Proof complexity and SAT solving

Jakob Nordström

University of Copenhagen and Lund University

SAT/SMT/AR Summer School
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Three Simple Problems...

Colouring

Does the graph $G = (V, E)$ have a colouring with k colours such that all neighbours have distinct colours?

Three Simple Problems. . .

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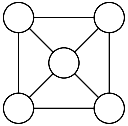

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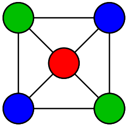

3-colouring? Yes

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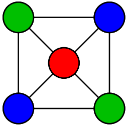

3-colouring? Yes, but no 2-colouring

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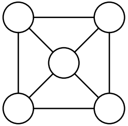

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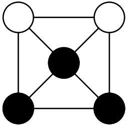

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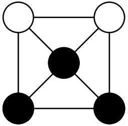

3-clique? Yes, but no 4-clique

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- Variables should be set to **true** or **false**
- Constraint $(x \lor \neg y \lor z)$: means x or z should be true or y false
- $\land$ means all constraints should hold simultaneously
- Is there a truth value assignment satisfying all constraints?

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Colouring:	frequency allocation for mobile base stations
Clique:	bioinformatics, computational chemistry
Sat:	easily models these and many other problems

. . . with Huge Practical Implications

- Some more examples of problems that can be encoded as propositional logic formulas:
  - computer hardware verification
  - computer software testing
  - artificial intelligence
  - operations research
  - cryptography
  - bioinformatics
  - et cetera. . .
- Leads to **humongous** formulas (100,000s or even 1,000,000s of variables)
- Can we use computers to solve these problems efficiently?

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  - COLOURING [Kho01, Zuc07]
  - CLIQUE [Hås99]
  - SAT [Hås01]

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  - COLOURING [Kho01, Zuc07]
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  - SAT [Hås01]
- Except that in practice, there are good algorithms for
  - COLOURING [DLMM08, DLMO09, DLMM11]
  - CLIQUE [Pro12, McC17]

  and amazing conflict-driven clause learning (CDCL) solvers [BS97, MS99, MMZ$^+$01] that solve huge SAT formulas

How can we understand real-world algorithms for NP-hard problems?

This talk: Use proof complexity (not only conceivable answer)

Algorithmic View of Proof Complexity

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1. Is there a short proof using rules in this proof system?
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Question 2: Topic for separate lecture(s) — lots of recent exciting progress; mostly negative (worst-case) results that proof search is hard, e.g., [AM20, GKMP20, dRGN$^+$21]

Applications of Proof Complexity

Three applied reasons for proof complexity:

1. Understand real-world applied algorithmic paradigms [**this lecture**]

2. Get ideas for algorithmic improvements
   [EN18, EN20, DGD$^+$21, DGN21, KBBN22]
   (See, e.g., tutorials https://www.youtube.com/watch?v=LZ8VztiplaQ and https://www.youtube.com/watch?v=wD_2tx1rTaw about RoundingSat)

3. Enhance algorithms to write machine-verifiable certificates of correctness
   [EGMN20, GMN20, GMM$^+$20, GN21, GMN22, GMNO22, VDB22, BGMN23, BBN$^+$23, MM23, GMM$^+$24, HOGN24, BBN$^+$24, DMM$^+$24, IOT$^+$24, MMN24]
   (See tutorial https://www.youtube.com/watch?v=s_5BIi4I22w about VeriPB)

Outline

1. DPLL, CDCL, and Resolution
   - Davis-Putnam-Logemann-Loveland (DPLL) Method
   - Conflict-Driven Clause Learning (CDCL)
   - Resolution Proof System
2. Algebraic and Semi-algebraic Approaches
   - Nullstellensatz
   - Gröbner Bases and Polynomial Calculus
   - Pseudo-Boolean Solving and Cutting Planes
3. Some More Advanced Proof Systems We Might Not Have Time for
   - Sherali-Adams and Sums of Squares
   - Stabbing Planes
   - Extended Resolution

Formal Description of SAT Problem

- Variable x: takes value **true** $(= 1)$ or **false** $(= 0)$
- Literal ℓ: variable x or its negation $\overline{x}$ (write $\overline{x}$ instead of $\neg x$)
- Clause $C = \ell_1 \lor \cdots \lor \ell_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- Conjunctive normal form (CNF) formula $F = C_1 \land \cdots \land C_m$: conjunction of clauses

The SATISFIABILITY (or just SAT) Problem

Given a CNF formula F, is it satisfiable?

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The SATISFIABILITY (or just SAT) Problem

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Here is our example formula again:

$$(x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u})$$
$$\land\ (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

The Same Problem in Three Different Shapes

$$
\begin{aligned}
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$$\begin{aligned}
(1-x)(1-z) &= 0 \\
(1-y)z &= 0 \\
(1-x)y(1-u) &= 0 \\
yu &= 0 \\
(1-u)(1-v) &= 0 \\
xv &= 0 \\
u(1-w) &= 0 \\
xuw &= 0
\end{aligned}$$

For **true** $= 1$ and **false** $= 0$, is there a $\{0,1\}$-valued solution?

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1 - x - z + xz &= 0 \\
z - yz &= 0 \\
y - xy - yu + xyu &= 0 \\
yu &= 0 \\
1 - u - v + uv &= 0 \\
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$1 - x - z + xz = 0$	$x + z \geq 1$
$z - yz = 0$	$y + (1 - z) \geq 1$
$y - xy - yu + xyu = 0$	$x + (1 - y) + u \geq 1$
$yu = 0$	$(1 - y) + (1 - u) \geq 1$
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$1 - x - z + xz = 0$	$x + z \geq 1$
$z - yz = 0$	$y - z \geq 0$
$y - xy - yu + xyu = 0$	$x - y + u \geq 0$
$yu = 0$	$-y - u \geq -1$
$1 - u - v + uv = 0$	$u + v \geq 1$
$xv = 0$	$-x - v \geq -1$
$u - uw = 0$	$-u + w \geq 0$
$xuw = 0$	$-x - u - w \geq -2$

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DPLL: Attempting Smart Case Analysis

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4. Set $x = 0$, simplify F and make recursive call
5. Set $x = 1$, simplify F and make recursive call
6. If result in both cases "unsatisfiable", then report "unsatisfiable" and return

A DPLL Toy Example

$$\begin{aligned} F = \quad & (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \\ & \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w}) \end{aligned}$$

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Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

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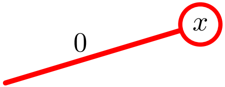

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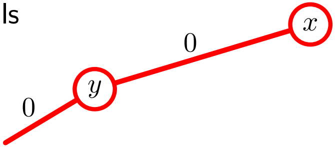

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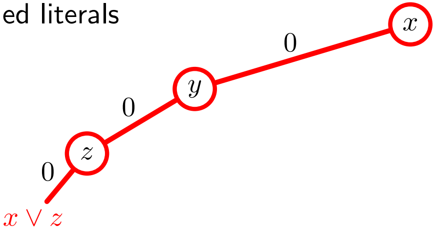

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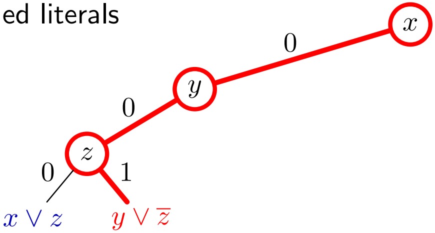

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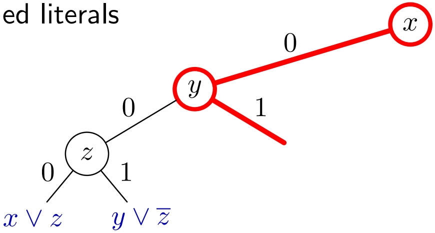

