

## NP and NP-completeness

$L \in NP$  if exists  $M_L(x, y)$  such that

- $M_L$  runs in polynomial time
- $x \in L \Leftrightarrow \exists y, |y| = \text{poly}(|x|)$ ,  
such that  $M_L(x, y) = 1$

$L$  is NP-HARD if for all  $L' \in NP$   
it holds that  $L' \leq_p L$

$L$  is NP-COMPLETE if

- $L \in NP$
- $L$  is NP-hard

### COOK-LEVIN THEOREM

SAT and 3-SAT are NP-complete

#### First proof idea

(i) Define Boolean function

$$f_x^L(y) = \begin{cases} 1 & \text{if } M_L(xy) = 1 \\ 0 & \text{otherwise} \end{cases}$$

(ii) Represent  $f_x^L$  as CNF formula

This formula has all the right properties, except it is unfortunately exponentially large

Define instead SNAPSHOTS  $z_i = \langle a_i, b_i, q_i \rangle$   
of Turing machine computation

- $a_i$  read from input tape }  
-  $b_i$  read from work tape } at time  $i$   
-  $q_i$  Turing machine state }

Can check correctness of  $z_i$  locally for

- $z_{i-1}$  for  $q_i$
- $\text{inputpos}(i)$  for  $a_i$
- $z_{\text{prev}}(i)$  for  $b_i$

Consistency check (given  $x$  and  $M_x$ )  
is Boolean function of constant number  
of bits

Write CNF Formula  $F_x$  with variables for  
 $y, z_1, z_2, \dots, z_T$

enforcing

- a) starting state  $z_1$  correct
- b)  $z_i$  is locally correct
- c)  $z_T$  is final state of accepting  
computation

Assignment to variables encode

- witness  $y$
- snapshots of accepting computation  
for  $M_x (x, y)$

Possible if and only if  $x \in L$

## Two observations

- ① If  $M_x$  runs in time  $T(1 \times 1)$ , then formula  $F_x$  (and time to compute it) can be made very small  $O(T \log T)$
- ② From satisfying assignment to  $F_x$ , can read off witness  $y$  for  $x$   
This is called a LEVIN REDUCTION

To prove language  $L$  NP-complete

- (i) Show  $L \in \text{NP}$  (usually easy)
- (ii) Reduce from SAT, 3-SAT or other NP-complete language to  $L$

Will see some such reductions

But first some more definitions

DEFINITION (COMPLEMENT CLASS)

For  $L \subseteq \Sigma^*$ , the COMPLEMENT LANGUAGE of  $L$  is  $\bar{L} = \Sigma^* \setminus L$

DEFINITION (coNP)

$$\text{coNP} = \{ L \mid \bar{L} \in \text{NP} \}$$

Technical aside: Usually exclude syntactically incorrect inputs from both  $L$  and  $\bar{L}$   
(Does not really matter — why?)

Note that  $\text{coNP}$  is not the complement of  $\text{NP}$   
The intersection is non-empty!

$$\mathcal{P} \subseteq \text{NP} \cap \text{coNP}$$

TAUTOLOGY is in  $\text{coNP}$  — if  $F$  is not a tautology, then this is witnessed by an assignment falsifying the formula

$$\boxed{\text{UNSAT}} = \{ F \mid F \text{ is an unsatisfiable CNF formula} \}$$

is also in  $\text{coNP}$

Both of these languages are  $\text{coNP}$ -complete — any other language  $L$  in  $\text{coNP}$  can be efficiently reduced to them

Proof sketch (for UNSAT) Given  $L \in \text{coNP}$ ,  
run Cook-Levin reduction on  $L$   
 $x \in L \iff x \notin \overline{L} \iff F_x \text{ unsatisfiable}$

How is  $\text{coNP}$  related to  $\text{NP}$ ?

Could there be short certificates for  $\text{UNSAT}$  that somehow "compress information about exponentially many failing assignments"?

