Certifying Combinatorial Solving Using Cutting Planes with Strengthening Rules

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Oberwolfach Workshop 2413
"Proof Complexity and Beyond"
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Based on joint work with Jeremias Berg, Bart Bogaerts, Jan Elffers, Ambros Gleixner, Stephan Gocht, Alexander Hoen, Hannes Ihalainen, Matti Järvisalo, Ciaran McCreesh, Matthew McIlree, Magnus O. Myreen, Andy Oertel, Yong Kiam Tan, and Dieter Vandesande

In a Galaxy Far, Far Away from Oberwolfach...

- Astounding progress last couple of decades on combinatorial solvers for, e.g.:
  - Boolean satisfiability (SAT) solving and optimization [BHvMW21]
  - Constraint programming [RvBW06]
  - Mixed integer linear programming [AW13, BR07]
  - Satisfiability modulo theories (SMT) solving [BHvMW21]
- Solvers very fast, but sometimes wrong (even best commercial ones) [BLB10, CKSW13, AGJ+18, GSD19, GS19, BMN22, BBN+23]
- Even get feasibility of solutions wrong (though this should be straightforward!)
- And how to check the absence of solutions?
- Or that a solution is optimal? (Even off-by-one mistakes can snowball into large errors if solver used as subroutine)

What Can Be Done About Solver Bugs?

- **Software testing**
  Hard to get good test coverage for sophisticated solvers
  Progress using fuzzing and delta debugging [BB09, BLB10, KB22, NPB22, PB23]
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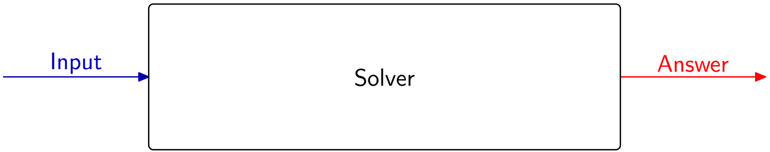
- **Proof logging**
  Make solver certifying [ABM$^+$11, MMNS11] by adding code so that it outputs
  1. not only answer but also
  2. simple, machine-verifiable proof that answer is correct

Proof Logging with Certifying Solvers: Workflow



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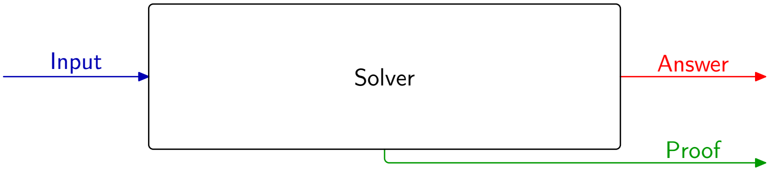

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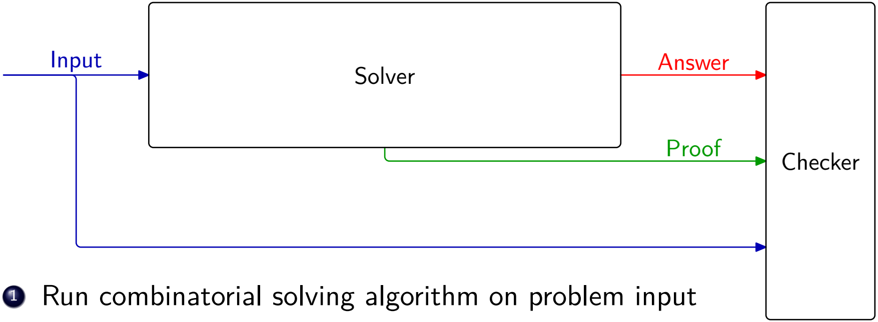

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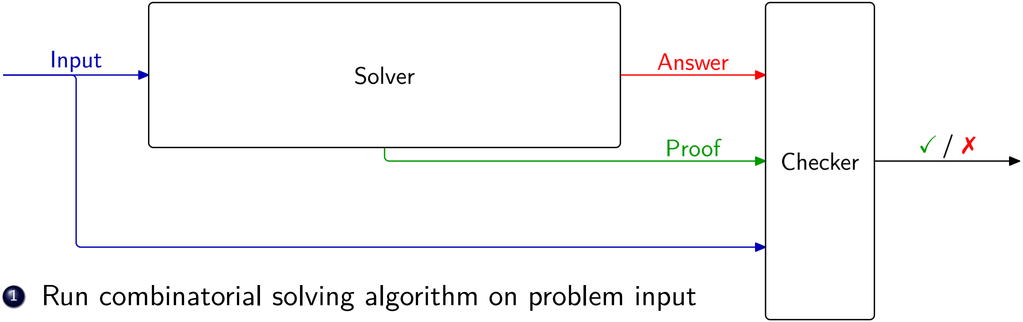
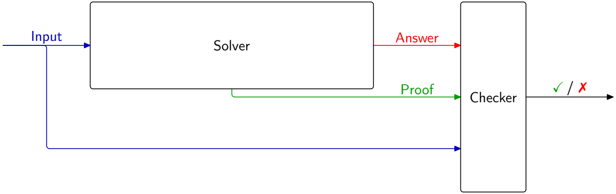


