=} 0$ ($p \vee \bar{u}$ is reason clause)

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Always propagate if possible, else decide

Add to assignment **trail**

Until satisfying assignment or **conflict**

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$$p \stackrel{d}{=} 0$$

$$u \stackrel{p \vee \bar{u}}{=} 0$$

$$q \stackrel{d}{=} 0$$

$$r \stackrel{q \vee r}{=} 1$$

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Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$

$$p \stackrel{d}{=} 0$$

$$u \stackrel{p \vee \bar{u}}{=} 0$$

$$q \stackrel{d}{=} 0$$

$$r \stackrel{q \vee r}{=} 1$$

$$w \stackrel{\bar{r} \vee w}{=} 1$$

$$x \stackrel{d}{=} 0$$

$$y \stackrel{u \vee x \vee y}{=} 1$$

$$z \stackrel{x \vee \bar{y} \vee z}{=} 1$$

Decision

Free choice to assign value to variable

Notation $p \stackrel{d}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \bar{u}$ forces $u = 0$

Notation $u \stackrel{p \vee \bar{u}}{=} 0$ ($p \vee \bar{u}$ is **reason clause**)

Always propagate if possible, else decide

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$$p \stackrel{d}{=} 0$$

decision
level 1

Decision

Free choice to assign value to variable

Notation $p \stackrel{d}{=} 0$

$$u \stackrel{p \vee \bar{u}}{=} 0$$

$$q \stackrel{d}{=} 0$$

decision
level 2

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \bar{u}$ forces $u = 0$

$$r \stackrel{q \vee r}{=} 1$$

$$w \stackrel{\bar{r} \vee w}{=} 1$$

$$x \stackrel{d}{=} 0$$

decision
level 3

Notation $u \stackrel{p \vee \bar{u}}{=} 0$ ($p \vee \bar{u}$ is **reason clause**)

$$y \stackrel{u \vee x \vee y}{=} 1$$

$$z \stackrel{x \vee \bar{y} \vee z}{=} 1$$

$$\bar{y} \vee \bar{z}$$

Always propagate if possible, else decide

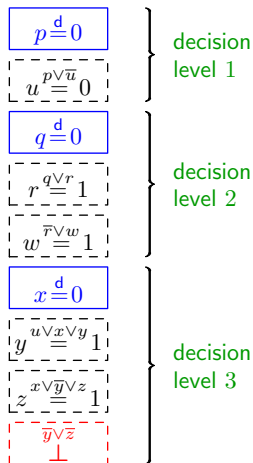
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Until satisfying assignment or **conflict**

Conflict Analysis

Time to analyse this conflict and learn from it!

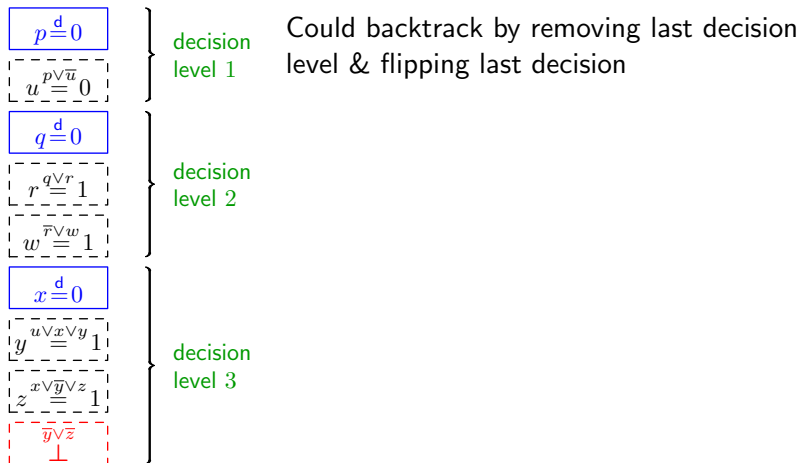
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$$\left. \begin{array}{l} q \stackrel{d}{=} 0 \\ r \stackrel{q \vee r}{=} 1 \\ w \stackrel{\bar{r} \vee w}{=} 1 \end{array} \right\} \begin{array}{l} \text{decision} \\ \text{level 2} \end{array}$$

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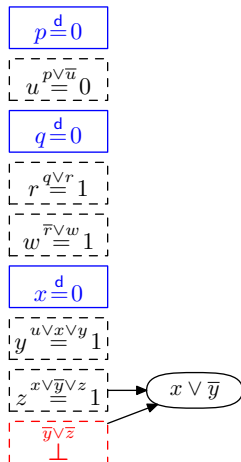
Could backtrack by removing last decision level & flipping last decision

But want to **learn** from conflict and cut away as much of search space as possible

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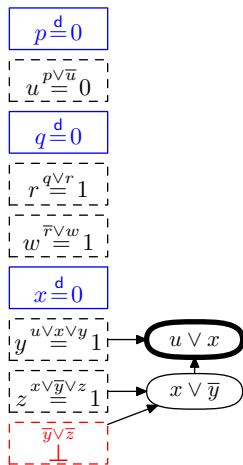
Case analysis over z for last two clauses:

