

# Combinatorial Solving with Provably Correct Results

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# Combinatorial Solving and Optimisation

- Revolution last couple of decades in **combinatorial solvers** for
    - Boolean satisfiability (SAT) solving [BHvMW21]<sup>1</sup>
    - Constraint programming (CP) [RvBW06]
    - Mixed integer linear programming (MIP) [AW13, BR07]
  - Solve NP-complete problems (or worse) very successfully in practice!
  - **Except solvers are sometimes wrong...** (Even best commercial ones)  
[BLB10, CKSW13, AGJ<sup>+</sup>18, GSD19, GS19, BMN22, BBN<sup>+</sup>23]
  - Even get feasibility of solutions wrong (though this should be straightforward!)
  - And how to check the absence of solutions?
  - Or that a solution is optimal? (Even off-by-one mistakes can snowball into large errors if solver used as subroutine)

<sup>1</sup>See end of slides for all references with bibliographic details.

# What Can Be Done About Solver Bugs?

## ■ Software testing

Hard to get good test coverage for sophisticated solvers

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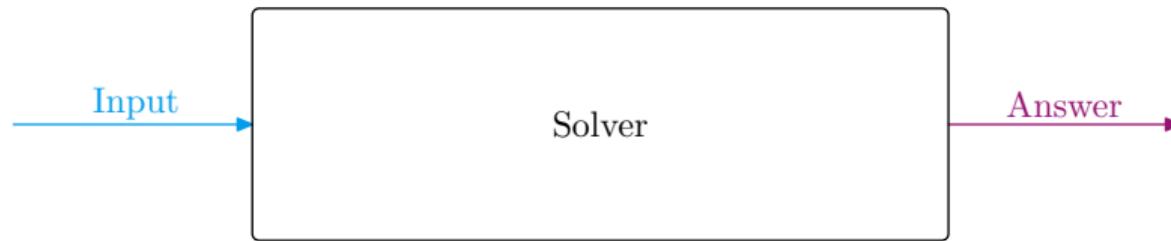
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### ■ Proof logging

Make solver certifying [ABM<sup>+</sup>11, MMNS11] by outputting

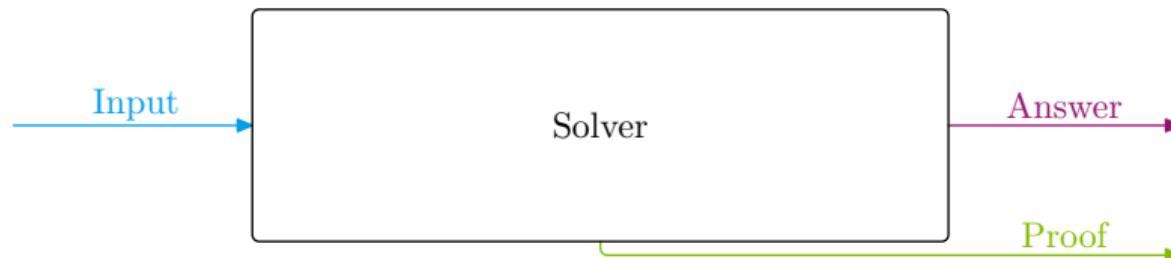
- 1 not only **answer** but also
  - 2 simple, machine-verifiable **proof** that answer is correct

# Proof Logging with Certifying Solvers: Workflow



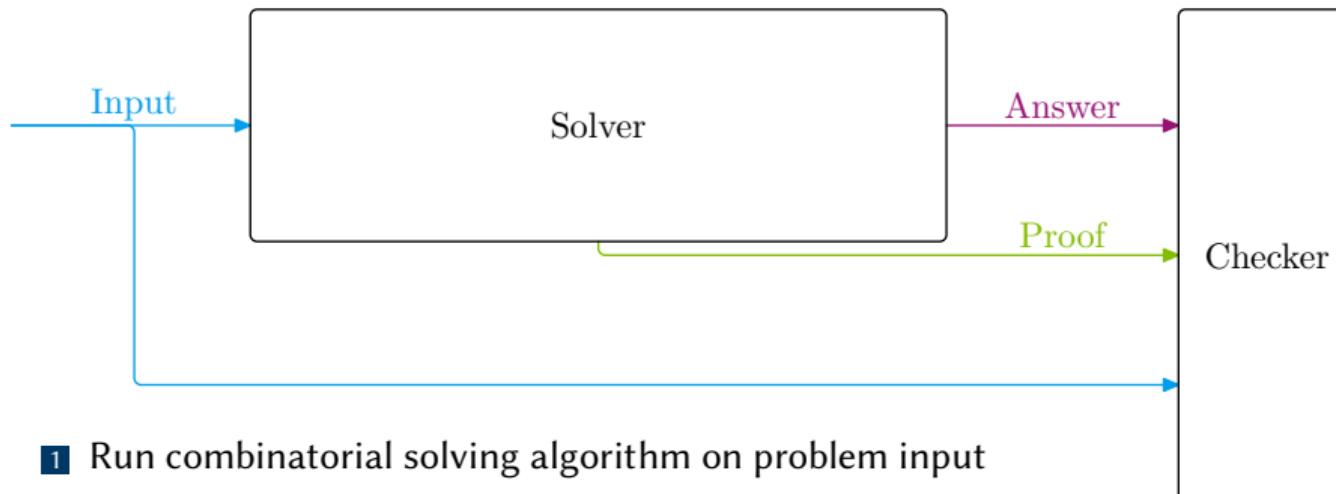
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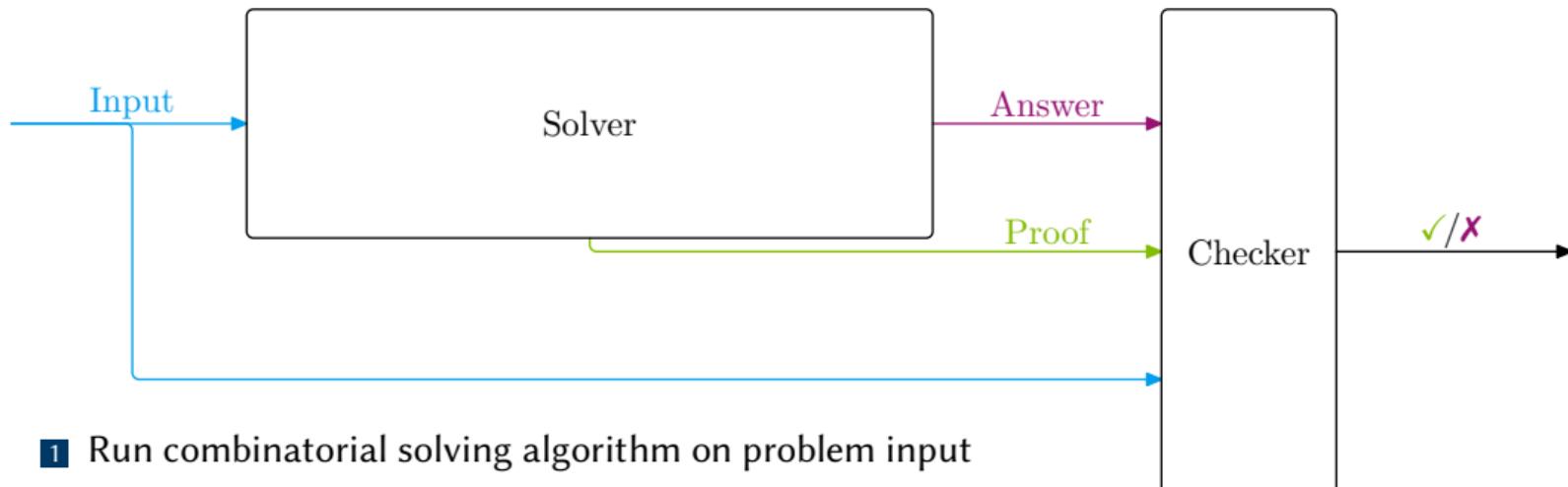
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  - 3 Feed input + answer + proof to proof checker

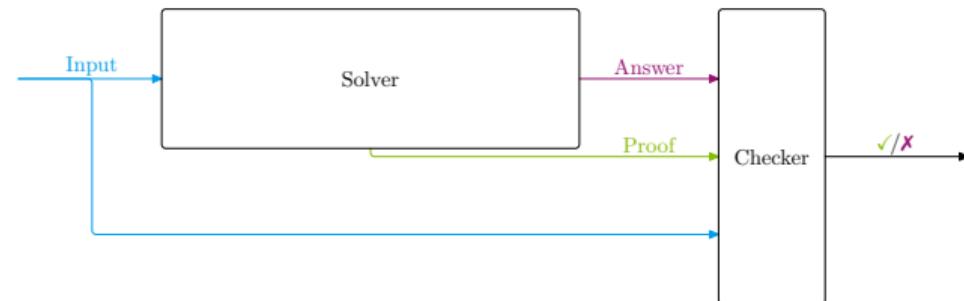
# Proof Logging with Certifying Solvers: Workflow



- 1 Run combinatorial solving algorithm on problem input
  - 2 Get as output not only answer but also proof
  - 3 Feed input + answer + proof to proof checker
  - 4 Verify that proof checker says answer is correct

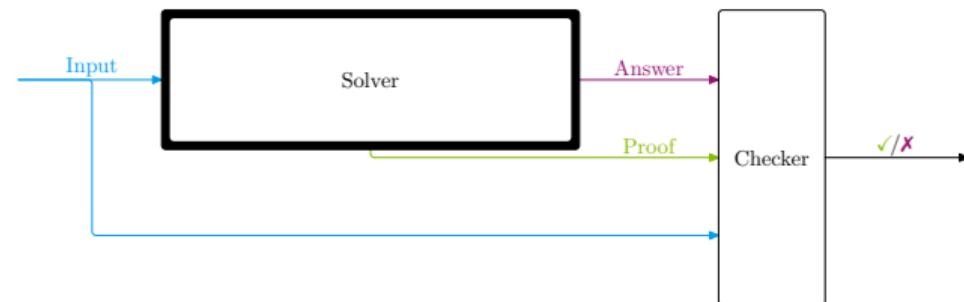
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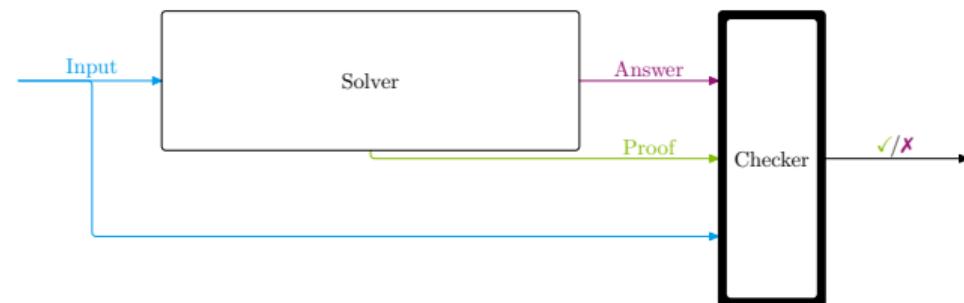
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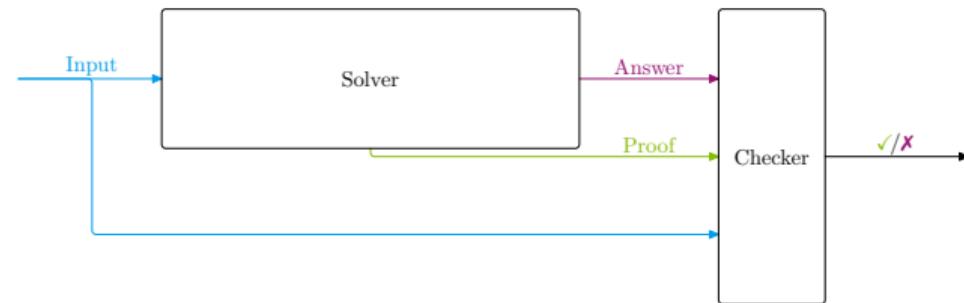
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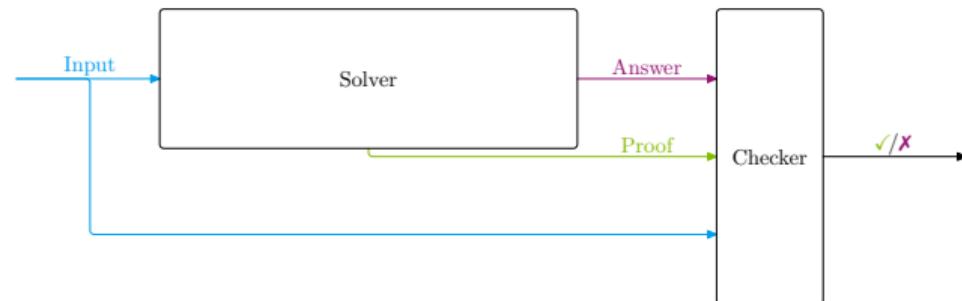


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Clear conflict expressivity vs. simplicity!

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## Clear conflict expressivity vs. simplicity!

Asking for both perhaps a little bit too good to be true?

# Take-Away Message from This Tutorial

Proof logging for combinatorial optimisation is possible with **single, unified method!**

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Proof logging for combinatorial optimisation is possible with **single, unified method!**

- Build on successes in proof logging for SAT solvers with proof formats such as DRAT [HHW13a, HHW13b, WHH14], GRIT [CMS17], LRAT [CHH<sup>+</sup>17], ...
- But represent constraints as **0–1 integer linear inequalities**
- Formalize reasoning using **cutting planes** [CCT87] proof system
- Add well-chosen **strengthening rules** [Goc22, GN21, BGMN23]
- Implemented in **VERIPB** (<https://gitlab.com/MIA0research/software/VeriPB>)

# The Sales Pitch For Proof Logging

- 1 Certifies correctness of computed results
- 2 Detects errors even if due to compiler bugs, hardware failures, or cosmic rays
- 3 Provides debugging support during development [EG21, GMM<sup>+</sup>20, KM21, BBN<sup>+</sup>23]
- 4 Facilitates performance analysis
- 5 Helps identify potential for further improvements
- 6 Enables auditability
- 7 Serves as stepping stone towards explainability

# The Rest of This Tutorial

Explain how to use **VERIPB** to do proof logging for

- SAT solving (including advanced techniques)
- SAT-based optimisation (MaxSAT)
- Subgraph algorithms
- Constraint programming
- Symmetry and dominance reasoning

in a unified way

# The SAT Problem

- Variable  $x$ : takes value **true** ( $= 1$ ) or **false** ( $= 0$ )
- Literal  $\ell$ : variable  $x$  or its negation  $\bar{x}$
- Clause  $C = \ell_1 \vee \cdots \vee \ell_k$ : disjunction of literals  
(Consider as sets, so no repetitions and order irrelevant)
- Conjunctive normal form (CNF) formula  $F = C_1 \wedge \cdots \wedge C_m$ : conjunction of clauses

## The SAT Problem

Given a CNF formula  $F$ , is it satisfiable?

For instance, what about:

$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge \\ (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$

# Proofs for SAT

For satisfiable instances: just specify satisfying assignment

For unsatisfiability: a sequence of clauses (CNF constraints)

- Each clause follows “obviously” from everything we know so far
- Final clause is empty, meaning contradiction (written  $\perp$ )
- Means original formula must be inconsistent

# What Is Obvious? Unit Propagation

## Unit Propagation

Clause  $C$  **unit propagates**  $\ell$  under partial assignment  $\rho$  if  $\rho$  falsifies all literals in  $C$  except  $\ell$

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Proof checker should know how to unit propagate until saturation

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DPLL [DP60, DLL62]: Assign variables and propagate; backtrack when clause violated

“Proof trace”: when backtracking, write negation of guesses made

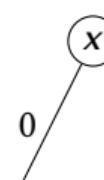
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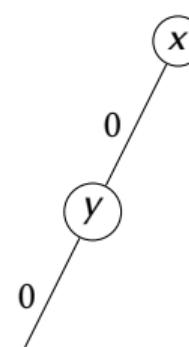


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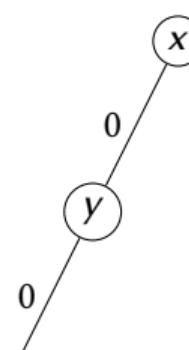


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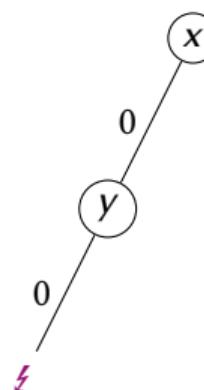
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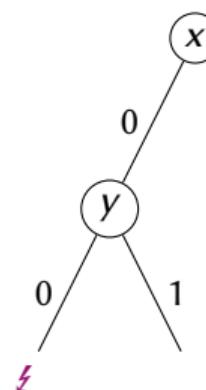
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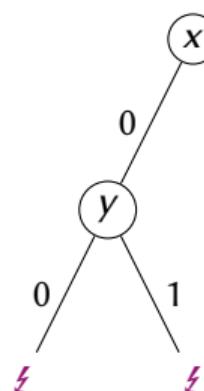
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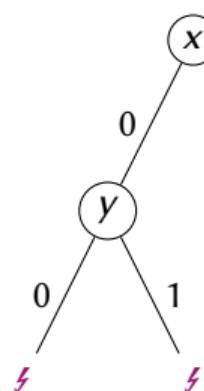
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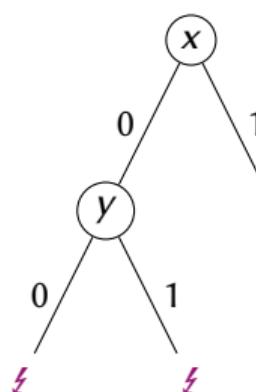
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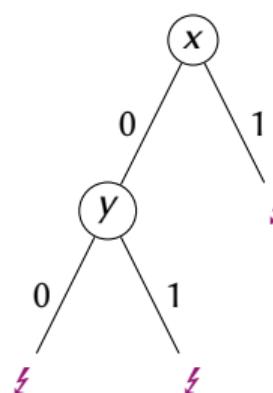
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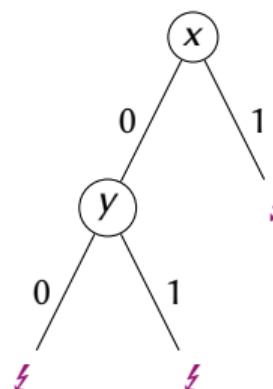
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- 4  $\bar{x}$
- 5  $\perp$



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## Fact

Backtrack clauses from DPLL solver generate a RUP proof

# What About Conflict-Driven Clause Learning (CDCL)?

Run CDCL [BS97, MS99, MMZ<sup>+</sup>01] on our favourite CNF formula:

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## Decision

Free choice to assign value to variable

Notation  $p \stackrel{d}{=} 0$

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$$p \stackrel{d}{=} 0$$

## Decision

Free choice to assign value to variable

$$\text{Notation } p \stackrel{d}{=} 0$$

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Forced choice to avoid falsifying clause