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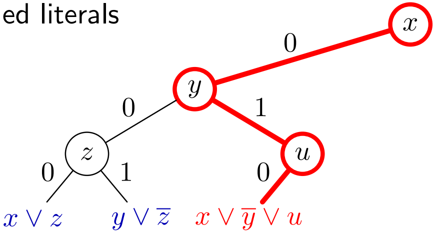

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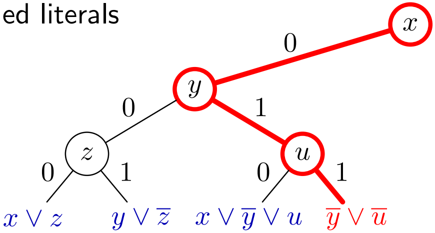

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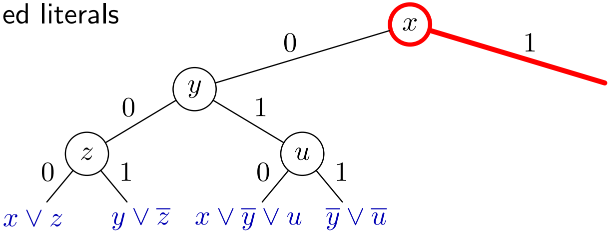

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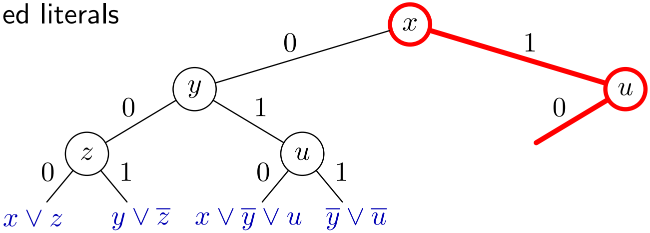
\begin{aligned}
F = \quad & (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \qquad) \land (\overline{y} \lor \overline{u}) \\
& \land (\qquad v) \land (\qquad \overline{v}) \land (\overline{u} \lor w) \land (\qquad \overline{u} \lor \overline{w})
\end{aligned}
$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals



A DPLL Toy Example

$$F = \quad (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \qquad) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\qquad \overline{v}) \land (\overline{u} \lor w) \land (\qquad \overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

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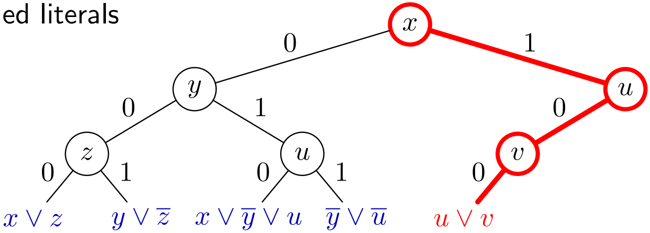

A DPLL Toy Example

$$F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \quad) \land (\overline{y} \lor \overline{u}) \land (\quad v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\quad \overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

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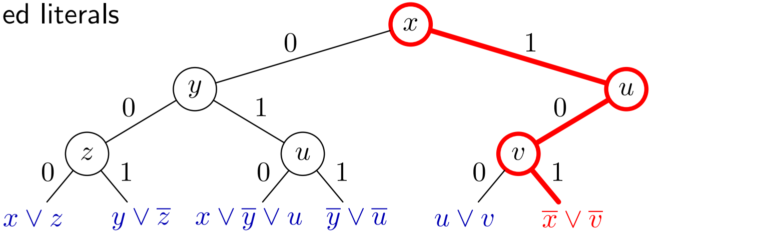


A DPLL Toy Example

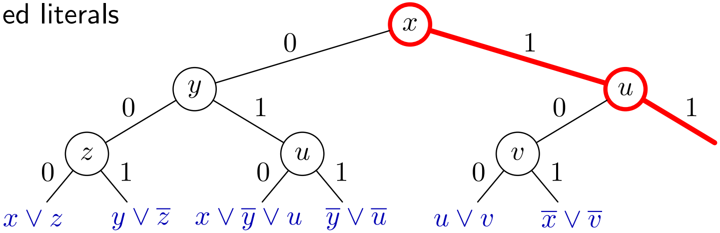
$$F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\overline{v}) \land (w) \land (\overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals



A DPLL Toy Example

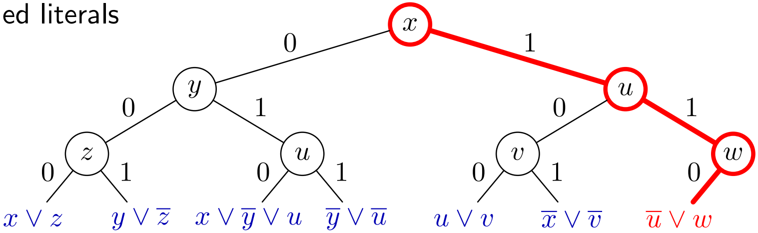
$$F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\overline{v}) \land (\overline{u} \lor w) \land (\overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals



A DPLL Toy Example

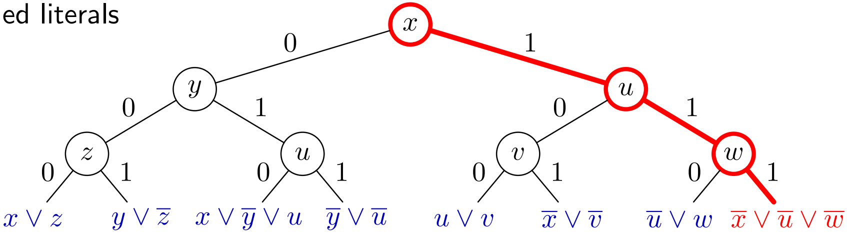
$$\begin{aligned} F = \quad & (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \\ & \land (u \lor v) \land (\qquad \overline{v}) \land (\qquad w) \land (\overline{x} \lor \overline{u} \lor \overline{w}) \end{aligned}$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals



A DPLL Toy Example

$$F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

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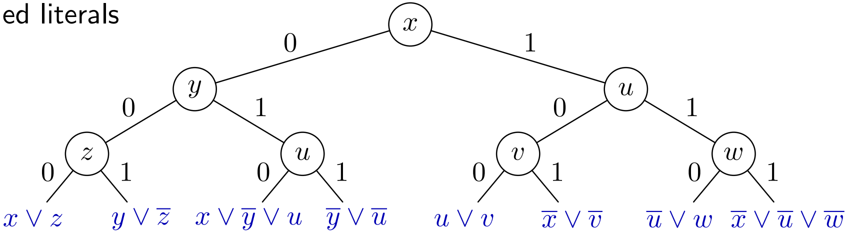

State-of-the-Art SAT Solving in One Slide

High-level description of modern conflict-driven clause learning (CDCL) SAT solving (as pioneered in [BS97, MS99, MMZ+01]):

- Try to build satisfying assignment for formula (branching or decision heuristic crucial)
- When partial assignment violates formula, compute explanation for conflict and add to formula as new clause (clause learning)
- Every once in a while, restart from beginning (but save computed info)

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

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Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \overset{\mathsf{d}}{=} 0}$

$u \overset{p \lor \overline{u}}{=} 0$

$\boxed{q \overset{\mathsf{d}}{=} 0}$

Decision

Free choice to assign value to variable

Notation $p \overset{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \overset{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

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Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

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Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \vee \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \vee r}{=} 1$

$w \stackrel{\overline{r} \vee w}{=} 1$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$y \stackrel{u \vee x \vee y}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \vee \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \vee r}{=} 1$

$w \stackrel{\overline{r} \vee w}{=} 1$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$y \stackrel{u \vee x \vee y}{=} 1$

$z \stackrel{x \vee \overline{y} \vee z}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \overset{\mathsf{d}}{=} 0}$

