Proof Complexity as a Computational Lens

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Three Simple Problems. . .

Colouring

Does the graph $G = (V, E)$ have a colouring with k colours such that all neighbours have distinct colours?

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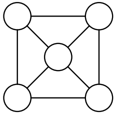

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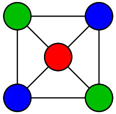

3-colouring? Yes

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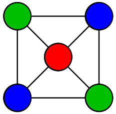

3-colouring? Yes, but no 2-colouring

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CLIQUE

Is there a clique in the graph $G = (V, E)$ with k vertices that are all pairwise connected by edges in E?

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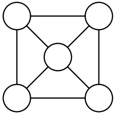

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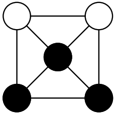

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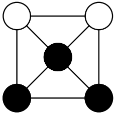

3-clique? Yes, but no 4-clique

Clique

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Given propositional logic formula, is there a satisfying assignment?

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Given propositional logic formula, is there a satisfying assignment?

$$(x \lor z) \land (y \lor \lnot z) \land (x \lor \lnot y \lor u) \land (\lnot y \lor \lnot u)$$
$$\land\ (u \lor v) \land (\lnot x \lor \lnot v) \land (\lnot u \lor w) \land (\lnot x \lor \lnot u \lor \lnot w)$$

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- Variables should be set to **true** or **false**
- Constraint $(x \lor \lnot y \lor z)$: means x or z should be true or y false
- $\land$ means all constraints should hold simultaneously
- Is there a truth value assignment satisfying all constraints?

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COLOURING:	frequency allocation for mobile base stations
CLIQUE:	bioinformatics, computational chemistry
SAT:	easily models these and many other problems

. . . with Huge Practical Implications

- Some more examples of problems that can be encoded as propositional logic formulas:
  - computer hardware verification
  - computer software testing
  - artificial intelligence
  - operations research
  - cryptography
  - bioinformatics
  - et cetera. . .
- Leads to **humongous** formulas (100,000s or even 1,000,000s of variables)
- Can we use computers to solve these problems efficiently?

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- SAT mentioned already in Gödel's famous letter in 1956 to von Neumann
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- Sat problem is NP-complete, so probably very hard [Coo71, Lev73]
- Assuming P $\neq$ NP, even impossible to meaningfully approximate
  - Colouring [Kho01, Zuc07]
  - Clique [Hås99]
  - Sat [Hås01]

Solving NP in Theory and Practice

- SAT mentioned already in Gödel's famous letter in 1956 to von Neumann
- Topic of intense research in computer science ever since 1960s
- SAT problem is NP-complete, so probably very hard [Coo71, Lev73]
- Assuming P $\neq$ NP, even impossible to meaningfully approximate
  - COLOURING [Kho01, Zuc07]
  - CLIQUE [Hås99]
  - SAT [Hås01]
- Except that in practice, there are good algorithms for
  - COLOURING [DLMM08, DLMO09, DLMM11]
  - CLIQUE [Pro12, McC17]

  and amazing conflict-driven clause learning (CDCL) solvers [BS97, MS99, MMZ^{+}01] that solve huge SAT problem instances

How can we understand real-world algorithms for NP-hard problems?

This lecture: Use proof complexity (not only conceivable answer)

Algorithmic View of Proof Complexity

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1. Is there a short proof of the right answer using rules in this proof system?
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Study infeasible problems — proofs of feasibility are trivial

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Question 2: Topic for separate lecture(s) — lots of recent exciting progress; mostly negative (worst-case) results that proof search is hard, e.g., [AM20, GKMP20, dRGN$^+$21]

Applications of Proof Complexity

Three applied reasons for proof complexity:

1. Understand real-world applied algorithmic paradigms [**this lecture**]

2. Get ideas for algorithmic improvements
[EN18, EN20, LBD+20, DGD+21, DGN21, KBBN22, MBGN23, MSB+25]
(See, e.g., tutorials youtu.be/VC0CHXoWnS4 and youtu.be/FIJ3k7HWpiQ about RoundingSat)

3. Enhance algorithms to write machine-verifiable certificates of correctness
[EGMN20, GMN20, GMM+20, GN21, GMN22, GMNO22, BBN+23, BGMN23, MM23, BBN+24, DMM+24, GMM+24, HOGN24, IOT+24, MMN24, DHN+25, KLM+25, MM25]
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Or just view this as a convenient excuse to study nice computational complexity problems for their own sake. . . ☺

Outline

Some Preliminaries

- Variable x: takes value **true** $(= 1)$ or **false** $(= 0)$
- Literal ℓ: variable x or its negation $\overline{x}$ (write $\overline{x}$ instead of $\neg x$)
- Clause $C = \ell_1 \vee \cdots \vee \ell_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- Conjunctive normal form (CNF) formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses

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- k-CNF formula: CNF formula with clauses of size $\leq k$ (typically k constant)
- Refer to clauses of CNF formula as axioms (as opposed to derived clauses)
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- Refer to clauses of CNF formula as axioms (as opposed to derived clauses)
- N denotes size of formula (# literals counted with repetitions)
- $\mathcal{O}(f(N))$ grows at most as quickly as $f(N)$ asymptotically
  $\Omega(g(N))$ grows at least as quickly as $g(N)$ asymptotically
  $\Theta(h(N))$ grows equally quickly as $h(N)$ asymptotically

The SAT Problem

The SATISFIABILITY (or just SAT) Problem

Given a formula F in conjunctive normal form (CNF), is it satisfiable?

The Sat Problem

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Given a formula F in conjunctive normal form (CNF), is it satisfiable?

For instance, what about our example CNF formula?

$$\begin{aligned} &(x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \\ \land\ &(u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w}) \end{aligned}$$

The Same Problem in Three Different Shapes

$$\begin{gathered}(x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \\ \land\ (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})\end{gathered}$$

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$$\begin{aligned}
(1-x)(1-z) &= 0 \\
(1-y)z &= 0 \\
(1-x)y(1-u) &= 0 \\
yu &= 0 \\
(1-u)(1-v) &= 0 \\
xv &= 0 \\
u(1-w) &= 0 \\
xuw &= 0
\end{aligned}$$

For **true** $= 1$ and **false** $= 0$, is there a $\{0,1\}$-valued solution?

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$$\begin{aligned}
1 - x - z + xz &= 0 \\
z - yz &= 0 \\
y - xy - yu + xyu &= 0 \\
yu &= 0 \\
1 - u - v + uv &= 0 \\
xv &= 0 \\
u - uw &= 0 \\
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$$\begin{aligned}
1 - x - z + xz &= 0 & x + z &\geq 1 \\
z - yz &= 0 & y + (1 - z) &\geq 1 \\
y - xy - yu + xyu &= 0 & x + (1 - y) + u &\geq 1 \\
yu &= 0 & (1 - y) + (1 - u) &\geq 1 \\
1 - u - v + uv &= 0 & u + v &\geq 1 \\
xv &= 0 & (1 - x) + (1 - v) &\geq 1 \\
u - uw &= 0 & (1 - u) + w &\geq 1 \\
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$1 - x - z + xz = 0$	$x + z \geq 1$
$z - yz = 0$	$y - z \geq 0$
$y - xy - yu + xyu = 0$	$x - y + u \geq 0$
$yu = 0$	$-y - u \geq -1$
$1 - u - v + uv = 0$	$u + v \geq 1$
$xv = 0$	$-x - v \geq -1$
$u - uw = 0$	$-u + w \geq 0$
$xuw = 0$	$-x - u - w \geq -2$

For **true** $= 1$ and **false** $= 0$, is there a $\{0, 1\}$-valued solution?

Clique and Colouring as CNF Formulas

Clique formula

"The graph $G = (V, E)$ has an m-clique"

$q_{k,1} \lor q_{k,2} \lor \cdots \lor q_{k,n}$	$\lvert V\rvert = n;\ 1 \leq k \leq m$	[some vertex is kth member of clique]
$\overline{q}_{k,u} \lor \overline{q}_{k,v}$	$u \neq v \in V;\ 1 \leq k \leq m$	[clique members are uniquely defined]
$\overline{q}_{k,v} \lor \overline{q}_{k',v}$	$v \in V;\ 1 \leq k < k' \leq m$	[no vertex counted as clique member twice]
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(Smarter encodings are possible, but these are good enough for our discussion)

DPLL: Attempting Smart Case Analysis

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3. Otherwise pick some variable x in F
4. Set $x = 0$, simplify F and make recursive call
5. Set $x = 1$, simplify F and make recursive call
6. If result in both cases "unsatisfiable", then report "unsatisfiable" and return

A DPLL Toy Example

$$\begin{aligned} F = \quad & (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \\ & \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w}) \end{aligned}$$

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Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals

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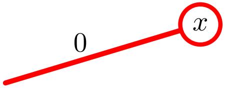

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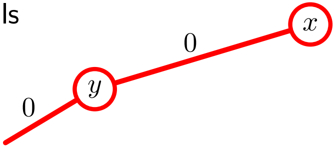


A DPLL Toy Example

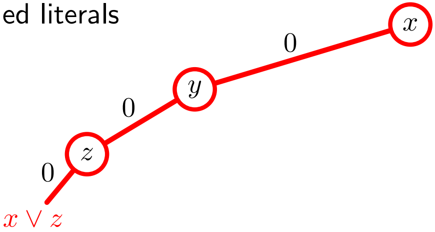
$$F = (x \lor z) \land (\overline{z}) \land (\overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals



A DPLL Toy Example

$$F = (\quad z) \land (y \lor \overline{z}) \land (\quad \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

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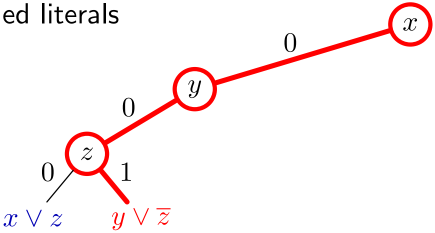

A DPLL Toy Example

$$F = (\quad z) \land (y \lor \overline{z}) \land (\quad u) \land (\quad \overline{u})$$
$$\land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

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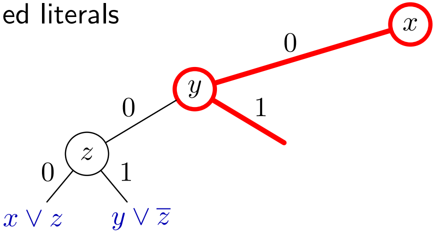

A DPLL Toy Example

$$
\begin{aligned}
F = \quad & (\quad z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\quad \overline{u}) \\
\land \; & (\quad v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})
\end{aligned}
$$

Visualize execution of DPLL algorithm as search tree

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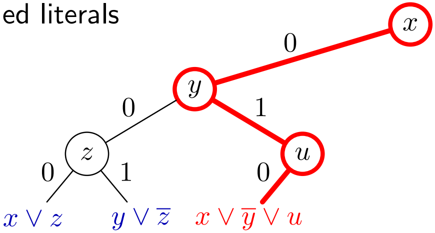

A DPLL Toy Example

$$F = (\quad z) \land (y \lor \overline{z}) \land (\quad u) \land (\overline{y} \lor \overline{u}) \\ \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\quad w) \land (\overline{x} \lor \quad \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

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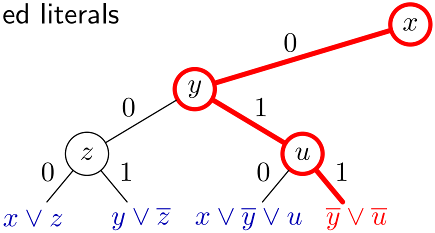

A DPLL Toy Example

$$F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\overline{v}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

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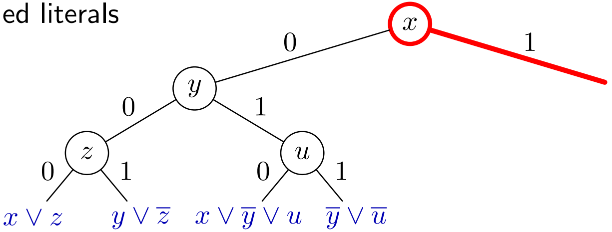


A DPLL Toy Example

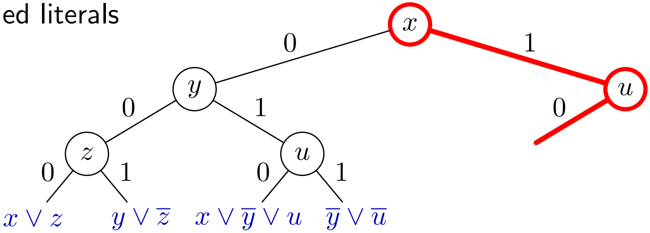
$$\begin{aligned} F = \quad & (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \qquad) \land (\overline{y} \lor \overline{u}) \\ & \land (\qquad v) \land (\qquad \overline{v}) \land (\overline{u} \lor w) \land (\qquad \overline{u} \lor \overline{w}) \end{aligned}$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals



