A One-Size-Fits-All Proof Logging System?

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1st International Workshop on Organizing and Optimizing Proof-logging Systems
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Based on joint work with Jeremias Berg, Bart Bogaerts, Jan Elffers, Ambros Gleixner, Stephan Gocht, Alexander Hoen, Hannes Ihalainen, Matti Järvisalo, Ciaran McCreesh, Matthew McIlree, Magnus O. Myreen, Andy Oertel, Yong Kiam Tan, and Dieter Vandesande

Combinatorial Solving and Optimization

- Astounding progress last couple of decades on combinatorial solvers for, e.g.:
  - Boolean satisfiability (SAT) solving and optimization [BHvMW21]
  - Constraint programming [RvBW06]
  - Mixed integer linear programming [AW13, BR07]
  - Satisfiability modulo theories (SMT) solving [BHvMW21]
- Solvers very fast, but sometimes wrong (even best commercial ones) [BLB10, CKSW13, AGJ$^+$18, GSD19, GS19, BMN22, BBN$^+$23]
- Even get feasibility of solutions wrong (though this should be straightforward!)
- And how to check the absence of solutions?
- Or that a solution is optimal? (Even off-by-one mistakes can snowball into large errors if solver used as subroutine)

What Can Be Done About Solver Bugs?

- **Software testing**
  Hard to get good test coverage for sophisticated solvers
  Progress using fuzzing and delta debugging [BB09, BLB10, KB22, NPB22, PB23]
  But inherently can only detect presence of bugs, not absence

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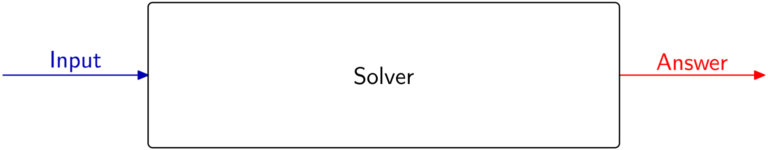
- **Proof logging**
  Make solver certifying [ABM+11, MMNS11] by adding code so that it outputs
  1. not only answer but also
  2. simple, machine-verifiable proof that answer is correct

Proof Logging with Certifying Solvers: Workflow



1. Run combinatorial solving algorithm on problem input

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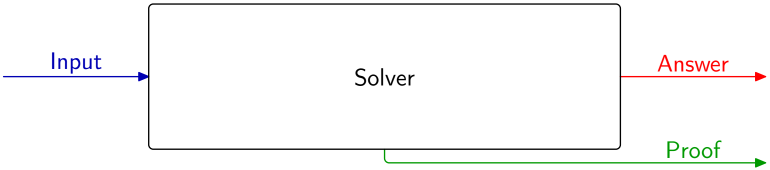

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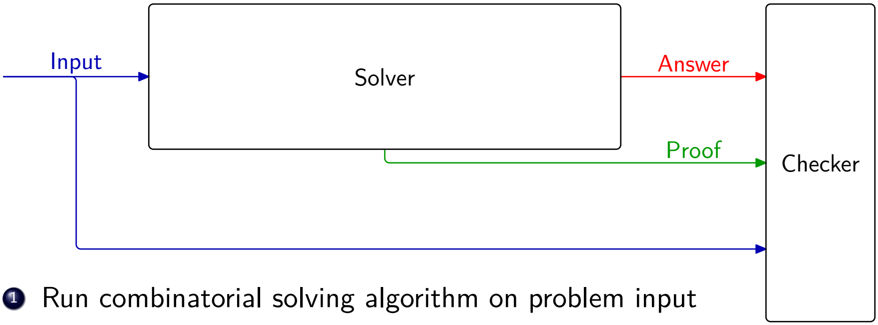

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3. Feed input + answer + proof to proof checker

