Proof complexity and SAT solving
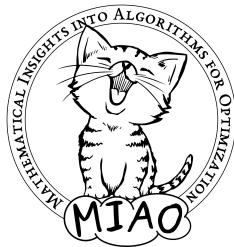
Jakob Nordström

University of Copenhagen and Lund University

SAT/SMT/AR Summer School
Nancy, France
June 29, 2024

COLOURING

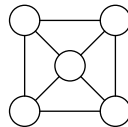Does the graph $G = (V, E)$ have a
colouring with k colours such that all
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Does the graph $G = (V, E)$ have a colouring with k colours such that all neighbours have distinct colours?

3-colouring? Yes, but no 2-colouring

CLIQUE

Is there a clique in the graph $G = (V, E)$ with k vertices that are all pairwise connected by edges in E?

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Is there a clique in the graph $G = (V, E)$ with k vertices that are all pairwise connected by edges in E?

3-clique? Yes, but no 4-clique

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Three Simple Problems. . .

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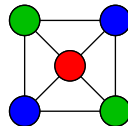
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SAT

Given propositional logic formula, is there a satisfying assignment?

Three Simple Problems...

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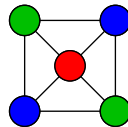Does the graph $G = (V, E)$ have a colouring with k colours such that all neighbours have distinct colours?

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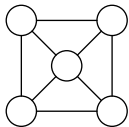Does the graph $G = (V, E)$ have a colouring with k colours such that all neighbours have distinct colours?

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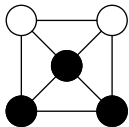
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- Variables should be set to **true** or **false**
- Constraint $(x \vee \neg y \vee z)$: means x or z should be true or y false
- $\wedge$ means all constraints should hold simultaneously
- Is there a truth value assignment satisfying all constraints?

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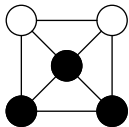
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COLOURING: frequency allocation for mobile base stations
CLIQUE: bioinformatics, computational chemistry
SAT: easily models these and many other problems

. . . with Huge Practical Implications

- Some more examples of problems that can be encoded as propositional logic formulas:
  - computer hardware verification
  - computer software testing
  - artificial intelligence
  - operations research
  - cryptography
  - bioinformatics
  - et cetera. . .

- Leads to **humongous** formulas (100,000s or even 1,000,000s of variables)

- Can we use computers to solve these problems efficiently?

Solving NP in Theory and Practice

- SAT mentioned already in Gödel's famous letter in 1956 to von Neumann
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  - COLOURING [Kho01, Zuc07]
  - CLIQUE [Hås99]
  - SAT [Hås01]

Solving NP in Theory and Practice

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- Assuming P $\neq$ NP, even impossible to meaningfully approximate
  - $\textsc{Colouring}$ [Kho01, Zuc07]
  - $\textsc{Clique}$ [Hås99]
  - $\textsc{Sat}$ [Hås01]
- Except that in practice, there are good algorithms for
  - $\textsc{Colouring}$ [DLMM08, DLMO09, DLMM11]
  - $\textsc{Clique}$ [Pro12, McC17]

  and amazing conflict-driven clause learning (CDCL) solvers [BS97, MS99, MMZ$^+$01]
  that solve huge $\textsc{Sat}$ formulas

How can we understand real-world algorithms for NP-hard problems?

This talk: Use proof complexity (not only conceivable answer)

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Study infeasible problems — proofs of feasibility are trivial

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Question 2: Topic for separate lecture(s) — lots of recent exciting progress; mostly negative (worst-case) results that proof search is hard, e.g., [AM20, GKMP20, dRGN$^+$21]

Applications of Proof Complexity

Three applied reasons for proof complexity:

1. Understand real-world applied algorithmic paradigms [**this talk**]

2. Get ideas for algorithmic improvements
   [EN18, EN20, DGD+21, DGN21, KBBN22]
   (See https://www.youtube.com/watch?v=LZ8VztiplaQ and
   https://www.youtube.com/watch?v=wD_2tx1rTaw)

3. Enhance algorithms to write machine-verifiable certificates of correctness
   [EGMN20, GMN20, GMM+20, GN21, GMN22, GMNO22, VDB22, BGMN23,
   BBN+23, MM23, GMM+24, HOGN24, BBN+24, DMM+24, IOT+24, MMN24]
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Formal Description of SAT Problem

- Variable x: takes value **true** $(= 1)$ or **false** $(= 0)$
- Literal ℓ: variable x or its negation $\overline{x}$ (write $\overline{x}$ instead of $\neg x$)
- Clause $C = \ell_1 \vee \cdots \vee \ell_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- Conjunctive normal form (CNF) formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses

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Here is our example formula again:

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The Same Problem in Three Different Shapes

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$$(1-x)(1-z) = 0$$
$$(1-y)z = 0$$
$$(1-x)y(1-u) = 0$$
$$yu = 0$$
$$(1-u)(1-v) = 0$$
$$xv = 0$$
$$u(1-w) = 0$$
$$xuw = 0$$

For **true** $= 1$ and **false** $= 0$, is there a $\{0,1\}$-valued solution?

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$$(x \lor z) \land (y \lor \neg z) \land (x \lor \neg y \lor u) \land (\neg y \lor \neg u)$$
$$\land \ (u \lor v) \land (\neg x \lor \neg v) \land (\neg u \lor w) \land (\neg x \lor \neg u \lor \neg w)$$

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$$1 - x - z + xz = 0 \qquad\qquad x + z \geq 1$$
$$z - yz = 0 \qquad\qquad y + (1 - z) \geq 1$$
$$y - xy - yu + xyu = 0 \qquad\qquad x + (1 - y) + u \geq 1$$
$$yu = 0 \qquad\qquad (1 - y) + (1 - u) \geq 1$$
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$$1 - u - v + uv = 0 \qquad\qquad u + v \geq 1$$
$$xv = 0 \qquad\qquad -x - v \geq -1$$
$$u - uw = 0 \qquad\qquad -u + w \geq 0$$
$$xuw = 0 \qquad\qquad -x - u - w \geq -2$$

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DPLL, CDCL, and Resolution
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Some More Advanced Proof Systems We Might Not Have Time for

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DPLL: Attempting Smart Case Analysis

The foundation of state-of-the-art SAT solvers is the DPLL method developed by Davis, Putnam, Logemann & Loveland [DP60, DLL62]

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6. If result in both cases "unsatisfiable", then report "unsatisfiable" and return

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$$F = \quad (x \vee z) \wedge (y \vee \overline{z}) \wedge (x \vee \overline{y} \vee u) \wedge (\overline{y} \vee \overline{u})$$
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Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing
- satisfied clauses
- falsified literals

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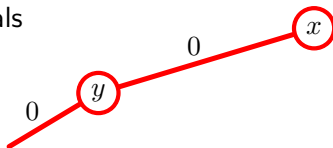- satisfied clauses
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$$\wedge \, (u \vee v) \wedge (\quad \overline{v}) \wedge (\overline{u} \vee w) \wedge (\quad \overline{u} \vee \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing
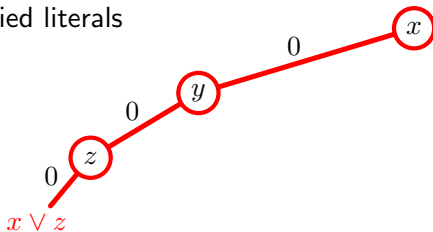- satisfied clauses
- falsified literals