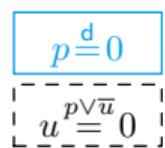
Given  $p = 0$ , clause  $p \vee \bar{u}$  forces  $u = 0$

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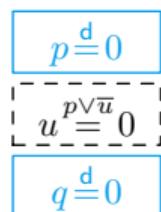
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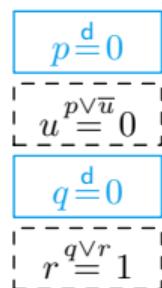
Add to assignment trail

Continue until satisfying assignment or conflict

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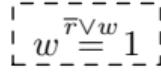
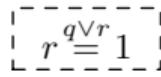
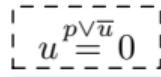
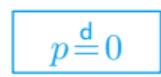
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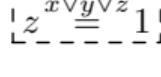
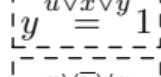
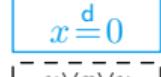
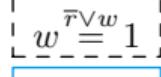
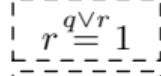
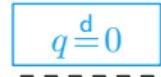
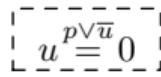
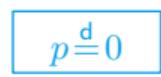
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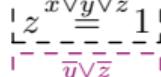
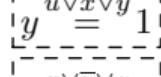
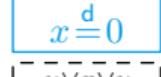
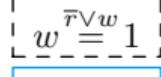
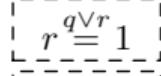
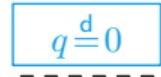
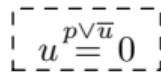
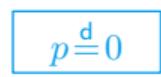
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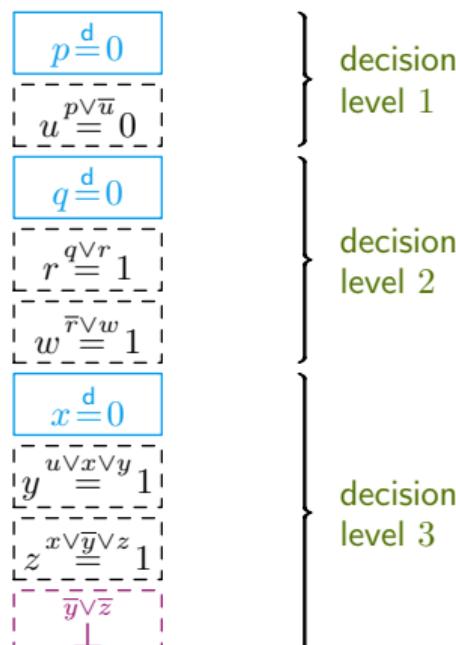
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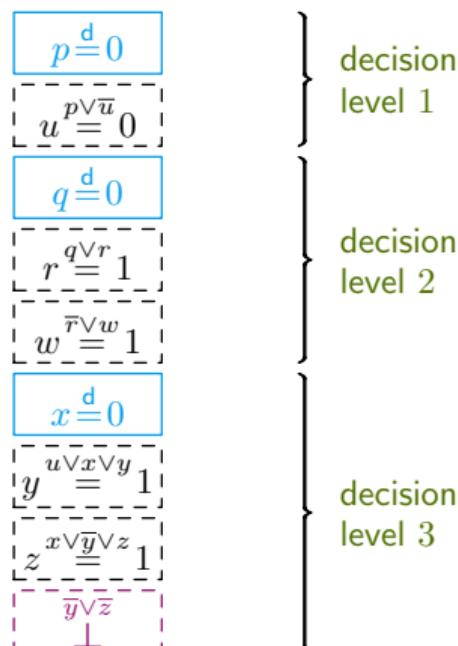
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# Conflict Analysis

Time to analyse this conflict and learn from it!

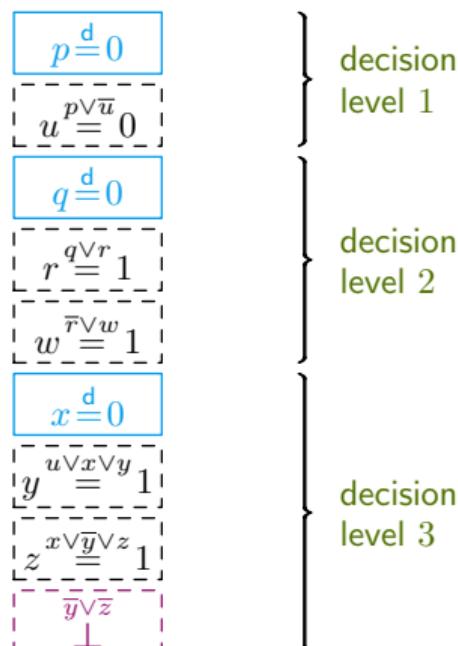
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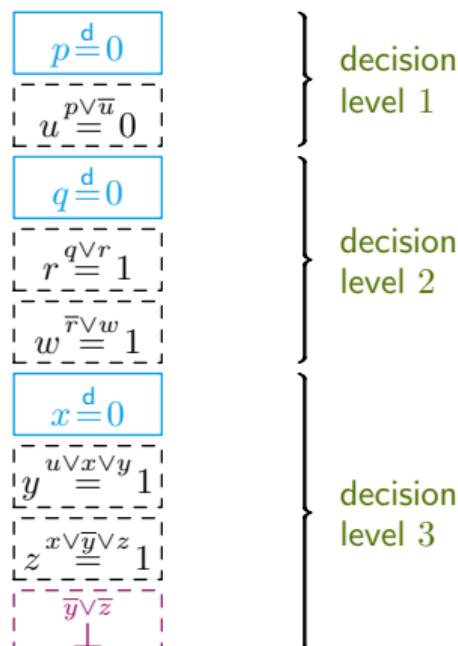


Could backtrack by erasing **conflict level** & flipping last decision

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$$\boxed{y \stackrel{u \vee x \vee y}{=} 1}$$

$$\boxed{z \stackrel{x \vee \bar{y} \vee z}{=} 1}$$

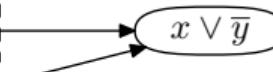
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Case analysis over  $z$  for last two clauses:

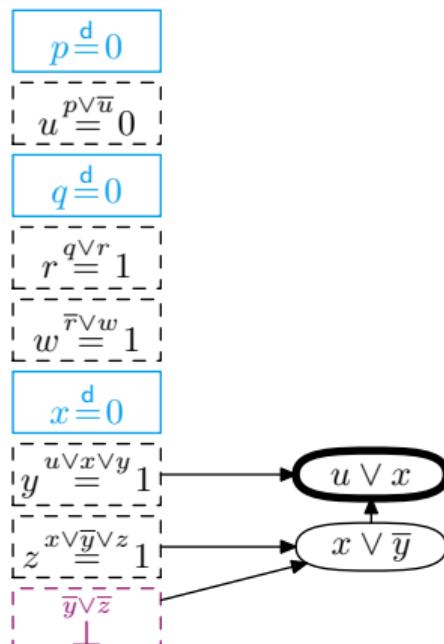
- $x \vee \bar{y} \vee z$  wants  $z = 1$
- $\bar{y} \vee \bar{z}$  wants  $z = 0$
- **Resolve** clauses by merging them & removing  $z$  — must satisfy  $x \vee \bar{y}$



# Conflict Analysis

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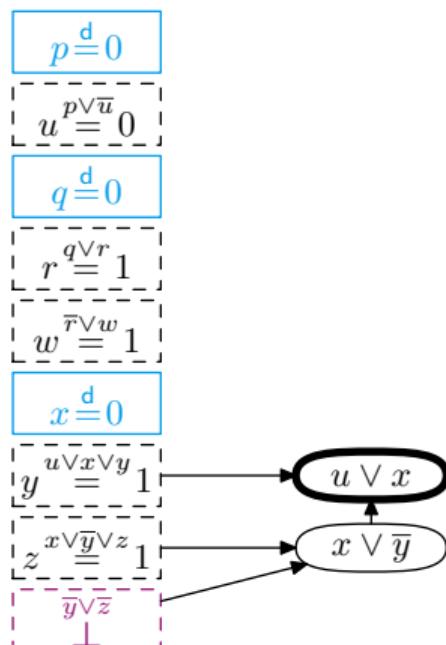
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Repeat until **UIP clause** with only 1 variable at conflict level after last decision — **learn** and **backjump**

# Complete Example of CDCL Execution

**Backjump:** undo max #decisions while learned clause propagates

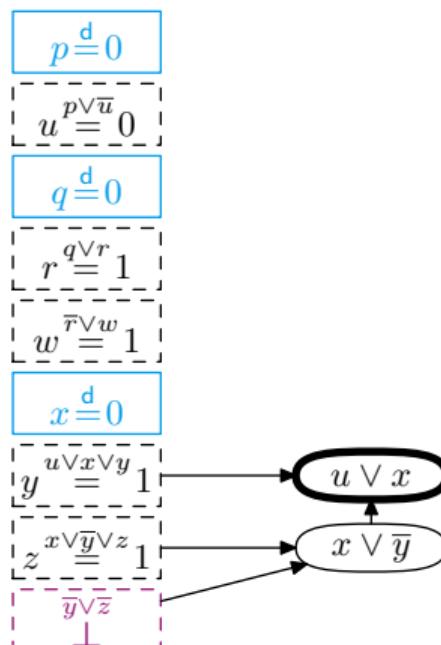
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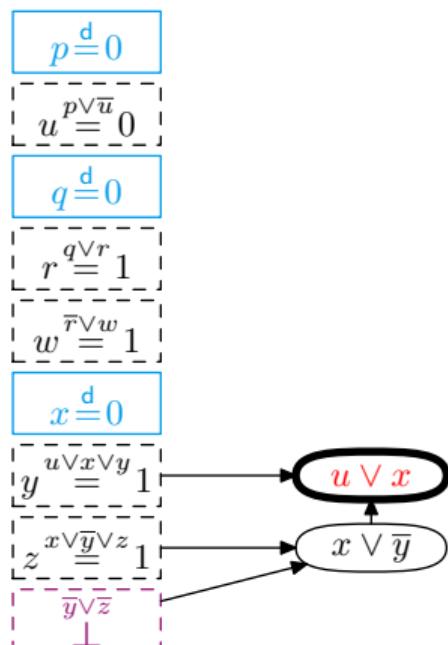


**Assertion level 1** (2nd largest level in learned clause) – trim trail to that level

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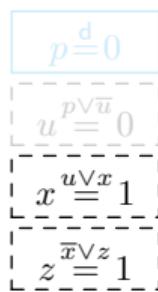
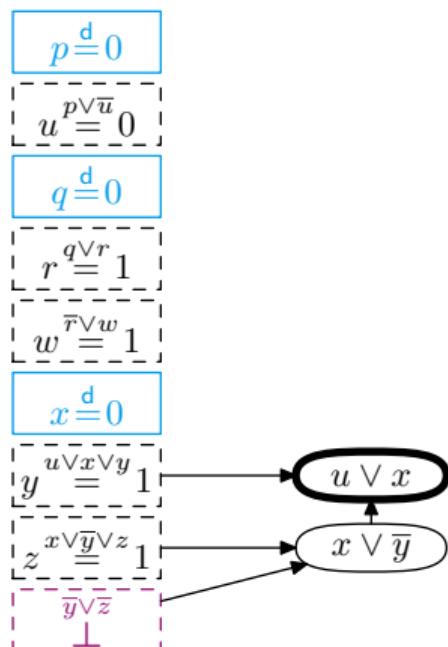
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Now UIP literal guaranteed to flip (**assert**) – but this is a **propagation**, not a decision

# Complete Example of CDCL Execution

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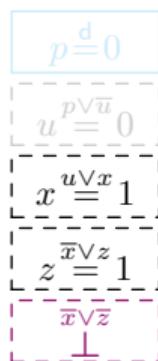
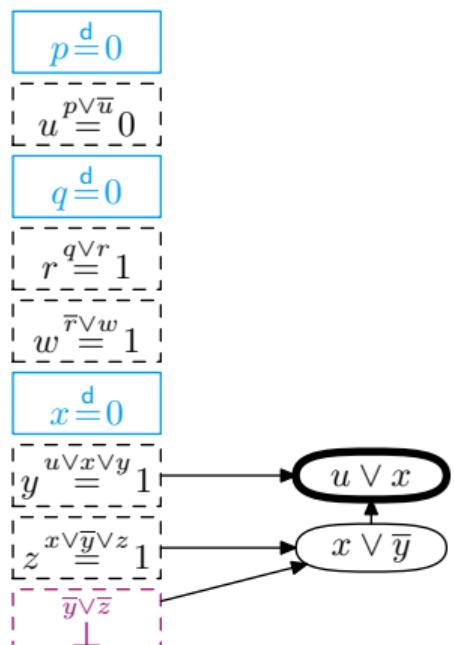
Now UIP literal guaranteed to flip (**assert**) – but this is a **propagation**, not a decision

Then continue as before...