$u \overset{p \lor \overline{u}}{=} 0$

$\boxed{q \overset{\mathsf{d}}{=} 0}$

$r \overset{q \lor r}{=} 1$

$w \overset{\overline{r} \lor w}{=} 1$

$\boxed{x \overset{\mathsf{d}}{=} 0}$

$y \overset{u \lor x \lor y}{=} 1$

$z \overset{x \lor \overline{y} \lor z}{=} 1$

$\overset{\overline{y} \lor \overline{z}}{\bot}$

Decision

Free choice to assign value to variable

Notation $p \overset{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \overset{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$p \stackrel{\mathsf{d}}{=} 0$
$u \stackrel{p \vee \overline{u}}{=} 0$
decision level 1

$q \stackrel{\mathsf{d}}{=} 0$
$r \stackrel{q \vee r}{=} 1$
$w \stackrel{\overline{r} \vee w}{=} 1$
decision level 2

$x \stackrel{\mathsf{d}}{=} 0$
$y \stackrel{u \vee x \vee y}{=} 1$
$z \stackrel{x \vee \overline{y} \vee z}{=} 1$
$\stackrel{\overline{y} \vee \overline{z}}{\bot}$
decision level 3

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

decision level 1

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

decision level 2

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

$\stackrel{\overline{y} \lor \overline{z}}{\bot}$

decision level 3

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \overset{\mathsf{d}}{=} 0}$
$u \overset{p \lor \overline{u}}{=} 0$
decision level 1

$\boxed{q \overset{\mathsf{d}}{=} 0}$
$r \overset{q \lor r}{=} 1$
$w \overset{\overline{r} \lor w}{=} 1$
decision level 2

$\boxed{x \overset{\mathsf{d}}{=} 0}$
$y \overset{u \lor x \lor y}{=} 1$
$z \overset{x \lor \overline{y} \lor z}{=} 1$
$\overset{\overline{y} \lor \overline{z}}{\bot}$
decision level 3

Could backtrack by erasing conflict level & flipping last decision

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

decision level 1

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

decision level 2

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

$\stackrel{\overline{y} \lor \overline{z}}{\bot}$

decision level 3

Could backtrack by erasing conflict level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

$\stackrel{\overline{y} \lor \overline{z}}{\bot}$

$x \lor \overline{y}$

Could backtrack by erasing conflict level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

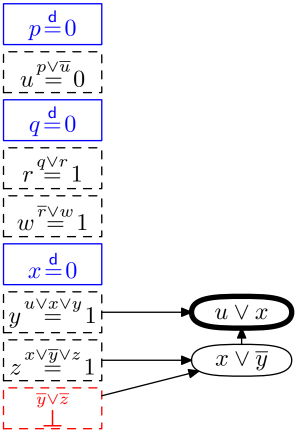
Case analysis over z for last two clauses:

- $x \lor \overline{y} \lor z$ wants $z = 1$
- $\overline{y} \lor \overline{z}$ wants $z = 0$
- Merge clauses & remove z — must satisfy $x \lor \overline{y}$

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



Could backtrack by erasing conflict level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

Case analysis over z for last two clauses:

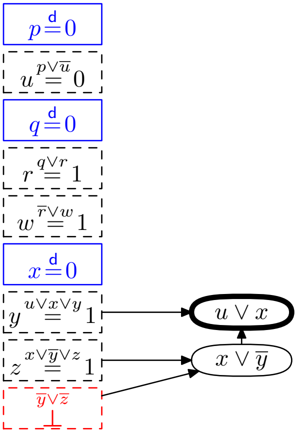
- $x \lor \overline{y} \lor z$ wants $z = 1$
- $\overline{y} \lor \overline{z}$ wants $z = 0$
- Merge clauses & remove z — must satisfy $x \lor \overline{y}$

Repeat until UIP clause with only 1 variable at conflict level after last decision — learn and backjump

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



Complete Example of CDCL Execution

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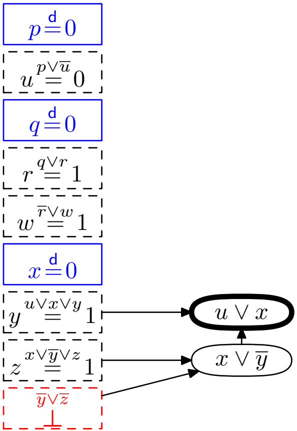

Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

Complete Example of CDCL Execution

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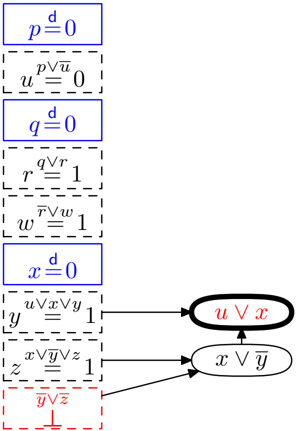

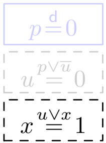

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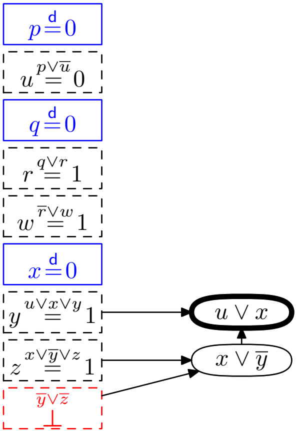

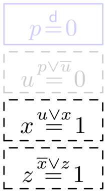

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Then continue as before. . .

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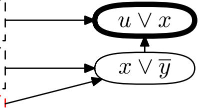

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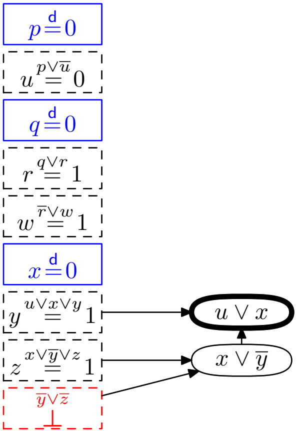

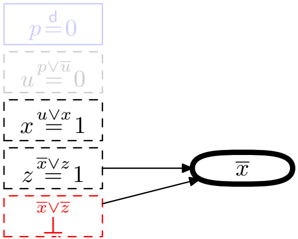

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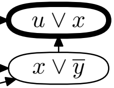

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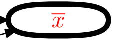

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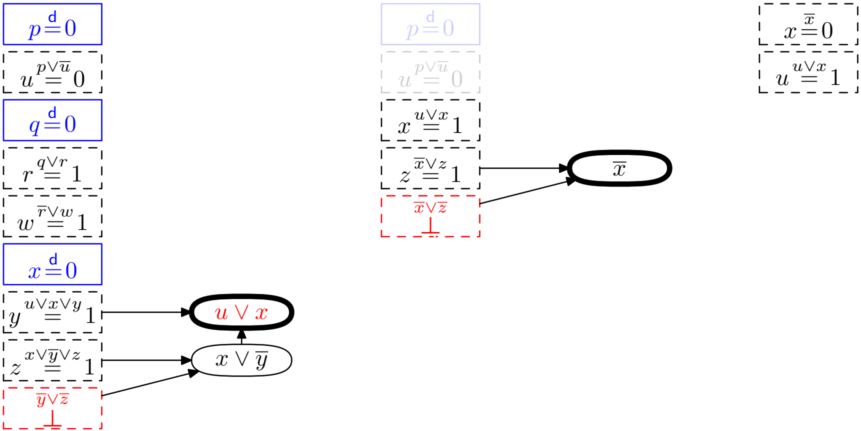

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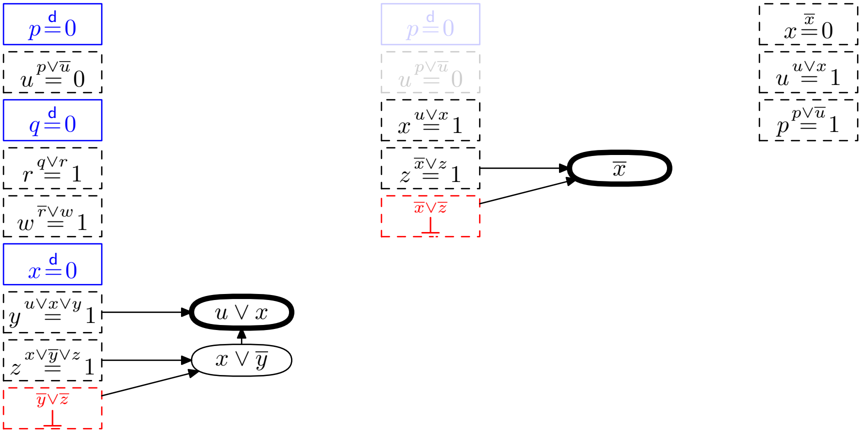

