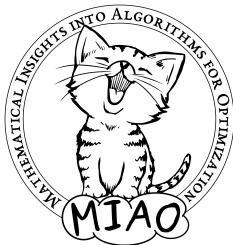


# Certified Symmetry and Dominance Breaking for Combinatorial Optimisation

Jakob Nordström

University of Copenhagen  
and Lund University

13th Pragmatics of SAT workshop  
Haifa, Israel  
August 1, 2022



Joint AAAI '22 paper with Bart Bogaerts, Stephan Gocht, and Ciaran McCreesh

# Combinatorial Solving and Optimisation

- Revolution last couple of decades in **combinatorial solvers** for
  - Boolean satisfiability (SAT) solving [BHvMW21]
  - Satisfiability modulo theories (SMT) solving [BHvMW21]
  - Constraint programming (CP) [RvBW06]
  - Mixed integer linear programming (MIP) [AW13, BR07]
- Solve NP problems (or worse) very successfully in practice!
- **Except solvers are sometimes wrong...** (Even best commercial ones) [BLB10, CKSW13, AGJ<sup>+</sup>18, GSD19, GS19]
- **Software testing** doesn't suffice to resolve this problem
- **Formal verification** techniques cannot deal with level of complexity of modern solvers

# Certified Results with Proof Logging

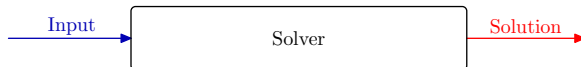
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- not only **solve problem** but also
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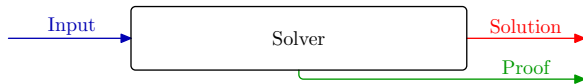
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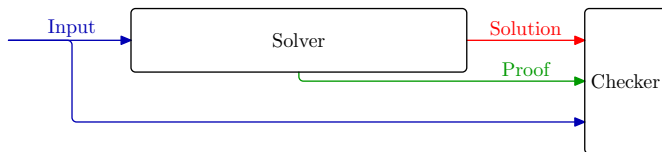
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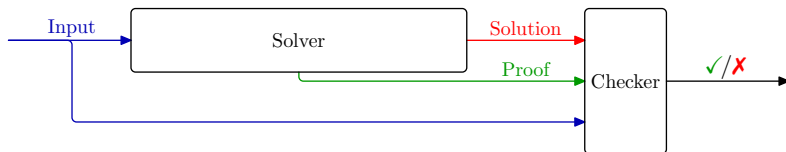
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## Workflow:

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- ③ Feed input + solution + proof to proof checker
- ④ Verify that proof checker says solution is correct

# Yet Another SAT Success Story

Many proof logging formats for SAT solving using CNF clausal format:

- DRAT [HHW13a, HHW13b, WHH14]
- GRIT [CMS17]
- LRAT [CHH<sup>+</sup>17]
- ...

Well established — required in main track of SAT competitions



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But efficient proof logging has remained out of reach for stronger paradigms

And, in fact, even for some advanced SAT solving techniques:

- cardinality reasoning
- Gaussian elimination
- symmetry handling

# Clausal Proof Logging Approaches

## **Cardinality and pseudo-Boolean reasoning** [SB06, BBH22]

Evaluated on fairly specific crafted benchmarks

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Problems with proof logging overhead and proof file size

## **Symmetry handling** [HHW15, TD20]

No fully general method for **symmetry breaking** (i.e., adding constraints to remove symmetric solutions)

Method for **symmetric learning** (i.e., adding symmetric versions of derived constraints) not compatible with SAT preprocessing

# Our Work: Efficient Proof Logging for Symmetry Breaking

Paper *Certified Symmetry and Dominance Breaking for Combinatorial Optimisation* at AAAI '22 [BGMN22]:

Implementation in proof checker VERIPB [Ver]

- First general & efficient proof logging method for **symmetry breaking**
- Supports also **pseudo-Boolean reasoning** and **Gaussian elimination**
- Based on **0-1 integer linear constraints** instead of clauses
- Uses **cutting planes method** [CCT87] with additional rules