# Complete Example of CDCL Execution

**Backjump:** undo max #decisions while learned clause propagates

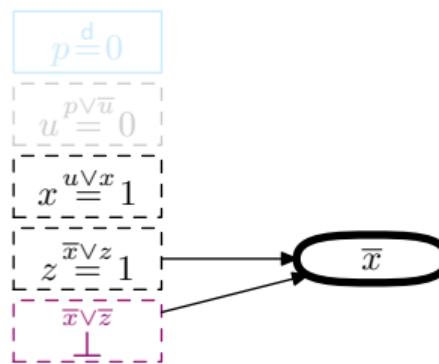
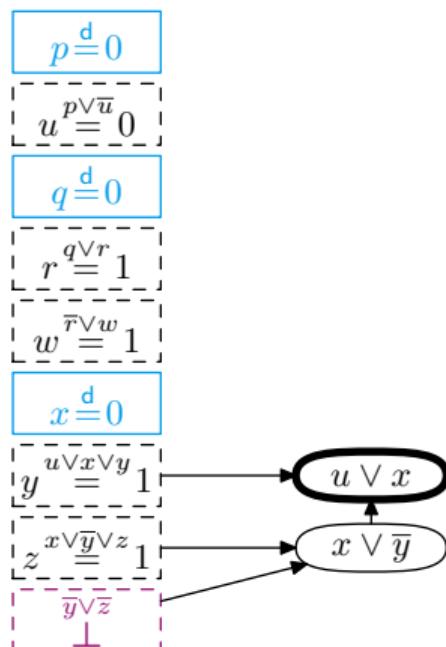
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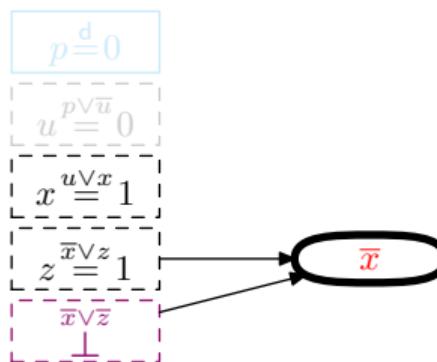
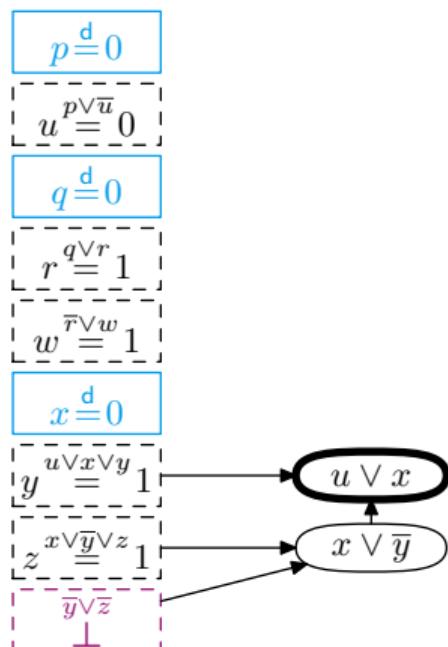
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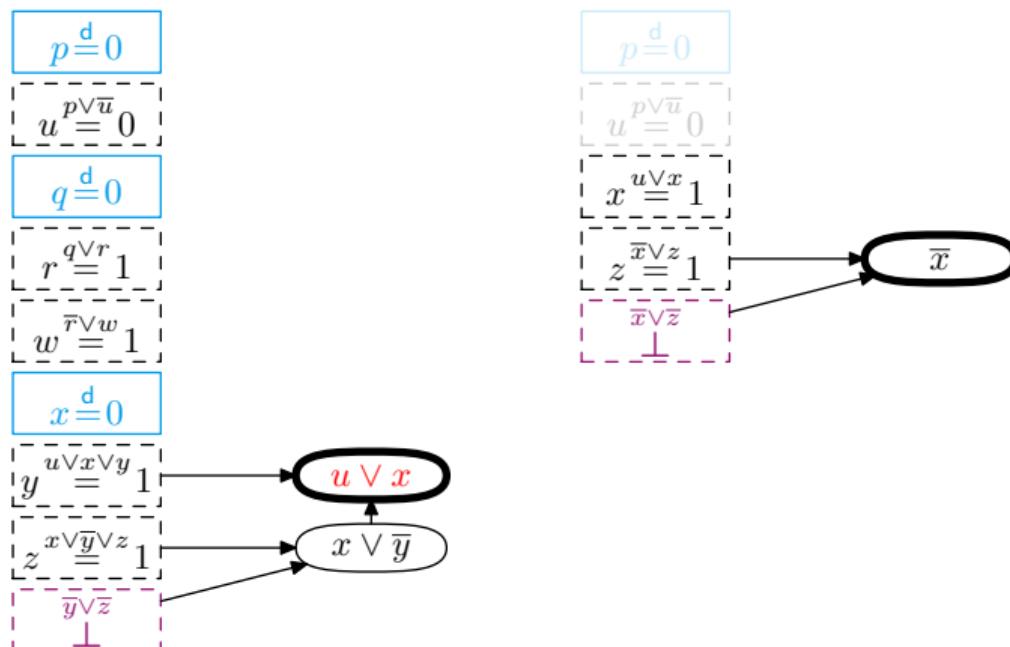
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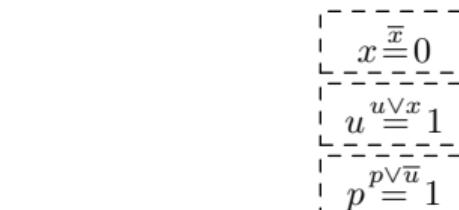
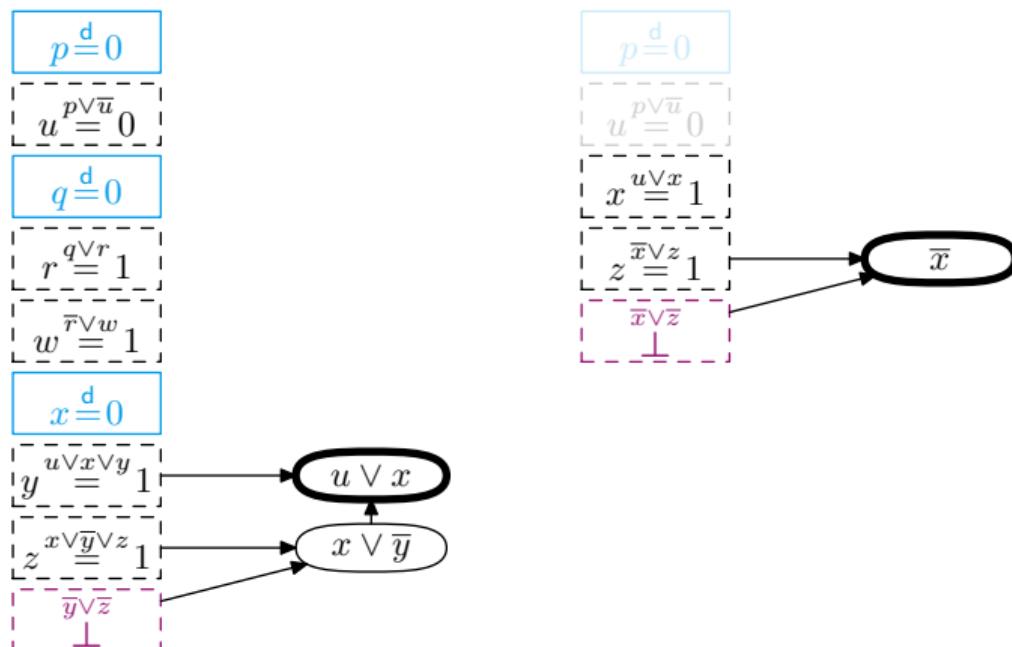
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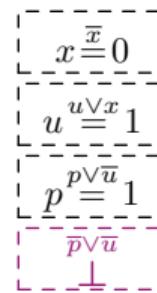
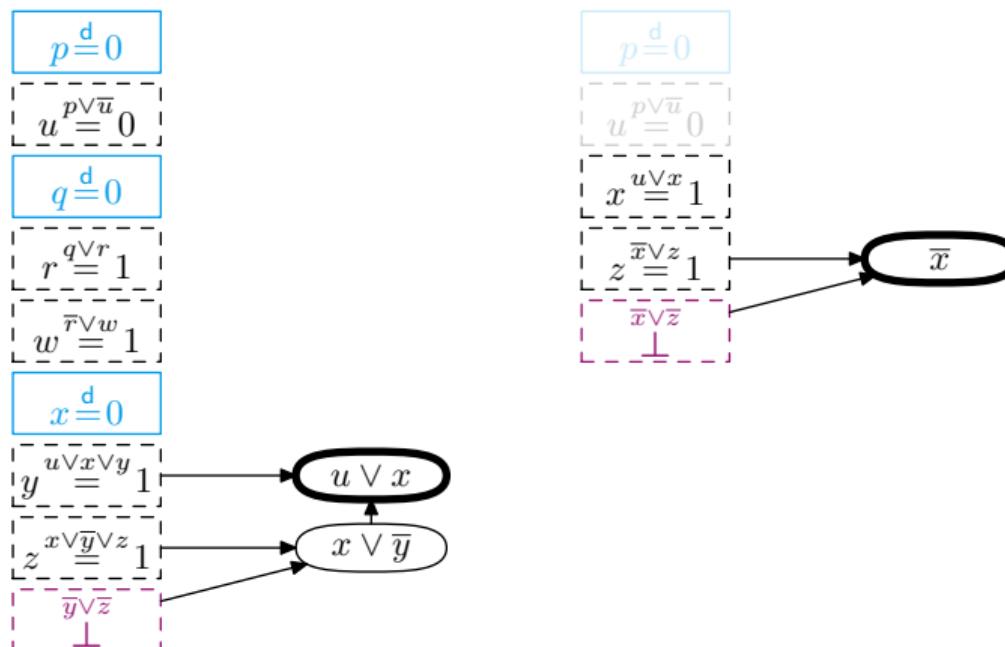
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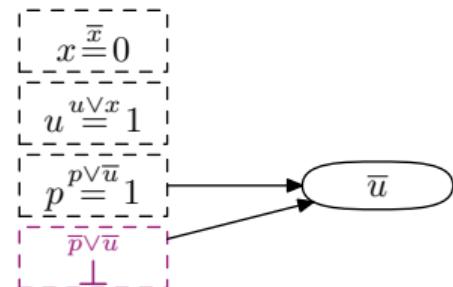
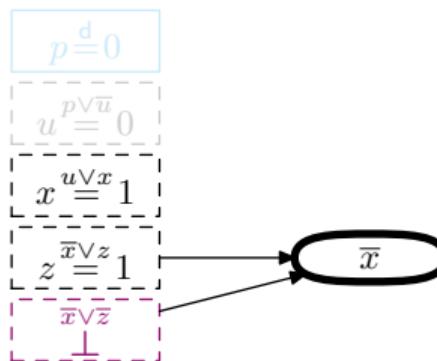
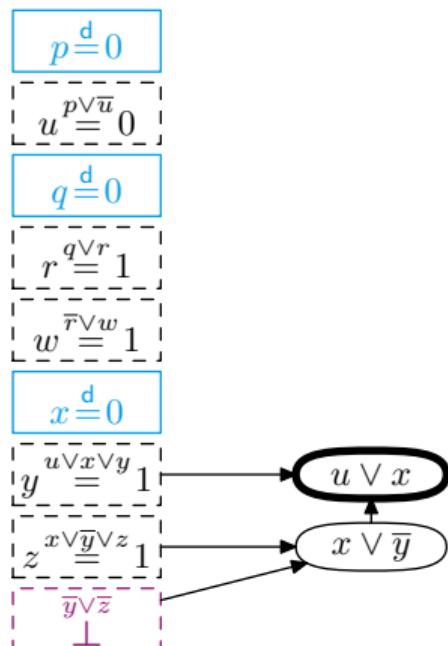
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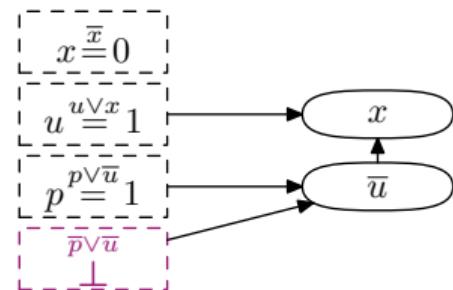
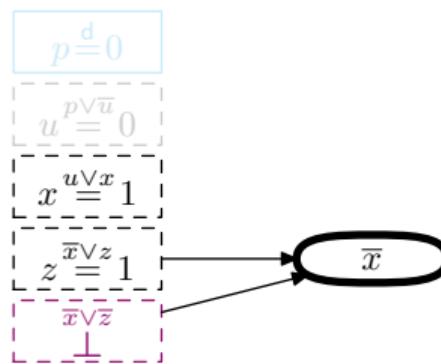
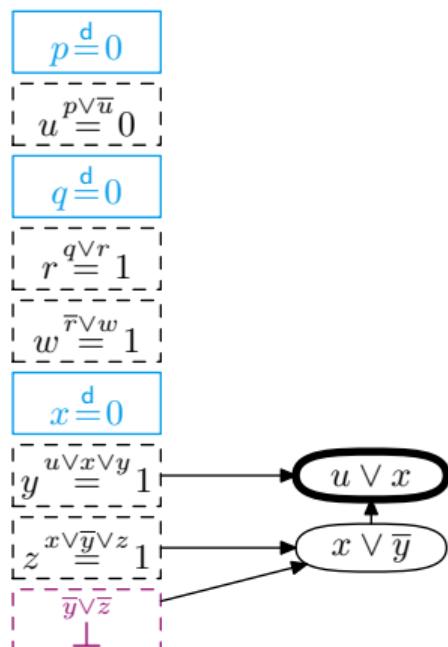
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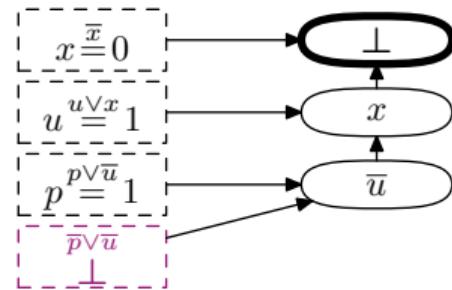
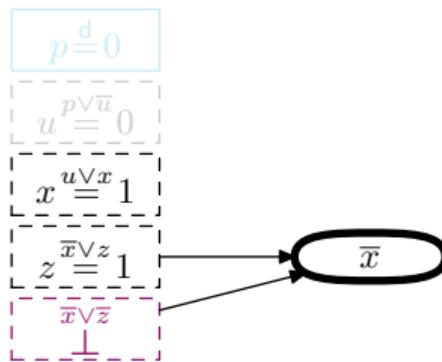
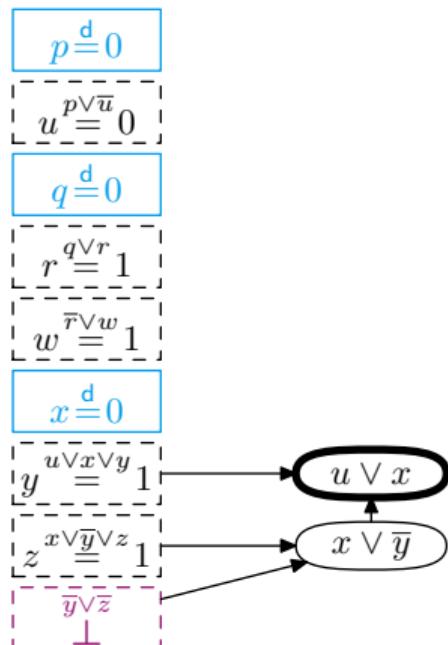
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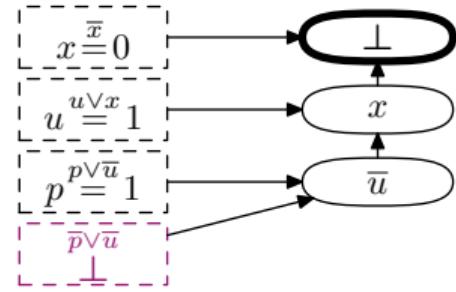
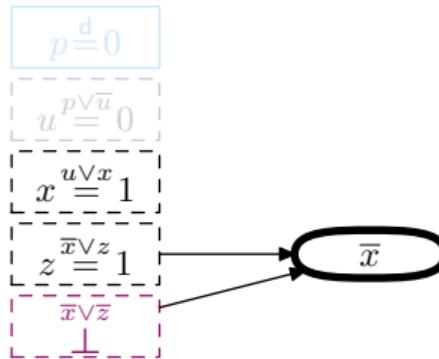
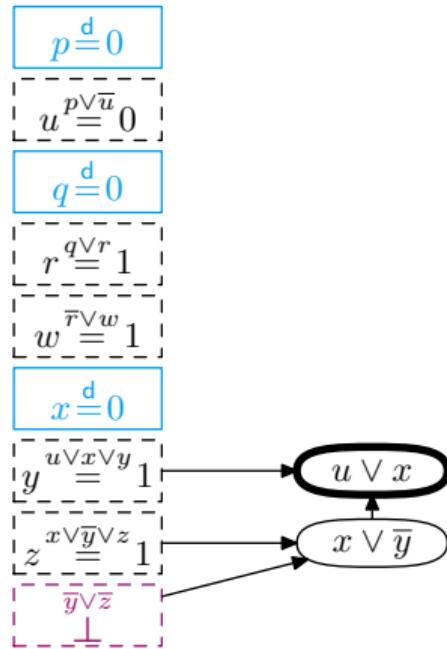
(\*) Ignores pre- and inprocessing, but we will get there...

# Resolution Proofs from CDCL Executions

Obtain resolution proof...

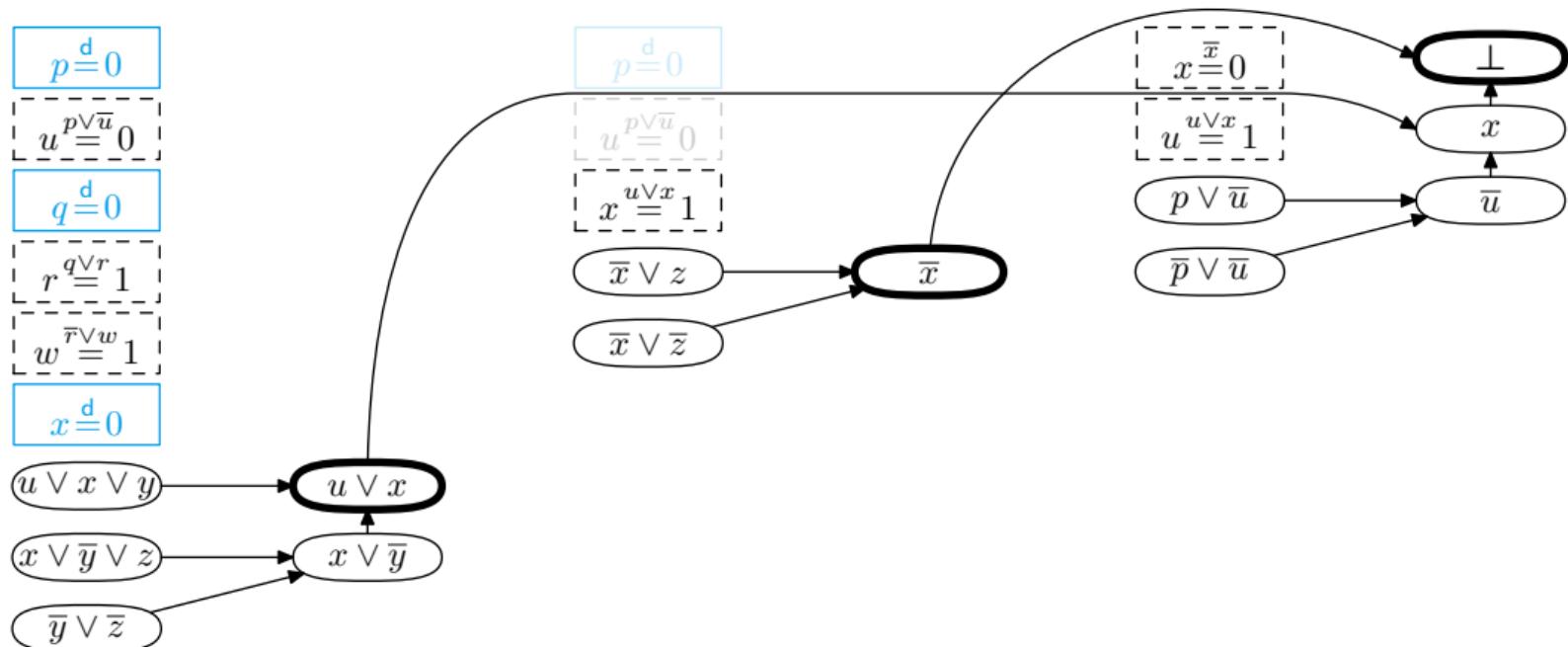
# Resolution Proofs from CDCL Executions

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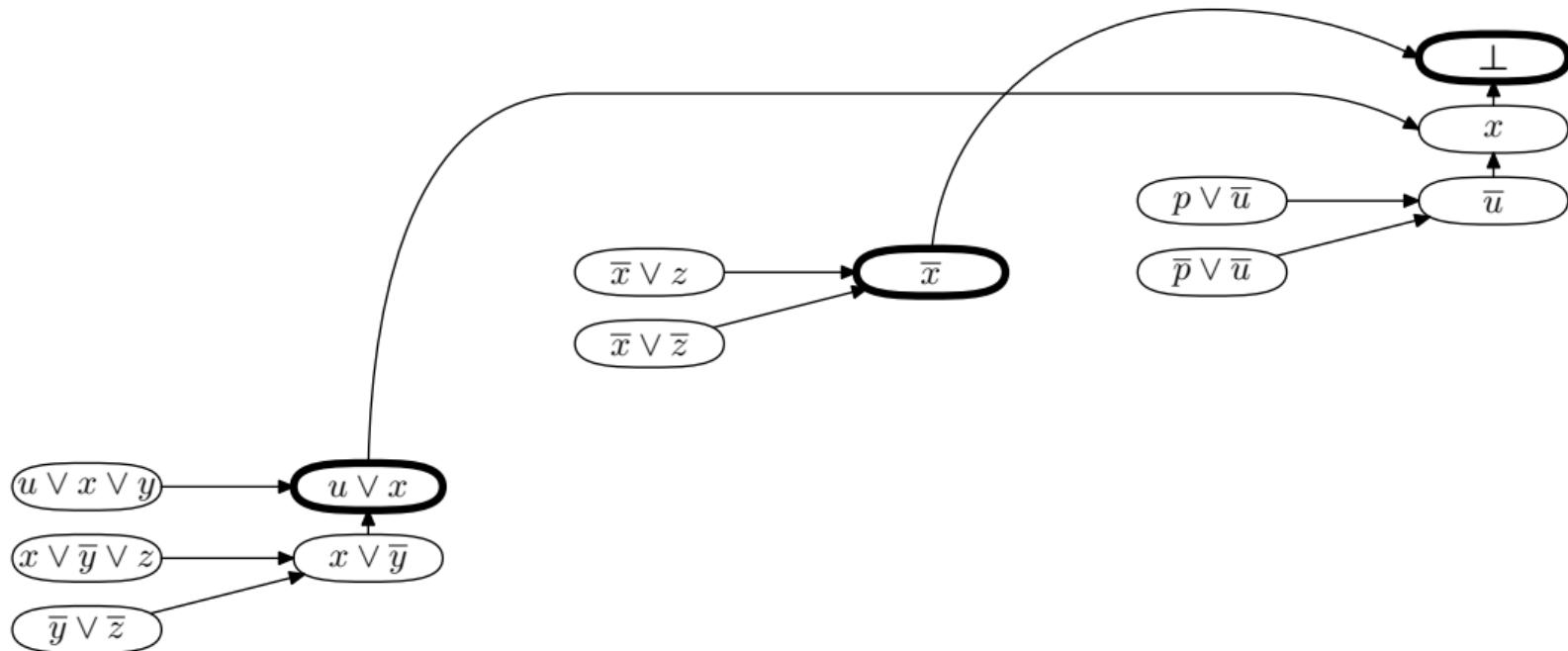
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is sequence of reverse unit propagation (RUP) clauses

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is sequence of reverse unit propagation (RUP) clauses

- 1  $u \vee \textcolor{green}{x}$
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$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (\textcolor{blue}{u} \vee \textcolor{red}{x} \vee y) \wedge (\textcolor{red}{x} \vee \bar{y} \vee z) \wedge (\textcolor{blue}{y} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\textcolor{blue}{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$

is sequence of reverse unit propagation (RUP) clauses

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# More Ingredients in Proof Logging for SAT

## Fact

RUP proofs can be viewed as shorthand for resolution proofs

See [BN21] for more on this and connections to SAT solving

But RUP and resolution are not enough for preprocessing, inprocessing, and some other kinds of reasoning

# Extension Variables, Part 1

Suppose we want a variable  $a$  encoding

$$a \Leftrightarrow (x \wedge y)$$

Extended resolution [Tse68]

Resolution rule plus [extension rule](#) introducing clauses

$$a \vee \bar{x} \vee \bar{y} \quad \bar{a} \vee x \quad \bar{a} \vee y$$

for fresh variable  $a$  (this is fine since  $a$  doesn't appear anywhere previously)

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Fact

Extended resolution (RUP + definition of new variables) is essentially equivalent to the DRAT proof logging system most commonly used for SAT solving

# Why Aren't We Done?

Practical limitations of current SAT proof logging technology:

- Difficulties dealing with stronger reasoning efficiently (even for SAT solving)
- Clausal proofs can't easily reflect what algorithms for other problems do

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Surprising claim: a slight change to **0-1 integer linear inequalities** does the job!