A DPLL Toy Example

$$
\begin{aligned}
F = \quad & (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \qquad) \land (\overline{y} \lor \overline{u}) \\
& \land (u \lor v) \land (\qquad \overline{v}) \land (\overline{u} \lor w) \land (\qquad \overline{u} \lor \overline{w})
\end{aligned}
$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

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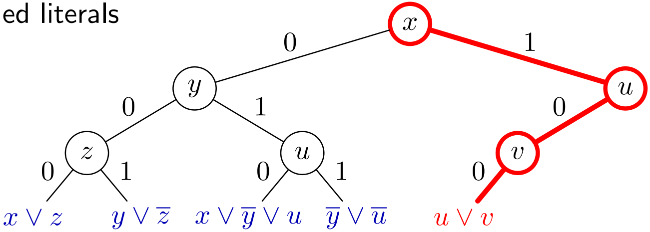
- falsified literals



A DPLL Toy Example

$$
\begin{aligned}
F = \quad & (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \quad) \land (\overline{y} \lor \overline{u}) \\
& \land (\quad v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\quad \overline{u} \lor \overline{w})
\end{aligned}
$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

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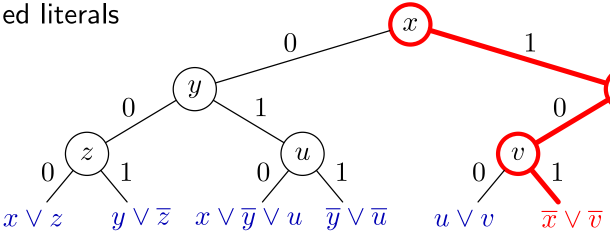


A DPLL Toy Example

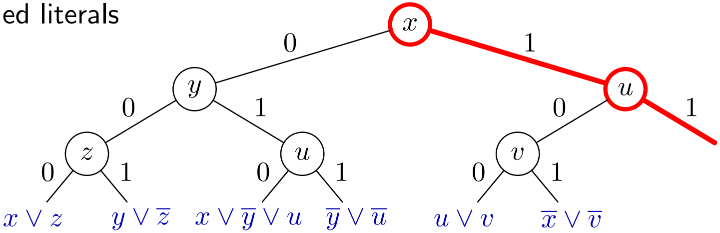
$$F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\quad \overline{v}) \land (\quad w) \land (\quad \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals



A DPLL Toy Example

$$F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\overline{v}) \land (\overline{u} \lor w) \land (\overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

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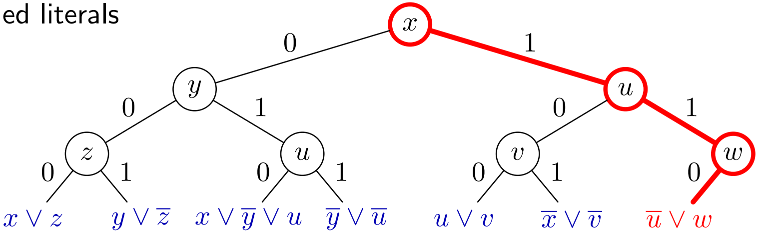
- satisfied clauses
- falsified literals



A DPLL Toy Example

$$F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\quad \overline{v}) \land (\quad w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
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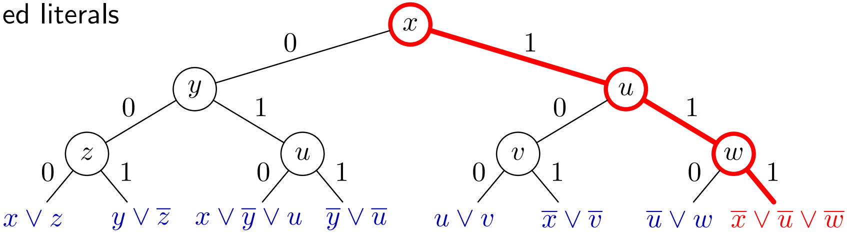

A DPLL Toy Example

$$
\begin{aligned}
F = \quad & (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \\
& \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})
\end{aligned}
$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

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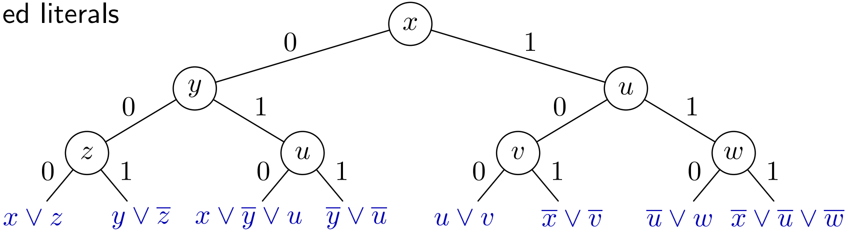

State-of-the-Art SAT Solving in One Slide

High-level description of modern conflict-driven clause learning (CDCL) SAT solving (as pioneered in [BS97, MS99, MMZ+01]):

- Try to build satisfying assignment for formula (branching or decision heuristic crucial)
- When partial assignment violates formula, compute explanation for conflict and add to formula as new clause (clause learning)
- Every once in a while, restart from beginning (but save computed info)

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \vee \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

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Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \overset{\mathsf{d}}{=} 0}$

$u \overset{p \lor \overline{u}}{=} 0$

$\boxed{q \overset{\mathsf{d}}{=} 0}$

$r \overset{q \lor r}{=} 1$

$w \overset{\overline{r} \lor w}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \overset{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \overset{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \vee \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \vee r}{=} 1$

$w \stackrel{\overline{r} \vee w}{=} 1$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$y \stackrel{u \lor x \lor y}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \overset{\mathsf{d}}{=} 0}$

$u \overset{p \lor \overline{u}}{=} 0$

$\boxed{q \overset{\mathsf{d}}{=} 0}$

$r \overset{q \lor r}{=} 1$

$w \overset{\overline{r} \lor w}{=} 1$

$\boxed{x \overset{\mathsf{d}}{=} 0}$

$y \overset{u \lor x \lor y}{=} 1$

$z \overset{x \lor \overline{y} \lor z}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \overset{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \overset{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$p \overset{\mathsf{d}}{=} 0$

$u \overset{p \vee \overline{u}}{=} 0$

$q \overset{\mathsf{d}}{=} 0$

$r \overset{q \vee r}{=} 1$

$w \overset{\overline{r} \vee w}{=} 1$

$x \overset{\mathsf{d}}{=} 0$

$y \overset{u \vee x \vee y}{=} 1$

$z \overset{x \vee \overline{y} \vee z}{=} 1$

$\overset{\overline{y} \vee \overline{z}}{\bot}$

Decision

Free choice to assign value to variable

Notation $p \overset{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \overset{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$p \stackrel{\mathsf{d}}{=} 0$
$u \stackrel{p \lor \overline{u}}{=} 0$
decision level 1

$q \stackrel{\mathsf{d}}{=} 0$
$r \stackrel{q \lor r}{=} 1$
$w \stackrel{\overline{r} \lor w}{=} 1$
decision level 2

$x \stackrel{\mathsf{d}}{=} 0$
$y \stackrel{u \lor x \lor y}{=} 1$
$z \stackrel{x \lor \overline{y} \lor z}{=} 1$
$\stackrel{\overline{y} \lor \overline{z}}{\bot}$
decision level 3

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$$\left.\begin{array}{l} \boxed{p \stackrel{\mathsf{d}}{=} 0} \\ u \stackrel{p \lor \overline{u}}{=} 0 \end{array}\right\} \text{decision level } 1$$

$$\left.\begin{array}{l} \boxed{q \stackrel{\mathsf{d}}{=} 0} \\ r \stackrel{q \lor r}{=} 1 \\ w \stackrel{\overline{r} \lor w}{=} 1 \end{array}\right\} \text{decision level } 2$$

$$\left.\begin{array}{l} \boxed{x \stackrel{\mathsf{d}}{=} 0} \\ y \stackrel{u \lor x \lor y}{=} 1 \\ z \stackrel{x \lor \overline{y} \lor z}{=} 1 \\ \stackrel{\overline{y} \lor \overline{z}}{\bot} \end{array}\right\} \text{decision level } 3$$

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

decision level 1

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

decision level 2

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

$\stackrel{\overline{y} \lor \overline{z}}{\bot}$

decision level 3

Could backtrack by erasing conflict level & flipping last decision

Conflict Analysis

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Trail	
$p \stackrel{\mathsf{d}}{=} 0$	decision level 1
$u \stackrel{p \lor \overline{u}}{=} 0$	
$q \stackrel{\mathsf{d}}{=} 0$	decision level 2
$r \stackrel{q \lor r}{=} 1$	
$w \stackrel{\overline{r} \lor w}{=} 1$	
$x \stackrel{\mathsf{d}}{=} 0$	decision level 3
$y \stackrel{u \lor x \lor y}{=} 1$	
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But want to learn from conflict and cut away as much of search space as possible

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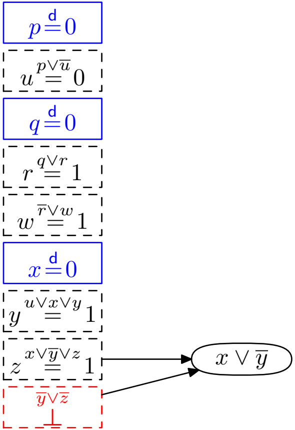

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Case analysis over z for last two clauses:

- $x \lor \overline{y} \lor z$ wants $z = 1$
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- Merge clauses & remove z — must satisfy $x \lor \overline{y}$

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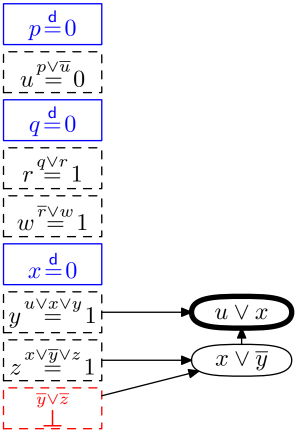

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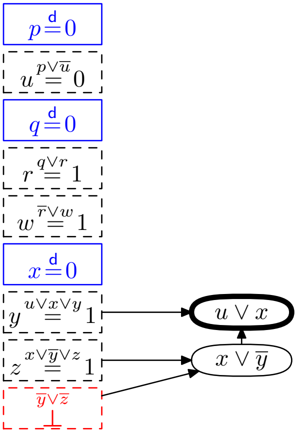
Repeat until UIP clause with only 1 variable at conflict level after last decision — learn and backjump

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$



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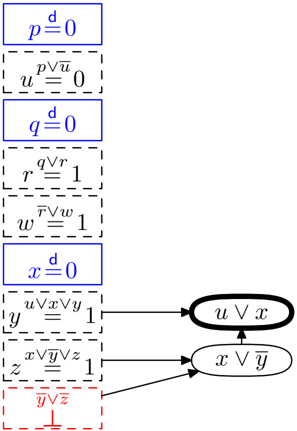

Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

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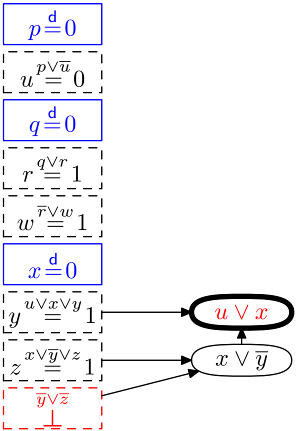

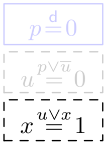

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Then continue as before. . .

