Certified Implicit Hitting Set Solving for Pseudo-Boolean Optimization

Benjamin Bogø Xiamin Chen Wietze Koops Pinyan Lu **Jakob Nordström**
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Dagstuhl Workshop 25371
Interactions in Constraint Optimization
September 11, 2025

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(*) *Thanks for the slides!*

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- This talk: 0-1 linear objective and inequalities (pseudo-Boolean in SAT-speak)
- IHS solving: Benders decomposition in OR-speak

Certified Solving using Proof Logging

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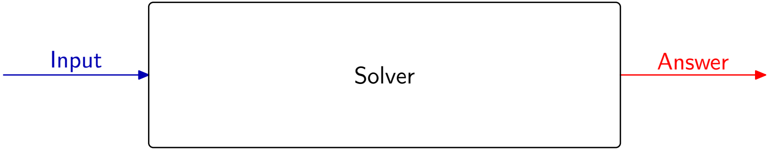
Certified Solving using Proof Logging

- Modern combinatorial solvers very fast, but sometimes wrong [BLB10, AGJ$^+$18, GSD19]
- Only currently feasible way of addressing this: Proof logging
  - Make solver certifying [ABM$^+$11, MMNS11] by adding code so that it outputs
  - not only answer but also
  - simple, machine-verifiable proof that answer is correct

Proof Logging with Certifying Solvers: Workflow



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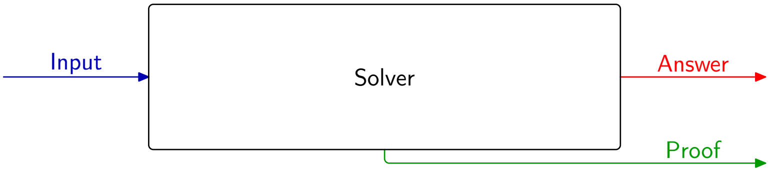

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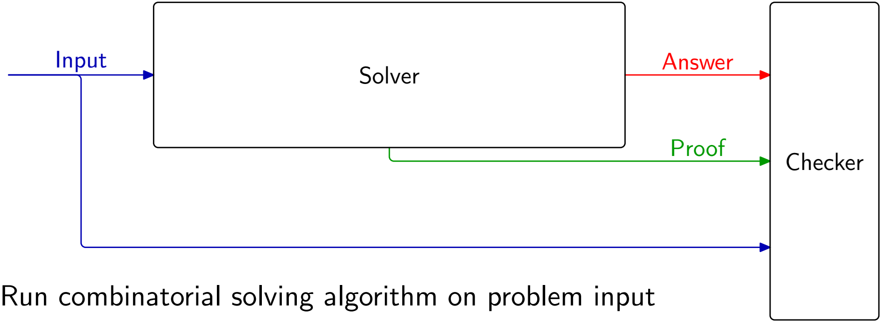

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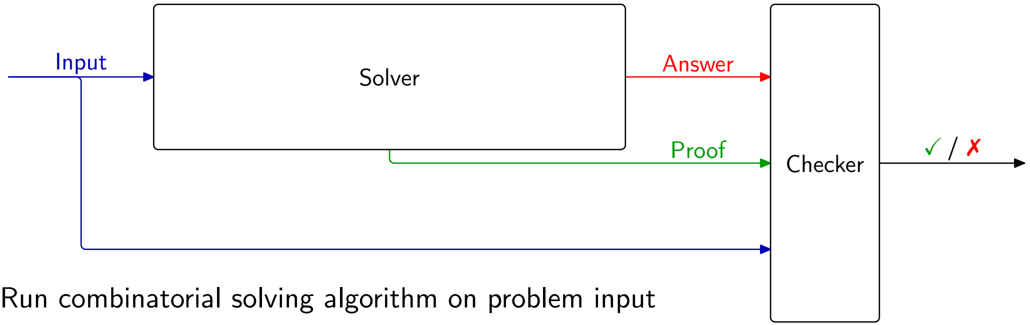