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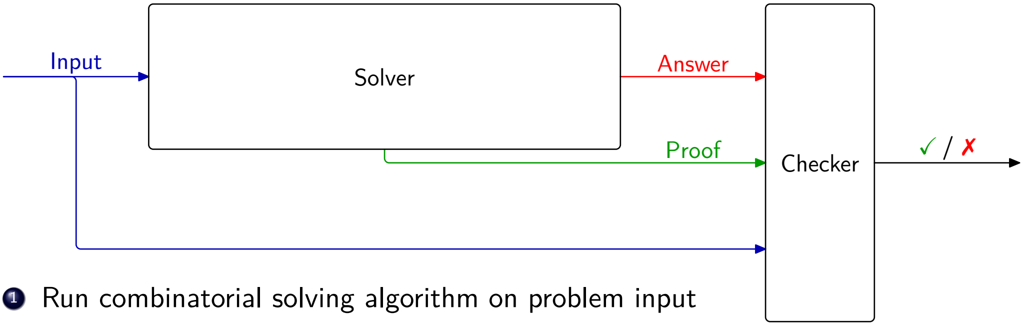
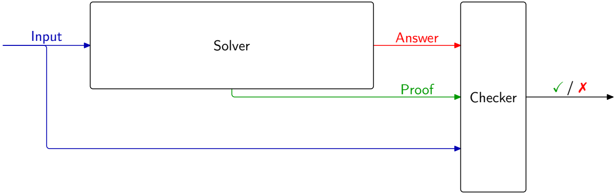


1. Run combinatorial solving algorithm on problem input
2. Get as output not only answer but also proof
3. Feed input + answer + proof to proof checker
4. Verify that proof checker says answer is correct

Proof Logging Desiderata

Proof format for certifying solver should be



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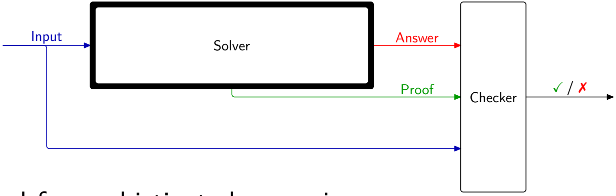

Proof format for certifying solver should be

- **very powerful:** minimal overhead for sophisticated reasoning

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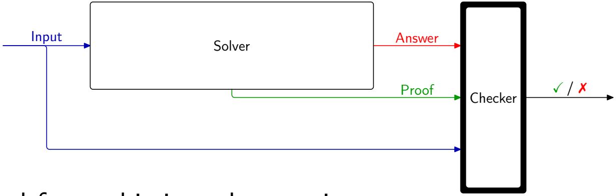

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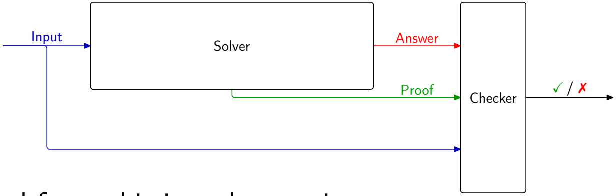

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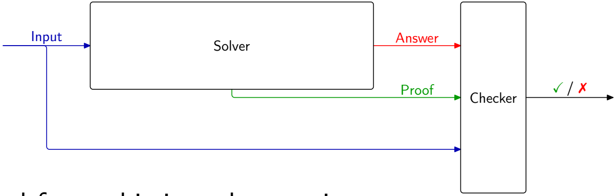

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Asking for both perhaps a little bit too good to be true?

Message of This Talk (and This Workshop)

Proof logging for combinatorial optimization is possible with single, unified method!

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- Build on successes in SAT solving with proof formats such as DRAT [HHW13a, HHW13b, WHH14], GRIT [CMS17], LRAT [CHH+17], ...
- But represent constraints as 0–1 integer linear inequalities
- Formalize reasoning using cutting planes [CCT87] proof system
- Add well-chosen strengthening rules [Goc22, GN21, BGMN23]
- Implemented in VeriPB (`https://gitlab.com/MIAOresearch/software/VeriPB`)

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1. Marketing pitch ☺

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Purpose of this talk:

1. Marketing pitch ☺
2. Overview of proof system behind VeriPB
3. Sample of applications and future challenges

Outline of This Talk

1. Proof Logging Principles
   - Pseudo-Boolean Basics
   - Proof Logging Goals
   - Workflow
2. Proof System
   - Cutting Planes Proof System and VeriPB Proof Format
   - Strengthening Rules and Deletion
   - Proofs for Decision and Optimization Problems
3. Example Applications and Future Directions
   - Advanced SAT Solving Techniques
   - Subgraph Isomorphism Solving
   - Further Challenges