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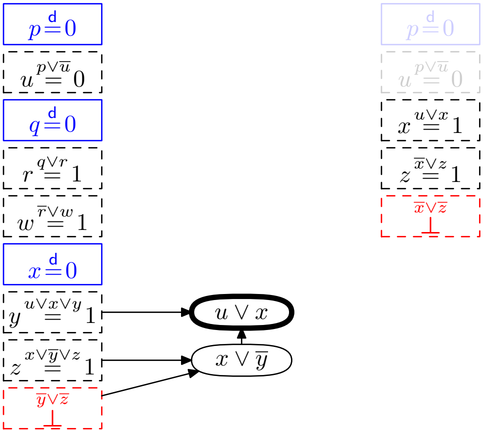

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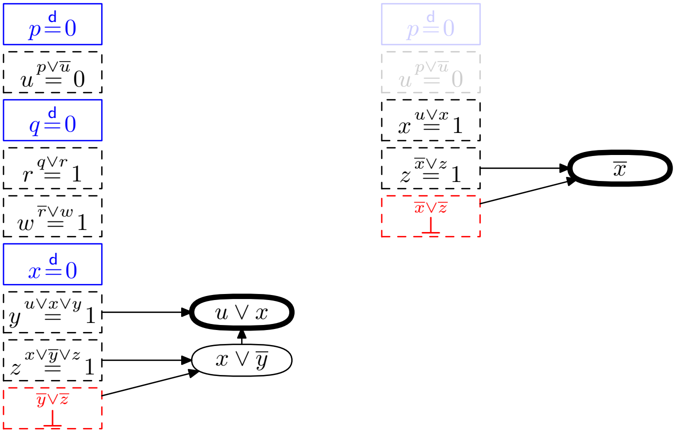

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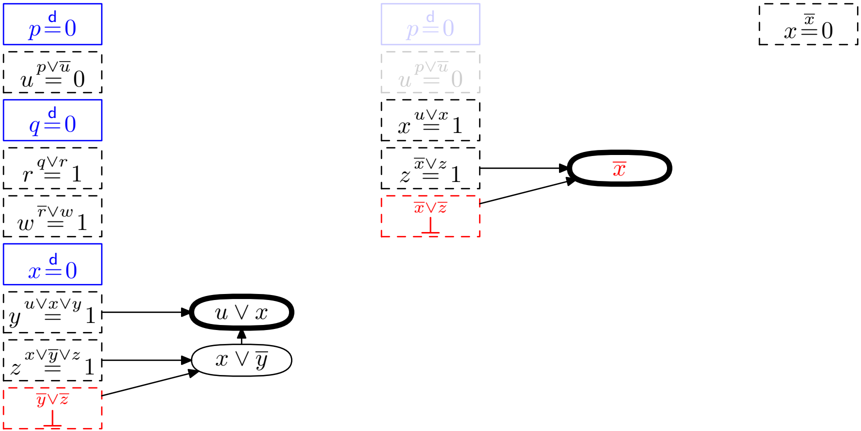

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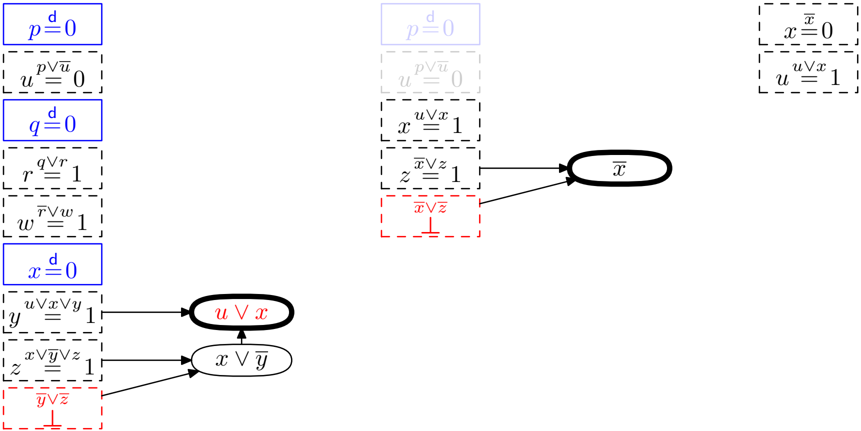

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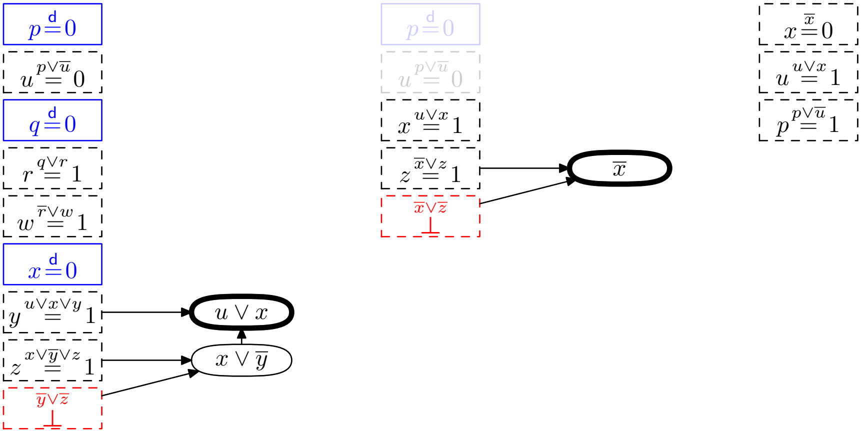

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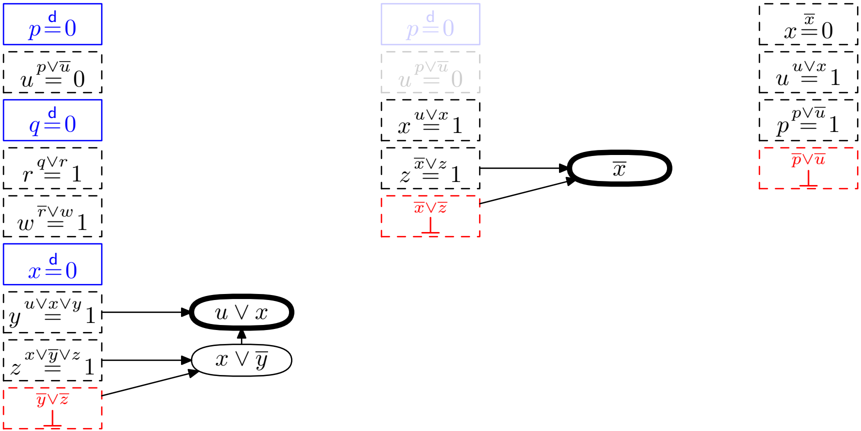

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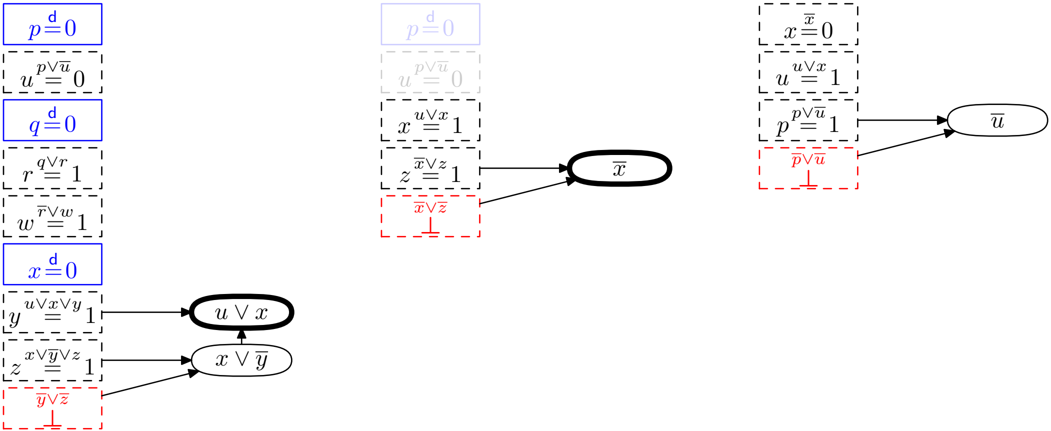

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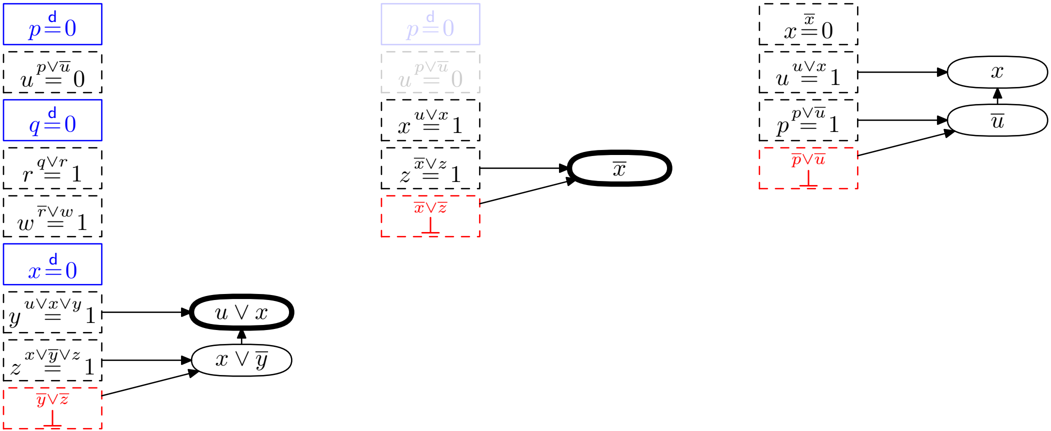

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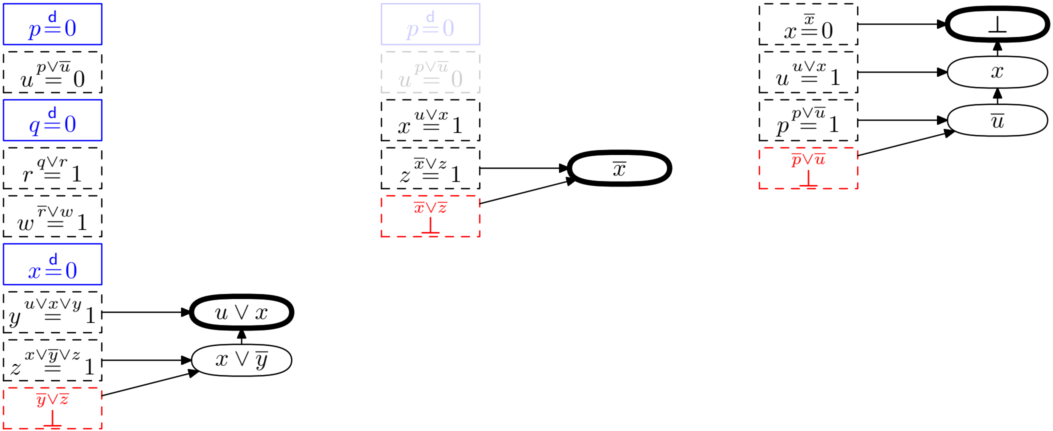

SAT Solver Analysis and the Resolution Proof System

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Many intricate, hard-to-understand heuristics
So focus instead on underlying method of reasoning

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Resolution proof system [Bla37, Rob65]

- Start with clauses of CNF formula (axioms)
- Derive new clauses by resolution rule

$$\frac{C_1 \lor x \qquad C_2 \lor \overline{x}}{C_1 \lor C_2}$$

Resolution Proofs by Contradiction

Resolution rule:

$$\frac{C_1 \lor x \qquad C_2 \lor \overline{x}}{C_1 \lor C_2}$$

Observation

If F is a satisfiable CNF formula and D is derived from clauses $D_1, D_2 \in F$ by the resolution rule, then $F \land D$ is satisfiable.

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If F is a satisfiable CNF formula and D is derived from clauses $D_1, D_2 \in F$ by the resolution rule, then $F \land D$ is satisfiable.

So can prove F unsatisfiable by deriving the unsatisfiable empty clause (denoted $\perp$) from F by resolution

Such proof by contradiction also called resolution refutation

DPLL and Resolution Proofs

A DPLL execution is essentially a resolution proof

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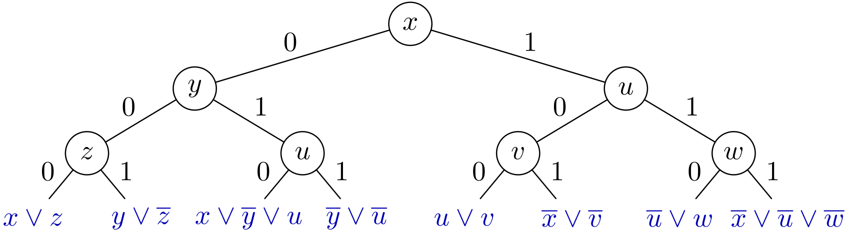
Look at our example again



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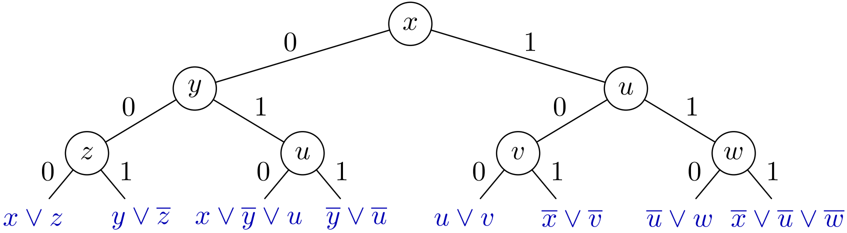

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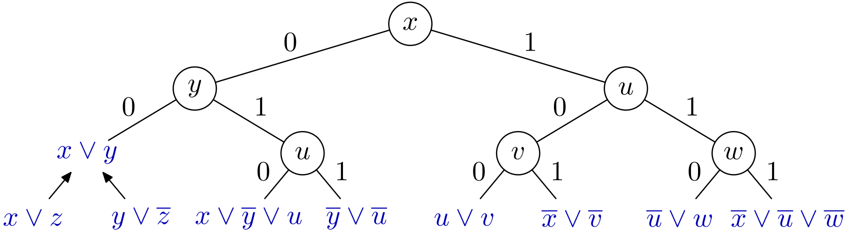

