Combinatorial Solving with Provably Correct Results

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Combinatorial Solving and Optimisation

- Revolution last couple of decades in combinatorial solvers for
  - Boolean satisfiability (SAT) solving [BHvMW21][1]
  - Constraint programming (CP) [RvBW06]
  - Mixed integer linear programming (MIP) [AW13, BR07]
- Solve NP-complete problems (or worse) very successfully in practice!
- Except solvers are sometimes wrong... (Even best commercial ones) [BLB10, CKSW13, AGJ$^+$18, GSD19, GS19, BMN22, BBN$^+$23]
- Even get feasibility of solutions wrong (though this should be straightforward!)
- And how to check the absence of solutions?
- Or that a solution is optimal? (Even off-by-one mistakes can snowball into large errors if solver used as subroutine)

[1] See end of slides for all references with bibliographic details

What Can Be Done About Solver Bugs?

- **Software testing**
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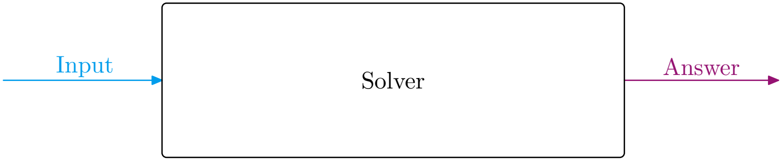
- **Proof logging**
  Make solver certifying [ABM+11, MMNS11] by outputting
  1. not only answer but also
  2. simple, machine-verifiable proof that answer is correct

Proof Logging with Certifying Solvers: Workflow



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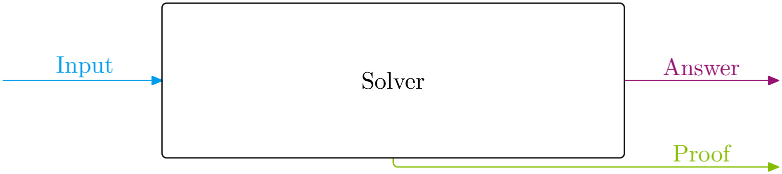

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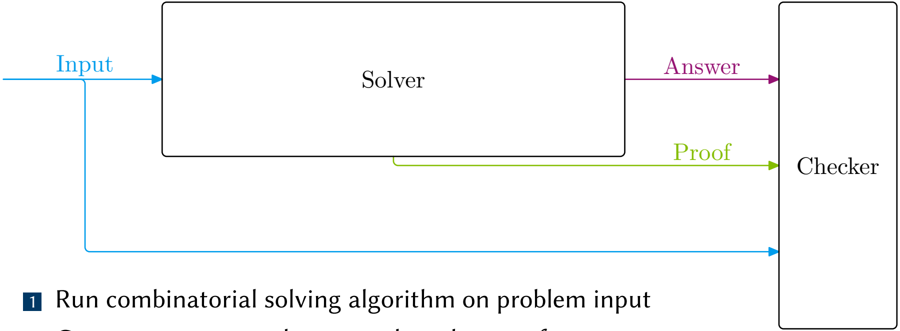

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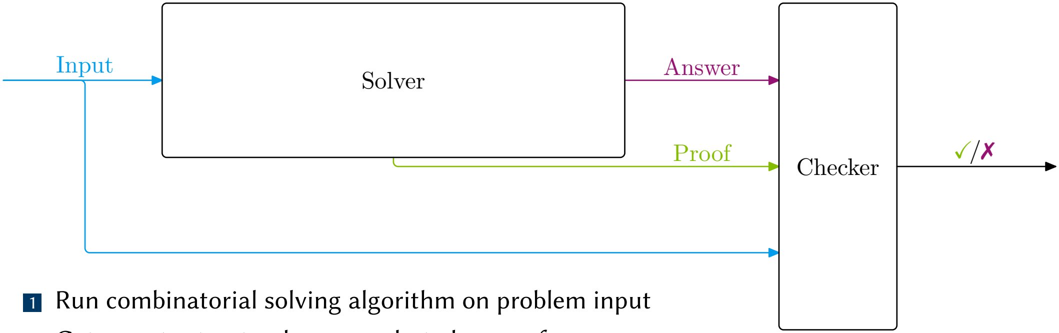

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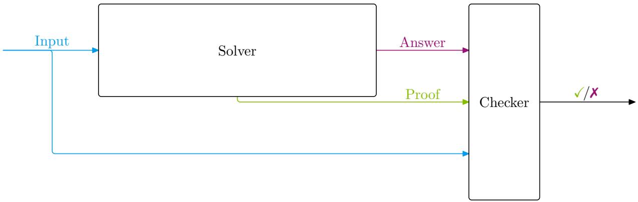
2. Get as output not only answer but also proof
3. Feed input + answer + proof to proof checker
4. Verify that proof checker says answer is correct

Proof Logging Desiderata



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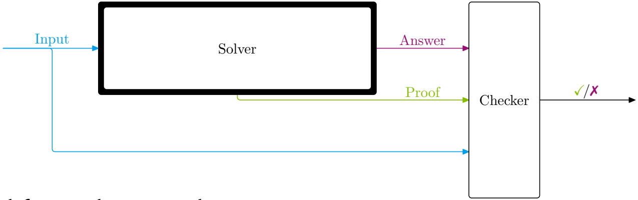

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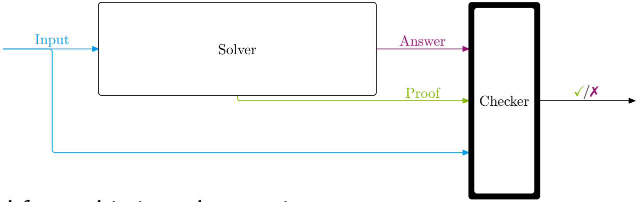

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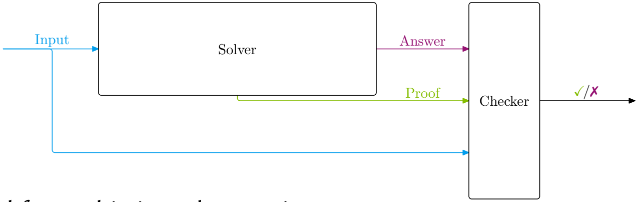

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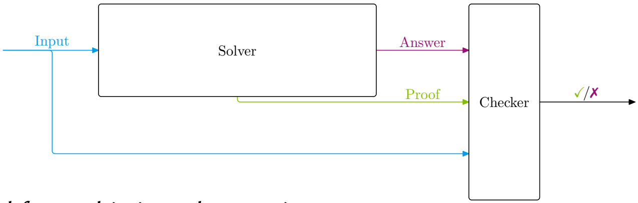

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Asking for both perhaps a little bit too good to be true?

Take-Away Message from This Tutorial

Proof logging for combinatorial optimisation is possible with single, unified method!

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Proof logging for combinatorial optimisation is possible with single, unified method!

- Build on successes in proof logging for SAT solvers with proof formats such as DRAT [HHW13a, HHW13b, WHH14], GRIT [CMS17], LRAT [CHH+17], . . .
- But represent constraints as 0–1 integer linear inequalities
- Formalize reasoning using cutting planes [CCT87] proof system
- Add well-chosen strengthening rules [Goc22, GN21, BGMN23]
- Implemented in VeriPB (`https://gitlab.com/MIAOresearch/software/VeriPB`)

The Sales Pitch For Proof Logging

1. Certifies correctness of computed results
2. Detects errors even if due to compiler bugs, hardware failures, or cosmic rays
3. Provides debugging support during development [EG21, GMM+20, KM21, BBN+23]
4. Facilitates performance analysis
5. Helps identify potential for further improvements
6. Enables auditability
7. Serves as stepping stone towards explainability

The Rest of This Tutorial

Explain how to use VeriPB to do proof logging for

- SAT solving (including advanced techniques)
- SAT-based optimisation (MaxSAT)
- Subgraph algorithms
- Constraint programming
- Symmetry and dominance reasoning

in a unified way

The SAT Problem

- Variable x: takes value **true** $(=1)$ or **false** $(=0)$
- Literal ℓ: variable x or its negation $\overline{x}$
- Clause $C = \ell_1 \vee \cdots \vee \ell_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- Conjunctive normal form (CNF) formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses

The SAT Problem

Given a CNF formula F, is it satisfiable?

For instance, what about:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge$$
$$(x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

Proofs for SAT

For satisfiable instances: just specify satisfying assignment

For unsatisfiability: a sequence of clauses (CNF constraints)

- Each clause follows “obviously” from everything we know so far
- Final clause is empty, meaning contradiction (written $\perp$)
- Means original formula must be inconsistent

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Unit Propagation

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Proof checker should know how to unit propagate until saturation

Davis-Putman-Logemann-Loveland (DPLL)

DPLL [DP60, DLL62]: Assign variables and propagate; backtrack when clause violated

"Proof trace": when backtracking, write negation of guesses made

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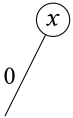

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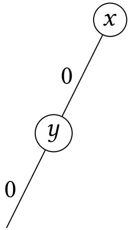

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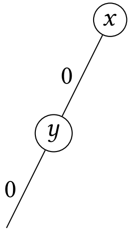

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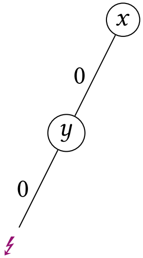

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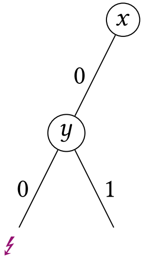

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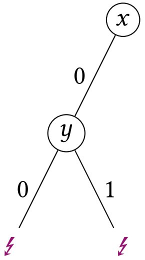

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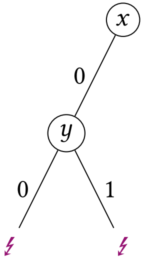

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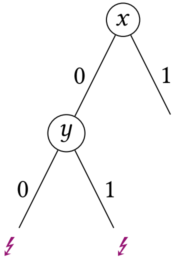

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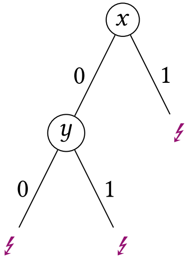

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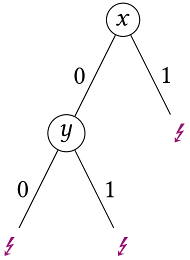
5. $\bot$



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Fact

Backtrack clauses from DPLL solver generate a RUP proof

What About Conflict-Driven Clause Learning (CDCL)?

Run CDCL [BS97, MS99, MMZ+01] on our favourite CNF formula:

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Notation $p \stackrel{\mathrm{d}}{=} 0$

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Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \overset{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

What About Conflict-Driven Clause Learning (CDCL)?

Run CDCL [BS97, MS99, MMZ+01] on our favourite CNF formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathrm{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathrm{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

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$\boxed{p \overset{\mathsf{d}}{=} 0}$

$\boxed{u \overset{p \vee \overline{u}}{=} 0}$

$\boxed{q \overset{\mathsf{d}}{=} 0}$

Decision

Free choice to assign value to variable

Notation $p \overset{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \overset{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

What About Conflict-Driven Clause Learning (CDCL)?

Run CDCL [BS97, MS99, MMZ+01] on our favourite CNF formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \overset{\mathrm{d}}{=} 0}$

$u \overset{p \lor \overline{u}}{=} 0$

$\boxed{q \overset{\mathrm{d}}{=} 0}$

$r \overset{q \lor r}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \overset{\mathrm{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \overset{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

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$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathrm{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathrm{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathrm{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

What About Conflict-Driven Clause Learning (CDCL)?

Run CDCL [BS97, MS99, MMZ+01] on our favourite CNF formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$\boxed{u \stackrel{p \vee \overline{u}}{=} 0}$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$\boxed{r \stackrel{q \vee r}{=} 1}$

$\boxed{w \stackrel{\overline{r} \vee w}{=} 1}$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

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Run CDCL [BS97, MS99, MMZ+01] on our favourite CNF formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$y \stackrel{u \lor x \lor y}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

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Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

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$p \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{p \lor \overline{u}}{=} 0$

$q \stackrel{\mathsf{d}}{=} 0$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

$x \stackrel{\mathsf{d}}{=} 0$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

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$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

$\stackrel{\overline{y} \lor \overline{z}}{\bot}$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

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$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$p \stackrel{\mathsf{d}}{=} 0$	decision level 1
$u \stackrel{p \lor \overline{u}}{=} 0$	
$q \stackrel{\mathsf{d}}{=} 0$	decision level 2
$r \stackrel{q \lor r}{=} 1$	
$w \stackrel{\overline{r} \lor w}{=} 1$	
$x \stackrel{\mathsf{d}}{=} 0$	decision level 3
$y \stackrel{u \lor x \lor y}{=} 1$	
$z \stackrel{x \lor \overline{y} \lor z}{=} 1$	
$\stackrel{\overline{y} \lor \overline{z}}{\bot}$	

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$
$u \stackrel{p \vee \overline{u}}{=} 0$
decision level 1

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$
$r \stackrel{q \vee r}{=} 1$
$w \stackrel{\overline{r} \vee w}{=} 1$
decision level 2

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$
$y \stackrel{u \vee x \vee y}{=} 1$
$z \stackrel{x \vee \overline{y} \vee z}{=} 1$
$\stackrel{\overline{y} \vee \overline{z}}{\bot}$
decision level 3

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

decision level 1

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

decision level 2

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

$\stackrel{\overline{y} \lor \overline{z}}{\bot}$

decision level 3

Could backtrack by erasing conflict level & flipping last decision

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$$\left.\begin{array}{l} \boxed{p \stackrel{\mathsf{d}}{=} 0} \\ u \stackrel{p \vee \overline{u}}{=} 0 \end{array}\right\} \text{decision level 1}$$

$$\left.\begin{array}{l} \boxed{q \stackrel{\mathsf{d}}{=} 0} \\ r \stackrel{q \vee r}{=} 1 \\ w \stackrel{\overline{r} \vee w}{=} 1 \end{array}\right\} \text{decision level 2}$$

$$\left.\begin{array}{l} \boxed{x \stackrel{\mathsf{d}}{=} 0} \\ y \stackrel{u \vee x \vee y}{=} 1 \\ z \stackrel{x \vee \overline{y} \vee z}{=} 1 \\ \stackrel{\overline{y} \vee \overline{z}}{\bot} \end{array}\right\} \text{decision level 3}$$

Could backtrack by erasing conflict level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$p \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{p \vee \overline{u}}{=} 0$

$q \stackrel{\mathsf{d}}{=} 0$

$r \stackrel{q \vee r}{=} 1$

$w \stackrel{\overline{r} \vee w}{=} 1$

$x \stackrel{\mathsf{d}}{=} 0$

$y \stackrel{u \vee x \vee y}{=} 1$

$z \stackrel{x \vee \overline{y} \vee z}{=} 1$

$\stackrel{\overline{y} \vee \overline{z}}{\perp}$

$x \vee \overline{y}$

Could backtrack by erasing conflict level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

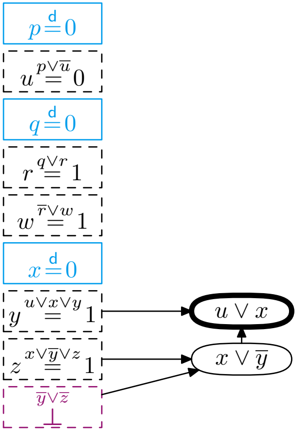
Case analysis over z for last two clauses:

- $x \vee \overline{y} \vee z$ wants $z = 1$
- $\overline{y} \vee \overline{z}$ wants $z = 0$
- Resolve clauses by merging them & removing z — must satisfy $x \vee \overline{y}$

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



Could backtrack by erasing conflict level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

Case analysis over z for last two clauses:

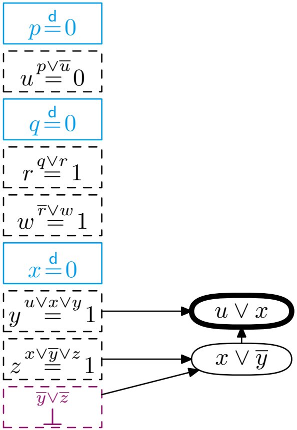
- $x \lor \overline{y} \lor z$ wants $z = 1$
- $\overline{y} \lor \overline{z}$ wants $z = 0$
- Resolve clauses by merging them & removing z — must satisfy $x \lor \overline{y}$

Repeat until UIP clause with only 1 variable at conflict level after last decision — learn and backjump

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

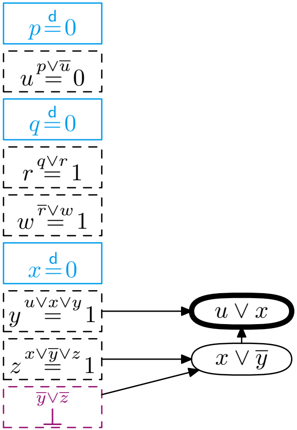
$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

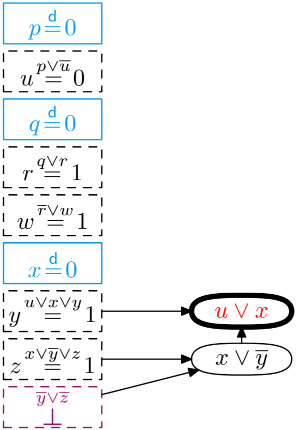


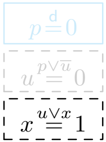
Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$





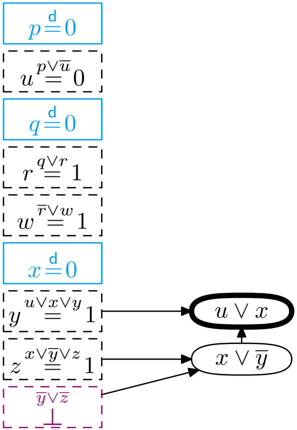
Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

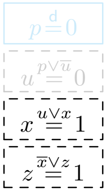
Now UIP literal guaranteed to flip (assert) — but this is a propagation, not a decision

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$





Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

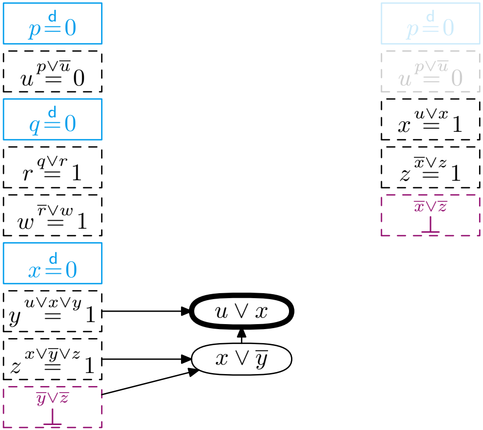
Now UIP literal guaranteed to flip (assert) — but this is a propagation, not a decision

Then continue as before…

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

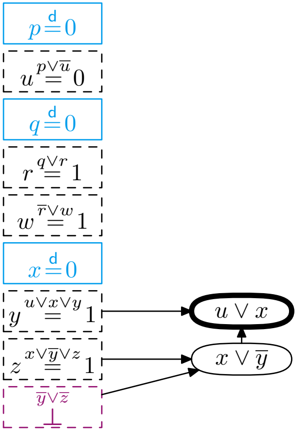
$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

