A Unified Proof System for Discrete Combinatorial Problems

Jakob Nordström

University of Copenhagen and Lund University

Dagstuhl Seminar 23471
"The Next Generation of Deduction Systems:
From Composition to Compositionality"
November 24, 2023

Based on joint work with Bart Bogaerts, Stephan Gocht, Ciaran McCreesh, Magnus O. Myreen, Andy Oertel, and Yong Kiam Tan

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The Success of Combinatorial Solving (and the Dirty Little Secret)

- Astounding progress last couple of decades on combinatorial solvers for, e.g.:
  - Boolean satisfiability (SAT) solving and optimization [BHvMW21]
  - Constraint programming [RvBW06]
  - Mixed integer linear programming [AW13, BR07]
  - Satisfiability modulo theories (SMT) solving [BHvMW21]
- Solvers very fast, but sometimes wrong (even best commercial ones) [BLB10, CKSW13, AGJ+18, GSD19, GS19, BMN22, BBN+23]
- Even get feasibility of solutions wrong (though this should be straightforward!)
- And how to check the absence of solutions?
- Or that a solution is optimal? (Even off-by-one mistakes can snowball into large errors if solver used as subroutine)

What Can Be Done About Solver Bugs?

- **Software testing**
  Hard to get good test coverage for sophisticated solvers
  Progress using fuzzing and delta debugging [BB09, BLB10, KB22, NPB22, PB23]
  But inherently can only detect presence of bugs, not absence

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  Prove that solver implementation adheres to formal specification
  Current techniques cannot scale to this level of complexity

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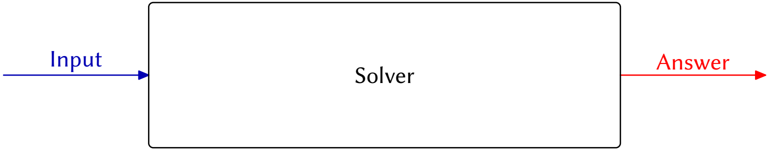
- **Proof logging**
  Make solver certifying [ABM+11, MMNS11] by adding code so that it outputs
  1. not only answer but also
  2. simple, machine-verifiable proof that answer is correct

Proof Logging with Certifying Solvers: Workflow



1. Run combinatorial solving algorithm on problem input

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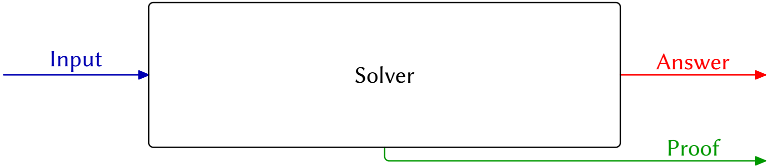

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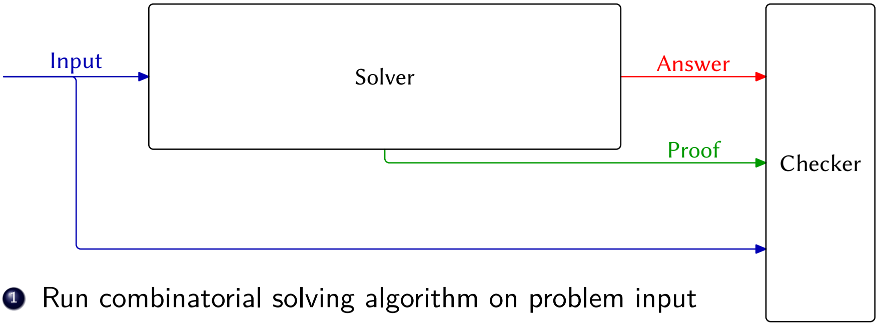

