Leveraging Computational Complexity Theory for Provably Correct Combinatorial Optimization

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The Success Story of Combinatorial Solving and Optimization

- Rich field of mathematics and computer science
- Impact in other areas of science and also industry, e.g.:
  - airline scheduling
  - hardware verification
  - donor-recipients matching for kidney transplants [MO12, BvdKM^{+}21]
- Computationally very challenging problems (NP-complete or worse)
- Lots of effort last couple of decades spent on developing sophisticated so-called combinatorial solvers that often work surprisingly well in practice
  - Boolean satisfiability (SAT) solving [BHvMW21]
  - Constraint programming [RvBW06]
  - Mixed integer linear programming [AW13, BR07]
  - Satisfiability modulo theories (SMT) solving [BHvMW21]

The Dirty Little Secret...

- Solvers very fast, but sometimes wrong (even best commercial ones) [BLB10, CKSW13, AGJ+18, GSD19, GS19, BMN22]
- Even worse: No way of knowing for sure when errors happen
- Solvers even get feasibility of solutions wrong (though this should be straightforward!)
- But how to check the absence of solutions?
- Or that a solution is optimal? (Even off-by-one mistakes can snowball into large errors if solver used as subroutine)

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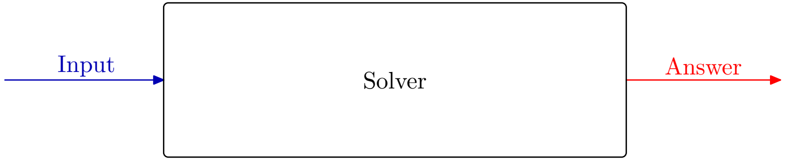
- **Proof logging**
  Make solver certifying [ABM$^+$11, MMNS11] by outputting
  1. not only answer but also
  2. simple, machine-verifiable proof that answer is correct

Proof Logging with Certifying Solvers: Workflow



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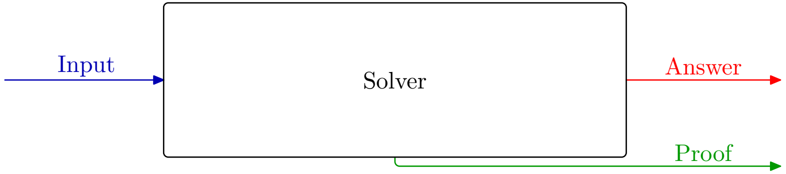

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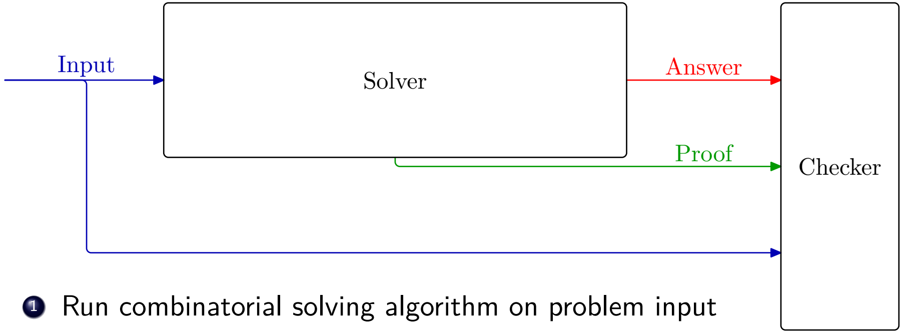