- Enables proof logging for **advanced SAT techniques** so far beyond reach for efficient DRAT proof logging:
  - Cardinality reasoning
  - Gaussian elimination
  - Symmetry breaking
- Supports use of SAT solvers for **optimisation problems (MaxSAT)**
- Can justify **graph reasoning** without knowing what a graph is
- Can justify **constraint programming** inference without knowing what an integer variable is

# Pseudo-Boolean Constraints

0-1 integer linear inequalities or (linear) pseudo-Boolean constraints:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals  $\ell_i$ :  $x_i$  or  $\bar{x}_i$  (where  $x_i + \bar{x}_i = 1$ )

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Sometimes convenient to use normalized form [Bar95] with all  $a_i, A$  positive (without loss of generality)

# Some Types of Pseudo-Boolean Constraints

## 1 Clauses

$$x_1 \vee \bar{x}_2 \vee x_3 \Leftrightarrow x_1 + \bar{x}_2 + x_3 \geq 1$$

## 2 Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

## 3 General pseudo-Boolean constraints

$$x_1 + 2\bar{x}_2 + 3x_3 + 4\bar{x}_4 + 5x_5 \geq 7$$

# Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

**Input/model axioms**

From the input

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$$\frac{\sum_i a_i \ell_i \geq A \quad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i) \ell_i \geq A + B}$$

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**Division** for any  $c \in \mathbb{N}^+$   
(assumes normalized form)

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \lceil \frac{a_i}{c} \rceil \ell_i \geq \lceil \frac{A}{c} \rceil}$$

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Multiply by 2

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$$\begin{array}{rcl} & w + 2x + y \geq 2 & \\ \text{Multiply by 2} & \hline & 2w + 4x + 2y \geq 4 \\ & w + 2x + 4y + 2z \geq 5 & \\ \text{Add} & \hline & 3w + 6x + 6y + 2z \geq 9 \end{array}$$

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$$\begin{array}{rcl} & w + 2x + y \geq 2 & \\ \text{Multiply by 2} & \hline & 2w + 4x + 2y \geq 4 \\ & w + 2x + 4y + 2z \geq 5 & \\ \text{Add} & \hline & 3w + 6x + 6y + 2z \geq 9 \\ & & \end{array} \quad \begin{array}{rcl} & \bar{z} \geq 0 & \\ \hline & 2\bar{z} \geq 0 & \\ & \text{Multiply by 2} & \end{array}$$

# Cutting Planes Toy Example

$$\begin{array}{c} w + 2x + y \geq 2 \\ \hline \text{Multiply by 2} & 2w + 4x + 2y \geq 4 & w + 2x + 4y + 2z \geq 5 & \bar{z} \geq 0 \\ \hline \text{Add} & \hline & 3w + 6x + 6y + 2z \geq 9 & 2\bar{z} \geq 0 \\ \hline \text{Add} & \hline & 3w + 6x + 6y + 2z + 2\bar{z} \geq 9 & \text{Multiply by 2} \end{array}$$

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$$\begin{array}{c} w + 2x + y \geq 2 \\ \hline \text{Multiply by 2} & 2w + 4x + 2y \geq 4 & w + 2x + 4y + 2z \geq 5 & \bar{z} \geq 0 \\ \hline \text{Add} & \hline & 3w + 6x + 6y + 2z \geq 9 & 2\bar{z} \geq 0 \\ \hline \text{Add} & \hline & 3w + 6x + 6y & \geq 7 \\ \hline & & & \end{array}$$

Multiply by 2

# Cutting Planes Toy Example

$$\begin{array}{c} w + 2x + y \geq 2 \\ \text{Multiply by 2} \quad \hline \\ 2w + 4x + 2y \geq 4 \\ \text{Add} \quad \hline \\ 3w + 6x + 6y + 2z \geq 9 \\ \text{Add} \quad \hline \\ 3w + 6x + 6y \geq 7 \\ \text{Divide by 3} \quad \hline \\ w + 2x + 2y \geq 2\frac{1}{3} \end{array}$$
$$\begin{array}{c} w + 2x + 4y + 2z \geq 5 \\ \hline \\ \bar{z} \geq 0 \\ \hline \\ 2\bar{z} \geq 0 \end{array}$$

Multiply by 2

# Cutting Planes Toy Example

$$\begin{array}{c} w + 2x + y \geq 2 \\ \hline \text{Multiply by 2} & 2w + 4x + 2y \geq 4 & w + 2x + 4y + 2z \geq 5 & \bar{z} \geq 0 \\ \hline \text{Add} & 3w + 6x + 6y + 2z \geq 9 & \hline & 2\bar{z} \geq 0 \\ \hline \text{Add} & 3w + 6x + 6y & \geq 7 & \text{Multiply by 2} \\ \hline \text{Divide by 3} & w + 2x + 2y & \geq 3 & \end{array}$$

# Cutting Planes Toy Example

$$\begin{array}{c} w + 2x + y \geq 2 \\ \text{Multiply by 2} \quad \hline 2w + 4x + 2y \geq 4 \\ \text{Add} \quad \hline 3w + 6x + 6y + 2z \geq 9 \\ \text{Add} \quad \hline 3w + 6x + 6y \quad \geq 7 \\ \text{Divide by 3} \quad \hline w + 2x + 2y \geq 3 \\ \text{Multiply by 2} \quad \hline 2\bar{z} \geq 0 \\ \bar{z} \geq 0 \end{array}$$

Naming constraints by integers and literal axioms by the literal involved (with  $\sim$  for negation) as

$$\text{Constraint 1} \doteq 2x + y + w \geq 2$$

$$\text{Constraint 2} \doteq 2x + 4y + 2z + w \geq 5$$

$$\sim z \doteq \bar{z} \geq 0$$

# Cutting Planes Toy Example

$$\begin{array}{c} w + 2x + y \geq 2 \\ \text{Multiply by 2} \\ \hline 2w + 4x + 2y \geq 4 \\ \text{Add} \\ \hline 3w + 6x + 6y + 2z \geq 9 \\ \text{Add} \\ \hline 3w + 6x + 6y \geq 7 \\ \text{Divide by 3} \\ \hline w + 2x + 2y \geq 3 \end{array} \quad \begin{array}{c} w + 2x + 4y + 2z \geq 5 \\ \text{Multiply by 2} \\ \hline 2\bar{z} \geq 0 \\ \text{Multiply by 2} \\ \hline 2\bar{z} \geq 0 \end{array}$$

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such a calculation is written in the proof log in reverse Polish notation as

pol 1 2 \* 2 + ~z 2 \* + 3 d

# Resolution and Cutting Planes

To simulate resolution step such as

$$\frac{\overline{y} \vee \overline{z} \quad x \vee \overline{y} \vee z}{x \vee \overline{y}}$$

we can perform the cutting planes steps

$$\begin{array}{c} \text{Add} \quad \frac{\overline{y} + \overline{z} \geq 1 \quad x + \overline{y} + z \geq 1}{x + 2\overline{y} \geq 1} \\ \text{Divide by 2} \quad \frac{}{x + \overline{y} \geq 1} \end{array}$$

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Given that the premises are clauses 7 and 5 in our example CNF formula, using references

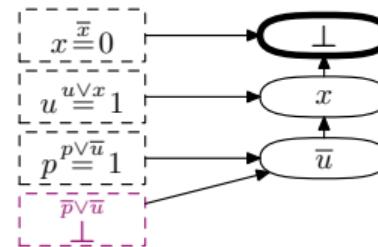
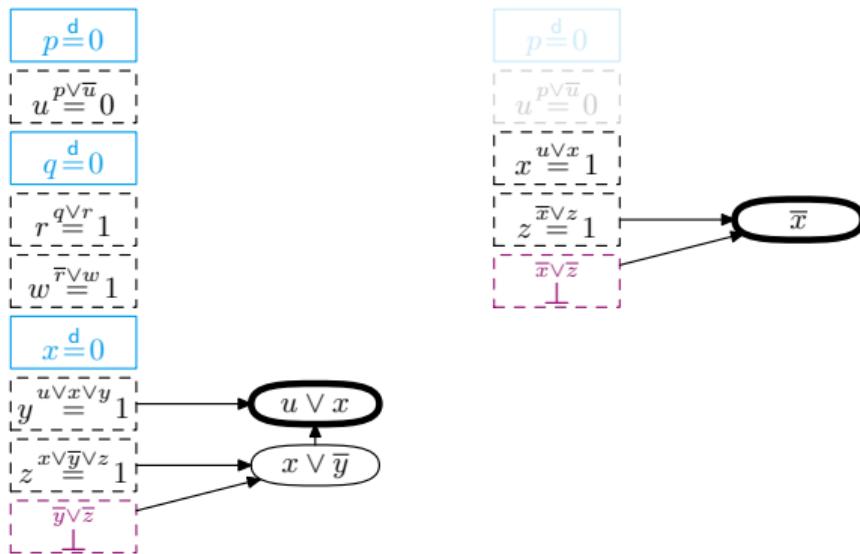
$$\text{Constraint 7} \doteq \bar{y} + \bar{z} \geq 1$$

$$\text{Constraint 5} \doteq x + \bar{y} + z \geq 1$$

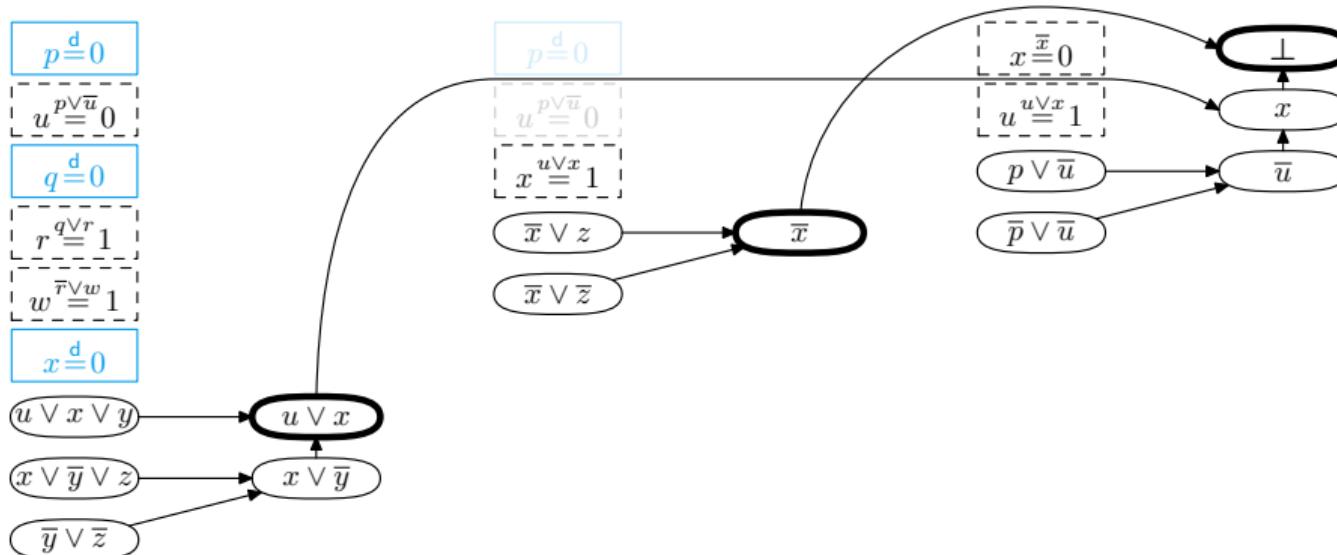
we can write this in the proof log as

pol 7 5 + 2 d

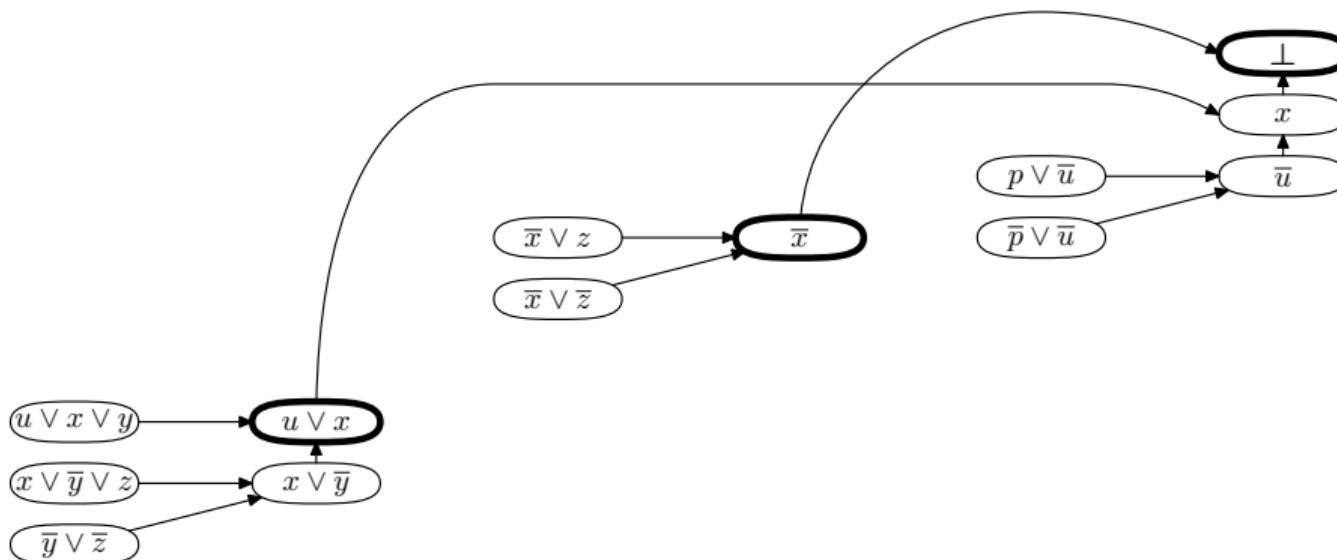
# Pseudo-Boolean Proof Logging for Example CDCL Conflict Analyses



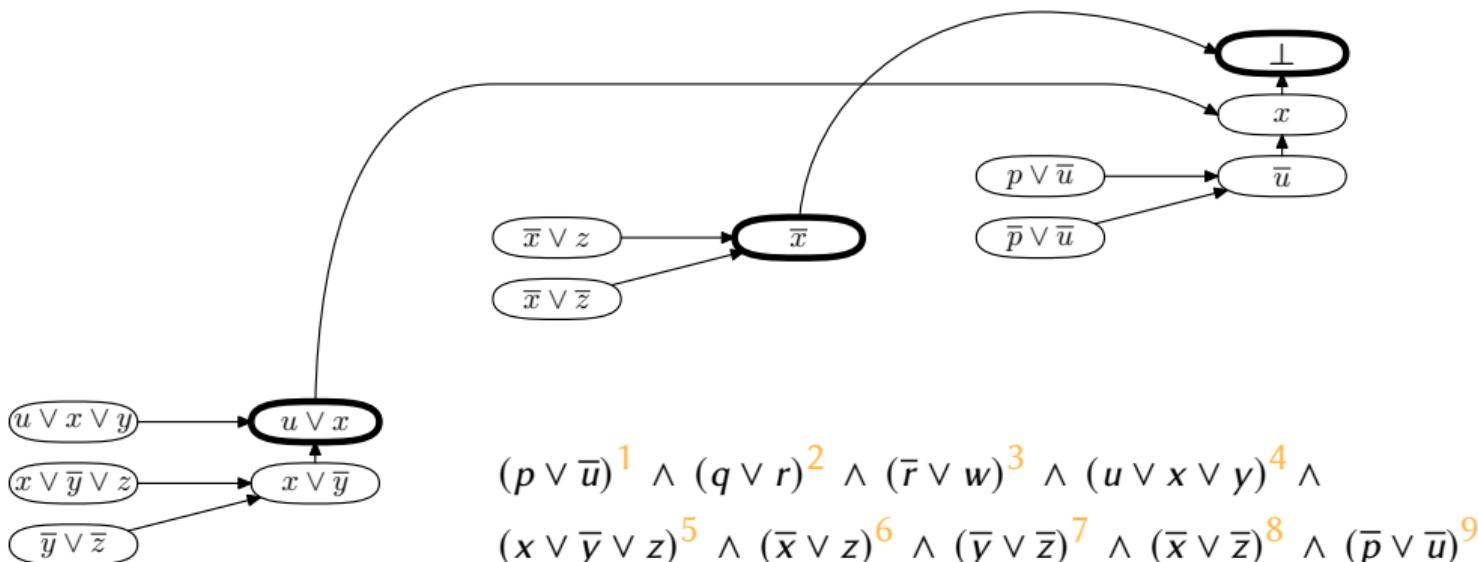
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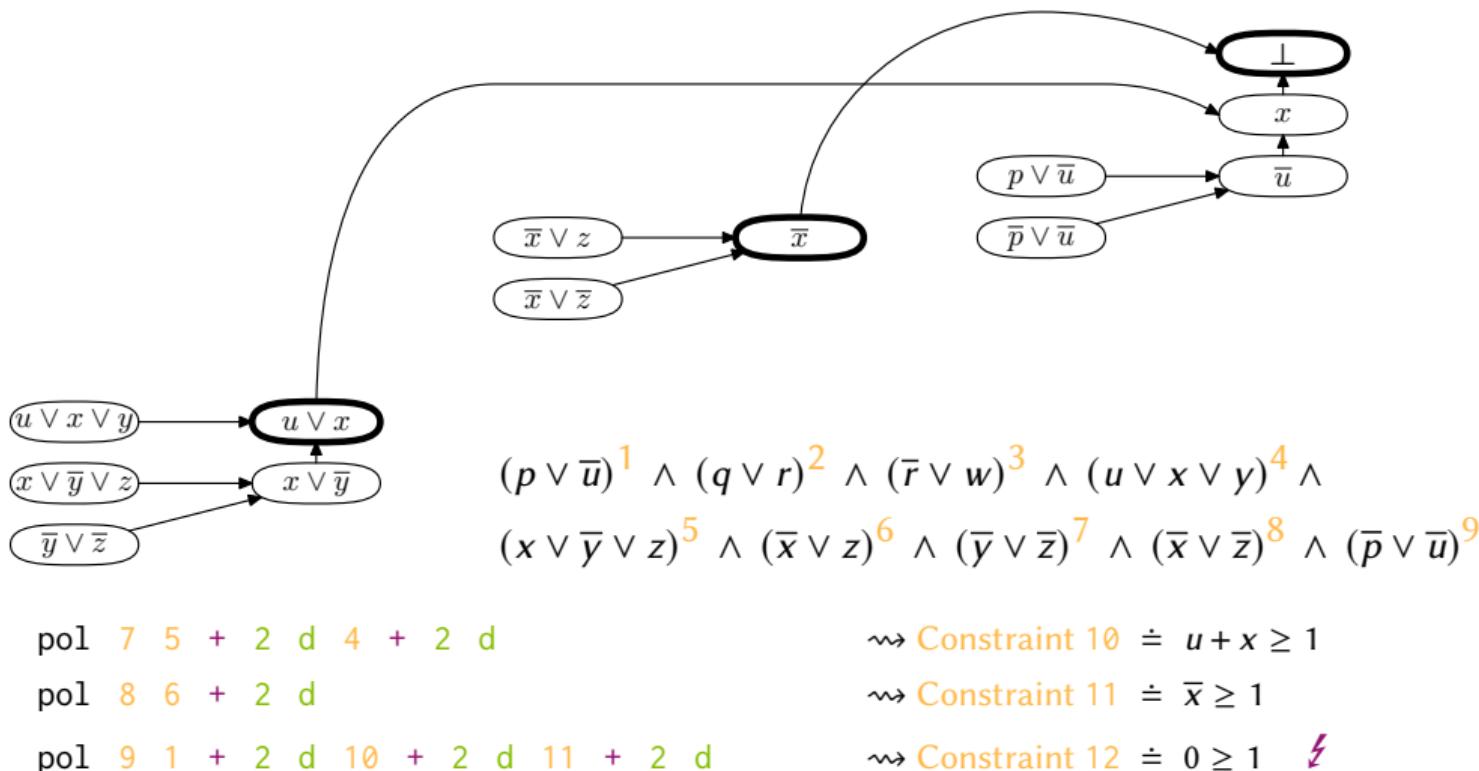
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# Pseudo-Boolean Proof Logging for Example CDCL Conflict Analyses