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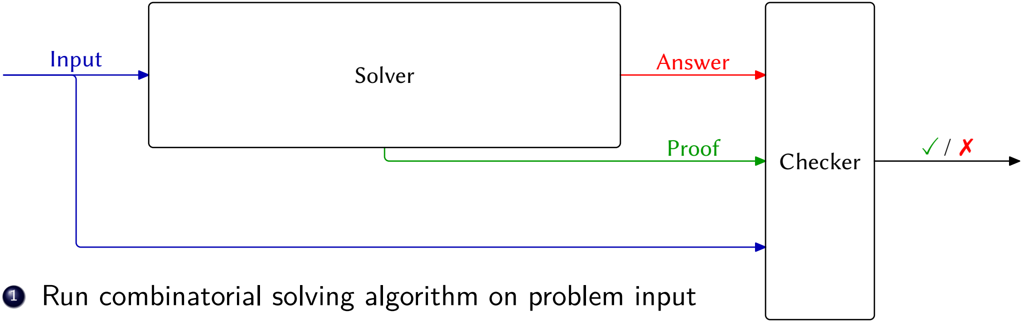
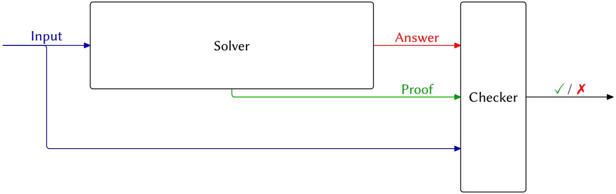


1. Run combinatorial solving algorithm on problem input
2. Get as output not only answer but also proof
3. Feed input + answer + proof to proof checker
4. Verify that proof checker says answer is correct

Proof Logging Desiderata

Proof format for certifying solver should be



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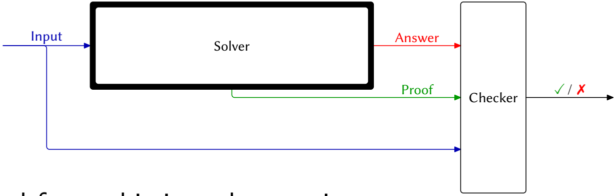

Proof format for certifying solver should be

- **very powerful:** minimal overhead for sophisticated reasoning

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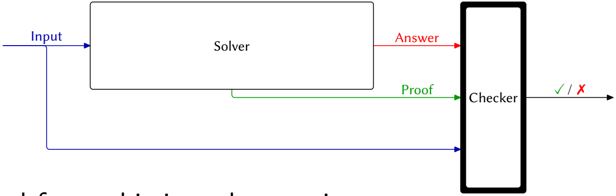

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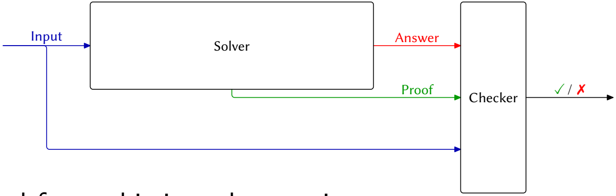

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Clear conflict expressivity vs. simplicity!

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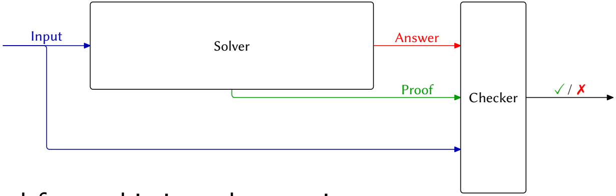

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Asking for both perhaps a little bit too good to be true?

This Talk

Proof logging for combinatorial optimization is possible with single, unified method!

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- Build on successes in SAT solving with proof formats such as DRAT [HHW13a, HHW13b, WHH14], GRIT [CMS17], LRAT [CHH+17], ...
- But represent constraints as 0–1 integer linear inequalities
- Formalize reasoning using cutting planes [CCT87] proof system
- Add well-chosen strengthening rules [Goc22, GN21, BGMN23]
- Implemented in VeriPB (`https://gitlab.com/MIAOresearch/software/VeriPB`)

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Purpose of this talk:

1. Marketing pitch ☺

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2. Explore potential connections with more challenging settings such as SMT, first-order logic, . . .