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A DPLL Toy Example

$$F = \quad (x \vee z) \wedge (y \vee \overline{z}) \wedge (x \vee \overline{y} \qquad) \wedge (\overline{y} \vee \overline{u})$$
$$\wedge (\qquad v) \wedge (\qquad \overline{v}) \wedge (\overline{u} \vee w) \wedge (\qquad \overline{u} \vee \overline{w})$$

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"Simplify formula" by (mentally) removing
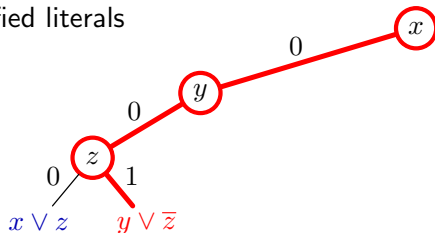- satisfied clauses
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$$\wedge (u \vee v) \wedge (\quad \overline{v}) \wedge (\overline{u} \vee w) \wedge (\quad \overline{u} \vee \overline{w})$$

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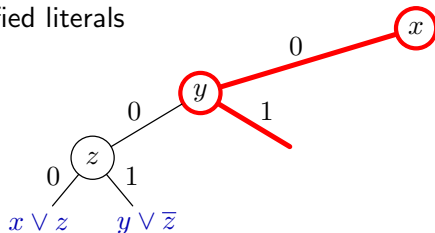- satisfied clauses
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A DPLL Toy Example

$$F = \quad (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \quad\quad) \land (\overline{y} \lor \overline{u})$$
$$\land (\quad\quad v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\quad\quad \overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

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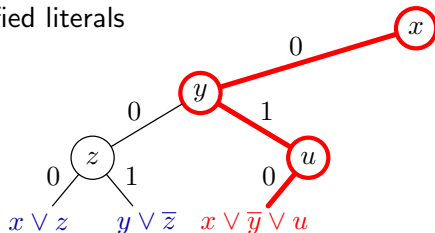
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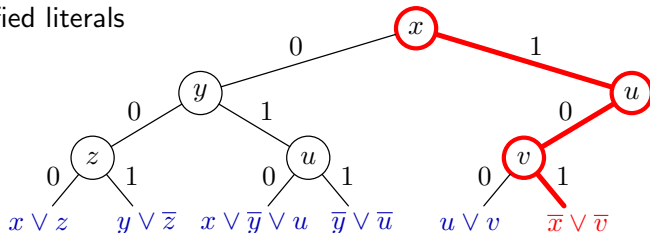
A DPLL Toy Example

$$F = \quad (x \vee z) \wedge (y \vee \overline{z}) \wedge (x \vee \overline{y} \vee u) \wedge (\overline{y} \vee \overline{u})$$
$$\wedge \; (u \vee v) \wedge (\quad \overline{v}) \wedge (\quad w) \wedge (\quad \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing
- satisfied clauses
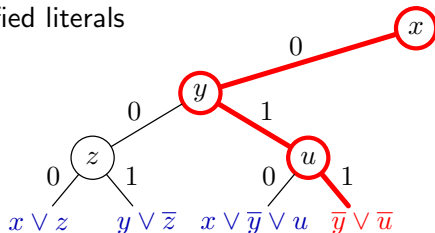- falsified literals
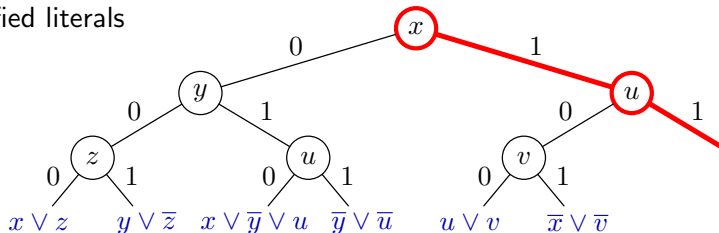
A DPLL Toy Example

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Visualize execution of DPLL algorithm as search tree

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DPLL, CDCL, and Resolution
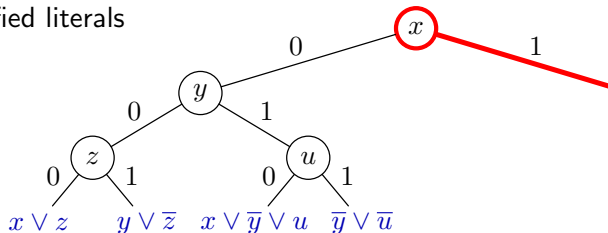Algebraic and Semi-algebraic Approaches
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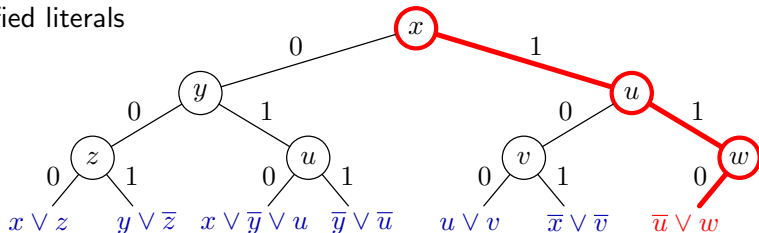
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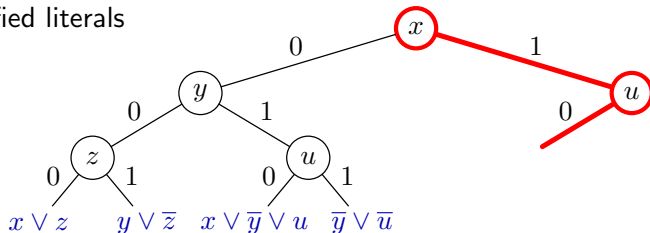
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DPLL, CDCL, and Resolution
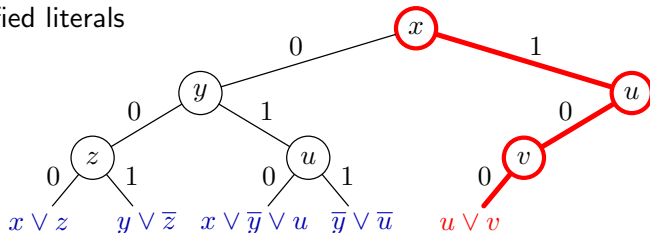Algebraic and Semi-algebraic Approaches
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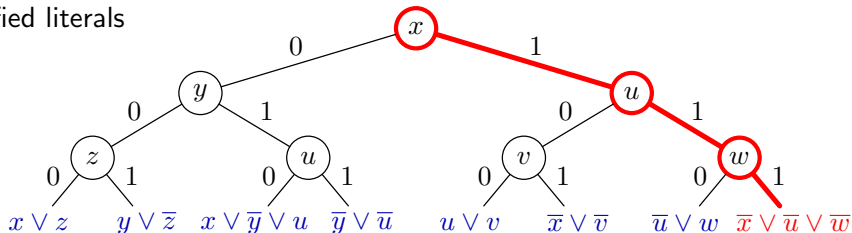
State-of-the-Art SAT Solving in One Slide

High-level description of modern conflict-driven clause learning (CDCL) SAT solving (as pioneered in [BS97, MS99, MMZ$^+$01]):

- Try to build satisfying assignment for formula (branching or decision heuristic crucial)

- When partial assignment violates formula, compute explanation for conflict and add to formula as new clause (clause learning)

- Every once in a while, restart from beginning (but save computed info)