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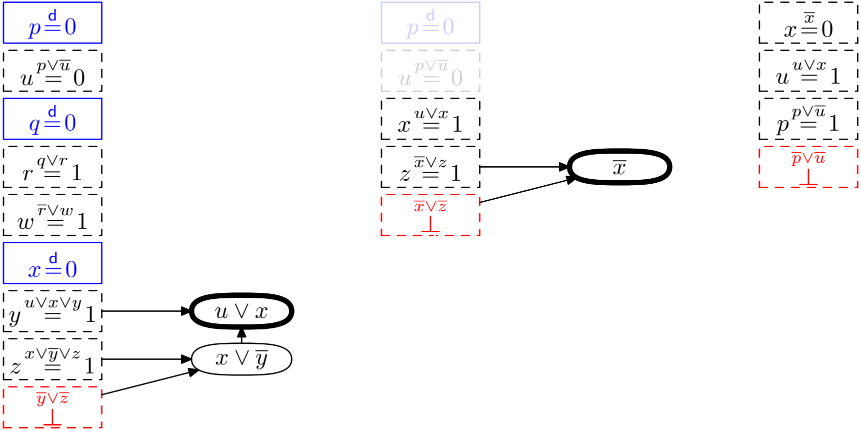

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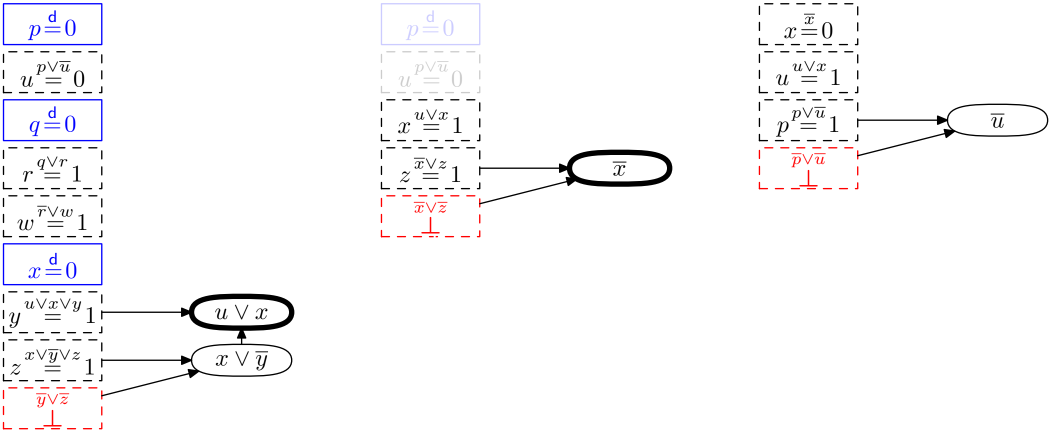

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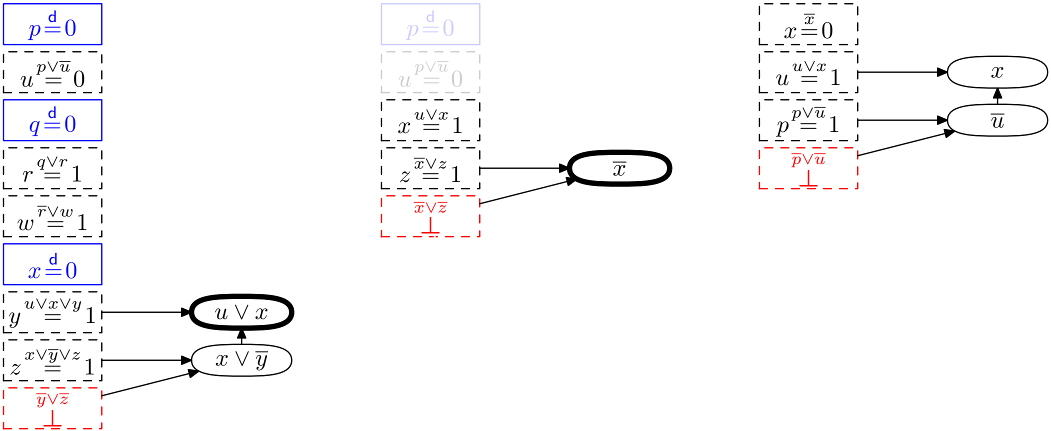

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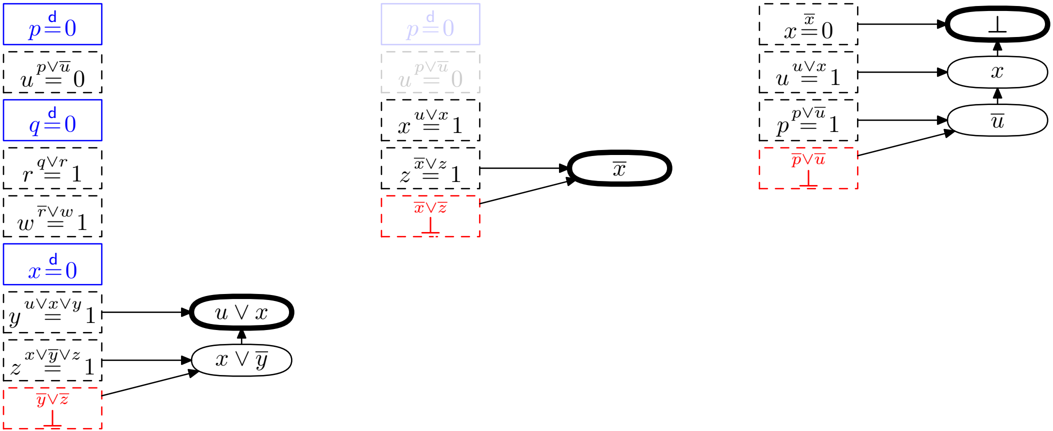

SAT Solver Analysis and the Resolution Proof System

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So focus instead on underlying method of reasoning

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Resolution proof system [Bla37, Rob65]

- Start with clauses of CNF formula (axioms)
- Derive new clauses by resolution rule

$$\frac{C_1 \lor x \qquad C_2 \lor \overline{x}}{C_1 \lor C_2}$$

Resolution Proofs by Contradction

Resolution rule:

$$\frac{C_1 \lor x \qquad C_2 \lor \overline{x}}{C_1 \lor C_2}$$

Observation

If F is a satisfiable CNF formula and D is derived from clauses $D_1, D_2 \in F$ by the resolution rule, then $F \land D$ is satisfiable.

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If F is a satisfiable CNF formula and D is derived from clauses $D_1, D_2 \in F$ by the resolution rule, then $F \land D$ is satisfiable.