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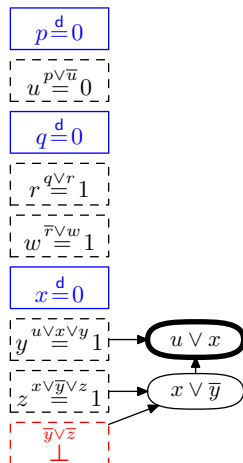
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Repeat until **UIP clause** with only 1 variable after last decision — **learn** and **backjump**

Complete Toy Example for CDCL Execution

Backjump: undo max #decisions while learned clause propagates

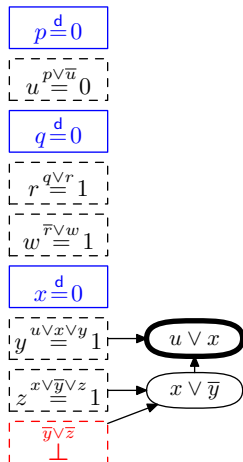
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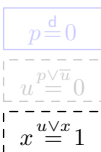
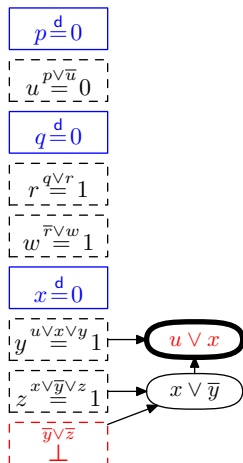


Assertion level 1 (max for non-UIP literal in learned clause) — keep trail to that level

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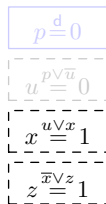
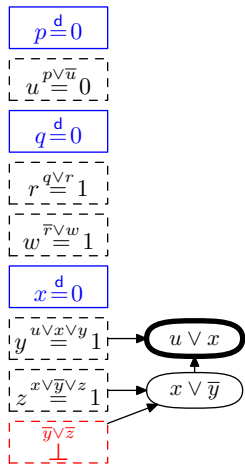
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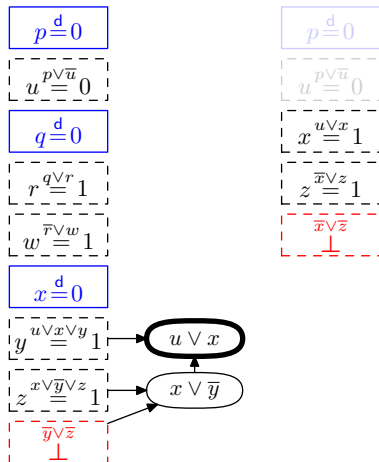
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Then continue as before...

