Tutorial on Conflict-Driven Pseudo-Boolean Solving

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Pseudo-Boolean?

Pseudo-Boolean (PB) function: $f : \{0,1\}^n \rightarrow \mathbb{R}$

Studied since 1960s in operations research and 0-1 integer linear programming [BH02]

Such a function f can always be represented as polynomial

Restriction for these lectures: f represented as linear form

Many problems expressible as optimizing value of linear pseudo-Boolean function under linear pseudo-Boolean constraints

Pseudo-Boolean vs. SAT

- PB format richer than conjunctive normal form (CNF)

Compare

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 3$$

and

$$\begin{aligned} & (x_1 \vee x_2 \vee x_3 \vee x_4) \wedge (x_1 \vee x_2 \vee x_3 \vee x_5) \wedge (x_1 \vee x_2 \vee x_3 \vee x_6) \\ \wedge\ & (x_1 \vee x_2 \vee x_4 \vee x_5) \wedge (x_1 \vee x_2 \vee x_4 \vee x_6) \wedge (x_1 \vee x_2 \vee x_5 \vee x_6) \\ \wedge\ & (x_1 \vee x_3 \vee x_4 \vee x_5) \wedge (x_1 \vee x_3 \vee x_4 \vee x_6) \wedge (x_1 \vee x_3 \vee x_5 \vee x_6) \\ \wedge\ & (x_1 \vee x_4 \vee x_5 \vee x_6) \wedge (x_2 \vee x_3 \vee x_4 \vee x_5) \wedge (x_2 \vee x_3 \vee x_4 \vee x_6) \\ \wedge\ & (x_2 \vee x_3 \vee x_5 \vee x_6) \wedge (x_2 \vee x_4 \vee x_5 \vee x_6) \wedge (x_3 \vee x_4 \vee x_5 \vee x_6) \end{aligned}$$

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- Yet close enough to SAT to benefit from SAT solving advances
- Also possible synergies with 0-1 integer linear programming (ILP)

Outline of Lecture on Pseudo-Boolean Solving

1. Preliminaries
   - Pseudo-Boolean Constraints
   - Pseudo-Boolean Solving and Optimization
2. Conflict-Driven Pseudo-Boolean Solving
   - The Conflict-Driven Paradigm
   - Pseudo-Boolean Reasoning Using Saturation
   - Pseudo-Boolean Reasoning Using Division
3. Going Beyond the State of the Art?
   - Challenges for Efficient PB Solving
   - Some Further References

Pseudo-Boolean Constraints and Normalized Form

For us, pseudo-Boolean constraints are always 0-1 integer linear constraints

$$\sum_i a_i \ell_i \bowtie A$$

- $\bowtie \in \{\geq, \leq, =, >, <\}$
- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
- variables x_i take values $0 = \mathit{false}$ or $1 = \mathit{true}$

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Convenient to use normalized form [Bar95] (without loss of generality)

$$\sum_i a_i \ell_i \geq A$$

- constraint always greater-than-or-equal
- $a_i, A \in \mathbb{N}$
- $A = \mathit{deg}(\sum_i a_i \ell_i \geq A)$ referred to as degree (of falsity)

Some Types of Pseudo-Boolean Constraints

1. Clauses are pseudo-Boolean constraints

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3. General constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Conversion to Normalized Form: Example

Normalized form used for convenience and without loss of generality

$$-x_1 + 2x_2 - 3x_3 + 4x_4 - 5x_5 < 0$$

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3. Replace $-\ell$ by $-(1-\overline{\ell})$ [where we define $\overline{\overline{x}} \doteq x$]

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4. Replace "$=$" by two inequalities "$\geq$" and "$\leq$"

Formulas, Decision Problems, and Optimization Problems

Pseudo-Boolean (PB) formula

Conjunction of pseudo-Boolean constraints

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Find satisfying assignment to F minimizing objective function $\sum_i w_i \ell_i$
(Maximization: minimize $-\sum_i w_i \ell_i$)

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This lecture:

- Focus on pseudo-Boolean solving
- But not hard to extend to (simple) optimization algorithm

Some Problems Expressed as PBO (1/2)

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Note: In all of these examples, the problem is to

- optimize a linear function
- subject to a CNF formula (all constraints are clausal)

Already expressive framework!