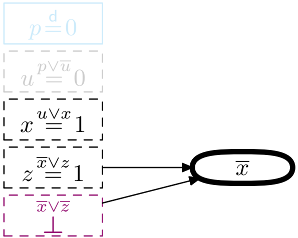


Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

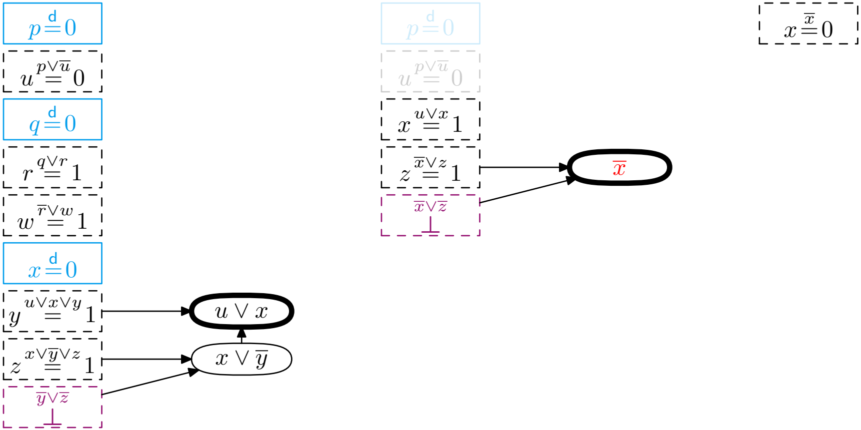




Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

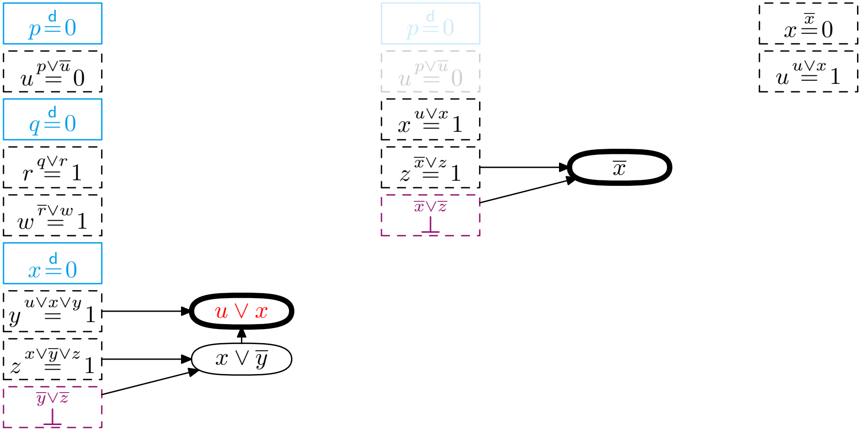
$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

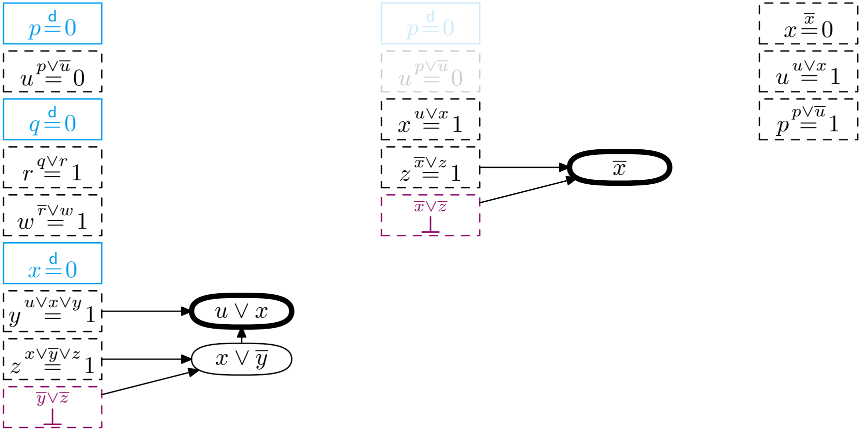
$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

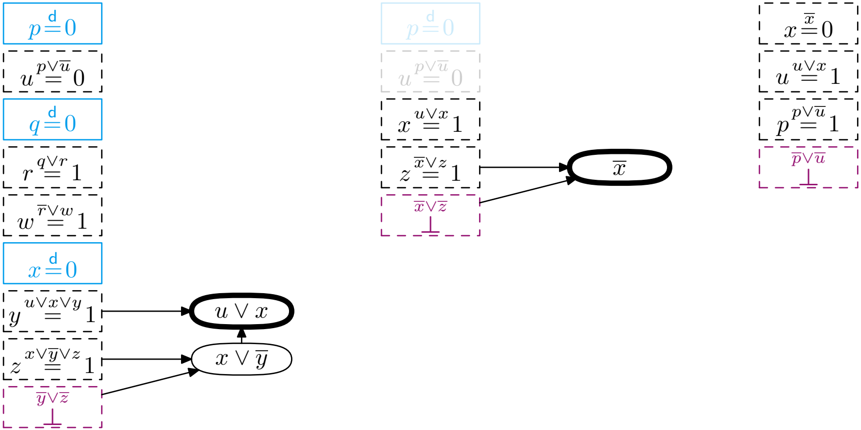
$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$



Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

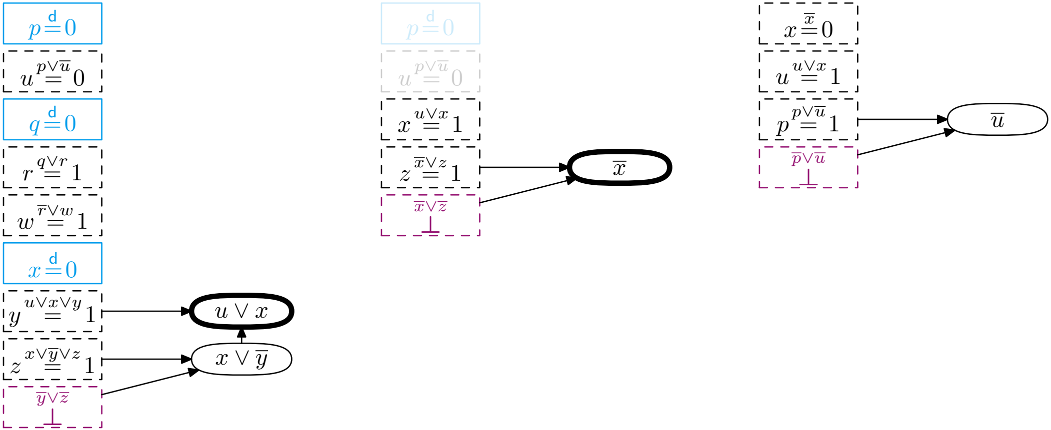
$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$



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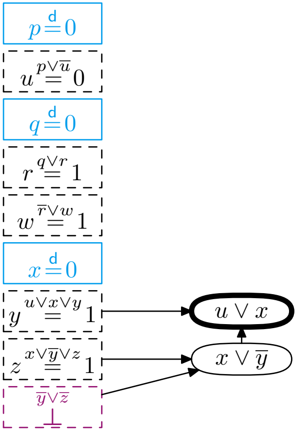

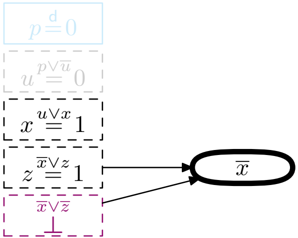

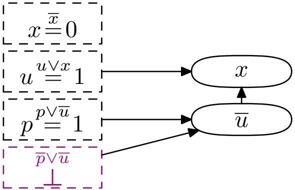

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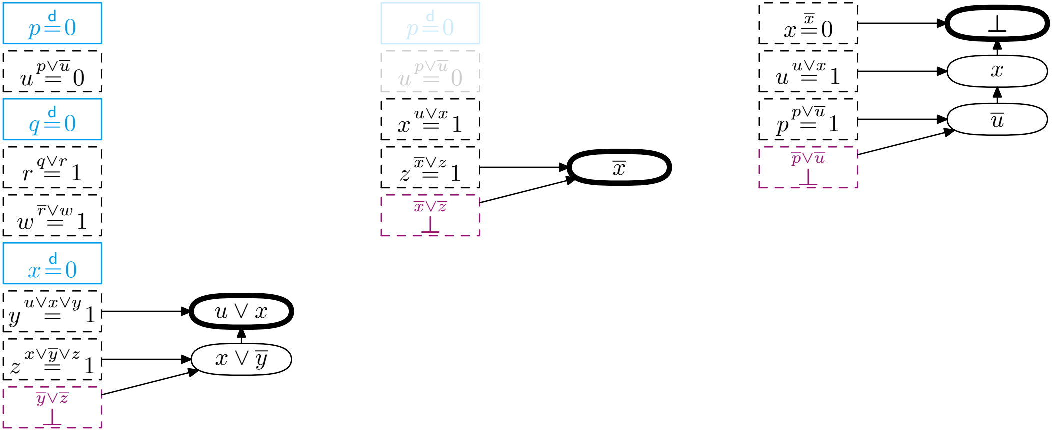

CDCL Reasoning and the Resolution Proof System

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Resolution proof system [Bla37, Rob65]

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$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

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Resolution Proofs from CDCL Executions

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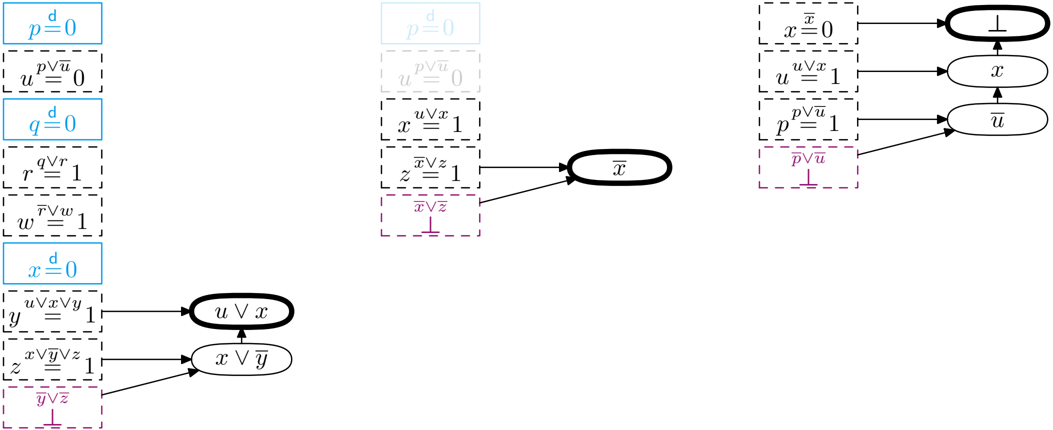

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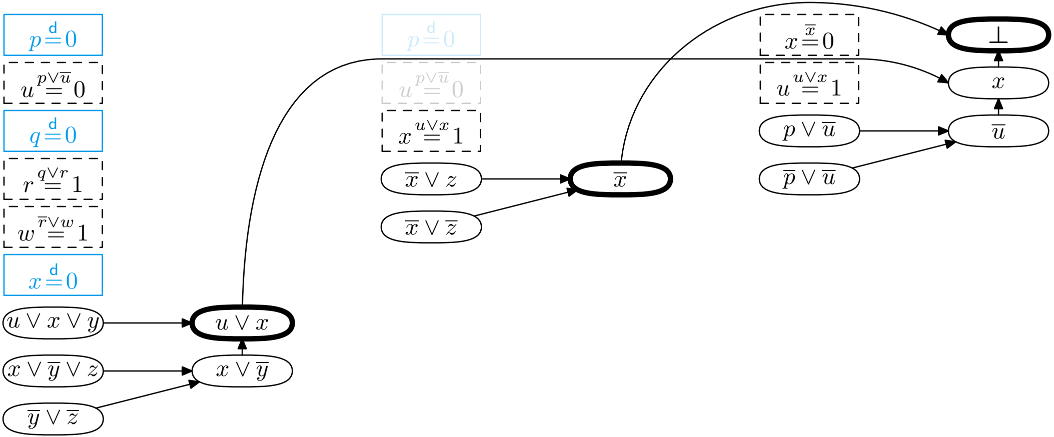
Obtain resolution proof from our example CDCL execution by stringing together conflict analyses:



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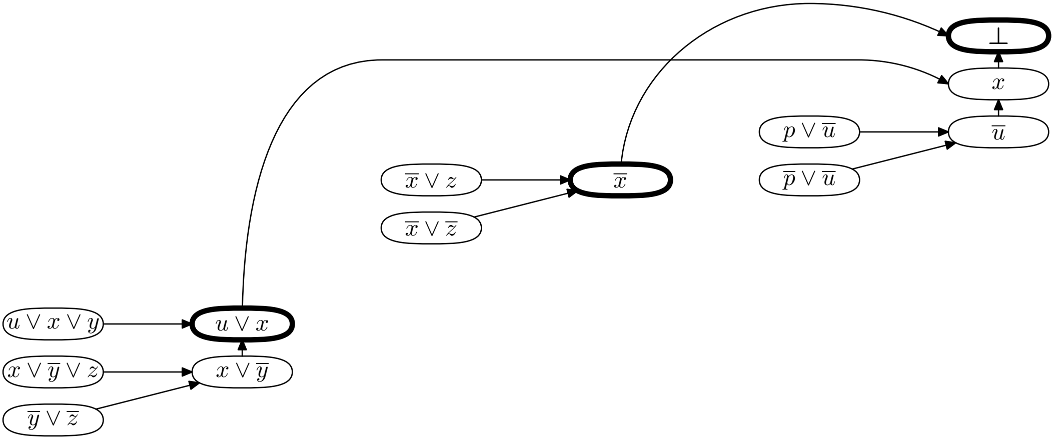

RUP Proofs and CDCL

But it turns out we can be lazier. . .

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All learned clauses generated by CDCL solver are RUP clauses

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3. $\perp$

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More Ingredients in Proof Logging for SAT

Fact

RUP proofs can be viewed as shorthand for resolution proofs

See [BN21] for more on this and connections to SAT solving

But RUP and resolution are not enough for preprocessing, inprocessing, and some other kinds of reasoning

Extension Variables, Part 1

Suppose we want a variable a encoding

$$a \Leftrightarrow (x \wedge y)$$

Extended resolution [Tse68]

Resolution rule plus extension rule introducing clauses

$$a \vee \overline{x} \vee \overline{y} \qquad \overline{a} \vee x \qquad \overline{a} \vee y$$

for fresh variable a (this is fine since a doesn't appear anywhere previously)

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Fact

Extended resolution (RUP + definition of new variables) is essentially equivalent to the DRAT proof logging system most commonly used for SAT solving

Why Aren’t We Done?

Practical limitations of current SAT proof logging technology:

- Difficulties dealing with stronger reasoning efficiently (even for SAT solving)
- Clausal proofs can’t easily reflect what algorithms for other problems do

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Surprising claim: a slight change to 0-1 integer linear inequalities does the job!

- Enables proof logging for advanced SAT techniques so far beyond reach for efficient DRAT proof logging:
  - Cardinality reasoning
  - Gaussian elimination
  - Symmetry breaking
- Supports use of SAT solvers for optimisation problems (MaxSAT)
- Can justify graph reasoning without knowing what a graph is
- Can justify constraint programming inference without knowing what an integer variable is

Pseudo-Boolean Constraints

0–1 integer linear inequalities or (linear) pseudo-Boolean constraints:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)

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Sometimes convenient to use normalized form [Bar95] with all a_i, A positive (without loss of generality)

Some Types of Pseudo-Boolean Constraints

1. Clauses

$$x_1 \lor \overline{x}_2 \lor x_3 \quad \Leftrightarrow \quad x_1 + \overline{x}_2 + x_3 \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input/model axioms From the input

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Input/model axioms From the input

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input/model axioms	From the input
Literal axioms	$\dfrac{}{\ell_i \geq 0}$
Addition	$\dfrac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i)\ell_i \geq A + B}$

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Multiplication for any $c \in \mathbb{N}^+$

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i c a_i \ell_i \geq cA}$$

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Division for any $c \in \mathbb{N}^+$
(assumes normalized form)

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \left\lceil \frac{a_i}{c} \right\rceil \ell_i \geq \left\lceil \frac{A}{c} \right\rceil}$$

Cutting Planes Toy Example

$$w + 2x + y \geq 2$$

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$$\text{Multiply by 2}\ \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

Cutting Planes Toy Example

$$\text{Multiply by } 2 \; \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5$$

Cutting Planes Toy Example

$$\dfrac{\text{Add}\quad \dfrac{\text{Multiply by 2}\quad w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}$$

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\dfrac{
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Naming constraints by integers and literal axioms by the literal involved (with ~ for negation) as

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\begin{aligned}
\text{Constraint 1} &\doteq 2x + y + w \geq 2 \\
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\sim z &\doteq \overline{z} \geq 0
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such a calculation is written in the proof log in reverse Polish notation as

```
pol 1 2 * 2 + ~z 2 * + 3 d
```

Resolution and Cutting Planes

To simulate resolution step such as

$$\frac{\overline{y} \lor \overline{z} \qquad x \lor \overline{y} \lor z}{x \lor \overline{y}}$$

we can perform the cutting planes steps

$$\text{Add}\ \frac{\overline{y} + \overline{z} \geq 1 \qquad x + \overline{y} + z \geq 1}{\text{Divide by 2}\ \dfrac{x + 2\overline{y} \geq 1}{x + \overline{y} \geq 1}}$$

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Given that the premises are clauses 7 and 5 in our example CNF formula, using references

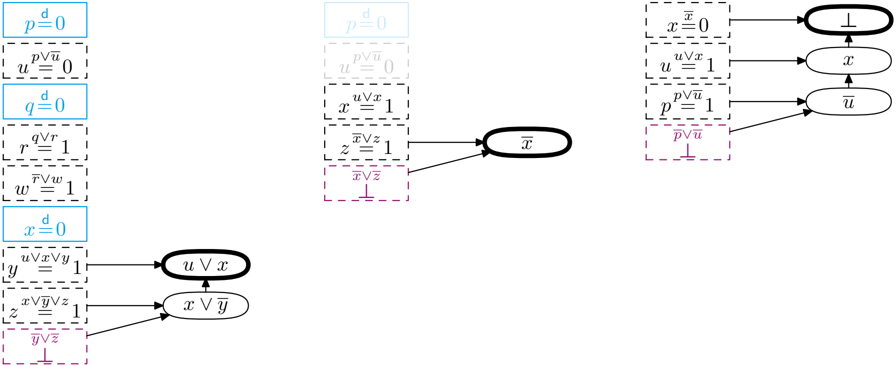
$$\text{Constraint 7} \doteq \overline{y} + \overline{z} \geq 1$$

$$\text{Constraint 5} \doteq x + \overline{y} + z \geq 1$$

we can write this in the proof log as

```
pol 7 5 + 2 d
```

Pseudo-Boolean Proof Logging for Example CDCL Conflict Analyses



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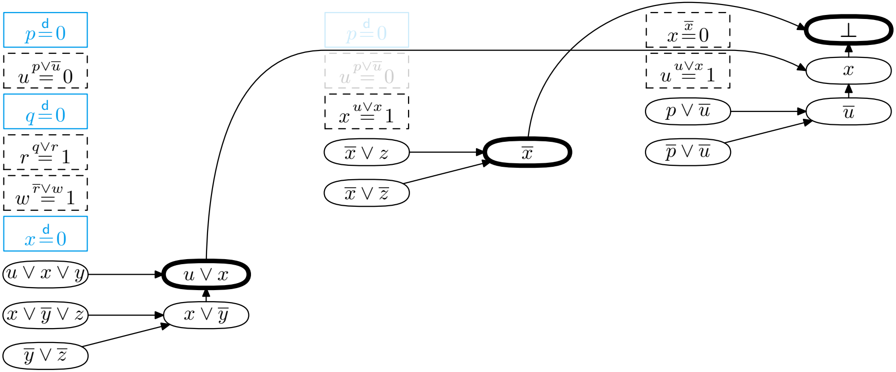