1. Run combinatorial solving algorithm on problem input
2. Get as output not only answer but also proof
3. Feed input + answer + proof to proof checker
4. Verify that proof checker says answer is correct

IHS Proof Logging

- Proof logging implemented for state-of-the-art solvers for other optimization paradigms
  - Solution-improving search [BBN+24]
  - Core-guided search [VDB22, BBN+23]

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- Successful IHS implementations for both MaxSAT [DB11] and pseudo-Boolean optimization [SBJ21, SBJ22], but so far no proof logging for IHS
  - Mixed integer programming (MIP) solver used for hitting set problem
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  - Also, not known how to do proof logging for all MIP solving techniques

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- Possible approaches to get certified IHS solving:
  1. Use pseudo-Boolean solver with proof logging for hitting set problem
  2. Use local search to find solutions for hitting set problem
  3. Find optimal solution with MIP, then let other certifying solver prove lower bound

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- Study if and why MIP technique crucial for implicit hitting set solving
- Compare pros and cons from point of view certified solving
- Explore ways of integrating IHS in “hybrid methods” using also other optimization paradigms (cf. [DGD+21, DGN21])

Pseudo-Boolean Optimization (PBO) Problem

- Pseudo-Boolean formula $\mathcal{F}$: collection of 0-1 integer linear inequalities

Example

$$
\begin{aligned}
x_1 + x_2 + 2\,\overline{x_4} &\geq 2 \\
x_1 + 2\,x_3 + \overline{x_5} &\geq 2 \\
x_4 + x_5 &\geq 1
\end{aligned}
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- 0-1 linear objective function $\mathcal{O}$ to minimize

Example

$$\text{min: } x_1 + x_2 + 3\,x_3$$

Implicit Hitting Set Solving in More Detail

- Split PBO problem $(\mathcal{F}, \mathcal{O})$ into two subproblems

PBO formula

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\text{min: } x_1 + x_2 + 3\,x_3 \\
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  - decision subproblem $\mathcal{F}$ (all constraints)

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Decision subproblem

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Implicit Hitting Set Solving in More Detail

- Split PBO problem $(\mathcal{F}, \mathcal{O})$ into two subproblems
  - decision subproblem $\mathcal{F}$ (all constraints)
  - hitting set subproblem (core constraints over objective variables only)

PBO formula

$$\begin{aligned} \text{min: } & x_1 + x_2 + 3\,x_3 \\ x_1 + x_2 + 2\,\overline{x_4} &\geq 2 \\ x_1 + 2\,x_3 + \overline{x_5} &\geq 2 \\ x_4 + x_5 &\geq 1 \end{aligned}$$

Decision subproblem

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Hitting set subproblem

$$\text{min: } x_1 + x_2 + 3\,x_3$$

(core constraints over x_1, x_2, x_3)

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Example

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Decision subproblem

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Example

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$$\begin{aligned} \text{min: } x_1 + x_2 + 3\,x_3 \\ x_1 + x_3 \geq 1 \\ x_2 + x_3 \geq 1 \end{aligned}$$

Example

$$\{\overline{x_1}, \overline{x_2}, \overline{x_3}\} \rightarrow x_1 + x_3 \geq 1$$
$$\{x_1, \overline{x_2}, \overline{x_3}\} \rightarrow x_2 + x_3 \geq 1$$
$$\{x_1, x_2, \overline{x_3}\}$$

Decision subproblem

$$\begin{aligned} x_1 + x_2 + 2\,\overline{x_4} \geq 2 \\ x_1 + 2\,x_3 + \overline{x_5} \geq 2 \\ x_4 + x_5 \geq 1 \end{aligned}$$

Implicit Hitting Set Solving

1. Find optimal solution α to current core constraints
2. Try to extend α to solution for all constraints in $\mathcal{F}$
   a. Solution extended: Optimum found!
   b. Otherwise: Extract new core & go to 1