The Sales Pitch For Proof Logging

1. Certifies correctness of computed results
2. Detects errors even if due to compiler bugs, hardware failures, or cosmic rays
3. Provides debugging support during software development [EG21, GMM+20, KM21, BBN+23]
4. Facilitates performance analysis
5. Helps identify potential for further improvements
6. Enables auditability
7. Serves as stepping stone towards explainability

Design Principles for Proof Logging

Proof logging implementation

- Don't change solver
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- Proof logging overhead small constant fraction ($\lessapprox 10\%$)
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Proof system

- Keep proof language maximally simple
- Reason about XOR constraints, CP propagators, symmetries, etc within language
- Combine proof logging with formally verified proof checker

Pseudo-Boolean Constraints

Proof consists of 0-1 integer linear inequalities or pseudo-Boolean constraints:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
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Sometimes convenient to use normalized form [Bar95] with all a_i, A positive (without loss of generality)

Some Types of Pseudo-Boolean Constraints

1. Clauses

$$x \lor \overline{y} \lor z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Pseudo-Boolean Proof Logging Wishlist

Paradigms

- SAT solving
- (linear) pseudo-Boolean solving
- graph solving
- constraint programming
- automated planning
- mixed integer linear programming
- SMT solving

Problem types

- decision / feasibility
- optimization
- multi-objective optimization
- projected model enumeration
- projected model counting
- preprocessing / problem reformulation

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Pseudo-Boolean Proof Logging — How and Why?

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1. 0-1 ILP expressive formalism for combinatorial problems (including objective)
2. Powerful reasoning capturing many combinatorial arguments
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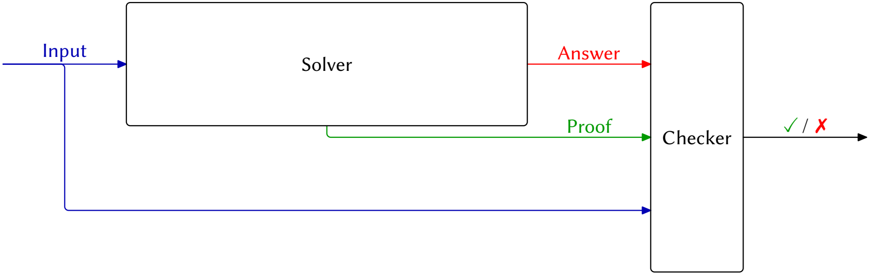
$$r \Leftarrow x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7 \qquad 9r + \overline{x}_1 + 2x_2 + 3\overline{x}_3 + 4x_4 + 5\overline{x}_5 \geq 9$$

Proof Logging with Formally Verified Checking: Full Workflow



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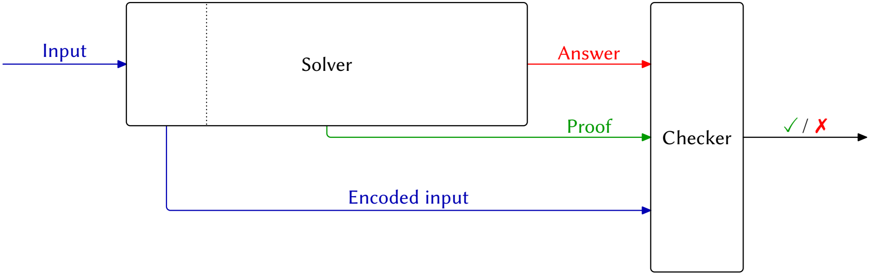

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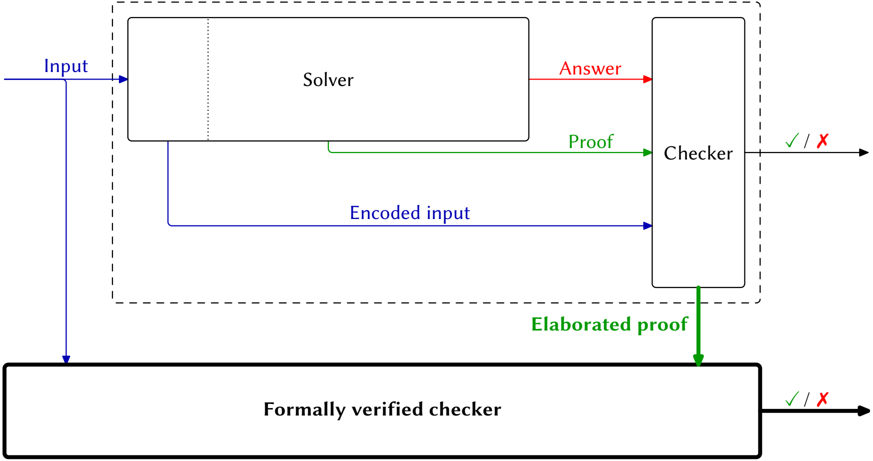