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Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$

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Decision

Free choice to assign value to variable

Notation $p \overset{\mathsf{d}}{=} 0$

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$$p \stackrel{\mathsf{d}}{=} 0$$

Decision

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$$\boxed{p \stackrel{\mathsf{d}}{=} 0}$$

Decision
Free choice to assign value to variable
Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation
Forced choice to avoid falsifying clause
Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$
Notation $u \stackrel{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

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$$\boxed{p \overset{\mathsf{d}}{=} 0}$$
$$\overline{u \overset{p \vee \overline{u}}{=} 0}$$

Decision
Free choice to assign value to variable
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$p \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{p \vee \overline{u}}{=} 0$

$q \stackrel{\mathsf{d}}{=} 0$

Decision
Free choice to assign value to variable
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Always propagate if possible, otherwise decide
Add to assignment trail
Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$$p \overset{\mathsf{d}}{=} 0$$

$$u \overset{p \vee \overline{u}}{=} 0$$

$$q \overset{\mathsf{d}}{=} 0$$

$$r \overset{q \vee r}{=} 1$$

Decision
Free choice to assign value to variable
Notation $p \overset{\mathsf{d}}{=} 0$

Unit propagation
Forced choice to avoid falsifying clause
Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$
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$$p \stackrel{\mathsf{d}}{=} 0$$

$$u \stackrel{p \vee \overline{u}}{=} 0$$

$$q \stackrel{\mathsf{d}}{=} 0$$

$$r \stackrel{q \vee r}{=} 1$$

$$w \stackrel{\overline{r} \vee w}{=} 1$$

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$$p \stackrel{\mathsf{d}}{=} 0$$

$$u \stackrel{p \vee \overline{u}}{=} 0$$

$$q \stackrel{\mathsf{d}}{=} 0$$

$$r \stackrel{q \vee r}{=} 1$$

$$w \stackrel{\overline{r} \vee w}{=} 1$$

$$x \stackrel{\mathsf{d}}{=} 0$$

Decision
Free choice to assign value to variable
Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation
Forced choice to avoid falsifying clause
Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$
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$$p \stackrel{\mathsf{d}}{=} 0$$

$$u \stackrel{p \vee \overline{u}}{=} 0$$

$$q \stackrel{\mathsf{d}}{=} 0$$

$$r \stackrel{q \vee r}{=} 1$$

$$w \stackrel{\overline{r} \vee w}{=} 1$$

$$x \stackrel{\mathsf{d}}{=} 0$$

$$y \stackrel{u \vee x \vee y}{=} 1$$

Decision
Free choice to assign value to variable
Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation
Forced choice to avoid falsifying clause
Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$
Notation $u \stackrel{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

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Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$

$p \overset{\mathsf{d}}{=} 0$

$u \overset{p \lor \overline{u}}{=} 0$

$q \overset{\mathsf{d}}{=} 0$

$r \overset{q \lor r}{=} 1$

$w \overset{\overline{r} \lor w}{=} 1$

$x \overset{\mathsf{d}}{=} 0$

$y \overset{u \lor x \lor y}{=} 1$

$z \overset{x \lor \overline{y} \lor z}{=} 1$

Decision
Free choice to assign value to variable
Notation $p \overset{\mathsf{d}}{=} 0$

Unit propagation
Forced choice to avoid falsifying clause
Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$
Notation $u \overset{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

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$p \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{p \vee \overline{u}}{=} 0$

$q \stackrel{\mathsf{d}}{=} 0$

$r \stackrel{q \vee r}{=} 1$

$w \stackrel{\overline{r} \vee w}{=} 1$

$x \stackrel{\mathsf{d}}{=} 0$

$y \stackrel{u \vee x \vee y}{=} 1$

$z \stackrel{x \vee \overline{y} \vee z}{=} 1$

$\overline{y} \vee \overline{z}$

$\perp$

Decision
Free choice to assign value to variable
Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation
Forced choice to avoid falsifying clause
Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$
Notation $u \stackrel{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide
Add to assignment trail
Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$p \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{p \vee \overline{u}}{=} 0$

decision level 1

Decision
Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

$q \stackrel{\mathsf{d}}{=} 0$

$r \stackrel{q \vee r}{=} 1$

$w \stackrel{\overline{r} \vee w}{=} 1$

decision level 2

Unit propagation
Forced choice to avoid falsifying clause
Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

$x \stackrel{\mathsf{d}}{=} 0$

$y \stackrel{u \vee x \vee y}{=} 1$

$z \stackrel{x \vee \overline{y} \vee z}{=} 1$

$\overline{y} \vee \overline{z}$
$\bot$

decision level 3

Always propagate if possible, otherwise decide
Add to assignment trail
Continue until satisfying assignment or conflict

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Conflict Analysis

Time to analyse this conflict and learn from it!

$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$

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$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

Could backtrack by erasing conflict level & flipping last decision

$p \overset{\text{d}}{=} 0$

$u \overset{p \vee \overline{u}}{=} 0$
} decision level 1

$q \overset{\text{d}}{=} 0$

$r \overset{q \vee r}{=} 1$

$w \overset{\overline{r} \vee w}{=} 1$
} decision level 2

$x \overset{\text{d}}{=} 0$

$y \overset{u \vee x \vee y}{=} 1$

$z \overset{x \vee \overline{y} \vee z}{=} 1$

$\overset{\overline{y} \vee \overline{z}}{\perp}$
} decision level 3

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Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

decision level 1

decision level 2

decision level 3

Could backtrack by erasing conflict level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

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$$p \stackrel{\mathsf{d}}{=} 0$$

$$u \stackrel{p \vee \overline{u}}{=} 0$$

$$q \stackrel{\mathsf{d}}{=} 0$$

$$r \stackrel{q \vee r}{=} 1$$

$$w \stackrel{\overline{r} \vee w}{=} 1$$

$$x \stackrel{\mathsf{d}}{=} 0$$

$$y \stackrel{u \vee x \vee y}{=} 1$$

$$z \stackrel{x \vee \overline{y} \vee z}{=} 1 \longrightarrow \boxed{x \vee \overline{y}}$$

$$\stackrel{\overline{y} \vee \overline{z}}{\perp}$$

Could backtrack by erasing conflict level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

Case analysis over z for last two clauses:

- $x \vee \overline{y} \vee z$ wants $z = 1$
- $\overline{y} \vee \overline{z}$ wants $z = 0$
- Merge clauses & remove z — must satisfy $x \vee \overline{y}$

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Repeat until UIP clause with only 1 variable at conflict level after last decision — learn and backjump

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Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

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Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

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Now UIP literal guaranteed to flip (assert) — but this is a propagation, not a decision

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Then continue as before. . .