1. Run combinatorial solving algorithm on problem input
2. Get as output not only answer but also proof
3. Feed input + answer + proof to proof checker
4. Verify that proof checker says answer is correct

Proof Logging Desiderata

Proof format for certifying solver should be



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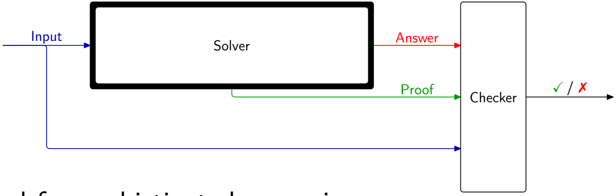

Proof format for certifying solver should be

- **very powerful:** minimal overhead for sophisticated reasoning

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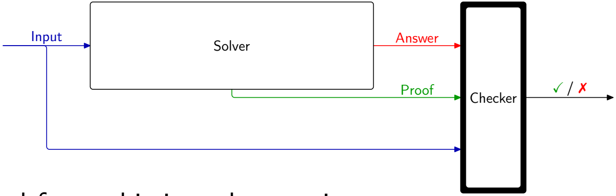

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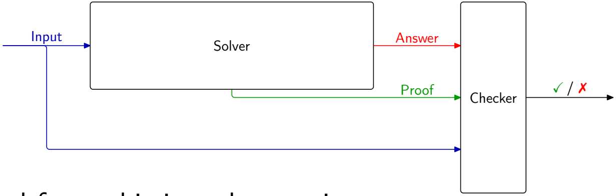

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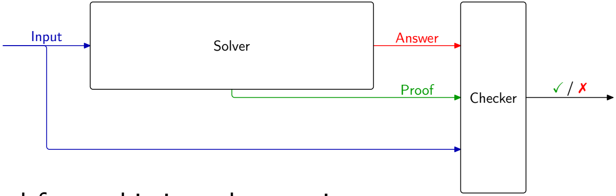

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Asking for both perhaps a little bit too good to be true?

This Talk

Proof logging for combinatorial optimization is possible with single, unified method!

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- Build on successes in SAT solving with proof formats such as DRAT [HHW13a, HHW13b, WHH14], GRIT [CMS17], LRAT [CHH+17], ...
- But represent constraints as 0–1 integer linear inequalities
- Formalize reasoning using cutting planes [CCT87] proof system
- Add well-chosen strengthening rules [Goc22, GN21, BGMN23]
- Implemented in VeriPB (`https://gitlab.com/MIAOresearch/software/VeriPB`)

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1. Marketing pitch ☺
2. Highlight some interesting related questions in proof complexity

The Sales Pitch For Proof Logging

1. Certifies correctness of computed results
2. Detects errors even if due to compiler bugs, hardware failures, or cosmic rays
3. Provides debugging support during software development [EG21, GMM$^+$20, KM21, BBN$^+$23]
4. Facilitates performance analysis
5. Helps identify potential for further improvements
6. Enables auditability
7. Serves as stepping stone towards explainability

Proof Language: Pseudo-Boolean Constraints

Proof consists of 0-1 integer linear inequalities or pseudo-Boolean constraints:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
- variables x_i take values $0 = \mathit{false}$ or $1 = \mathit{true}$

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Sometimes convenient to use normalized form [Bar95] with all a_i, A positive (without loss of generality)

Some Types of Pseudo-Boolean Constraints

1. Clauses

$$x \vee \overline{y} \vee z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Pseudo-Boolean Proof Logging Wishlist

Paradigms

- SAT solving
- (linear) pseudo-Boolean solving
- graph solving
- constraint programming
- automated planning
- mixed integer linear programming
- SMT solving

Problem types

- decision / feasibility
- optimization
- multi-objective optimization
- projected model enumeration
- projected model counting
- preprocessing / problem reformulation

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Design Principles for Proof Logging

Proof logging implementation

- Don't change solver
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Performance goals

- Proof logging overhead small constant fraction ($\lessapprox 10\%$)
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Proof system

- Keep proof language maximally simple
- Reason about XOR constraints, CP propagators, symmetries, etc within language
- Combine proof logging with formally verified proof checker

Pseudo-Boolean Proof Logging — How and Why?