The Sales Pitch For Proof Logging

1. Certifies correctness of computed results
2. Detects errors even if due to compiler bugs, hardware failures, or cosmic rays
3. Provides debugging support during software development [EG21, GMM$^+$20, KM21, BBN$^+$23]
4. Facilitates performance analysis
5. Helps identify potential for further improvements
6. Enables auditability
7. Serves as stepping stone towards explainability

Proof Language: Pseudo-Boolean Constraints

Proof consists of 0-1 integer linear inequalities or pseudo-Boolean constraints:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
- variables x_i take values $0 = \mathit{false}$ or $1 = \mathit{true}$

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Sometimes convenient to use normalized form [Bar95] with all a_i, A positive (without loss of generality)

Some Types of Pseudo-Boolean Constraints

1. Clauses

$$x \vee \overline{y} \vee z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Pseudo-Boolean Proof Logging Wishlist

Paradigms

- SAT solving
- (linear) pseudo-Boolean solving
- graph solving
- constraint programming
- automated planning
- mixed integer linear programming
- SMT solving

Problem types

- decision / feasibility
- optimization
- multi-objective optimization
- projected model enumeration
- projected model counting
- preprocessing / problem reformulation

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Design Principles for Proof Logging

Proof logging implementation

- Don't change solver
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- Proof logging overhead small constant fraction ($\lessapprox 10\%$)
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Proof system

- Keep proof language maximally simple
- Reason about XOR constraints, CP propagators, symmetries, etc within language
- Combine proof logging with formally verified proof checker

Pseudo-Boolean Proof Logging — How and Why?

If problem is (special case of) 0-1 integer linear program

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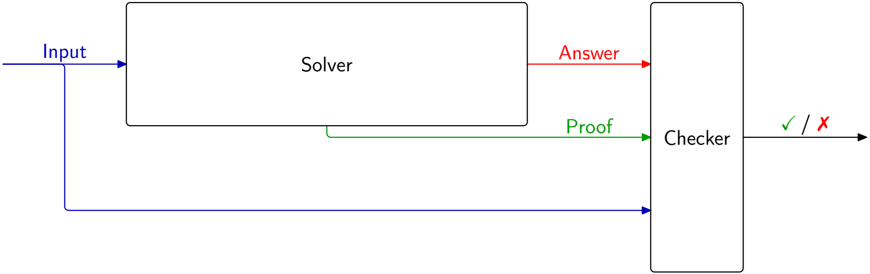
$$r \Rightarrow x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7 \qquad 7\overline{r} + x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

$$r \Leftarrow x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7 \qquad 9r + \overline{x}_1 + 2x_2 + 3\overline{x}_3 + 4x_4 + 5\overline{x}_5 \geq 9$$

Proof Logging with Formally Verified Checking: Full Workflow



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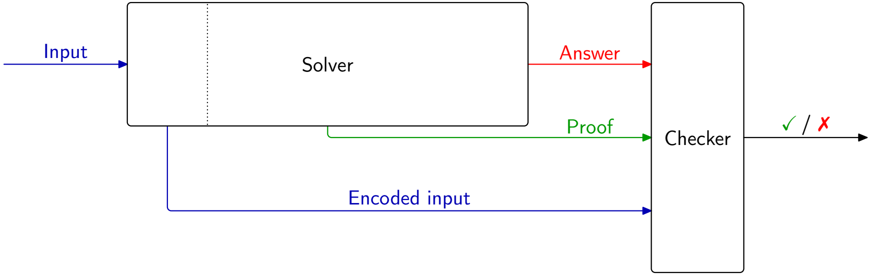

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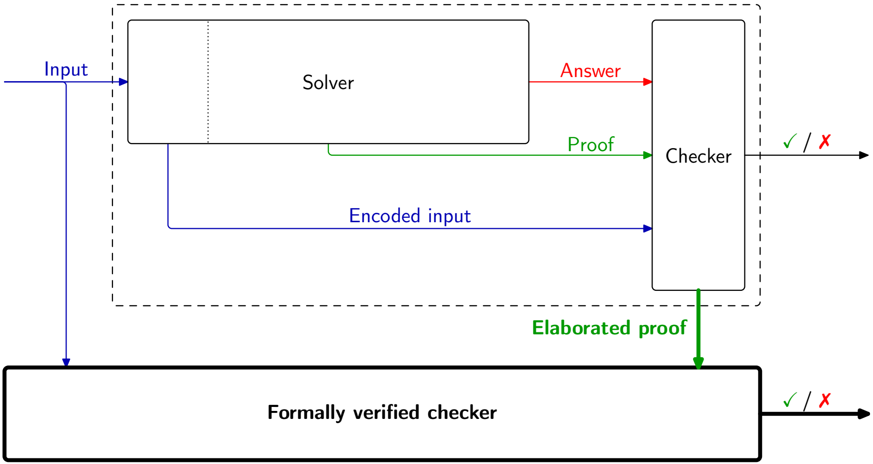