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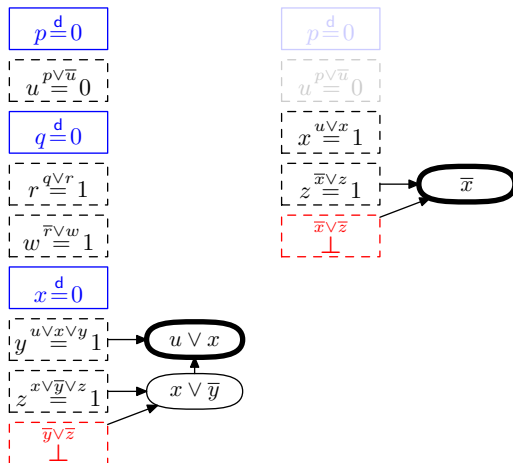
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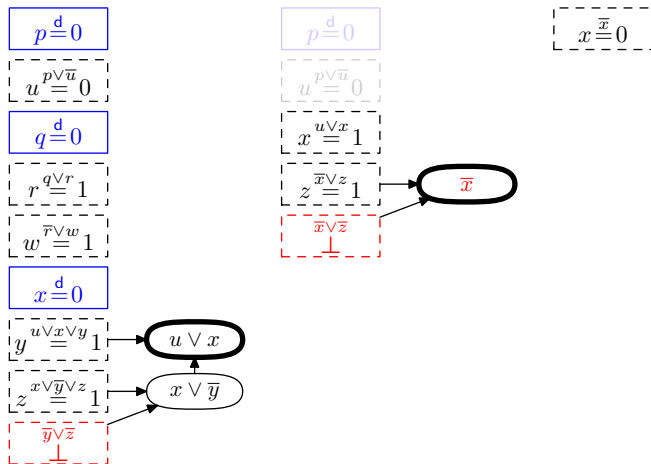
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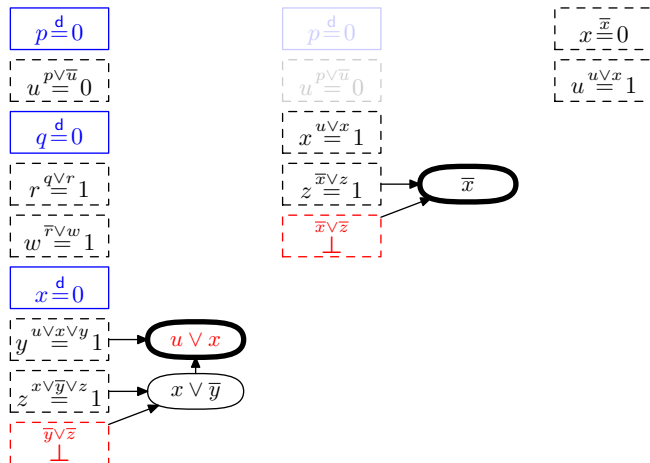
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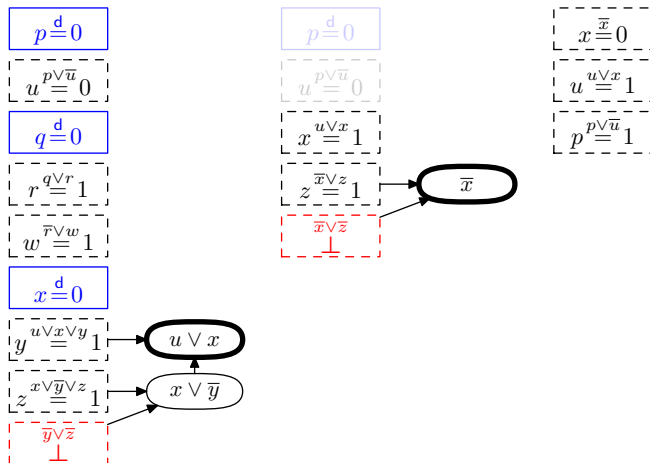
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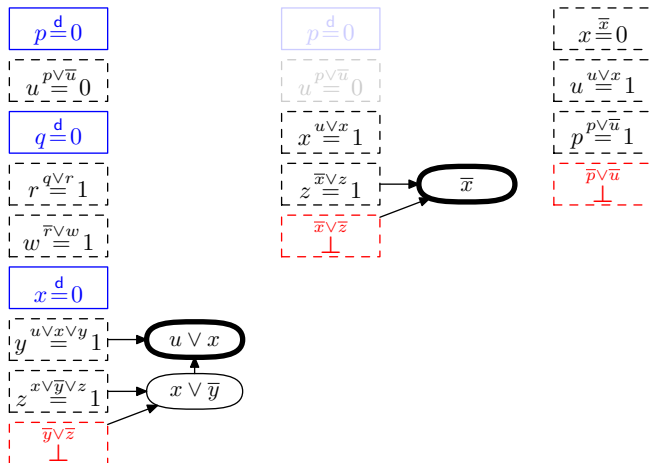
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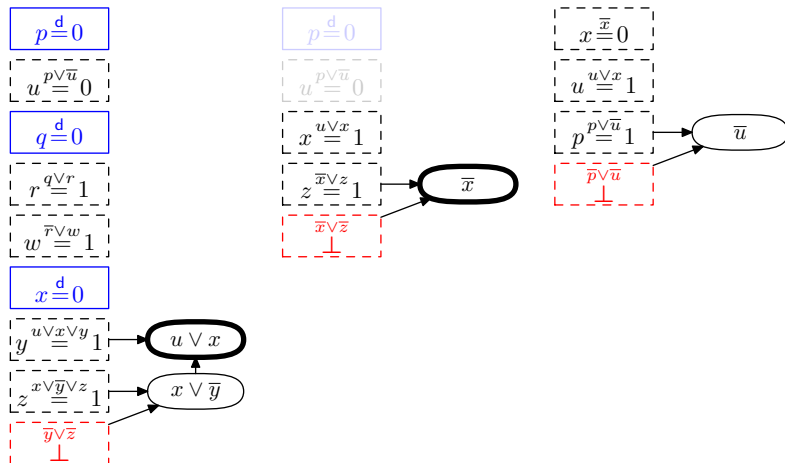
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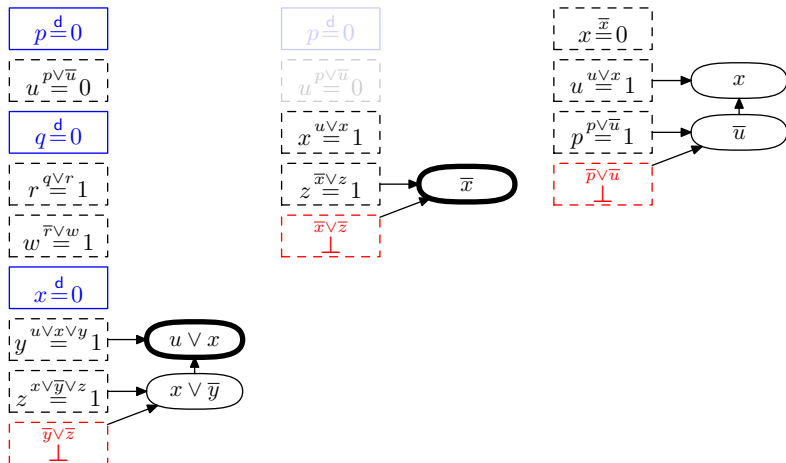
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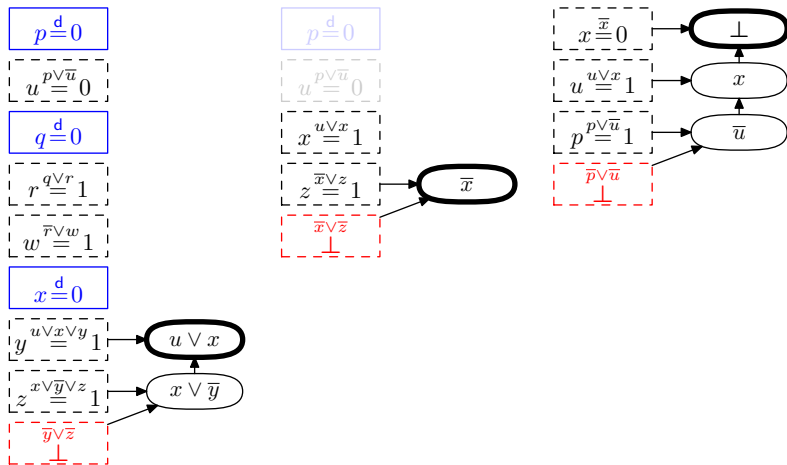
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SAT Solver Analysis and the Resolution Proof System