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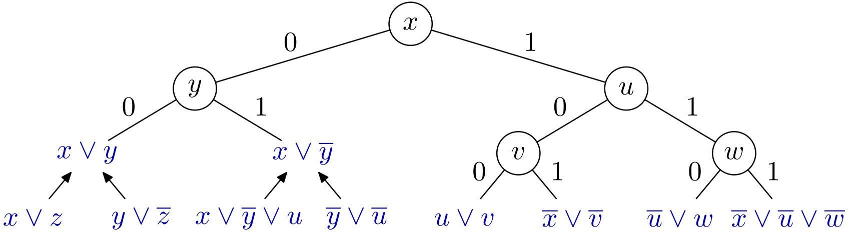

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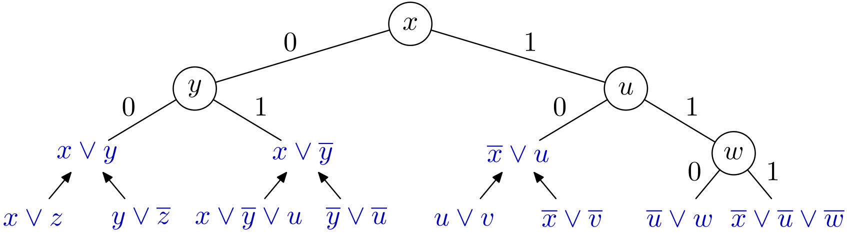

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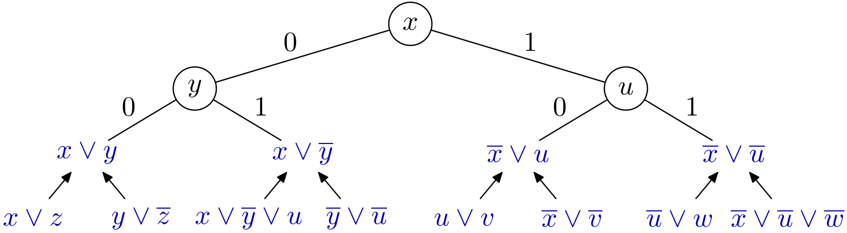

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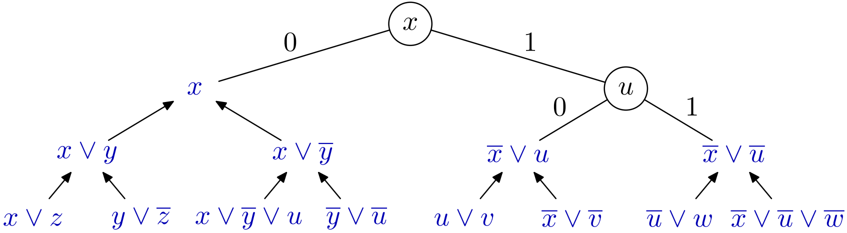

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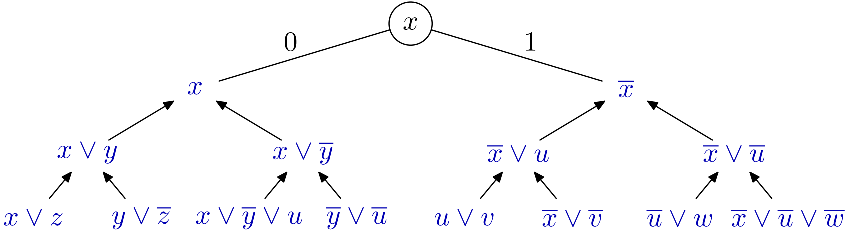

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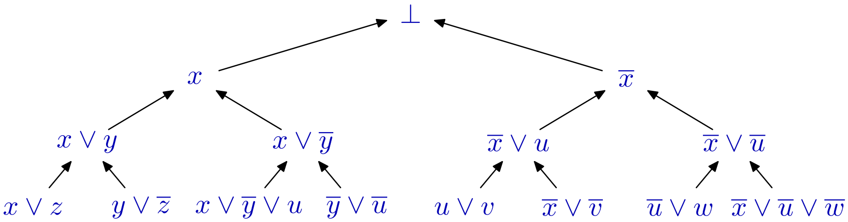

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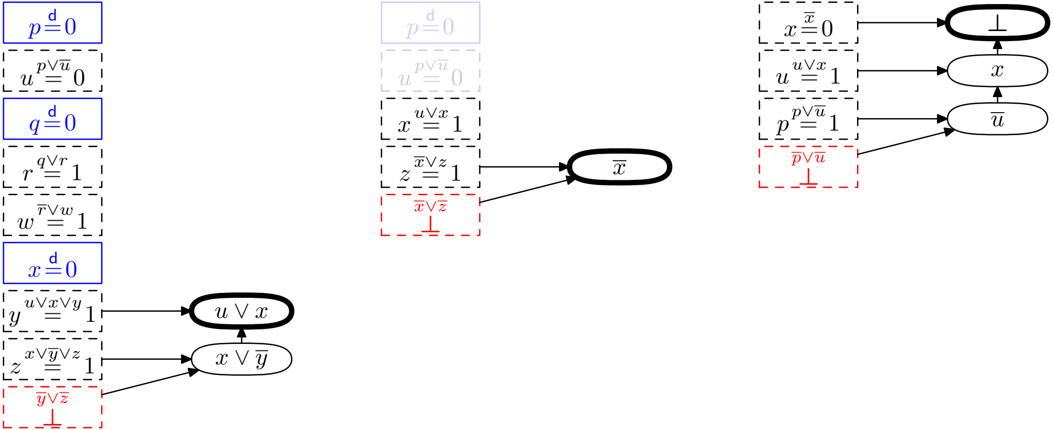
- Conflict-driven clause learning adds "shortcut edges" in tree, but still yields resolution proof

CDCL and Resolution Proofs

Obtain resolution proof. . .

CDCL and Resolution Proofs

Obtain resolution proof from our example CDCL execution...



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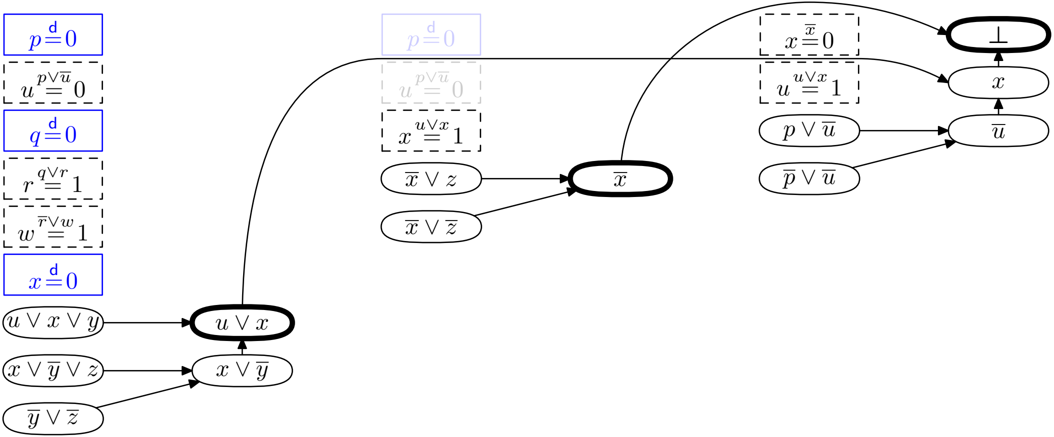
Obtain resolution proof from our example CDCL execution by stringing together conflict analyses:



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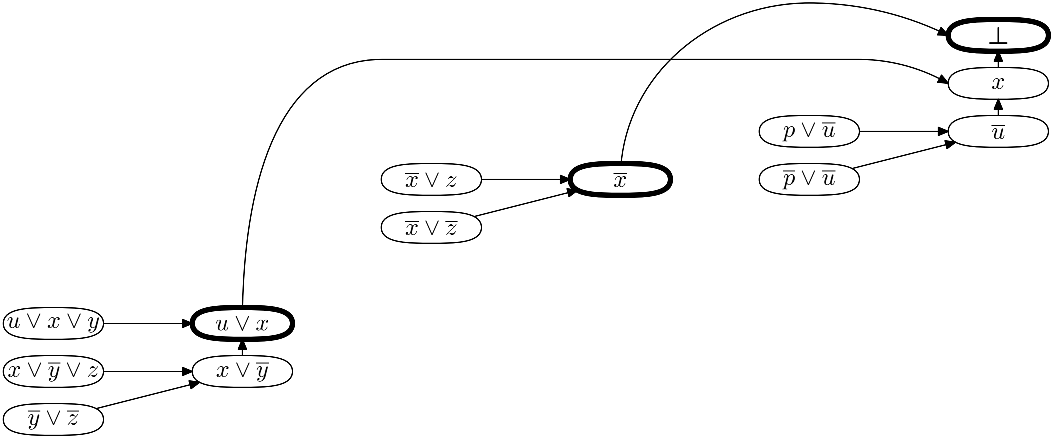

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- But ensuring this in practice is NP-hard [AM20]

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- If there are short proofs, then CDCL can run fast in principle [AFT11, PD11]
- But ensuring this in practice is NP-hard [AM20]

(*) Except for some preprocessing techniques, which is an important omission, but this gets complicated and we don't have time to go into details. . .

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  - Why are applied instances easy?
- Paradox: resolution quite weak proof system; many strong proof complexity lower bounds for (seemingly) "obvious" formulas

Examples of Hard Formulas for Resolution (1/3)

Pigeonhole principle (PHP) formulas [Hak85]

"$n + 1$ pigeons don't fit into n holes"

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$p_{i,1} \vee p_{i,2} \vee \cdots \vee p_{i,n}$	[every pigeon i gets a hole]
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Can also add "functionality" and "onto" axioms

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Even onto functional PHP hard — **"resolution cannot count"**

Resolution proof requires $\exp(\Omega(n)) = \exp\left(\Omega\left(\sqrt[3]{N}\right)\right)$ clauses
(measured in terms of formula size N, i.e., total number of literals in formula)

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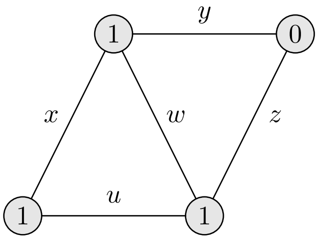

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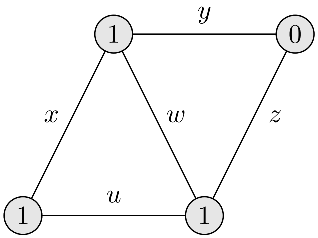

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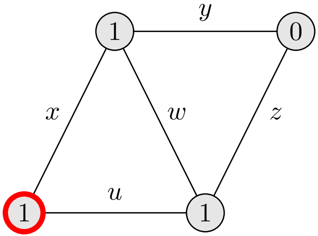

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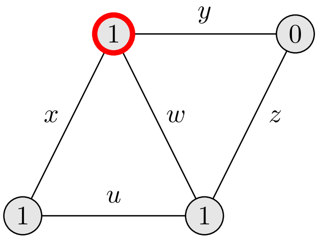

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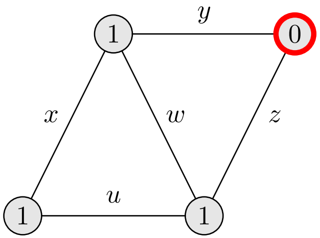

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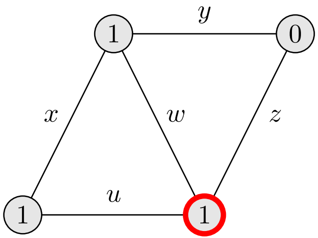

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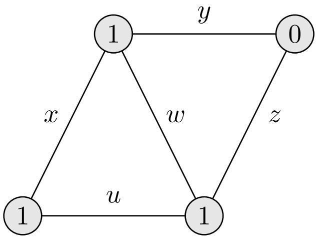

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Requires proof size $\exp(\Omega(N))$ on well-connected so-called expander graphs — **"resolution cannot count** $\bmod 2$**"**

Examples of Hard Formulas for Resolution (3/3)

Random k-CNF formulas [CS88]

Δn randomly sampled k-clauses over n variables

($\Delta \gtrsim 4.5$ sufficient to get unsatisfiable 3-CNF almost surely)

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And more...

- COLOURING [BCMM05]
- Zero-one designs [Spe10, VS10, MN14]
- Et cetera... (See, e.g., [BN21] for overview)

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But not CLIQUE!