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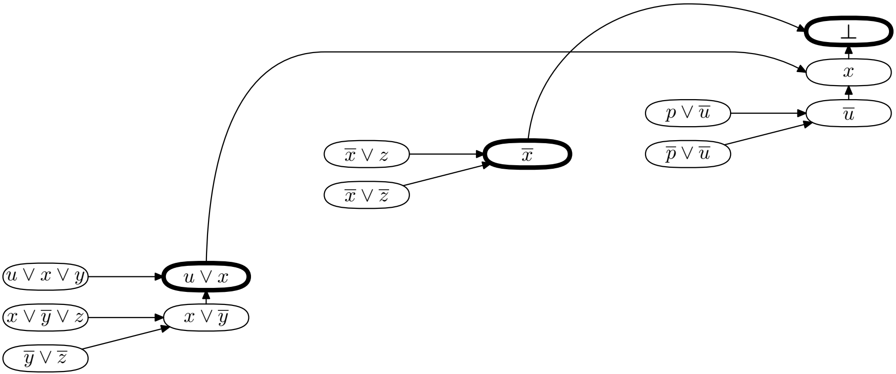

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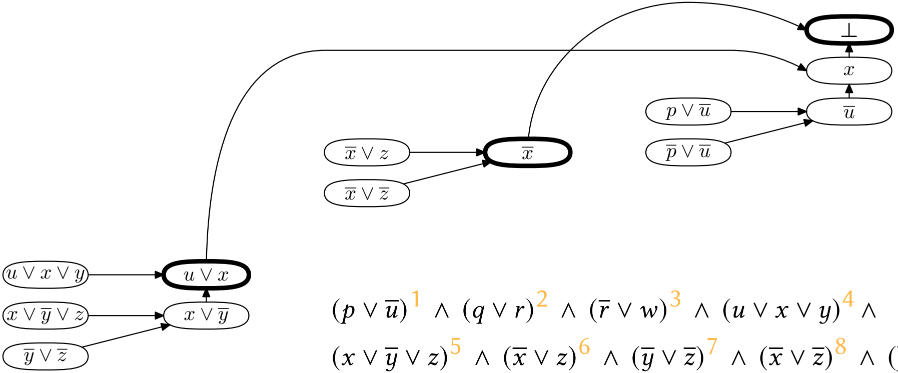

$$(p \vee \overline{u})^{1} \wedge (q \vee r)^{2} \wedge (\overline{r} \vee w)^{3} \wedge (u \vee x \vee y)^{4} \wedge$$
$$(x \vee \overline{y} \vee z)^{5} \wedge (\overline{x} \vee z)^{6} \wedge (\overline{y} \vee \overline{z})^{7} \wedge (\overline{x} \vee \overline{z})^{8} \wedge (\overline{p} \vee \overline{u})^{9}$$

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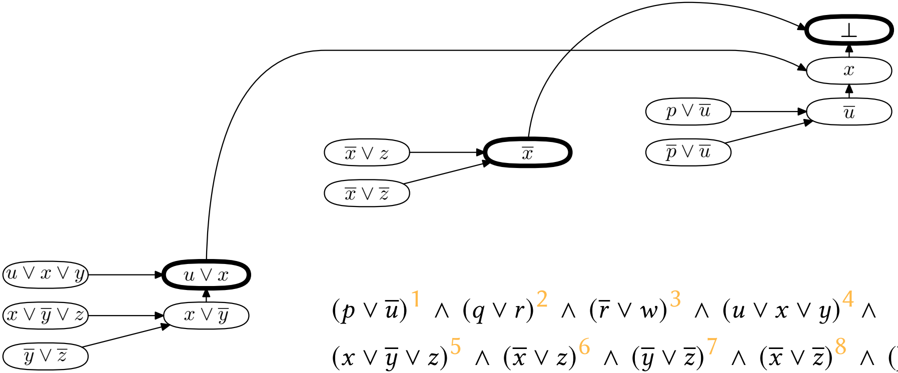

$$(p \lor \overline{u})^{1} \land (q \lor r)^{2} \land (\overline{r} \lor w)^{3} \land (u \lor x \lor y)^{4} \land$$
$$(x \lor \overline{y} \lor z)^{5} \land (\overline{x} \lor z)^{6} \land (\overline{y} \lor \overline{z})^{7} \land (\overline{x} \lor \overline{z})^{8} \land (\overline{p} \lor \overline{u})^{9}$$

`pol 7 5 + 2 d 4 + 2 d` $\rightsquigarrow$ Constraint 10 $\doteq u + x \geq 1$

`pol 8 6 + 2 d` $\rightsquigarrow$ Constraint 11 $\doteq \overline{x} \geq 1$

`pol 9 1 + 2 d 10 + 2 d 11 + 2 d` $\rightsquigarrow$ Constraint 12 $\doteq 0 \geq 1$ ↯

RUP Revisited

Can define (reverse) unit propagation in a pseudo-Boolean setting

Constraint C propagates variable x if setting x to “wrong value” would make C unsatisfiable

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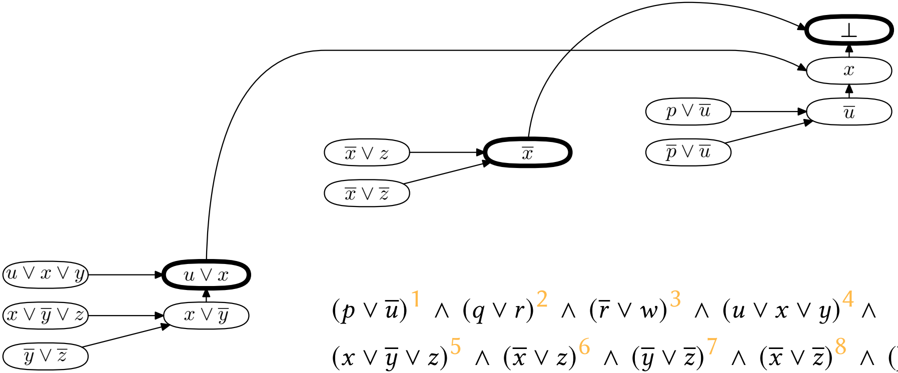
Risk for confusion:

- Constraint programming people might call this (reverse) integer bounds consistency
  - Does the same thing if we’re working with clauses
  - More interesting for general pseudo-Boolean constraints
- SAT people beware: constraints can propagate multiple times and multiple variables

Pseudo-Boolean Proof Logging for Example CDCL Execution with RUP



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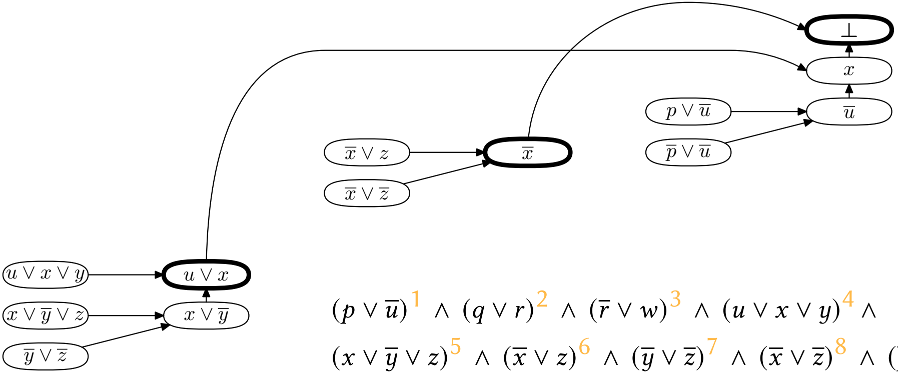