How to make **rigorous** analysis of SAT solver performance?

Many intricate, hard-to-understand heuristics

So focus instead on **underlying method of reasoning**

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Resolution proof system [Bla37, Rob65]

- Start with clauses of CNF formula (**axioms**)
- Derive new clauses by **resolution rule**

$$\frac{C_1 \vee x \quad C_2 \vee \bar{x}}{C_1 \vee C_2}$$

Resolution Proofs by Contradiction

Resolution rule:

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Observation

If F is a satisfiable CNF formula and D is derived from clauses $D_1, D_2 \in F$ by the resolution rule, then $F \wedge D$ is satisfiable.

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So can prove F **unsatisfiable** by deriving the unsatisfiable empty clause (denoted \perp) from F by resolution

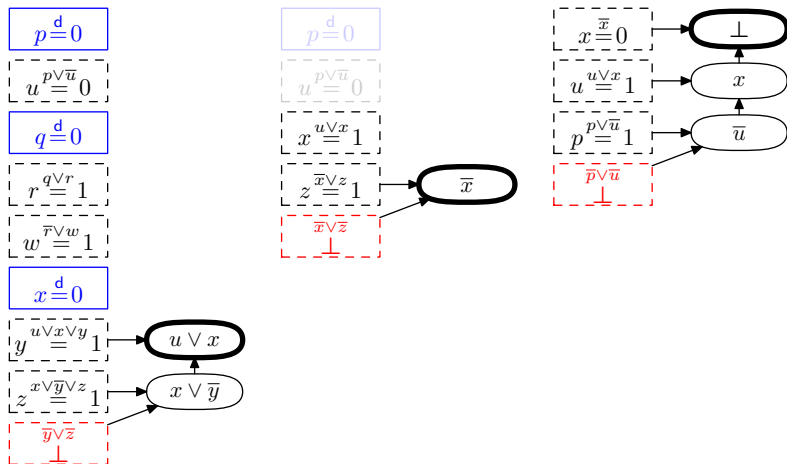
Such proof by contradiction also called **resolution refutation**

CDCL and Resolution Proofs

Obtain resolution proof. . .

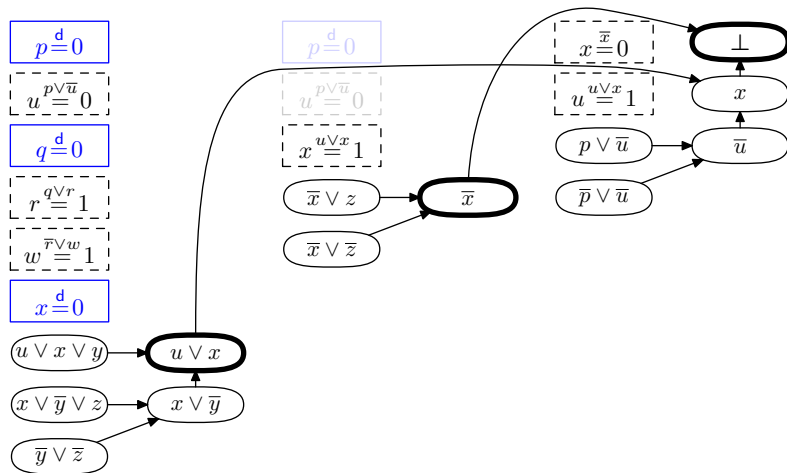
CDCL and Resolution Proofs

Obtain resolution proof from our example CDCL execution...