So can prove F unsatisfiable by deriving the unsatisfiable empty clause (denoted $\perp$) from F by resolution

Such proof by contradiction also called resolution refutation

DPLL and Resolution Proofs

A DPLL execution is essentially a resolution proof

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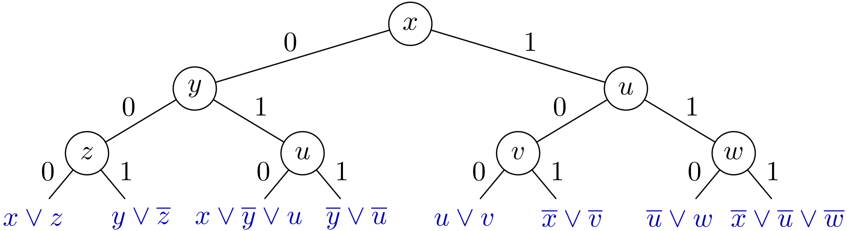
Look at our example again



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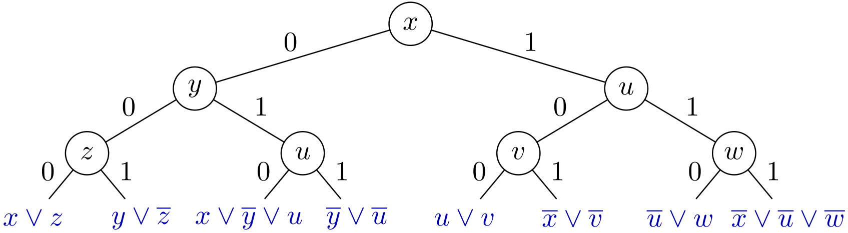

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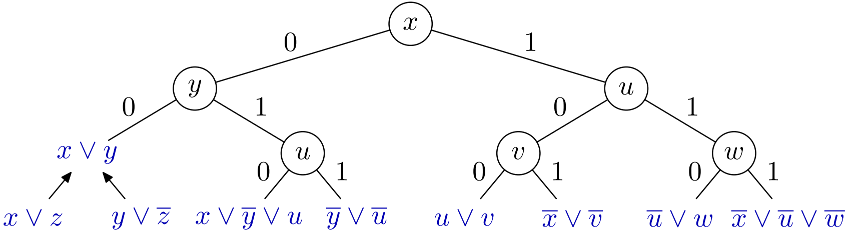

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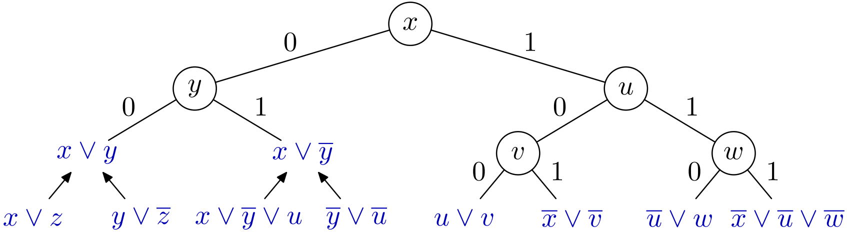

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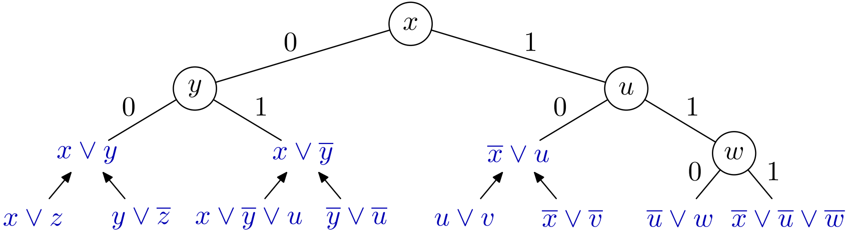

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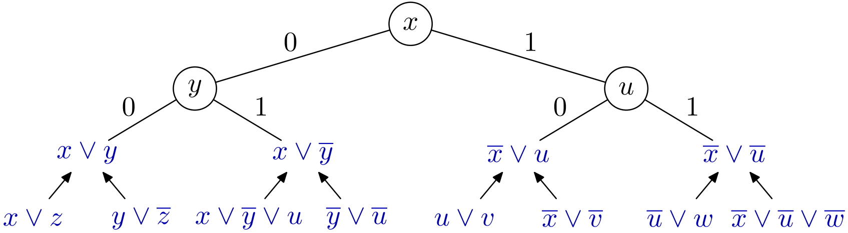

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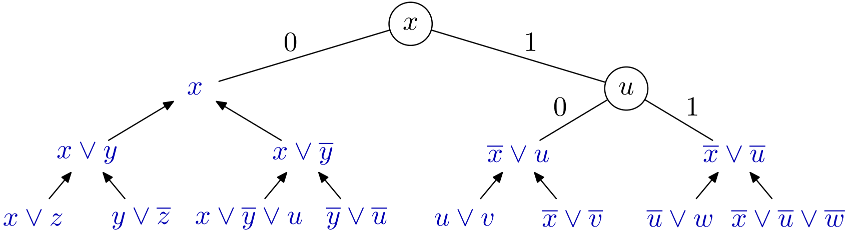

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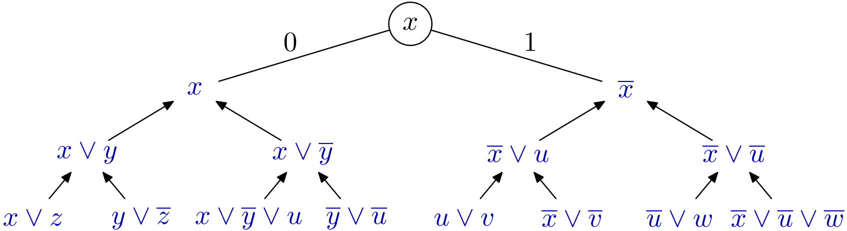

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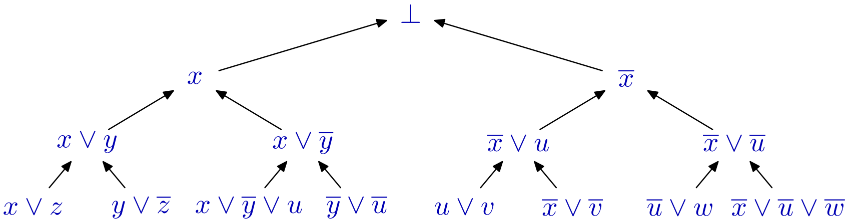

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- Conflict-driven clause learning adds "shortcut edges" in tree, but still yields resolution proof

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Obtain resolution proof...

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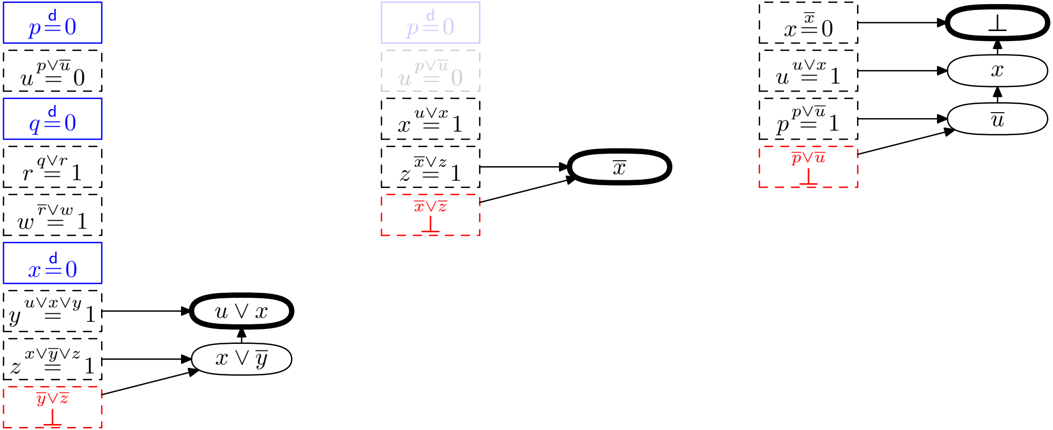
Obtain resolution proof from our example CDCL execution...