VeriPB Proof Configuration (Slightly Simplified)

Core set $\mathcal{C}$

- Contains input formula at the start
- Maintains "equivalence" with input formula

Derived set $\mathcal{D}$

- All constraints derived during search
- Also intermediate constraints used in proof logging [but not used by solver]

Objective $f = \sum_i w_i \ell_i + k$

- 0–1 linear function to minimize
- Or $f = 0$ for decision problem
- Keep track of best known bound; initialize to ∞

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input axioms From the input

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Literal axioms $$\frac{}{\ell_i \geq 0}$$

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Addition

$$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i) \ell_i \geq A + B}$$

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Multiplication for any $c \in \mathbb{N}^+$

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i c a_i \ell_i \geq cA}$$

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Division for any $c \in \mathbb{N}^+$
(constraint in normalized form)

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \left\lceil \frac{a_i}{c} \right\rceil \ell_i \geq \left\lceil \frac{A}{c} \right\rceil}$$

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Saturation
(constraint in normalized form)

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \min(a_i, A) \cdot \ell_i \geq A}$$

Cutting Planes Toy Example

$$w + 2x + y \geq 2$$

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Multiply by 2 $\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$

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$$\dfrac{\text{Multiply by } 2 \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} \text{ Add}$$

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Divide by 3: $\dfrac{3w + 6x + 6y \geq 7}{w + 2x + 2y \geq 2\frac{1}{3}}$

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$$\text{Divide by } 3\;\dfrac{\text{Add}\;\dfrac{\text{Add}\;\dfrac{\text{Multiply by } 2\;\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} \qquad \dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\;\text{Multiply by } 2}{3w + 6x + 6y \geq 7}}{w + 2x + 2y \geq 3}$$

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By naming constraints by integers and literal axioms by the literal involved as

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\begin{aligned}
\text{Constraint 1} &\doteq 2x + y + w \geq 2 \\
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$$

such a calculation is written in the proof log in reverse Polish notation as

```
pol 1 2 * 2 + ~z 2 * + 3 d
```

Redundance-Based Strengthening

C is redundant with respect to F if F and $F \wedge C$ are equisatisfiable

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C is redundant with respect to F if and only if there is a substitution ω (mapping variables to truth values or literals), called a witness, for which

$$F \wedge \neg C \models (F \wedge C)\restriction_{\omega}$$

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- Proof sketch for interesting direction: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \land C$

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Redundance-based strengthening ([BT19, GN21], inspired by [JHB12])

C is redundant with respect to F if and only if there is a substitution ω (mapping variables to truth values or literals), called a witness, for which

$$F \land \neg C \models (F \land C)\restriction_{\omega}$$

- Proof sketch for interesting direction: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \land C$
- In a proof, the implication needs to be efficiently verifiable — every $D \in (F \land C)\restriction_{\omega}$ should follow from $F \land \neg C$ either
  1. "obviously" or
  2. by explicitly presented derivation

Redundance and Dominance Rules in VeriPB (Slightly Simplified)

Redundance-based strengthening, optimization version [BGMN23]

Add constraint C to derived set $\mathcal{D}$ if exists witness substitution ω such that

$$\mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \models (\mathcal{C} \cup \mathcal{D} \cup \{C\})\restriction_\omega \cup \{f\restriction_\omega \leq f\}$$

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Can be more aggressive if witness ω strictly improves solution

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- Applying ω should strictly decrease f
- If so, don't need to show that $(\mathcal{D} \cup \{C\})\!\restriction_\omega$ implied!

Soundness of Dominance Rule

Dominance-based strengthening

Add constraint C to derived set $\mathcal{D}$ if exists witness substitution ω such that

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Why is this sound? Let $\mathcal{D} = \emptyset$ for simplicity

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7. ...