Most researchers believe

$$\boxed{\text{NP} \neq \text{coNP}}$$

but this is a wide-open problem

THEOREM

O-1 LINEAR PROGRAMMING IS  
NP-complete

Proof: Containment in NP easy - just  
plug in solution and check

NP-hardness: Reduce from SAT

Translate clauses to linear inequalities

$$x \mapsto x$$

$$\bar{x} \mapsto 1 - x$$

$$x \vee \bar{y} \vee z \mapsto x + (1 - y) + z \geq 1$$

$$\text{or } x - y + z \geq 0$$

Linear inequality satisfied

precisely where clause satisfied

Clearly polynomial-time reduction

Interesting counterfactual question:

What would complexity theory have  
looked like if the first NP-complete  
problem would have been O-1 linear  
programming rather than  
Boolean satisfiability?

3-COLOURING = { $G \mid$  Undirected graph  $G$  has 3-colouring}

3-colouring: Colour all vertices with 3 colours

$$\chi : V \rightarrow \{\text{red, blue, green}\}$$

Colouring  $\chi$  valid if for every edge  $(u, v) \in E(G)$  it holds that  $\chi(u) \neq \chi(v)$

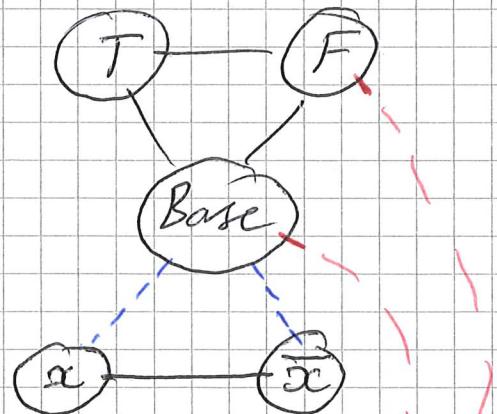
THEOREM | 3-COLOURING is NP-complete

Proof: Let witness be colouring