Approaches for Pseudo-Boolean Problems

What we will discuss in the coming lectures:

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Rough conceptual difference:

- **PB/SAT:** Focus on integral solutions, try to find optimal one
- **ILP/MIP:** Find optimal non-integer solution; search for integral solutions nearby

Basic trade-off: Inference power vs. inference speed

A Quick Recap of Modern SAT Solving

DPLL method [DP60, DLL62]

- Assign values to variables (in some smart way)
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$\mathcal{D} \leftarrow F$; `// initialize clause database to contain formula`
$\rho \leftarrow \emptyset$; `// initialize assignment trail to empty`
forever do
    if *ρ falsifies some clause $C \in \mathcal{D}$* **then**
        $A \leftarrow$ analyzeConflict$(\mathcal{D}, \rho, C)$;
        if $A = \perp$ **then** output `UNSATISFIABLE` and exit;
        else
            add A to $\mathcal{D}$ and backjump by shrinking ρ ;
    else if *exists clause $C \in \mathcal{D}$ unit propagating x to $b \in \{0,1\}$ under ρ* **then**
        add propagated assignment $x \overset{D}{=} b$ to ρ ;
    else if *time to restart* **then** $\rho \leftarrow \emptyset$;
    else if *time for clause database reduction* **then**
        erase (roughly) half of learned clauses in $\mathcal{D} \setminus F$ from $\mathcal{D}$
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Conflict Analysis Pseudocode

analyzeConflict$(\mathcal{D}, \rho, C_{\mathrm{confl}})$

$C_{\mathrm{learn}} \leftarrow C_{\mathrm{confl}}$;
while C_{learn} *not UIP clause* **and** $C_{\mathrm{learn}} \neq \bot$ **do**
  $\ell \leftarrow$ literal assigned last on trail ρ;
  if ℓ *propagated* **and** $\overline{\ell}$ *occurs in* C_{learn} **then**
    $C_{\mathrm{reason}} \leftarrow$ reason$(\ell, \rho, \mathcal{D})$;
    $C_{\mathrm{learn}} \leftarrow$ resolve$(C_{\mathrm{learn}}, C_{\mathrm{reason}})$;
  $\rho \leftarrow \rho \setminus \{\ell\}$;
return C_{learn};

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- At conflict
  1. Do conflict analysis to derive new constraint
  2. Add new constraint to constraint database
  3. Backjump by rolling back decisions so that learned constraint propagates asserting literal (flipping it to opposite value)

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$\{\overline{x}_5\}$	3	propagates $\overline{x}_4$ (coefficient $>$ slack)
$\{\overline{x}_5, \overline{x}_4\}$	3	propagation doesn't change slack

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ρ	$slack(C; \rho)$	comment
$\{\}$	8	
$\{\overline{x}_5\}$	3	propagates $\overline{x}_4$ (coefficient $>$ slack)
$\{\overline{x}_5, \overline{x}_4\}$	3	propagation doesn't change slack
$\{\overline{x}_5, \overline{x}_4, \overline{x}_3, x_2\}$	-2	conflict (slack < 0)

Propagation, Conflict, and Slack

Let ρ current assignment of solver (a.k.a. trail)
Represent as $\rho = \{$(ordered) set of literals assigned true$\}$

Slack measures how far ρ is from falsifying $\sum_i a_i \ell_i \geq A$

$$slack(\textstyle\sum_i a_i \ell_i \geq A; \rho) = \displaystyle\sum_{\ell_i \text{ not falsified by } \rho} a_i - A$$

Consider $C \doteq x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$

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$\{\overline{x}_5, \overline{x}_4, \overline{x}_3, x_2\}$	-2	conflict (slack < 0)

Note: constraint can be conflicting though not all variables assigned

Conflict Analysis Invariant

Consider example CDCL conflict analysis from SAT solving lecture

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$p \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{p \lor \overline{u}}{=} 0$

$q \stackrel{\mathsf{d}}{=} 0$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

$x \stackrel{\mathsf{d}}{=} 0$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

$\stackrel{\overline{y} \lor \overline{z}}{\bot}$

Conflict Analysis Invariant

Consider example CDCL conflict analysis from SAT solving lecture

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \vee \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \vee r}{=} 1$

$w \stackrel{\overline{r} \vee w}{=} 1$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$y \stackrel{u \vee x \vee y}{=} 1$

$z \stackrel{x \vee \overline{y} \vee z}{=} 1$

$\stackrel{\overline{y} \vee \overline{z}}{\perp}$

Assignment "left on trail" always falsifies derived clause

Conflict Analysis Invariant

Consider example CDCL conflict analysis from SAT solving lecture

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

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Assignment "left on trail" always falsifies derived clause