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6. Otherwise $((\alpha \circ \omega) \circ \omega) \circ \omega$ satisfies $\mathcal{C}$ and $f(((\alpha \circ \omega) \circ \omega) \circ \omega) < f((\alpha \circ \omega) \circ \omega)$
7. ...
8. Can't go on forever, so finally reach α' satisfying $\mathcal{C} \cup \{C\}$

Soundness of Dominance Rule (Continued)

Dominance-based strengthening

Add constraint C to derived set $\mathcal{D}$ if exists witness substitution ω such that

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- Same inductive proof as before, but also nested forward induction over derivation
- Or pick α satisfying $\mathcal{C} \cup \mathcal{D}$ and minimizing f and argue by contradiction

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Further extensions:

- Define dominance rule with respect to order independent of objective function
- Switch between different orders in same proof
- See [BGMN23] for details

Three Pseudo-Boolean Proof Logging Vignettes

1. Advanced SAT solving techniques [GN21, BGMN23]
2. Graph solving (subgraph isomorphism) [GMN20, GMM⁺20]
3. Constraint programming [EGMN20, GMN22, MM23]

Parity (XOR) Reasoning in SAT Solving

Given clauses

$$x \lor y \lor z$$
$$x \lor \overline{y} \lor \overline{z}$$
$$\overline{x} \lor y \lor \overline{z}$$
$$\overline{x} \lor \overline{y} \lor z$$

and

$$y \lor z \lor w$$
$$y \lor \overline{z} \lor \overline{w}$$
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want to derive

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This is just parity reasoning:

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$$x + y + z = 1 \pmod 2$$
$$y + z + w = 1 \pmod 2$$

imply

$$x + w = 0 \pmod 2$$

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But used in CryptoMiniSat [Cry]

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Could add XORs to language, but prefer to keep things super-simple

Pseudo-Boolean Proof Logging for XOR Reasoning

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Use redundance rule with fresh variables a, b to derive

$$x + y + z + 2a = 3$$
$$y + z + w + 2b = 3$$

("$=$" syntactic sugar for "$\geq$" plus "$\leq$")

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From this can extract

$$x + \overline{w} \geq 1$$
$$\overline{x} + w \geq 1$$

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VeriPB can certify XOR reasoning [GN21]

Symmetry Breaking in SAT Solving

1. Pretend to solve optimisation problem minimizing $f \doteq \sum_{i=1}^{n} 2^{n-i} \cdot x_i$ (search for lexicographically smallest assignment satisfying formula)

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3. Derive symmetry breaking clauses from this PB constraint:

$$y_0 \qquad\qquad \overline{y}_j \vee \overline{\sigma(x_j)} \vee x_j$$

$$\overline{y}_{j-1} \vee \overline{x}_j \vee \sigma(x_j) \qquad\qquad y_j \vee \overline{y}_{j-1} \vee \overline{x}_j$$

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VeriPB can certify fully general SAT symmetry breaking [BGMN23]

The Subgraph Isomorphism Problem

Input

- Pattern graph $\mathcal{P}$ with vertices $V(\mathcal{P}) = \{a, b, c, \ldots\}$
- Target graph $\mathcal{T}$ with vertices $V(\mathcal{T}) = \{u, v, w, \ldots\}$

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Task

- Find all subgraph isomorphisms $\varphi : V(\mathcal{P}) \to V(\mathcal{T})$
- I.e., if
  1. $\varphi(a) = u$
  2. $\varphi(b) = v$
  3. $(a, b) \in E(\mathcal{P})$

  then must have $(u, v) \in E(\mathcal{T})$

Pseudo-Boolean Proof Logging for Subgraph Isomorphism Solving

All reasoning steps in Glasgow Subgraph Solver [ADH⁺19, GSS] can be formalized efficiently in the cutting planes proof system [GMN20]

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Means that

1. Solver can justify each step by writing local formal derivation
2. Local derivations can be chained into global correctness proof
3. Proof checkable by stand-alone verifier that knows nothing about graphs
4. Strong correctness guarantees:
   - Even for buggy solver, a correct proof is always accepted
   - Even for formally verified solver that gets whacked by cosmic radiation/hardware failure, wrong proof will always be rejected