If problem is (special case of) 0-1 integer linear program

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- do proof logging for 0-1 ILP formulation [but solver still works with original input]

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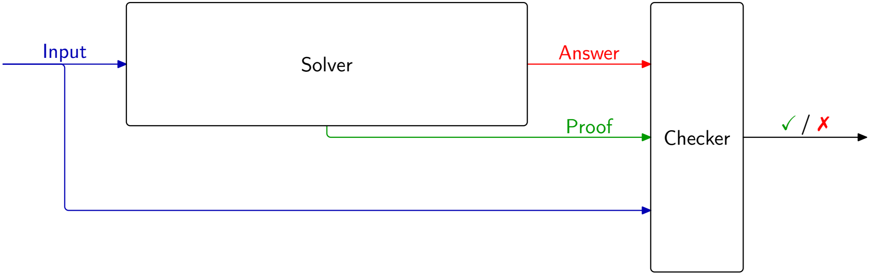
Goldilocks compromise between expressivity and simplicity:

1. 0-1 ILP expressive formalism for combinatorial problems (including objective)
2. Powerful reasoning capturing many combinatorial arguments

Proof Logging with Formally Verified Checking: Full Workflow



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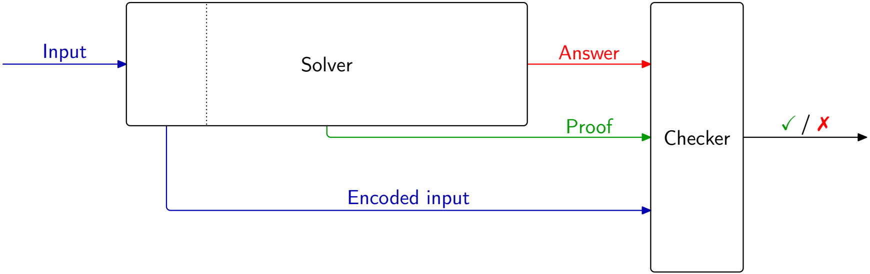

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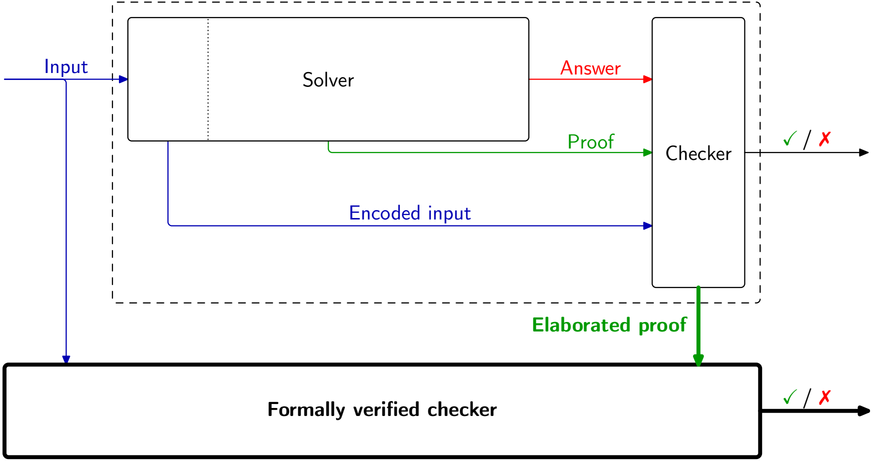

VeriPB Proof Configuration (Slightly Simplified)

Core set $\mathcal{C}$

- Contains input formula at the start
- Maintains "equivalence" with input formula

Derived set $\mathcal{D}$

- All constraints derived during search
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Order $\mathcal{O}$

- Pseudo-Boolean formula encoding pre-order (reflexive and transitive)
- Syntactic proof of properties required
- Applied to specified variable set $\vec{z}$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input axioms From the input

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Saturation
(constraint in normalized form)
$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \min(a_i, A) \cdot \ell_i \geq A}$$

Cutting Planes Toy Example

$$w + 2x + y \geq 2$$

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Multiply by 2 $$\frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

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$$\dfrac{\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}\;\text{Multiply by } 2 \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}\;\text{Add}$$

Cutting Planes Toy Example

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By naming constraints by integers and literal axioms by the literal involved as

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$$

such a calculation is written in the proof log in reverse Polish notation as

```
pol 1 2 * 2 + ~z 2 * + 3 d
```

Open Problem: Division Versus Saturation

Division $$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \left\lceil \frac{a_i}{c} \right\rceil \ell_i \geq \left\lceil \frac{A}{c} \right\rceil}$$

Saturation $$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \min(a_i, A) \cdot \ell_i \geq A}$$

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How do division and saturation rules compare?