VeriPB Proof Structure

1. **Preamble**
   Load input formula
   Specify settings
2. **Derivation section**
   Derivations of new constraints
   Logging of solutions
3. **Output section**
   Listing of constraints currently in database
   Input to next stage (or for debugging)
4. **Conclusions section**
   Specification of what was established
   - satisfiability / unsatisfiability
   - optimality (or upper and lower bounds)
   - other types of conclusions to be added

VeriPB Proof Configuration

Core set $\mathcal{C}$

- Contains input formula at the start
- Maintains "equivalence" with input formula

Derived set $\mathcal{D}$

- All constraints derived during search
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Objective $f = \sum_i w_i \ell_i + k$

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Order $\mathcal{O}$

- Pseudo-Boolean formula encoding pre-order (reflexive and transitive)
- Syntactic proof of properties required
- Applied to specified variable set $\vec{z}$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input axioms From the input

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Multiplication for any $c \in \mathbb{N}^+$ $$\frac{\sum_i a_i \ell_i \geq A}{\sum_i c a_i \ell_i \geq cA}$$

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Division for any $c \in \mathbb{N}^+$
(constraint in normalized form)

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \left\lceil \frac{a_i}{c} \right\rceil \ell_i \geq \left\lceil \frac{A}{c} \right\rceil}$$

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Saturation
(constraint in normalized form)

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \min(a_i, A) \cdot \ell_i \geq A}$$

Cutting Planes Toy Example

$$w + 2x + y \geq 2$$

Cutting Planes Toy Example

Multiply by 2 $\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$

Cutting Planes Toy Example

$$\text{Multiply by } 2 \ \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5$$

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$$\dfrac{\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}\ \text{Multiply by } 2 \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}\ \text{Add}$$

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\qquad \overline{z} \geq 0
$$

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$$

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Multiply by 2: $\dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}$

Add: $\dfrac{3w + 6x + 6y + 2z \geq 9 \qquad 2\overline{z} \geq 0}{3w + 6x + 6y \geq 7}$

Divide by 3: $\dfrac{3w + 6x + 6y \geq 7}{w + 2x + 2y \geq 3}$

Cutting Planes Toy Example

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By naming constraints by integers and literal axioms by the literal involved as

$$\begin{aligned}
\text{Constraint 1} &\doteq 2x+y+w\geq 2\\
\text{Constraint 2} &\doteq 2x+4y+2z+w\geq 5\\
\sim\text{z} &\doteq \overline{z}\geq 0
\end{aligned}$$

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\cfrac{
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\text{~z} &\doteq \overline{z} \geq 0
\end{aligned}
$$

such a calculation is written in the proof log in reverse Polish notation as

```
pol 1 2 * 2 + ~z 2 * + 3 d
```

More About VeriPB Proofs

Variables

- start with a letter in `A-Z` or `a-z`
- continue with characters in `A-Z`, `a-z`, `0-9`, or `[]{}-_^`
  (square and curly brackets, hyphen, underscore, and caret)
- contain at least two characters

Constraints

Are referred to by positive integers (constraint IDs)

Derivation rules and requirements

Come in two flavours

1. kernel format for formally verified proof checker
2. augmented format with convenience rules such as reverse unit propagation (RUP)

Open Problem: Division Versus Saturation

Division $$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \left\lceil \frac{a_i}{c} \right\rceil \ell_i \geq \left\lceil \frac{A}{c} \right\rceil}$$

Saturation $$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \min(a_i, A) \cdot \ell_i \geq A}$$

How do division and saturation rules compare?

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How do division and saturation rules compare?