Subgraph Isomorphism as a Pseudo-Boolean Formula

- Pattern graph $\mathcal{P}$ with $V(\mathcal{P}) = \{a, b, c, \ldots\}$
- Target graph $\mathcal{T}$ with $V(\mathcal{T}) = \{u, v, w, \ldots\}$
- No loops (for simplicity)

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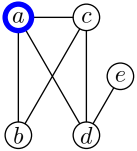
Pseudo-Boolean encoding

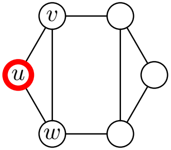
$$\sum_{v \in V(\mathcal{T})} x_{a \mapsto v} = 1 \qquad \text{[every } a \text{ maps somewhere]}$$

$$\sum_{b \in V(\mathcal{P})} \overline{x}_{b \mapsto u} \geq |V(\mathcal{P})| - 1 \qquad \text{[mapping is one-to-one]}$$

$$\overline{x}_{a \mapsto u} + \sum_{v \in N(u)} x_{b \mapsto v} \geq 1 \qquad \text{[edge } (a, b) \text{ maps to edge } (u, v)\text{]}$$

Pseudo-Boolean Proof Logging Example: Degree Preprocessing





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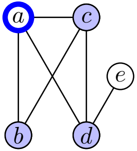

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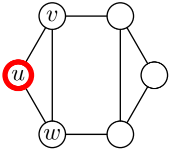

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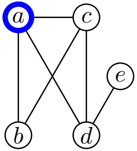

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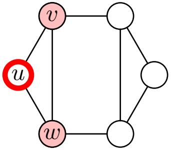

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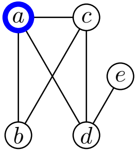

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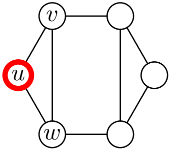

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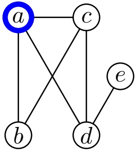

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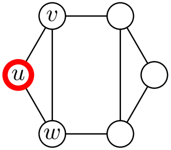

Sum up all constraints & divide by 3 to obtain

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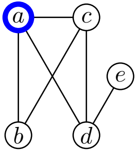

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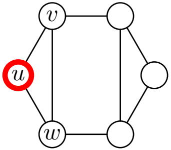

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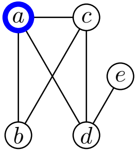

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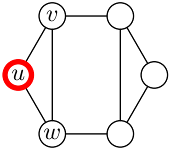

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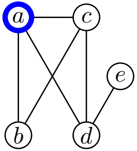

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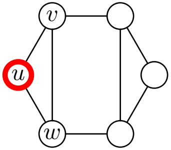

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Integer Variables in Constraint Programming (1/2)

How to deal with integer variables?

Given $A \in \{-3 \dots 9\}$, the direct encoding is:

$$
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a_{=-3} + a_{=-2} + a_{=-1} + a_{=0} + a_{=1} + a_{=2} + a_{=3} \\
+ a_{=4} + a_{=5} + a_{=6} + a_{=7} + a_{=8} + a_{=9} = 1
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This doesn't work for large domains. . .

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This doesn't work for large domains...

We can instead use a binary encoding:

$$\begin{aligned} -16a_{\text{neg}} + 1a_{\text{b0}} + 2a_{\text{b1}} + 4a_{\text{b2}} + 8a_{\text{b3}} &\geq -3 \qquad \text{and} \\ 16a_{\text{neg}} + -1a_{\text{b0}} + -2a_{\text{b1}} + -4a_{\text{b2}} + -8a_{\text{b3}} &\geq -9 \end{aligned}$$

Doesn't propagate much, but that isn't a problem for proof logging

Integer Variables in Constraint Programming (2/2)