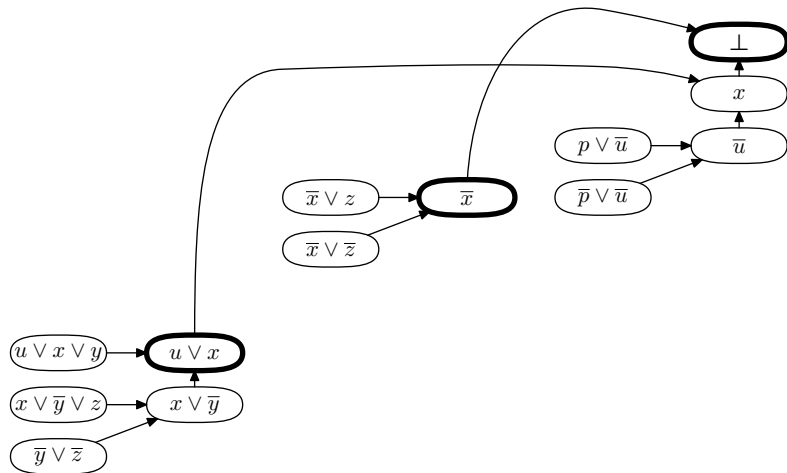
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(*) Except for some **preprocessing techniques**, which is an important omission, but this gets complicated and we don't have time to go into details. . .

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 - Why are applied instances easy?
- Paradox: resolution quite weak proof system; many strong proof complexity lower bounds for (seemingly) “obvious” formulas

Examples of Hard Formulas For Resolution (1/3)

Pigeonhole principle (PHP) formulas [Hak85]

“ $n + 1$ pigeons don't fit into n holes”

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$$p_{i,1} \vee p_{i,2} \vee \cdots \vee p_{i,n}$$

every pigeon i gets a hole

$$\bar{p}_{i,j} \vee \bar{p}_{i',j}$$

no hole j gets two pigeons $i \neq i'$

Can also add “functionality” and “onto” axioms

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Even onto functional PHP hard — **“resolution cannot count”**

Resolution proof requires $\exp(\Omega(n)) = \exp(\Omega(\sqrt[3]{N}))$ clauses
(measured in terms of formula size N)

Examples of Hard Formulas For Resolution (2/3)

Tseitin formulas [Urq87]

“Sum of degrees of vertices in graph is even”

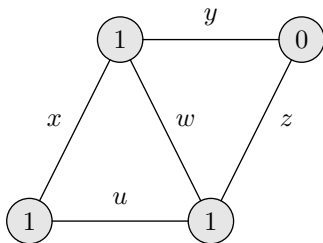
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Tseitin formulas [Urq87]

“Sum of degrees of vertices in graph is even”

Variables = edges (in undirected graph of bounded degree)

- Label every vertex 0/1 so that sum of labels odd
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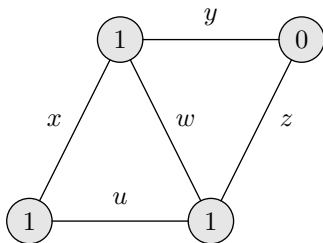
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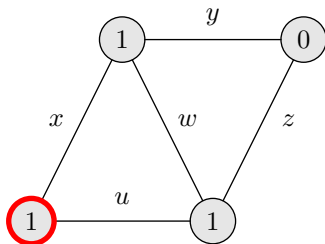
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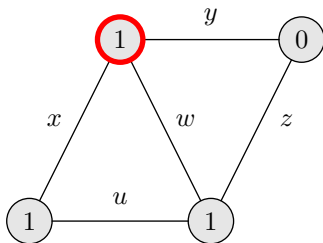
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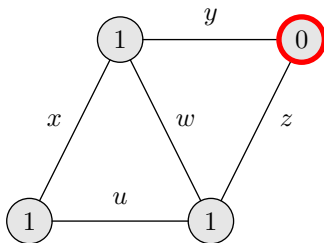
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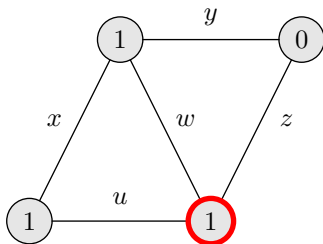
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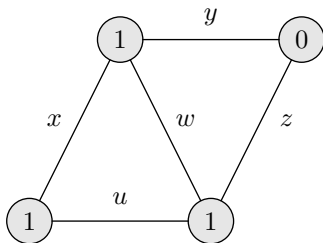
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Requires **proof size** $\exp(\Omega(N))$ on well-connected so-called **expander graphs** — **"resolution cannot count mod 2"**

Examples of Hard Formulas for Resolution (3/3)

Random k -CNF formulas [CS88]

Δn randomly sampled k -clauses over n variables

($\Delta \gtrsim 4.5$ sufficient to get unsatisfiable 3-CNF almost surely)

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And more. . .

- COLOURING [BCMM05]
- Zero-one designs [Spe10, VS10, MN14]
- Et cetera. . . (See, e.g., [BN21] for overview)

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- Et cetera... (See, e.g., [BN21] for overview)

But not CLIQUE!