- Strengths of rules as such incomparable [GNY19]
- Cutting planes with division can be exponentially stronger than cutting planes with saturation
- Unknown whether cutting planes with saturation can be stronger than cutting planes with division

Redundance-Based Strengthening

C is redundant with respect to F if F and $F \cup \{C\}$ are equisatisfiable

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- Proof sketch for interesting direction: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \cup \{C\}$
- In a proof, the implication needs to be efficiently verifiable — every $D \in (F \cup \{C\})\restriction_{\omega}$ should follow from $F \cup \{\neg C\}$ either
  1. "obviously" or
  2. by explicitly presented derivation

Example: Deriving $a \leftrightarrow (x \wedge y)$ Using the Redundance Rule

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Open Problems: Strength of Restricted Redundance Rules?

Adding redundance rule $\Rightarrow$ proof system polynomially equivalent to extended Frege

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Adding redundance rule $\Rightarrow$ proof system polynomially equivalent to extended Frege

1. What is the power of the redundance rule if we forbid new variables?
   For resolution + redundance known that:
   - Pigeonhole principle formulas easy
   - Tseitin formulas easy
2. What is the power of resolution with redundance if we only allow new variables $z \leftrightarrow C$ for previously derived clauses C?
   - Corresponds (kind of) to reasoning in core-guided MaxSAT solvers

Redundance and Dominance Rules in VeriPB (Slightly Simplified)

Redundance-based strengthening, optimization version [BGMN23]

Add constraint C to derived set $\mathcal{D}$ if exists witness substitution ω such that

$$\mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \models (\mathcal{C} \cup \mathcal{D} \cup \{C\})\restriction_\omega \cup \{f\restriction_\omega \leq f\}$$

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- Applying ω should strictly decrease f
- If so, don't need to show that $(\mathcal{D} \cup \{C\})\restriction_\omega$ implied!

Soundness of Dominance Rule

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7. ...
8. Can't go on forever, so finally reach α' satisfying $\mathcal{C} \cup \{C\}$

Soundness of Dominance Rule (Continued)

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- Same inductive proof as before, but also nested forward induction over derivation
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Further extensions:

- Define dominance rule with respect to order $\mathcal{O}$ independent of objective function
- Switch between different orders in same proof
- See [BGMN23] for details

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Could it be even stronger?!

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Plausibly yes [KT23] — talk by Neil after the break

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Important to allow deletions of constraints from database

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But powerful strengthening rules create problems:

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- Satisfiable formulas can turn unsatisfiable(!)

Solution: distinguish between deletion from core set $\mathcal{C}$ and derived set $\mathcal{D}$

Deletion, Core Transfer, and Order Change

Deletion

1. Deletion of constraint C always OK from derived set $\mathcal{D}$
2. OK from core set $\mathcal{C}$ only if C can be rederived from $\mathcal{C} \setminus \{C\}$ with redundance rule (otherwise unchecked deletion — special conditions apply)

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Core transfer

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Change of order

Possible to change order only if $\mathcal{D} = \emptyset$

Parity (XOR) Reasoning in SAT Solving

Given clauses

$$x \lor y \lor z$$
$$x \lor \overline{y} \lor \overline{z}$$
$$\overline{x} \lor y \lor \overline{z}$$
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Could add XORs to language, but prefer to keep things super-simple

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Use redundance rule with fresh variables a, b to derive

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(“$=$” syntactic sugar for “$\geq$” plus “$\leq$”)

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VeriPB can certify XOR reasoning [GN21]

Symmetry Breaking in SAT Solving

1. Pretend to solve optimisation problem minimizing $f \doteq \sum_{i=1}^{n} 2^{n-i} \cdot x_i$ (search for lexicographically smallest assignment satisfying formula)

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$$f \leq f\restriction_\sigma \quad \doteq \quad \sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

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3. Derive symmetry breaking clauses from this PB constraint:

$$y_0 \qquad\qquad \overline{y}_j \vee \overline{\sigma(x_j)} \vee x_j$$

$$\overline{y}_{j-1} \vee \overline{x}_j \vee \sigma(x_j) \qquad\qquad y_j \vee \overline{y}_{j-1} \vee \overline{x}_j$$

$$\overline{y}_j \vee y_{j-1} \qquad\qquad y_j \vee \overline{y}_{j-1} \vee \sigma(x_j)$$

Symmetry Breaking in SAT Solving

1. Pretend to solve optimisation problem minimizing $f \doteq \sum_{i=1}^{n} 2^{n-i} \cdot x_i$ (search for lexicographically smallest assignment satisfying formula)
2. Derive (for proof log only) pseudo-Boolean version of lex-leader constraint

$$f \leq f\restriction_\sigma \quad \doteq \quad \sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

3. Derive symmetry breaking clauses from this PB constraint:

$$y_0 \geq 1 \qquad\qquad \overline{y}_j + \overline{\sigma(x_j)} + x_j \geq 1$$

$$\overline{y}_{j-1} + \overline{x}_j + \sigma(x_j) \geq 1 \qquad\qquad y_j + \overline{y}_{j-1} + \overline{x}_j \geq 1$$

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VeriPB can certify fully general SAT symmetry breaking [BGMN23]

Open Problem: Symmetry Breaking with Redundance Rule?