$\overline{y} \lor \overline{z}$ falsified by trail $\rho = \{\overline{p}, \overline{u}, \overline{q}, r, w, \overline{x}, y, z\}$

Conflict Analysis Invariant

Consider example CDCL conflict analysis from SAT solving lecture

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

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- $r \stackrel{q \lor r}{=} 1$
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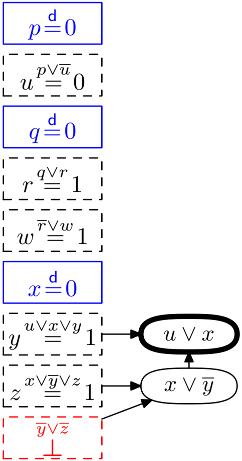
Conflict Analysis Invariant

Consider example CDCL conflict analysis from SAT solving lecture

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$



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Assignment "left on trail" always falsifies derived clause

$\Rightarrow$ derived clause "explains" conflict

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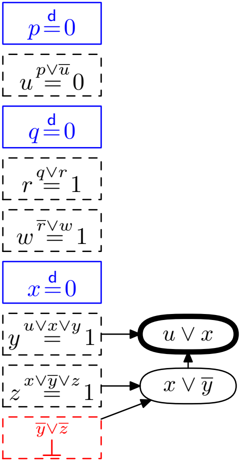

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$\Rightarrow$ derived clause "explains" conflict

Terminate analysis when explanation "looks nice"

Conflict Analysis Invariant

Consider example CDCL conflict analysis from SAT solving lecture

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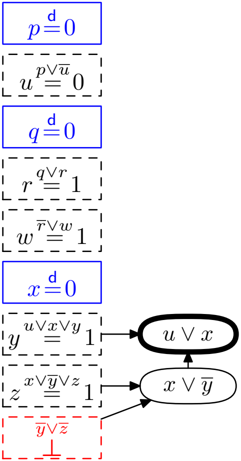

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Assignment “left on trail” always falsifies derived clause

$\Rightarrow$ derived clause “explains” conflict

Terminate analysis when explanation “looks nice”

Namely: after backjump, some variable guaranteed to flip

Generalized Resolution

Can mimic resolution step

$$\frac{x \lor \overline{y} \lor z \qquad \overline{y} \lor \overline{z}}{x \lor \overline{y}}$$

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by adding clauses as pseudo-Boolean constraints

$$\frac{x + \overline{y} + z \geq 1 \qquad \overline{y} + \overline{z} \geq 1}{x + 2\overline{y} \geq 1}$$

(Recall $z + \overline{z} = 1$)

Generalized Resolution

Can mimic resolution step

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(Recall $z + \overline{z} = 1$)

Generalized resolution rule (from [Hoo88, Hoo92])
Positive linear combination so that some variable cancels

$$\frac{a_1 x_1 + \sum_{i \geq 2} a_i \ell_i \geq A \qquad b_1 \overline{x}_1 + \sum_{i \geq 2} b_i \ell_i \geq B}{\sum_{i \geq 2} \left(\frac{c}{a_1} a_i + \frac{c}{b_1} b_i\right) \ell_i \geq \frac{c}{a_1} A + \frac{c}{b_1} B - c} \; [c = \mathrm{lcm}(a_1, b_1)]$$

Saturation

Actually, not quite the right constraint in mimicking of resolution

$$\frac{x + \overline{y} + z \geq 1 \qquad \overline{y} + \overline{z} \geq 1}{x + 2\overline{y} \geq 1}$$

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Saturation rule

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \min\{a_i, A\} \cdot \ell_i \geq A}$$

Sound over integers, not over reals (need such rules for SAT solving)

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Sound over integers, not over reals (need such rules for SAT solving)

[Generalized resolution as defined in [Hoo88, Hoo92] includes fix above, but convenient here to make the two separate steps explicit]

Analyze Conflict with Generalized Resolution + Saturation!

$$C_1 \doteq 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4$$
$$C_2 \doteq 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3$$

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Trail $\rho = \{x_1 \stackrel{\mathsf{d}}{=} 0, x_2 \stackrel{C_1}{=} 1, x_3 \stackrel{C_1}{=} 1\}$ $\Rightarrow$ Conflict with C_2
(Note: same constraint can propagate several times!)