Containment in NP - obvious

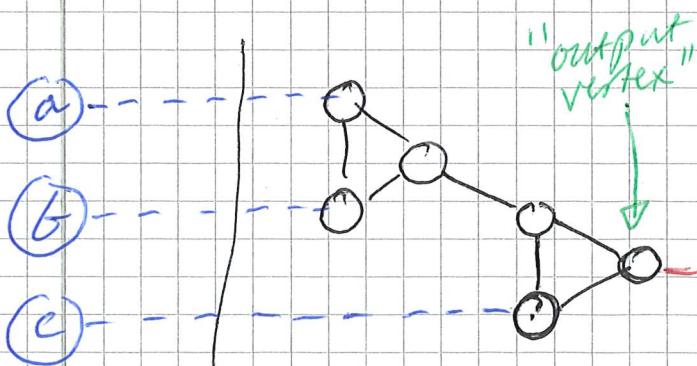
NP-hardness: Reduce from 3-SAT

Gadgets: ① Assignment triangle



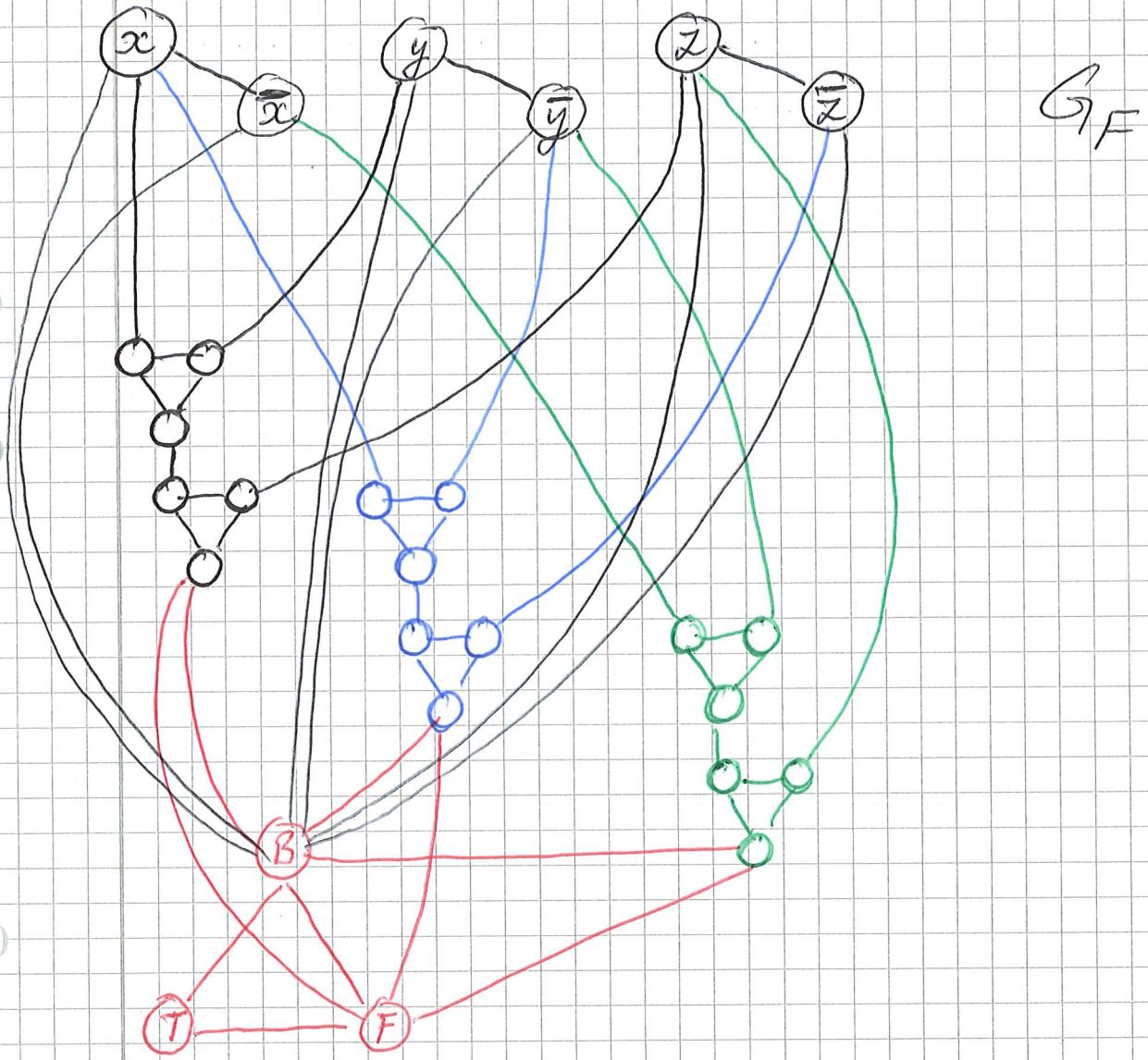
② For every variable  $x$

③ For every clause  
 $a \vee b \vee c$



Example

$$\begin{aligned} & (x \vee y \vee z) \\ \wedge & (x \vee \bar{y} \vee \bar{z}) \\ \wedge & (\bar{x} \vee \bar{y} \vee z) \end{aligned} \quad \left. \right\} F$$



Need to prove

$(\Rightarrow)$   $F$  satisfiable  $\Rightarrow G_F$  3-colourable

$(\Leftarrow)$   $G_F$  3-colourable  $\Rightarrow F$  satisfiable

(It is clear that  $G_F$  can be constructed from  $F$  in polynomial time.)

$(\Rightarrow)$  let  $\alpha$  be satisfying assignment  
for  $F$

Colours  $T$  - green

$F$  - red

(Base) - blue

true literals - green

false literals - red

If a clause contains a true literal,  
then "output vertex" can be coloured  
green / true

$(\Leftarrow)$  Given colouring of  $G_F$

Suppose wlog (by permuting colours  
if necessary) that

$T$  - green

$F$  - red

(Base) - blue

Note that no literal is coloured blue  
because of variable gadget.