- Strengths of rules as such incomparable [GNY19]
- Cutting planes with division can be exponentially stronger than cutting planes with saturation
- Unknown whether cutting planes with saturation can be stronger than cutting planes with division

Redundance-Based Strengthening

C is redundant with respect to F if F and $F \cup \{C\}$ are equisatisfiable

Want to allow adding such "redundant" constraints

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C is redundant with respect to F if and only if there is a substitution ω (mapping variables to truth values or literals), called a witness, for which

$$F \cup \{\neg C\} \models (F \cup \{C\})\restriction_\omega$$

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- Proof sketch for interesting direction: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \cup \{C\}$
- In a proof, the implication needs to be efficiently verifiable — every $D \in (F \cup \{C\})\restriction_\omega$ should follow from $F \cup \{\neg C\}$ either
  1. "obviously" (e.g., by so-called weakening or unit propagation) or
  2. by explicitly presented derivation

Example: Deriving $a \leftrightarrow (x \wedge y)$ Using the Redundance Rule

Want to derive

$$2\overline{a} + x + y \geq 2 \qquad\qquad a + \overline{x} + \overline{y} \geq 1$$

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2. $F \cup \{2\overline{a} + x + y \geq 2, \neg(a + \overline{x} + \overline{y} \geq 1)\} \models$
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   Choose $\omega = \{a \mapsto 1\}$ — F untouched; new constraint satisfied
   $\neg(a + \overline{x} + \overline{y} \geq 1)$ forces $x \mapsto 1$ and $y \mapsto 1$, hence $2\overline{a} + x + y \geq 2$ remains satisfied after forcing a to be true

Redundance and Dominance Rules in VeriPB (Slightly Simplified)

Redundance-based strengthening, optimization version [BGMN23]

Add constraint C to derived set $\mathcal{D}$ if exists witness substitution ω such that

$$\mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \models (\mathcal{C} \cup \mathcal{D} \cup \{C\})\restriction_\omega \cup \{f\restriction_\omega \leq f\}$$

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- Applying ω should strictly decrease f
- If so, don't need to show that $(\mathcal{D} \cup \{C\})\restriction_\omega$ implied!

Soundness of Dominance Rule

Dominance-based strengthening

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8. Can't go on forever, so finally reach α' satisfying $\mathcal{C} \cup \{C\}$

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- Same inductive proof as before, but also nested forward induction over derivation
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Further extensions:

- Define dominance rule with respect to order $\mathcal{O}$ independent of objective function
- Switch between different orders in same proof
- See [BGMN23] for details

Strengthening Rules in Their (Almost) Full Formal Glory

Witness ω: substitution mapping variables to truth values or literals

Redundance-based strengthening (witness ω show how to "patch assignment")

Derive constraint C from $\mathcal{C} \cup \mathcal{D}$ if exists witness ω such that

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Strengthening Rules: Proof Format

`red` $\langle$`Constraint` $C\rangle$ `;` $\langle$*`var1`*$\rangle$ `->` $\langle$*`val1`*$\rangle$ `...` $\langle$*`varN`*$\rangle$ `->` $\langle$*`valN`*$\rangle$ `; begin`
    subproofs for proof goals
`end`

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- Witness ω should be explicitly specified in proof log
- Subproofs of proof goals should also be explicit
- But can be skipped for proof goals "obvious" to proof checker (like by weakening or unit propagation)

The Problem of Deleting Constraints

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Solution: distinguish between deletion from core set $\mathcal{C}$ and derived set $\mathcal{D}$

Deletion, Core Transfer, and Order Change

Deletion

1. Deletion of constraint C always OK from derived set $\mathcal{D}$
2. OK from core set $\mathcal{C}$ only if C can be rederived from $\mathcal{C} \setminus \{C\}$ with redundance rule (otherwise unchecked deletion — special conditions apply)

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Change of order

Possible to change order only if $\mathcal{D} = \emptyset$

Conclusions for Decision Problems

`NONE`

Status is undetermined

`SAT [ :` ⟨*assignment*⟩`]`

Propagate given assignment w.r.t. database, then check against original formula

If no assignment given, then

- solution should have been logged
- no unchecked deletion must have occurred

`UNSAT [ :` ⟨*constraint ID*⟩`]`

Only valid if no solution has been logged

Check that specified constraint is contradictory (technically: negative slack)