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```
rup 1 u 1 x >= 1 ;
rup 1 ~x >= 1 ;
rup >= 1 ;
```

$\rightsquigarrow$ Constraint 10 $\doteq$ $u + x \geq 1$

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$\rightsquigarrow$ Constraint 12 $\doteq$ $0 \geq 1$ ↯

Extension Variables, Part 2

Suppose we want new, fresh variable a encoding

$$a \Leftrightarrow (3x + 2y + z + w \geq 3)$$

This time, introduce constraints

$$3\overline{a} + 3x + 2y + z + w \geq 3 \qquad 5a + 3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \geq 5$$

Again, needs support from the proof system

Proof Logs for "Extended Cutting Planes"

For satisfiable instances: just specify a satisfying assignment.

For unsatisfiability: a sequence of pseudo-Boolean constraints in (slight extension of) OPB format [RM16]

- Each constraint follows "obviously" from what is known so far
- Either implicitly, by RUP…
- Or by an explicit cutting planes derivation…
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(*) Not actually implemented this way — details to come later...

Deleting Constraints

In practice, important to erase constraints to save memory and time during verification

Fairly straightforward to deal with from the point of view of proof logging

So ignored in this tutorial for simplicity and clarity

Enumeration and Optimisation Problems

Enumeration:

- When a solution is found, can log it
- Introduces a new constraint saying “not this solution”
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For optimisation:

- Define an objective $f = \sum_i w_i \ell_i$, $w_i \in \mathbb{Z}$, to minimise subject to the contraints in the formula
- To maximise, negate objective
- Log a solution α; get an objective-improving constraint $\sum_i w_i \ell_i \leq -1 + \sum_i w_i \alpha(\ell_i)$
- Semantics for proof of optimality: “infeasible to find better solution than best so far”

Pseudo-Boolean Proof Logging — How and Why?

If problem is (special case of) 0–1 integer linear program (ILP)

- just do proof logging

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$$r \Rightarrow x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

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$$r \Rightarrow x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7 \qquad 7\overline{r} + x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

$$r \Leftarrow x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7 \qquad 9r + \overline{x}_1 + 2x_2 + 3\overline{x}_3 + 4x_4 + 5\overline{x}_5 \geq 9$$

The VeriPB Format and Tool

`https://gitlab.com/MIAOresearch/software/VeriPB`

Released under MIT Licence

Various features to help development:

- Extended variable name syntax allowing human-readable names
- Proof tracing
- "Trust me" assertions for incremental proof logging

Documentation:

- Description of VeriPB checker [BMM+23] used in SAT 2023 competition (`https://satcompetition.github.io/2023/checkers.html`)
- Specific details on different proof logging techniques covered in research papers [EGMN20, GMN20, GMM+20, GN21, GMN22, GMNO22, VDB22, BBN+23, BGMN23, MM23]
- Lots of concrete example files at `https://gitlab.com/MIAOresearch/software/VeriPB`

Parity (XOR) Reasoning

Given clauses

$$x \lor y \lor z$$
$$x \lor \overline{y} \lor \overline{z}$$
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Could add XORs to language, but prefer to keep things super-simple

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Introduce extension variables a, b and derive

$$x + y + z + 2a = 3$$
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("=" syntactic sugar for "≥" plus "≤")

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VeriPB can certify XOR reasoning [GN21]

CDCL Solvers on Pseudo-Boolean Inputs

Can re-encode to CNF and run CDCL:

- *MiniSat+* [ES06]
- *Open-WBO* [MML14]
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E.g., encode pseudo-Boolean constraint

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

to clauses with extension variables

$$s_{i,k} \Leftrightarrow \sum_{j=1}^{i} x_j \geq k$$

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$$k \cdot \overline{s}_{i,k} + \sum_{j=1}^{i} x_j \geq k$$
$$(i-k+1) \cdot s_{i,k} + \sum_{j=1}^{i} \overline{x}_j \geq i-k+1$$

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$$(i - k + 1) \cdot s_{i,k} + \sum_{j=1}^{i} \bar{x}_j \geq i - k + 1$$

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How to know translation is correct?

VeriPB can certify pseudo-Boolean-to-CNF rewriting [GMNO22, VDB22]

Certified Maximum Satisfiability (MaxSAT) Solving

Minimize linear objective subject to satisfying formula in conjunctive normal form (CNF)

$$\begin{aligned} \min\ & 2x_1 + x_2 \\ \text{s.t.}\ & x_1 \lor \overline{z} \\ & z \lor x_2 \end{aligned} \longrightarrow \boxed{\text{MaxSAT solver}} \longrightarrow \text{Result: optimum } 1$$

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Does not work Only proves answer correct, not reasoning within solver!

MaxSAT Solvers

Three main categories:

- Linear SAT-UNSAT search
  1. Call SAT solver to find some solution
  2. Add clauses encoding "I want a better solution"
  3. Repeat (last found solution is optimal)

MaxSAT Solvers

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- Linear SAT-UNSAT search
  1. Call SAT solver to find some solution
  2. Add clauses encoding "I want a better solution"
  3. Repeat (last found solution is optimal)

  VeriPB-based proof logging available [VDB22, Van23]

MaxSAT Solvers

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- Core-guided search
  1. Call SAT solver to find solution under most optimistic assumptions
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- Implicit Hitting Set
  1. Call SAT solver to find solution under most optimistic assumptions
  2. Use hitting set solver (MIP solver) to recompute what most possible optimistic assumptions are
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  3. Repeat (first solution is optimal)

  No proof logging available yet

MaxSAT example (LSU search)

Objective: $min \sum_i r_i$

VeriPB proof:

derived	justification

$\overline{x}_1 \lor x_2$

$x_1 \lor \overline{x}_2$

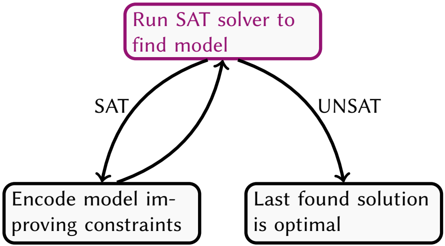
$\overline{x}_2 \lor x_3$

$\overline{x}_3 \lor x_4$

$\overline{x}_1 \lor \overline{x}_2 \lor r_1$

$x_1 \lor x_2 \lor r_2$

$x_2 \lor x_4 \lor r_3$



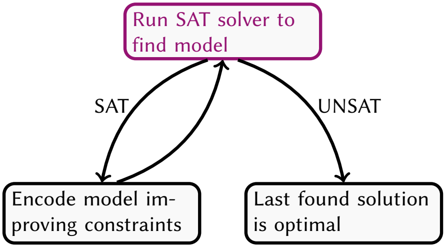
MaxSAT example (LSU search)

Objective: $min \sum_i r_i$

VeriPB proof:

derived	justification

$$\begin{array}{ll}
\overline{x}_1 \lor x_2 & \overline{x}_1 \lor \overline{x}_2 \lor r_1 \\
x_1 \lor \overline{x}_2 & x_1 \lor x_2 \lor r_2 \\
\overline{x}_2 \lor x_3 & x_2 \lor x_4 \lor r_3 \\
\overline{x}_3 \lor x_4 & x_2 \lor r_2
\end{array}$$



MaxSAT example (LSU search)

Objective: *min* $\sum_i r_i$

VeriPB proof:

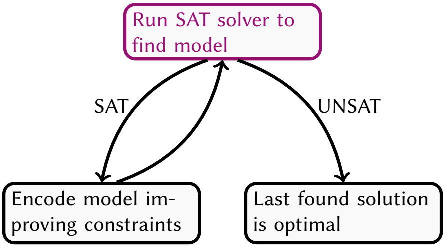
derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation

$\overline{x}_1 \vee x_2$ $\quad$ $\overline{x}_1 \vee \overline{x}_2 \vee r_1$

$x_1 \vee \overline{x}_2$ $\quad$ $x_1 \vee x_2 \vee r_2$

$\overline{x}_2 \vee x_3$ $\quad$ $x_2 \vee x_4 \vee r_3$

$\overline{x}_3 \vee x_4$ $\quad$ $x_2 \vee r_2$



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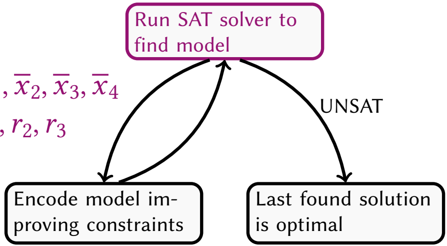
$x_1 \vee \overline{x}_2$ $\quad$ $x_1 \vee x_2 \vee r_2$

$\overline{x}_2 \vee x_3$ $\quad$ $x_2 \vee x_4 \vee r_3$

$\overline{x}_3 \vee x_4$ $\quad$ $x_2 \vee r_2$

$\overline{x}_1, \overline{x}_2, \overline{x}_3, \overline{x}_4$

$\overline{r}_1, r_2, r_3$



MaxSAT example (LSU search)

Objective: *min* $\sum_i r_i$

VeriPB proof:

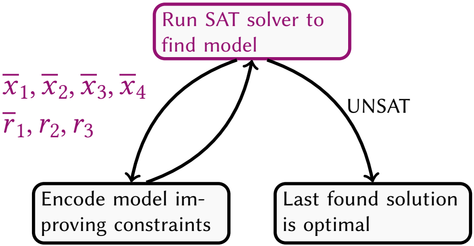
derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \ldots, \overline{x}_4, \overline{r}_1, r_2, r_3\}$	Incumbent solution

$\overline{x}_1 \lor x_2$ $\quad$ $\overline{x}_1 \lor \overline{x}_2 \lor r_1$

$x_1 \lor \overline{x}_2$ $\quad$ $x_1 \lor x_2 \lor r_2$

$\overline{x}_2 \lor x_3$ $\quad$ $x_2 \lor x_4 \lor r_3$

$\overline{x}_3 \lor x_4$ $\quad$ $x_2 \lor r_2$



MaxSAT example (LSU search)

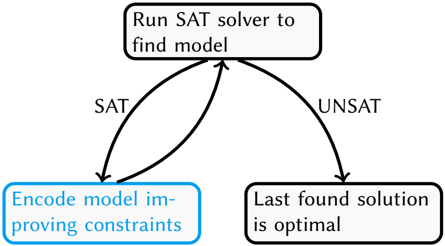
Objective: $min \sum_i r_i$

VeriPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \ldots, \overline{x}_4, \overline{r}_1, r_2, r_3\}$	Incumbent solution
$\sum_i r_i \leq 1$	Objective Improvement Rule

$\overline{x}_1 \vee x_2$
$x_1 \vee \overline{x}_2$
$\overline{x}_2 \vee x_3$
$\overline{x}_3 \vee x_4$

$\overline{x}_1 \vee \overline{x}_2 \vee r_1$
$x_1 \vee x_2 \vee r_2$
$x_2 \vee x_4 \vee r_3$
$x_2 \vee r_2$



MaxSAT example (LSU search)

Objective: *min* $\sum_i r_i$

VeriPB proof:

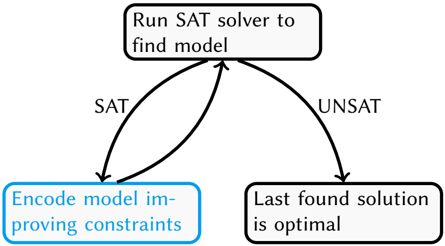
derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \ldots, \overline{x}_4, \overline{r}_1, r_2, r_3\}$	Incumbent solution
$\sum_i r_i \leq 1$	Objective Improvement Rule
$\mathrm{PB}(p_1 \Leftrightarrow (\sum_i r_i \geq 1))$	Fresh variable (RBS)
$\mathrm{PB}(p_2 \Leftrightarrow (\sum_i r_i \geq 2))$	

$\overline{x}_1 \vee x_2$ $\qquad$ $\overline{x}_1 \vee \overline{x}_2 \vee r_1$

$x_1 \vee \overline{x}_2$ $\qquad$ $x_1 \vee x_2 \vee r_2$

$\overline{x}_2 \vee x_3$ $\qquad$ $x_2 \vee x_4 \vee r_3$

$\overline{x}_3 \vee x_4$ $\qquad$ $x_2 \vee r_2$



MaxSAT example (LSU search)

Objective: $min \sum_i r_i$

VeriPB proof:

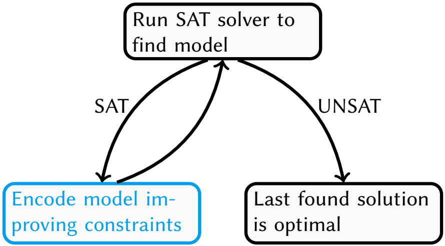
derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \ldots, \overline{x}_4, \overline{r}_1, r_2, r_3\}$	Incumbent solution
$\sum_i r_i \leq 1$	Objective Improvement Rule
$j \cdot \overline{p}_j + \sum_i r_i \geq j$	Fresh variable (RBS)
$(4 - j) \cdot p_j + \sum_i \overline{r}_i \geq 4 - j$	

$\overline{x}_1 \vee x_2$ $\quad$ $\overline{x}_1 \vee \overline{x}_2 \vee r_1$

$x_1 \vee \overline{x}_2$ $\quad$ $x_1 \vee x_2 \vee r_2$

$\overline{x}_2 \vee x_3$ $\quad$ $x_2 \vee x_4 \vee r_3$

$\overline{x}_3 \vee x_4$ $\quad$ $x_2 \vee r_2$



MaxSAT example (LSU search)

Objective: *min* $\sum_i r_i$

VeriPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \ldots, \overline{x}_4, \overline{r}_1, r_2, r_3\}$	Incumbent solution
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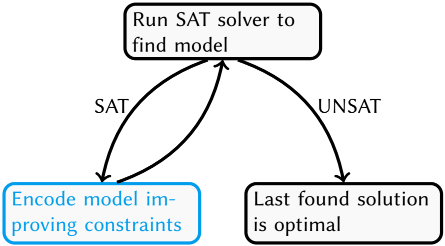
$j \cdot \overline{p}_j + \sum_i r_i \geq j$	Fresh variable (RBS)
$(4 - j) \cdot p_j + \sum_i \overline{r}_i \geq 4 - j$	
$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation

$\overline{x}_1 \vee x_2$ $\quad$ $\overline{x}_1 \vee \overline{x}_2 \vee r_1$
$x_1 \vee \overline{x}_2$ $\quad$ $x_1 \vee x_2 \vee r_2$
$\overline{x}_2 \vee x_3$ $\quad$ $x_2 \vee x_4 \vee r_3$
$\overline{x}_3 \vee x_4$ $\quad$ $x_2 \vee r_2$



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Objective: *min* $\sum_i r_i$

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derived	justification
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$\sum_i r_i \leq 1$	Objective Improvement Rule
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$(4 - j) \cdot p_j + \sum_i \overline{r}_i \geq 4 - j$	
$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation

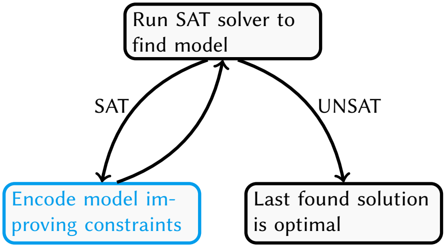
$\overline{x}_1 \vee x_2$ $\overline{x}_1 \vee \overline{x}_2 \vee r_1$

$x_1 \vee \overline{x}_2$ $x_1 \vee x_2 \vee r_2$

$\overline{x}_2 \vee x_3$ $x_2 \vee x_4 \vee r_3$

$\overline{x}_3 \vee x_4$ $x_2 \vee r_2$

$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$



MaxSAT example (LSU search)

Objective: $min \sum_i r_i$

VeriPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \ldots, \overline{x}_4, \overline{r}_1, r_2, r_3\}$	Incumbent solution
$\sum_i r_i \leq 1$	Objective Improvement Rule
$j \cdot \overline{p}_j + \sum_i r_i \geq j$	Fresh variable (RBS)
$(4-j) \cdot p_j + \sum_i \overline{r}_i \geq 4-j$	
$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\overline{p}_2 \geq 1$	Explicit CP derivation

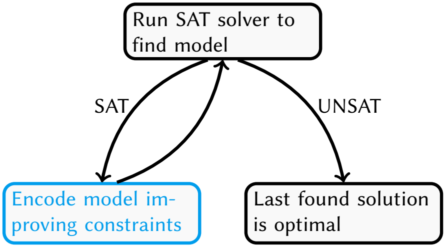
$\overline{x}_1 \vee x_2$ $\quad$ $\overline{x}_1 \vee \overline{x}_2 \vee r_1$

$x_1 \vee \overline{x}_2$ $\quad$ $x_1 \vee x_2 \vee r_2$

$\overline{x}_2 \vee x_3$ $\quad$ $x_2 \vee x_4 \vee r_3$

$\overline{x}_3 \vee x_4$ $\quad$ $x_2 \vee r_2$

$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$



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derived	justification
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$\{\overline{x}_1, \ldots, \overline{x}_4, \overline{r}_1, r_2, r_3\}$	Incumbent solution
$\sum_i r_i \leq 1$	Objective Improvement Rule
$j \cdot \overline{p}_j + \sum_i r_i \geq j$	Fresh variable (RBS)
$(4 - j) \cdot p_j + \sum_i \overline{r}_i \geq 4 - j$	
$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\overline{p}_2 \geq 1$	Explicit CP derivation

$\overline{x}_1 \vee x_2$ $\quad$ $\overline{x}_1 \vee \overline{x}_2 \vee r_1$

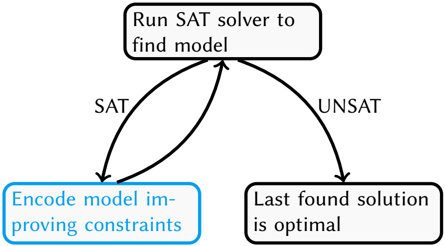
$x_1 \vee \overline{x}_2$ $\quad$ $x_1 \vee x_2 \vee r_2$

$\overline{x}_2 \vee x_3$ $\quad$ $x_2 \vee x_4 \vee r_3$

$\overline{x}_3 \vee x_4$ $\quad$ $x_2 \vee r_2$

$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$

$\overline{p}_2$



MaxSAT example (LSU search)

Objective: $min \sum_i r_i$

VeriPB proof:

derived	justification
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$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\overline{p}_2 \geq 1$	Explicit CP derivation

$\overline{x}_1 \lor x_2$ $\quad$ $\overline{x}_1 \lor \overline{x}_2 \lor r_1$

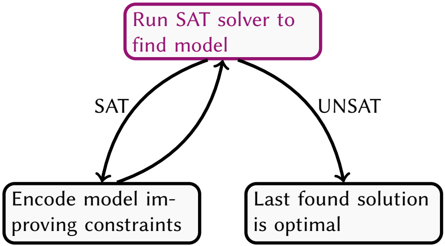
$x_1 \lor \overline{x}_2$ $\quad$ $x_1 \lor x_2 \lor r_2$

$\overline{x}_2 \lor x_3$ $\quad$ $x_2 \lor x_4 \lor r_3$

$\overline{x}_3 \lor x_4$ $\quad$ $x_2 \lor r_2$

$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$

$\overline{p}_2$



MaxSAT example (LSU search)

Objective: $min \sum_i r_i$

VeriPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \ldots, \overline{x}_4, \overline{r}_1, r_2, r_3\}$	Incumbent solution
$\sum_i r_i \leq 1$	Objective Improvement Rule
$j \cdot \overline{p}_j + \sum_i r_i \geq j$	Fresh variable (RBS)
$(4 - j) \cdot p_j + \sum_i \overline{r}_i \geq 4 - j$	
$\text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\overline{p}_2 \geq 1$	Explicit CP derivation
$x_4 \geq 1$	Reverse Unit Propagation

$\overline{x}_1 \vee x_2$ $\quad$ $\overline{x}_1 \vee \overline{x}_2 \vee r_1$

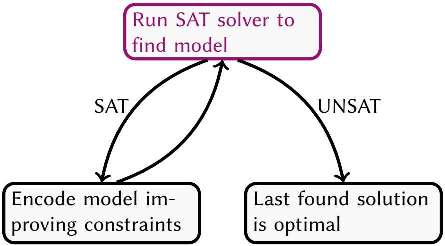
$x_1 \vee \overline{x}_2$ $\quad$ $x_1 \vee x_2 \vee r_2$

$\overline{x}_2 \vee x_3$ $\quad$ $x_2 \vee x_4 \vee r_3$

$\overline{x}_3 \vee x_4$ $\quad$ $x_2 \vee r_2$

$\text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$

$\overline{p}_2$ $\quad$ x_4



MaxSAT example (LSU search)

Objective: $min \sum_i r_i$

VeriPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \ldots, \overline{x}_4, \overline{r}_1, r_2, r_3\}$	Incumbent solution
$\sum_i r_i \leq 1$	Objective Improvement Rule
$j \cdot \overline{p}_j + \sum_i r_i \geq j$	Fresh variable (RBS)
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$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\overline{p}_2 \geq 1$	Explicit CP derivation
$x_4 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \overline{x}_2, \overline{x}_3, x_4, \overline{r}_1, r_2, \overline{r}_3\}$	Incumbent solution

$\overline{x}_1 \vee x_2$

$x_1 \vee \overline{x}_2$

$\overline{x}_2 \vee x_3$

$\overline{x}_3 \vee x_4$

$\overline{x}_1 \vee \overline{x}_2 \vee r_1$

$x_1 \vee x_2 \vee r_2$

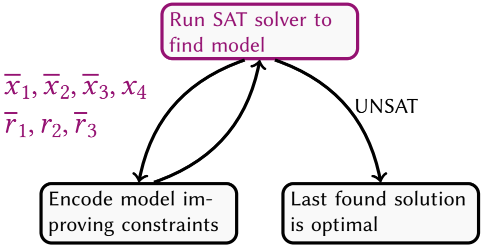
$x_2 \vee x_4 \vee r_3$

$x_2 \vee r_2$

$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$

$\overline{p}_2$

x_4



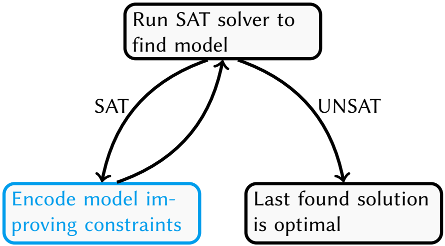
MaxSAT example (LSU search)

Objective: $min \sum_i r_i$

VeriPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \ldots, \overline{x}_4, \overline{r}_1, r_2, r_3\}$	Incumbent solution
$\sum_i r_i \leq 1$	Objective Improvement Rule
$j \cdot \overline{p}_j + \sum_i r_i \geq j$	Fresh variable (RBS)
$(4 - j) \cdot p_j + \sum_i \overline{r}_i \geq 4 - j$	
$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\overline{p}_2 \geq 1$	Explicit CP derivation
$x_4 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \overline{x}_2, \overline{x}_3, x_4, \overline{r}_1, r_2, \overline{r}_3\}$	Incumbent solution
$\sum_i r_i \leq 0$	Objective Improvement Rule

$\overline{x}_1 \vee x_2$	$\overline{x}_1 \vee \overline{x}_2 \vee r_1$
$x_1 \vee \overline{x}_2$	$x_1 \vee x_2 \vee r_2$
$\overline{x}_2 \vee x_3$	$x_2 \vee x_4 \vee r_3$
$\overline{x}_3 \vee x_4$	$x_2 \vee r_2$
$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	
$\overline{p}_2$	x_4



MaxSAT example (LSU search)

Objective: $min \sum_i r_i$

VeriPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \ldots, \overline{x}_4, \overline{r}_1, r_2, r_3\}$	Incumbent solution
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$j \cdot \overline{p}_j + \sum_i r_i \geq j$	Fresh variable (RBS)
$(4 - j) \cdot p_j + \sum_i \overline{r}_i \geq 4 - j$	
$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\overline{p}_2 \geq 1$	Explicit CP derivation
$x_4 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \overline{x}_2, \overline{x}_3, x_4, \overline{r}_1, r_2, \overline{r}_3\}$	Incumbent solution
$\sum_i r_i \leq 0$	Objective Improvement Rule
$\overline{p}_1 \geq 1$	Explicit CP derivation

$\overline{x}_1 \lor x_2$ $\quad$ $\overline{x}_1 \lor \overline{x}_2 \lor r_1$

$x_1 \lor \overline{x}_2$ $\quad$ $x_1 \lor x_2 \lor r_2$

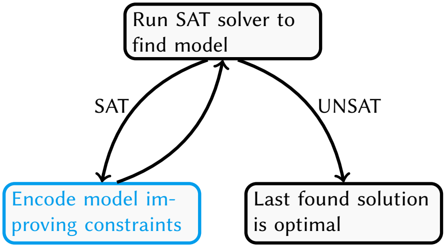
$\overline{x}_2 \lor x_3$ $\quad$ $x_2 \lor x_4 \lor r_3$

$\overline{x}_3 \lor x_4$ $\quad$ $x_2 \lor r_2$

$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$

$\overline{p}_2$ $\quad$ x_4

$\overline{p}_1$



MaxSAT example (LSU search)

Objective: *min* $\sum_i r_i$

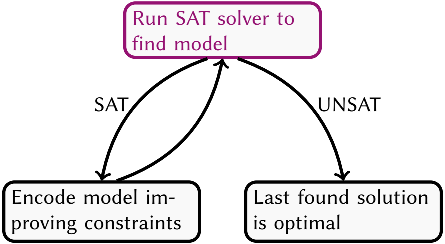
VeriPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \ldots, \overline{x}_4, \overline{r}_1, r_2, r_3\}$	Incumbent solution
$\sum_i r_i \leq 1$	Objective Improvement Rule
$j \cdot \overline{p}_j + \sum_i r_i \geq j$	Fresh variable (RBS)
$(4-j) \cdot p_j + \sum_i \overline{r}_i \geq 4-j$	
$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\overline{p}_2 \geq 1$	Explicit CP derivation
$x_4 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \overline{x}_2, \overline{x}_3, x_4, \overline{r}_1, r_2, \overline{r}_3\}$	Incumbent solution
$\sum_i r_i \leq 0$	Objective Improvement Rule
$\overline{p}_1 \geq 1$	Explicit CP derivation

$\overline{x}_1 \vee x_2$	$\overline{x}_1 \vee \overline{x}_2 \vee r_1$
$x_1 \vee \overline{x}_2$	$x_1 \vee x_2 \vee r_2$
$\overline{x}_2 \vee x_3$	$x_2 \vee x_4 \vee r_3$
$\overline{x}_3 \vee x_4$	$x_2 \vee r_2$

$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$

$\overline{p}_2$	x_4
$\overline{p}_1$	



MaxSAT example (LSU search)

Objective: $min \sum_i r_i$

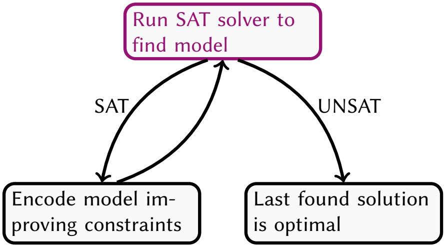
VeriPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \ldots, \overline{x}_4, \overline{r}_1, r_2, r_3\}$	Incumbent solution
$\sum_i r_i \leq 1$	Objective Improvement Rule
$j \cdot \overline{p}_j + \sum_i r_i \geq j$	Fresh variable (RBS)
$(4 - j) \cdot p_j + \sum_i \overline{r}_i \geq 4 - j$	
$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\overline{p}_2 \geq 1$	Explicit CP derivation
$x_4 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \overline{x}_2, \overline{x}_3, x_4, \overline{r}_1, r_2, \overline{r}_3\}$	Incumbent solution
$\sum_i r_i \leq 0$	Objective Improvement Rule
$\overline{p}_1 \geq 1$	Explicit CP derivation
$0 \geq 1$	Reverse Unit Propagation

$\overline{x}_1 \lor x_2$	$\overline{x}_1 \lor \overline{x}_2 \lor r_1$
$x_1 \lor \overline{x}_2$	$x_1 \lor x_2 \lor r_2$
$\overline{x}_2 \lor x_3$	$x_2 \lor x_4 \lor r_3$
$\overline{x}_3 \lor x_4$	$x_2 \lor r_2$

$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$

$\overline{p}_2$	x_4
$\overline{p}_1$	$\perp$



MaxSAT example (LSU search)

Objective: $min \sum_i r_i$

VeriPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \ldots, \overline{x}_4, \overline{r}_1, r_2, r_3\}$	Incumbent solution
$\sum_i r_i \leq 1$	Objective Improvement Rule
$j \cdot \overline{p}_j + \sum_i r_i \geq j$	Fresh variable (RBS)
$(4 - j) \cdot p_j + \sum_i \overline{r}_i \geq 4 - j$	
$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\overline{p}_2 \geq 1$	Explicit CP derivation
$x_4 \geq 1$	Reverse Unit Propagation
$\{\overline{x}_1, \overline{x}_2, \overline{x}_3, x_4, \overline{r}_1, r_2, \overline{r}_3\}$	Incumbent solution
$\sum_i r_i \leq 0$	Objective Improvement Rule
$\overline{p}_1 \geq 1$	Explicit CP derivation
$0 \geq 1$	Reverse Unit Propagation

$\overline{x}_1 \vee x_2$ $\quad$ $\overline{x}_1 \vee \overline{x}_2 \vee r_1$

$x_1 \vee \overline{x}_2$ $\quad$ $x_1 \vee x_2 \vee r_2$

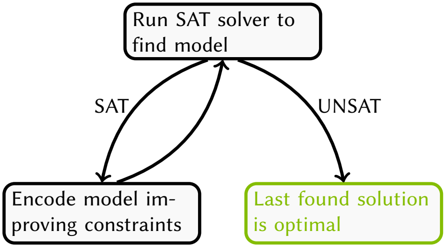
$\overline{x}_2 \vee x_3$ $\quad$ $x_2 \vee x_4 \vee r_3$

$\overline{x}_3 \vee x_4$ $\quad$ $x_2 \vee r_2$

$\mathrm{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$

$\overline{p}_2$ $\quad$ x_4

$\overline{p}_1$ $\quad$ $\perp$



Progress So Far

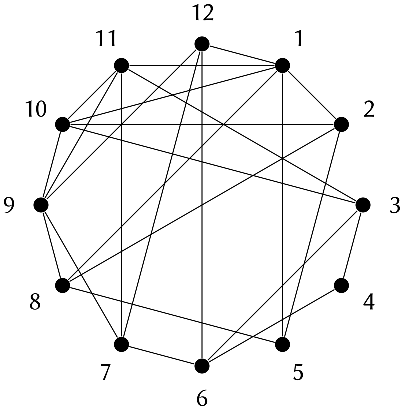
We've seen proof logging, and how it works for SAT

We've learned about

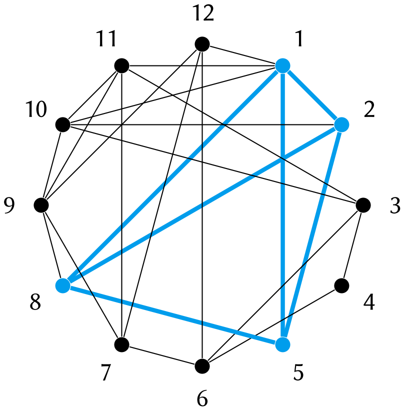
- pseudo-Boolean constraints (0–1 linear inequalities)
- cutting planes reasoning
- VeriPB

Coming next, some worked examples from dedicated graph solvers

The Maximum Clique Problem



The Maximum Clique Problem



Maximum Clique Solvers

There are a lot of dedicated solvers for clique problems

But there are issues:

- “State-of-the-art” solvers have been buggy.
- Often undetected: error rate of around 0.1 [MPP19]

Often used inside other solvers

- An off-by-one result can cause much larger errors

A Brief and Incomplete Guide to Clique Solving (1/4)

Recursive maximum clique algorithm:

- Pick a vertex v
- Either v is in the clique…
  - Throw away every vertex not adjacent to v
  - If vertices remain, recurse
- …or v is not in the clique
  - Throw v away and pick another vertex

A Brief and Incomplete Guide to Clique Solving (2/4)

Key data structures:

- Growing clique C
- Set of potential vertices P
  - All the vertices we haven't thrown away yet
  - Every $v \in P$ is adjacent to every $w \in C$

A Brief and Incomplete Guide to Clique Solving (2/4)

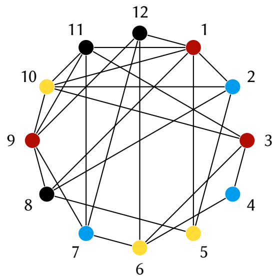
Key data structures:

- Growing clique C
- Set of potential vertices P
  - All the vertices we haven't thrown away yet
  - Every $v \in P$ is adjacent to every $w \in C$

Branch and bound:

- Remember the biggest clique $C^\star$ found so far
- If $|C| + |P| \leq \left|C^\star\right|$, no need to keep going

A Brief and Incomplete Guide to Clique Solving (3/4)



Given a k-colouring of a subgraph, that subgraph cannot have a clique of more than k vertices

We can use $|C| + \#colours(P)$ as a bound, for any colouring

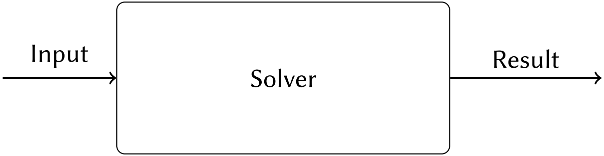
A Brief and Incomplete Guide to Clique Solving (4/4)

- This brings us to 1997
- Many improvements since then
  - better bound functions
  - clever vertex selection heuristics
  - efficient data structures
  - local search
  - ...
- But key ideas for proof logging can be explained without worrying about such things

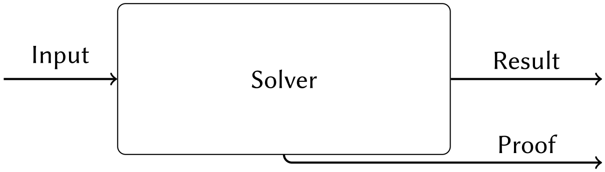
Making a Proof Logging Clique Solver

1. Output a pseudo-Boolean encoding of the problem
   - Clique problems have several standard file formats
2. Make the solver log its search tree
   - Output a small header
   - Output something on every backtrack
   - Output something every time a solution is found
   - Output a small footer
3. Figure out how to log the bound function

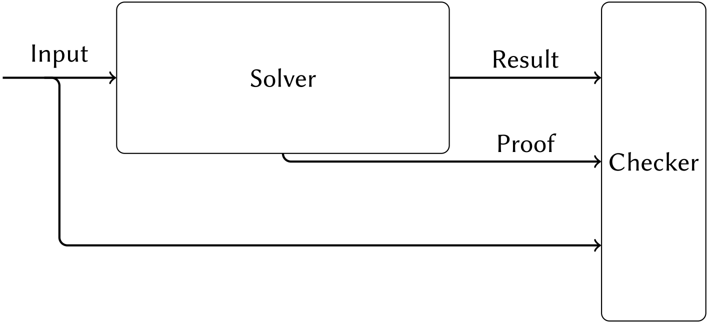
A Slightly Different Proof Logging Workflow



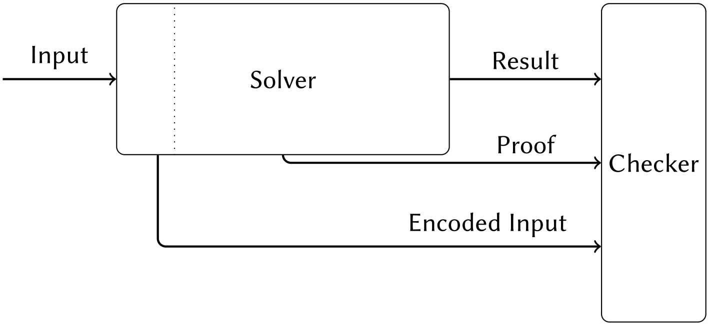
A Slightly Different Proof Logging Workflow



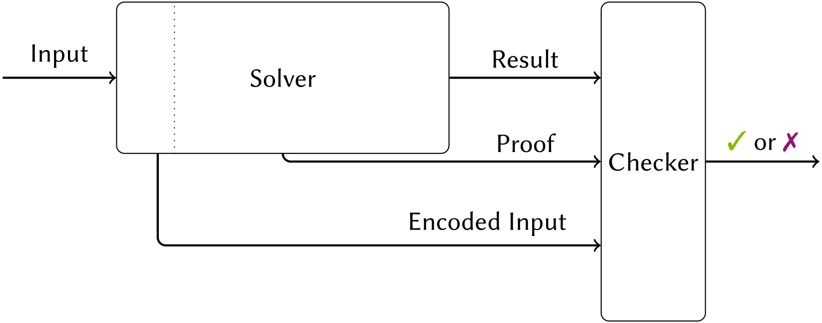
A Slightly Different Proof Logging Workflow



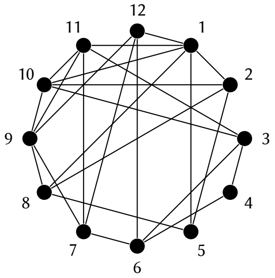
A Slightly Different Proof Logging Workflow



A Slightly Different Proof Logging Workflow



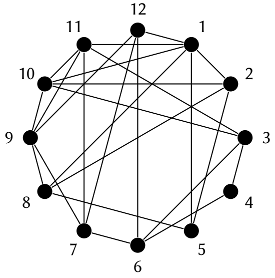
A Pseudo-Boolean Encoding for Clique (in OPB Format)