- Refuting existence of k -clique should require proof size $n^{\Omega(k)}$
- Only known for restricted so-called regular resolution [ABdR⁺21]

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- Add Boolean axioms

$$x_j^2 - x_j = 0$$

for all variables

Hilbert's Nullstellensatz

Consider any system of polynomial equations

$$p_1(x_1, \dots, x_n) = 0$$

$$x_1^2 - x_1 = 0$$

$$p_2(x_1, \dots, x_n) = 0$$

$$x_2^2 - x_2 = 0$$

$$\vdots$$

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Hilbert's Nullstellensatz

System infeasible \Leftrightarrow exist $q_i, r_j \in \mathbb{F}[x_1, \dots, x_n]$ such that

$$\sum_{i=1}^m q_i(x_1, \dots, x_n) \cdot p_i(x_1, \dots, x_n) + \sum_{j=1}^n r_j(x_1, \dots, x_n) \cdot (x_j^2 - x_j) = 1$$

Nullstellensatz Proof System [BIK⁺94]

Nullstellensatz refutation of

$$\begin{array}{ll} p_i(x_1, \dots, x_n) = 0 & i \in [m] \\ x_j^2 - x_j = 0 & j \in [n] \end{array}$$

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Complexity measures of refutations:

- **Size**: number of monomials (when all polynomials expanded out)
- **Degree**: highest total degree of any polynomial

Nullstellensatz Example (Not Expanded out)

$$\begin{aligned} & (x \vee z) \wedge (y \vee \neg z) \wedge (x \vee \neg y \vee u) \wedge (\neg y \vee \neg u) \\ & \wedge (u \vee v) \wedge (\neg x \vee \neg v) \wedge (\neg u \vee w) \wedge (\neg x \vee \neg u \vee \neg w) \end{aligned}$$

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$$(1 - x)(1 - z)$$

$$(1 - y)z$$

$$(1 - x)y(1 - u)$$

$$yu$$

$$(1 - u)(1 - v)$$

$$xv$$

$$u(1 - w)$$

$$xuw$$

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(No use of Boolean axioms)

Nullstellensatz Proof Search

- Solve linear system of equations with coefficients of polynomials q_i, r_j as unknowns
- Used successfully to solve, e.g., graph colouring problems [DLMM08, DLMO09, DLMM11]
- Running time grows exponentially with degree, though high-degree refutations can be very small [BCIP02, dRMNR21]

Dual Variables

- Annoying problem: $x_1 \vee x_2 \vee x_3$ translates to polynomial

$$(1-x_1)(1-x_2)(1-x_3) = 1-x_1-x_2-x_3+x_1x_2+x_1x_3+x_2x_3-x_1x_2x_3$$

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- Doesn't affect degree (obviously), but can decrease size exponentially [dRLNS21] (also for other algebraic proof systems)

Dynamic Construction of Nullstellensatz Certificates

Nullstellensatz again

Infeasibility of

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- **Ideal** \mathcal{I} :

- ① $p, q \in \mathcal{I} \Rightarrow p + q \in \mathcal{I}$

- ② $p \in \mathcal{I} \Rightarrow r \cdot p \in \mathcal{I}$ for any r

- Compute polynomials in this ideal \mathcal{I} step by step

- Use “multivariate division” to check whether 1 lies in ideal or not

Gröbner Bases: Admissible Orderings and Leading Terms

Admissible ordering \preceq on monomials m, m', t :

- ① $m \preceq m' \Rightarrow t \cdot m \preceq t \cdot m'$
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Examples:

- Lexicographic
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Can write $p = \text{lt}(p) + p'$ for $\text{lt}(p)$ leading term (largest w.r.t. \preceq)

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If $\text{lt}(p) = t \cdot \text{lt}(q)$, can **reduce** $p \bmod q$ by computing $p - t \cdot q$

"Multivariate division": Reduce p modulo all q in set of polynomials \mathcal{G} until no further reductions possible

\mathcal{G} is a **Gröbner basis** if final result uniquely determined

Gröbner Bases: Buchberger's Algorithm

Buchberger's algorithm for computing Gröbner bases (**very** rough)

- 1 Let $\mathcal{G} :=$ all axioms
- 2 Pick unprocessed pair $p, q \in \mathcal{G}$ or terminate if none exists
- 3 Compute $p' = t_p \cdot p$ and $q' = t_q \cdot q$ to make leading terms cancel
- 4 Set $S := p' - q'$; reduce $S \bmod \mathcal{G}$ with multivariate division; add result to \mathcal{G} if non-zero
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Facts:

- Buchberger's algorithm computes Gröbner basis
- At termination, $1 \in \mathcal{G} \Leftrightarrow$ polynomial equations infeasible

Polynomial Calculus [CEI96, ABRW02]