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- Resolve reason$(x_3, \rho) = C_1$ with C_2 over x_3 to get resolve(C_1, C_2, x_3)

$$\frac{2x_1 + 2x_2 + 2x_3 + x_4 \geq 4 \qquad 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3}{x_4 \geq 1}$$

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- Applying saturate$(x_4 \geq 1)$ does nothing
- Non-negative slack w.r.t. $\rho' = \{x_1 \stackrel{\mathsf{d}}{=} 0, x_2 \stackrel{C_1}{=} 1\}$
  Not conflicting! Does not explain mistake in assignment

What Went Wrong? And What to Do About It?

Accident report

- Generalized resolution sound over the reals
- Given $\rho' = \{x_1 = 0, x_2 = 1\}$, over the reals have
  - $C_1 \doteq 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4$ propagates $x_3 \geq \frac{1}{2}$
  - $C_2 \doteq 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3$ satisfied by $x_3 \leq \frac{1}{2}$
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Remedial action

- Strengthen propagation to $x_3 \geq 1$ also over the reals
- I.e., want reason C with $slack(C; \rho') = 0$
- **Fix** (non-obvious): Apply weakening

  $$\text{weaken}\left(\textstyle\sum_i a_i\ell_i \geq A, \ell_j\right) \doteq \textstyle\sum_{i \neq j} a_i\ell_i \geq A - a_j$$

  to reason constraint and then saturate
- Approach in [CK05] (goes back to observations in [Wil76])

Try to Reduce the Reason Constraint

$$C_1 \doteq 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4$$
$$C_2 \doteq 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3$$

Trail $\rho = \{x_1 \stackrel{\mathsf{d}}{=} 0, x_2 \stackrel{C_1}{=} 1, x_3 \stackrel{C_1}{=} 1\}$ $\Rightarrow$ Conflict with C_2

Let's try to

1. Weaken reason on non-falsified literal (but not last propagated)
2. Saturate weakened constraint
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$$\dfrac{\text{saturate}\ \dfrac{\text{weaken } x_2\ \dfrac{2x_1 + 2x_2 + 2x_3 + x_4 \geq 4}{2x_1 + 2x_3 + x_4 \geq 2}}{2x_1 + 2x_3 + x_4 \geq 2} \qquad 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3}{2\overline{x}_2 + x_4 \geq 1}\ \text{resolve } x_3$$

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$$\text{resolve } x_3 \dfrac{\text{saturate } \dfrac{\text{weaken } x_2 \dfrac{2x_1 + 2x_2 + 2x_3 + x_4 \geq 4}{2x_1 + 2x_3 + x_4 \geq 2}}{2x_1 + 2x_3 + x_4 \geq 2} \qquad 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3}{2\overline{x}_2 + x_4 \geq 1}$$

Bummer! Still non-negative slack — not conflicting

Try Again to Reduce the Reason Constraint...

$$C_1 \doteq 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4$$
$$C_2 \doteq 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3$$

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$$\dfrac{\dfrac{\dfrac{2x_1 + 2x_2 + 2x_3 + x_4 \geq 4}{2x_1 + 2x_3 \geq 1}\,\text{weaken } \{x_2, x_4\}}{x_1 + x_3 \geq 1}\,\text{saturate} \qquad 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3}{2\overline{x}_2 \geq 1}\,\text{resolve } x_3$$

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$$\dfrac{\text{resolve } x_3 \dfrac{\text{saturate } \dfrac{\text{weaken } \{x_2, x_4\} \dfrac{2x_1 + 2x_2 + 2x_3 + x_4 \geq 4}{2x_1 + 2x_3 \geq 1}}{x_1 + x_3 \geq 1}\qquad 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3}{2\overline{x}_2 \geq 1}$$

Negative slack — conflicting! Shows setting x_2 true was a mistake

Try Again to Reduce the Reason Constraint...