```
* #variable= 12 #constraint= 41
min: -1 x1 -1 x2 -1 x3 -1 x4 ... and so on... -1 x11 -1 x12 ;
1 ~x3 1 ~x1 >= 1 ;
1 ~x3 1 ~x2 >= 1 ;
1 ~x4 1 ~x1 >= 1 ;
* ... and a further 38 similar lines for the remaining non-edges
```

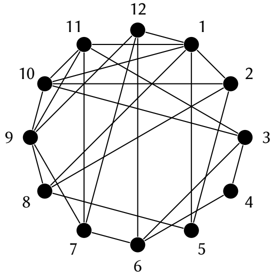
First Attempt at a Proof

```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



First Attempt at a Proof

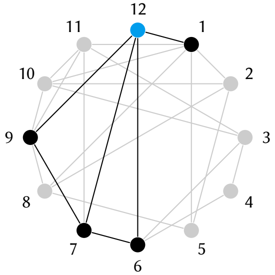
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Start with a header
Load the 41 problem axioms

First Attempt at a Proof

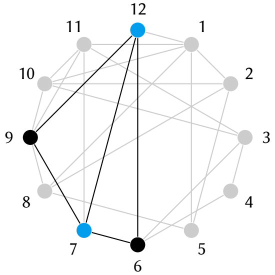
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Branch accepting 12
Throw away non-adjacent vertices

First Attempt at a Proof

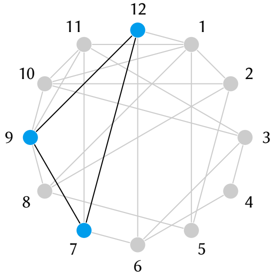
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Branch also accepting 7
Throw away non-adjacent vertices

First Attempt at a Proof

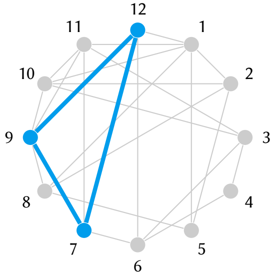
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Branch also accepting 9
Throw away non-adjacent vertices

First Attempt at a Proof

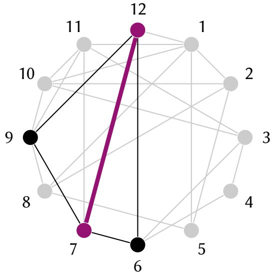
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



We branched on 12, 7, 9
Found a new incumbent
Now looking for a ≥ 4 vertex clique

First Attempt at a Proof

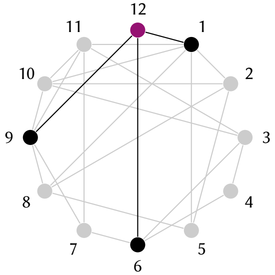
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Backtrack from 12, 7
9 explored already, only 6 feasible
No ≥ 4 vertex clique possible
Effectively this deletes the 7–12 edge

First Attempt at a Proof

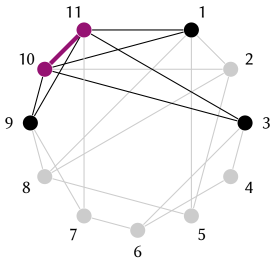
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Backtrack from 12
Only 1, 6 and 9 feasible (1-colourable)
No ≥ 4 vertex clique possible
Effectively this deletes vertex 12

First Attempt at a Proof

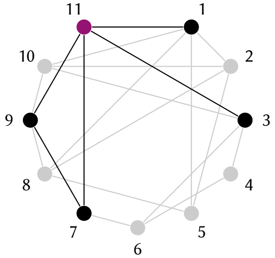
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Branch on 11 then 10
Only 1, 3 and 9 feasible (1-colourable)
No ≥ 4 vertex clique possible
Backtrack, deleting the edge

First Attempt at a Proof

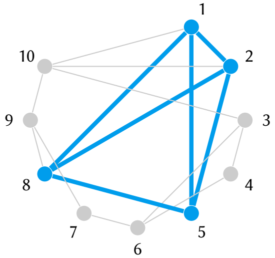
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Backtrack from 11
2-colourable, so no ≥ 4 clique
Delete the vertex

First Attempt at a Proof

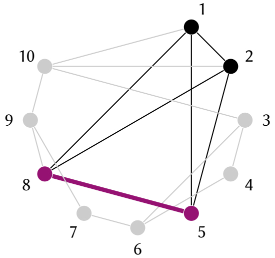
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Branch on 8, 5, 1, 2
Find a new incumbent
Now looking for a ≥ 5 vertex clique

First Attempt at a Proof

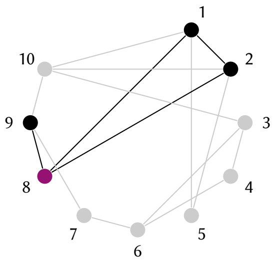
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Backtrack from 8, 5
Only 4 vertices; can't have a ≥ 5 clique
Delete the edge

First Attempt at a Proof

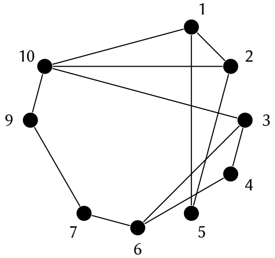
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Backtrack from 8
Still not enough vertices
Delete the vertex

First Attempt at a Proof

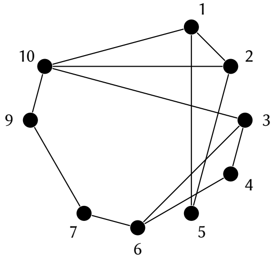
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Remaining graph is 3-colourable
Backtrack from root node

First Attempt at a Proof

```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



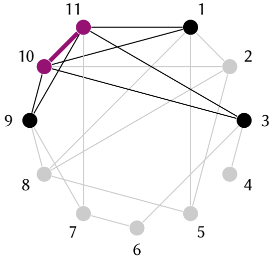
Finish with what we've concluded
We specify a lower and an upper bound
Remember we're minimising $\sum_v -1 \times v$, so a 4-clique has an objective value of -4

Verifying This Proof (Or Not. . .)

```
$ veripb clique.opb clique-attempt-one.veripb
Verification failed.
Failed in proof file line 6.
Hint: Failed to show '1 ~x10 1 ~x11 >= 1' by reverse unit propagation.
```

Verifying This Proof (Or Not...)

```
$ veripb clique.opb clique-attempt-one.veripb
Verification failed.
Failed in proof file line 6.
Hint: Failed to show '1 ~x10 1 ~x11 >= 1' by reverse unit propagation.
```



Verifying This Proof (Or Not...)

```
$ veripb --trace clique.opb clique-attempt-one.veripb
line 002: f 41
  ConstraintId 001: 1 ~x1 1 ~x3 >= 1
  ConstraintId 002: 1 ~x2 1 ~x3 >= 1
...
  ConstraintId 041: 1 ~x11 1 ~x12 >= 1
line 003: soli x7 x9 x12 ~x1 ~x2 ~x3 ~x4 ~x5 ~x6 ~x8 ~x10 ~x11
  ConstraintId 042: 1 x1 1 x2 1 x3 1 x4 1 x5 1 x6 1 x7 1 x8 1 x9 1 x10 1 x11 1 x12 >= 4
line 004: rup 1 ~x12 1 ~x7 >= 1 ;
  ConstraintId 043: 1 ~x7 1 ~x12 >= 1
line 005: rup 1 ~x12 >= 1 ;
  ConstraintId 044: 1 ~x12 >= 1
line 006: rup 1 ~x11 1 ~x10 >= 1 ;
Verification failed.
Failed in proof file line 6.
Hint: Failed to show '1 ~x10 1 ~x11 >= 1' by reverse unit propagation.
```

Dealing With Colourings

The colour bound doesn't follow by RUP…

But we can lazily recover at-most-one constraints for each colour class!

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$$
\begin{aligned}
&(\overline{x}_1 + \overline{x}_6 \geq 1) \\
+\ &(\overline{x}_1 + \overline{x}_9 \geq 1) && = 2\overline{x}_1 + \overline{x}_6 + \overline{x}_9 \geq 2 \\
+\ &(\overline{x}_6 + \overline{x}_9 \geq 1) && = 2\overline{x}_1 + 2\overline{x}_6 + 2\overline{x}_9 \geq 3 \\
&/\ 2 && = \overline{x}_1 + \overline{x}_6 + \overline{x}_9 \geq 2 \\
& && \text{i.e. } x_1 + x_6 + x_9 \leq 1
\end{aligned}
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&/\,2 & &= \overline{x}_1 + \overline{x}_6 + \overline{x}_9 \geq 2 \\
& & &\text{i.e. } x_1 + x_6 + x_9 \leq 1
\end{aligned}
$$

This generalises to colour classes of any size v

- Each non-edge is used exactly once, $v(v-1)$ additions
- $v-3$ multiplications and $v-2$ divisions

Solvers don't need to "understand" cutting planes to write this derivation to proof log

What This Looks Like in the Proof Log

pseudo-Boolean proof version 2.0
f 41
soli x12 x7 x9
rup 1 ~x12 1 ~x7 >= 1 ;
* bound, colour classes [x1 x6 x9]
pol $7_{1 \not\sim 6}$ $19_{1 \not\sim 9}$ + $24_{6 \not\sim 9}$ + 2 d
pol 42_{obj} -1 +
rup 1 ~x12 >= 1 ;
* bound, colour classes [x1 x3 x9]
pol $1_{1 \not\sim 3}$ $19_{1 \not\sim 9}$ + $21_{3 \not\sim 9}$ + 2 d
pol 42_{obj} -1 +
rup 1 ~x11 1 ~x10 >= 1 ;
* bound, colour classes [x1 x3 x7]
* [x9]
pol $1_{1 \not\sim 3}$ $10_{1 \not\sim 7}$ + $12_{3 \not\sim 7}$ + 2 d
pol 42_{obj} -1 +
rup 1 ~x11 >= 1 ;
soli x8 x5 x2 x1
rup 1 ~x8 1 ~x5 >= 1 ;
* bound, colour classes [x1 x9] [x2]
pol 53_{obj} $19_{1 \not\sim 9}$ +
rup 1 ~x8 >= 1 ;
* bound, colour classes [x1 x3 x7]
* [x2 x4 x9] [x5 x6 x10]
pol $1_{1 \not\sim 3}$ $10_{1 \not\sim 7}$ + $12_{3 \not\sim 7}$ + 2 d
pol 53_{obj} -1 +
pol $4_{2 \not\sim 4}$ $20_{2 \not\sim 9}$ + $22_{4 \not\sim 9}$ + 2 d
pol 53_{obj} -3 + -1 +
pol $9_{5 \not\sim 6}$ $26_{5 \not\sim 10}$ + $27_{6 \not\sim 10}$ + 2 d
pol 53_{obj} -5 + -3 + -1 +
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof

Verifying This Proof (For Real, This Time)

```
$ veripb --trace clique.opb clique-attempt-two.veripb
=== begin trace ===
line 002: f 41
  ConstraintId 001: 1 ~x1 1 ~x3 >= 1
  ConstraintId 002: 1 ~x2 1 ~x3 >= 1
...
  ConstraintId 041: 1 ~x11 1 ~x12 >= 1
line 003: soli x7 x9 x12 ~x1 ~x2 ~x3 ~x4 ~x5 ~x6 ~x8 ~x10 ~x11
  ConstraintId 042: 1 x1 1 x2 1 x3 1 x4 1 x5 1 x6 1 x7 1 x8 1 x9 1 x10 1 x11 1 x12 >= 4
line 004: rup 1 ~x12 1 ~x7 >= 1 ;
  ConstraintId 043: 1 ~x7 1 ~x12 >= 1
line 005: * bound, colour classes [ x1 x6 x9 ]
line 006: pol 7 19 + 24 + 2 d
  ConstraintId 044: 1 ~x1 1 ~x6 1 ~x9 >= 2
line 007: pol 42 43 +
  ConstraintId 045: 1 x1 1 x2 1 x3 1 x4 1 x5 1 x6 1 x8 1 x9 1 x10 1 x11 >= 3
...
  ConstraintId 061: 1 ~x5 1 ~x6 1 ~x10 >= 2
line 028: pol 53 57 + 59 + 61 +
  ConstraintId 062: 1 x8 1 x11 1 x12 >= 2
line 029: rup >= 1 ;
  ConstraintId 063: >= 1
line 030: output NONE
line 031: conclusion BOUNDS -4 -4
line 032: end pseudo-Boolean proof
=== end trace ===