- Compute polynomials in ideal \mathcal{I} generated by p_i , $x_j^2 - x_j$, and $x_j + x'_j - 1$ step by step:
 - $p_i \in \mathcal{I}$, $x_j^2 - x_j \in \mathcal{I}$, and $x_j + x'_j - 1 \in \mathcal{I}$
(axioms)
 - If $p, q \in \mathcal{I}$, then $\alpha p + \beta q \in \mathcal{I}$ for any $\alpha, \beta \in \mathbb{F}$
(linear combination)
 - If $p \in \mathcal{I}$, then $m \cdot p \in \mathcal{I}$ for any monomial $m = \prod_j x_j$
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(multiplication)
- A refutation is a derivation ending with the polynomial 1
- Complexity measures:
 - **Size**: total number of monomials in all polynomials in derivation expanded out
 - **Degree**: highest total degree of any polynomial
- Polynomial calculus (much) stronger than Nullstellensatz w.r.t. both size and degree

Polynomial Calculus Can Simulate Resolution

Polynomial calculus **can always simulate resolution proofs efficiently**
step by step

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simulated by polynomial calculus derivation

$$\frac{x'yz' \quad \frac{\frac{yz}{x'yz} \quad \frac{z + z' - 1}{x'yz + x'yz' - x'y}}{-x'yz' + x'y}}{x'y}$$

Polynomial Calculus is Strictly Stronger than Resolution

Polynomial calculus **can be exponentially stronger** than resolution

For instance:

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Other hard formulas:

- Tseitin-like formulas for counting mod p if $p \neq \text{field characteristic}$ [BGIP01]
- Random k -CNF formulas
 - all characteristics except 2 [BI99]
 - all characteristics [AR03]

COLOURING and CLIQUE for Polynomial Calculus

COLOURING

- Exponential worst-case lower bounds in [LN17]
- Exponential **average-case** lower bounds in [CdRN⁺23]

CLIQUE

Essentially nothing known!

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- Use **dual variables!** [KBBN22]

Gröbner bases: Some Problems and Questions

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- ❷ Dual variables increase reasoning power exponentially [dRLNS21]
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- ❸ Analysis of polynomial calculus uses degree-lexicographic ordering
In computational algebra, many other orderings used
Prove proof complexity separation results for different orderings?

SAT as System of 0–1 Integer Linear Inequalities

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to 0-1 integer linear inequalities

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- Add variable axioms

$$\begin{aligned} x_j &\geq 0 \\ -x_j &\geq -1 \end{aligned}$$

for all variables

Cutting Planes Proof System [CCT87]

Cutting planes introduced in [CCT87] to model integer linear programming algorithm in [Gom63, Chv73]

Can be applied to any system of 0-1 integer linear inequalities

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Cutting planes derivation rules

$$\text{Multiplication} \quad \frac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA} \quad c \in \mathbb{N}^+$$

$$\text{Addition} \quad \frac{\sum a_i x_i \geq A \quad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$$

$$\text{Division} \quad \frac{\sum a_i x_i \geq A}{\sum \lceil a_i / c \rceil x_i \geq \lceil A / c \rceil} \quad c \in \mathbb{N}^+$$

Cutting Planes Derivations and Refutations

- A cutting planes derivation is a sequence of 0-1 integer linear inequalities derived using
 - Axioms (input inequalities and variable bounds)
 - Derivation rules applied to previous inequalities
- A refutation ends with the inequality $0 \geq 1$
- Complexity measures:
 - **Length**: # inequalities
 - **Size**: Count also bit size of representing all coefficients

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- Cutting planes **can simulate resolution reasoning efficiently** and **can be exponentially stronger** (e.g., for PHP, just count and argue that $\# \text{pigeons} > \# \text{holes}$)
- And **0-1 linear inequalities** are similar to but **much more concise** than CNF

Compare

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 3$$

and

$$\begin{aligned} & (x_1 \vee x_2 \vee x_3 \vee x_4) \wedge (x_1 \vee x_2 \vee x_3 \vee x_5) \wedge (x_1 \vee x_2 \vee x_3 \vee x_6) \\ & \wedge (x_1 \vee x_2 \vee x_4 \vee x_5) \wedge (x_1 \vee x_2 \vee x_4 \vee x_6) \wedge (x_1 \vee x_2 \vee x_5 \vee x_6) \\ & \wedge (x_1 \vee x_3 \vee x_4 \vee x_5) \wedge (x_1 \vee x_3 \vee x_4 \vee x_6) \wedge (x_1 \vee x_3 \vee x_5 \vee x_6) \\ & \wedge (x_1 \vee x_4 \vee x_5 \vee x_6) \wedge (x_2 \vee x_3 \vee x_4 \vee x_5) \wedge (x_2 \vee x_3 \vee x_4 \vee x_6) \\ & \wedge (x_2 \vee x_3 \vee x_5 \vee x_6) \wedge (x_2 \vee x_4 \vee x_5 \vee x_6) \wedge (x_3 \vee x_4 \vee x_5 \vee x_6) \end{aligned}$$

Hard Formulas for Cutting Planes

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Variables

- $p_{i,j}$ indicators of the edges in graph; $1 \leq i < j \leq n$
- $q_{k,i}$ identify members of m -clique; $1 \leq k \leq m, 1 \leq i \leq n$
- $r_{i,\ell}$ specify colouring of vertices; $1 \leq \ell \leq m - 1, 1 \leq i \leq n$