If no constraint given, check that database unit propagates to contradiction

Optimization Problems

Any solution α found is logged with `soli` "log solution and improve" command

- provided solution α checked against current core set $\mathcal{C}$
- Objective-improving constraint $\sum_i w_i \ell_i \leq -1 + \sum_i w_i \cdot \alpha(\ell_i)$ added to core set (forces search for better solutions)

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Proof of optimality: Contradiction derived from objective-improving constraint

Proof format supports not just optimality, but also non-tight upper and lower bounds

Conclusions for Optimization Problems

`NONE`
No solution or lower bound found

`BOUNDS` $\langle \mathit{LB} \rangle$ `[ :` $\langle \mathit{constraint\ ID} \rangle$ `]` $\langle \mathit{UB} \rangle$ `[ :` $\langle \mathit{assignment} \rangle$ `]`
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Upper bound
Propagate given assignment w.r.t. database, then check against original formula
If no assignment given, then

- solution with value $\langle UB \rangle$ should have been logged
- no unchecked deletion must have occurred

Parity (XOR) Reasoning in SAT Solving

Given clauses

$$x \lor y \lor z$$
$$x \lor \overline{y} \lor \overline{z}$$
$$\overline{x} \lor y \lor \overline{z}$$
$$\overline{x} \lor \overline{y} \lor z$$

and

$$y \lor z \lor w$$
$$y \lor \overline{z} \lor \overline{w}$$
$$\overline{y} \lor z \lor \overline{w}$$
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want to derive

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This is just parity reasoning:

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This is just parity reasoning:

$$x + y + z = 1 \pmod 2$$
$$y + z + w = 1 \pmod 2$$

imply

$$x + w = 0 \pmod 2$$

Parity (XOR) Reasoning in SAT Solving

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DRAT proof logging like [PR16] too inefficient in practice!

Could add XORs to language, but prefer to keep things super-simple

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Use redundance rule with fresh variables a, b to derive

$$x + y + z + 2a = 3$$
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("$=$" syntactic sugar for "$\geq$" plus "$\leq$")

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VeriPB can certify XOR reasoning [GN21]

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1. Pretend to solve optimisation problem minimizing $f \doteq \sum_{i=1}^{n} 2^{n-i} \cdot x_i$ (search for lexicographically smallest assignment satisfying formula)

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VeriPB can certify fully general SAT symmetry breaking [BGMN23]

Open Problem: Symmetry Breaking with Redundance Rule?

Is the dominance rule really needed for fully general symmetry breaking?

Or could the redundance rule be enough?

Weaker DRAT strengthening rule sufficient for "pigeonhole-style" symmetries [HHW15]

Open Problem: Efficient Substitution Proofs?

Can cutting planes with redundance and dominance support proofs with lemmas/substitution efficiently?

Special case: symmetric learning in SAT solving [DBB17]

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Can cutting planes with redundance and dominance support proofs with lemmas/substitution efficiently?

Special case: symmetric learning in SAT solving [DBB17]

Can be done in principle, but seems very finicky...

Extension and substitution proof systems don't mix well

The Subgraph Isomorphism Problem

Input

- Pattern graph $\mathcal{P}$ with vertices $V(\mathcal{P}) = \{a, b, c, \ldots\}$
- Target graph $\mathcal{T}$ with vertices $V(\mathcal{T}) = \{u, v, w, \ldots\}$

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Task

- Find all subgraph isomorphisms $\varphi : V(\mathcal{P}) \to V(\mathcal{T})$
- I.e., if
  1. $\varphi(a) = u$
  2. $\varphi(b) = v$
  3. $(a, b) \in E(\mathcal{P})$

  then must have $(u, v) \in E(\mathcal{T})$

Pseudo-Boolean Proof Logging for Subgraph Isomorphism Solving

All reasoning steps in Glasgow Subgraph Solver [ADH$^+$19, GSS] can be formalized efficiently in the cutting planes proof system [GMN20]

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1. Solver can justify each step by writing local formal derivation
2. Local derivations can be chained into global correctness proof
3. Proof checkable by stand-alone verifier that knows nothing about graphs
4. With end-to-end fully formally verified result [GMM+24]

