Graph Colouring Is Hard on Average for Polynomial Calculus and Nullstellensatz

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University of Copenhagen and Lund University

Milestones and Motifs in the Theory of Proofs, Algebraic Computation, and Lower Bounds
IIT Gandhinagar
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Joint work with Jonas Conneryd, Susanna de Rezende, Shuo Pang, and Kilian Risse

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Thanks for the slides!

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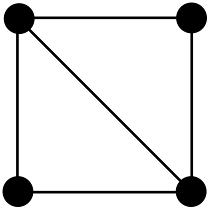
Can vertices of graph G be coloured with k colours so that all neighbours get distinct colours?

One of Karp's 21 NP-complete problems



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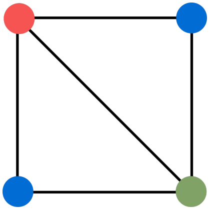

Is Graph Colouring Hard?

Colouring seems hard even to approximate:

- If G k-colourable, best efficient algorithm uses $k \cdot \widetilde{\Omega}(n)$ colours [Halldorsson 93]
- If G 3-colourable, best algorithm uses $n^{0.199\cdots}$ colours [Kawarabayashi–Thorup 17]
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However, applied algorithms appear to do well:

- Backtracking and SAT-based algorithms
  [San Segundo 12, Hebrard–Katsirelos 20, Heule–Karahalios–van Hoeve 22]
- Integer programming
  [Mehortra–Trick 95, Gualandi–Malucelli 12]
- Algebraic algorithms
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Can we prove that k-colouring is hard for these algorithms?

Hardness for Algebraic Algorithms

- Exponential lower bounds known for explicit graphs
  [Lauria–Nordström 17, Atserias–Ochremiak 19]
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Perhaps graph colouring is *easy on most graphs?*

To rule this out, want average-case hardness results

SAT-based algorithms **[Beame–Culberson–Mitchell–Moore 05]**

Conflict-driven clause learning (CDCL) SAT solvers need exponential time for k-colouring on random graphs for $k \geq 3$

Our Result

Theorem

Algorithms based on Hilbert Nullstellensatz and/or Gröbner bases require exponential time to solve k-colouring on random graphs for $k \geq 3$

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Established via proof complexity:

- Formalise reasoning method in algorithm as a proof system
- Fast execution for non-k-colourable graph G yields short proof of statement "G is not k-colourable"
- Show that such short proofs do not exist

Nullstellensatz Proof System

To show polynomials $p_1, \ldots, p_m$ in $\mathbb{F}[\vec{x}]$, have no common root in $\mathbb{F}$, suffices to find polynomials $q_1, \ldots, q_m$ in $\mathbb{F}[\vec{x}]$ such that

$$\sum_{i=1}^{m} q_i(\vec{x}) \cdot p_i(\vec{x}) = 1$$

This is a Nullstellensatz proof of unsatisfiability

[Beame–Impagliazzo–Krajíček–Pitassi–Pudlák 96]

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Soundness: if such polynomials q_i exist, then clearly $\{p_i\}$ have no common root

Completeness (Boolean variables): special case of Hilbert's Nullstellensatz

Polynomial Calculus Proof System [Clegg–Edmonds–Impagliazzo 96]

Dynamic version: given $\{p_1, \ldots, p_m\}$, derive new polynomials using two rules

(linear combination)	$$\frac{p \quad q}{\alpha p + \beta q}$$	$\alpha, \beta \in \mathbb{F}$
(multiplication)	$$\frac{p}{x \cdot p}$$	x variable

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Polynomial calculus proof system models Gröbner basis computations

- **Proof size:** # of monomials in derivation
  Make proof system stronger by allowing dual variables $\overline{x}_i$ for negative literals
  [Alekhnovich–Ben-Sasson–Razborov–Wigderson 02]
- **Proof degree:** max total degree of polynomial in derivation

Encoding k-Colouring as Polynomials

Variables $x_{v,i}$ = "vertex v gets colour i", $v \in V(G)$, $i \in [k]$

Axiom polynomials for graph G:

Each vertex gets a colour	$\sum_{i=1}^{k} x_{v,i} - 1$	
Colours are unique	$x_{v,i} \cdot x_{v,i'}$	$i \neq i'$
Distinct colours for neighbours	$x_{u,i} \cdot x_{v,i}$	$(u,v) \in E(G)$
Variables are Boolean	$x_{v,i}^2 - x_{v,i}$	

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Other important encoding used in computational algebra [Bayer 82]:

- Colours X_v are kth roots of unity $\{1, \zeta, \zeta^2, \cdots, \zeta^{k-1}\}$ (assuming $\mathrm{char}(\mathbb{F}) \nmid k$)
- Linear substitution from X_v to $x_{v,1}, \ldots, x_{v,k}$ $\Rightarrow$ (roughly) same proof degree

More Formal Statement of Result

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For G random sparse graph on n vertices, with probability $1 - o(1)$ any polynomial calculus proof of fact "G is not 3-colourable" has size $\exp(\Omega(n))$

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- For both random regular graphs and Erdős–Rényi random graphs (with appropriately chosen parameters)
- Obtained by showing $\Omega(n)$ degree lower bound
- Implies exponential size lower bound for Boolean encoding

[Impagliazzo–Pudlák–Sgall 99]

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Task: separate 1 from $\{$polynomials derivable in degree $D\}$

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[Razborov 98]: suffices to find linear $R : \mathbb{F}[\vec{x}] \longrightarrow \mathbb{F}[\vec{x}]$ such that

1. $R(\text{axiom}) = 0$
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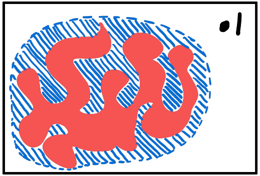

Derivable in degree D

$\ker(R)$

R overapproximates what is derivable in degree D

Quick Recap: Polynomial Ideals

Given set of polynomials $\mathcal{P}$, ideal $\langle \mathcal{P} \rangle$ is smallest set such that

- $\mathcal{P} \subseteq \langle \mathcal{P} \rangle$
- $p, q \in \langle \mathcal{P} \rangle \Rightarrow p + q \in \langle \mathcal{P} \rangle$
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Connection to polynomial calculus:

- $\langle \mathcal{P} \rangle$ contains all polynomial implied by $\mathcal{P}$
- Which is exactly what is derivable by polynomial calculus
- $1 \in \langle \mathcal{P} \rangle \Leftrightarrow \mathcal{P}$ is unsatisfiable

Polynomial Ideal Reductions

- Impose total order on monomials (with 1 smallest)
- Order polynomials by largest monomial (leading monomial)
- **Reduction modulo ideal $\langle \mathcal{P} \rangle$:** Operator $R_{\langle \mathcal{P} \rangle} : \mathbb{F}[\vec{x}] \to \mathbb{F}[\vec{x}]$ defined as

$$R_{\langle \mathcal{P} \rangle}(q) := \text{minimum polynomial in } \{q - r \mid r \in \langle \mathcal{P} \rangle\}$$

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Properties of $R_{\langle \mathcal{P} \rangle}$:

- well-defined
- linear
- $\ker\left(R_{\langle \mathcal{P} \rangle}\right) = \langle \mathcal{P} \rangle$
- $R^2_{\langle \mathcal{P} \rangle} = R_{\langle \mathcal{P} \rangle}$

Example of Polynomial Reduction

Consider $\mathbb{F}[x, y]$ and ideal generated by $\{x + y\}$.