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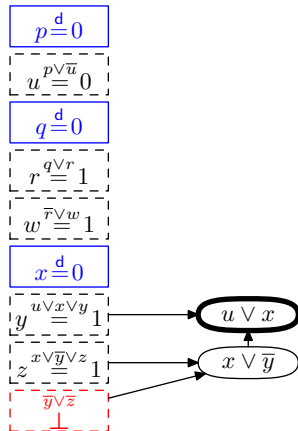
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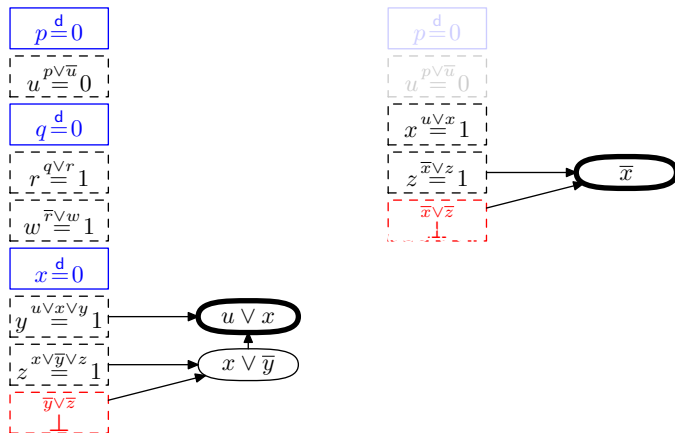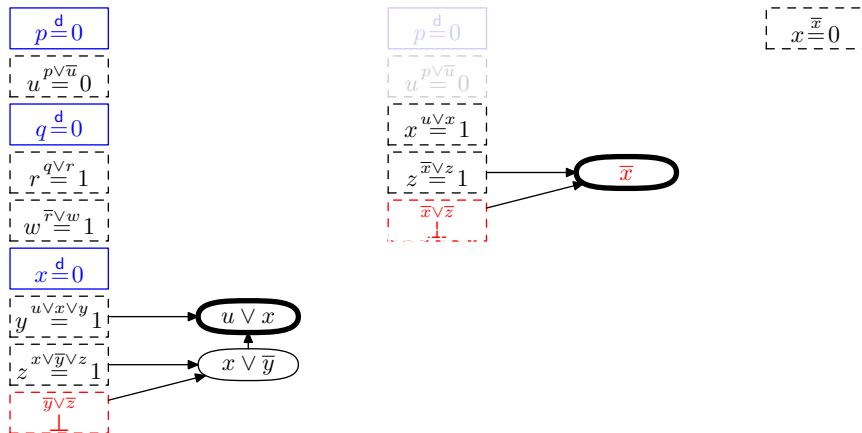
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SAT Solver Analysis and the Resolution Proof System

How to make rigorous analysis of SAT solver performance?

Many intricate, hard-to-understand heuristics

So focus instead on underlying method of reasoning

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SAT Solver Analysis and the Resolution Proof System

How to make rigorous analysis of SAT solver performance?
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So focus instead on underlying method of reasoning

Resolution proof system [Bla37, Rob65]

- Start with clauses of CNF formula (axioms)
- Derive new clauses by resolution rule

$$\frac{C_1 \vee x \qquad C_2 \vee \overline{x}}{C_1 \vee C_2}$$

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Resolution Proofs by Contradction

Resolution rule:

$$\frac{C_1 \vee x \qquad C_2 \vee \overline{x}}{C_1 \vee C_2}$$

Observation

If F is a satisfiable CNF formula and D is derived from clauses $D_1, D_2 \in F$ by the resolution rule, then $F \wedge D$ is satisfiable.

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Observation

If F is a satisfiable CNF formula and D is derived from clauses $D_1, D_2 \in F$ by the resolution rule, then $F \wedge D$ is satisfiable.

So can prove F unsatisfiable by deriving the unsatisfiable empty clause (denoted $\perp$) from F by resolution

Such proof by contradiction also called resolution refutation

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DPLL and Resolution Proofs

A DPLL execution is essentially a resolution proof

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and apply resolution rule $\frac{C_1 \vee x \quad C_2 \vee \overline{x}}{C_1 \vee C_2}$ bottom-up

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DPLL Running Time and Tree-Like Resolution Proof Size

- Can extract resolution proof from any DPLL execution

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- Hence, lower bounds on tree-like proof size in resolution $\Rightarrow$
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  lower bounds on DPLL running time

- Conflict-driven clause learning adds "shortcut edges" in tree, but still yields
  resolution proof

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CDCL and Resolution Proofs

Obtain resolution proof. . .

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CDCL and Resolution Proofs

Obtain resolution proof from our example CDCL execution...

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Obtain resolution proof from our example CDCL execution by stringing together conflict analyses:

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- Lower (and upper) bounds for different methods of reasoning about propositional logic formulas studied in proof complexity

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(*) Except for some preprocessing techniques, which is an important omission, but this gets complicated and we don't have time to go into details. . .

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Current State of Affairs in SAT Solving

- State-of-the-art CDCL solvers often perform amazingly well
  ("SAT is easy in practice")

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Resolution Proof System

Current State of Affairs in SAT Solving

- State-of-the-art CDCL solvers often perform amazingly well
  ("SAT is easy in practice")

- Very poor theoretical understanding:
    - Why do heuristics work?
    - Why are applied instances easy?

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- Paradox: resolution quite weak proof system; many strong proof complexity lower
  bounds for (seemingly) "obvious" formulas

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Examples of Hard Formulas For Resolution (1/3)

Pigeonhole principle (PHP) formulas [Hak85]

"$n + 1$ pigeons don't fit into n holes"

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"$n + 1$ pigeons don't fit into n holes"

Variables $p_{i,j} = $ "pigeon $i \to$ hole j"; $1 \leq i \leq n + 1$; $1 \leq j \leq n$

$$p_{i,1} \vee p_{i,2} \vee \cdots \vee p_{i,n} \qquad \text{every pigeon } i \text{ gets a hole}$$
$$\overline{p}_{i,j} \vee \overline{p}_{i',j} \qquad \text{no hole } j \text{ gets two pigeons } i \neq i'$$

Can also add "functionality" and "onto" axioms

$$\overline{p}_{i,j} \vee \overline{p}_{i,j'} \qquad \text{no pigeon } i \text{ gets two holes } j \neq j'$$
$$p_{1,j} \vee p_{2,j} \vee \cdots \vee p_{n+1,j} \qquad \text{every hole } j \text{ gets a pigeon}$$

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$\quad p_{i,1} \vee p_{i,2} \vee \cdots \vee p_{i,n}$ every pigeon i gets a hole

$\quad \overline{p}_{i,j} \vee \overline{p}_{i',j}$ no hole j gets two pigeons $i \neq i'$

Can also add "functionality" and "onto" axioms

$\quad \overline{p}_{i,j} \vee \overline{p}_{i,j'}$ no pigeon i gets two holes $j \neq j'$

$\quad p_{1,j} \vee p_{2,j} \vee \cdots \vee p_{n+1,j}$ every hole j gets a pigeon

Even onto functional PHP hard — **"resolution cannot count"**

Resolution proof requires $\exp(\Omega(n)) = \exp(\Omega(\sqrt[3]{N}))$ clauses
(measured in terms of formula size N)

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Examples of Hard Formulas For Resolution (2/3)

Tseitin formulas [Urq87]
"Sum of degrees of vertices in graph is even"

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Examples of Hard Formulas For Resolution (2/3)

Tseitin formulas [Urq87]
"Sum of degrees of vertices in graph is even"

Variables = edges (in undirected graph of bounded degree)

- Label every vertex $0/1$ so that sum of labels odd
- Write CNF requiring parity of # true incident edges = label

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$$
\begin{aligned}
& (u \vee x) && \wedge (y \vee \overline{z}) \\
& \wedge (\overline{u} \vee \overline{x}) && \wedge (\overline{y} \vee z) \\
& \wedge (w \vee x \vee y) && \wedge (u \vee w \vee z) \\
& \wedge (w \vee \overline{x} \vee \overline{y}) && \wedge (u \vee \overline{w} \vee \overline{z}) \\
& \wedge (\overline{w} \vee x \vee \overline{y}) && \wedge (\overline{u} \vee w \vee \overline{z}) \\
& \wedge (\overline{w} \vee \overline{x} \vee y) && \wedge (\overline{u} \vee \overline{w} \vee z)
\end{aligned}
$$