# RUP Revisited

Can define (reverse) unit propagation in a pseudo-Boolean setting

Constraint  $C$  propagates variable  $x$  if setting  $x$  to “wrong value” would make  $C$  unsatisfiable

# RUP Revisited

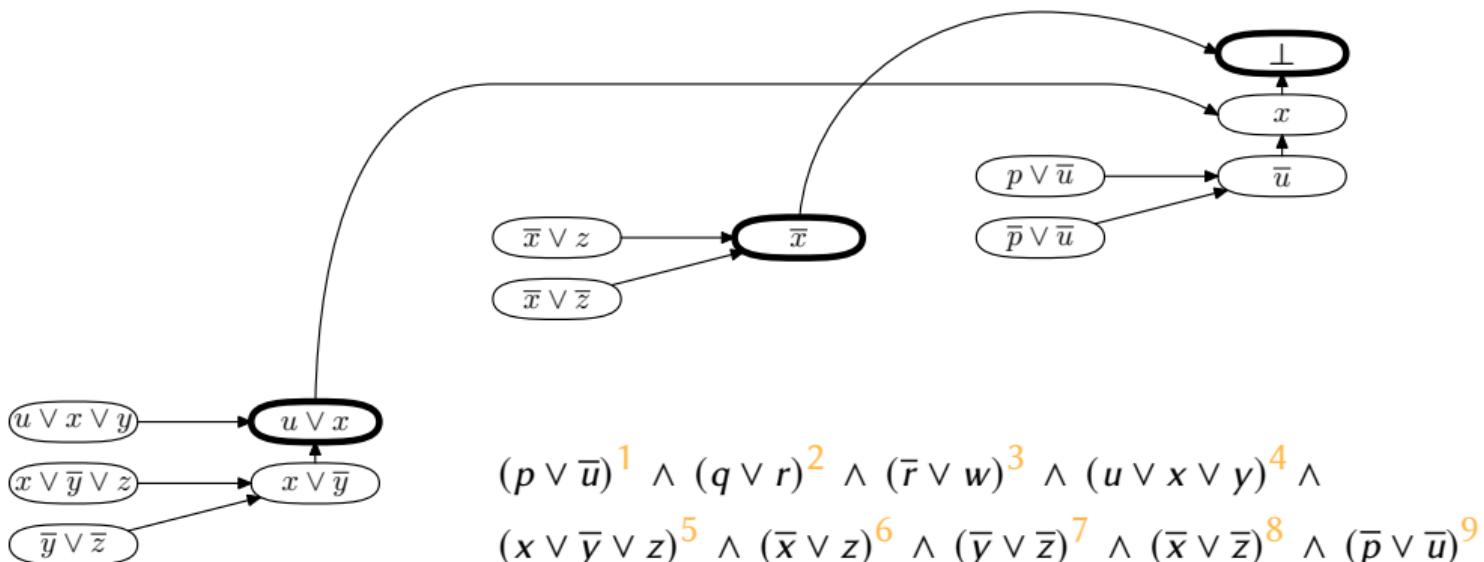
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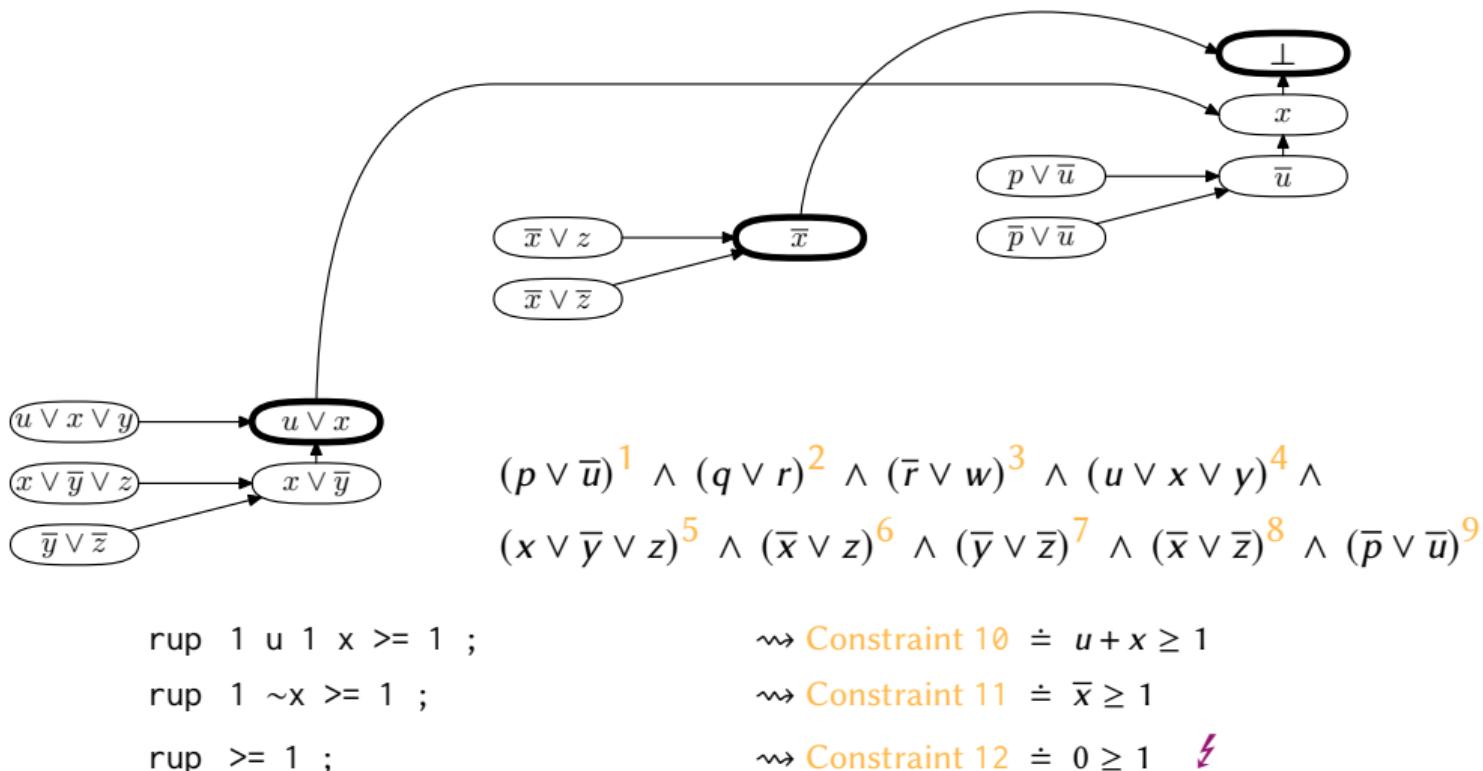
Risk for confusion:

- Constraint programming people might call this (reverse) integer bounds consistency
  - Does the same thing if we’re working with clauses
  - More interesting for general pseudo-Boolean constraints
- SAT people beware: constraints can propagate multiple times and multiple variables

# Pseudo-Boolean Proof Logging for Example CDCL Execution with RUP



# Pseudo-Boolean Proof Logging for Example CDCL Execution with RUP



## Extension Variables, Part 2

Suppose we want new, fresh variable  $a$  encoding

$$a \Leftrightarrow (3x + 2y + z + w \geq 3)$$

This time, introduce constraints

$$3\bar{a} + 3x + 2y + z + w \geq 3 \quad 5a + 3\bar{x} + 2\bar{y} + \bar{z} + \bar{w} \geq 5$$

Again, needs support from the proof system

# Proof Logs for “Extended Cutting Planes”

For satisfiable instances: just specify a satisfying assignment.

For unsatisfiability: a sequence of **pseudo-Boolean constraints** in (slight extension of) OPB format [RM16]

- Each constraint follows “obviously” from what is known so far
- Either implicitly, by RUP...
- Or by an explicit cutting planes derivation...
- Or as an extension variable reifying a new constraint\*
- Final constraint is  $0 \geq 1$

(\*) Not actually implemented this way — details to come later...

# Deleting Constraints

In practice, important to erase constraints to save memory and time during verification

Fairly straightforward to deal with from the point of view of proof logging

So ignored in this tutorial for simplicity and clarity

# Enumeration and Optimisation Problems

Enumeration:

- When a solution is found, can log it
- Introduces a new constraint saying “not this solution”
- So the proof semantics is “infeasible, except for all the solutions I told you about”

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For optimisation:

- Define an objective  $f = \sum_i w_i \ell_i$ ,  $w_i \in \mathbb{Z}$ , to minimise subject to the constraints in the formula
- To maximise, negate objective
- Log a solution  $\alpha$ ; get an objective-improving constraint  $\sum_i w_i \ell_i \leq -1 + \sum_i w_i \alpha(\ell_i)$
- Semantics for proof of optimality: “infeasible to find better solution than best so far”

# Pseudo-Boolean Proof Logging – How and Why?

If problem is (special case of) 0–1 integer linear program (ILP)

- just do proof logging

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**Goldilocks compromise** between expressivity and simplicity:

- 1 0–1 ILP **expressive formalism** for combinatorial problems (including objective)
- 2 **Powerful reasoning** capturing many combinatorial arguments (even for SAT)
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$$r \Rightarrow x_1 + 2\bar{x}_2 + 3x_3 + 4\bar{x}_4 + 5x_5 \geq 7$$

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$$7\bar{r} + x_1 + 2\bar{x}_2 + 3x_3 + 4\bar{x}_4 + 5x_5 \geq 7$$

$$9r + \bar{x}_1 + 2x_2 + 3\bar{x}_3 + 4x_4 + 5\bar{x}_5 \geq 9$$

# Parity (XOR) Reasoning

Given clauses

$$x \vee y \vee z$$

$$x \vee \bar{y} \vee \bar{z}$$

$$\bar{x} \vee y \vee \bar{z}$$

$$\bar{x} \vee \bar{y} \vee z$$

and

$$y \vee z \vee w$$

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want to derive

$$x \vee \bar{w}$$

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This is just parity reasoning:

and

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$x + y + z = 1 \pmod{2}$

$y + z + w = 1 \pmod{2}$

imply

$x + w = 0 \pmod{2}$

and

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Exponentially hard for CDCL [Urq87]  
But used in *CryptoMiniSat* [Cry]

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But used in *CryptoMiniSat* [Cry]

DRAT proof logging like [PR16] too inefficient in practice!

Could add XORs to language, but prefer to keep things super-simple

want to derive

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$\bar{x} \vee w$

# Pseudo-Boolean Proof Logging for XOR Reasoning

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Introduce extension variables  $a, b$  and derive

$x + y + z + 2a = 3$

$y + z + w + 2b = 3$

(“=” syntactic sugar for “ $\geq$ ” plus “ $\leq$ ”)

and

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Add to get

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From this can extract

$x + \bar{w} \geq 1$

$\bar{x} + w \geq 1$

want to derive

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VERIPB can certify XOR reasoning [GN21]

# CDCL Solvers on Pseudo-Boolean Inputs

Can re-encode to CNF and run CDCL:

- *MiniSat+* [ES06]
- *Open-WBO* [MML14]
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E.g., encode pseudo-Boolean constraint

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

to clauses with extension variables

$$s_{i,k} \Leftrightarrow \sum_{j=1}^i x_j \geq k$$

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How to know translation is correct?

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$$s_{i,k} \Leftrightarrow \sum_{j=1}^i x_j \geq k$$

$$k \cdot \bar{s}_{i,k} + \sum_{j=1}^i x_j \geq k$$

$$(i - k + 1) \cdot s_{i,k} + \sum_{j=1}^i \bar{x}_j \geq i - k + 1$$

$$\begin{aligned}\bar{s}_{1,1} \vee x_1 \\ \bar{s}_{2,1} \vee s_{1,1} \vee x_2 \\ \bar{s}_{2,2} \vee s_{1,1} \\ \bar{s}_{2,2} \vee x_2 \\ \bar{s}_{3,1} \vee s_{2,1} \vee x_3 \\ \bar{s}_{3,2} \vee s_{2,1} \\ \bar{s}_{3,2} \vee s_{2,2} \vee x_3 \\ \bar{s}_{4,1} \vee s_{3,1} \vee x_4 \\ \bar{s}_{4,2} \vee s_{3,1} \\ \bar{s}_{4,2} \vee s_{3,2} \vee x_4 \\ s_{4,2}\end{aligned}$$

How to know translation is correct?

# CDCL Solvers on Pseudo-Boolean Inputs

Can re-encode to CNF and run CDCL:

- *MiniSat+* [ES06]
- *Open-WBO* [MML14]
- *NaPS* [SN15]

E.g., encode pseudo-Boolean constraint

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

to clauses with extension variables

$$s_{i,k} \Leftrightarrow \sum_{j=1}^i x_j \geq k$$

$$k \cdot \bar{s}_{i,k} + \sum_{j=1}^i x_j \geq k$$

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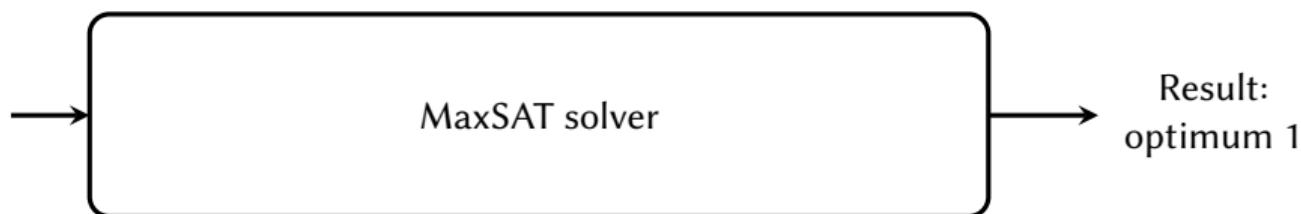
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VERIPB can certify **pseudo-Boolean-to-CNF rewriting** [GMNO22, VDB22]

# Certified Maximum Satisfiability (MaxSAT) Solving

Minimize linear objective subject to satisfying formula in conjunctive normal form (CNF)

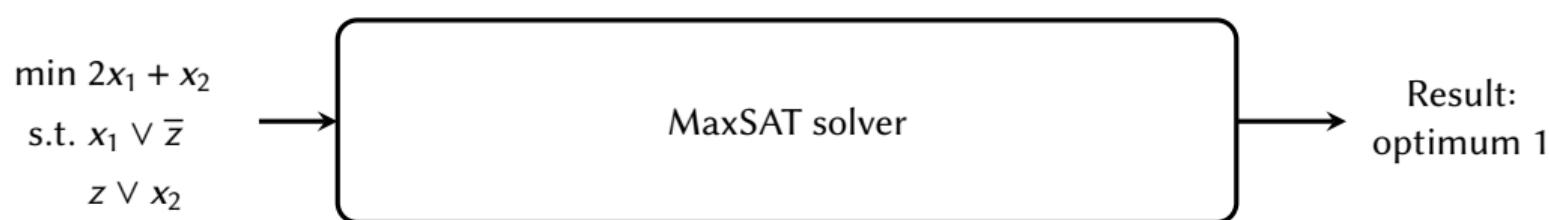
$$\begin{aligned} \text{min } & 2x_1 + x_2 \\ \text{s.t. } & x_1 \vee \bar{z} \\ & z \vee x_2 \end{aligned}$$



Many MaxSAT solvers internally make use of SAT solver.