Verification succeeded.
```

Different Clique Algorithms

Different search orders?

✓ Irrelevant for proof logging

Using local search to initialise?

✓ Just log the incumbent

Different bound functions?

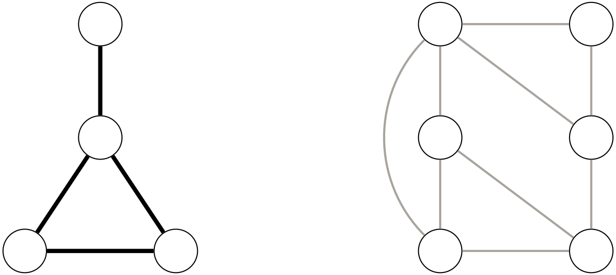
- Is cutting planes strong enough to justify every useful bound function ever invented?
- So far, seems like it…

Weighted cliques?

✓ Multiply a colour class by its largest weight

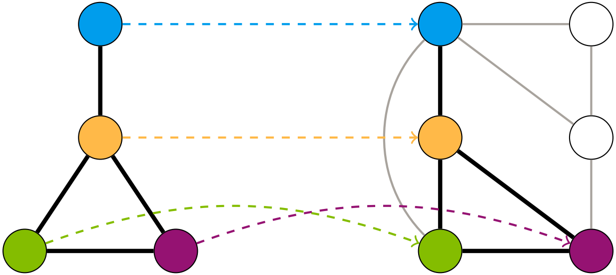
✓ Also works for vertices "split between colour classes"

Subgraph Isomorphism



- Find the pattern inside the target
- Applications in compilers, biochemistry, model checking, pattern recognition, …
- Often want to find all matches

Subgraph Isomorphism



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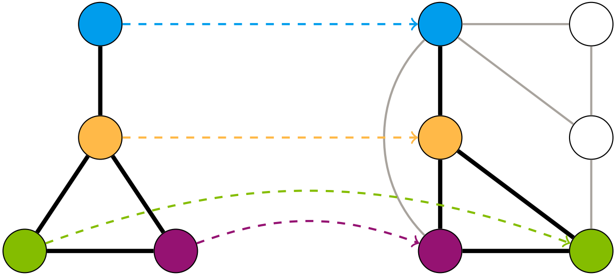

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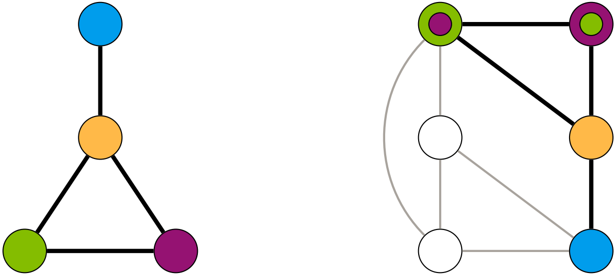

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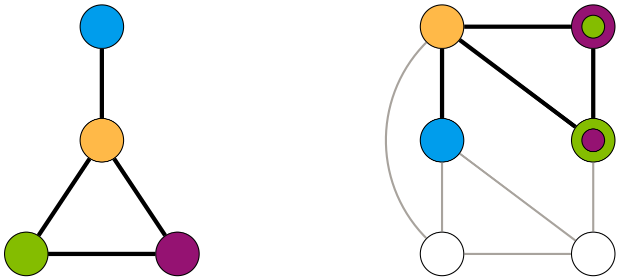

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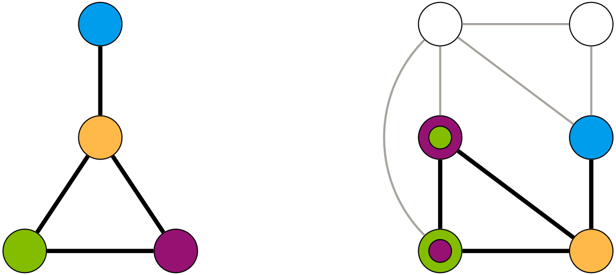

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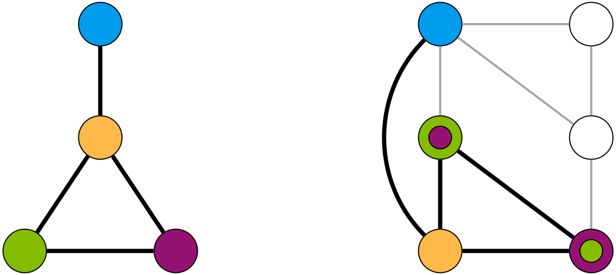

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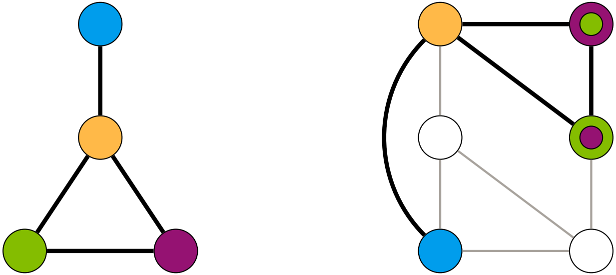

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Subgraph Isomorphism in Pseudo-Boolean Form

Each pattern vertex gets a target vertex:

$$\sum_{t \in \mathrm{V}(T)} x_{p,t} = 1 \qquad\qquad p \in \mathrm{V}(P)$$

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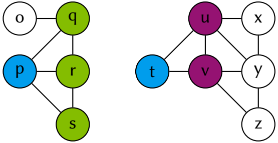
Adjacency constraints, if p is mapped to t, then p's neighbours must be mapped to t's neighbours:

$$\overline{x}_{p,t} + \sum_{u \in \mathrm{N}(t)} x_{q,u} \geq 1 \qquad\qquad p \in \mathrm{V}(P),\ q \in \mathrm{N}(p),\ t \in \mathrm{V}(T)$$

Degree Reasoning in Cutting Planes

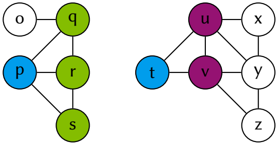


Pattern vertex p of degree $\deg(p)$ can never be mapped to target vertex t of degree $< \deg(p)$ in any subgraph isomorphism

Observe $N(p) = \{q, r, s\}$ and $N(t) = \{u, v\}$

We wish to derive $\overline{x}_{p,t} \geq 1$

Degree Reasoning in Cutting Planes



Adjacency:

$$\overline{x}_{p,t} + x_{q,u} + x_{q,v} \geq 1$$
$$\overline{x}_{p,t} + x_{r,u} + x_{r,v} \geq 1$$
$$\overline{x}_{p,t} + x_{s,u} + x_{s,v} \geq 1$$

Injectivity:

$$-x_{o,u} + -x_{p,u} + -x_{q,u} + -x_{r,u} + -x_{s,u} \geq -1$$
$$-x_{o,v} + -x_{p,v} + -x_{q,v} + -x_{r,v} + -x_{s,v} \geq -1$$

Literal axioms:

$$x_{o,u} \geq 0$$
$$x_{o,v} \geq 0$$
$$x_{p,u} \geq 0$$
$$x_{p,v} \geq 0$$

Add these together and divide by 3 to get

$$\overline{x}_{p,t} \geq 1$$

Degree Reasoning in VeriPB

```
pol 18_{p~t:q} 19_{p~t:r} + 20_{p~t:s} +    * sum adjacency constraints
  12_{inj(u)} + 13_{inj(v)} +                * sum injectivity constraints
  xo_u + xo_v +                             * cancel stray xo_*
  xp_u + xp_v +                             * cancel stray xp_*
  3 d                                       * divide, and we're done
```

Or we can ask VeriPB to do the last bit of simplification automatically:

```
pol 18_{p~t:q} 19_{p~t:r} + 20_{p~t:s} +    * sum adjacency constraints
  12_{inj(u)} + 13_{inj(v)} +                * sum injectivity constraints
ia -1 : 1 ~xp_t >= 1 ;                      * desired conclusion is implied
```

Other Forms of Reasoning

We can also log all of the other things state of the art subgraph solvers do:

- Injectivity reasoning and filtering
- Distance filtering
- Neighbourhood degree sequences
- Path filtering
- Supplemental graphs

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- Supplemental graphs

Proof steps are "efficient" using cutting planes.

- Length of proof $\approx$ time complexity of the reasoning algorithms
- Most proof steps require only trivial additional computations

Limitations

Why trust the encoding?

- Correctness of encoding can be formally verified! Work in progress. . .

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Works up to moderately-sized hard instances

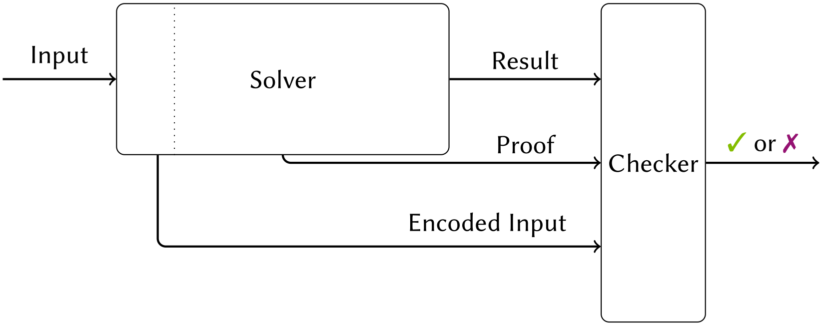
- Even an $O(n^3)$ encoding is painful
- Particularly bad when the pseudo-Boolean encoding talks about “non-edges” but large sparse graphs are “easy”

Code for Proof Logging Subgraph Solver

`https://github.com/ciaranm/glasgow-subgraph-solver`

Released under MIT Licence

Recap (1/2)



Recap (2/2)

Proof logging implementation

- Don't change solver
- Just add proof logging statements (plus some book-keeping)

Performance goals

Want linear(ish) scaling in terms of solver running time for

- proof size
- proof checking time

What About Constraint Programming?

Non-Boolean variables?

Constraints?

- Encoding constraints in pseudo-Boolean form?
- Justifying inferences?

Reformulations?

Compiling CP Variables (1/2)

Given $A \in \{-3 \ldots 9\}$, the direct encoding is:

$$\begin{aligned} a_{=-3} + a_{=-2} + a_{=-1} + a_{=0} + a_{=1} + a_{=2} + a_{=3} \\ + a_{=4} + a_{=5} + a_{=6} + a_{=7} + a_{=8} + a_{=9} = 1 \end{aligned}$$

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This doesn't work for large domains…

We could use a binary encoding:

$$\begin{aligned} -16a_{\text{neg}} + 1a_{\text{b0}} + 2a_{\text{b1}} + 4a_{\text{b2}} + 8a_{\text{b3}} \geq -3 \text{ and} \\ 16a_{\text{neg}} + -1a_{\text{b0}} + -2a_{\text{b1}} + -4a_{\text{b2}} + -8a_{\text{b3}} \geq -9 \end{aligned}$$

This doesn't propagate much, but that isn't a problem for proof logging

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This doesn't propagate much, but that isn't a problem for proof logging

Convention in what follows:

- Upper-case A, B, C are CP variables;
- Lower-case a, b, c are corresponding Boolean variables in PB encoding

Compiling CP Variables (2/2)

We can mix binary and an order encoding! Where needed, define:

$$a_{\geq 4} \Leftrightarrow -16a_{\mathrm{neg}} + 1a_{\mathrm{b0}} + 2a_{\mathrm{b1}} + 4a_{\mathrm{b2}} + 8a_{\mathrm{b3}} \geq 4$$

$$a_{\geq 5} \Leftrightarrow -16a_{\mathrm{neg}} + 1a_{\mathrm{b0}} + 2a_{\mathrm{b1}} + 4a_{\mathrm{b2}} + 8a_{\mathrm{b3}} \geq 5$$

$$a_{=4} \Leftrightarrow a_{\geq 4} \land \overline{a}_{\geq 5}$$

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$$
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a_{=4} &\Leftrightarrow a_{\geq 4} \land \overline{a}_{\geq 5}
\end{aligned}
$$

When creating $a_{\geq i}$, also introduce pseudo-Boolean constraints encoding

$$a_{\geq i} \Rightarrow a_{\geq j} \quad \text{and} \quad a_{\geq h} \Rightarrow a_{\geq i}$$

for the closest values $j < i < h$ that already exist

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for the closest values $j < i < h$ that already exist

We can do this:

- Inside the pseudo-Boolean model, where needed
- Otherwise lazily during proof logging

Compiling Constraints

- Also need to compile every constraint to pseudo-Boolean form
- Doesn’t need to be a propagating encoding
- Can use additional variables

Compiling Linear Inequalities

Given inequality

$$2A + 3B + 4C \geq 42$$

where $A, B, C \in \{-3 \dots 9\}$

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Encode in pseudo-Boolean form as

$$\begin{aligned} -32a_{\text{neg}} + 2a_{\text{b0}} + 4a_{\text{b1}} + 8a_{\text{b2}} + 16a_{\text{b3}} \\ + -48b_{\text{neg}} + 3b_{\text{b0}} + 6b_{\text{b1}} + 12b_{\text{b2}} + 24b_{\text{b3}} \\ + -64c_{\text{neg}} + 4c_{\text{b0}} + 8c_{\text{b1}} + 16c_{\text{b2}} + 32c_{\text{b3}} \geq 42 \end{aligned}$$

Compiling Table Constraints

Constraints can be specified extensionally as list of feasible tuples, called a table
Variable assignments must match some row in table

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Given table constraint

$$(A, B, C) \in [(1, 2, 3), (1, 3, 4), (2, 2, 5)]$$

define

$$3\bar{t}_0 + a_{=1} + b_{=2} + c_{=3} \geq 3 \qquad \text{i.e.,} \quad t_0 \Rightarrow (a_{=1} \wedge b_{=2} \wedge c_{=3})$$
$$3\bar{t}_1 + a_{=1} + b_{=4} + c_{=4} \geq 3 \qquad \text{i.e.,} \quad t_1 \Rightarrow (a_{=1} \wedge b_{=4} \wedge c_{=4})$$
$$3\bar{t}_2 + a_{=2} + b_{=2} + c_{=5} \geq 3 \qquad \text{i.e.,} \quad t_2 \Rightarrow (a_{=2} \wedge b_{=2} \wedge c_{=5})$$

using tuple selector variables

$$t_0 + t_1 + t_2 = 1$$

Encoding Constraint Definitions

Already know how to do it for any constraint with a sane encoding using some combination of

- CNF
- Integer linear inequalities
- Table constraints
- Auxiliary variables

Simplicity is important, propagation strength isn't

Justifying Search

Mostly this works as in earlier examples

Restarts are easy

No need to justify guesses or decisions — only justify backtracking

Justifying Inference

Key idea

Anything the constraint programming solver knows must follow from unit propagation of guessed assignments on constraints in proof log

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If it follows from unit propagation on the encoding, nothing needed

Some propagators and encodings need RUP steps for inferences

- A lot of propagators are effectively "doing a little bit of lookahead" but in an efficient way

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Some propagators and encodings need RUP steps for inferences

- A lot of propagators are effectively "doing a little bit of lookahead" but in an efficient way

A few need explicit cutting planes justifications written to the proof log

- Linear inequalities just need to multiply and add
- All-different needs a bit more

Justifying All-Different Failures

$$
\begin{array}{rl}
V \in & \{\,1 \quad\quad\quad\quad 4 \quad 5\,\} \\
W \in & \{\,1 \quad 2 \quad 3 \quad\quad\quad\quad\,\} \\
X \in & \{\quad\quad 2 \quad 3 \quad\quad\quad\quad\,\} \\
Y \in & \{\,1 \quad\quad\quad 3 \quad\quad\quad\quad\,\} \\
Z \in & \{\,1 \quad\quad\quad 3 \quad\quad\quad\quad\,\}
\end{array}
$$

Justifying All-Different Failures

$$
\begin{aligned}
V &\in \{\, 1 \quad \quad \quad 4 \quad 5 \,\} \\
W &\in \{\, 1 \quad 2 \quad 3 \quad \quad \quad \,\} \\
X &\in \{\, \quad 2 \quad 3 \quad \quad \quad \,\} \\
Y &\in \{\, 1 \quad \quad 3 \quad \quad \quad \,\} \\
Z &\in \{\, 1 \quad \quad 3 \quad \quad \quad \,\}
\end{aligned}
$$

Justifying All-Different Failures

$$
\begin{array}{lllllll}
V \in \{ 1 & & & 4 & 5 \} & & \\
W \in \{ 1 & 2 & 3 & & \} & w_{=1} + w_{=2} + w_{=3} \geq 1 & [\, W \text{ takes some value} \,] \\
X \in \{ & 2 & 3 & & \} & & \\
Y \in \{ 1 & & 3 & & \} & & \\
Z \in \{ 1 & & 3 & & \} & &
\end{array}
$$

Justifying All-Different Failures

$$
\begin{array}{rlllll}
V \in \{\, 1 & & & 4 & 5\,\} & \\
W \in \{\, 1 & 2 & 3 & & \,\} & w_{=1} + w_{=2} + w_{=3} \geq 1 \quad [\ W \text{ takes some value}\ ] \\
X \in \{\, & 2 & 3 & & \,\} & x_{=2} + x_{=3} \geq 1 \quad [\ X \text{ takes some value}\ ] \\
Y \in \{\, 1 & & 3 & & \,\} & y_{=1} + y_{=3} \geq 1 \quad [\ Y \text{ takes some value}\ ] \\
Z \in \{\, 1 & & 3 & & \,\} & z_{=1} + z_{=3} \geq 1 \quad [\ Z \text{ takes some value}\ ]
\end{array}
$$

Justifying All-Different Failures

$$
\begin{array}{llllll}
V \in \{1, 4, 5\} & & & \\
W \in \{1, 2, 3\} & w_{=1} + w_{=2} + w_{=3} & \geq 1 & [\ W \text{ takes some value}\ ] \\
X \in \{2, 3\} & x_{=2} + x_{=3} & \geq 1 & [\ X \text{ takes some value}\ ] \\
Y \in \{1, 3\} & y_{=1} + y_{=3} & \geq 1 & [\ Y \text{ takes some value}\ ] \\
Z \in \{1, 3\} & z_{=1} + z_{=3} & \geq 1 & [\ Z \text{ takes some value}\ ] \\
\\
\rightarrow & -v_{=1} + -w_{=1} + -y_{=1} + -z_{=1} & \geq -1 & [\ \text{At most one variable} = 1\ ] \\
\rightarrow & -w_{=2} + -x_{=2} & \geq -1 & [\ \text{At most one variable} = 2\ ] \\
\rightarrow & -w_{=3} + -x_{=3} + -y_{=3} + -z_{=3} & \geq -1 & [\ \text{At most one variable} = 3\ ]
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\begin{array}{lllllll}
V \in \{1 \quad\quad 4 \quad 5\} & & & & & & \\
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X \in \{\quad 2 \quad 3 \quad\quad\} & & x_{=2} + & x_{=3} & & \geq 1 & [\ X \text{ takes some value}\ ] \\
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$$
-v_{=1} \geq 1 \qquad [\ \text{Sum all constraints so far}\ ]
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$$
\begin{array}{lll}
-v_{=1} & \geq 1 & [\ \text{Sum all constraints so far}\ ] \\
v_{=1} & \geq 0 & [\ \text{Variable } v_{=1} \text{ non-negative}\ ]
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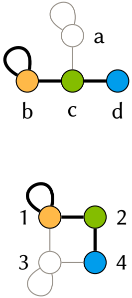
$$
\begin{array}{lll}
-v_{=1} & \geq 1 & [\text{ Sum all constraints so far}] \\
v_{=1} & \geq 0 & [\text{ Variable } v_{=1} \text{ non-negative}] \\
\\
0 & \geq 1 & [\text{ Sum above two constraints}]
\end{array}
$$

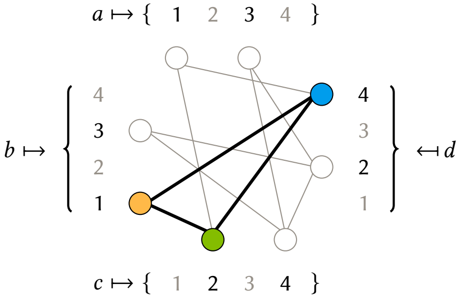
Reformulation

Auto-tabulation is possible

- Heavy use of extension variables

Can re-encode maximum common subgraph as a clique problem, without changing pseudo-Boolean encoding





High Level Modelling Languages?

High level modelling languages like MiniZinc and Essence have complicated compilers

How do we know we're giving a proof for the problem the user actually specified?

Future research…

Code

https://github.com/ciaranm/glasgow-constraint-solver

Released under MIT Licence

Supports proof logging for global constraints including:

- All-different
- Integer linear inequality (including for very large domains)
- Smart table and regular
- Minimum / maximum of an array
- Element
- Absolute value
- (Hamiltonian) Circuit

Details in [EGMN20, GMN22, MM23]

Strengthening Rules (And Truth About Extension Variables)

When is it allowed to derive a new constraint? If it is (clear that it is) implied?

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Sometimes weaker criterion needed — recall that to get variable a encoding

$$a \Leftrightarrow (3x + 2y + z + w \geq 3)(x \wedge y)$$

we introduced pseudo-Boolean constraints

$$3\overline{a} + 3x + 2y + z + w \geq 3 \qquad 5a + 3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \geq 5$$

Cutting planes method inherently cannot certify such constraints — they are not implied!

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Cutting planes method inherently cannot certify such constraints — they are not implied!

Wish to allow without-loss-of-generality arguments that can derive non-implied constraints

Redundance-Based Strengthening

C is redundant with respect to F if F and $F \wedge C$ are equisatisfiable

Adding redundant constraints should be OK

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C is redundant with respect to F iff there is a substitution ω (mapping variables to truth values or literals), called a witness, for which

$$F \wedge \neg C \models (F \wedge C)\restriction_\omega$$

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Fact

α satisfies $\phi\restriction_\omega$ iff $\alpha \circ \omega$ satisfies ϕ

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Proof sketch for interesting direction: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \wedge C$

Witness ω should be specified, and implication should be efficiently verifiable, which is the case for constraints in $(F \wedge C)\restriction_\omega$ that are, e.g.,

- Reverse unit propagation (RUP) constraints w.r.t. $F \wedge \neg C$
- Obviously implied by a single constraint among $F \wedge \neg C$

Toy example of Redundance Rule

Choose binary encoding of two integers in $[0, 15]$
that sum up to 25 and are equal modulo two

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$$1 \cdot x_0 + 2 \cdot x_1 + 4 \cdot x_2 + 8 \cdot x_3$$
$$+ 1 \cdot y_0 + 2 \cdot y_1 + 4 \cdot y_2 + 8 \cdot y_3 = 25$$

$$x_0 = y_0$$

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$$-1 \cdot x_0 - 2 \cdot x_1 - 4 \cdot x_2 - 8 \cdot x_3$$
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To derive without loss of generality $x \leq y$ (argument: we can always swap them)

$$1 \cdot x_0 + 2 \cdot x_1 + 4 \cdot x_2 + 8 \cdot x_3 \leq 1 \cdot y_0 + 2 \cdot y_1 + 4 \cdot y_2 + 8 \cdot y_3$$