For literal  $l$ , define  $\alpha(l) = \begin{cases} 1 & \text{if } \chi(l) = \text{green} \\ 0 & \text{if } \chi(l) = \text{red} \end{cases}$

Defines truth value assignment in view of  
variable gadget

Every output vertex in clause gadget coloured  
true / green. Only possible if some literal  
in clause coloured true - hence  
 $\alpha$  satisfies all clauses



$$\text{CLIQUE} = \{ \langle G, k \rangle \mid \begin{array}{l} G \text{ is an undirected graph} \\ \text{containing a } k\text{-clique} \end{array} \}$$

A  $k$ -clique in  $G$  is a set of  $k$  pairwise connected vertices  $v_1, v_2, \dots, v_k \in V(G)$

That is, all edges  $(v_i, v_j)$ ,  $1 \leq i < j \leq k$  are in  $E(G)$

THEOREM CLIQUE is NP-complete

We will show

(i) CLIQUE  $\in$  NP

(ii) 3-SAT  $\leq_p$  CLIQUE (CLIQUE is NP-hard)

For (i), let witnesses be the  $k$  vertices in a clique. Verifier checks that all edges between these vertices are in graph.

Can clearly be done in polynomial time on reasonable computer with reasonable programming language (which can be efficiently simulated by Turing machine).

For (ii), given a 3-CNF formula  $F$ , we need to construct a graph  $G_F$  and choose a parameter  $k_F$  such that

$F$  satisfiable  $\Leftrightarrow G_F$  has  $k_F$ -clique

And construction should be computable in polynomial time (in size of  $F$ )

Construction of  $G_F$  from  $F = \bigwedge_{i=1}^m C_i$

For every clause  $C_r = l_{r,1} \vee l_{r,2} \vee l_{r,3}$   
create 3 vertices  $v_{r,1}, v_{r,2}, v_{r,3}$

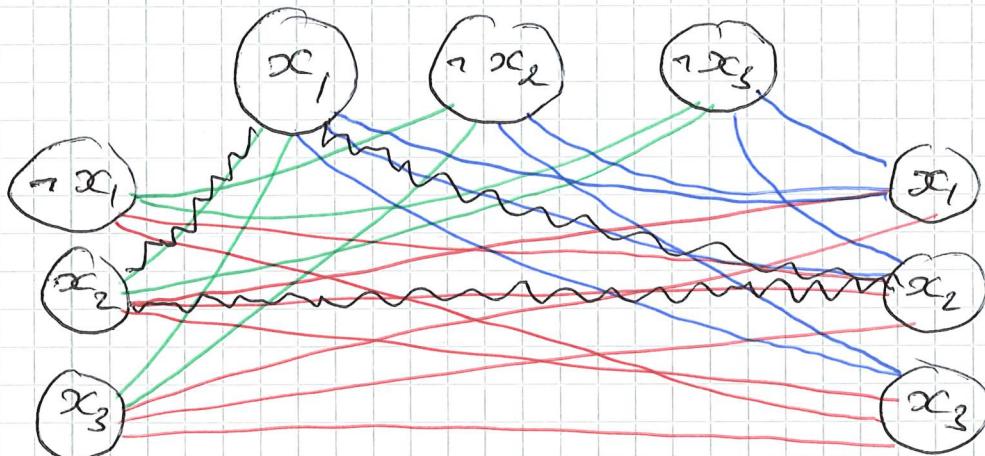
Add edge  $(v_{r,i}, v_{s,j})$  iff

- a)  $r \neq s$  (vertices come from different clauses)
- b)  $l_{r,i}$  and  $l_{s,j}$  are not negations of each other