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$$\cfrac{\cfrac{\cfrac{2x_1 + 2x_2 + 2x_3 + x_4 \geq 4}{2x_1 + 2x_3 \geq 1}\ \text{weaken } \{x_2, x_4\}}{x_1 + x_3 \geq 1}\ \text{saturate} \qquad 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3}{2\overline{x}_2 \geq 1}\ \text{resolve } x_3$$

Negative slack — conflicting! Shows setting x_2 true was a mistake

Backjump propagates to conflict without solver making any decisions
Done! Next conflict analysis will derive contradiction
(Or, in practice, terminate immediately at conflict without decisions)

Reason Reduction Using Saturation [CK05]

reduceSat($C_{\text{reason}}, C_{\text{learn}}, \ell, \rho$)

while slack(resolve($C_{\text{learn}}, C_{\text{reason}}, \ell$); ρ) ≥ 0 **do**
    $\ell' \leftarrow$ literal in $C_{\text{reason}} \setminus \{\ell\}$ not falsified by ρ;
    $C_{\text{reason}} \leftarrow$ saturate(weaken(C_{reason}, ℓ'));
return C_{reason};

Reason Reduction Using Saturation [CK05]

reduceSat$(C_{\mathrm{reason}}, C_{\mathrm{learn}}, \ell, \rho)$

while $\mathit{slack}(\text{resolve}(C_{\mathrm{learn}}, C_{\mathrm{reason}}, \ell); \rho) \geq 0$ **do**
  $\ell' \leftarrow$ literal in $C_{\mathrm{reason}} \setminus \{\ell\}$ not falsified by ρ;
  $C_{\mathrm{reason}} \leftarrow \text{saturate}(\text{weaken}(C_{\mathrm{reason}}, \ell'))$;
return C_{reason};

Why does this work?

- Slack is subadditive

$$\mathit{slack}(c \cdot C + d \cdot D; \rho) \leq c \cdot \mathit{slack}(C; \rho) + d \cdot \mathit{slack}(D; \rho)$$

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- By invariant have $\mathit{slack}(C_{\text{learn}}; \rho) < 0$
- Weakening leaves $\mathit{slack}(C_{\text{reason}}; \rho)$ unchanged
- Saturation decreases slack — hit 0 when max #literals weakened

Pseudo-Boolean Conflict Analysis Pseudocode

analyzePBconflict($\mathcal{D}, \rho, C_{\text{confl}}$)

$C_{\text{learn}} \leftarrow C_{\text{confl}}$;
while C_{learn} *not asserting* **and** $C_{\text{learn}} \neq \bot$ **do**
  $\ell \leftarrow$ literal assigned last on trail ρ;
  if ℓ *propagated* **and** $\overline{\ell}$ *occurs in* C_{learn} **then**
    $C_{\text{reason}} \leftarrow$ reason($\ell, \rho, \mathcal{D}$);
    $\boldsymbol{C_{\text{reason}} \leftarrow}$ **reduceSat**$\boldsymbol{(C_{\text{reason}}, C_{\text{learn}}, \ell, \rho)}$;
    $C_{\text{learn}} \leftarrow$ resolve($C_{\text{learn}}, C_{\text{reason}}, \ell$);
    $C_{\text{learn}} \leftarrow$ saturate(C_{learn});
  $\rho \leftarrow \rho \setminus \{\ell\}$;
return C_{learn};

Reduction of reason new compared to CDCL — otherwise the same

Essentially conflict analysis used in SAT4J [LP10]

Some Problems Compared to CDCL

- Compared to clauses harder to detect propagation for constraints like
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$\Rightarrow$ lots of lcm computations
$\Rightarrow$ coefficient sizes can explode (expensive arithmetic)
- For CNF inputs, degenerates to resolution!
$\Rightarrow$ CDCL but with super-expensive data structures

The Cutting Planes Proof System

Cutting planes as defined in theory literature [CCT87] doesn't use saturation but instead division (a.k.a. Chvátal-Gomory cut)

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Linear combination $$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i) \ell_i \geq c_A A + c_B B}$$

Division $$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \lceil a_i / c \rceil \ell_i \geq \lceil A/c \rceil}$$

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- Cutting planes with division implicationally complete
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- Can division yield stronger conflict analysis?
  (Used for integer linear programming in CutSat [JdM13])

Using Division to Reduce the Reason

$$
\begin{aligned}
C_1 &\doteq 2x_1 + 2x_2 + 2x_3 + x_4 \geq 4 \\
C_2 &\doteq 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3
\end{aligned}
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Trail $\rho = \{x_1 \overset{\mathsf{d}}{=} 0, x_2 \overset{C_1}{=} 1, x_3 \overset{C_1}{=} 1\}$ $\Rightarrow$ Conflict with C_2

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Terminate immediately!

Reason Reduction Using Division [EN18]

reduceDiv$(C_{\mathrm{reason}}, C_{\mathrm{learn}}, \ell, \rho)$

$c \leftarrow \mathit{coeff}(C_{\mathrm{reason}}, \ell)$;
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So now why does **this** work?