Is the dominance rule really needed for fully general symmetry breaking?

Or could the redundance rule be enough?

Weaker DRAT strengthening rule sufficient for "pigeonhole-style" symmetries [HHW15]

Open Problem: Efficient Substitution Proofs?

Can cutting planes with redundance and dominance support proofs with lemmas/substitution efficiently?

Special case: symmetric learning in SAT solving [DBB17]

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Special case: symmetric learning in SAT solving [DBB17]

Seems very finicky...

Extension and substitution proof systems don't mix well

Future Research Directions

Performance and reliability of pseudo-Boolean proof logging

- Trim proof while verifying (as in DRAT-TRIM [HHW13a])
- Compress proof file using binary format
- Design formally verified proof checker *(work in progress [BMM$^+$23])*

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Proof logging for other combinatorial problems and techniques

- Model counting
- Mixed integer linear programming *(work on SCIP in [CGS17, EG21, DEGH23])*
- Satisfiability modulo theories (SMT) solving *(work on CVC5, Z3, ... [BBC+23])*

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And more. . .

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- Lots of other challenging problems and interesting ideas

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And more...

- Use proof logs for algorithm analysis or explainability purposes
- Lots of other challenging problems and interesting ideas
- We're hiring! Talk to me to join the pseudo-Boolean proof logging revolution! ☺

VeriPB Documentation

VeriPB tutorial at *CP '22* [BMN22]

- video at `youtu.be/s_5BIi4I22w`
- updated slides for *IJCAI '23* tutorial [BMN23]

Description of VeriPB and CakePB [BMM+23] for SAT 2023 competition

- Available at `satcompetition.github.io/2023/checkers.html`

Specific details on different proof logging techniques covered in research papers [EGMN20, GMN20, GMM+20, GN21, GMN22, GMNO22, VDB22, BBN+23, BGMN23, MM23, GMM+24, HOGN24, IOT+24, MMN24]

Lots of concrete example files at `gitlab.com/MIAOresearch/software/VeriPB`

Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity
- **Open:** Quite a few intriguing proof complexity questions (both upper and lower bounds)

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Thank you for your attention!

References I

[ABM$^+$11] Eyad Alkassar, Sascha Böhme, Kurt Mehlhorn, Christine Rizkallah, and Pascal Schweitzer. An introduction to certifying algorithms. *it - Information Technology Methoden und innovative Anwendungen der Informatik und Informationstechnik*, 53(6):287–293, December 2011.

[AGJ$^+$18] Özgür Akgün, Ian P. Gent, Christopher Jefferson, Ian Miguel, and Peter Nightingale. Metamorphic testing of constraint solvers. In *Proceedings of the 24th International Conference on Principles and Practice of Constraint Programming (CP '18)*, volume 11008 of *Lecture Notes in Computer Science*, pages 727–736. Springer, August 2018.

[AW13] Tobias Achterberg and Roland Wunderling. Mixed integer programming: Analyzing 12 years of progress. In Michael Jünger and Gerhard Reinelt, editors, *Facets of Combinatorial Optimization*, pages 449–481. Springer, 2013.

[Bar95] Peter Barth. A Davis-Putnam based enumeration algorithm for linear pseudo-Boolean optimization. Technical Report MPI-I-95-2-003, Max-Planck-Institut für Informatik, January 1995.

[BB09] Robert Brummayer and Armin Biere. Fuzzing and delta-debugging SMT solvers. In *Proceedings of the 7th International Workshop on Satisfiability Modulo Theories (SMT '09)*, pages 1–5, August 2009.

References II

[BBC$^+$23] Haniel Barbosa, Clark Barrett, Byron Cook, Bruno Dutertre, Gereon Kremer, Hanna Lachnitt, Aina Niemetz, Andres Nötzli, Alex Ozdemir, Mathias Preiner, Andrew Reynolds, Cesare Tinelli, and Yoni Zohar. Generating and exploiting automated reasoning proof certificates. *Communications of the ACM*, 66(10):86—95, October 2023.

[BBN$^+$23] Jeremias Berg, Bart Bogaerts, Jakob Nordström, Andy Oertel, and Dieter Vandesande. Certified core-guided MaxSAT solving. In *Proceedings of the 29th International Conference on Automated Deduction (CADE-29)*, volume 14132 of *Lecture Notes in Computer Science*, pages 1–22. Springer, July 2023.