- Order $x \succ y$ extended to all monomials (lexicographically, say)
- $\mathcal{R}_{\langle x+y \rangle} : \; x^a y^b \mapsto (-1)^a y^{a+b}$

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- For each monomial m, reduce m modulo ideal of subset $S(m)$ of axioms
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Intuition:

- $S(m)$ contains axioms "closely related" to variables in m
- R indistinguishable from polynomial ideal reduction in low degree, but $R(1) \neq 0$
- Think of R as pseudo-reduction modulo fake ideal claiming that $\mathcal{P}$ is satisfiable

From Pseudo-reductions to Degree Lower Bounds

Recall that we want three properties from linear operator R:

1. $R(\text{axiom}) = 0$
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This would show:

- All input axioms in $\mathcal{P}$ are in $\ker(R)$
- All polynomials derivable from $\mathcal{P}$ in degree $\leq D$ are in $\ker(R)$
- But $1 \notin \ker(R)$
- So degree lower bound $> D$ follows

Getting Pseudo-reductions to Behave Well

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- Dream scenario: Show that there exists ideal $\mathcal{I}$ such that
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- Then

$$R(p) = R_{\langle S(m_1)\rangle}(m_1) + R_{\langle S(m_2)\rangle}(m_2) = R_{\mathcal{I}}(m_1) + R_{\mathcal{I}}(m_2) = R_{\mathcal{I}}(m_1 + m_2) = 0$$

Why Aren't We Done Already?

- All of this is old news...
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- Crucial new ideas in [Romero-Tunçel 22] — more about that later

Degree Lower Bounds for Colouring

- For colouring, associate to each monomial m a vertex set V_m
- Define

$$R\left(\sum_i c_i m_i \right) := \sum_i c_i \underbrace{R_{V_{m_i}}}_{\text{reduction modulo ideal of “}G[V_{m_i}]\text{ is }k\text{-colourable”}} (m_i)$$

- Technical challenge: construct V_m so that R satisfies required properties

Vertex Set V_m

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- Define closure $\mathrm{Cl}(U) \supseteq U$ of vertex sets U
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Desired properties of closure:

1. **Subset-preserving:** $U' \subseteq \mathrm{Cl}(U) \Rightarrow \mathrm{Cl}(U') \subseteq \mathrm{Cl}(U)$
2. **Size-preserving:** $|U| \leq D \Rightarrow |\mathrm{Cl}(U)| = O(D)$
3. **Reduction-preserving:** For any monomial m mentioning only vertices in $\mathrm{Cl}(U)$ and any vertex set J of size $O(D)$ it holds that

$$R_{\mathrm{Cl}(U)}(m) = R_{\mathrm{Cl}(U) \cup J}(m)$$

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Reduction lemma [CdRNPR 23]

For fixed order on vertices (and variables), can achieve this property if:

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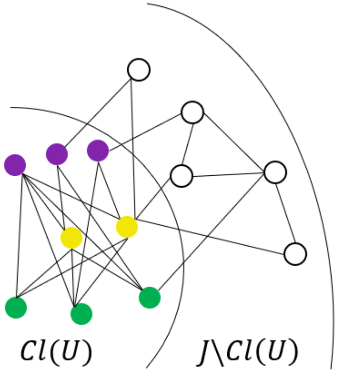

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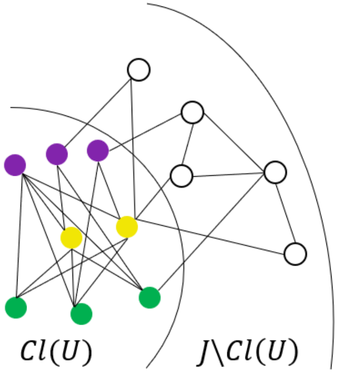
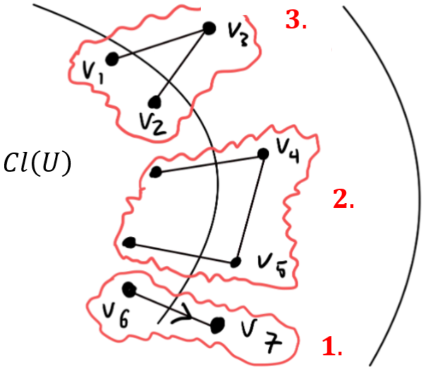
- ... in **order-decreasing** way: for each v in $J \setminus \mathsf{Cl}(U)$, colour can be determined based on colouring of $\{w \in \mathsf{Cl}(U) : w < v\}$