```
pseudo-Boolean proof version 2.0
f 4
red 1 y0 2 y1 4 y2 8 y3
→   -1 x0 -2 x1 -4 x2 -8 x3  >= 0 ;
→   y0 -> x0 x0 -> y0 y1 -> x1 x1 -> y1
→   y2 -> x2 x2 -> y2 y3 -> x3 x3 -> y3
```

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Choose binary encoding of two integers in $[0, 15]$ that sum up to 25 and are equal modulo two

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```

Why does this work? Need to show $F \wedge \neg C \models (F \wedge C)\restriction_\omega$

- $F\restriction_\omega$ equals F (swaps last two constraints)
- $C\restriction_\omega$ says $y \leq x$ while $\neg C$ says $y < x$

Deriving $a \Leftrightarrow (3x + 2y + z + w \geq 3)$ Using the Redundance Rule

Want to derive

$$3\overline{a} + 3x + 2y + z + w \geq 3 \qquad 5a + 3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \geq 5$$

using condition $F \wedge \neg C \models (F \wedge C)\restriction_{\omega}$

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1. $F \wedge \neg(3\overline{a} + 3x + 2y + z + w \geq 3) \models \big(F \wedge (3\overline{a} + 3x + 2y + z + w \geq 3)\big)\restriction_{\omega}$
   Choose $\omega = \{a \mapsto 0\}$ — F untouched; new constraint satisfied

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   $\big(F \wedge (3\overline{a} + 3x + 2y + z + w \geq 3) \wedge (5a + 3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \geq 5)\big)\restriction_\omega$
   Choose $\omega = \{a \mapsto 1\}$ — F untouched; new constraint satisfied
   $\neg(5a + 3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \geq 5$ forces $3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \leq 4$
   This is the same constraint as $3\overline{a} + 3x + 2y + z + w \geq 3$
   And VeriPB can automatically detect this implication

Redundance and Dominance Rules for Optimisation

Redundance-based strengthening, optimisation version

Add constraint C to formula F if exists witness substitution ω s.t.

$$F \land \neg C \models (F \land C)\restriction_\omega \ \land \ f\restriction_\omega \leq f$$

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- Applying ω should strictly decrease f
- If so, don't need to show that $C\restriction_\omega$ holds!

Soundness of Dominance Rule

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7. ...
8. Can't go on forever, so finally reach α' satisfying $F \land C$

Strength of Dominance Rule

Dominance-based strengthening (stronger, still simplified)

If $C_1, C_2, \ldots, C_{m-1}$ have been derived from F (maybe using dominance), then can derive C_m if exists witness substitution ω s.t.

$$F \wedge \bigwedge_{i=1}^{m-1} C_i \wedge \neg C_m \models F{\restriction_\omega} \wedge f{\restriction_\omega} < f$$

Only consider F — no need to show that any $C_i{\restriction_\omega}$ implied!

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- Same inductive proof as before, but nested
- Or pick solution α minimizing f and argue by contradiction

Strength of Dominance Rule

Dominance-based strengthening (stronger, still simplified)

If $C_1, C_2, \ldots, C_{m-1}$ have been derived from F (maybe using dominance), then can derive C_m if exists witness substitution ω s.t.

$$F \wedge \bigwedge_{i=1}^{m-1} C_i \wedge \neg C_m \models F\restriction_\omega \wedge f\restriction_\omega < f$$

Only consider F — no need to show that any $C_i\restriction_\omega$ implied!

Now why is *this* sound?

- Same inductive proof as before, but nested
- Or pick solution α minimizing f and argue by contradiction

Further extensions:

- Define dominance rule w.r.t. order independent of objective
- Switch between different orders in same proof
- See [BGMN23] for details

Using the Dominance Rule for Symmetry Handling

Dominance rule very powerful; can be used for symmetry and dominance breaking

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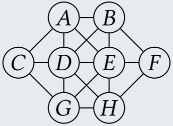
Examples:

1. Symmetries in constraint programming (manual symmetry breaking)
2. Vertex dominance in clique solving (automatic dominance breaking during search)
3. Symmetries in SAT solving (automatic symmetry breaking in preprocessing)

Symmetry Elimination (CP)

The Crystal Maze Puzzle



Place numbers 1 to 8 without repetition; adjacent circles cannot have consecutive numbers

Symmetry Elimination (CP)

Human modellers might add:

- $A < G$ (mirror vertically)
- $A < B$ (mirror horizontally)
- $A \leq 4$ (value symmetry)

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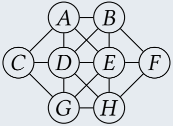

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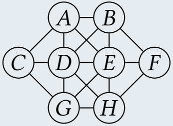

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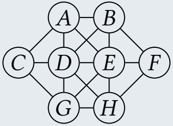

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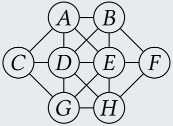

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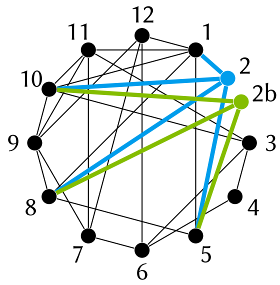
Research challenge: Constraint programming toolchain supporting this

Lazy Global Domination for Maximum Clique [MP16]



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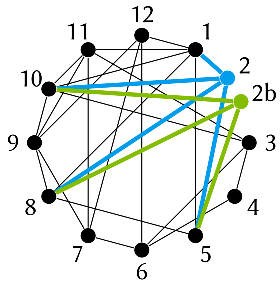

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Dominance rule can justify this

- Even when detected dynamically during search

Strategy for SAT Symmetry Breaking in SAT Solving

1. Pretend to solve optimisation problem minimizing $f \doteq \sum_{i=1}^{n} 2^{n-i} \cdot x_i$ (search for lexicographically smallest assignment satisfying formula)

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Symmetry Breaking: Example

Example: Pigeonhole principle (PHP) formula

- Variables p_{ij} $(1 \leq i \leq 4, 1 \leq j \leq 3)$ true iff pigeon i in hole j
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Order: "Pick smallest hole for pigeon 1, then smallest for pigeon 2, ..."

$$f \doteq 2^{11} \cdot p_{13} + 2^{10} \cdot p_{12} + 2^{9} \cdot p_{11} + 2^{8} \cdot p_{23} + \cdots + 1 \cdot p_{41}$$

Breaking a Single Simple Symmetry (Example)

- F is a formula expressing PHP constraints with $F\restriction_{\sigma_{(12)}} = F$
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Similar to DRAT symmetry breaking [HHW15]

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This idea does not generalize

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Can satisfy this constraint by applying $\sigma_{(1234)}$ once, twice, or thrice

Breaking Symmetries with the Dominance Rule (1/2)

Definition

Given a symmetry σ, the (pseudo-Boolean) breaking constraint of σ is

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Theorem ([BGMN23])

C_σ can be derived from F using dominance with witness σ

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Why does it work?

- Witness need not satisfy all derived constraints
- Sufficient to just produce “better” assignment

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(derivable with redundance rule)

If y_{k-1} is true, x_k is at most $\sigma(x_k)$
(derivable from the PB breaking constraint)

Back to Our Pigeons — Setting up the Pretend Optimisation Problem

Start the proof and load input formula

```
pseudo-Boolean proof version 2.0
f 22
pre_order exp
  vars
    left  u1 u2 u3 u4 u5 u6 u7 u8 u9 u10 u11 u12
    right v1 v2 v3 v4 v5 v6 v7 v8 v9 v10 v11 v12
    aux
  end
  def
    -1 u12 1 v12 -2 u11 2 v11 [...] -1024 u2 1024 v2 -2048 u1 2048 v1 >= 0;
  end
  transitivity
    vars
      fresh_right w1 w2 w3 w4 w5 w6 w7 w8 w9 w10 w11 w12
    end
  proof
    proofgoal #1
      pol  1 2 + 3 +
      qed -1
    qed
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(Actually defining an order — see [BGMN23] for details)

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load_order exp p13 p12 p11 p23 p22 p21 p31 p32 p33 p41 p42 p43
```

Back to Our Pigeons — Deriving the Constraints

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
  qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
red 1 ~y1  1  y0       >= 1 ; y1 -> 0
red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1  p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Pseudo-Boolean breaking constraint

Back to Our Pigeons — Deriving the Constraints

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
  qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
red 1 ~y1  1  y0        >= 1 ; y1 -> 0
red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1  p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Pseudo-Boolean breaking constraint

Use dominance with withness $\sigma = (p_{11}p_{21})(p_{12}p_{22})(p_{13}p_{23})$

Back to Our Pigeons — Deriving the Constraints

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + \\ 2^{10} \cdot (p_{22} - p_{12}) + \\ \cdots \geq 0$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
 proofgoal #2
   pol -1 -2 +
 qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
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red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
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red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Pseudo-Boolean breaking constraint

Use dominance with withness $\sigma = (p_{11}p_{21})(p_{12}p_{22})(p_{13}p_{23})$

$$F \wedge \neg C \models F\restriction_\omega \;\wedge\; (f\restriction_\omega < f)$$

VeriPB fills in all missing subproofs except for $\neg C \wedge C \models \bot$

Back to Our Pigeons — Deriving the Constraints

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

$y_0 \geq 1$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
  qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
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red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
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red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by redundance with witness $\omega = \{y_0 \mapsto 1\}$

$$F \wedge \mathcal{D} \wedge \neg(y_0 \geq 1) \models (F \wedge \mathcal{D})\restriction_\omega \wedge (y_0 \geq 1)\restriction_\omega$$

Back to Our Pigeons — Deriving the Constraints

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

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dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
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red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by redundance with witness $\omega = \{y_0 \mapsto 1\}$

$$F \wedge \mathcal{D} \wedge \neg(y_0 \geq 1) \models (F \wedge \mathcal{D})\restriction_\omega \wedge (y_0 \geq 1)\restriction_\omega$$

$$F \wedge \mathcal{D} \wedge (\overline{y}_0 \geq 1) \models (F \wedge \mathcal{D}) \wedge (1 \geq 1)$$

Back to Our Pigeons — Deriving the Constraints

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

$$y_0 \geq 1$$

$$\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13}) \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
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red 1 y0  >= 1 ; y0 -> 1
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red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1  p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by RUP

$$F \wedge \mathcal{D} \wedge \neg(\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13}) \geq 1)$$

Back to Our Pigeons — Deriving the Constraints

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

$$y_0 \geq 1$$

$$\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13}) \geq 1$$

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dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
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    pol -1 -2 +
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red 1 y0  >= 1 ; y0 -> 1
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red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by RUP

$$F \wedge \mathcal{D} \wedge \neg(\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13}) \geq 1) \models F \wedge \mathcal{D} \wedge (y_0 \geq 1) \wedge (p_{13} \geq 1) \wedge (\overline{p}_{23} \geq 1)$$

Back to Our Pigeons — Deriving the Constraints

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

$y_0 \geq 1$

$\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13}) \geq 1$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
  qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0   1 ~p13 1 p23 >= 1 ;
red 1 ~y1   1  y0         >= 1 ; y1 -> 0
red 1 ~y1   1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1  p23 1 ~y0   1 y1   >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0   1 y1   >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1   1 ~p12 1 p22 >= 1 ;
```

Derivable by RUP

$$F \wedge \mathcal{D} \wedge \neg(\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13}) \geq 1) \models F \wedge \mathcal{D} \wedge (y_0 \geq 1) \wedge (p_{13} \geq 1) \wedge (\overline{p}_{23} \geq 1)$$

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

Back to Our Pigeons — Deriving the Constraints

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

$$y_0 \geq 1$$

$$\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13}) \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
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end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
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red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1  p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by RUP

$$F \wedge \mathcal{D} \wedge \neg(\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13}) \geq 1) \models F \wedge \mathcal{D} \wedge (y_0 \geq 1) \wedge (p_{13} \geq 1) \wedge (\overline{p}_{23} \geq 1)$$

$$2^{11} \cdot (\quad -1 \quad) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

Back to Our Pigeons — Deriving the Constraints

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

$$y_0 \geq 1$$

$$\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13}) \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
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end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
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red 1  p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by RUP

$$F \wedge \mathcal{D} \wedge \neg(\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13}) \geq 1) \models F \wedge \mathcal{D} \wedge (y_0 \geq 1) \wedge (p_{13} \geq 1) \wedge (\overline{p}_{23} \geq 1)$$

$$2^{11} \cdot (\quad -1 \quad) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

where $\sum_{i=1}^{10} 2^i < 2^{11}$

Back to Our Pigeons — Deriving the Constraints

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

$$y_0 \geq 1$$

$$\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\overline{y}_1 + y_0 \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
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end
red 1 y0  >= 1 ; y0 -> 1
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red 1 ~y1  1  y0         >= 1 ; y1 -> 0
red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1  p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by redundance with witness $\omega = \{y_1 \mapsto 0\}$

$$F \wedge \mathcal{D} \wedge \neg(\overline{y}_1 + y_0 \geq 1) \models (F \wedge \mathcal{D})\restriction_\omega \wedge (\overline{y}_1 + y_0 \geq 1)\restriction_\omega$$

Back to Our Pigeons — Deriving the Constraints

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

$$y_0 \geq 1$$

$$\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\overline{y}_1 + y_0 \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
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    pol -1 -2 +
  qed -1
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red 1 y0  >= 1 ; y0 -> 1
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red 1 ~y1  1  y0          >= 1 ; y1 -> 0
red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1  p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by redundance with witness $\omega = \{y_1 \mapsto 0\}$

$$F \wedge \mathcal{D} \wedge \neg(\overline{y}_1 + y_0 \geq 1) \models (F \wedge \mathcal{D})\restriction_\omega \wedge (\overline{y}_1 + y_0 \geq 1)\restriction_\omega$$

$$F \wedge \mathcal{D} \wedge (y_1 + \overline{y}_0 \geq 2) \models (F \wedge \mathcal{D}) \wedge (1 + y_0 \geq 1)$$

Back to Our Pigeons — Deriving the Constraints

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

$$y_0 \geq 1$$

$$\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\overline{y}_1 + y_0 \geq 1$$

$$\overline{y}_1 + \overline{\sigma(p_{13})} + p_{13} \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
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red 1 y0  >= 1 ; y0 -> 1
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red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
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red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by redundance with witness $\omega = \{y_1 \mapsto 0\}$ (essentially same argument)

Back to Our Pigeons — Deriving the Constraints

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

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$$y_1 + \overline{y}_0 + \overline{p}_{13} \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
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red 1 y0  >= 1 ; y0 -> 1
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pol 26 32 2048 * +
del id 26
rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by redundance with witness $\omega = \{y_1 \mapsto 1\}$

$$F \wedge \mathcal{D} \wedge \neg(y_1 + \overline{y}_0 + \overline{p}_{13} \geq 1) \models (F \wedge \mathcal{D})\restriction_\omega \wedge (y_1 + \overline{y}_0 + \overline{p}_{13} \geq 1)\restriction_\omega$$

Back to Our Pigeons — Deriving the Constraints

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

$$y_0 \geq 1$$

$$\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\overline{y}_1 + y_0 \geq 1$$

$$\overline{y}_1 + \overline{\sigma(p_{13})} + p_{13} \geq 1$$

$$y_1 + \overline{y}_0 + \overline{p}_{13} \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
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red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by redundance with witness $\omega = \{y_1 \mapsto 1\}$

$$F \wedge \mathcal{D} \wedge \neg(y_1 + \overline{y}_0 + \overline{p}_{13} \geq 1) \models (F \wedge \mathcal{D})\restriction_\omega \wedge (y_1 + \overline{y}_0 + \overline{p}_{13} \geq 1)\restriction_\omega$$

$$F \wedge \mathcal{D} \wedge (\overline{y}_1 + y_0 + p_{13} \geq 3) \models \cdots \wedge \mathcal{D}\restriction_\omega \wedge \ldots$$

Back to Our Pigeons — Deriving the Constraints

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

$$y_0 \geq 1$$

$$\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13}) \geq 1$$

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$$\overline{y}_1 + \overline{\sigma(p_{13})} + p_{13} \geq 1$$

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dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
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red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by redundance with witness $\omega = \{y_1 \mapsto 1\}$

$$F \land \mathcal{D} \land \neg(y_1 + \overline{y}_0 + \overline{p}_{13} \geq 1) \models (F \land \mathcal{D})\restriction_\omega \land (y_1 + \overline{y}_0 + \overline{p}_{13} \geq 1)\restriction_\omega$$

$$F \land \mathcal{D} \land (\overline{y}_1 + y_0 + p_{13} \geq 3) \models \cdots \land \mathcal{D}\restriction_\omega \land \ldots$$

Back to Our Pigeons — Deriving the Constraints

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

$$y_0 \geq 1$$

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$$\overline{y}_1 + \overline{\sigma(p_{13})} + p_{13} \geq 1$$

$$y_1 + \overline{y}_0 + \overline{p}_{13} \geq 1$$

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dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
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red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by redundance with witness $\omega = \{y_1 \mapsto 1\}$

$$F \wedge \mathcal{D} \wedge \neg(y_1 + \overline{y}_0 + \overline{p}_{13} \geq 1) \models (F \wedge \mathcal{D})\restriction_\omega \wedge (y_1 + \overline{y}_0 + \overline{p}_{13} \geq 1)\restriction_\omega$$

$$F \wedge \mathcal{D} \wedge (\overline{y}_1 + y_0 + p_{13} \geq 3) \models \cdots \wedge \mathcal{D}\restriction_\omega \wedge \ldots$$

Back to Our Pigeons — Deriving the Constraints

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

$$y_0 \geq 1$$

$$\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13}) \geq 1$$

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$$y_1 + \overline{y}_0 + \overline{p}_{13} \geq 1$$

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dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
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red 1 y0  >= 1 ; y0 -> 1
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pol 26 32 2048 * +
del id 26
rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by redundance with witness $\omega = \{y_1 \mapsto 1\}$
(same argument)

Back to Our Pigeons — Deriving the Constraints

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

$$y_0 \geq 1$$

$$\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\overline{y}_1 + y_0 \geq 1$$

$$\overline{y}_1 + \overline{\sigma(p_{13})} + p_{13} \geq 1$$

$$y_1 + \overline{y}_0 + \overline{p}_{13} \geq 1$$

$$y_1 + \overline{y}_0 + \sigma(p_{13}) \geq 1$$

$$2^{11} \cdot \overline{y}_1 + 2^{10} \cdot (p_{22} - p_{12}) \ldots \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
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  qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
red 1 ~y1  1  y0        >= 1 ; y1 -> 0
red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1  p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Simplify the pseudo-Boolean breaking constraint and delete original constraint

Back to Our Pigeons — Deriving the Constraints

Derived constraints ($\mathcal{D}$):

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \cdots \geq 0$$

$$y_0 \geq 1$$

$$\overline{y}_0 + \overline{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\overline{y}_1 + y_0 \geq 1$$

$$\overline{y}_1 + \overline{\sigma(p_{13})} + p_{13} \geq 1$$

$$y_1 + \overline{y}_0 + \overline{p}_{13} \geq 1$$

$$y_1 + \overline{y}_0 + \sigma(p_{13}) \geq 1$$

$$2^{11} \cdot \overline{y}_1 + 2^{10} \cdot (p_{22} - p_{12}) \ldots \geq 1$$

$$\overline{y}_1 + \overline{p}_{12} + \sigma(p_{22}) \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
  qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
red 1 ~y1  1  y0        >= 1 ; y1 -> 0
red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1  p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Continue in the same way for following y_i-variables

...