Set  $k_F = m = \# \text{ clauses in } F$

Example

$$\boxed{\begin{aligned} F &= (x_1 \vee \neg x_2 \vee \neg x_3) \\ &\wedge (\neg x_1 \vee x_2 \vee \neg x_3) \\ &\wedge (x_1 \vee x_2 \vee x_3) \end{aligned}} \quad \begin{array}{l} C_1 \\ C_2 \\ C_3 \end{array}$$



Clearly possible to build  $G_F$  &  $k_F$  from  $F$  in polynomial time

Need to show

- $$\begin{aligned} (\Rightarrow) \quad F \text{ unsatisfiable} &\implies G_F \text{ has } k_F \text{-clique} \\ (\Leftarrow) \quad F \text{ satisfiable} &\Leftarrow G_F \text{ has } k_F \text{-clique} \end{aligned}$$

( $\Rightarrow$ ) Let  $\alpha$  be satisfying assignment  
 $\alpha$  satisfies at least one literal  $l_{r,i}$  per clause  $C_r$   
Pick one such literal per clause.  
Since literals are all true, none is negation of other, so clique  
One vertex per clause  $\Rightarrow k_F$ -clique

Example  $\alpha = \{x_1 \mapsto 1, x_2 \mapsto 1, x_3 \mapsto 0\}$

Pick  $x_1$  from  $C_1$ ,  $x_2$  from  $C_2$ ,  $x_3$  from  $C_3$

( $\Leftarrow$ ) Suppose  $G_F$  has  $k_F$ -clique.  
Then at least one (precisely one) vertex  $v_{r,i}$   
per clause  $C_r$   
All literals  $l_{r,i}$  can be assigned true, since  
there are edges between them.  
Yields partial truth value assignment  
satisfying all clauses.  
(Assign any remaining variables  
arbitrarily)

Vertex cover of  $G = (V, E)$  is subset  $V' \subseteq V$  such that every edge  $(u, v) \in E$  has an endpoint in  $V'$  (i.e.,  $\{u, v\} \cap V' \neq \emptyset$ )

$\text{VERTEXCOVER} = \{\langle G, k \rangle \mid G \text{ has vertex cover of size } k\}$

THEOREM  $\text{VERTEXCOVER}$  is  $\text{NP}$ -complete

We will show

already known to be

(i)  $\text{VERTEXCOVER} \in \text{NP}$

$\text{NP}$ -complete, so  
can reduce from it!

(ii)  $\text{Clique} \leq_p \text{VERTEXCOVER}$  ( $\text{VERTEXCOVER}$  is  
 $\text{NP-hard}$ )

For (i), suitable witness consists of

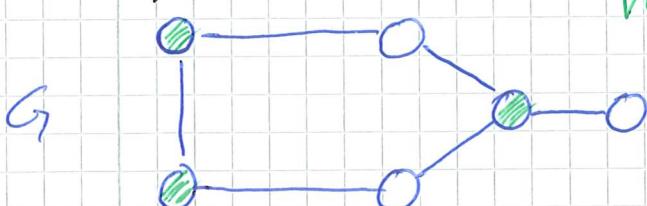
$$V' = (v_1, v_2, \dots, v_k)$$

Verifier checks for all edges  $(u, v)$  that

$$\{u, v\} \cap \{v_1, \dots, v_k\} \neq \emptyset$$

Can clearly be done in polynomial time

Example

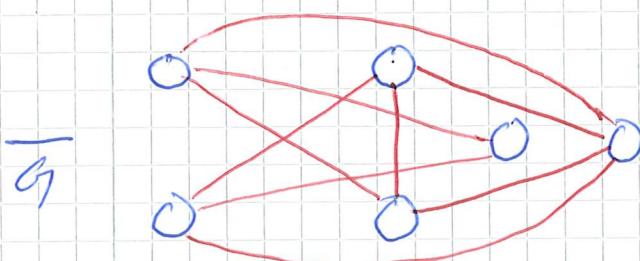


Vertex cover of size 3

For (ii), translate  $\langle G, k \rangle$  to  $\langle \bar{G}, |V| - k \rangle$

where  $\boxed{\bar{G}}$  is complement graph with same vertex set but edges  $(V \times V) \setminus E$