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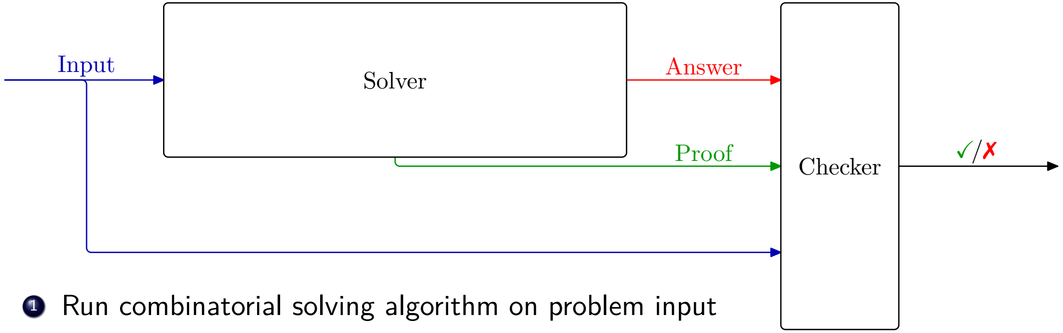

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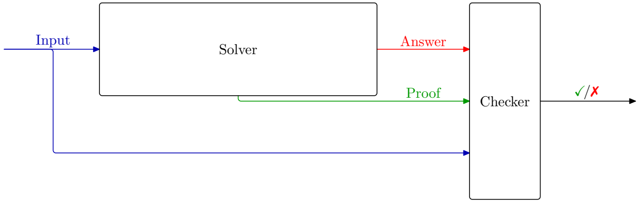
2. Get as output not only answer but also proof
3. Feed input + answer + proof to proof checker
4. Verify that proof checker says answer is correct

Proof Logging Wishlist

Proof format for certifying solver should be



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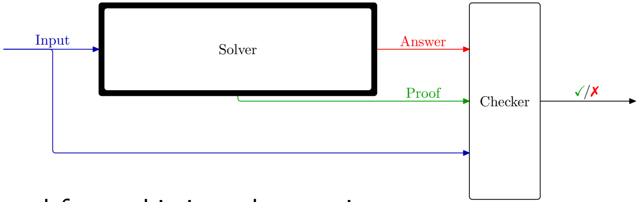

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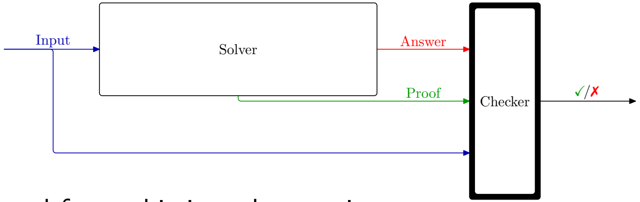

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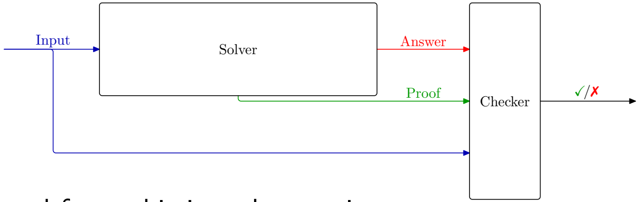

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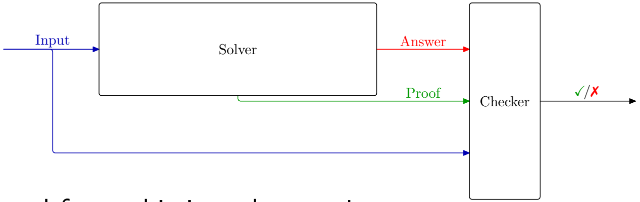

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Asking for both perhaps a little bit too good to be true?

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- Only need propositional logic
- Represent constraints as 0–1 integer linear inequalities
- Formalize reasoning using extended resolution [Tse68] and cutting planes [CCT87] proof systems
- Add well-chosen strengthening rules [Goc22, GN21, BGMN22]
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Making constructive use of computational complexity theory!