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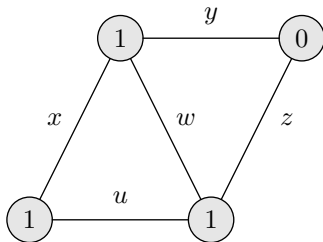Resolution Proof System

Examples of Hard Formulas For Resolution (2/3)

Tseitin formulas [Urq87]
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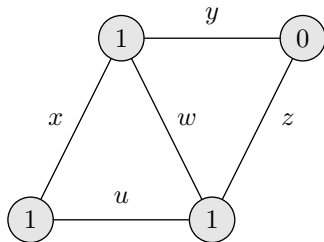- Write CNF requiring parity of $\#$ true incident edges = label

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$$\wedge (\overline{w} \vee x \vee \overline{y}) \qquad \wedge (\overline{u} \vee w \vee \overline{z})$$
$$\wedge (\overline{w} \vee \overline{x} \vee y) \qquad \wedge (\overline{u} \vee \overline{w} \vee z)$$

Requires proof size $\exp(\Omega(N))$ on well-connected so-called expander graphs —
"resolution cannot count $\bmod 2$**"**

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Examples of Hard Formulas for Resolution (3/3)

Random k-CNF formulas [CS88]
Δn randomly sampled k-clauses over n variables
($\Delta \gtrsim 4.5$ sufficient to get unsatisfiable 3-CNF almost surely)

Again lower bound $\exp(\Omega(N))$

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Again lower bound $\exp(\Omega(N))$

And more...

- COLOURING [BCMM05]
- Zero-one designs [Spe10, VS10, MN14]
- Et cetera... (See, e.g., [BN21] for overview)

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And more...

- Colouring [BCMM05]
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- Et cetera... (See, e.g., [BN21] for overview)

But no such strong lower bounds known for Clique!

- Refuting existence of k-clique should require proof size $n^{\Omega(k)}$
- Only known for restricted so-called regular resolution [ABdR$^+$21]

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Algebraic and Semi-algebraic Approaches   Gröbner Bases and Polynomial Calculus
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SAT as System of Polynomial Equations

- Given CNF formula $F = \bigwedge_{i=1}^{m} C_i$

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- Given CNF formula $F = \bigwedge_{i=1}^{m} C_i$

- Translate clauses

$$C = \bigvee_{i \in \mathcal{P}} x_i \vee \bigvee_{j \in \mathcal{N}} \overline{x}_j$$

to polynomial equations

$$\prod_{i \in \mathcal{P}} (1 - x_i) \cdot \prod_{j \in \mathcal{N}} x_j = 0$$

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to polynomial equations

$$\prod_{i \in \mathcal{P}} (1 - x_i) \cdot \prod_{j \in \mathcal{N}} x_j = 0$$

- Add Boolean axioms

$$x_j^2 - x_j = 0$$

for all variables

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Hilbert's Nullstellensatz

Consider any system of polynomial equations

$$p_1(x_1, \ldots, x_n) = 0 \qquad\qquad x_1^2 - x_1 = 0$$
$$p_2(x_1, \ldots, x_n) = 0 \qquad\qquad x_2^2 - x_2 = 0$$
$$\vdots \qquad\qquad\qquad\qquad \vdots$$
$$p_m(x_1, \ldots, x_n) = 0 \qquad\qquad x_n^2 - x_n = 0$$

in polynomial ring over field $\mathbb{F}$

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$$
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p_2(x_1, \ldots, x_n) &= 0 & x_2^2 - x_2 &= 0 \\
&\vdots & &\vdots \\
p_m(x_1, \ldots, x_n) &= 0 & x_n^2 - x_n &= 0
\end{aligned}
$$

in polynomial ring over field $\mathbb{F}$

Hilbert's Nullstellensatz

System infeasible $\Leftrightarrow$ exist $q_i, r_j \in \mathbb{F}[x_1, \ldots, x_n]$ such that

$$
\sum_{i=1}^{m} q_i(x_1, \ldots, x_n) \cdot p_i(x_1, \ldots, x_n) + \sum_{j=1}^{n} r_j(x_1, \ldots, x_n) \cdot (x_j^2 - x_j) = 1
$$

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Nullstellensatz Proof System [BIK+94]

Nullstellensatz refutation of

$$p_i(x_1, \ldots, x_n) = 0 \qquad\qquad i \in [m]$$
$$x_j^2 - x_j = 0 \qquad\qquad j \in [n]$$

is (syntactic) equality

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Complexity measures of refutations:

- Size: number of monomials (when all polynomials expanded out)
- Degree: highest total degree of any polynomial

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Nullstellensatz Example (Not Expanded out)

$$(x \lor z) \land (y \lor \neg z) \land (x \lor \neg y \lor u) \land (\neg y \lor \neg u)$$
$$\land (u \lor v) \land (\neg x \lor \neg v) \land (\neg u \lor w) \land (\neg x \lor \neg u \lor \neg w)$$

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$$(x \vee z) \wedge (y \vee \neg z) \wedge (x \vee \neg y \vee u) \wedge (\neg y \vee \neg u)$$
$$\wedge (u \vee v) \wedge (\neg x \vee \neg v) \wedge (\neg u \vee w) \wedge (\neg x \vee \neg u \vee \neg w)$$

$(1-x)(1-z)$

$(1-y)z$

$(1-x)y(1-u)$

yu

$(1-u)(1-v)$

xv

$u(1-w)$

xuw

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$$\wedge (u \vee v) \wedge (\neg x \vee \neg v) \wedge (\neg u \vee w) \wedge (\neg x \vee \neg u \vee \neg w)$$

$$(1 - y) \cdot (1 - x)(1 - z)$$
$$+ \quad (1 - x) \cdot (1 - y)z$$
$$+ \qquad 1 \cdot (1 - x)y(1 - u)$$
$$+ \quad (1 - x) \cdot yu$$
$$+ \qquad x \cdot (1 - u)(1 - v)$$
$$+ \quad (1 - u) \cdot xv$$
$$+ \qquad x \cdot u(1 - w)$$
$$+ \qquad 1 \cdot xuw$$

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$$\wedge (u \vee v) \wedge (\neg x \vee \neg v) \wedge (\neg u \vee w) \wedge (\neg x \vee \neg u \vee \neg w)$$

$$
\begin{aligned}
& (1-y) \cdot (1-x)(1-z) \\
&+ (1-x) \cdot (1-y)z \\
&+ 1 \cdot (1-x)y(1-u) \\
&+ (1-x) \cdot yu \\
&+ x \cdot (1-u)(1-v) \\
&+ (1-u) \cdot xv \\
&+ x \cdot u(1-w) \\
&+ 1 \cdot xuw \\
&= 1
\end{aligned}
$$

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$$\wedge (u \vee v) \wedge (\neg x \vee \neg v) \wedge (\neg u \vee w) \wedge (\neg x \vee \neg u \vee \neg w)$$

$$
\begin{aligned}
& (1-y) \cdot (1-x)(1-z) \\
+\ & (1-x) \cdot (1-y)z \\
+\ & \qquad 1 \cdot (1-x)y(1-u) \\
+\ & (1-x) \cdot yu \\
+\ & \qquad x \cdot (1-u)(1-v) \\
+\ & (1-u) \cdot xv \\
+\ & \qquad x \cdot u(1-w) \\
+\ & \qquad 1 \cdot xuw \\
=\ & \qquad 1
\end{aligned}
$$