[BGMN23] Bart Bogaerts, Stephan Gocht, Ciaran McCreesh, and Jakob Nordström. Certified dominance and symmetry breaking for combinatorial optimisation. *Journal of Artificial Intelligence Research*, 77:1539–1589, August 2023. Preliminary version in *AAAI '22*.

[BHvMW21] Armin Biere, Marijn J. H. Heule, Hans van Maaren, and Toby Walsh, editors. *Handbook of Satisfiability*, volume 336 of *Frontiers in Artificial Intelligence and Applications*. IOS Press, 2nd edition, February 2021.

References III

[BLB10] Robert Brummayer, Florian Lonsing, and Armin Biere. Automated testing and debugging of SAT and QBF solvers. In *Proceedings of the 13th International Conference on Theory and Applications of Satisfiability Testing (SAT '10)*, volume 6175 of *Lecture Notes in Computer Science*, pages 44–57. Springer, July 2010.

[BMM+23] Bart Bogaerts, Ciaran McCreesh, Magnus O. Myreen, Jakob Nordström, Andy Oertel, and Yong Kiam Tan. Documentation of VeriPB and CakePB for the SAT competition 2023. Available at `https://satcompetition.github.io/2023/checkers.html`, March 2023.

[BMN22] Bart Bogaerts, Ciaran McCreesh, and Jakob Nordström. Solving with provably correct results: Beyond satisfiability, and towards constraint programming. Tutorial at the *28th International Conference on Principles and Practice of Constraint Programming*. Slides available at `http://www.jakobnordstrom.se/presentations/`, August 2022.

[BMN23] Bart Bogaerts, Ciaran McCreesh, and Jakob Nordström. Combinatorial solving with provably correct results. Tutorial at the *32nd International Joint Conference on Artificial Intelligence*. Slides available at `http://www.jakobnordstrom.se/presentations/`, August 2023.

References IV

[BR07] Robert Bixby and Edward Rothberg. Progress in computational mixed integer programming—A look back from the other side of the tipping point. *Annals of Operations Research*, 149(1):37–41, February 2007.

[BT19] Samuel R. Buss and Neil Thapen. DRAT proofs, propagation redundancy, and extended resolution. In *Proceedings of the 22nd International Conference on Theory and Applications of Satisfiability Testing (SAT '19)*, volume 11628 of *Lecture Notes in Computer Science*, pages 71–89. Springer, July 2019.

[CCT87] William Cook, Collette Rene Coullard, and György Turán. On the complexity of cutting-plane proofs. *Discrete Applied Mathematics*, 18(1):25–38, November 1987.

[CGS17] Kevin K. H. Cheung, Ambros M. Gleixner, and Daniel E. Steffy. Verifying integer programming results. In *Proceedings of the 19th International Conference on Integer Programming and Combinatorial Optimization (IPCO '17)*, volume 10328 of *Lecture Notes in Computer Science*, pages 148–160. Springer, June 2017.

References V

[CHH$^+$17] Luís Cruz-Filipe, Marijn J. H. Heule, Warren A. Hunt Jr., Matt Kaufmann, and Peter Schneider-Kamp. Efficient certified RAT verification. In *Proceedings of the 26th International Conference on Automated Deduction (CADE-26)*, volume 10395 of *Lecture Notes in Computer Science*, pages 220–236. Springer, August 2017.

[CKSW13] William Cook, Thorsten Koch, Daniel E. Steffy, and Kati Wolter. A hybrid branch-and-bound approach for exact rational mixed-integer programming. *Mathematical Programming Computation*, 5(3):305–344, September 2013.

[CMS17] Luís Cruz-Filipe, João P. Marques-Silva, and Peter Schneider-Kamp. Efficient certified resolution proof checking. In *Proceedings of the 23rd International Conference on Tools and Algorithms for the Construction and Analysis of Systems (TACAS '17)*, volume 10205 of *Lecture Notes in Computer Science*, pages 118–135. Springer, April 2017.

[Cry] CryptoMiniSat SAT solver. `https://github.com/msoos/cryptominisat/`.

[DBB17] Jo Devriendt, Bart Bogaerts, and Maurice Bruynooghe. Symmetric explanation learning: Effective dynamic symmetry handling for SAT. In *Proceedings of the 20th International Conference on Theory and Applications of Satisfiability Testing (SAT '17)*, volume 10491 of *Lecture Notes in Computer Science*, pages 83–100. Springer, August 2017.

References VI

[DEGH23] Jasper van Doornmalen, Leon Eifler, Ambros Gleixner, and Christopher Hojny. A proof system for certifying symmetry and optimality reasoning in integer programming. Technical Report 2311.03877, arXiv.org, November 2023.