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$$q_{k,1} \vee q_{k,2} \vee \cdots \vee q_{k,n}$$

some vertex is the k th member of clique

$$\bar{q}_{k,i} \vee \bar{q}_{k',i}$$

clique members are uniquely defined ($k \neq k'$)

$$p_{i,j} \vee \bar{q}_{k,i} \vee \bar{q}_{k',j}$$

clique members are connected by edges

$$r_{i,1} \vee r_{i,2} \vee \cdots \vee r_{i,m-1}$$

every vertex i has a colour

$$\bar{p}_{i,j} \vee \bar{r}_{i,\ell} \vee \bar{r}_{j,\ell}$$

neighbours have distinct colours

More Hard Formulas for Cutting Planes?

Lower bound for clique-colouring formulas uses **interpolation** and **circuit complexity**

- From small cutting planes proof, build small circuit of special type that can decide whether graph has clique
- Prove separately that no such small circuits can exist
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Some more recent developments in

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Nothing known for COLOURING or CLIQUE

Surprisingly, Tseitin formulas are at most quasi-polynomially hard for cutting planes [DT20]!

SAT Solvers Based on Cutting Planes?

So-called **pseudo-Boolean (PB) solvers** using (subset of) cutting planes reasoning developed in, e.g., [CK05, SS06, LP10, EN18]

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Challenge 1: Conjunctive normal form

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Is it truly harder to build good pseudo-Boolean solvers?

Or has just so much more work has been put into CDCL solvers?

Division Versus Saturation

Normalized form: Use negated literals as needed to make positive linear combination with all $a_i, A > 0$

Boolean derivation rules for 0–1 integer linear inequalities

$$\text{Division} \quad \frac{\sum a_i \ell_i \geq A}{\sum \lceil a_i/c \rceil \ell_i \geq \lceil A/c \rceil} \quad c \in \mathbb{N}^+$$

$$\text{Saturation} \quad \frac{\sum a_i \ell_i \geq A}{\sum \min\{a_i, A\} \cdot \ell_i \geq A}$$

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- ... And most often also in practice [EN18]

Sherali-Adams (SA) and Sums of Squares (SoS)

Refutation of $p_i \in \mathbb{R}[x_1, \dots, x_n]$, $i \in [m]$, and $x_j^2 - x_j$, $j \in [n]$

Nullstellensatz

$$\sum_{i=1}^m q_i \cdot p_i + \sum_{j=1}^n r_j \cdot (x_j^2 - x_j) = 1$$

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$$\sum_{i=1}^m q_i \cdot p_i + \sum_{j=1}^n r_j \cdot (x_j^2 - x_j) + \sum_{k=1}^s s_k^2 = -1$$

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Sums of squares is strictly stronger than polynomial calculus (over \mathbb{R}) while Sherali-Adams and polynomial calculus are incomparable [Ber18]

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Sums of squares very strong proof system, except it cannot do parity reasoning efficiently [GV01, Gri01]

Survey [FKP19] is recommended for more reading

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Stabbing planes with polynomial-size coefficient can be simulated by cutting planes with quasi-polynomial overhead [DT20, FGI⁺21]

Extended Resolution [Tse68]

Resolution rule

$$\frac{C_1 \vee x \quad C_2 \vee \bar{x}}{C_1 \vee C_2}$$

Extension rule introducing clauses

$$a \vee \bar{x} \vee \bar{y} \quad \bar{a} \vee x \quad \bar{a} \vee y$$

for fresh variable a (encoding that $a \leftrightarrow (x \wedge y)$ must hold)

Extended Resolution and SAT Solving

- Closely related (and equivalent) to *DRAT* system used to justify correctness of some SAT preprocessing techniques [JHB12]
- *DRAT* also used for SAT solver proof logging
- Attempts to combine extended resolution with CDCL in, e.g., [AKS10, Hua10]
- Without restrictions, corresponds to extremely strong **extended Frege system** [CR79] — pretty much no lower bounds known
- To analyse solvers using extended resolution, would need to:
 - Describe heuristics/rules actually used
 - See if possible to reason about such restricted proof system

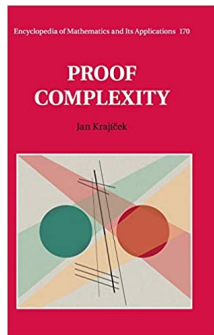
Some More References for Further Reading

Handbook of Satisfiability (Especially chapter 7 😊)



[BHvMW21]

Proof Complexity by Jan Krajíček



[Kra19]

Summing up This Presentation

Overview of some proof systems used in combinatorial solving:

- Resolution \longleftrightarrow Conflict-driven clause learning
- Nullstellensatz and polynomial calculus \longleftrightarrow Gröbner bases
- Cutting planes \longleftrightarrow pseudo-Boolean solving

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Thank you for your attention!

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