Outline of This Talk

Combinatorial Problems (1/2)

Boolean satisfiability (SAT)
Decide if exists satisfying assignment to conjunctive normal form (CNF) formula

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land$$
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SAT-based optimization (MaxSAT)
Minimize 0–1 linear expression

$$p + q + 2\overline{r} + 3u + 5\overline{w}$$

subject to constraints in a CNF formula

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Minimize $\sum_i w_i x_i$ subject to $A\mathbf{x} \geq \mathbf{b}$
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Satisfiability modulo theories (SMT)
Propositional logic formula with variables express statements in theories, e.g.:

- uninterpreted functions
- linear arithmetic
- arrays

Computational Complexity of Combinatorial Problems

- These problems are NP-complete [Coo71, Lev73] or worse
- Believed to require exponential time in the worst case [IP01, CIP09]
- Proving such lower bounds is the goal of computational complexity theory [GW08]
- Has not stopped practitioners from solving problems very efficiently in practice (often, not always)

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- "$25957 = 101 \cdot 257$; check yourself that these are primes."
  Concise! Primality easy to check [Mil76, Rab80, AKS04]

Key demand: Proofs should be short but efficiently verifiable

Proof System

Proof system for formal language L [CR79] of "true claims":

Deterministic algorithm $P(x, \pi)$ that runs in time polynomial in $|x|$ and $|\pi|$ such that

- for all $x \in L$ there exists a string π (a proof) such that $P(x, \pi) = 1$
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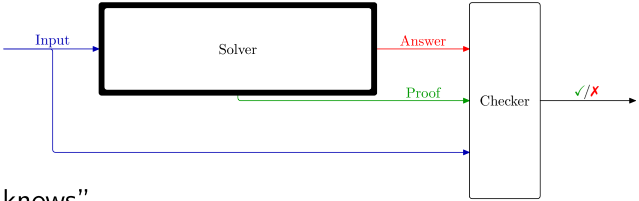
Note that proof π can be very large compared to x
Only have to achieve polynomial running time in $|x| + |\pi|$
Goal of proof complexity: establish lower bounds on proof size

Proof Logging with Certifying Solvers: Practical Requirements



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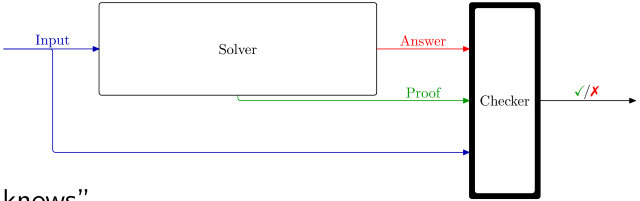

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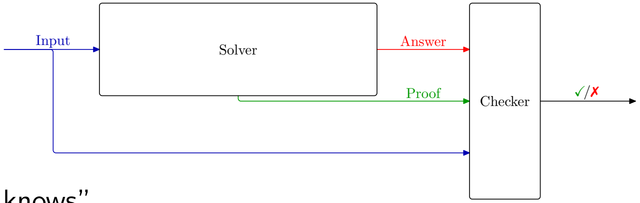

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Fully automated process — no proof assistants
Higher-order logics too complicated and/or too slow(?)

The Sales Pitch for Proof Logging

1. ***Certifies correctness*** of solver output
2. ***Detects errors*** even if due to compiler bugs, hardware failures, or cosmic rays
3. Helps with ***debugging*** during development [EG21, GMM^{+}20, KM21, BBN^{+}23]
4. Facilitates ***performance analysis***
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8. Can ***validate computer-generated proofs*** in mathematics [HK17]

The Proof Logging Story So Far

Huge success for Boolean satisfiability (SAT) solving

- Proof formats such as
  - DRAT [HHW13a, HHW13b, WHH14]
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And, in fact, even for advanced SAT solving techniques such as

- cardinality detection
- parity reasoning
- symmetry breaking

The Boolean Satisfiability (SAT) Problem

- Variable x: takes value **true** $(=1)$ or **false** $(=0)$
- Literal ℓ: variable x or its negation $\overline{x}$
- Clause $C = \ell_1 \lor \cdots \lor \ell_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- Conjunctive normal form (CNF) formula $F = C_1 \land \cdots \land C_m$: conjunction of clauses

The SAT problem

Given a CNF formula F, is it satisfiable?