Construction of Closure (1/2)

Sufficient to prevent certain structures in neighbourhood outside Cl(U)



Construction of Closure (1/2)

Sufficient to prevent certain structures in neighbourhood outside $\mathrm{Cl}(U)$

1. Vertex with a larger neighbour in $\mathrm{Cl}(U)$
2. Edge between neighbours of $\mathrm{Cl}(U)$
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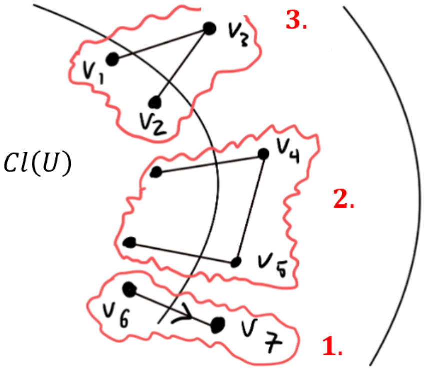

Construction of Closure (1/2)

Sufficient to prevent certain structures in neighbourhood outside $\mathsf{Cl}(U)$

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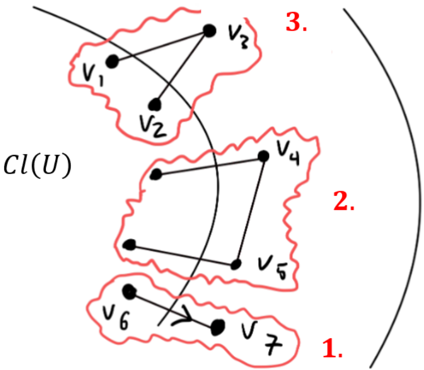
Same structures identified in [Romero-Tunçel 22] *in colouring lower bound for large-girth graphs!*



Construction of Closure (2/2)

Sufficient to prevent certain structures in neighbourhood outside Cl(U)

Constructing the closure of a set U

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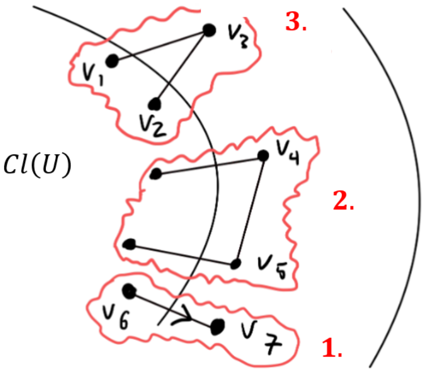

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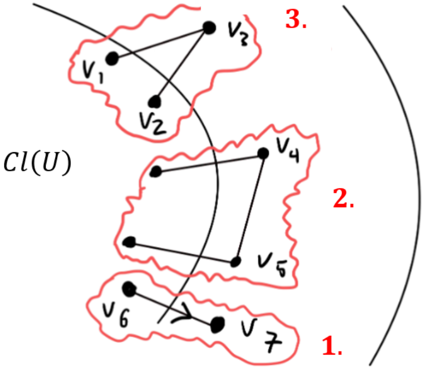

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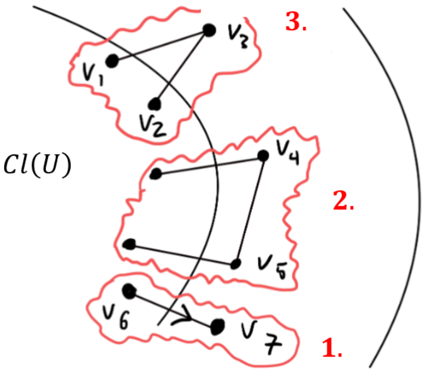