Size 27
Degree 3
(No use of Boolean axioms)

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Nullstellensatz Proof Search

- Solve linear system of equations with coefficients of polynomials q_i, r_j as unknowns

- Used successfully to solve, e.g., graph colouring problems
  [DLMM08, DLMO09, DLMM11]

- Running time grows exponentially with degree, though high-degree refutations can be very small [BCIP02, dRMNR21]

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Dual Variables

- Annoying problem: $x_1 \vee x_2 \vee x_3$ translates to polynomial

$$(1 - x_1)(1 - x_2)(1 - x_3) = 1 - x_1 - x_2 - x_3 + x_1 x_2 + x_1 x_3 + x_2 x_3 - x_1 x_2 x_3$$

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Dual Variables

- Annoying problem: $x_1 \lor x_2 \lor x_3$ translates to polynomial

$$(1 - x_1)(1 - x_2)(1 - x_3) = 1 - x_1 - x_2 - x_3 + x_1 x_2 + x_1 x_3 + x_2 x_3 - x_1 x_2 x_3$$

- More generally, exponential blow-up in # positive literals

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- More generally, exponential blow-up in # positive literals
- Fix: introduce dual variables x_i' and axioms $x_i + x_i' - 1 = 0$

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- More generally, exponential blow-up in # positive literals

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- Translate $C = \bigvee_{i \in \mathcal{P}} x_i \vee \bigvee_{j \in \mathcal{N}} \overline{x}_j$ to polynomial equations

$$\prod_{i \in \mathcal{P}} x_i' \cdot \prod_{j \in \mathcal{N}} x_j = 0$$

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$$\prod_{i \in \mathcal{P}} x_i' \cdot \prod_{j \in \mathcal{N}} x_j = 0$$

- Doesn't affect degree (obviously), but can decrease size exponentially [dRLNS21] (also for other algebraic proof systems)

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Dynamic Construction of Nullstellensatz Certificates

Nullstellensatz again

Infeasibility of

$$p_i(x_1, \ldots, x_n) = 0 \qquad i \in [m]$$
$$x_j^2 - x_j = 0 \qquad j \in [n]$$
$$x_j + x_j' - 1 = 0 \qquad j \in [n]$$

$$\Updownarrow$$

1 lies in polynomial ideal $\mathcal{I}$ generated by these polynomials

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Nullstellensatz again

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$$p_i(x_1, \ldots, x_n) = 0 \qquad\qquad i \in [m]$$

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$$x_j + x_j' - 1 = 0 \qquad\qquad j \in [n]$$

$$\Updownarrow$$

1 lies in polynomial ideal $\mathcal{I}$ generated by these polynomials

- Ideal $\mathcal{I}$:
  1. $p, q \in \mathcal{I} \Rightarrow p + q \in \mathcal{I}$
  2. $p \in \mathcal{I} \Rightarrow r \cdot q \in \mathcal{I}$ for any r
- Compute polynomials in this ideal $\mathcal{I}$ step by step
- Use "multivariate division" to check whether 1 lies in ideal or not

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Gröbner Bases: Admissible Orderings and Leading Terms

Admissible ordering $\preceq$ on monomials m, m', t:

1. $m \preceq m' \Rightarrow t \cdot m \preceq t \cdot m'$
2. $m \preceq t \cdot m$

Examples:

- Lexicographic
- Degree-lexicographic

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1. $m \preceq m' \Rightarrow t \cdot m \preceq t \cdot m'$
2. $m \preceq t \cdot m$

Examples:

- Lexicographic
- Degree-lexicographic

Can write $p = \mathrm{lt}(p) + p'$ for $\mathrm{lt}(p)$ leading term (largest w.r.t. $\preceq$)

If $\mathrm{lt}(p) = t \cdot \mathrm{lt}(q)$, can reduce $p \bmod q$ by computing $p - t \cdot q$

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Gröbner Bases: Admissible Orderings and Leading Terms

Admissible ordering $\preceq$ on monomials m, m', t:

1. $m \preceq m' \Rightarrow t \cdot m \preceq t \cdot m'$
2. $m \preceq t \cdot m$

Examples:

- Lexicographic
- Degree-lexicographic

Can write $p = \text{lt}(p) + p'$ for $\text{lt}(p)$ leading term (largest w.r.t. $\preceq$)

If $\text{lt}(p) = t \cdot \text{lt}(q)$, can reduce $p \bmod q$ by computing $p - t \cdot q$

"Multivariate division": Reduce p modulo all q in set of polynomials $\mathcal{G}$ until no further reductions possible

$\mathcal{G}$ is a Gröbner basis if final result uniquely determined

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Gröbner Bases: Buchberger's Algorithm

Buchberger's algorithm for computing Gröbner bases (**very** rough)

1. Let $\mathcal{G} :=$ all axioms
2. Pick unprocessed pair $p, q \in \mathcal{G}$ or terminate if none exists
3. Compute $p' = t_p \cdot p$ and $q' = t_q \cdot q$ to make leading terms cancel
4. Set $S := p' - q'$; reduce $S \bmod \mathcal{G}$ with multivariate division; add result to $\mathcal{G}$ if non-zero
5. Go to 2

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Facts:

- Buchberger's algorithm computes Gröbner basis
- At termination, $1 \in \mathcal{G} \Leftrightarrow$ polynomial equations infeasible

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Polynomial Calculus [CEI96, ABRW02]

- Compute polynomials in ideal $\mathcal{I}$ generated by p_i, $x_j^2 - x_j$, and $x_j + x_j' - 1$ step by step:
  - $p_i \in \mathcal{I}$, $x_j^2 - x_j \in \mathcal{I}$, and $x_j + x_j' - 1 \in \mathcal{I}$ (axioms)
  - If $p, q \in \mathcal{I}$, then $\alpha p + \beta q \in \mathcal{I}$ for any $\alpha, \beta \in \mathbb{F}$ (linear combination)
  - If $p \in \mathcal{I}$, then $m \cdot p \in \mathcal{I}$ for any monomial $m = \prod_j x_j$ (multiplication)

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- A refutation is a derivation ending with the polynomial 1

- Complexity measures:
  - Size: total number of monomials in all polynomials in derivation expanded out
  - Degree: highest total degree of any polynomial

- Polynomial calculus (much) stronger than Nullstellensatz w.r.t. both size and degree

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Polynomial Calculus Can Simulate Resolution

Polynomial calculus can always simulate resolution proofs efficiently step by step

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Example: Resolution step

$$\frac{x \vee \overline{y} \vee z \qquad \overline{y} \vee \overline{z}}{x \vee \overline{y}}$$

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Example: Resolution step

$$\frac{x \vee \overline{y} \vee z \qquad \overline{y} \vee \overline{z}}{x \vee \overline{y}}$$

simulated by polynomial calculus derivation

$$\frac{yz \qquad \dfrac{z + z' - 1}{x'yz \qquad x'yz + x'yz' - x'y}}{\dfrac{x'yz' \qquad -x'yz' + x'y}{x'y}}$$

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Polynomial Calculus is Strictly Stronger than Resolution

Polynomial calculus can be exponentially stronger than resolution

For instance:

- Tseitin formulas on expander graphs if $\mathbb{F} = \mathrm{GF}(2)$
- Onto functional pigeonhole principle over any field [Rii93]

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But other versions of pigeonhole principle formulas remain hard:

- "vanilla" PHP [Raz98, AR03]
- onto PHP [AR03]
- functional PHP [MN15]

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Other hard formulas:

- Tseitin-like formulas for counting $\mod p$ if $p \neq$ field characteristic [BGIP01]
- Random k-CNF formulas
  - all characteristics except 2 [BI99]
  - all characteristics [AR03]

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COLOURING and CLIQUE for Polynomial Calculus

COLOURING

- Exponential worst-case lower bounds in [LN17]
- Exponential average-case lower bounds in [CdRN+23]

CLIQUE

Essentially nothing known!