- Refuting existence of k-clique should require proof size $n^{\Omega(k)}$
- Only known for restricted so-called regular resolution [ABdR^{+}21]

Other Complexity Measures for Resolution

The exponential size lower bounds mentioned can all be proven by studying width, i.e., the size of a largest clause in the proof

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If any resolution refutation of k-CNF formula F over n variables requires width linear in n, then refuting F in resolution requires size exponential in n

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If any resolution refutation of k-CNF formula F over n variables requires width linear in n, then refuting F in resolution requires size exponential in n

There are also other complexity measures of interest such as

- space: memory needed for self-contained presentation of refutation
- depth: longest path in refutation represented as directed acyclic graph (DAG)

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$$\prod_{i \in \mathcal{P}} (1 - x_i) \cdot \prod_{j \in \mathcal{N}} x_j = 0$$
- Add Boolean axioms
$$x_j^2 - x_j = 0$$
  for all variables

Hilbert's Nullstellensatz

Consider any system of polynomial equations

$$\begin{aligned} p_1(x_1, \ldots, x_n) &= 0 \qquad\qquad & x_1^2 - x_1 &= 0 \\ p_2(x_1, \ldots, x_n) &= 0 & x_2^2 - x_2 &= 0 \\ &\vdots & &\vdots \\ p_m(x_1, \ldots, x_n) &= 0 & x_n^2 - x_n &= 0 \end{aligned}$$

in polynomial ring over field $\mathbb{F}$

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Hilbert's Nullstellensatz

System infeasible $\Leftrightarrow$ exist $q_i, r_j \in \mathbb{F}[x_1,\ldots,x_n]$ such that

$$\sum_{i=1}^{m} q_i(x_1,\ldots,x_n)\cdot p_i(x_1,\ldots,x_n) + \sum_{j=1}^{n} r_j(x_1,\ldots,x_n)\cdot(x_j^2 - x_j) = 1$$

Nullstellensatz Proof System [BIK+94]

Nullstellensatz refutation of

$$p_i(x_1, \ldots, x_n) = 0 \qquad i \in [m]$$
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Complexity measures of refutations:

- Size: number of monomials (when all polynomials expanded out)
- Degree: highest total degree of any polynomial

Nullstellensatz Example

$$(x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u})$$
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$$(1-x)(1-z)$$
$$(1-y)z$$
$$(1-x)y(1-u)$$
$$yu$$
$$(1-u)(1-v)$$
$$xv$$
$$u(1-w)$$
$$xuw$$

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+\ &1 \cdot (1-x)y(1-u) \\
+\ &(1-x) \cdot yu \\
+\ &x \cdot (1-u)(1-v) \\
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$$
\begin{aligned}
& 1 - x - y - z + xy + xz + yz - xyz \\
+\ & z - xz - yz + xyz \\
+\ & y - yu - xy + xyu \\
+\ & yu - xyu \\
+\ & x - xu - xv + xuv \\
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Size 27
Degree 3
(No use of Boolean axioms)

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Nullstellensatz Proof Search

- Solve linear system of equations with coefficients of polynomials q_i, r_j as unknowns
- Used successfully to solve, e.g., graph colouring problems [DLMM08, DLMO09, DLMM11]
- Running time grows exponentially with degree, though high-degree refutations can be very small [BCIP02, dRMNR21]

Dual Variables

- Annoying problem: $x_1 \lor x_2 \lor x_3$ translates to polynomial

$$(1 - x_1)(1 - x_2)(1 - x_3) = 1 - x_1 - x_2 - x_3 + x_1x_2 + x_1x_3 + x_2x_3 - x_1x_2x_3$$

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- Doesn't affect degree (obviously), but can decrease size exponentially [dRLNS21] (also for other algebraic proof systems)

Dynamic Construction of Nullstellensatz Certificates

Nullstellensatz again

Infeasibility of

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$$\Updownarrow$$

1 lies in polynomial ideal $\mathcal{I}$ generated by these polynomials

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- Ideal $\mathcal{I}$:
  1. $p, q \in \mathcal{I} \Rightarrow p + q \in \mathcal{I}$
  2. $p \in \mathcal{I} \Rightarrow r \cdot p \in \mathcal{I}$ for any r
- Compute polynomials in this ideal $\mathcal{I}$ step by step
- Use "multivariate division" to check whether 1 lies in ideal or not

Gröbner Bases: Admissible Orderings and Leading Terms

Admissible ordering $\preceq$ on monomials m, m', t:

1. $m \preceq m' \Rightarrow t \cdot m \preceq t \cdot m'$
2. $m \preceq t \cdot m$

Examples:

- Lexicographic
- Degree-lexicographic

Gröbner Bases: Admissible Orderings and Leading Terms

Admissible ordering $\preceq$ on monomials m, m', t:

1. $m \preceq m' \Rightarrow t \cdot m \preceq t \cdot m'$
2. $m \preceq t \cdot m$

Examples:

- Lexicographic
- Degree-lexicographic

Can write $p = \mathrm{lt}(p) + p'$ for $\mathrm{lt}(p)$ leading term (largest w.r.t. $\preceq$)

If $\mathrm{lt}(p) = t \cdot \mathrm{lt}(q)$, can reduce $p \bmod q$ by computing $p - t \cdot q$

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"Multivariate division": Reduce p modulo all q in set of polynomials $\mathcal{G}$ until no further reductions possible

$\mathcal{G}$ is a Gröbner basis if final result uniquely determined

Gröbner Bases: Buchberger's Algorithm

Buchberger's algorithm for computing Gröbner bases (very rough)

1. Let $\mathcal{G} :=$ all axioms
2. Pick unprocessed pair $p, q \in \mathcal{G}$ or terminate if none exists
3. Compute $p' = t_p \cdot p$ and $q' = t_q \cdot q$ to make leading terms cancel
4. Set $S := p' - q'$; reduce $S \bmod \mathcal{G}$ with multivariate division; add result to $\mathcal{G}$ if non-zero
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Facts:

- Buchberger's algorithm computes Gröbner basis
- At termination, $1 \in \mathcal{G} \Leftrightarrow$ polynomial equations infeasible

Polynomial Calculus [CEI96, ABRW02]

- Compute polynomials in ideal $\mathcal{I}$ generated by p_i, $x_j^2 - x_j$, and $x_j + x'_j - 1$ step by step:
  - $p_i \in \mathcal{I}$, $x_j^2 - x_j \in \mathcal{I}$, and $x_j + x'_j - 1 \in \mathcal{I}$ (axioms)
  - If $p, q \in \mathcal{I}$, then $\alpha p + \beta q \in \mathcal{I}$ for any $\alpha, \beta \in \mathbb{F}$ (linear combination)
  - If $p \in \mathcal{I}$, then $m \cdot p \in \mathcal{I}$ for any monomial $m = \prod_j x_j$ (multiplication)

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  - If $p \in \mathcal{I}$, then $m \cdot p \in \mathcal{I}$ for any monomial $m = \prod_j x_j$ (multiplication)
- A polynomial calculus refutation is a derivation ending with the polynomial 1
- Complexity measures:
  - Size: total number of monomials in all polynomials in derivation expanded out
  - Degree: highest total degree of any polynomial
- Polynomial calculus (much) stronger than Nullstellensatz w.r.t. both size and degree

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simulated by polynomial calculus derivation

$$\cfrac{x'yz' \qquad \cfrac{\cfrac{yz}{x'yz} \qquad \cfrac{z + z' - 1}{x'yz + x'yz' - x'y}}{-x'yz' + x'y}}{x'y}$$

Polynomial Calculus is Strictly Stronger than Resolution

Polynomial calculus can be exponentially stronger than resolution

For instance:

- Tseitin formulas on expander graphs if $\mathbb{F} = \mathrm{GF}(2)$
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Other hard formulas:

- Tseitin-like formulas for counting $\bmod p$ if $p \neq$ field characteristic [BGIP01]
- Random k-CNF formulas
    - all characteristics except 2 [BI99]
    - all characteristics [AR03]

COLOURING and CLIQUE for Polynomial Calculus

COLOURING

- Exponential worst-case lower bounds in [LN17]
- Exponential average-case lower bounds in [CdRN$^+$23]

CLIQUE

Almost nothing known! (Except lower bounds for very large cliques)

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If any polynomial calculus refutation of k-CNF formula F over n variables requires degree linear in n, then refuting F in polynomial calculus requires size exponential in n

- Other complexity measures analogous to those for resolution are also studied
- Many results analogous to resolution hold, but are much harder to prove
- Some analogous results are *believed* to hold, but remain open

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- Use dual variables! [KBBN22]

Gröbner bases: Some Problems and Questions

1. Buchberger not a great SAT solving algorithm
   Slow and memory-intensive, and computes too much info (#solutions)
   Possible to use conflict-driven paradigm?!

2. Dual variables increase reasoning power exponentially [dRLNS21]
   But are immediately eliminated by multivariate division
   Possible to design dual-variable-aware Buchberger?!

3. Analysis of polynomial calculus uses degree-lexicographic ordering
   In computational algebra, many other orderings used
   Prove proof complexity separation results for different orderings?

SAT as System of 0–1 Integer Linear Inequalities

- Given CNF formula $F = \bigwedge_{i=1}^{m} C_i$

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$$C = \bigvee_{i \in \mathcal{P}} x_i \vee \bigvee_{j \in \mathcal{N}} \overline{x}_j$$

to 0-1 integer linear inequalities

$$\sum_{i \in \mathcal{P}} x_i + \sum_{j \in \mathcal{N}} (1 - x_j) \geq 1$$

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$$\sum_{i \in \mathcal{P}} x_i + \sum_{j \in \mathcal{N}} (1 - x_j) \geq 1$$

- Add variable axioms

$$x_j \geq 0$$
$$-x_j \geq -1$$

for all variables

Cutting Planes Proof System [CCT87]

Cutting planes proof system introduced in [CCT87] to model integer linear programming algorithm in [Gom63, Chv73]

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Cutting planes derivation rules

Multiplication $$\frac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA} \quad c \in \mathbb{N}^+$$

Addition $$\frac{\sum a_i x_i \geq A \qquad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$$

Division $$\frac{\sum a_i x_i \geq A}{\sum \lceil a_i / c \rceil x_i \geq \lceil A/c \rceil} \quad c \in \mathbb{N}^+$$

Cutting Planes Derivations and Refutations

- A cutting planes derivation is a sequence of 0-1 integer linear inequalities derived using
  - Axioms (clauses and variable bounds)
  - Multiplication $\sum a_i x_i \geq A \Rightarrow \sum c a_i x_i \geq cA$
  - Addition $\sum a_i x_i \geq A$, $\sum b_i x_i \geq B \Rightarrow \sum (a_i + b_i) x_i \geq A + B$
  - Division $\sum a_i x_i \geq A \Rightarrow \sum \lceil a_i / c \rceil x_i \geq \lceil A/c \rceil$
- A cutting planes refutation ends with the inequality $0 \geq 1$
- Complexity measures:
  - Length: # inequalities
  - Size: Count also bit size of representing all coefficients

Cutting Planes vs. Resolution

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- Cutting planes can simulate resolution reasoning efficiently and can be exponentially stronger (e.g., for PHP, just count and argue that #pigeons $>$ #holes)
- And 0-1 linear inequalities are similar to but much more concise than CNF

Compare

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 3$$

and

$$\begin{aligned}
&(x_1 \lor x_2 \lor x_3 \lor x_4) \land (x_1 \lor x_2 \lor x_3 \lor x_5) \land (x_1 \lor x_2 \lor x_3 \lor x_6) \\
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Hard Formulas for Cutting Planes

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$q_{k,1} \lor q_{k,2} \lor \cdots \lor q_{k,n}$	$1 \leq k \leq m$	[some vertex is kth member of clique]
$\overline{q}_{k,v} \lor \overline{q}_{k',v}$	$1 \leq v \leq n;\ k \neq k'$	[no vertex counted as clique member twice]
$p_{u,v} \lor \overline{q}_{k,u} \lor \overline{q}_{k',v}$	$1 \leq u < v \leq n;\ k \neq k'$	[clique members are neighbours]
$r_{v,1} \lor r_{v,2} \lor \cdots \lor r_{v,m-1}$	$1 \leq v \leq n;$	[every vertex has a colour]
$\overline{p}_{u,v} \lor \overline{r}_{u,\ell} \lor \overline{r}_{v,\ell}$	$1 \leq u < v \leq n;\ 1 \leq \ell \leq m-1$	[neighbours have distinct colours]

More Hard Formulas for Cutting Planes?

Lower bound for clique-colouring formulas uses interpolation and circuit complexity

- From small cutting planes proof, build small circuit of special type that can decide whether graph has clique
- Prove separately that no such small circuits can exist
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Some recent developments in [dRMN^{+}20, HP17, FPPR22, GGKS20, Sok24]

- Random $\mathcal{O}(\log n)$-CNF formulas exponentially hard [HP17, FPPR22]
- Lower bound for random k-CNF formulas open
- Surprisingly, Tseitin formulas have refutations of quasi-polynomial size [DT20]!
- Nothing known for COLOURING or CLIQUE

SAT Solvers Based on Cutting Planes?

So-called pseudo-Boolean (PB) solvers using (subset of) cutting planes reasoning developed in, e.g., [CK05, SS06, LP10, EN18]

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Is it truly harder to build good pseudo-Boolean solvers?
Or has just so much more work has been put into CDCL solvers?