Hitting set subproblem

$$\text{min: } x_1 + x_2 + 3\,x_3$$
$$x_1 + x_3 \geq 1$$
$$x_2 + x_3 \geq 1$$

Example

$$\{\overline{x_1}, \overline{x_2}, \overline{x_3}\} \to x_1 + x_3 \geq 1$$
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$$\{x_1, x_2, \overline{x_3}\} \to \{x_1, x_2, \overline{x_3}, x_4, \overline{x_5}\}$$

Decision subproblem

$$x_1 + x_2 + 2\,\overline{x_4} \geq 2$$
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$$x_4 + x_5 \geq 1$$

Proof Logging for IHS Solving in More Detail

- Reasoning for decision subproblem
  - Conflict-driven search — use pseudo-Boolean proof logging [KLM+25]
  - Core extraction — just special case of conflict analysis
    (so-called decision learning scheme)

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- Reasoning for hitting set subproblem
  - More challenging
  - Incremental problem — new core constraints keep getting added

Proof Logging for Implicit Hitting Set Subproblem

- Optimization solvers use found solutions to trim search space
  - Infer new constraints from requirement to improve solution further
  - Solution with value $v \Rightarrow$ add objective-improving constraint $\mathcal{O} \leq v - 1$

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  - As hitting set subproblem grows, optimal solution gets worse
  - Previous objective-improving constraints too optimistic
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  1. Start hitting set optimizer over from scratch each time
  2. Manual book-keeping of valid constraints
  3. Automatic book-keeping via reified constraints

Book-keeping for Invalidated Constraints

Hitting set subproblem

$$\text{min: } x_1 + x_2 + 3\,x_3$$

Book-keeping for Invalidated Constraints

Hitting set subproblem

$$\text{min: } x_1 + x_2 + 3\,x_3$$

$$x_1 + x_2 + 3\,x_3 \leq 4 \qquad \text{Solution: } 5 \tag{1}$$

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$$x_1 + x_2 + 3\,x_3 \leq 4 \qquad \text{Solution: } 5 \qquad (1)$$

$$x_1 + x_2 + 3\,x_3 \leq -1 \qquad \text{Optimum: } 0 \qquad (2)$$

Book-keeping for Invalidated Constraints

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$x_1 + x_2 + 3\,x_3 \leq 4$	Solution: 5	(1)
$x_1 + x_2 + 3\,x_3 \leq -1$	Optimum: 0	(2)
$x_1 + x_3 \geq 1$	Add core constraint	(3)

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$x_1 + \overline{x_2} \geq 1$	Infer by (3) and (4)	(5)

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$x_1 + x_2 + 3\,x_3 \leq 0$	Optimum: 1	(6)

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$x_1 + x_2 + 3\,x_3 \leq 1$	Optimum: 2	(8)

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$x_1 + x_2 + 3\,x_3 \leq 1$	Optimum: 2	(8)

Book-keeping for Invalidated Constraints

Hitting set subproblem

$$\text{min: } x_1 + x_2 + 3\,x_3$$

$-\overline{s_5} + x_1 + x_2 + 3\,x_3 \leq 4$	Solution: 5	(1)
$x_1 + x_2 + 3\,x_3 \leq -1$	Optimum: 0	(2)
$x_1 + x_3 \geq 1$	Add core constraint	(3)
$x_1 + x_2 + 3\,x_3 \leq 3$	Solution: 4	(4)
$x_1 + \overline{x_2} \geq 1$	Infer by (3) and (4)	(5)
$x_1 + x_2 + 3\,x_3 \leq 0$	Optimum: 1	(6)
$x_2 + x_3 \geq 1$	Add core constraint	(7)
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Hitting set subproblem

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$$
\begin{aligned}
-\,\overline{s_5} + x_1 + x_2 + 3\,x_3 &\leq 4 && \text{Solution: } 5 && (1)\\
-6\,\overline{s_0} + x_1 + x_2 + 3\,x_3 &\leq -1 && \text{Optimum: } 0 && (2)\\
x_1 + x_3 &\geq 1 && \text{Add core constraint} && (3)\\
-2\,\overline{s_4} + x_1 + x_2 + 3\,x_3 &\leq 3 && \text{Solution: } 4 && (4)\\
x_1 + \overline{x_2} &\geq 1 && \text{Infer by (3) and (4)} && (5)\\
-5\,\overline{s_1} + x_1 + x_2 + 3\,x_3 &\leq 0 && \text{Optimum: } 1 && (6)\\
x_2 + x_3 &\geq 1 && \text{Add core constraint} && (7)\\
-4\,\overline{s_2} + x_1 + x_2 + 3\,x_3 &\leq 1 && \text{Optimum: } 2 && (8)
\end{aligned}
$$

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What About Performance?