[EG21] Leon Eifler and Ambros Gleixner. A computational status update for exact rational mixed integer programming. In *Proceedings of the 22nd International Conference on Integer Programming and Combinatorial Optimization (IPCO '21)*, volume 12707 of *Lecture Notes in Computer Science*, pages 163–177. Springer, May 2021.

[EGMN20] Jan Elffers, Stephan Gocht, Ciaran McCreesh, and Jakob Nordström. Justifying all differences using pseudo-Boolean reasoning. In *Proceedings of the 34th AAAI Conference on Artificial Intelligence (AAAI '20)*, pages 1486–1494, February 2020.

[GMM$^+$20] Stephan Gocht, Ross McBride, Ciaran McCreesh, Jakob Nordström, Patrick Prosser, and James Trimble. Certifying solvers for clique and maximum common (connected) subgraph problems. In *Proceedings of the 26th International Conference on Principles and Practice of Constraint Programming (CP '20)*, volume 12333 of *Lecture Notes in Computer Science*, pages 338–357. Springer, September 2020.

References VII

[GMM+24] Stephan Gocht, Ciaran McCreesh, Magnus O. Myreen, Jakob Nordström, Andy Oertel, and Yong Kiam Tan. End-to-end verification for subgraph solving. In *Proceedings of the 368h AAAI Conference on Artificial Intelligence (AAAI '24)*, pages 8038–8047, February 2024.

[GMN20] Stephan Gocht, Ciaran McCreesh, and Jakob Nordström. Subgraph isomorphism meets cutting planes: Solving with certified solutions. In *Proceedings of the 29th International Joint Conference on Artificial Intelligence (IJCAI '20)*, pages 1134–1140, July 2020.

[GMN22] Stephan Gocht, Ciaran McCreesh, and Jakob Nordström. An auditable constraint programming solver. In *Proceedings of the 28th International Conference on Principles and Practice of Constraint Programming (CP '22)*, volume 235 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 25:1–25:18, August 2022.

[GMNO22] Stephan Gocht, Ruben Martins, Jakob Nordström, and Andy Oertel. Certified CNF translations for pseudo-Boolean solving. In *Proceedings of the 25th International Conference on Theory and Applications of Satisfiability Testing (SAT '22)*, volume 236 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 16:1–16:25, August 2022.

References VIII

[GN21] Stephan Gocht and Jakob Nordström. Certifying parity reasoning efficiently using pseudo-Boolean proofs. In *Proceedings of the 35th AAAI Conference on Artificial Intelligence (AAAI '21)*, pages 3768–3777, February 2021.

[GNY19] Stephan Gocht, Jakob Nordström, and Amir Yehudayoff. On division versus saturation in pseudo-Boolean solving. In *Proceedings of the 28th International Joint Conference on Artificial Intelligence (IJCAI '19)*, pages 1711–1718, August 2019.

[Goc22] Stephan Gocht. *Certifying Correctness for Combinatorial Algorithms by Using Pseudo-Boolean Reasoning*. PhD thesis, Lund University, June 2022. Available at `https://portal.research.lu.se/en/publications/certifying-correctness-for-combinatorial-algorithms-by-using-pseu`.

[GS19] Graeme Gange and Peter Stuckey. Certifying optimality in constraint programming. Presentation at KTH Royal Institute of Technology. Slides available at `https://www.kth.se/polopoly_fs/1.879851.1550484700!/CertifiedCP.pdf`, February 2019.

References IX

[GSD19] Xavier Gillard, Pierre Schaus, and Yves Deville. SolverCheck: Declarative testing of constraints. In *Proceedings of the 25th International Conference on Principles and Practice of Constraint Programming (CP '19)*, volume 11802 of *Lecture Notes in Computer Science*, pages 565–582. Springer, October 2019.

[HHW13a] Marijn J. H. Heule, Warren A. Hunt Jr., and Nathan Wetzler. Trimming while checking clausal proofs. In *Proceedings of the 13th International Conference on Formal Methods in Computer-Aided Design (FMCAD '13)*, pages 181–188, October 2013.

[HHW13b] Marijn J. H. Heule, Warren A. Hunt Jr., and Nathan Wetzler. Verifying refutations with extended resolution. In *Proceedings of the 24th International Conference on Automated Deduction (CADE-24)*, volume 7898 of *Lecture Notes in Computer Science*, pages 345–359. Springer, June 2013.

[HHW15] Marijn J. H. Heule, Warren A. Hunt Jr., and Nathan Wetzler. Expressing symmetry breaking in DRAT proofs. In *Proceedings of the 25th International Conference on Automated Deduction (CADE-25)*, volume 9195 of *Lecture Notes in Computer Science*, pages 591–606. Springer, August 2015.