Division Versus Saturation

Use negated literals as needed to get all a_i, A positive (normalized form)

Boolean derivation rules for 0–1 integer linear inequalities

$$\text{Division}\quad \frac{\sum a_i \ell_i \geq A}{\sum \lceil a_i / c \rceil \ell_i \geq \lceil A/c \rceil} \quad c \in \mathbb{N}^+$$

$$\text{Saturation}\quad \frac{\sum a_i \ell_i \geq A}{\sum \min\{a_i, A\} \cdot \ell_i \geq A}$$

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- . . . And most often also in practice [EN18], though not always [LBD$^+$20]

Separating Division from Saturation?

Even colouring formulas [Mar06]

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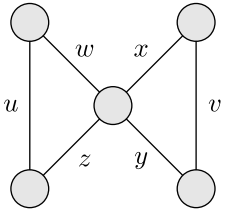

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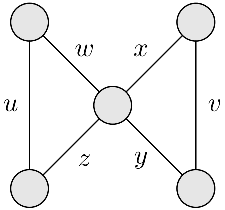

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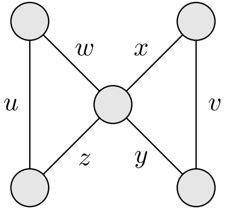

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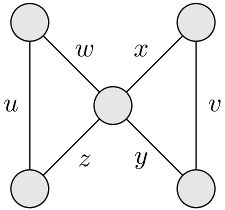

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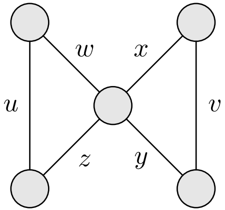

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- Pseudo-Boolean encoding hard in practice for pseudo-Boolean solvers [EGNV18]

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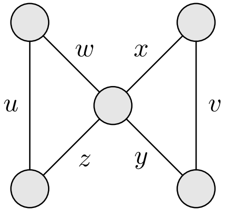

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v + y &\geq 1 & -v - y &\geq -1
\end{aligned}
$$

Unsatisfiable $\Leftrightarrow$ total $\#$ edges odd

- Very easy for (tree-like) cutting planes
- CNF encoding exponentially hard for resolution (and PB solvers) for expander graphs
- Pseudo-Boolean encoding hard in practice for pseudo-Boolean solvers [EGNV18]
- Possible to prove lower bounds for cutting planes with saturation instead of division?

The Subgraph Isomorphism Problem

Input

- Pattern graph $\mathcal{P}$ with vertices $V(\mathcal{P}) = \{a, b, c, \ldots\}$
- Target graph $\mathcal{T}$ with vertices $V(\mathcal{T}) = \{u, v, w, \ldots\}$
- No loops (for simplicity)

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Task

- Find all subgraph isomorphisms $\varphi : V(\mathcal{P}) \to V(\mathcal{T})$
- I.e., one-to-one mappings φ such that if
  1. $\varphi(a) = u$
  2. $\varphi(b) = v$
  3. $(a, b) \in E(\mathcal{P})$

  then must have $(u, v) \in E(\mathcal{T})$

Cutting Planes Lower Bounds for Subgraph Isomorphism?

Subgraph isomorphism formula

$\sum_{v \in V(\mathcal{T})} x_{a,v} \geq 1$	[every pattern vertex $a \in V(\mathcal{P})$ maps somewhere]
$\sum_{v \in V(\mathcal{T})} -x_{a,v} \geq -1$	[... but only to one target vertex $u \in V(\mathcal{T})$]
$\sum_{b \in V(\mathcal{P})} -x_{b,u} \geq -1$	[mapping is one-to-one]
$-x_{a,u} + \sum_{v \in N(u)} x_{b,v} \geq 0$	[edge $(a,b) \in E(\mathcal{P})$ maps to edge $(u,v) \in E(\mathcal{T})$]

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- All reasoning steps in Glasgow Subgraph Solver [ADH+19, GSS] can be formalized efficiently in cutting planes [GMN20, GMM+24]
- So lower bounds for any graph pairs $(\mathcal{P}, \mathcal{T})$ would establish theoretical limitations on state-of-the-art algorithms

Sherali–Adams (SA) and Sum of Squares (SoS)

Refutation of $p_i \in \mathbb{R}[x_1, \ldots, x_n]$, $i \in [m]$, and $x_j^2 - x_j$, $j \in [n]$

Nullstellensatz

$$\sum_{i=1}^{m} q_i \cdot p_i + \sum_{j=1}^{n} r_j \cdot (x_j^2 - x_j) = 1$$

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Sherali–Adams (SA) $(\alpha_k \in \mathbb{R}^+)$

$$\sum_{i=1}^{m} q_i \cdot p_i + \sum_{j=1}^{n} r_j \cdot (x_j^2 - x_j) + \sum_{k=1}^{t} \alpha_k \prod_{i \in \mathcal{P}_t} (1 - x_i) \cdot \prod_{j \in \mathcal{N}_t} x_j = -1$$

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Sum of squares (SoS) $(s_k \in \mathbb{R}[x_1, \ldots, x_n])$

$$\sum_{i=1}^{m} q_i \cdot p_i + \sum_{j=1}^{n} r_j \cdot (x_j^2 - x_j) + \sum_{k=1}^{s} s_k^2 = -1$$

Sherali–Adams, Sum of Squares, and Relations to Other Proof Systems

Sherali–Adams models linear programming (LP) hierarchies
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Strict hierarchy (over $\mathbb{R}$):

- Nullstellensatz
- Sherali–Adams
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Sum of squares is strictly stronger than polynomial calculus (over $\mathbb{R}$)
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Survey [FKP19] recommended for more reading

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Intended to model modern 0-1 integer linear programming

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Complexity measures:

- Length: $\#$ branching nodes / sets $\mathcal{S}$
- Size: Count also bit size for representing all coefficients

Stabbing Planes and Cutting Planes

Stabbing planes efficiently simulates cutting planes [BFI+18]

Stabbing planes probably much stronger!?

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Recent news: Interpolation and circuit complexity can be used to get similar lower bounds for stabbing planes as for cutting planes! [GP24]

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Recent news: Interpolation and circuit complexity can be used to get similar lower bounds for stabbing planes as for cutting planes! [GP24]

Still possible that stabbing planes is exponentially more powerful than cutting planes, but hard to know what to believe

Extended Resolution [Tse68]

Resolution rule

$$\frac{C_1 \lor x \qquad C_2 \lor \overline{x}}{C_1 \lor C_2}$$

Extension rule introducing clauses

$$a \lor \overline{x} \lor \overline{y} \qquad \overline{a} \lor x \qquad \overline{a} \lor y$$

for fresh variable a (encoding that $a \leftrightarrow (x \land y)$ must hold)

Extended Resolution and SAT Solving

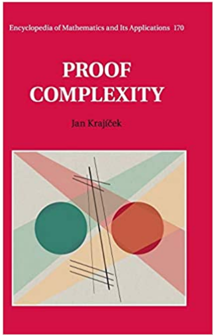
- Closely related (and equivalent) to *DRAT* proof system used to justify correctness of some SAT preprocessing techniques [JHB12]
- *DRAT* also used for SAT solver proof logging [HHW13a, HHW13b, WHH14]
- Attempts to combine extended resolution with CDCL in, e.g., [AKS10, Hua10]
- Without restrictions, as powerful as extremely strong extended Frege system [CR79] — pretty much no lower bounds known
- To analyse solvers using extended resolution, would need to:
  - Describe heuristics/rules actually used
  - See if possible to reason about such restricted proof system

Some More References for Further Reading

Handbook of Satisfiability
(Especially chapter 7 ☺)

[BHvMW21]

Proof Complexity
by Jan Krajíček



[Kra19]

Summing up This Lecture

Overview of some proof systems used in combinatorial solving:

- Resolution $\longleftrightarrow$ conflict-driven clause learning (CDCL)
- Nullstellensatz and polynomial calculus $\longleftrightarrow$ Gröbner bases
- Cutting planes $\longleftrightarrow$ pseudo-Boolean solving

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Very brief discussion of some other proof systems:

- Sherali–Adams and sums of squares $\longleftrightarrow$ LP and SDP hierarchies
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- Analyse state-of-the-art algorithms (and provide methods for certifying correctness!)
- Give ideas for new approaches
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Thank you for your attention!

References I

[ABdR+21] Albert Atserias, Ilario Bonacina, Susanna F. de Rezende, Massimo Lauria, Jakob Nordström, and Alexander Razborov. Clique is hard on average for regular resolution. *Journal of the ACM*, 68(4):23:1–23:26, August 2021. Preliminary version in *STOC '18*.

[ABRW02] Michael Alekhnovich, Eli Ben-Sasson, Alexander A. Razborov, and Avi Wigderson. Space complexity in propositional calculus. *SIAM Journal on Computing*, 31(4):1184–1211, April 2002. Preliminary version in *STOC '00*.

[ADH+19] Blair Archibald, Fraser Dunlop, Ruth Hoffmann, Ciaran McCreesh, Patrick Prosser, and James Trimble. Sequential and parallel solution-biased search for subgraph algorithms. In *Proceedings of the 16th International Conference on the Integration of Constraint Programming, Artificial Intelligence, and Operations Research (CPAIOR '19)*, volume 11494 of *Lecture Notes in Computer Science*, pages 20–38. Springer, June 2019.

[AFT11] Albert Atserias, Johannes Klaus Fichte, and Marc Thurley. Clause-learning algorithms with many restarts and bounded-width resolution. *Journal of Artificial Intelligence Research*, 40:353–373, January 2011. Preliminary version in *SAT '09*.

References II

[AKS10] Gilles Audemard, George Katsirelos, and Laurent Simon. A restriction of extended resolution for clause learning SAT solvers. In *Proceedings of the 24th AAAI Conference on Artificial Intelligence (AAAI '10)*, pages 15–20, July 2010.

[AM20] Albert Atserias and Moritz Müller. Automating resolution is NP-hard. *Journal of the ACM*, 67(5):31:1–31:17, October 2020. Preliminary version in *FOCS '19*.

[AR03] Michael Alekhnovich and Alexander A. Razborov. Lower bounds for polynomial calculus: Non-binomial case. *Proceedings of the Steklov Institute of Mathematics*, 242:18–35, 2003. Available at `http://people.cs.uchicago.edu/~razborov/files/misha.pdf`. Preliminary version in *FOCS '01*.

[BBN+23] Jeremias Berg, Bart Bogaerts, Jakob Nordström, Andy Oertel, and Dieter Vandesande. Certified core-guided MaxSAT solving. In *Proceedings of the 29th International Conference on Automated Deduction (CADE-29)*, volume 14132 of *Lecture Notes in Computer Science*, pages 1–22. Springer, July 2023.

References III

[BBN+24] Jeremias Berg, Bart Bogaerts, Jakob Nordström, Andy Oertel, Tobias Paxian, and Dieter Vandesande. Certifying without loss of generality reasoning in solution-improving maximum satisfiability. In *Proceedings of the 30th International Conference on Principles and Practice of Constraint Programming (CP '24)*, volume 307 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 4:1–4:28, September 2024.

[BCIP02] Joshua Buresh-Oppenheim, Matthew Clegg, Russell Impagliazzo, and Toniann Pitassi. Homogenization and the polynomial calculus. *Computational Complexity*, 11(3-4):91–108, 2002. Preliminary version in *ICALP '00*.

[BCMM05] Paul Beame, Joseph C. Culberson, David G. Mitchell, and Cristopher Moore. The resolution complexity of random graph k-colorability. *Discrete Applied Mathematics*, 153(1-3):25–47, December 2005.

[Ber18] Christoph Berkholz. The relation between polynomial calculus, Sherali-Adams, and sum-of-squares proofs. In *Proceedings of the 35th Symposium on Theoretical Aspects of Computer Science (STACS '18)*, volume 96 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 11:1–11:14, February 2018.

References IV

[BFI+18] Paul Beame, Noah Fleming, Russell Impagliazzo, Antonina Kolokolova, Denis Pankratov, Toniann Pitassi, and Robert Robere. Stabbing planes. In *Proceedings of the 9th Innovations in Theoretical Computer Science Conference (ITCS '18)*, volume 94 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 10:1–10:20, January 2018.

[BGIP01] Samuel R. Buss, Dima Grigoriev, Russell Impagliazzo, and Toniann Pitassi. Linear gaps between degrees for the polynomial calculus modulo distinct primes. *Journal of Computer and System Sciences*, 62(2):267–289, March 2001. Preliminary version in *CCC '99*.

[BGMN23] Bart Bogaerts, Stephan Gocht, Ciaran McCreesh, and Jakob Nordström. Certified dominance and symmetry breaking for combinatorial optimisation. *Journal of Artificial Intelligence Research*, 77:1539–1589, August 2023. Preliminary version in *AAAI '22*.

[BHvMW21] Armin Biere, Marijn J. H. Heule, Hans van Maaren, and Toby Walsh, editors. *Handbook of Satisfiability*, volume 336 of *Frontiers in Artificial Intelligence and Applications*. IOS Press, 2nd edition, February 2021.