For instance, what about:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land$$
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Proofs for SAT

For satisfiable instances: just specify a satisfying assignment

For unsatisfiability: a sequence of clauses

- Each clause follows "obviously" from everything we know so far
- Final clause is empty, meaning contradiction (written $\perp$)
- Means original formula must be inconsistent

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Unit propagation

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Proof checker should know how to unit propagate until saturation

Davis-Putman-Logemann-Loveland (DPLL)

DPLL [DP60, DLL62]: Assign variables and propagate; backtrack when clause violated

“Proof trace”: when backtracking, write negation of guesses made

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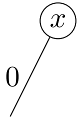

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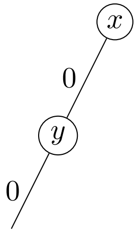

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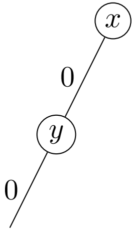

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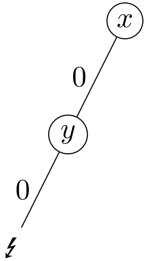
"Proof trace": when backtracking, write negation of guesses made

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

1. $x \lor y$



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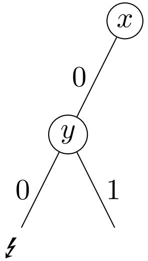

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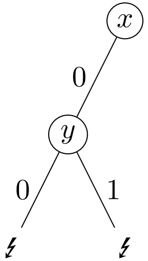

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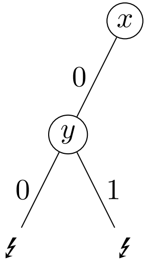
2. $x \lor \overline{y}$
3. x



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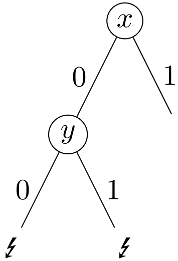

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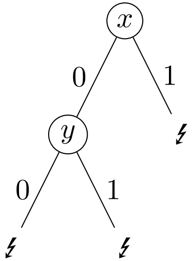

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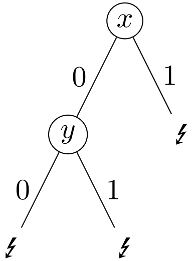
1. $x \lor y$
2. $x \lor \overline{y}$
3. x
4. $\overline{x}$
5. $\perp$



Reverse Unit Propagation (RUP)

To make this into a proof, need backtrack clauses to be easily verifiable

Reverse unit propagation (RUP) clause [GN03, Van08]

C is a reverse unit propagation (RUP) clause with respect to F if

- assigning C to false,
- then unit propagating on F until saturation
- leads to contradiction

If so, F clearly implies C, and this condition is easy to verify efficiently

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- Backtrack clauses from DPLL solver generate RUP proofs
- True also for learned clauses in modern conflict-driven clause learning (CDCL) SAT solvers [MS96, BS97, MMZ^{+}01]

Writing Proofs in the DRAT Format

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

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Formula in DIMACS

```
p cnf 8 9
1 -4 0
2 3 0
-2 5 0
4 6 7 0
6 -7 8 0
-6 8 0
-7 -8 0
-6 -8 0
-1 -4 0
```

Writing Proofs in the DRAT Format

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

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DPLL Proof in RUP

$x \lor y$

$x \lor \overline{y}$

x

$\overline{x}$

$\perp$

Writing Proofs in the DRAT Format

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```

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$x \lor y$

$x \lor \overline{y}$

x

$\overline{x}$

$\perp$

DPLL Proof in DRAT

```
6 7 0
6 -7 0
6 0
-6 0
0
```

More Ingredients in Proof Logging for SAT

Fact

RUP proofs are shorthand for so-called resolution proofs

See [BN21] for more on this and connections to SAT solving

But RUP and resolution aren't enough for preprocessing, inprocessing, and some other kinds of reasoning

Extension Variables

Suppose SAT solver preprocessor wants to introduce new, fresh variable a encoding

$$a \leftrightarrow (x \land y)$$

Extended resolution: allow to introduce clauses

$$a \lor \overline{x} \lor \overline{y} \qquad \overline{a} \lor x \qquad \overline{a} \lor y$$

Should be fine, so long as a doesn't appear anywhere previously

Fact

Extended resolution (RUP + definition of new variables) is essentially equivalent to the DRAT proof logging system

Why Aren’t We Done?