References X

[HOGN24] Alexander Hoen, Andy Oertel, Ambros Gleixner, and Jakob Nordström. Certifying MIP-based presolve reductions for 0–1 integer linear programs. In *Proceedings of the 21st International Conference on the Integration of Constraint Programming, Artificial Intelligence, and Operations Research (CPAIOR '24)*, volume 14742 of *Lecture Notes in Computer Science*, pages 310–328. Springer, May 2024.

[IOT+24] Hannes Ihalainen, Andy Oertel, Yong Kiam Tan, Jeremias Berg, Matti Järvisalo, Magnus O. Myreen, and Jakob Nordström. Certified MaxSAT preprocessing. In *Proceedings of the 12th International Joint Conference on Automated Reasoning (IJCAR '24)*, volume 14739 of *Lecture Notes in Computer Science*, pages 396–418. Springer, July 2024.

[JHB12] Matti Järvisalo, Marijn J. H. Heule, and Armin Biere. Inprocessing rules. In *Proceedings of the 6th International Joint Conference on Automated Reasoning (IJCAR '12)*, volume 7364 of *Lecture Notes in Computer Science*, pages 355–370. Springer, June 2012.

[KB22] Daniela Kaufmann and Armin Biere. Fuzzing and delta debugging and-inverter graph verification tools. In *Proceedings of the 16th International Conference on Tests and Proofs (TAP '22)*, volume 13361 of *Lecture Notes in Computer Science*, pages 69–88. Springer, July 2022.

References XI

[KM21] Sonja Kraiczy and Ciaran McCreesh. Solving graph homomorphism and subgraph isomorphism problems faster through clique neighbourhood constraints. In *Proceedings of the 30th International Joint Conference on Artificial Intelligence (IJCAI '21)*, pages 1396–1402, August 2021.

[KT23] Leszek Kołodziejczyk and Neil Thapen. Personal communication, 2023.

[MM23] Matthew McIlree and Ciaran McCreesh. Proof logging for smart extensional constraints. In *Proceedings of the 29th International Conference on Principles and Practice of Constraint Programming (CP '23)*, volume 280 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 26:1–26:17, August 2023.

[MMN24] Matthew McIlree, Ciaran McCreesh, and Jakob Nordström. Proof logging for the circuit constraint. In *Proceedings of the 21st International Conference on the Integration of Constraint Programming, Artificial Intelligence, and Operations Research (CPAIOR '24)*, volume 14743 of *Lecture Notes in Computer Science*, pages 38–55. Springer, May 2024.

[MMNS11] Ross M. McConnell, Kurt Mehlhorn, Stefan Näher, and Pascal Schweitzer. Certifying algorithms. *Computer Science Review*, 5(2):119–161, May 2011.

References XII

[NPB22] Aina Niemetz, Mathias Preiner, and Clark W. Barrett. Murxla: A modular and highly extensible API fuzzer for SMT solvers. In *Proceedings of the 34th International Conference on Computer Aided Verification (CAV '22)*, volume 13372 of *Lecture Notes in Computer Science*, pages 92–106. Springer, August 2022.

[PB23] Tobias Paxian and Armin Biere. Uncovering and classifying bugs in MaxSAT solvers through fuzzing and delta debugging. In *Proceedings of the 14th International Workshop on Pragmatics of SAT*, volume 3545 of *CEUR Workshop Proceedings*, pages 59–71. CEUR-WS.org, July 2023.

[PR16] Tobias Philipp and Adrián Rebola-Pardo. DRAT proofs for XOR reasoning. In *Proceedings of the 15th European Conference on Logics in Artificial Intelligence (JELIA '16)*, volume 10021 of *Lecture Notes in Computer Science*, pages 415–429. Springer, November 2016.

[RvBW06] Francesca Rossi, Peter van Beek, and Toby Walsh, editors. *Handbook of Constraint Programming*, volume 2 of *Foundations of Artificial Intelligence*. Elsevier, 2006.

[Urq87] Alasdair Urquhart. Hard examples for resolution. *Journal of the ACM*, 34(1):209–219, January 1987.

References XIII

[VDB22] Dieter Vandesande, Wolf De Wulf, and Bart Bogaerts. QMaxSATpb: A certified MaxSAT solver. In *Proceedings of the 16th International Conference on Logic Programming and Non-monotonic Reasoning (LPNMR '22)*, volume 13416 of *Lecture Notes in Computer Science*, pages 429–442. Springer, September 2022.

[WHH14] Nathan Wetzler, Marijn J. H. Heule, and Warren A. Hunt Jr. DRAT-trim: Efficient checking and trimming using expressive clausal proofs. In *Proceedings of the 17th International Conference on Theory and Applications of Satisfiability Testing (SAT '14)*, volume 8561 of *Lecture Notes in Computer Science*, pages 422–429. Springer, July 2014.