$(u, v)$  is edge in  $\bar{G} \iff (u, v)$  is not edge in  $G$



$\bar{G}$  can clearly be constructed from  $G$  in polynomial time

Need to show:

$$\langle G, k \rangle \in \text{CLIQUE} \iff \langle \bar{G}, |V| - k \rangle \in \text{VERTEXCOVER}$$

( $\Rightarrow$ ) Suppose  $C \subseteq V$  is clique in  $G$  of size  $|C| = k$

$$u, v \in C, u \neq v \Rightarrow (u, v) \text{ edge in } G \\ \Rightarrow (u, v) \text{ not edge in } \bar{G}$$

So  $C$  contains no edges in  $\bar{G}$

$\Rightarrow$  all edges in  $\bar{G}$  have an endpoint in  $V \setminus C$ , which is a vertex cover of size  $|V| - k$ .

( $\Leftarrow$ ) Suppose  $C^*$  vertex cover in  $\bar{G}$

Then  $V \setminus C^*$  contains no edges in  $\bar{G}$

Means that all edges in  $V \setminus C^*$  are present in  $G$

Hence if  $C^*$  vertex cover in  $\bar{G}$  of size  $|V| - k$ , then  $V \setminus C^*$  clique in  $G$  of size  $|V| - (|V| - k) = k$ .

## Subset sum

think of subset  $S'$  as indices of numbers

Given positive integers  $A_1, A_2, \dots, A_n, T$   
Is there subset  $S' \subseteq \{1, 2, \dots, n\}$  such that

$$\sum_{i \in S'} A_i = T ?$$

$$\text{SUBSET SUM} = \left\{ \langle A_1, \dots, A_n, T \rangle \mid \exists S' \text{ s.t. } \sum_{i \in S'} A_i = T \right\}$$

THEOREM SUBSET SUM is NP-complete.

We will show

(i) SUBSET SUM  $\in \text{NP}$

(ii) 3-SAT  $\leq_p$  SUBSET SUM (SUBSET SUM is NP-hard)

For (i), let witness be set of indices  $S'$ .

Verifier computes  $\sum_{i \in S'} A_i$  and checks if this sum equals  $T$ .

Can clearly be done in polynomial time.

For (ii), suppose we have 3-CNF formula  $F = \bigwedge_{i=1}^m C_i$  over variables  $x_1, \dots, x_n$ .

Create subset sum instance

$$\{A_1^T, A_1^F, A_2^T, A_2^F, \dots, A_n^T, A_n^F, B_1^1, B_1^2, \dots, B_m^1, B_m^2\}$$

as  $T$  as follows

- Each number has  $n+m$  decimal digits
- $n$  most significant digits associated with  $x_1, \dots, x_n$
- $m$  least significant digits associated with  $C_1, \dots, C_m$

Assume w.l.o.g.

- No clause contains both a variable and its negation (if so, remove this clause)
- All variable appear in  $F$  (remove & renumber otherwise)

## Construction of numbers

$A_i^T$ : digit  $x_i$  is 1  
 digit  $C_j$  is 1 if  $x_i \in C_j$   
 all other digits 0

$A_i^F$ : digit  $x_i$  is 1  
 digit  $C_j$  is 1 if  $x_i \in C_j$   
 all other digits 0

$B_j^1$ : digit  $C_j$  is 1  
 all other digits 0

$B_j^2$ : digit  $C_j$  is 2  
 all other digits 0

$T$ : all digits  $x_i$  are 1  
 all digits  $C_j$  are 4

( $\Rightarrow$ )

Suppose  $F$  satisfied by assignment  $\alpha$

- If  $\alpha(x_i) = 1$  pick  $A_i^T$ , otherwise  $A_i^F$   
 - If  $\alpha$  satisfies

- 1 literal in  $C_j$  - pick  $B_j^1$  and  $B_j^2$
- 2 literals in  $C_j$  - pick  $B_j^2$
- 3 literals in  $C_j$  - pick  $B_j^1$

This sums to  $T$

( $\Leftarrow$ ) Given solution  $S'$  to subset sum instance

Let  $\alpha$  set  $\alpha_i = 1$  iff  $A_i^T$  chosen in  $S'$

Why is this a satisfying assignment?