[BI99] Eli Ben-Sasson and Russell Impagliazzo. Random CNF's are hard for the polynomial calculus. In *Proceedings of the 40th Annual IEEE Symposium on Foundations of Computer Science (FOCS '99)*, pages 415–421, October 1999. Journal version in [BI10].

References V

[BI10] Eli Ben-Sasson and Russell Impagliazzo. Random CNF's are hard for the polynomial calculus. *Computational Complexity*, 19(4):501–519, 2010. Preliminary version in *FOCS '99*.

[BIK+94] Paul Beame, Russell Impagliazzo, Jan Krajíček, Toniann Pitassi, and Pavel Pudlák. Lower bounds on Hilbert's Nullstellensatz and propositional proofs. In *Proceedings of the 35th Annual IEEE Symposium on Foundations of Computer Science (FOCS '94)*, pages 794–806, November 1994.

[Bla37] Archie Blake. *Canonical Expressions in Boolean Algebra*. PhD thesis, University of Chicago, 1937.

[BLLM14] Armin Biere, Daniel Le Berre, Emmanuel Lonca, and Norbert Manthey. Detecting cardinality constraints in CNF. In *Proceedings of the 17th International Conference on Theory and Applications of Satisfiability Testing (SAT '14)*, volume 8561 of *Lecture Notes in Computer Science*, pages 285–301. Springer, July 2014.

[BN21] Samuel R. Buss and Jakob Nordström. Proof complexity and SAT solving. In Biere et al. [BHvMW21], chapter 7, pages 233–350.

[BS97] Roberto J. Bayardo Jr. and Robert Schrag. Using CSP look-back techniques to solve real-world SAT instances. In *Proceedings of the 14th National Conference on Artificial Intelligence (AAAI '97)*, pages 203–208, July 1997.

References VI

[BW01] Eli Ben-Sasson and Avi Wigderson. Short proofs are narrow—resolution made simple. *Journal of the ACM*, 48(2):149–169, March 2001. Preliminary version in *STOC '99*.

[CCT87] William Cook, Collette Rene Coullard, and György Turán. On the complexity of cutting-plane proofs. *Discrete Applied Mathematics*, 18(1):25–38, November 1987.

[CdRN+23] Jonas Conneryd, Susanna F. de Rezende, Jakob Nordström, Shuo Pang, and Kilian Risse. Graph colouring is hard on average for polynomial calculus and Nullstellensatz. In *Proceedings of the 64th Annual IEEE Symposium on Foundations of Computer Science (FOCS '23)*, pages 1–11, November 2023.

[CEI96] Matthew Clegg, Jeffery Edmonds, and Russell Impagliazzo. Using the Groebner basis algorithm to find proofs of unsatisfiability. In *Proceedings of the 28th Annual ACM Symposium on Theory of Computing (STOC '96)*, pages 174–183, May 1996.

[Chv73] Vašek Chvátal. Edmonds polytopes and a hierarchy of combinatorial problems. *Discrete Mathematics*, 4(1):305–337, 1973.

References VII

[CK05] Donald Chai and Andreas Kuehlmann. A fast pseudo-Boolean constraint solver. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 24(3):305–317, March 2005. Preliminary version in *DAC '03*.

[Coo71] Stephen A. Cook. The complexity of theorem-proving procedures. In *Proceedings of the 3rd Annual ACM Symposium on Theory of Computing (STOC '71)*, pages 151–158, May 1971.

[CR79] Stephen A. Cook and Robert A. Reckhow. The relative efficiency of propositional proof systems. *Journal of Symbolic Logic*, 44(1):36–50, March 1979. Preliminary version in *STOC '74*.

[CS88] Vašek Chvátal and Endre Szemerédi. Many hard examples for resolution. *Journal of the ACM*, 35(4):759–768, October 1988.

[DGD+21] Jo Devriendt, Stephan Gocht, Emir Demirović, Jakob Nordström, and Peter Stuckey. Cutting to the core of pseudo-Boolean optimization: Combining core-guided search with cutting planes reasoning. In *Proceedings of the 35th AAAI Conference on Artificial Intelligence (AAAI '21)*, pages 3750–3758, February 2021.

References VIII

[DGN21] Jo Devriendt, Ambros Gleixner, and Jakob Nordström. Learn to relax: Integrating 0-1 integer linear programming with pseudo-Boolean conflict-driven search. *Constraints*, 26(1–4):26–55, October 2021. Preliminary version in *CPAIOR '20*.

[DHN+25] Simon Dold, Malte Helmert, Jakob Nordström, Gabriele Röger, and Tanja Schindler. Pseudo-Boolean proof logging for optimal classical planning. In *Proceedings of the 35th International Conference on Automated Planning and Scheduling (ICAPS '25)*, November 2025. To appear.

[DLL62] Martin Davis, George Logemann, and Donald Loveland. A machine program for theorem proving. *Communications of the ACM*, 5(7):394–397, July 1962.

[DLMM08] Jesús A. De Loera, Jon Lee, Peter N. Malkin, and Susan Margulies. Hilbert's Nullstellensatz and an algorithm for proving combinatorial infeasibility. In *Proceedings of the 21st International Symposium on Symbolic and Algebraic Computation (ISSAC '08)*, pages 197–206, July 2008.

[DLMM11] Jesús A. De Loera, Jon Lee, Peter N. Malkin, and Susan Margulies. Computing infeasibility certificates for combinatorial problems through Hilbert's Nullstellensatz. *Journal of Symbolic Computation*, 46(11):1260–1283, November 2011.

References IX

[DLMO09] Jesús A. De Loera, Jon Lee, Susan Margulies, and Shmuel Onn. Expressing combinatorial problems by systems of polynomial equations and Hilbert's Nullstellensatz. *Combinatorics, Probability and Computing*, 18(4):551–582, July 2009.

[DMM+24] Emir Demirović, Ciaran McCreesh, Matthew McIlree, Jakob Nordström, Andy Oertel, and Konstantin Sidorov. Pseudo-Boolean reasoning about states and transitions to certify dynamic programming and decision diagram algorithms. In *Proceedings of the 30th International Conference on Principles and Practice of Constraint Programming (CP '24)*, volume 307 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 9:1–9:21, September 2024.

[DP60] Martin Davis and Hilary Putnam. A computing procedure for quantification theory. *Journal of the ACM*, 7(3):201–215, 1960.

[dRGN+21] Susanna F. de Rezende, Mika Göös, Jakob Nordström, Toniann Pitassi, Robert Robere, and Dmitry Sokolov. Automating algebraic proof systems is NP-hard. In *Proceedings of the 53rd Annual ACM Symposium on Theory of Computing (STOC '21)*, pages 209–222, June 2021.

References X

[dRLNS21] Susanna F. de Rezende, Massimo Lauria, Jakob Nordström, and Dmitry Sokolov. The power of negative reasoning. In *Proceedings of the 36th Annual Computational Complexity Conference (CCC '21)*, volume 200 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 40:1–40:24, July 2021.

[dRMN+20] Susanna F. de Rezende, Or Meir, Jakob Nordström, Toniann Pitassi, Robert Robere, and Marc Vinyals. Lifting with simple gadgets and applications to circuit and proof complexity. In *Proceedings of the 61st Annual IEEE Symposium on Foundations of Computer Science (FOCS '20)*, pages 24–30, November 2020.

[dRMNR21] Susanna F. de Rezende, Or Meir, Jakob Nordström, and Robert Robere. Nullstellensatz size-degree trade-offs from reversible pebbling. *Computational Complexity*, 30:4:1–4:45, February 2021. Preliminary version in *CCC '19*.

[DT20] Daniel Dadush and Samarth Tiwari. On the complexity of branching proofs. In *Proceedings of the 35th Annual Computational Complexity Conference (CCC '20)*, volume 169 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 34:1–34:35, July 2020.

References XI

[EGMN20] Jan Elffers, Stephan Gocht, Ciaran McCreesh, and Jakob Nordström. Justifying all differences using pseudo-Boolean reasoning. In *Proceedings of the 34th AAAI Conference on Artificial Intelligence (AAAI '20)*, pages 1486–1494, February 2020.

[EGNV18] Jan Elffers, Jesús Giráldez-Cru, Jakob Nordström, and Marc Vinyals. Using combinatorial benchmarks to probe the reasoning power of pseudo-Boolean solvers. In *Proceedings of the 21st International Conference on Theory and Applications of Satisfiability Testing (SAT '18)*, volume 10929 of *Lecture Notes in Computer Science*, pages 75–93. Springer, July 2018.

[EN18] Jan Elffers and Jakob Nordström. Divide and conquer: Towards faster pseudo-Boolean solving. In *Proceedings of the 27th International Joint Conference on Artificial Intelligence (IJCAI '18)*, pages 1291–1299, July 2018.

[EN20] Jan Elffers and Jakob Nordström. A cardinal improvement to pseudo-Boolean solving. In *Proceedings of the 34th AAAI Conference on Artificial Intelligence (AAAI '20)*, pages 1495–1503, February 2020.

[FGI+21] Noah Fleming, Mika Göös, Russell Impagliazzo, Toniann Pitassi, Robert Robere, Li-Yang Tan, and Avi Wigderson. On the power and limitations of branch and cut. In *Proceedings of the 36th Annual Computational Complexity Conference (CCC '21)*, volume 200 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 6:1–6:30, July 2021.

References XII

[FKP19] Noah Fleming, Pravesh Kothari, and Toniann Pitassi. Semialgebraic proofs and efficient algorithm design. *Foundations and Trends in Theoretical Computer Science*, 14(1–2):1–221, December 2019.

[FPPR22] Noah Fleming, Denis Pankratov, Toniann Pitassi, and Robert Robere. Random $\theta(\log n)$-CNFs are hard for cutting planes. *Journal of the ACM*, 69(3):19:1–19:32, June 2022. Preliminary version in *FOCS '17*.

[GGKS20] Ankit Garg, Mika Göös, Pritish Kamath, and Dmitry Sokolov. Monotone circuit lower bounds from resolution. *Theory of Computing*, 16(13):1–30, 2020. Preliminary version in *STOC '18*.

[GKMP20] Mika Göös, Sajin Koroth, Ian Mertz, and Toniann Pitassi. Automating cutting planes is NP-hard. In *Proceedings of the 52nd Annual ACM Symposium on Theory of Computing (STOC '20)*, pages 68–77, June 2020.

[GMM+20] Stephan Gocht, Ross McBride, Ciaran McCreesh, Jakob Nordström, Patrick Prosser, and James Trimble. Certifying solvers for clique and maximum common (connected) subgraph problems. In *Proceedings of the 26th International Conference on Principles and Practice of Constraint Programming (CP '20)*, volume 12333 of *Lecture Notes in Computer Science*, pages 338–357. Springer, September 2020.

References XIII

[GMM+24] Stephan Gocht, Ciaran McCreesh, Magnus O. Myreen, Jakob Nordström, Andy Oertel, and Yong Kiam Tan. End-to-end verification for subgraph solving. In *Proceedings of the 38th AAAI Conference on Artificial Intelligence (AAAI '24)*, pages 8038–8047, February 2024.

[GMN20] Stephan Gocht, Ciaran McCreesh, and Jakob Nordström. Subgraph isomorphism meets cutting planes: Solving with certified solutions. In *Proceedings of the 29th International Joint Conference on Artificial Intelligence (IJCAI '20)*, pages 1134–1140, July 2020.

[GMN22] Stephan Gocht, Ciaran McCreesh, and Jakob Nordström. An auditable constraint programming solver. In *Proceedings of the 28th International Conference on Principles and Practice of Constraint Programming (CP '22)*, volume 235 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 25:1–25:18, August 2022.

[GMNO22] Stephan Gocht, Ruben Martins, Jakob Nordström, and Andy Oertel. Certified CNF translations for pseudo-Boolean solving. In *Proceedings of the 25th International Conference on Theory and Applications of Satisfiability Testing (SAT '22)*, volume 236 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 16:1–16:25, August 2022.

References XIV

[GN21] Stephan Gocht and Jakob Nordström. Certifying parity reasoning efficiently using pseudo-Boolean proofs. In *Proceedings of the 35th AAAI Conference on Artificial Intelligence (AAAI '21)*, pages 3768–3777, February 2021.

[GNY19] Stephan Gocht, Jakob Nordström, and Amir Yehudayoff. On division versus saturation in pseudo-Boolean solving. In *Proceedings of the 28th International Joint Conference on Artificial Intelligence (IJCAI '19)*, pages 1711–1718, August 2019.

[Gom63] Ralph E. Gomory. An algorithm for integer solutions of linear programs. In R.L. Graves and P. Wolfe, editors, *Recent Advances in Mathematical Programming*, pages 269–302. McGraw-Hill, New York, 1963.

[GP24] Max Gläser and Marc E. Pfetsch. Sub-exponential lower bounds for branch-and-bound with general disjunctions via interpolation. In *Proceedings of the 35th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '24)*, pages 3747–3764, January 2024.