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What About Algebraic SAT Solvers?

- Excitement about Gröbner basis approach after [CEI96], but promise of performance improvement failed to deliver

- Meanwhile: the CDCL revolution in late 1990s...

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- Some current SAT solvers do Gaussian elimination, but this is only very limited form of polynomial calculus

- Is it harder to build good algebraic SAT solvers, or is it just that too little work has been done (or both)?

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- Use dual variables! [KBBN22]

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Gröbner bases: Some Problems and Questions

1. Buchberger not a great SAT solving algorithm
   Slow and memory-intensive, and computes too much info
   Possible to use conflict-driven paradigm?!

2. Dual variables increase reasoning power exponentially [dRLNS21]
   But are immediately eliminated by multivariate division
   Possible to design dual-variable-aware Buchberger?!

3. Analysis of polynomial calculus uses degree-lexicographic ordering
   In computational algebra, many other orderings used
   Prove proof complexity separation results for different orderings?

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SAT as System of 0–1 Integer Linear Inequalities

- Given CNF formula $F = \bigwedge_{i=1}^{m} C_i$

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SAT as System of 0–1 Integer Linear Inequalities

- Given CNF formula $F = \bigwedge_{i=1}^{m} C_i$
- Translate clauses

$$C = \bigvee_{i \in \mathcal{P}} x_i \vee \bigvee_{j \in \mathcal{N}} \overline{x}_j$$

to 0-1 integer linear inequalities

$$\sum_{i \in \mathcal{P}} x_i + \sum_{j \in \mathcal{N}} (1 - x_j) \geq 1$$

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  to 0-1 integer linear inequalities

$$\sum_{i \in \mathcal{P}} x_i + \sum_{j \in \mathcal{N}} (1 - x_j) \geq 1$$

- Add variable axioms

$$x_j \geq 0$$
$$-x_j \geq -1$$

  for all variables

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Cutting Planes Proof System [CCT87]

Cutting planes introduced in [CCT87] to model integer linear programming algorithm in [Gom63, Chv73]

Can be applied to any system of 0-1 integer linear inequalities

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Cutting planes derivation rules

$$\text{Multiplication } \frac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA} \quad c \in \mathbb{N}^+$$

$$\text{Addition } \frac{\sum a_i x_i \geq A \qquad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$$

$$\text{Division } \frac{\sum a_i x_i \geq A}{\sum \lceil a_i/c \rceil x_i \geq \lceil A/c \rceil} \quad c \in \mathbb{N}^+$$

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Cutting Planes Derivations and Refutations

- A cutting planes derivation is a sequence of 0-1 integer linear inequalities derived using
  - Axioms (clauses and variable bounds)
  - Multiplication $\sum a_i x_i \geq A \Rightarrow \sum c a_i x_i \geq cA$
  - Addition $\sum a_i x_i \geq A, \sum b_i x_i \geq B \Rightarrow \sum (a_i + b_i) x_i \geq A + B$
  - Division $\sum a_i x_i \geq A \Rightarrow \sum \lceil a_i / c \rceil x_i \geq \lceil A / c \rceil$

- A refutation ends with the inequality $0 \geq 1$

- Complexity measures:
  - Length: # inequalities
  - Size: Count also bit size of representing all coefficients

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Cutting Planes vs. Resolution

- Cutting planes can simulate resolution reasoning efficiently and can be exponentially stronger (e.g., for PHP, just count and argue that #pigeons > #holes)

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Cutting Planes vs. Resolution

- Cutting planes can simulate resolution reasoning efficiently and can be exponentially stronger (e.g., for PHP, just count and argue that #pigeons > #holes)

- And 0-1 linear inequalities are similar to but much more concise than CNF

Compare
$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 3$$

and

$$(x_1 \lor x_2 \lor x_3 \lor x_4) \land (x_1 \lor x_2 \lor x_3 \lor x_5) \land (x_1 \lor x_2 \lor x_3 \lor x_6)$$
$$\land (x_1 \lor x_2 \lor x_4 \lor x_5) \land (x_1 \lor x_2 \lor x_4 \lor x_6) \land (x_1 \lor x_2 \lor x_5 \lor x_6)$$
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$$\land (x_1 \lor x_4 \lor x_5 \lor x_6) \land (x_2 \lor x_3 \lor x_4 \lor x_5) \land (x_2 \lor x_3 \lor x_4 \lor x_6)$$
$$\land (x_2 \lor x_3 \lor x_5 \lor x_6) \land (x_2 \lor x_4 \lor x_5 \lor x_6) \land (x_3 \lor x_4 \lor x_5 \lor x_6)$$

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Hard Formulas for Cutting Planes

Clique-colouring formulas [Pud97]
"A graph with an m-clique is not $(m-1)$-colourable"

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"A graph with an m-clique is not $(m-1)$-colourable"

Variables

- $p_{i,j}$ indicators of the edges in graph; $1 \leq i < j \leq n$
- $q_{k,i}$ identify members of m-clique; $1 \leq k \leq m$, $1 \leq i \leq n$
- $r_{i,\ell}$ specify colouring of vertices; $1 \leq \ell \leq m-1$, $1 \leq i \leq n$

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$$q_{k,1} \vee q_{k,2} \vee \cdots \vee q_{k,n} \qquad \text{some vertex is the } k\text{th member of clique}$$

$$\overline{q}_{k,i} \vee \overline{q}_{k',i} \qquad \text{clique members are uniquely defined } (k \neq k')$$

$$p_{i,j} \vee \overline{q}_{k,i} \vee \overline{q}_{k',j} \qquad \text{clique members are connected by edges}$$

$$r_{i,1} \vee r_{i,2} \vee \cdots \vee r_{i,m-1} \qquad \text{every vertex } i \text{ has a colour}$$

$$\overline{p}_{i,j} \vee \overline{r}_{i,\ell} \vee \overline{r}_{j,\ell} \qquad \text{neighbours have distinct colours}$$

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More Hard Formulas for Cutting Planes?

Lower bound for clique-colouring formulas uses interpolation and circuit complexity

- From small cutting planes proof, build small circuit of special type that can decide whether graph has clique
- Prove separately that no such small circuits can exist
- Hence, no small cutting planes proofs can exist either

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Cutting planes not well understood at all
Clear need for development of new analysis methods
Some exciting contributions in [HP17, FPPR22, GGKS20, Sok23]

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Nothing known for COLOURING or CLIQUE
Surprisingly, Tseitin formulas are at most quasi-polynomially hard for cutting planes [DT20]!

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SAT Solvers Based on Cutting Planes?