We can mix binary and order encodings! Define linear inequalities encoding

$$a_{\geq 4} \Leftrightarrow -16a_{\mathrm{neg}} + 1a_{\mathrm{b0}} + 2a_{\mathrm{b1}} + 4a_{\mathrm{b2}} + 8a_{\mathrm{b3}} \geq 4$$

$$a_{\geq 5} \Leftrightarrow -16a_{\mathrm{neg}} + 1a_{\mathrm{b0}} + 2a_{\mathrm{b1}} + 4a_{\mathrm{b2}} + 8a_{\mathrm{b3}} \geq 5$$

$$a_{=4} \Leftrightarrow a_{\geq 4} \wedge \overline{a}_{\geq 5}$$

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When creating $a_{\geq i}$, also introduce pseudo-Boolean constraints encoding

$$a_{\geq i} \Rightarrow a_{\geq j} \quad \text{and} \quad a_{\geq h} \Rightarrow a_{\geq i}$$

for the closest values $j < i < h$ that already exist

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We can do this:

- Inside the pseudo-Boolean model where needed
- Otherwise lazily during proof logging

Table Constraints

Constraints can be specified extensionally as list of feasible tuples, called a table
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Given table constraint

$$(A, B, C) \in [(1, 2, 3), (1, 3, 4), (2, 2, 5)]$$

define

$$3\bar{t}_1 + a_{=1} + b_{=2} + c_{=3} \geq 3 \qquad \text{i.e.,} \quad t_1 \Rightarrow (a_{=1} \wedge b_{=2} \wedge c_{=3})$$
$$3\bar{t}_2 + a_{=1} + b_{=4} + c_{=4} \geq 3 \qquad \text{i.e.,} \quad t_2 \Rightarrow (a_{=1} \wedge b_{=4} \wedge c_{=4})$$
$$3\bar{t}_3 + a_{=2} + b_{=2} + c_{=5} \geq 3 \qquad \text{i.e.,} \quad t_3 \Rightarrow (a_{=2} \wedge b_{=2} \wedge c_{=5})$$

using tuple selector variables

$$t_1 + t_2 + t_3 = 1$$

A Constraint Programming Solver with Pseudo-Boolean Proof Logging

Proof-of-concept constraint programming solver at

`https://github.com/ciaranm/glasgow-constraint-solver`

Supports proof logging for global constraints including:

- All-different
- Integer linear inequality (including for very large domains)
- Smart table and regular
- Minimum / maximum of an array
- Element (kind of array indexing)
- Absolute value
- (Hamiltonian) Circuit

Details in [EGMN20, GMN22, MM23]

Using VeriPB for SAT Solving

1. Use dedicated tools for Gaussian elimination [GN21], symmetry breaking [BGMN23], PB-to-CNF translation [GMNO22], et cetera
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Short dictionary for DRAT-to-VeriPB translations

DRAT	VeriPB
`1`	`x1`
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`1 -2 3 0`	`1 x1 1 ~x2 1 x3 >= 1 ;`
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3. But LRAT syntactically rewritten for VeriPB should allow way faster proof checking — see latest version of CaDiCaL [CaD]

VeriPB Documentation

VeriPB tutorial at *CP '22* [BMN22]

- video at `youtu.be/s_5BIi4I22w`
- updated slides for *IJCAI '23* tutorial [BMN23]

Description of VeriPB and CakePB [BMM+23] for SAT 2023 competition

- Available at `satcompetition.github.io/2023/checkers.html`

Specific details on different proof logging techniques covered in research papers [EGMN20, GMN20, GMM+20, GN21, GMN22, GMNO22, VDB22, BBN+23, BGMN23, MM23]

Lots of concrete example files at `gitlab.com/MIAOresearch/software/VeriPB`

Future Research Directions

Performance and reliability of pseudo-Boolean proof logging

- Trim proof while verifying (as in DRAT-trim [HHW13a])
- Compress proof file using binary format
- Design formally verified proof checker *(work in progress [BMM⁺23])*

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Proof logging for other combinatorial problems and techniques

- Model counting
- Symmetric learning and recycling (substitution) of subproofs
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- Satisfiability modulo theories (SMT) solving *(work on* cvc5, Z3, … *[BBC+23])*

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And more. . .

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And more. . .

- Use proof logs for algorithm analysis or explainability purposes
- Lots of other challenging problems and interesting ideas
- We're hiring! Talk to me to join the pseudo-Boolean proof logging revolution! ☺

Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity
- **Action point:** What problems can VeriPB solve for you? ☺

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Thank you for your attention!

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