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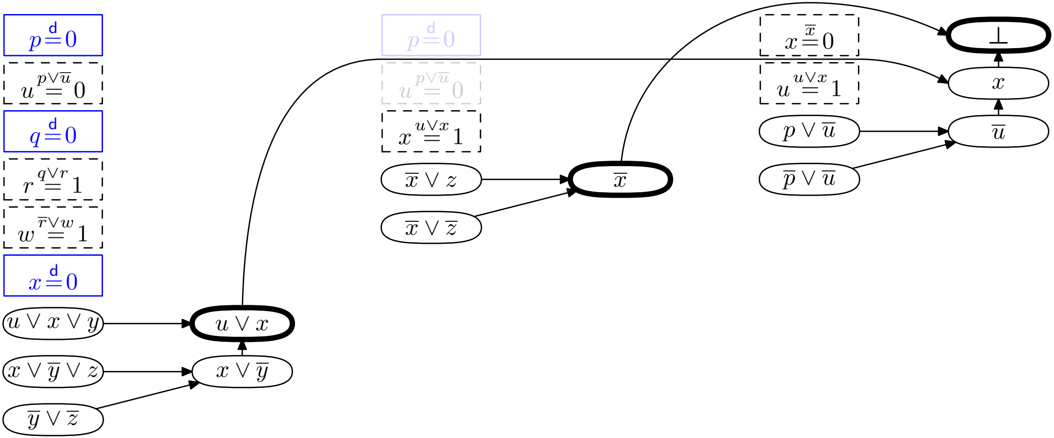

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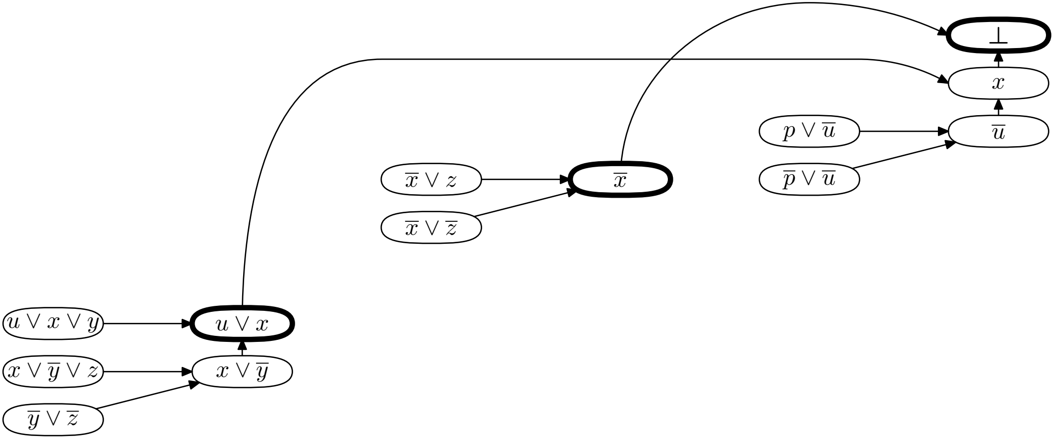

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(*) Except for some preprocessing techniques, which is an important omission, but this gets complicated and we don't have time to go into details. . .

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- Very poor theoretical understanding:
  - Why do heuristics work?
  - Why are applied instances easy?
- Paradox: resolution quite weak proof system; many strong proof complexity lower bounds for (seemingly) "obvious" formulas

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Pigeonhole principle (PHP) formulas [Hak85]

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Even onto functional PHP hard — **"resolution cannot count"**

Resolution proof requires $\exp(\Omega(n)) = \exp\bigl(\Omega\bigl(\sqrt[3]{N}\bigr)\bigr)$ clauses
(measured in terms of formula size N)

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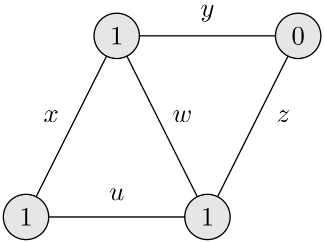

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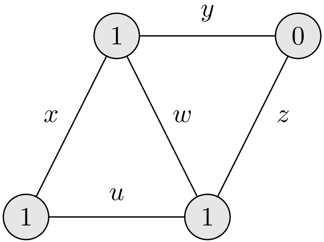

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(u \lor x) & \land\ (y \lor \overline{z}) \\
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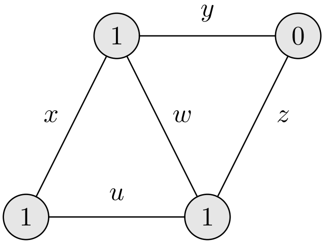
Examples of Hard Formulas For Resolution (2/3)

Tseitin formulas [Urq87]

"Sum of degrees of vertices in graph is even"

Variables = edges (in undirected graph of bounded degree)

- Label every vertex 0/1 so that sum of labels odd
- Write CNF requiring parity of # true incident edges = label



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\land\ (\overline{w} \lor \overline{x} \lor y) & \land\ (\overline{u} \lor \overline{w} \lor z)
\end{array}
$$

Requires proof size $\exp(\Omega(N))$ on well-connected so-called expander graphs —
"resolution cannot count $\mathrm{mod}\ 2$**"**

Examples of Hard Formulas for Resolution (3/3)

Random k-CNF formulas [CS88]

Δn randomly sampled k-clauses over n variables
($\Delta \gtrsim 4.5$ sufficient to get unsatisfiable 3-CNF almost surely)

Again lower bound $\exp(\Omega(N))$

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And more...

- COLOURING [BCMM05]
- Zero-one designs [Spe10, VS10, MN14]
- Et cetera... (See, e.g., [BN21] for overview)

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But no such strong lower bounds known for CLIQUE!

- Refuting existence of k-clique should require proof size $n^{\Omega(k)}$
- Only known for restricted so-called regular resolution [ABdR$^+$21]

SAT as System of Polynomial Equations

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to polynomial equations
$$\prod_{i \in \mathcal{P}} (1 - x_i) \cdot \prod_{j \in \mathcal{N}} x_j = 0$$
- Add Boolean axioms
$$x_j^2 - x_j = 0$$
for all variables

Hilbert's Nullstellensatz

Consider any system of polynomial equations

$$\begin{aligned} p_1(x_1,\ldots,x_n) &= 0 \\ p_2(x_1,\ldots,x_n) &= 0 \\ &\vdots \\ p_m(x_1,\ldots,x_n) &= 0 \end{aligned} \qquad\qquad \begin{aligned} x_1^2 - x_1 &= 0 \\ x_2^2 - x_2 &= 0 \\ &\vdots \\ x_n^2 - x_n &= 0 \end{aligned}$$

in polynomial ring over field $\mathbb{F}$

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\end{aligned}
$$

in polynomial ring over field $\mathbb{F}$

Hilbert's Nullstellensatz

System infeasible $\Leftrightarrow$ exist $q_i, r_j \in \mathbb{F}[x_1,\ldots,x_n]$ such that

$$\sum_{i=1}^{m} q_i(x_1,\ldots,x_n) \cdot p_i(x_1,\ldots,x_n) + \sum_{j=1}^{n} r_j(x_1,\ldots,x_n) \cdot (x_j^2 - x_j) = 1$$

Nullstellensatz Proof System [BIK+94]

Nullstellensatz refutation of

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is (syntactic) equality

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Complexity measures of refutations:

- Size: number of monomials (when all polynomials expanded out)
- Degree: highest total degree of any polynomial

Nullstellensatz Example (Not Expanded out)

$$(x \lor z) \land (y \lor \neg z) \land (x \lor \neg y \lor u) \land (\neg y \lor \neg u)$$
$$\land\ (u \lor v) \land (\neg x \lor \neg v) \land (\neg u \lor w) \land (\neg x \lor \neg u \lor \neg w)$$

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$$(1-x)(1-z)$$
$$(1-y)z$$
$$(1-x)y(1-u)$$
$$yu$$
$$(1-u)(1-v)$$
$$xv$$
$$u(1-w)$$
$$xuw$$

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$$\begin{aligned}
& (1-y) \cdot (1-x)(1-z) \\
+ \;& (1-x) \cdot (1-y)z \\
+ \;& 1 \cdot (1-x)y(1-u) \\
+ \;& (1-x) \cdot yu \\
+ \;& x \cdot (1-u)(1-v) \\
+ \;& (1-u) \cdot xv \\
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+\ & x \cdot u(1-w) \\
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=\ & 1
\end{aligned}$$