Practical limitations of SAT proof logging technology:

- Difficulties dealing with stronger reasoning efficiently
- Clausal proofs can’t easily reflect what other algorithms do

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Surprising claim: a slight change to 0-1 integer linear inequalities does the job!
Can support proof logging for

- Graph reasoning without knowing what a graph is
- Constraint programming without knowing, e.g., what an integer variable is
- Advanced SAT techniques so far beyond reach for efficient DRAT proof logging

Pseudo-Boolean Constraints

0-1 integer linear inequalities or pseudo-Boolean constraints:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)

Some Types of Pseudo-Boolean Constraints

1. Clauses

$$x \vee \overline{y} \vee z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Cutting Planes Proof System [CCT87]

Input axioms From the input

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Literal axioms $$\frac{}{\ell_i \geq 0}$$

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Input axioms From the input

Literal axioms

$$\frac{}{\ell_i \geq 0}$$

Addition

$$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i) \ell_i \geq A + B}$$

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Multiplication
for any $c \in \mathbb{N}^+$

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for any $c \in \mathbb{N}^+$

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Division
for any $c \in \mathbb{N}^+$

$$\frac{\sum_i c a_i \ell_i \geq A}{\sum_i a_i \ell_i \geq \left\lceil \frac{A}{c} \right\rceil}$$

Cutting Planes Toy Example

$$w + 2x + y \geq 2$$

Cutting Planes Toy Example

Mul by 2 $$\frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

Cutting Planes Toy Example

$$\text{Mul by } 2 \quad \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5$$

Cutting Planes Toy Example

$$\text{Add}\ \dfrac{\text{Mul by } 2\ \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}$$

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$$

Cutting Planes Toy Example

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$$

Cutting Planes Toy Example

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Such a calculation can be written in a proof line assuming handles

$$\begin{aligned} C_1 &\doteq 2x+y+w \geq 2 \\ C_2 &\doteq 2x+4y+2z+w \geq 5 \\ Ax(\overline{z}) &\doteq \overline{z} \geq 0 \end{aligned}$$

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using postfix notation something like

$$C_1\;\;2\;\;\texttt{Mul}\;\;C_2\;\;\texttt{Add}\;\;Ax(\overline{z})\;\;2\;\;\texttt{Mul}\;\;\texttt{Add}\;\;3\;\;\texttt{Div}$$

Pseudo-Boolean Proofs

For satisfiable instances: just specify a satisfying assignment

For unsatisfiability: a sequence of pseudo-Boolean constraints in (slight extension of) OPB format [RM16]

Each constraint follows "obviously" from what is known so far

- Either implicitly, by (generalization of) RUP...
- Or by an explicit cutting planes derivation...
- Or by (generalization of) extension rule

Final constraint is $0 \geq 1$

Proof Logging for Graph Solving and Constraint Programming

Pseudo-Boolean proof logging can also certify reasoning in

- graph solving for clique, subgraph isomorphism, and maximum common connected subgraph [GMN20, GMM$^+$20] without knowing anything about
  - vertices
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Caveat: Need input pre-translated into 0–1 integer linear program
Such translations should be formally verified (work in progress)

Integer Variables

Represent integer a as sum of bits $\sum_i 2^i \cdot a_i$

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Use extension rule to introduce new variables

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$$a_{=k} \Leftrightarrow (a_{\geq k} \wedge \overline{a}_{\geq k+1})$$

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$$k \cdot \overline{a}_{\geq k} + \sum_i 2^i \cdot a_i \geq k$$

$$\left(\sum_i 2^i - k + 1\right) \cdot a_{\geq k} + \sum_i 2^i \cdot \overline{a}_i \geq \sum_i 2^i - k + 1$$

$$a_{=k} \Leftrightarrow (a_{\geq k} \wedge \overline{a}_{\geq k+1})$$

$$2 \cdot \overline{a}_{=k} + a_{\geq k} + \overline{a}_{\geq k+1} \geq 2$$

$$a_{=k} + \overline{a}_{\geq k} + a_{\geq k+1} \geq 1$$

(with definitions represented as 0–1 inequalities)