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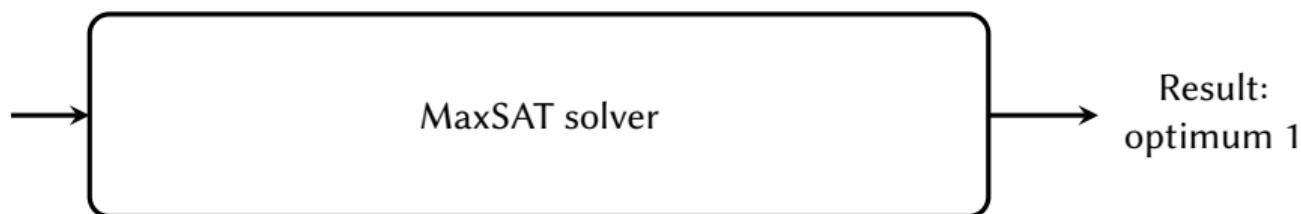
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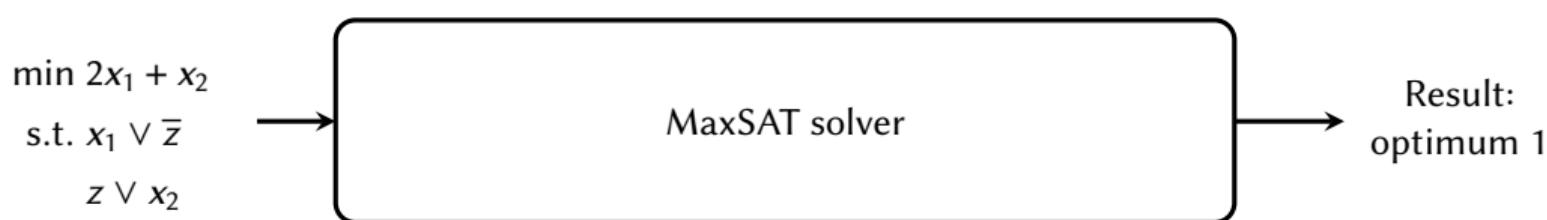
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## Does not work

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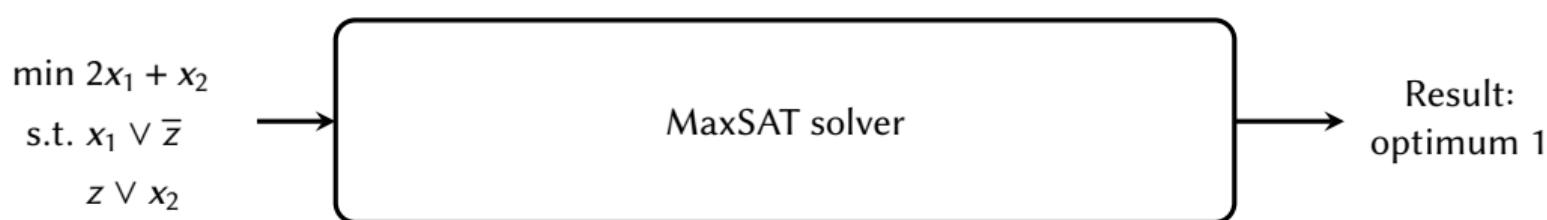
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Causes serious overhead

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Many MaxSAT solvers internally make use of SAT solver. Idea:

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  - Causes serious overhead

**Does not work** Only proves answer correct, not reasoning within solver!

# MaxSAT Solvers

Three main categories:

- Linear SAT-UNSAT search
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  - 3 Repeat (last found solution is optimal)

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No proof logging available **yet**

# MaxSAT example (LSU search)

Objective:  $\min \sum_i r_i$

VERIPB proof:

derived

justification

---

$$\bar{x}_1 \vee x_2$$

$$x_1 \vee \bar{x}_2$$

$$\bar{x}_2 \vee x_3$$

$$\bar{x}_3 \vee x_4$$

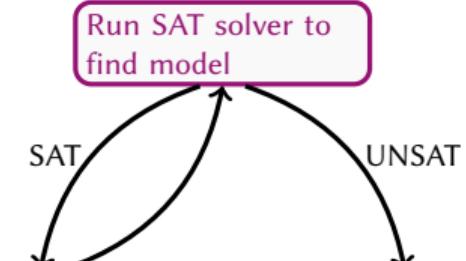
$$\bar{x}_1 \vee \bar{x}_2 \vee r_1$$

$$x_1 \vee x_2 \vee r_2$$

$$x_2 \vee x_4 \vee r_3$$

Encode model improving constraints

Last found solution is optimal



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$$x_2 \vee r_2$$



# MaxSAT example (LSU search)

Objective:  $\min \sum_i r_i$

VERIPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation

$$\bar{x}_1 \vee x_2$$

$$x_1 \vee \bar{x}_2$$

$$\bar{x}_2 \vee x_3$$

$$\bar{x}_3 \vee x_4$$

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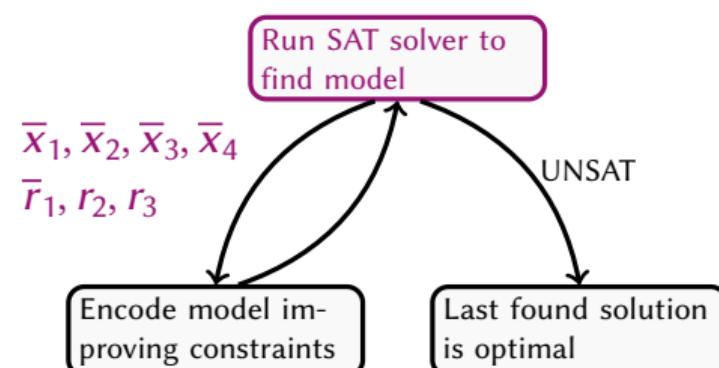
$$\bar{x}_3 \vee x_4$$

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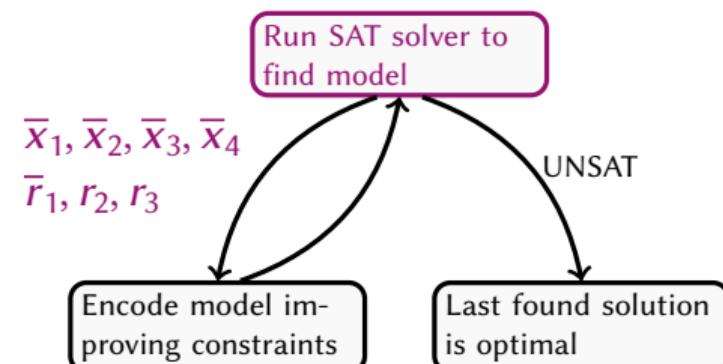
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$\{\bar{x}_1, \dots, \bar{x}_4, \bar{r}_1, r_2, r_3\}$	Incumbent solution

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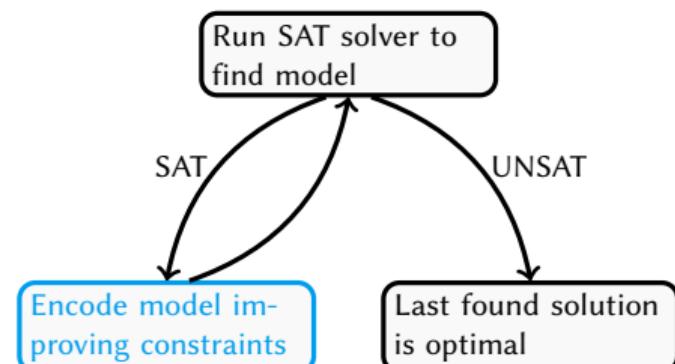
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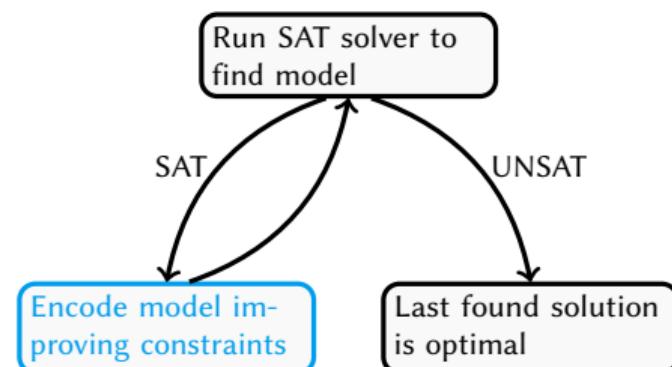
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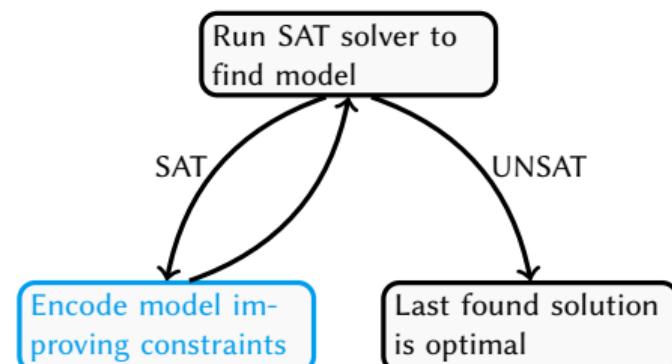
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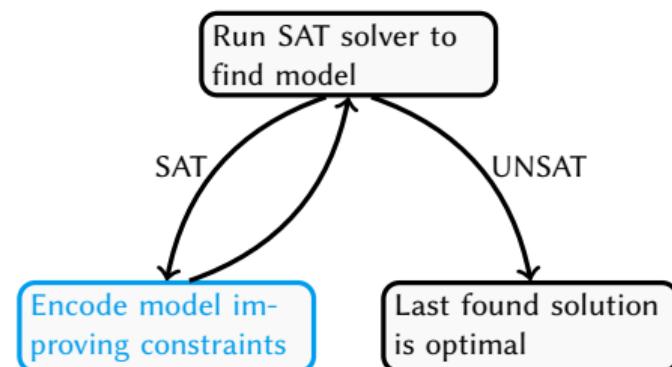
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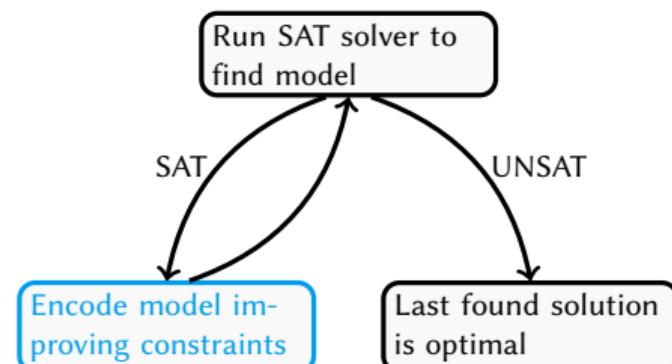
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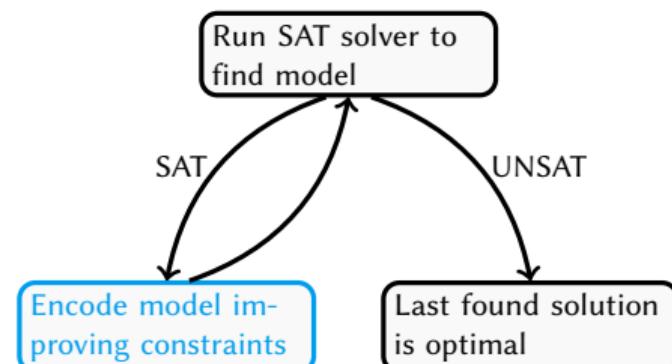
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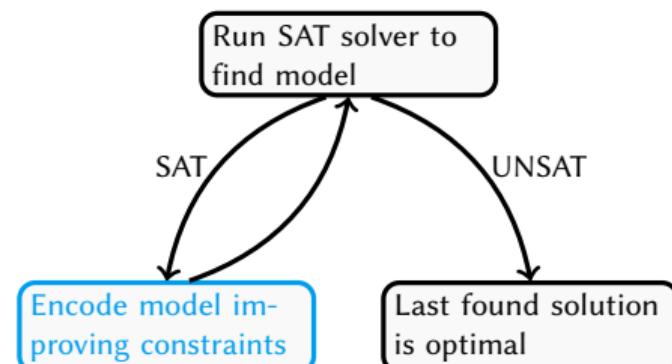
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 \bar{p}_2 &
 \end{array}$$



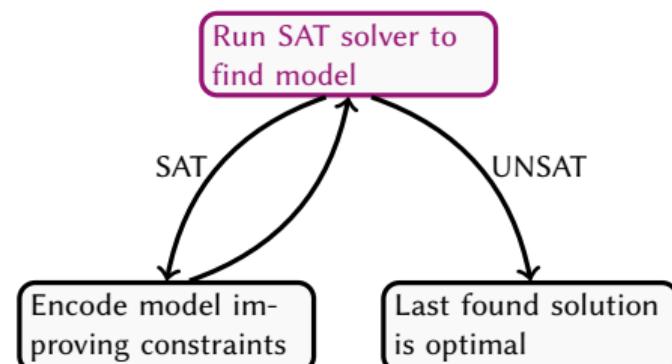
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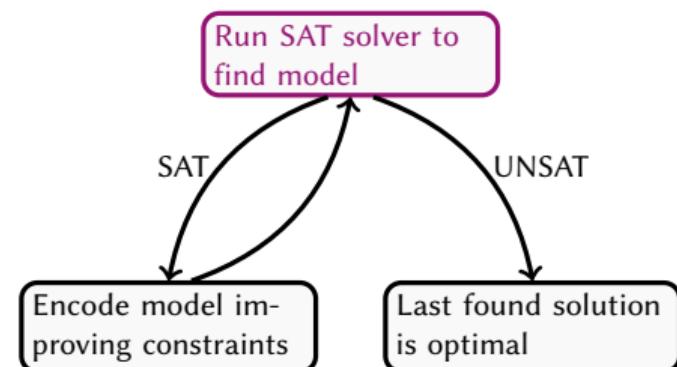
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$\bar{p}_2 \geq 1$	Reverse Unit Propagation
$x_4 \geq 1$	

$$\begin{array}{ll}
 \bar{x}_1 \vee x_2 & \bar{x}_1 \vee \bar{x}_2 \vee r_1 \\
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 \bar{x}_2 \vee x_3 & x_2 \vee x_4 \vee r_3 \\
 \bar{x}_3 \vee x_4 & x_2 \vee r_2 \\
 \text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j)) & \\
 \bar{p}_2 & x_4
 \end{array}$$



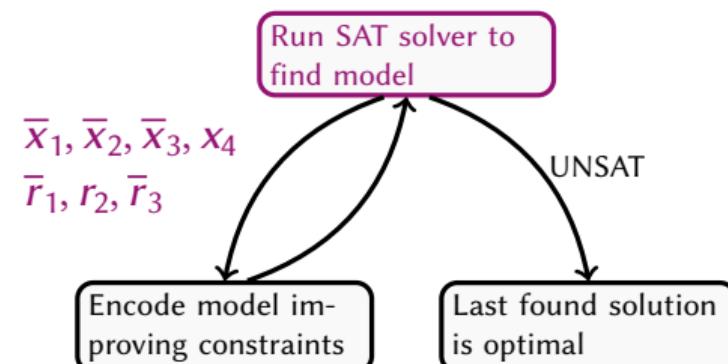
# MaxSAT example (LSU search)

Objective:  $\min \sum_i r_i$

VERIPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\bar{x}_1, \dots, \bar{x}_4, \bar{r}_1, r_2, r_3\}$	Incumbent solution
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$\bar{p}_2 \geq 1$	Reverse Unit Propagation
$x_4 \geq 1$	Incumbent solution
$\{\bar{x}_1, \bar{x}_2, \bar{x}_3, x_4, \bar{r}_1, r_2, \bar{r}_3\}$	

$$\begin{array}{ll}
 \bar{x}_1 \vee x_2 & \bar{x}_1 \vee \bar{x}_2 \vee r_1 \\
 x_1 \vee \bar{x}_2 & x_1 \vee x_2 \vee r_2 \\
 \bar{x}_2 \vee x_3 & x_2 \vee x_4 \vee r_3 \\
 \bar{x}_3 \vee x_4 & x_2 \vee r_2 \\
 \text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j)) & \\
 \bar{p}_2 & x_4
 \end{array}$$



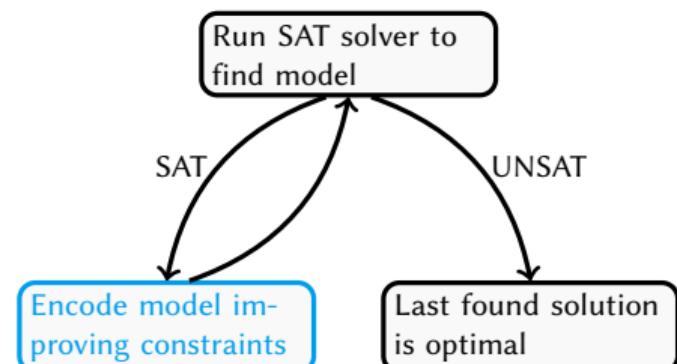
# MaxSAT example (LSU search)

Objective:  $\min \sum_i r_i$

VERIPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\bar{x}_1, \dots, \bar{x}_4, \bar{r}_1, r_2, r_3\}$	Incumbent solution
$\sum_i r_i \leq 1$	Objective Improvement Rule
$j \cdot \bar{p}_j + \sum_i r_i \geq j$	Fresh variable (RBS)
$(4 - j) \cdot p_j + \sum_i \bar{r}_i \geq 4 - j$	Explicit CP derivation
$\text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\bar{p}_2 \geq 1$	Reverse Unit Propagation
$x_4 \geq 1$	Incumbent solution
$\{\bar{x}_1, \bar{x}_2, \bar{x}_3, x_4, \bar{r}_1, r_2, \bar{r}_3\}$	Objective Improvement Rule
$\sum_i r_i \leq 0$	

$$\begin{array}{ll}
 \bar{x}_1 \vee x_2 & \bar{x}_1 \vee \bar{x}_2 \vee r_1 \\
 x_1 \vee \bar{x}_2 & x_1 \vee x_2 \vee r_2 \\
 \bar{x}_2 \vee x_3 & x_2 \vee x_4 \vee r_3 \\
 \bar{x}_3 \vee x_4 & x_2 \vee r_2 \\
 \text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j)) & \\
 \bar{p}_2 & x_4
 \end{array}$$



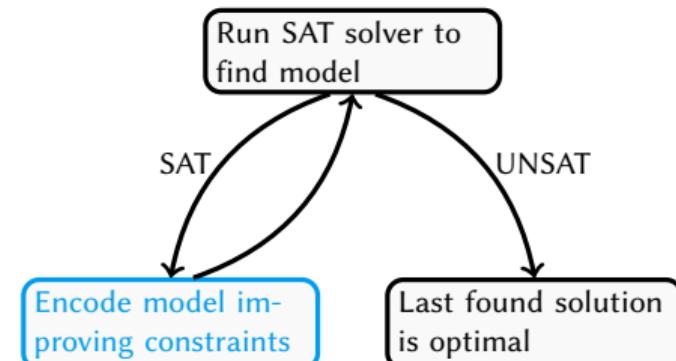
# MaxSAT example (LSU search)

Objective:  $\min \sum_i r_i$

VERIPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\bar{x}_1, \dots, \bar{x}_4, \bar{r}_1, r_2, r_3\}$	Incumbent solution
$\sum_i r_i \leq 1$	Objective Improvement Rule
$j \cdot \bar{p}_j + \sum_i r_i \geq j$	Fresh variable (RBS)
$(4 - j) \cdot p_j + \sum_i \bar{r}_i \geq 4 - j$	Explicit CP derivation
$\text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\bar{p}_2 \geq 1$	Reverse Unit Propagation
$x_4 \geq 1$	Incumbent solution
$\{\bar{x}_1, \bar{x}_2, \bar{x}_3, x_4, \bar{r}_1, r_2, \bar{r}_3\}$	Objective Improvement Rule
$\sum_i r_i \leq 0$	Explicit CP derivation
$\bar{p}_1 \geq 1$	

$$\begin{array}{ll}
 \bar{x}_1 \vee x_2 & \bar{x}_1 \vee \bar{x}_2 \vee r_1 \\
 x_1 \vee \bar{x}_2 & x_1 \vee x_2 \vee r_2 \\
 \bar{x}_2 \vee x_3 & x_2 \vee x_4 \vee r_3 \\
 \bar{x}_3 \vee x_4 & x_2 \vee r_2 \\
 \text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j)) & \\
 \bar{p}_2 & x_4 \\
 \bar{p}_1 &
 \end{array}$$