Size 27
Degree 3
(No use of Boolean axioms)

Nullstellensatz Proof Search

- Solve linear system of equations with coefficients of polynomials q_i, r_j as unknowns
- Used successfully to solve, e.g., graph colouring problems [DLMM08, DLMO09, DLMM11]
- Running time grows exponentially with degree, though high-degree refutations can be very small [BCIP02, dRMNR21]

Dual Variables

- Annoying problem: $x_1 \lor x_2 \lor x_3$ translates to polynomial

$$(1 - x_1)(1 - x_2)(1 - x_3) = 1 - x_1 - x_2 - x_3 + x_1x_2 + x_1x_3 + x_2x_3 - x_1x_2x_3$$

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- More generally, exponential blow-up in $\#$ positive literals
- Fix: introduce dual variables x'_i and axioms $x_i + x'_i - 1 = 0$
- Translate $C = \bigvee_{i \in \mathcal{P}} x_i \lor \bigvee_{j \in \mathcal{N}} \overline{x}_j$ to polynomial equations

$$\prod_{i \in \mathcal{P}} x'_i \cdot \prod_{j \in \mathcal{N}} x_j = 0$$

- Doesn't affect degree (obviously), but can decrease size exponentially [dRLNS21] (also for other algebraic proof systems)

Dynamic Construction of Nullstellensatz Certificates

Nullstellensatz again

Infeasibility of

$$\begin{aligned} p_i(x_1, \ldots, x_n) &= 0 & \qquad i &\in [m] \\ x_j^2 - x_j &= 0 & \qquad j &\in [n] \\ x_j + x_j' - 1 &= 0 & \qquad j &\in [n] \end{aligned}$$

$$\Updownarrow$$

1 lies in polynomial ideal $\mathcal{I}$ generated by these polynomials

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- Ideal $\mathcal{I}$:
  1. $p, q \in \mathcal{I} \Rightarrow p + q \in \mathcal{I}$
  2. $p \in \mathcal{I} \Rightarrow r \cdot p \in \mathcal{I}$ for any r
- Compute polynomials in this ideal $\mathcal{I}$ step by step
- Use "multivariate division" to check whether 1 lies in ideal or not

Gröbner Bases: Admissible Orderings and Leading Terms

Admissible ordering $\preceq$ on monomials m, m', t:

1. $m \preceq m' \Rightarrow t \cdot m \preceq t \cdot m'$
2. $m \preceq t \cdot m$

Examples:

- Lexicographic
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Can write $p = \mathrm{lt}(p) + p'$ for $\mathrm{lt}(p)$ leading term (largest w.r.t. $\preceq$)

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"Multivariate division": Reduce p modulo all q in set of polynomials $\mathcal{G}$ until no further reductions possible

$\mathcal{G}$ is a Gröbner basis if final result uniquely determined

Gröbner Bases: Buchberger's Algorithm

Buchberger's algorithm for computing Gröbner bases (**very** rough)

1. Let $\mathcal{G} :=$ all axioms
2. Pick unprocessed pair $p, q \in \mathcal{G}$ or terminate if none exists
3. Compute $p' = t_p \cdot p$ and $q' = t_q \cdot q$ to make leading terms cancel
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Facts:

- Buchberger's algorithm computes Gröbner basis
- At termination, $1 \in \mathcal{G} \Leftrightarrow$ polynomial equations infeasible

Polynomial Calculus [CEI96, ABRW02]

- Compute polynomials in ideal $\mathcal{I}$ generated by p_i, $x_j^2 - x_j$, and $x_j + x_j' - 1$ step by step:
  - $p_i \in \mathcal{I}$, $x_j^2 - x_j \in \mathcal{I}$, and $x_j + x_j' - 1 \in \mathcal{I}$ (axioms)
  - If $p, q \in \mathcal{I}$, then $\alpha p + \beta q \in \mathcal{I}$ for any $\alpha, \beta \in \mathbb{F}$ (linear combination)
  - If $p \in \mathcal{I}$, then $m \cdot p \in \mathcal{I}$ for any monomial $m = \prod_j x_j$ (multiplication)

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- A refutation is a derivation ending with the polynomial 1
- Complexity measures:
  - Size: total number of monomials in all polynomials in derivation expanded out
  - Degree: highest total degree of any polynomial
- Polynomial calculus (much) stronger than Nullstellensatz w.r.t. both size and degree

Polynomial Calculus Can Simulate Resolution

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$$\frac{x \lor \overline{y} \lor z \qquad \overline{y} \lor \overline{z}}{x \lor \overline{y}}$$

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Example: Resolution step

$$\frac{x \lor \overline{y} \lor z \qquad \overline{y} \lor \overline{z}}{x \lor \overline{y}}$$

simulated by polynomial calculus derivation

$$\frac{x'yz' \qquad \dfrac{\dfrac{yz}{x'yz} \qquad \dfrac{z + z' - 1}{x'yz + x'yz' - x'y}}{-x'yz' + x'y}}{x'y}$$

Polynomial Calculus is Strictly Stronger than Resolution

Polynomial calculus can be exponentially stronger than resolution

For instance:

- Tseitin formulas on expander graphs if $\mathbb{F} = \mathrm{GF}(2)$
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Other hard formulas:

- Tseitin-like formulas for counting $\bmod p$ if $p \neq$ field characteristic [BGIP01]
- Random k-CNF formulas
  - all characteristics except 2 [BI99]
  - all characteristics [AR03]

COLOURING and CLIQUE for Polynomial Calculus

COLOURING

- Exponential worst-case lower bounds in [LN17]
- Exponential average-case lower bounds in [CdRN$^+$23]

CLIQUE

Essentially nothing known!

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- Use dual variables! [KBBN22]

Gröbner bases: Some Problems and Questions

1. Buchberger not a great SAT solving algorithm
   Slow and memory-intensive, and computes too much info
   Possible to use conflict-driven paradigm?!

2. Dual variables increase reasoning power exponentially [dRLNS21]
   But are immediately eliminated by multivariate division
   Possible to design dual-variable-aware Buchberger?!

3. Analysis of polynomial calculus uses degree-lexicographic ordering
   In computational algebra, many other orderings used
   Prove proof complexity separation results for different orderings?

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$$\sum_{i \in \mathcal{P}} x_i + \sum_{j \in \mathcal{N}} (1 - x_j) \geq 1$$

- Add variable axioms

$$x_j \geq 0$$
$$-x_j \geq -1$$

for all variables

Cutting Planes Proof System [CCT87]

Cutting planes introduced in [CCT87] to model integer linear programming algorithm in [Gom63, Chv73]

Can be applied to any system of 0-1 integer linear inequalities

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Cutting planes derivation rules

Multiplication $$\frac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA} \quad c \in \mathbb{N}^+$$

Addition $$\frac{\sum a_i x_i \geq A \qquad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$$

Division $$\frac{\sum a_i x_i \geq A}{\sum \lceil a_i / c \rceil x_i \geq \lceil A/c \rceil} \quad c \in \mathbb{N}^+$$

Cutting Planes Derivations and Refutations

- A cutting planes derivation is a sequence of 0-1 integer linear inequalities derived using
  - Axioms (clauses and variable bounds)
  - Multiplication $\sum a_i x_i \geq A \Rightarrow \sum c a_i x_i \geq cA$
  - Addition $\sum a_i x_i \geq A$, $\sum b_i x_i \geq B \Rightarrow \sum (a_i + b_i) x_i \geq A + B$
  - Division $\sum a_i x_i \geq A \Rightarrow \sum \lceil a_i / c \rceil x_i \geq \lceil A/c \rceil$
- A refutation ends with the inequality $0 \geq 1$
- Complexity measures:
  - Length: $\#$ inequalities
  - Size: Count also bit size of representing all coefficients