- Sufficient to get reason with slack 0 since
  1. $\mathit{slack}(C_{\text{learn}}; \rho) < 0$
  2. slack is subadditive

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- Slack same after weakening $\Rightarrow$ always $0 \leq \mathit{slack}(C_{\mathrm{reason}}; \rho) < c$
- After max #weakenings have $0 \leq \mathit{slack}(\mathsf{divide}(C_{\mathrm{reason}}, c); \rho) < 1$

Division vs. Saturation

- Higher conflict speed when PB reasoning doesn't help [EN18]
- Seems to perform better when PB reasoning crucial [EGNV18]
- Keeps coefficients small — can (often) do fixed-precision arithmetic
- But SAT4J still better for some circuit verification problems [LBD+20]
- And still equally hard to detect propagation
- Also, still degenerates to resolution for CNF inputs
- Sometimes very poor performance even on infeasible 0-1 LPs!

Some PB Solving Challenges I: Input Format

1. CNF: PB solvers degenerate to CDCL for CNF inputs — how to harness power of cutting planes in this setting?
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   - Use presolver PaPILO [PaP] from mixed integer linear programming (MIP) solver SCIP [SCI]?
4. Robustness: Make PB solvers less sensitive to presence of extra constraints (anecdotally, CDCL solvers seem more stable)

Some PB Solving Challenges II: Conflict Analysis

1. Choice of Boolean rule:
   - Division, saturation, or select adaptively?
   - Or some other cut rule from ILP?
   - Try to avoid irrelevant literals? [LMMW20]

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2. Many more degrees of freedom than in CDCL:
    - Skip resolution steps when slack very negative?
    - How aggressively to weaken reason in reduction step? [LMW20]
    - Learn general PB constraints or more limited form?
    - How far to backjump when choice of several levels?
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3. Do constraint minimization à la [SB09, HS09]?
4. How to assess quality of learned constraints?
5. Theoretical potential & limitations poorly understood [VEG+18]
   - Separations in power between different methods of PB reasoning?
   - In particular, is division-based reasoning stronger than saturation-based reasoning? [GNY19]

Some PB Solving Challenges III: Solver Heuristics

Many heuristics more or less copied from CDCL — maybe tailor more carefully to PB setting?

1. Variable selection: VSIDS [MMZ+01] or VMTF [Rya04] or something else?

Some PB Solving Challenges III: Solver Heuristics

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3. Phase saving: Standard as in [PD07], multiple phases [BF20], or something else?
4. Different "modes" for SAT-focused and UNSAT-focused search?

See [Wal20] for a first in-depth investigation of some of these questions

Some PB Solving Challenges IV: Efficiency and Correctness

1. Efficient unit propagation for PB constraints is a major challenge — latest news in [Dev20], but still much left to do

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Some PB Solving Challenges IV: Efficiency and Correctness

1. Efficient unit propagation for PB constraints is a major challenge — latest news in [Dev20], but still much left to do
2. Efficient detection of assertiveness during conflict analysis
3. Efficient and concise proof logging for pseudo-Boolean solving (shameless self-plug: ongoing work on pseudo-Boolean proof checker VeriPB [Ver, GMN20b] in [EGMN20, GMN20a, GMM+20, GN21, BGMN22, GMNO22])

Some References for Further Reading (and Watching)

Handbook of Satisfiability [BHvMW21]

- Chapter 7: Proof Complexity and SAT Solving
- Chapter 23: MaxSAT, Hard and Soft Constraints
- Chapter 24: Maximum Satisfiability
- Chapter 28: Pseudo-Boolean and Cardinality Constraints

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- Chapter 28: Pseudo-Boolean and Cardinality Constraints

Video tutorials on pseudo-Boolean solving

From the *Satisfiability: Theory, Practice, and Beyond* program at UC Berkeley in spring 2021
`https://tinyurl.com/PBSATtutorial`
[Try to cover as much of this as possible today]

Summing up

- Pseudo-Boolean framework expressive and powerful
- Can be approached using successful conflict-driven paradigm from SAT solving
- In theory, potential for exponential increase in performance
- In practice, some highly nontrivial challenges regarding
  - Algorithm design
  - Efficient implementation
  - Theoretical understanding
- But maybe also quite a bit of low-hanging fruit?
  (And clause-based SAT solving took 50+ years to get right)
- In any case, lots of fun questions to work on! ☺

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Thank you for your attention!

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