So-called pseudo-Boolean (PB) solvers using (subset of) cutting planes reasoning developed in, e.g., [CK05, SS06, LP10, EN18]

Perhaps counter-intuitively, hard to make competitive with CDCL

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Challenge 1: Conjunctive normal form

- Pseudo-Boolean solvers terrible for CNF input
- Solvers can rewrite CNF to more helpful 0-1 linear inequalities [BLLM14, EN20], but this doesn't work so well in practice
- Better to encode problem with 0-1 inequalities from the start

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Challenge 2: Increased degrees of freedom(!?)

- Cutting planes much smarter method of reasoning
- But this makes it trickier to design smart search algorithms

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Is it truly harder to build good pseudo-Boolean solvers?

Or has just so much more work has been put into CDCL solvers?

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Division Versus Saturation

Use negated literals as needed to get all a_i, A positive

Boolean derivation rules for 0–1 integer linear inequalities

$$\text{Division } \frac{\sum a_i \ell_i \geq A}{\sum \lceil a_i/c \rceil \ell_i \geq \lceil A/c \rceil} \quad c \in \mathbb{N}^+$$

$$\text{Saturation } \frac{\sum a_i \ell_i \geq A}{\sum \min\{a_i, A\} \cdot \ell_i \geq A}$$

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$$\text{Saturation } \frac{\sum a_i \ell_i \geq A}{\sum \min\{a_i, A\} \cdot \ell_i \geq A}$$

- Complexity literature of cutting planes uses division [CCT87]
- Pseudo-Boolean solvers instead adopted saturation [CK05, LP10]
- Open how the two variants compare, but clear that division can sometimes be better in theory [GNY19]

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Division Versus Saturation

Use negated literals as needed to get all a_i, A positive

Boolean derivation rules for 0–1 integer linear inequalities

$$\text{Division} \quad \frac{\sum a_i \ell_i \geq A}{\sum \lceil a_i/c \rceil \ell_i \geq \lceil A/c \rceil} \quad c \in \mathbb{N}^+$$

$$\text{Saturation} \quad \frac{\sum a_i \ell_i \geq A}{\sum \min\{a_i, A\} \cdot \ell_i \geq A}$$

- Complexity literature of cutting planes uses division [CCT87]
- Pseudo-Boolean solvers instead adopted saturation [CK05, LP10]
- Open how the two variants compare, but clear that division can sometimes be better in theory [GNY19]
- ... And most often also in practice [EN18], though not always [LBD$^+$20]

DPLL, CDCL, and Resolution   **Sherali-Adams and Sums of Squares**
Algebraic and Semi-algebraic Approaches   Stabbing Planes
Some More Advanced Proof Systems We Might Not Have Time for   Extended Resolution

Sherali–Adams (SA) and Sums of Squares (SoS)

Refutation of $p_i \in \mathbb{R}[x_1, \ldots, x_n]$, $i \in [m]$, and $x_j^2 - x_j$, $j \in [n]$

Nullstellensatz

$$\sum_{i=1}^{m} q_i \cdot p_i + \sum_{j=1}^{n} r_j \cdot (x_j^2 - x_j) = 1$$

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$$\sum_{i=1}^{m} q_i \cdot p_i + \sum_{j=1}^{n} r_j \cdot (x_j^2 - x_j) = -1$$

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Sherali–Adams (SA) $(\alpha_k \in \mathbb{R}^+)$

$$\sum_{i=1}^m q_i \cdot p_i + \sum_{j=1}^n r_j \cdot (x_j^2 - x_j) + \sum_{k=1}^t \alpha_k \prod_{i \in \mathcal{P}_t} (1 - x_i) \cdot \prod_{j \in \mathcal{N}_t} x_j = -1$$

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Sums of squares (SoS) $(s_k \in \mathbb{R}[x_1, \ldots, x_n])$

$$\sum_{i=1}^{m} q_i \cdot p_i + \sum_{j=1}^{n} r_j \cdot (x_j^2 - x_j) + \sum_{k=1}^{s} s_k^2 = -1$$

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Sherali–Adams, Sums of Squares, and Relations to Other Proof Systems

Sherali–Adams models linear programming (LP) hierarchies

Sums of squares models semidefinite programming (SDP) hierarchies

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Strict hierarchy (over $\mathbb{R}$):

- Nullstellensatz
- Sherali–Adams
- Sums of squares

Sums of squares is strictly stronger than polynomial calculus (over $\mathbb{R}$)
Sherali-Adams and polynomial calculus are incomparable [Ber18]

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Sums of squares very strong proof system (e.g., can reason about PHP)
But can't do parity reasoning efficiently [GV01, Gri01]

Survey [FKP19] recommended for more reading

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Intended to model modern 0-1 integer linear programming

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4. Recurse with $\mathcal{S} := \mathcal{S} \cup \left\{ \sum_i a_i \ell_i \leq A - 1 \right\}$

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4. Recurse with $\mathcal{S} := \mathcal{S} \cup \left\{ \sum_i a_i \ell_i \leq A - 1 \right\}$

Complexity measures:

- Length: # branching nodes / sets $\mathcal{S}$
- Size: Count also bit size for representing all coefficients

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Stabbing Planes and Cutting Planes

Stabbing planes efficiently simulates cutting planes [BFI+18]

Stabbing planes probably much stronger!?

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Or maybe not... Stabbing planes with polynomial-size coefficient can be simulated by cutting planes with quasi-polynomial overhead [DT20, FGI+21]

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Very recent news: Interpolation and circuit complexity can be used to get similar lower bounds for stabbing planes as for cutting planes! [GP24]

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Still possible that stabbing planes is exponentially more powerful than cutting planes, but hard to know what to believe

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Extended Resolution [Tse68]

Resolution rule

$$\frac{C_1 \vee x \qquad C_2 \vee \overline{x}}{C_1 \vee C_2}$$

Extension rule introducing clauses

$$a \vee \overline{x} \vee \overline{y} \qquad \overline{a} \vee x \qquad \overline{a} \vee y$$

for fresh variable a (encoding that $a \leftrightarrow (x \wedge y)$ must hold)

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Extended Resolution and SAT Solving

- Closely related (and equivalent) to *DRAT* system used to justify correctness of some SAT preprocessing techniques [JHB12]

- *DRAT* also used for SAT solver proof logging

- Attempts to combine extended resolution with CDCL in, e.g., [AKS10, Hua10]

- Without restrictions, corresponds to extremely strong extended Frege system [CR79] — pretty much no lower bounds known

- To analyse solvers using extended resolution, would need to:
    - Describe heuristics/rules actually used
    - See if possible to reason about such restricted proof system

Some More References for Further Reading

Handbook of Satisfiability
(Especially chapter 7 ☺)

[BHvMW21]

Proof Complexity
by Jan Krajíček
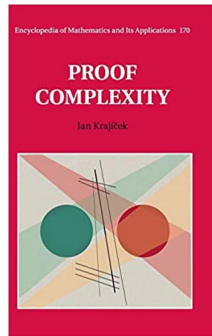
[Kra19]

Summing up This Presentation

Overview of some proof systems used in combinatorial solving:

- Resolution $\longleftrightarrow$ conflict-driven clause learning (CDCL)
- Nullstellensatz and polynomial calculus $\longleftrightarrow$ Gröbner bases
- Cutting planes $\longleftrightarrow$ pseudo-Boolean solving

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- Analyse state-of-the-art algorithms (and provide methods for certifying correctness!)
- Give ideas for new approaches
- Provide a fun playground for theory-practice interaction!

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Thank you for your attention!

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