Cutting Planes vs. Resolution

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- And 0-1 linear inequalities are similar to but much more concise than CNF

Compare

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 3$$

and

$$\begin{aligned} &(x_1 \lor x_2 \lor x_3 \lor x_4) \land (x_1 \lor x_2 \lor x_3 \lor x_5) \land (x_1 \lor x_2 \lor x_3 \lor x_6) \\ \land &(x_1 \lor x_2 \lor x_4 \lor x_5) \land (x_1 \lor x_2 \lor x_4 \lor x_6) \land (x_1 \lor x_2 \lor x_5 \lor x_6) \\ \land &(x_1 \lor x_3 \lor x_4 \lor x_5) \land (x_1 \lor x_3 \lor x_4 \lor x_6) \land (x_1 \lor x_3 \lor x_5 \lor x_6) \\ \land &(x_1 \lor x_4 \lor x_5 \lor x_6) \land (x_2 \lor x_3 \lor x_4 \lor x_5) \land (x_2 \lor x_3 \lor x_4 \lor x_6) \\ \land &(x_2 \lor x_3 \lor x_5 \lor x_6) \land (x_2 \lor x_4 \lor x_5 \lor x_6) \land (x_3 \lor x_4 \lor x_5 \lor x_6) \end{aligned}$$

Hard Formulas for Cutting Planes

Clique-colouring formulas [Pud97]

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Variables

- $p_{i,j}$ indicators of the edges in graph; $1 \leq i < j \leq n$
- $q_{k,i}$ identify members of m-clique; $1 \leq k \leq m$, $1 \leq i \leq n$
- $r_{i,\ell}$ specify colouring of vertices; $1 \leq \ell \leq m-1$, $1 \leq i \leq n$

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$q_{k,1} \lor q_{k,2} \lor \cdots \lor q_{k,n}$	some vertex is the kth member of clique
$\overline{q}_{k,i} \lor \overline{q}_{k',i}$	clique members are uniquely defined $(k \neq k')$
$p_{i,j} \lor \overline{q}_{k,i} \lor \overline{q}_{k',j}$	clique members are connected by edges
$r_{i,1} \lor r_{i,2} \lor \cdots \lor r_{i,m-1}$	every vertex i has a colour
$\overline{p}_{i,j} \lor \overline{r}_{i,\ell} \lor \overline{r}_{j,\ell}$	neighbours have distinct colours

More Hard Formulas for Cutting Planes?

Lower bound for clique-colouring formulas uses interpolation and circuit complexity

- From small cutting planes proof, build small circuit of special type that can decide whether graph has clique
- Prove separately that no such small circuits can exist
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Clear need for development of new analysis methods
Some recent developments in [dRMN$^+$20, HP17, FPPR22, GGKS20, Sok23]

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Nothing known for COLOURING or CLIQUE
Surprisingly, Tseitin formulas are at most quasi-polynomially hard for cutting planes [DT20]!

SAT Solvers Based on Cutting Planes?

So-called pseudo-Boolean (PB) solvers using (subset of) cutting planes reasoning developed in, e.g., [CK05, SS06, LP10, EN18]

Perhaps counter-intuitively, hard to make competitive with CDCL

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Challenge 1: Conjunctive normal form

- Pseudo-Boolean solvers terrible for CNF input
- Solvers can rewrite CNF to more helpful 0-1 linear inequalities [BLLM14, EN20], but this doesn't work so well in practice
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Is it truly harder to build good pseudo-Boolean solvers?
Or has just so much more work has been put into CDCL solvers?

Division Versus Saturation

Use negated literals as needed to get all a_i, A positive

Boolean derivation rules for 0–1 integer linear inequalities

Division
$$\frac{\sum a_i \ell_i \geq A}{\sum \lceil a_i / c \rceil \ell_i \geq \lceil A/c \rceil} \quad c \in \mathbb{N}^+$$

Saturation
$$\frac{\sum a_i \ell_i \geq A}{\sum \min\{a_i, A\} \cdot \ell_i \geq A}$$

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- Pseudo-Boolean solvers instead adopted saturation [CK05, LP10]
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- . . . And most often also in practice [EN18], though not always [LBD$^+$20]

Sherali–Adams (SA) and Sums of Squares (SoS)

Refutation of $p_i \in \mathbb{R}[x_1, \ldots, x_n]$, $i \in [m]$, and $x_j^2 - x_j$, $j \in [n]$

Nullstellensatz

$$\sum_{i=1}^{m} q_i \cdot p_i + \sum_{j=1}^{n} r_j \cdot (x_j^2 - x_j) = 1$$

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$$\sum_{i=1}^{m} q_i \cdot p_i + \sum_{j=1}^{n} r_j \cdot (x_j^2 - x_j) + \sum_{k=1}^{t} \alpha_k \prod_{i \in \mathcal{P}_t} (1 - x_i) \cdot \prod_{j \in \mathcal{N}_t} x_j = -1$$

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Sums of squares (SoS) $(s_k \in \mathbb{R}[x_1, \ldots, x_n])$

$$\sum_{i=1}^{m} q_i \cdot p_i + \sum_{j=1}^{n} r_j \cdot (x_j^2 - x_j) + \sum_{k=1}^{s} s_k^2 = -1$$

Sherali–Adams, Sums of Squares, and Relations to Other Proof Systems

Sherali–Adams models linear programming (LP) hierarchies

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Strict hierarchy (over $\mathbb{R}$):

- Nullstellensatz
- Sherali–Adams
- Sums of squares

Sums of squares is strictly stronger than polynomial calculus (over $\mathbb{R}$)
Sherali-Adams and polynomial calculus are incomparable [Ber18]

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Sums of squares very strong proof system (e.g., can reason about PHP)
But can't do parity reasoning efficiently [GV01, Gri01]

Survey [FKP19] recommended for more reading

Stabbing Planes [BFI+18]

Intended to model modern 0-1 integer linear programming

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Stabbing planes refutation of set of 0-1 integer linear inequalities $\mathcal{S}$

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Complexity measures:

- Length: $\#$ branching nodes / sets $\mathcal{S}$
- Size: Count also bit size for representing all coefficients

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Still possible that stabbing planes is exponentially more powerful than cutting planes, but hard to know what to believe

Extended Resolution [Tse68]

Resolution rule

$$\frac{C_1 \lor x \qquad C_2 \lor \overline{x}}{C_1 \lor C_2}$$

Extension rule introducing clauses

$$a \lor \overline{x} \lor \overline{y} \qquad \overline{a} \lor x \qquad \overline{a} \lor y$$

for fresh variable a (encoding that $a \leftrightarrow (x \land y)$ must hold)

Extended Resolution and SAT Solving

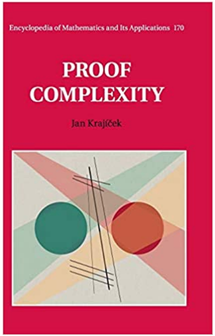
- Closely related (and equivalent) to *DRAT* system used to justify correctness of some SAT preprocessing techniques [JHB12]
- *DRAT* also used for SAT solver proof logging
- Attempts to combine extended resolution with CDCL in, e.g., [AKS10, Hua10]
- Without restrictions, corresponds to extremely strong extended Frege system [CR79] — pretty much no lower bounds known
- To analyse solvers using extended resolution, would need to:
  - Describe heuristics/rules actually used
  - See if possible to reason about such restricted proof system

Some More References for Further Reading

Handbook of Satisfiability
(Especially chapter 7 ☺)

[BHvMW21]

Proof Complexity
by Jan Krajíček



[Kra19]

Summing up This Presentation

Overview of some proof systems used in combinatorial solving:

- Resolution $\longleftrightarrow$ conflict-driven clause learning (CDCL)
- Nullstellensatz and polynomial calculus $\longleftrightarrow$ Gröbner bases
- Cutting planes $\longleftrightarrow$ pseudo-Boolean solving

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Thank you for your attention!

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