- Work in progress — so far, so crappy...
- Book-keeping with reified objective-improving constraints involves serious challenges
- But the solver works!
- First certifying IHS solver with proofs that can be checked (somewhat) efficiently
- Submitted to standard and certified tracks of Pseudo-Boolean Competition 2025 [Pse25]
- Not great competition results, but not the worst solver either
  (which is a bit of a miracle given how many features are missing)

Limited Experimental Evaluation

Set-up:

- Benchmarks: Pseudo-Boolean Competition 2024 OPT-LIN optimization instances [Pse24]
- Memory: 16 GB
- Timeout: 3600s (1 hour)

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  - RoundingSat for both decision and hitting set subproblems (two different solvers)

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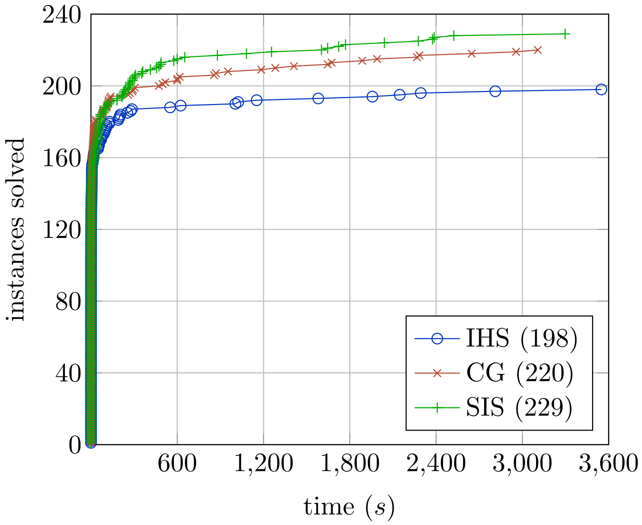
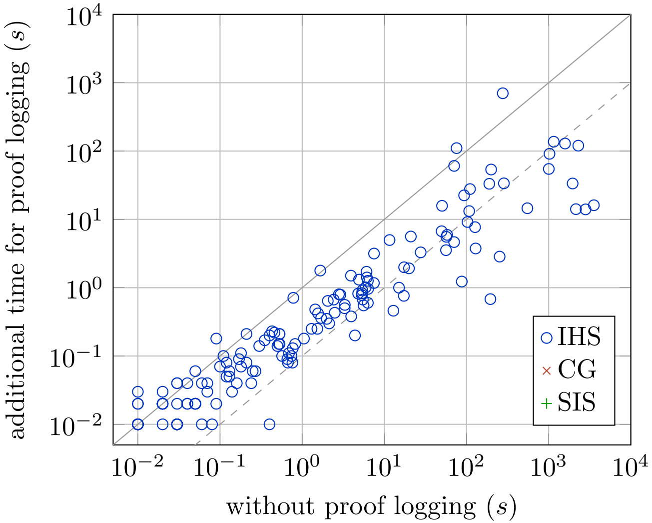
Evaluate

- Pure implicit hitting set (IHS) solving
  - RoundingSat for both decision and hitting set subproblems (two different solvers)
- Compared to core-guided (CG) and solution-improving search (SIS) [KLM$^+$25]
  - Both as implemented in RoundingSat
  - . . . Which uses LP solver SoPlex as important subroutine