Subgraph Isomorphism as a Pseudo-Boolean Formula

- Pattern graph $\mathcal{P}$ with $V(\mathcal{P}) = \{a, b, c, \ldots\}$
- Target graph $\mathcal{T}$ with $V(\mathcal{T}) = \{u, v, w, \ldots\}$
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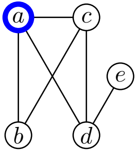
Pseudo-Boolean encoding

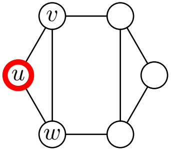
$$\sum_{v \in V(\mathcal{T})} x_{a \mapsto v} = 1 \qquad \text{[every } a \text{ maps somewhere]}$$

$$\sum_{b \in V(\mathcal{P})} \overline{x}_{b \mapsto u} \geq |V(\mathcal{P})| - 1 \qquad \text{[mapping is one-to-one]}$$

$$\overline{x}_{a \mapsto u} + \sum_{v \in N(u)} x_{b \mapsto v} \geq 1 \qquad \text{[edge } (a, b) \text{ maps to edge } (u, v)\text{]}$$

Pseudo-Boolean Proof Logging Example: Degree Preprocessing





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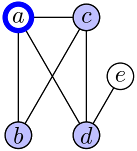

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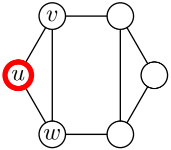

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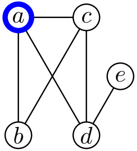

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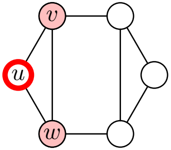

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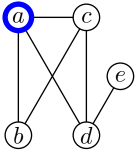

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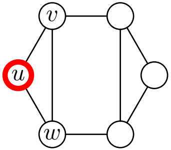

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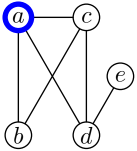

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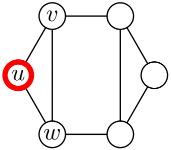

Sum up all constraints & divide by 3 to obtain

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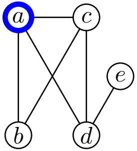

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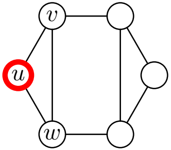

Sum up all constraints & divide by 3 to obtain

$$3\overline{x}_{a\mapsto u} + 10 \geq 11$$

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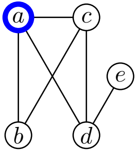

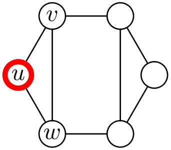

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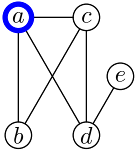

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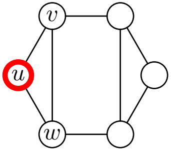

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Future Research Directions

Performance and reliability of pseudo-Boolean proof logging

- Trim proof while verifying (as in DRAT-TRIM [HHW13a])
- Compress proof file using binary format
- Design formally verified proof checking backends [BMM+23, GMM+24, IOT+24]

Future Research Directions

Performance and reliability of pseudo-Boolean proof logging

- Trim proof while verifying (as in DRAT-TRIM [HHW13a])
- Compress proof file using binary format
- Design formally verified proof checking backends [BMM+23, GMM+24, IOT+24]

Proof logging for other combinatorial problems and techniques

- Model enumeration and counting
- SMT solving *(work on* CVC5, SMTINTERPOL, Z3, ... *[BBC+23, HS22])*
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And more. . .

- Use proof logs for algorithm analysis or explainability purposes
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- Lots of other challenging problems and interesting ideas
- We're hiring! Talk to me to join the pseudo-Boolean proof logging revolution! ☺

VeriPB Documentation

VeriPB tutorial at *CP '22* [BMN22]

- video at `youtu.be/s_5BIi4I22w`
- updated slides for *IJCAI '23* tutorial [BMN23]

Description of VeriPB and CakePB [BMM+23] for SAT 2023 competition

- Available at `satcompetition.github.io/2023/checkers.html`

Specific details on different proof logging techniques covered in research papers [EGMN20, GMN20, GMM+20, GN21, GMN22, GMNO22, VDB22, BBN+23, BGMN23, MM23, GMM+24, HOGN24, IOT+24, MMN24]

Lots of concrete example files at `gitlab.com/MIAOresearch/software/VeriPB`

Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity
- **Action point:** What problems can VeriPB solve for you? ☺

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Thank you for your attention!

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