# Outline of Presentation

What I hope to cover in the rest of this presentation:

- Basics of proof logging with 0-1 linear constraints
- New rule for symmetry and dominance breaking
- Application to symmetry breaking for SAT (and some other problems)
- Some future research directions

# 0-1 Integer Linear (a.k.a. Pseudo-Boolean) Constraints

Pseudo-Boolean (PB) constraints are 0-1 integer linear constraints

$$C \doteq \sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals  $\ell_i$ :  $x_i$  or  $\bar{x}_i$  (where  $x_i + \bar{x}_i = 1$ )
- variables  $x_i$  take values  $0 = \text{false}$  or  $1 = \text{true}$



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Pseudo-Boolean formulas  $F \doteq \bigwedge_{i=1}^m C_i$  are conjunctions of pseudo-Boolean constraints

# Some Types of Pseudo-Boolean Constraints

## 1 Clauses

$$x \vee \bar{y} \vee z \quad \Leftrightarrow \quad x + \bar{y} + z \geq 1$$

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## 3 General pseudo-Boolean constraints

$$x_1 + 2\bar{x}_2 + 3x_3 + 4\bar{x}_4 + 5x_5 \geq 7$$

# Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

**Literal axioms**  $\frac{}{\ell_i \geq 0}$

**Linear combination**  $\frac{\sum_i a_i \ell_i \geq A \quad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i) \ell_i \geq c_A A + c_B B} \quad [c_A, c_B \in \mathbb{N}]$

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- Generalize **reverse unit propagation (RUP)** rule [GN03, Van08] to PB constraints — just convenient shorthand for derivation
- Also need **extension** rule (analogue of RAT [JHB12]) to deal with, e.g., preprocessing

## Extension Rule: Redundance-Based Strengthening

$C$  is **redundant** with respect to  $F$  if  $F$  and  $F \wedge C$  are **equisatisfiable**

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- Implication should be **efficiently verifiable** — every  $D \in (F \wedge C) \upharpoonright_{\omega}$  should follow from  $F \wedge \neg C$  by, e.g.,
  - 1 weakening (addition of literal axioms  $\ell_i \geq 0$ )
  - 2 reverse unit propagation (RUP)
  - 3 explicit derivation presented in proof log

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(SAT talk Thu Aug 4 at 15:00)

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Don't miss CP tutorial Tue Aug 2 at 14:00 *Solving with Provably Correct Results: Beyond Satisfiability, and Towards Constraint Programming*

# The Challenge of Symmetries

(Syntactic) symmetry: substitution  $\sigma$  preserving  $F$  ( $F|_{\sigma} \doteq F$ )

- Show up in some hard SAT benchmarks
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Allow to add all symmetric versions of learned constraint [DBB17]



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Not supported by standard SAT proof logging!

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Deal with **symmetry breaking** by switching focus to **optimisation**  
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- $F \wedge (\sum_i w_i l_i < \sum_i w_i \cdot \alpha(l_i))$  is infeasible

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- $F \wedge (\sum_i w_i l_i < \sum_i w_i \cdot \alpha(l_i))$  is infeasible

Note that  $\sum_i w_i l_i < \sum_i w_i \cdot \alpha(l_i)$  means  $\sum_i w_i l_i \leq -1 + \sum_i w_i \cdot \alpha(l_i)$

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Redundance-based strengthening, optimisation version

Add constraint  $C$  to formula  $F$  if exists witness substitution  $\omega$  such that

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- If so, don't need to show that  $C|_{\omega}$  implied!