Time vs Solved Instances



Solving Time vs Proof Logging Time



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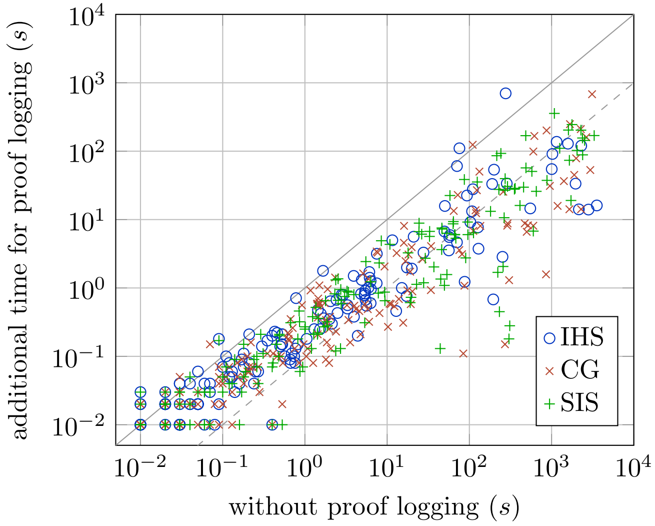
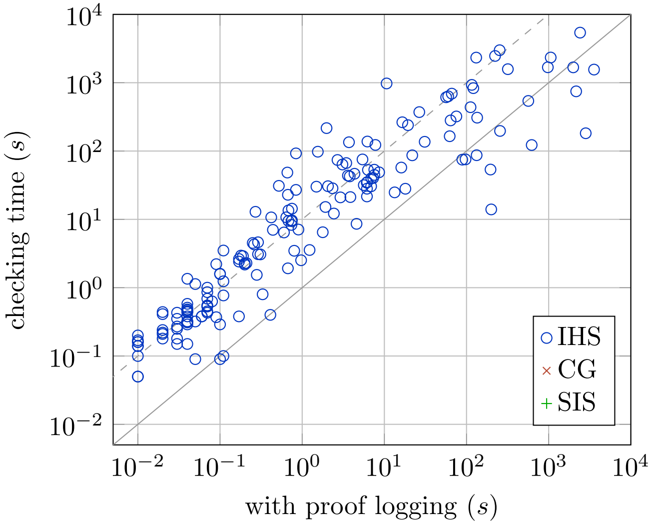


Solving and Proof Logging Time vs Checking Time



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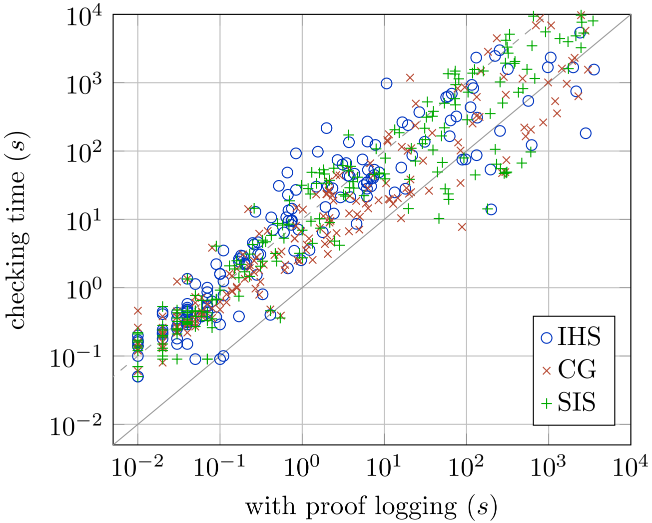

Future Work

- Pseudo-Boolean (PB) solving
  - More efficient book-keeping (with or without reified variables)

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- Investigate trade-offs between MIP usage and proof logging by comparing
  - MIP solver for hitting set + PB solver generating proof for claimed lower bound
  - PB hitting set optimizer with book-keeping for objective-improving constraints

Conclusion

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  - Improve performance of book-keeping for objective-improving constraints
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  - Understand if and why MIP solving is crucial
  - Make certified IHS solving competitive with other optimization approaches (by making it part of hybrid methods)

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Thank you for your attention!

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