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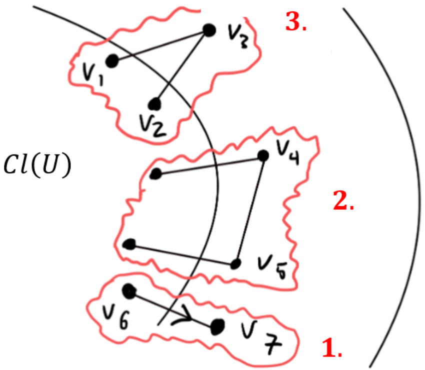

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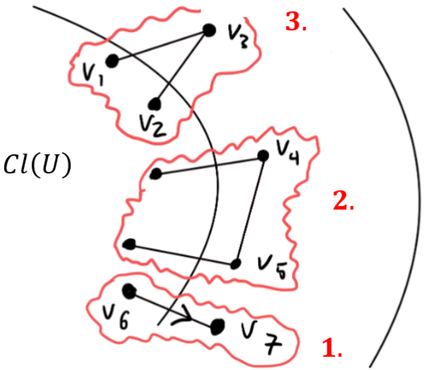

Not hard to show $\mathrm{Cl}(U)$ well-defined

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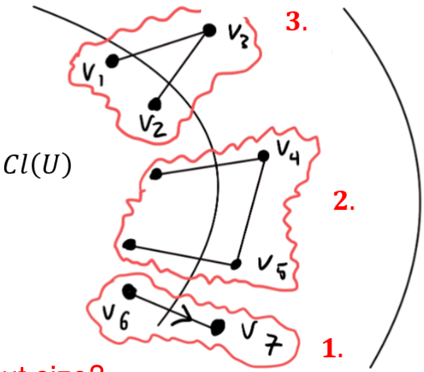

Not hard to show Cl(U) well-defined, but what about size?

Keeping the Closure Small Enough

Size lemma [CdRNPR 23]

For random n-vertex graph with max vertex degree d, it holds for any vertex set U with $|U| \leq 2^{-d^{O(1)}} \cdot n$ that

$$|\mathrm{Cl}(U)| = O(|U|)$$

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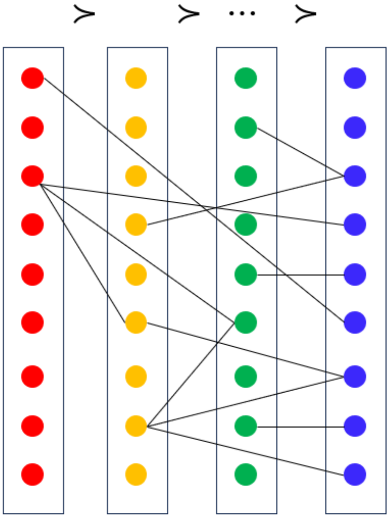
- Proof relies on "good" vertex order introduced by [Romero-Tunçel 22]:

  Order vertices according to a valid colouring of G

- Chromatic number of random graph G is $\chi(G) = O(d/\log d) = O(1)$
  $\Longrightarrow$ order-decreasing paths have length $O(1)$



Use any vertex order that respects colour classes

Completing the Proof (Sketch) of the Colouring Lower Bound

Size lemma: $|\mathrm{Cl}(U)| = O(|U|)$ for all U of small size

- Intuition: Closure $\mathrm{Cl}(U)$ obtained from sequence of vertex sets $U \subset U_1 \subset U_2 \subset \ldots$ of increasing edge density
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Pseudo-reduction operator properties:

- $R(\text{axiom}) = 0$ since each axiom p mentions vertex set U_p of size ≤ 2 and $R(m) = R_{\mathrm{Cl}(U_p)}(m)$ for each monomial m in p
- $R(xp) = R(xR(p))$ for all p of degree $\leq D - 1$ since closure is size- and reduction-preserving
- $R(1) = 1$ since $\mathrm{Cl}(\emptyset) = \emptyset$ and $R_{\mathrm{Cl}(1)}(\cdot)$ hence does nothing

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4. Connections between pseudo-reductions and other lower bound operators
   - Designs for Nullstellensatz
   - Pseudo-expectations for sums-of-squares

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Thank you for your attention!