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- 7 ...
- 8 Can't go on forever, so finally reach  $\alpha'$  satisfying  $F \wedge C$



# Strength of Dominance Rule

## Dominance-based strengthening (stronger, still simplified)

If  $C_1, C_2, \dots, C_{m-1}$  have been derived from  $F$  (maybe using dominance), then can derive also  $C_m$  if exists witness substitution  $\omega$  such that

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Further extensions:

- Define dominance rule w.r.t. order independent of objective function
- Switch between different orders in same proof
- See [BGMN22] for details

# Strategy for SAT Symmetry Breaking

- 1 Pretend to **solve optimisation problem** minimizing  $f \doteq \sum_{i=1}^n 2^{n-i} \cdot x_i$   
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- 3 Derive **CNF encoding** of lex-leader constraints from PB constraint  
(in same spirit as [GMNO22])

$y_0$	$\overline{y_j} \vee \overline{\sigma(x_j)} \vee x_j$
$\overline{y_{j-1}} \vee \overline{x_j} \vee \sigma(x_j)$	$y_j \vee \overline{y_{j-1}} \vee \overline{x_j}$
$\overline{y_j} \vee y_{j-1}$	$y_j \vee \overline{y_{j-1}} \vee \sigma(x_j)$

# Breaking Symmetries With the Dominance Rule (1/2)

## Theorem

$C_\sigma \doteq f \leq f|_\sigma$  can be derived from  $F$  using dominance with witness  $\sigma$

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Redundance-based strengthening can be used analogously to [HHW15]



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- but only guaranteed to work for breaking **single symmetry**  $\sigma$
- if  $\sigma$  is **involution** (i.e., its own inverse)
- not known how to deal with symmetries that are complex or interact

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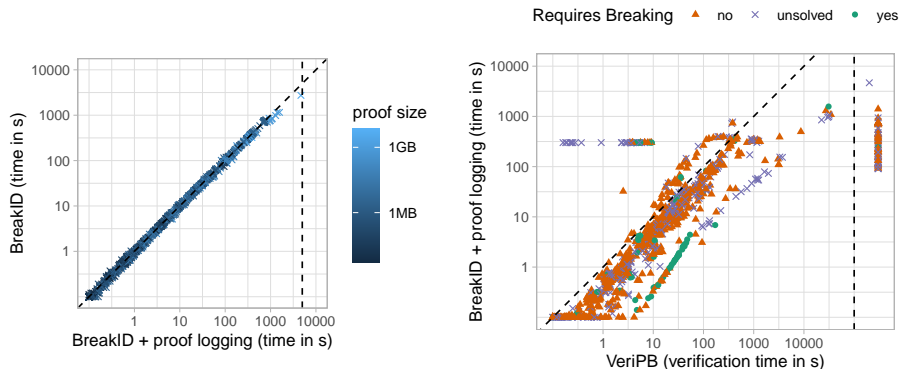
Why does it work?

- Witness need not satisfy all derived constraints
- Sufficient to just produce “better” assignment



# Experimental Evaluation

- Evaluated on SAT competition benchmarks
- BREAKID [DBBD16, Bre] used to find and break symmetries



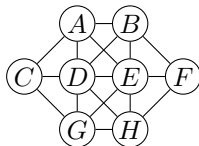
- proof logging overhead negligible
- verification at most 20 times slower than solving for 95% of instances

# Symmetry Breaking for Constraint Programming

## Crystal Maze puzzle

Place numbers 1 to 8 without repetition

Adjacent circles mustn't have consecutive numbers

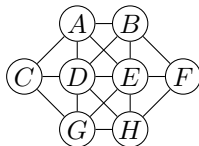


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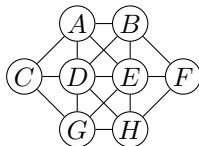
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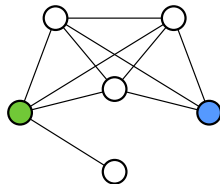
*Technical challenge:* integer-valued variables

See [GMN22] for more detailed discussion

# Dominance Breaking for Maximum Clique Solving

## Maximum clique solving

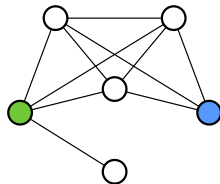
Find largest fully connected component



# Dominance Breaking for Maximum Clique Solving

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## Lazy global domination [MP16]

Only consider green and not blue vertex

(since every neighbour of blue is also neighbour of green)

*Technical challenge:* vertex domination detected only lazily during search  
Dominance rule (rather than redundancy rule) really helpful here

# Future Research Directions

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- **We're hiring!** Talk to me to join the proof logging revolution!

# Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity
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*Thank you for your attention!*

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