Future Research Directions

Performance and reliability of pseudo-Boolean proof logging

- Trim proof while verifying (as in *DRAT-Trim* [HHW13a])
- Compress proof file using binary format
- Design formally verified proof checker *(work in progress [BMM+23])*

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Proof logging for other combinatorial problems and techniques

- Symmetric learning and recycling (substitution) of subproofs
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And more…

- Use proof logs for algorithm analysis or explainability purposes
- Lots of other challenging problems and interesting ideas

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And more…

- Use proof logs for algorithm analysis or explainability purposes
- Lots of other challenging problems and interesting ideas
- Talk to us if you want to join the proof logging revolution! ☺
  We're happy to collaborate, and we're hiring

Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity

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The end.

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- **Action point:** What problems can VeriPB solve for you?

The end. Or rather, the beginning!

References for Getting Started with VeriPB

`https://gitlab.com/MIAOresearch/software/VeriPB`

Released under MIT Licence

Various features to help development:

- Extended variable name syntax allowing human-readable names
- Proof tracing
- “Trust me” assertions for incremental proof logging

Documentation:

- Description of VeriPB checker [BMM+23] used in SAT 2023 competition (`https://satcompetition.github.io/2023/checkers.html`)
- Specific details on different proof logging techniques covered in research papers [EGMN20, GMN20, GMM+20, GN21, GMN22, GMNO22, VDB22, BBN+23, BGMN23, MM23]
- Lots of concrete example files at `https://gitlab.com/MIAOresearch/software/VeriPB`

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https://gitlab.com/MIAOresearch/software/VeriPB

https://github.com/ciaranm/glasgow-constraint-solver

https://github.com/ciaranm/glasgow-subgraph-solver

https://bitbucket.org/krr/breakid

Parity Reasoning: Experiments

Implemented parity reasoning and PB proof logging engine[2]

Also DRAT proof logging as described in [PR16]

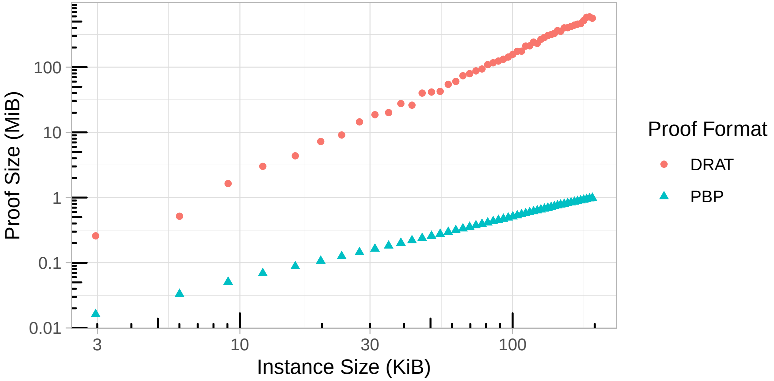
Experiments with MiniSat[3]

Set-up:[4]

- Intel Core i5-1145G7 @2.60GHz $\times$ 4
- Memory limit 8GiB
- Disk write speed roughly 200 MiB/s
- Read speed of 2 GiB/s

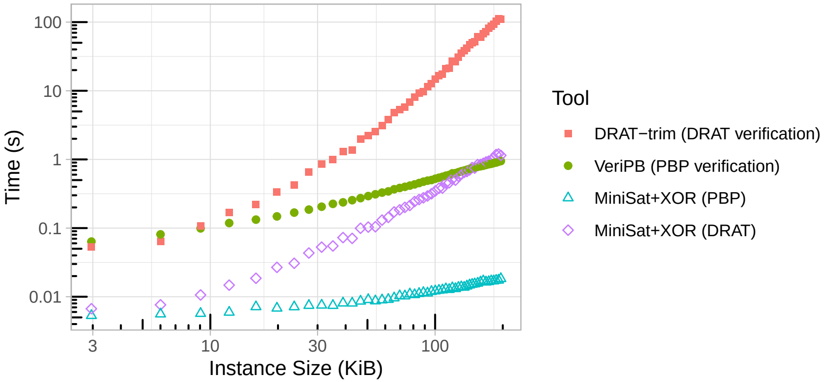
[2] `https://gitlab.com/MIAOresearch/tools-and-utlities/xorengine`
[3] `http://minisat.se/`
[4] Tools, benchmarks, data and evaluation scripts available at `https://doi.org/10.5281/zenodo.7083485`

Parity Reasoning: Proof Size



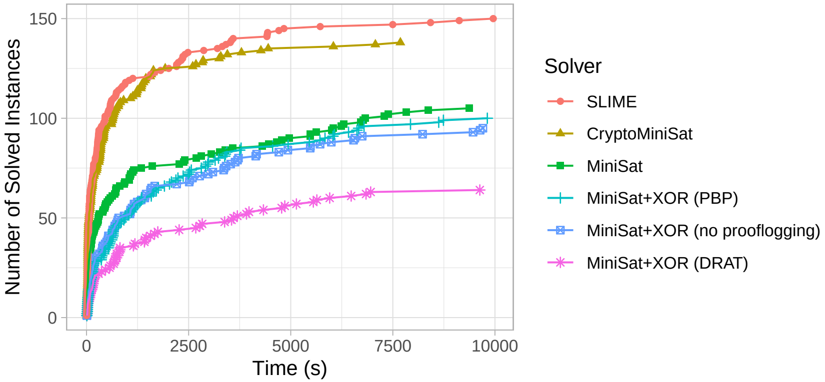
Proof sizes for Tseitin formulas using DRAT and PB proof logging

Parity Reasoning: Solving and Verification Time



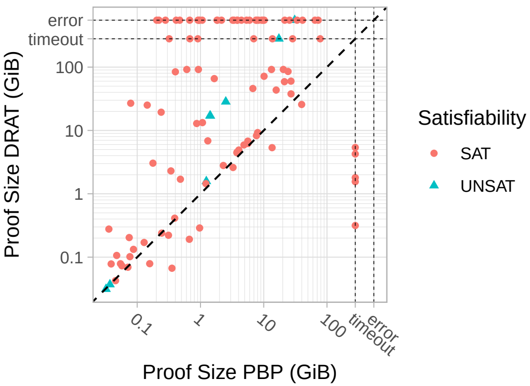
Solving and verification time for Tseitin formulas

Parity Reasoning: Crypto Track of SAT 2021 Competition



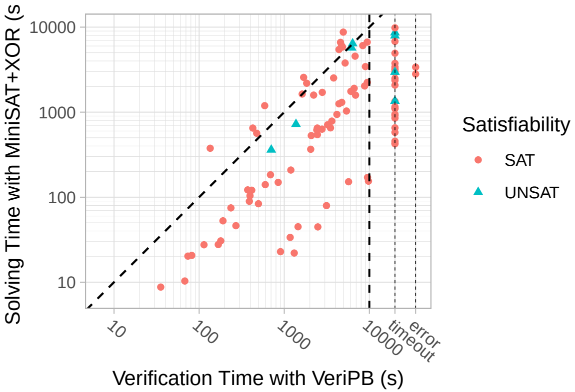
Cumulative plot for the crypto track of the SAT Competition 2021

Parity Reasoning: Crypto Track Proof Size



DRAT and PB proof sizes for crypto track of SAT Competition 2021

Parity Reasoning: Crypto Track Verification Time



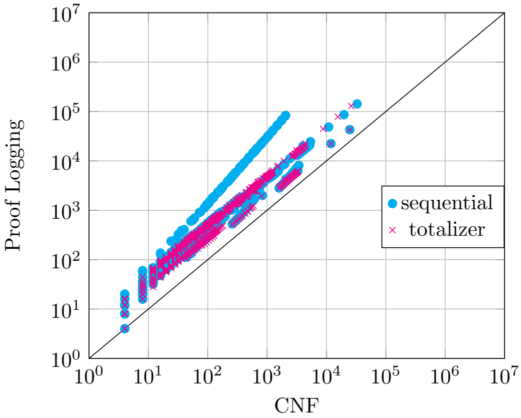
Time required for solving and verifying crypto instances

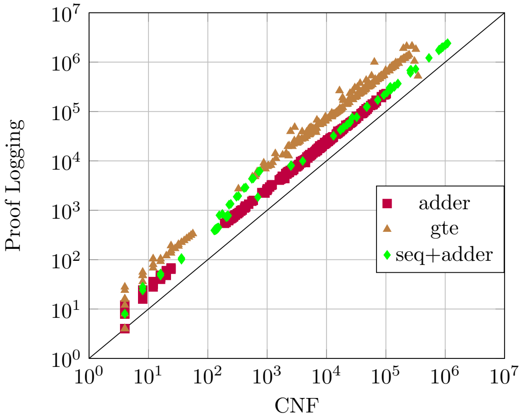
PB-to-CNF Translation: Experiments

- Certified translations for the following CNF encodings:[5]
  - Sequential counter [Sin05]
  - Totalizer [BB03]
  - Generalized totalizer [JMM15]
  - Adder network [ES06]
- Proof verified by proof checker VeriPB
- Benchmarks from PB 2016 Evaluation:[6]
  - SMALLINT decision benchmarks without purely clausal formulas
  - 3 subclasses of benchmarks:
    - Only cardinality constraints (sequential counter, totalizer)
    - Only general 0-1 ILP constraints (generalized totalizer, adder network)
    - Mixed cardinality & general 0-1 ILP constraints (sequential counter + adder network)

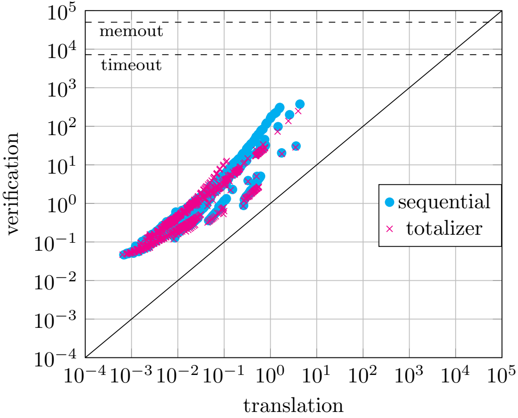
[5] `https://github.com/forge-lab/VeritasPBLib`
[6] `http://www.cril.univ-artois.fr/PB16/`

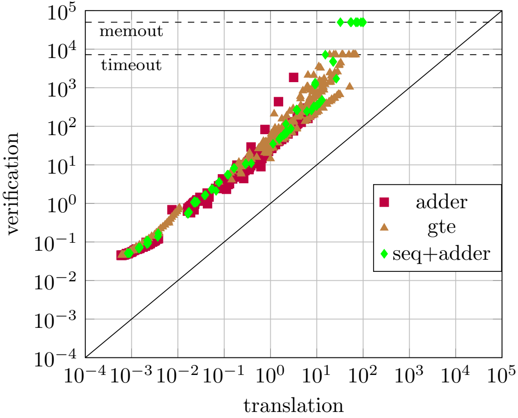
PB-to-CNF: CNF Size vs Proof Size in KiB





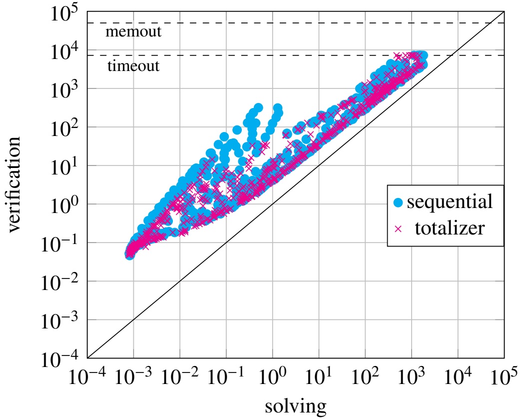
PB-to-CNF: Translation vs Verification Time in Seconds

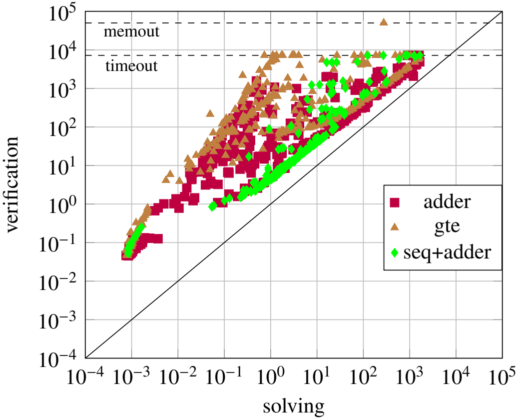




- Translation just generates clauses and proof
- Verification slower, as reasoning has to be performed

PB-to-CNF: Solving Time vs Verification Time in Seconds





- Solved with fork of Kissat[7] syntactically modified to output pseudo-Boolean proofs
- Room for improvement, but clearly shows approach is viable

PB-to-CNF: Future Work

Improving performance:

- Cutting Planes derivations instead of reverse unit propagations [VDB22]
- Backwards checking/trimming for verification (as in DRAT-trim [HHW13a])

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Extend proof logging further:

- Sorting networks like odd-even mergesort, bitonic sorter [Bat68]
- MaxSAT solving

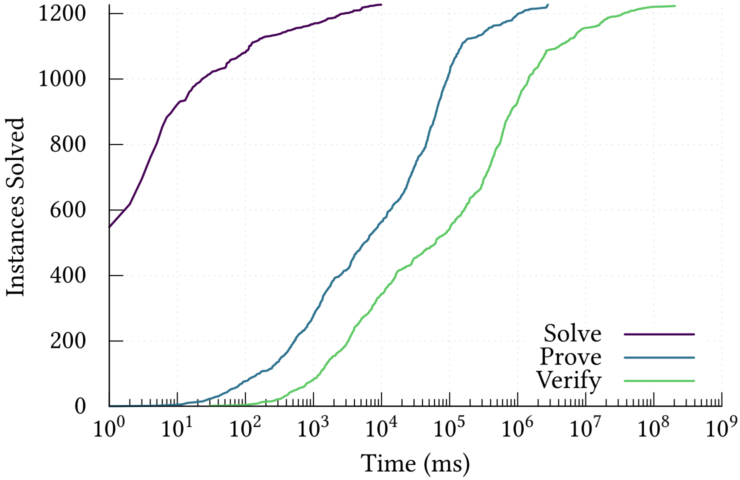
Clique Results

- Implemented in the Glasgow Subgraph Solver.
  - Bit-parallel, can perform a colouring and recursive call in under a microsecond.
- 59 of the 80 DIMACS instances take under 1,000 seconds to solve without logging.
- Produced and verified proofs for 57 of these 59 instances (the other two reached 1TByte disk space).
- Mean slowdown from proof logging is 80.1 (due to disk I/O).
- Mean verification slowdown a further 10.1.
- Approximate implementation effort: one Masters student.

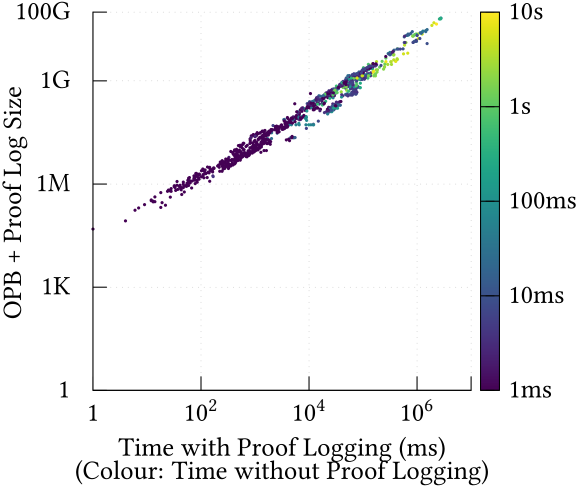
Subgrap Isomorphism Results

- The Pseudo-Boolean models can be large: had to restrict to instances with no more than 260 vertices in the target graph.
- Took enumeration instances which could be solved without proof logging in under ten seconds.
- 1,227 instances from Solnon's benchmark collection:
  - 789 unsatisfiable, up to 50,635,140 solutions in the rest.
  - 498 instances solved without guessing.
  - Hardest solved satisfiable and unsatisfiable instances required 53,605,482 and 2,074,386 recursive calls.

Subgrap Isomorphism Results



Subgrap Isomorphism Results



How Expensive is Proof Logging? (1/2)

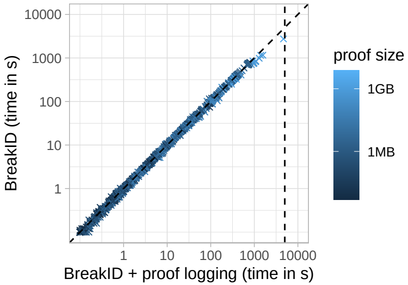
- Laurent D. Michel, Pierre Schaus, Pascal Van Hentenryck: *MiniCP: A Lightweight Solver for Constraint Programming.* [MSH21]
- Five benchmark problems allowing comparison of solvers "doing the same thing":
  - Simple models
  - Fixed search order and well-defined propagation consistency levels
  - Few global constraints
- Probably close to the worst case for proof logging performance
- Also: Crystal Maze and World's Hardest Sudoku

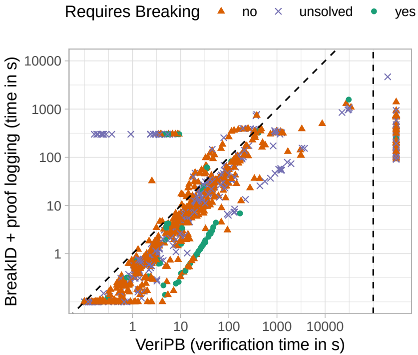
How Expensive is Proof Logging? (2/2)

- Our solver: faster than the fastest of *MiniCP*, *OscaR*, and *Choco*
- Proof logging slowdown: between 8.4 and 61.1 factor
  - 800,000 to 3,000,000 inferences per second
  - Proof logs can be hundreds of GBytes
  - No effort put into making the proof-writing code run fast
- Verification slowdown: a further factor 10 to 100
  - Probably possible to reduce this substantially if we are prepared to put more care into writing proofs

Experimental Evaluation of SAT Symmetry Breaking

- Evaluated on SAT competition benchmarks
- *BreakID* [DBBD16, Bre] used to find and break symmetries





- proof logging overhead negligible
- verification at most 20 times slower than solving for 95% of instances