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(with definitions represented as 0–1 inequalities)

Go back and forth between representations to support efficient proof logging

All-Different Propagator

$$
\begin{array}{rl}
V \in \{ & 1 \quad \quad \quad 4 \quad 5 \, \} \\
W \in \{ & 1 \quad 2 \quad 3 \quad \quad \, \} \\
X \in \{ & \quad 2 \quad 3 \quad \quad \, \} \\
Y \in \{ & 1 \quad \quad 3 \quad \quad \, \} \\
Z \in \{ & 1 \quad \quad 3 \quad \quad \, \}
\end{array}
$$

All-Different Propagator

$$\begin{array}{rclcccccl}
V & \in & \{ & 1 & & & 4 & 5 & \} \\
W & \in & \{ & 1 & 2 & 3 & & & \} \\
X & \in & \{ & & 2 & 3 & & & \} \\
Y & \in & \{ & 1 & & 3 & & & \} \\
Z & \in & \{ & 1 & & 3 & & & \}
\end{array}$$

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\end{array}
\qquad w_{=1} + w_{=2} + w_{=3} \geq 1
$$

All-Different Propagator

$$\begin{array}{llllllll}
V \in \{ & 1 & & & 4 & 5 \} & & \\
W \in \{ & 1 & 2 & 3 & & \} & w_{=1} + w_{=2} + w_{=3} & \geq 1 \\
X \in \{ & & 2 & 3 & & \} & x_{=2} + x_{=3} & \geq 1 \\
Y \in \{ & 1 & & 3 & & \} & y_{=1} + y_{=3} & \geq 1 \\
Z \in \{ & 1 & & 3 & & \} & z_{=1} + z_{=3} & \geq 1
\end{array}$$

All-Different Propagator

$$\begin{array}{llllllllllllllll}
V \in \{ & 1 & & & 4 & 5 \} & & & & & & & & & & \\
W \in \{ & 1 & 2 & 3 & & \} & w_{=1} & + & w_{=2} & + & w_{=3} & & & & & \geq 1 \\
X \in \{ & & 2 & 3 & & \} & & & x_{=2} & + & x_{=3} & & & & & \geq 1 \\
Y \in \{ & 1 & & 3 & & \} & y_{=1} & & & + & y_{=3} & & & & & \geq 1 \\
Z \in \{ & 1 & & 3 & & \} & z_{=1} & & & + & z_{=3} & & & & & \geq 1
\end{array}$$

$$\begin{array}{ll}
\rightarrow & -v_{=1} + -w_{=1} + \quad -y_{=1} + -z_{=1} \geq -1 \\
\quad\rightarrow & -w_{=2} + -x_{=2} \geq -1 \\
\qquad\rightarrow & -w_{=3} + -x_{=3} + -y_{=3} + -z_{=3} \geq -1
\end{array}$$

All-Different Propagator

$$\begin{array}{rl}
V \in \{1 \quad\quad 4 \quad 5\} & \\
W \in \{1 \quad 2 \quad 3 \quad\quad\} & w_{=1} + w_{=2} + w_{=3} \geq 1 \\
X \in \{\quad 2 \quad 3 \quad\quad\} & x_{=2} + x_{=3} \geq 1 \\
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\end{array}$$

$$\begin{array}{rl}
\rightarrow & -v_{=1} + -w_{=1} + -y_{=1} + -z_{=1} \geq -1 \\
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$$-v_{=1} \geq 1$$

All-Different Propagator

$$\begin{array}{lllllllllll}
V \in \{ 1 & & & 4 & 5 \} & & & & & & \\
W \in \{ 1 & 2 & 3 & & \} & w_{=1} + & w_{=2} + & w_{=3} & & & \geq 1 \\
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Other Constraint Programming Reasoning

Efficient proof logging support for

- Table constraints
- Arrays
- Problem reformulations
- Backtracking during search
- Et cetera. . .