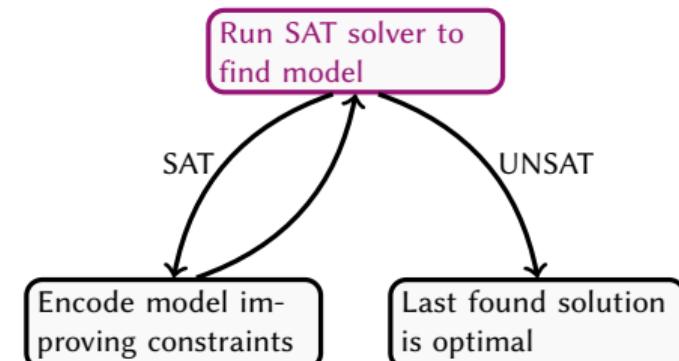
# MaxSAT example (LSU search)

Objective:  $\min \sum_i r_i$

VERIPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\bar{x}_1, \dots, \bar{x}_4, \bar{r}_1, r_2, r_3\}$	Incumbent solution
$\sum_i r_i \leq 1$	Objective Improvement Rule
$j \cdot \bar{p}_j + \sum_i r_i \geq j$	Fresh variable (RBS)
$(4 - j) \cdot p_j + \sum_i \bar{r}_i \geq 4 - j$	Explicit CP derivation
$\text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\bar{p}_2 \geq 1$	Reverse Unit Propagation
$x_4 \geq 1$	Incumbent solution
$\{\bar{x}_1, \bar{x}_2, \bar{x}_3, x_4, \bar{r}_1, r_2, \bar{r}_3\}$	Objective Improvement Rule
$\sum_i r_i \leq 0$	Explicit CP derivation
$\bar{p}_1 \geq 1$	

$$\begin{array}{ll}
 \bar{x}_1 \vee x_2 & \bar{x}_1 \vee \bar{x}_2 \vee r_1 \\
 x_1 \vee \bar{x}_2 & x_1 \vee x_2 \vee r_2 \\
 \bar{x}_2 \vee x_3 & x_2 \vee x_4 \vee r_3 \\
 \bar{x}_3 \vee x_4 & x_2 \vee r_2 \\
 \text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j)) & \\
 \bar{p}_2 & x_4 \\
 \bar{p}_1 &
 \end{array}$$



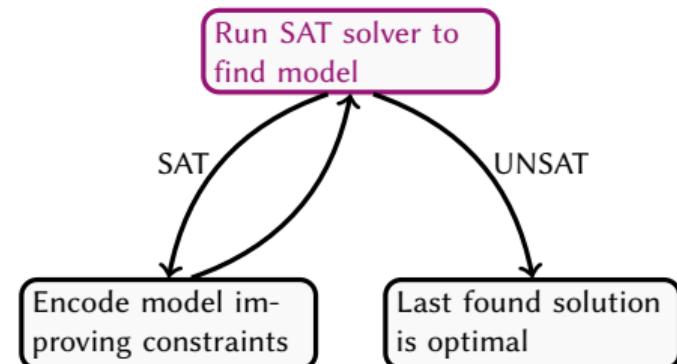
# MaxSAT example (LSU search)

Objective:  $\min \sum_i r_i$

VERIPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\bar{x}_1, \dots, \bar{x}_4, \bar{r}_1, r_2, r_3\}$	Incumbent solution
$\sum_i r_i \leq 1$	Objective Improvement Rule
$j \cdot \bar{p}_j + \sum_i r_i \geq j$	Fresh variable (RBS)
$(4 - j) \cdot p_j + \sum_i \bar{r}_i \geq 4 - j$	Explicit CP derivation
$\text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\bar{p}_2 \geq 1$	Reverse Unit Propagation
$x_4 \geq 1$	Incumbent solution
$\{\bar{x}_1, \bar{x}_2, \bar{x}_3, x_4, \bar{r}_1, r_2, \bar{r}_3\}$	Objective Improvement Rule
$\sum_i r_i \leq 0$	Explicit CP derivation
$\bar{p}_1 \geq 1$	Reverse Unit Propagation
$0 \geq 1$	

$$\begin{array}{ll}
 \bar{x}_1 \vee x_2 & \bar{x}_1 \vee \bar{x}_2 \vee r_1 \\
 x_1 \vee \bar{x}_2 & x_1 \vee x_2 \vee r_2 \\
 \bar{x}_2 \vee x_3 & x_2 \vee x_4 \vee r_3 \\
 \bar{x}_3 \vee x_4 & x_2 \vee r_2 \\
 \text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j)) & \\
 \bar{p}_2 & x_4 \\
 \bar{p}_1 & \perp
 \end{array}$$



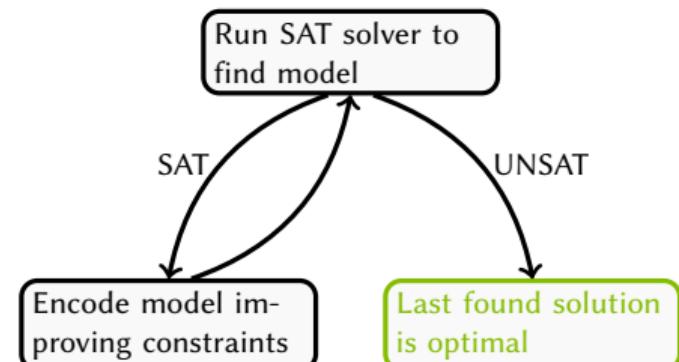
# MaxSAT example (LSU search)

Objective:  $\min \sum_i r_i$

VERIPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\bar{x}_1, \dots, \bar{x}_4, \bar{r}_1, r_2, r_3\}$	Incumbent solution
$\sum_i r_i \leq 1$	Objective Improvement Rule
$j \cdot \bar{p}_j + \sum_i r_i \geq j$	Fresh variable (RBS)
$(4 - j) \cdot p_j + \sum_i \bar{r}_i \geq 4 - j$	Explicit CP derivation
$\text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\bar{p}_2 \geq 1$	Reverse Unit Propagation
$x_4 \geq 1$	Incumbent solution
$\{\bar{x}_1, \bar{x}_2, \bar{x}_3, x_4, \bar{r}_1, r_2, \bar{r}_3\}$	Objective Improvement Rule
$\sum_i r_i \leq 0$	Explicit CP derivation
$\bar{p}_1 \geq 1$	Reverse Unit Propagation
$0 \geq 1$	

$$\begin{array}{ll}
 \bar{x}_1 \vee x_2 & \bar{x}_1 \vee \bar{x}_2 \vee r_1 \\
 x_1 \vee \bar{x}_2 & x_1 \vee x_2 \vee r_2 \\
 \bar{x}_2 \vee x_3 & x_2 \vee x_4 \vee r_3 \\
 \bar{x}_3 \vee x_4 & x_2 \vee r_2 \\
 \text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j)) & \\
 \bar{p}_2 & x_4 \\
 \bar{p}_1 & \perp
 \end{array}$$



# Progress So Far

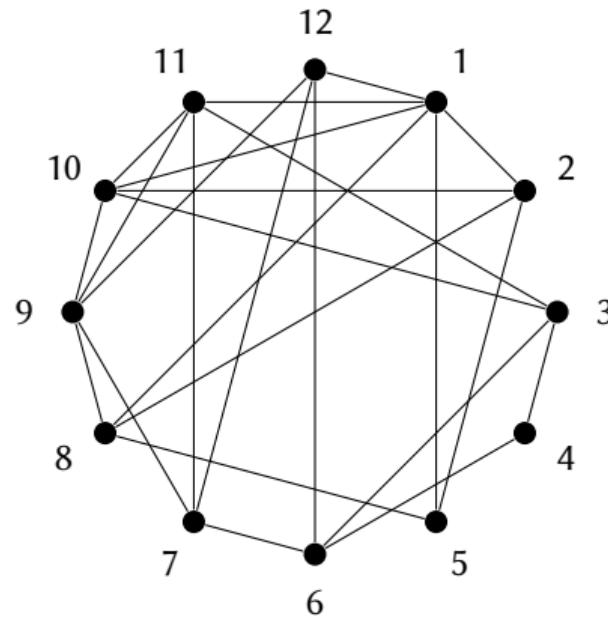
We've seen proof logging, and how it works for SAT.

We've learned about

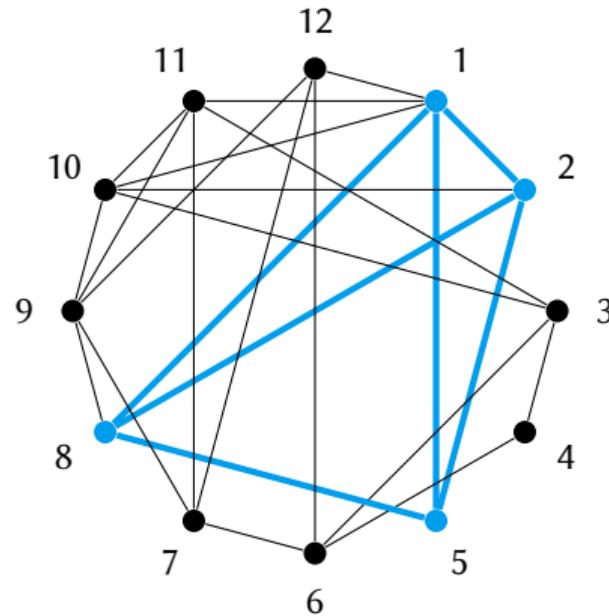
- pseudo-Boolean constraints (0-1 linear inequalities),
- cutting planes reasoning, and
- VERIPB.

Coming next, some worked examples from dedicated graph solvers.

## The Maximum Clique Problem



## The Maximum Clique Problem



# Maximum Clique Solvers

There are a lot of dedicated solvers for clique problems.

But there are issues:

- “State of the art” solvers have been buggy.
- Often undetected: error rate of around 0.1% [MPP19].

Often used inside other solvers.

- An off-by-one result can cause much larger errors.

# A Brief and Incomplete Guide to Clique-Solving

Recursive maximum clique algorithm:

- Pick a vertex  $v$ .
- Either  $v$  is in the clique...
  - Throw away every vertex not adjacent to  $v$ .
  - If vertices remain, recurse.
- ...or  $v$  is not in the clique.
  - Throw  $v$  away and pick another vertex.

# A Brief and Incomplete Guide to Clique-Solving

Key data structures:

- A growing clique  $C$ .
- A set of potential vertices  $P$ .
  - All the vertices we haven't thrown away yet.
  - Every  $v \in P$  is adjacent to every  $w \in C$ .

# A Brief and Incomplete Guide to Clique-Solving

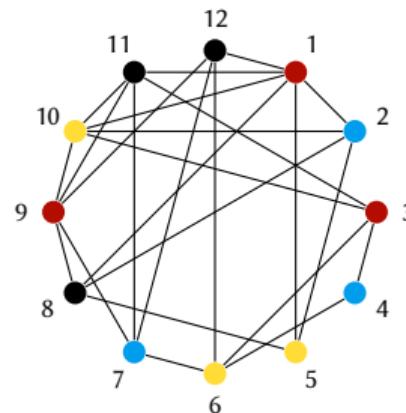
Key data structures:

- A growing clique  $C$ .
- A set of potential vertices  $P$ .
  - All the vertices we haven't thrown away yet.
  - Every  $v \in P$  is adjacent to every  $w \in C$ .

Branch and bound:

- Remember the biggest clique found so far,  $C^*$ .
- If  $|C| + |P| \leq |C^*|$ , no need to keep going.

# A Brief and Incomplete Guide to Clique-Solving



Given a  $k$ -colouring of a subgraph, that subgraph cannot have a clique of more than  $k$  vertices.  
We can use  $|C| + \#colours(P)$  as a bound, for any colouring.

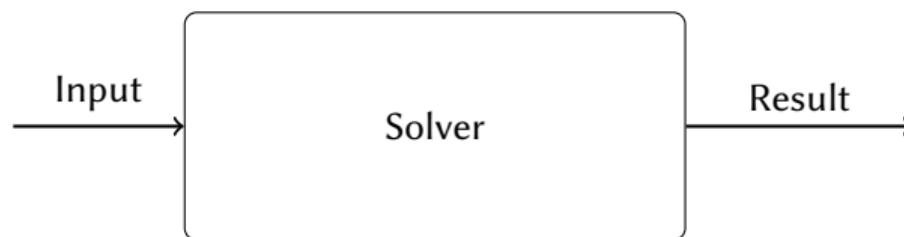
# A Brief and Incomplete Guide to Clique-Solving

- This brings us to 1997.
- In practice, better bound functions, clever vertex selection heuristics, efficient data structures, local search, ...

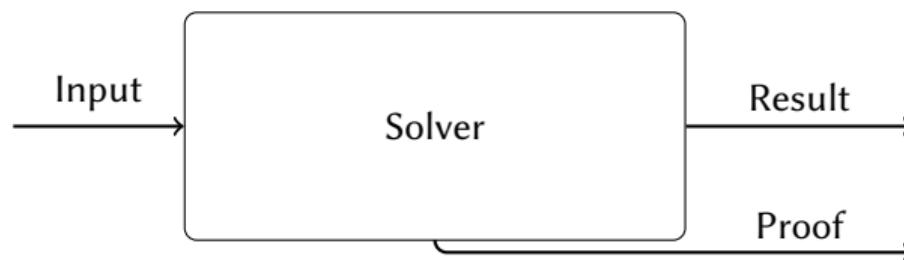
# Making a Proof-Logging Clique Solver

- 1** Output a pseudo-Boolean encoding of the problem.
  - Clique problems have several standard file formats.
- 2** Make the solver log its search tree.
  - Output a small header.
  - Output something on every backtrack.
  - Output something every time a solution is found.
  - Output a small footer.
- 3** Figure out how to log the bound function.

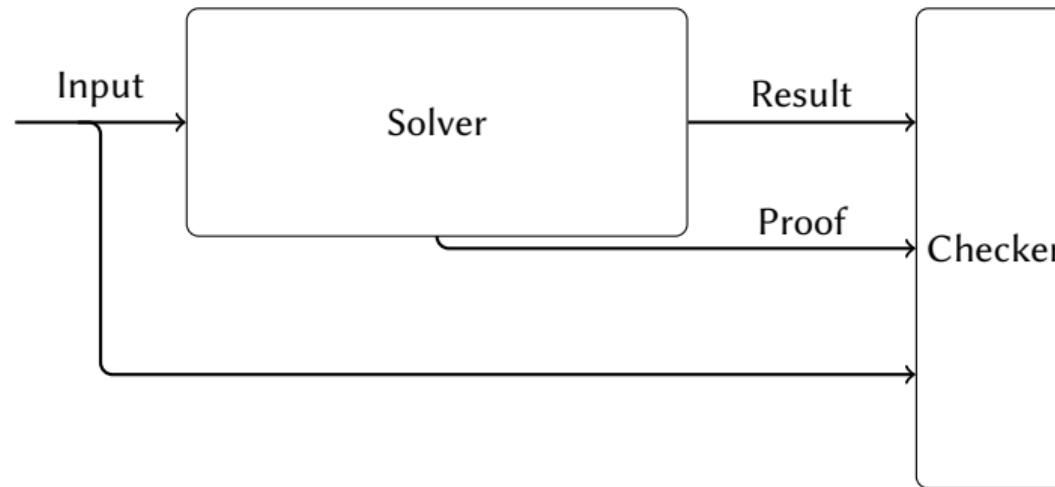
# A Slightly Different Workflow



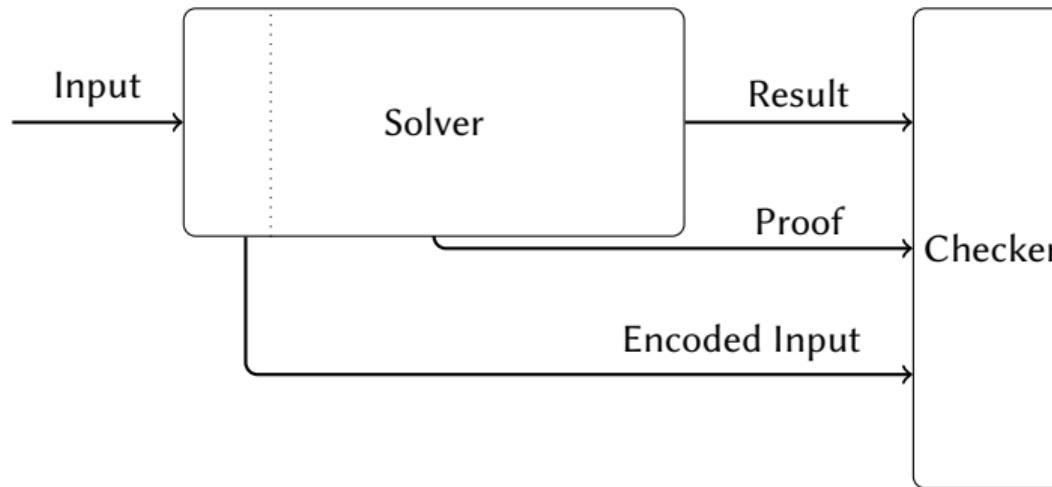
# A Slightly Different Workflow



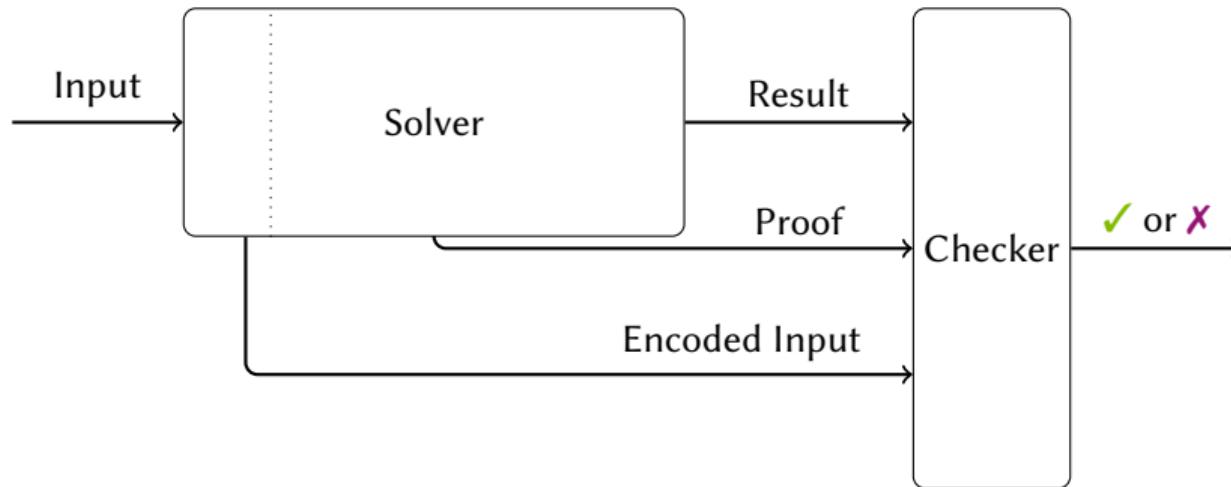
# A Slightly Different Workflow



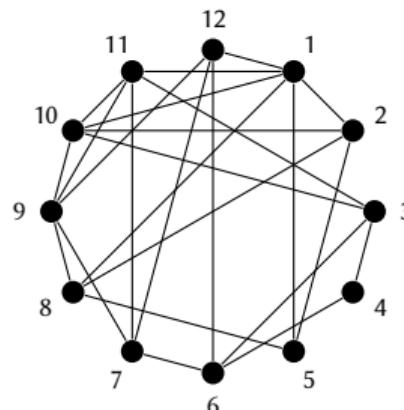
# A Slightly Different Workflow



# A Slightly Different Workflow



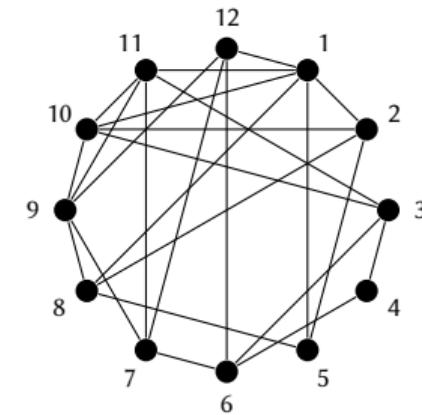
# A Pseudo-Boolean Encoding for Clique (in OPB Format)



```
* #variable= 12 #constraint= 41
min: -1 x1 -1 x2 -1 x3 -1 x4 . . . and so on. . . -1 x11 -1 x12 ;
1 ~x3 1 ~x1 >= 1 ;
1 ~x3 1 ~x2 >= 1 ;
1 ~x4 1 ~x1 >= 1 ;
* . . . and a further 38 similar lines for the remaining non-edges
```