[Gri01] Dima Grigoriev. Linear lower bound on degrees of Positivstellensatz calculus proofs for the parity. *Theoretical Computer Science*, 259(1–2):613–622, May 2001.

[GSS] The Glasgow subgraph solver. `https://github.com/ciaranm/glasgow-subgraph-solver`.

References XV

[GV01] Dima Grigoriev and Nicolai Vorobjov. Complexity of Null- and Positivstellensatz proofs. *Annals of Pure and Applied Logic*, 113(1–3):153–160, December 2001.

[Hak85] Armin Haken. The intractability of resolution. *Theoretical Computer Science*, 39(2-3):297–308, August 1985.

[Hås99] Johan Håstad. Clique is hard to approximate within $n^{1-\epsilon}$. *Acta Mathematica*, 182:105–142, 1999. Preliminary version in *FOCS '96*.

[Hås01] Johan Håstad. Some optimal inapproximability results. *Journal of the ACM*, 48(4):798–859, July 2001. Preliminary version in *STOC '97*.

[HHW13a] Marijn J. H. Heule, Warren A. Hunt Jr., and Nathan Wetzler. Trimming while checking clausal proofs. In *Proceedings of the 13th International Conference on Formal Methods in Computer-Aided Design (FMCAD '13)*, pages 181–188, October 2013.

[HHW13b] Marijn J. H. Heule, Warren A. Hunt Jr., and Nathan Wetzler. Verifying refutations with extended resolution. In *Proceedings of the 24th International Conference on Automated Deduction (CADE-24)*, volume 7898 of *Lecture Notes in Computer Science*, pages 345–359. Springer, June 2013.

References XVI

[HOGN24] Alexander Hoen, Andy Oertel, Ambros Gleixner, and Jakob Nordström. Certifying MIP-based presolve reductions for 0–1 integer linear programs. In *Proceedings of the 21st International Conference on the Integration of Constraint Programming, Artificial Intelligence, and Operations Research (CPAIOR '24)*, volume 14742 of *Lecture Notes in Computer Science*, pages 310–328. Springer, May 2024.

[HP17] Pavel Hrubeš and Pavel Pudlák. Random formulas, monotone circuits, and interpolation. In *Proceedings of the 58th Annual IEEE Symposium on Foundations of Computer Science (FOCS '17)*, pages 121–131, October 2017.

[Hua10] Jinbo Huang. Extended clause learning. *Artificial Intelligence*, 174(15):1277–1284, October 2010.

[IOT$^+$24] Hannes Ihalainen, Andy Oertel, Yong Kiam Tan, Jeremias Berg, Matti Järvisalo, Magnus O. Myreen, and Jakob Nordström. Certified MaxSAT preprocessing. In *Proceedings of the 12th International Joint Conference on Automated Reasoning (IJCAR '24)*, volume 14739 of *Lecture Notes in Computer Science*, pages 396–418. Springer, July 2024.

[IPS99] Russell Impagliazzo, Pavel Pudlák, and Jiří Sgall. Lower bounds for the polynomial calculus and the Gröbner basis algorithm. *Computational Complexity*, 8(2):127–144, 1999.

References XVII

[JHB12] Matti Järvisalo, Marijn J. H. Heule, and Armin Biere. Inprocessing rules. In *Proceedings of the 6th International Joint Conference on Automated Reasoning (IJCAR '12)*, volume 7364 of *Lecture Notes in Computer Science*, pages 355–370. Springer, June 2012.

[KB20] Daniela Kaufmann and Armin Biere. Nullstellensatz-proofs for multiplier verification. In *Proceedings of the 22nd International Workshop on Computer Algebra in Scientific Computing (CASC' 20)*, volume 12291 of *Lecture Notes in Computer Science*, pages 368–389. Springer, September 2020.

[KB21] Daniela Kaufmann and Armin Biere. AMulet 2.0 for verifying multiplier circuits. In *Proceedings of the 27th International Conference on Tools and Algorithms for the Construction and Analysis of Systems (TACAS '21)*, volume 12652 of *Lecture Notes in Computer Science*, pages 357–364. Springer, March-April 2021.

[KBBN22] Daniela Kaufmann, Paul Beame, Armin Biere, and Jakob Nordström. Adding dual variables to algebraic reasoning for circuit verification. In *Proceedings of the 25th Design, Automation and Test in Europe Conference (DATE '22)*, pages 1435–1440, March 2022.

[KBK20a] Daniela Kaufmann, Armin Biere, and Manuel Kauers. From DRUP to PAC and back. In *Proceedings of the Design, Automation & Test in Europe Conference & Exhibition (DATE '20)*, pages 654–657, March 2020.

References XVIII

[KBK20b] Daniela Kaufmann, Armin Biere, and Manuel Kauers. Incremental column-wise verifiation of arithmetic circuits using computer algebra. *Formal Methods in Systems Design*, 56(1–3):22–54, 2020. Preliminary version in *FMCAD '17*.

[KFB20] Daniela Kaufmann, Mathias Fleury, and Armin Biere. The proof checkers Pacheck and Pastèque for the practical algebraic calculus. In *Proceedings of the 20th Conference on Formal Methods in Computer-Aided Design (FMCAD '20)*, pages 264–269, September 2020.

[Kho01] Subhash Khot. Improved inapproximability results for MaxClique, chromatic number and approximate graph coloring. In *Proceedings of the 42nd Annual IEEE Symposium on Foundations of Computer Science (FOCS '01)*, pages 600–609, October 2001.

[KLM$^+$25] Wietze Koops, Daniel Le Berre, Magnus O. Myreen, Jakob Nordström, Andy Oertel, Yong Kiam Tan, and Marc Vinyals. Practically feasible proof logging for pseudo-Boolean optimization. In *Proceedings of the 31st International Conference on Principles and Practice of Constraint Programming (CP '25)*, volume 340 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 21:1–21:27, August 2025.

[Kra19] Jan Krajíček. *Proof Complexity*, volume 170 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, March 2019.

References XIX

[LBD+20] Vincent Liew, Paul Beame, Jo Devriendt, Jan Elffers, and Jakob Nordström. Verifying properties of bit-vector multiplication using cutting planes reasoning. In *Proceedings of the 20th Conference on Formal Methods in Computer-Aided Design (FMCAD '20)*, pages 194–204, September 2020.

[Lev73] Leonid A. Levin. Universal sequential search problems. *Problemy peredachi informatsii*, 9(3):115–116, 1973. In Russian. Available at `http://mi.mathnet.ru/ppi914`.

[LN17] Massimo Lauria and Jakob Nordström. Graph colouring is hard for algorithms based on Hilbert's Nullstellensatz and Gröbner bases. In *Proceedings of the 32nd Annual Computational Complexity Conference (CCC '17)*, volume 79 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 2:1–2:20, July 2017.

[LP10] Daniel Le Berre and Anne Parrain. The Sat4j library, release 2.2. *Journal on Satisfiability, Boolean Modeling and Computation*, 7:59–64, July 2010.

[Mar06] Klas Markström. Locality and hard SAT-instances. *Journal on Satisfiability, Boolean Modeling and Computation*, 2(1-4):221–227, 2006.

References XX

[MBGN23] Gioni Mexi, Timo Berthold, Ambros Gleixner, and Jakob Nordström. Improving conflict analysis in MIP solvers by pseudo-Boolean reasoning. In *Proceedings of the 29th International Conference on Principles and Practice of Constraint Programming (CP '23)*, volume 280 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 27:1–26:19, August 2023.

[McC17] Ciaran McCreesh. *Solving Hard Subgraph Problems in Parallel*. PhD thesis, University of Glasgow, 2017.

[MM23] Matthew McIlree and Ciaran McCreesh. Proof logging for smart extensional constraints. In *Proceedings of the 29th International Conference on Principles and Practice of Constraint Programming (CP '23)*, volume 280 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 26:1–26:17, August 2023.

[MM25] Matthew McIlree and Ciaran McCreesh. Certifying bounds propagation for integer multiplication constraints. In *Proceedings of the 39th AAAI Conference on Artificial Intelligence (AAAI '25)*, pages 11309–11317, February-March 2025.

References XXI

[MMN24] Matthew McIlree, Ciaran McCreesh, and Jakob Nordström. Proof logging for the circuit constraint. In *Proceedings of the 21st International Conference on the Integration of Constraint Programming, Artificial Intelligence, and Operations Research (CPAIOR '24)*, volume 14743 of *Lecture Notes in Computer Science*, pages 38–55. Springer, May 2024.

[MMZ+01] Matthew W. Moskewicz, Conor F. Madigan, Ying Zhao, Lintao Zhang, and Sharad Malik. Chaff: Engineering an efficient SAT solver. In *Proceedings of the 38th Design Automation Conference (DAC '01)*, pages 530–535, June 2001.

[MN14] Mladen Mikša and Jakob Nordström. Long proofs of (seemingly) simple formulas. In *Proceedings of the 17th International Conference on Theory and Applications of Satisfiability Testing (SAT '14)*, volume 8561 of *Lecture Notes in Computer Science*, pages 121–137. Springer, July 2014.

[MN24] Mladen Mikša and Jakob Nordström. A generalized method for proving polynomial calculus degree lower bounds. *Journal of the ACM*, 71(6):37:1–37:43, November 2024. Preliminary version in *CCC '15*.

[MS99] João P. Marques-Silva and Karem A. Sakallah. GRASP: A search algorithm for propositional satisfiability. *IEEE Transactions on Computers*, 48(5):506–521, May 1999. Preliminary version in *ICCAD '96*.

References XXII

[MSB+25] Gioni Mexi, Felipe Serrano, Timo Berthold, Ambros Gleixner, and Jakob Nordström. Cut-based conflict analysis in mixed integer programming. *INFORMS Journal on Computing*, 2025. To appear.

[PD11] Knot Pipatsrisawat and Adnan Darwiche. On the power of clause-learning SAT solvers as resolution engines. *Artificial Intelligence*, 175(2):512–525, February 2011. Preliminary version in *CP '09*.

[Pro12] Patrick Prosser. Exact algorithms for maximum clique: A computational study. *Algorithms*, 5(4):545–587, November 2012.

[Pud97] Pavel Pudlák. Lower bounds for resolution and cutting plane proofs and monotone computations. *Journal of Symbolic Logic*, 62(3):981–998, September 1997.

[Raz98] Alexander A. Razborov. Lower bounds for the polynomial calculus. *Computational Complexity*, 7(4):291–324, December 1998.

[Rii93] Søren Riis. *Independence in Bounded Arithmetic*. PhD thesis, University of Oxford, 1993.

[Rob65] John Alan Robinson. A machine-oriented logic based on the resolution principle. *Journal of the ACM*, 12(1):23–41, January 1965.

References XXIII

[Sch08] Grant Schoenebeck. Linear level Lasserre lower bounds for certain k-CSPs. In *Proceedings of the 49th Annual IEEE Symposium on Foundations of Computer Science (FOCS '08)*, pages 593–602, October 2008.

[Sok24] Dmitry Sokolov. Random $(\log n)$-CNF are hard for cutting planes (again). In *Proceedings of the 56th Annual ACM Symposium on Theory of Computing (STOC '24)*, pages 2008–2015, June 2024.

[Spe10] Ivor Spence. sgen1: A generator of small but difficult satisfiability benchmarks. *Journal of Experimental Algorithmics*, 15:1.2:1–1.2:15, March 2010.

[SS06] Hossein M. Sheini and Karem A. Sakallah. Pueblo: A hybrid pseudo-Boolean SAT solver. *Journal on Satisfiability, Boolean Modeling and Computation*, 2(1-4):165–189, March 2006. Preliminary version in *DATE '05*.

[Tse68] Grigori Tseitin. On the complexity of derivation in propositional calculus. In A. O. Silenko, editor, *Structures in Constructive Mathematics and Mathematical Logic, Part II*, pages 115–125. Consultants Bureau, New York-London, 1968.

[Urq87] Alasdair Urquhart. Hard examples for resolution. *Journal of the ACM*, 34(1):209–219, January 1987.

References XXIV

[VS10] Allen Van Gelder and Ivor Spence. Zero-one designs produce small hard SAT instances. In *Proceedings of the 13th International Conference on Theory and Applications of Satisfiability Testing (SAT '10)*, volume 6175 of *Lecture Notes in Computer Science*, pages 388–397. Springer, July 2010.

[WHH14] Nathan Wetzler, Marijn J. H. Heule, and Warren A. Hunt Jr. DRAT-trim: Efficient checking and trimming using expressive clausal proofs. In *Proceedings of the 17th International Conference on Theory and Applications of Satisfiability Testing (SAT '14)*, volume 8561 of *Lecture Notes in Computer Science*, pages 422–429. Springer, July 2014.

[Zuc07] David Zuckerman. Linear degree extractors and the inapproximability of max clique and chromatic number. *Theory of Computing*, 3(6):103–128, August 2007. Preliminary version in *STOC '06*.