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Not at all trivial to implement
Lots of work left to get to full-fledged constraint programming solver
But so far everything has been possible to do [EGMN20, GMN22]

Actual Extension Rule: Redundance-Based Strengthening

C is redundant with respect to F if F and $F \wedge C$ are equisatisfiable

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Proof sketch for interesting direction: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \wedge C$

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Witness ω should be specified, and implication be efficiently verifiable (which is the case, e.g., if all constraints in $(F \wedge C)\restriction_{\omega}$ are RUP)

Deriving $a \leftrightarrow (x \wedge y)$ Using the Redundance Rule

Want to derive

$$2\overline{a} + x + y \geq 2 \qquad a + \overline{x} + \overline{y} \geq 1$$

using condition $F \wedge \neg C \models (F \wedge C)\restriction_{\omega}$

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   $\neg(a + \overline{x} + \overline{y} \geq 1)$ forces $x \mapsto 1$ and $y \mapsto 1$, hence $2\overline{a} + x + y \geq 2$ remains satisfied after forcing a to be true

Optimization Problems

Pseudo-Boolean optimization

Minimize $f = \sum_i w_i \ell_i$ (for $w_i \in \mathbb{N}$) subject to constraints in F

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Note that $\sum_i w_i \ell_i < \sum_i w_i \cdot \alpha(\ell_i)$ means $\sum_i w_i \ell_i \leq -1 + \sum_i w_i \cdot \alpha(\ell_i)$

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Add constraint C to formula F if exists witness substitution ω such that

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Can be more aggressive if witness ω strictly improves solution

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Add constraint D to formula F if exists witness substitution ω such that

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7. ...

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7. $\ldots$
8. Can't go on forever, so finally reach α' satisfying $F \land D$

Strength of Dominance Rule

Dominance-based strengthening (stronger, still simplified) [BGMN22]

If $D_1, D_2, \ldots, D_{m-1}$ have been derived from F (maybe using dominance), then can derive also D_m if exists witness substitution ω such that

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- Switch between different orders in same proof

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Show up also in hard SAT benchmarks

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Symmetric learning

- Allow to add all symmetric versions of learned clause [DBB17]
- Adding rules for symmetric reasoning as in [TD20] breaks extension rule

Viewing Symmetry as an Optimization Problem

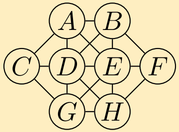
Deal with symmetries by switching focus to optimization

Invent objective function $\sum_{i=1}^{n} 2^i \cdot x_i$) corresponding to lexicographic order

Now dominance-based strengthening $=$ symmetry breaking!

Symmetry Elimination Example: Crystal Maze Puzzle

The Crystal Maze Puzzle



Place numbers 1 to 8 without repetition; adjacent circles cannot have consecutive numbers

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Human modellers might add:

- $A < G$ (mirror vertically)
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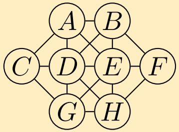

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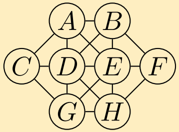

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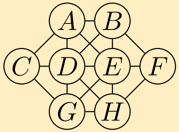

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Can derive these constraints inside the proof rather than adding to input

- Witness ω: symmetry
- Order: Lexicographic $(A, B, \ldots, H)$
- No group theory required!

Directions for Future Research

Proof logging for combinatorial optimization

- Pseudo-Boolean optimization and MaxSAT solving (work in [GMNO22, VDB22, BBN^{+}23])
- General constraint programming (work in [EGMN20, GMN22])
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- SMT proof logging using stronger logics?

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- We're hiring! Talk to me to join the proof logging revolution!

Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like a promising approach
- Requires powerful but simple proof systems — need for “computationally efficient logic”
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Thank you for your attention!

References I

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