Clearly possible  
 to construct such  
 a subset sum instance  
 in polynomial time  
 in the size of  $F$

Example  $F = (x_1 \vee \neg x_2 \vee \neg x_3) \wedge (\neg x_1 \vee x_2 \vee x_3) \wedge (x_1 \vee x_2 \vee x_3)$

$n = 3$

$m = 3$

Solution corresponding

to  $x_1 \mapsto 1, x_2 \mapsto 1, x_3 \mapsto 0$

$A_2^T = \underline{100} \underline{101} *$

$A_1^F = \underline{100} 010$

$A_2^T = \underline{010} 011 *$

$A_2^F = \underline{010} 100$

$A_3^T = \underline{001} 011$

$A_3^F = \underline{001} 100 *$

$B_1^1 = \underline{000} 100$

$B_1^2 = \underline{000} 200 *$   $C_1$

$B_2^1 = \underline{000} 010 *$

$B_2^2 = \underline{000} 020 *$

$B_3^1 = \underline{000} 001$

$B_3^2 = \underline{000} 002 *$   $C_3$

$T = \underline{111} \underline{444}$

First  $n$  digits of  $T$  enforce that exactly one of  $x_i$  and  $\neg x_i$  chosen

For last  $m$  digits, can only result 4 if at least one literal in clause is chosen to be true.

QUESTION: What happens if we flip  $x_2$  to false and consider  $x_1 \mapsto 1$ ,  $x_2 \mapsto 0$ ,  $x_3 \mapsto 0$ ?

## DECISION VS. SEARCH

### THEOREM

If  $P = NP$ , then for every  $L \in NP$  we can find a polynomial-time Turing machine  $M$  that finds certificates for  $x \in L$  (i.e., "solves  $x$ ")

### Proof sketch

Start with SAT.

Given CNF formula  $F(x_1, \dots, x_n)$

set  $x_1 = 1$ , and simplify  $F$  to

$F/x_1 = 1$  - remove satisfied clauses

- then remove falsified literals

Check if  $F/x_1 = 1$  is satisfiable; if so set  $b = 1$

Else set  $x_1 = 0$  and simplify  $F$  to  $F/x_1 = 0$

Check if  $F/x_1 = 0$  satisfiable; if so set  $b = 0$

otherwise report UNSATISFIABLE

Remember  $x_1 = b$  and make recursive call on  $F/x_1 = b$

Return satisfying assignment to  $x_1, \dots, x_n$  for

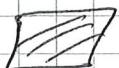
$F/x_1 = b$  plus  $x_1 = b$

For general  $L \in NP$ , run reduction

$x \mapsto F_{xc}$  in proof of Cook-Levin

Find satisfying assignment to  $F_{xc}$

Read off witness  $y$  for  $x$  from this satisfying assignment.



SAT is DOWNWARD SELF-REDUCIBLE

Given efficient algorithm that solves SAT on input size  $< n$  can solve also instances of size  $n$

ALL NP-complete problems/languages have this property (follows from Cook-Lerma)

### SO FAR

Focus on  $P$ ,  $NP$ ,  $NP$ -complements

You should have seen most of this before, but

- Perhaps not presented in this way
- Or in this level of detail

### GOING FORWARD

New material (most likely)

- Can we separate any complexity classes? Is  $P$  distinct from  $\text{EXP}$ ?
- Can we say anything about the landscape between  $P$  and  $NP$  (if  $P \neq NP$ , that is)?