# First Attempt at a Proof

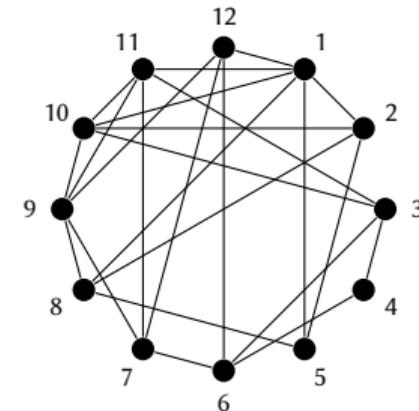
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



# First Attempt at a Proof

pseudo-Boolean proof version 2.0  
f 41

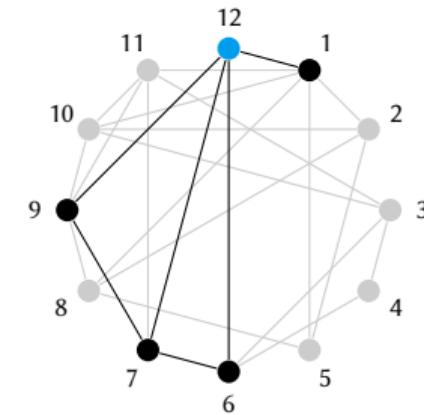
```
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Start with a header.  
Load the 41 problem axioms.

# First Attempt at a Proof

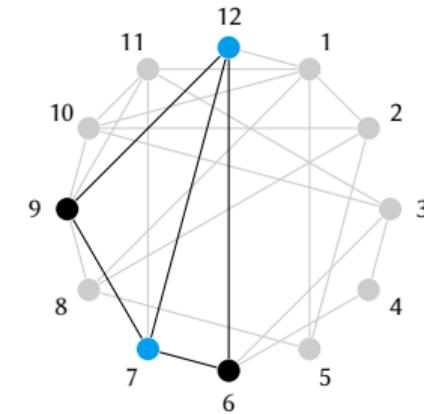
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Branch accepting 12.  
Throw away non-adjacent vertices.

# First Attempt at a Proof

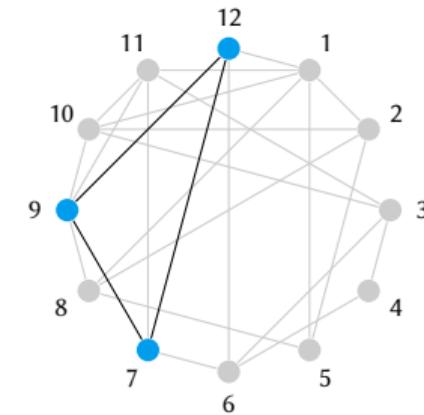
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Branch also accepting 7.  
Throw away non-adjacent vertices.

# First Attempt at a Proof

```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Branch also accepting 9.  
Throw away non-adjacent vertices.

# First Attempt at a Proof

pseudo-Boolean proof version 2.0

f 41

soli x7 x9 x12

rup 1 ~x12 1 ~x7 >= 1 ;

rup 1 ~x12 >= 1 ;

rup 1 ~x11 1 ~x10 >= 1 ;

rup 1 ~x11 >= 1 ;

soli x1 x2 x5 x8

rup 1 ~x8 1 ~x5 >= 1 ;

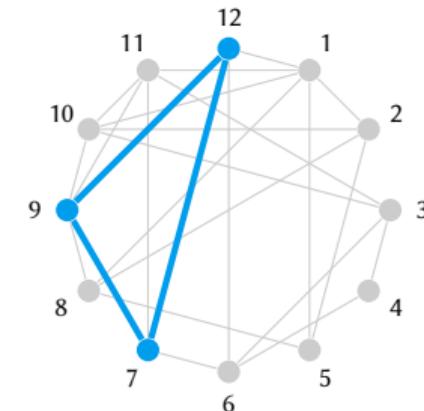
rup 1 ~x8 >= 1 ;

rup >= 1 ;

output NONE

conclusion BOUNDS -4 -4

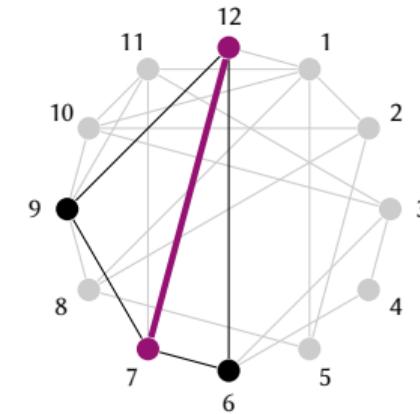
end pseudo-Boolean proof



We branched on 12, 7, 9.  
Found a new incumbent.  
Now looking for a  $\geq 4$  vertex clique.

# First Attempt at a Proof

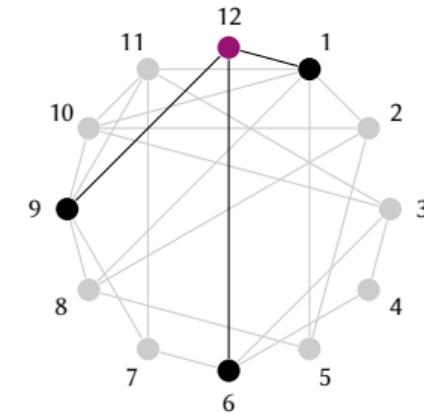
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Backtrack from 12, 7.  
9 explored already, only 6 feasible.  
No  $\geq 4$  vertex clique possible.  
Effectively this deletes the 7–12 edge.

# First Attempt at a Proof

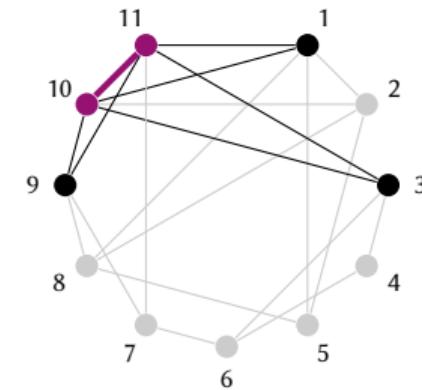
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Backtrack from 12.  
Only 1, 6 and 9 feasible (1-colourable).  
No  $\geq 4$  vertex clique possible.  
Effectively this deletes vertex 12.

## First Attempt at a Proof

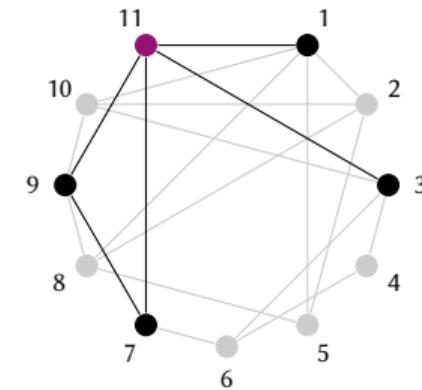
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Branch on 11 then 10.  
Only 1, 3 and 9 feasible (1-colourable).  
No  $\geq 4$  vertex clique possible.  
Backtrack, deleting the edge.

# First Attempt at a Proof

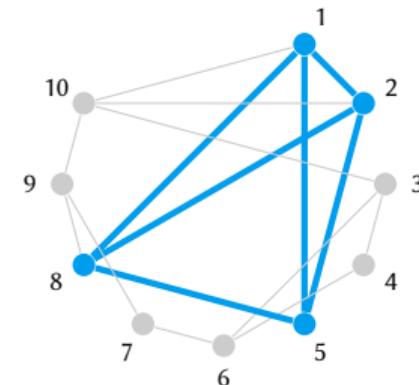
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Backtrack from 11.  
2-colourable, so no  $\geq 4$  clique.  
Delete the vertex.

## First Attempt at a Proof

```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Branch on 8, 5, 1, 2.  
Find a new incumbent.  
Now looking for a  $\geq 5$  vertex clique.

# First Attempt at a Proof

pseudo-Boolean proof version 2.0

f 41

soli x7 x9 x12

rup 1 ~x12 1 ~x7 >= 1 ;

rup 1 ~x12 >= 1 ;

rup 1 ~x11 1 ~x10 >= 1 ;

rup 1 ~x11 >= 1 ;

soli x1 x2 x5 x8

rup 1 ~x8 1 ~x5 >= 1 ;

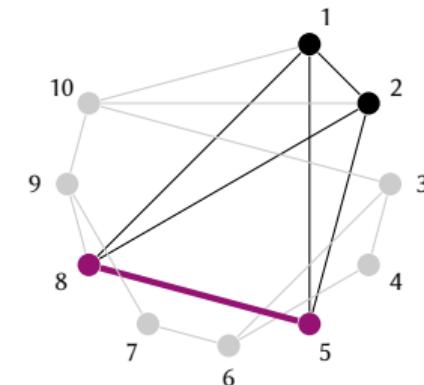
rup 1 ~x8 >= 1 ;

rup >= 1 ;

output NONE

conclusion BOUNDS -4 -4

end pseudo-Boolean proof



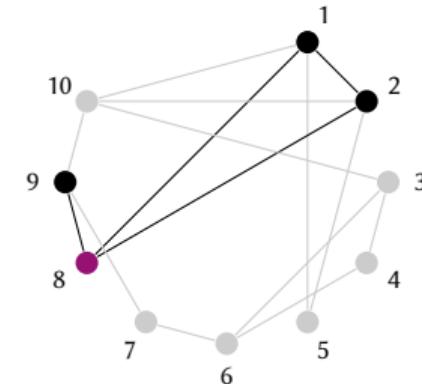
Backtrack from 8, 5.

Only 4 vertices, can't have a  $\geq 5$  clique.

Delete the edge.

# First Attempt at a Proof

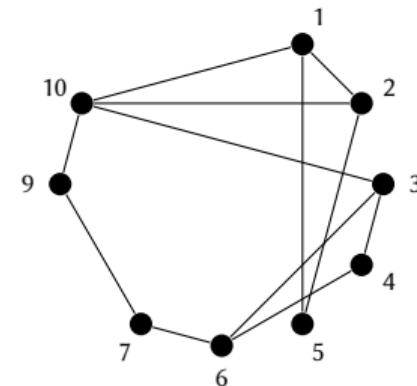
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Backtrack from 8.  
Still not enough vertices.  
Delete the vertex.

# First Attempt at a Proof

```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Remaining graph is 3-colourable.  
Backtrack from root node.

# First Attempt at a Proof

pseudo-Boolean proof version 2.0

f 41

soli x7 x9 x12

rup 1 ~x12 1 ~x7 >= 1 ;

rup 1 ~x12 >= 1 ;

rup 1 ~x11 1 ~x10 >= 1 ;

rup 1 ~x11 >= 1 ;

soli x1 x2 x5 x8

rup 1 ~x8 1 ~x5 >= 1 ;

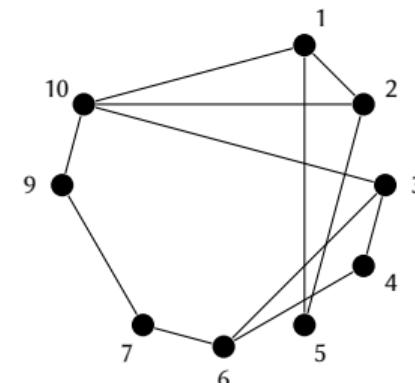
rup 1 ~x8 >= 1 ;

rup >= 1 ;

output NONE

conclusion BOUNDS -4 -4

end pseudo-Boolean proof



Finish with what we've concluded.

We specify a lower and an upper bound.

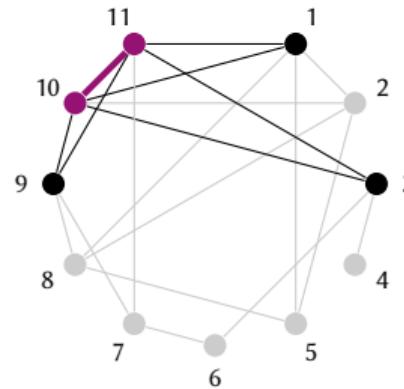
Remember we're minimising  $\sum_v -1 \times v$ , so a 4-clique has an objective value of -4.

# Verifying This Proof (Or Not...)

```
$ veripb clique.opb clique-attempt-one.veripb
Verification failed.
Failed in proof file line 6.
Hint: Failed to show '1 ~x10 1 ~x11 >= 1' by reverse unit propagation.
```

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Verification failed.
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```



# Verifying This Proof (Or Not...)

```
$ veripb --trace clique.opb clique-attempt-one.veripb
line 002: f 41
  ConstraintId 001: 1 ~x1 1 ~x3 >= 1
  ConstraintId 002: 1 ~x2 1 ~x3 >= 1
...
  ConstraintId 041: 1 ~x11 1 ~x12 >= 1
line 003: soli x7 x9 x12 ~x1 ~x2 ~x3 ~x4 ~x5 ~x6 ~x8 ~x10 ~x11
  ConstraintId 042: 1 x1 1 x2 1 x3 1 x4 1 x5 1 x6 1 x7 1 x8 1 x9 1 x10 1 x11 1 x12 >= 4
line 004: rup 1 ~x12 1 ~x7 >= 1 ;
  ConstraintId 043: 1 ~x7 1 ~x12 >= 1
line 005: rup 1 ~x12 >= 1 ;
  ConstraintId 044: 1 ~x12 >= 1
line 006: rup 1 ~x11 1 ~x10 >= 1 ;
Verification failed.
Failed in proof file line 6.
Hint: Failed to show '1 ~x10 1 ~x11 >= 1' by reverse unit propagation.
```

# Dealing With Colourings

The colour bound doesn't follow by RUP!

We can lazily recover at-most-one constraints for each colour class.

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$$\begin{aligned} & (\bar{x}_1 + \bar{x}_6 \geq 1) \\ & + (\bar{x}_1 + \bar{x}_9 \geq 1) & = 2\bar{x}_1 + \bar{x}_6 + \bar{x}_9 \geq 2 \\ & + (\bar{x}_6 + \bar{x}_9 \geq 1) & = 2\bar{x}_1 + 2\bar{x}_6 + 2\bar{x}_9 \geq 3 \\ & / 2 & = \bar{x}_1 + \bar{x}_6 + \bar{x}_9 \geq 2 \\ & & \text{i.e. } x_1 + x_6 + x_9 \leq 1 \end{aligned}$$

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This generalises for arbitrarily large colour classes.

- Each non-edge is used exactly once,  $v(v - 1)$  additions.
- $v - 3$  multiplications and  $v - 2$  divisions.

Solvers don't need to “understand” cutting planes to write this out.

# What This Looks Like

```
pseudo-Boolean proof version 2.0
f 41
soli x12 x7 x9
rup 1 ~x12 1 ~x7 >= 1 ;
* bound, colour classes [ x1 x6 x9 ]
pol 71~6 191~9 + 246~9 + 2 d
pol 42obj -1 +
rup 1 ~x12 >= 1 ;
* bound, colour classes [ x1 x3 x9 ]
pol 11~3 191~9 + 213~9 + 2 d
pol 42obj -1 +
rup 1 ~x11 1 ~x10 >= 1 ;
* bound, colour classes [ x1 x3 x7 ]
* [ x9 ]
pol 11~3 101~7 + 123~7 + 2 d
pol 42obj -1 +
rup 1 ~x11 >= 1 ;
```

```
soli x8 x5 x2 x1
rup 1 ~x8 1 ~x5 >= 1 ;
* bound, colour classes [ x1 x9 ] [ x2 ]
pol 53obj 191~9 +
rup 1 ~x8 >= 1 ;
* bound, colour classes [ x1 x3 x7 ]
* [ x2 x4 x9 ] [ x5 x6 x10 ]
pol 11~3 101~7 + 123~7 + 2 d
pol 53obj -1 +
pol 42~4 202~9 + 224~9 + 2 d
pol 53obj -3 + -1 +
pol 95~6 265~10 + 276~10 + 2 d
pol 53obj -5 + -3 + -1 +
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```

# Verifying This Proof (For Real, This Time)

```
$ veripb --trace clique.opb clique-attempt-two.veripb
== begin trace ==
line 002: f 41
  ConstraintId 001: 1 ~x1 1 ~x3 >= 1
  ConstraintId 002: 1 ~x2 1 ~x3 >= 1
...
  ConstraintId 041: 1 ~x11 1 ~x12 >= 1
line 003: soli x7 x9 x12 ~x1 ~x2 ~x3 ~x4 ~x5 ~x6 ~x8 ~x10 ~x11
  ConstraintId 042: 1 x1 1 x2 1 x3 1 x4 1 x5 1 x6 1 x7 1 x8 1 x9 1 x10 1 x11 1 x12 >= 4
line 004: rup 1 ~x12 1 ~x7 >= 1 ;
  ConstraintId 043: 1 ~x7 1 ~x12 >= 1
line 005: * bound, colour classes [ x1 x6 x9 ]
line 006: pol 7 19 + 24 + 2 d
  ConstraintId 044: 1 ~x1 1 ~x6 1 ~x9 >= 2
line 007: pol 42 43 +
  ConstraintId 045: 1 x1 1 x2 1 x3 1 x4 1 x5 1 x6 1 x8 1 x9 1 x10 1 x11 >= 3
...
  ConstraintId 061: 1 ~x5 1 ~x6 1 ~x10 >= 2
line 028: pol 53 57 + 59 + 61 +
  ConstraintId 062: 1 x8 1 x11 1 x12 >= 2
line 029: rup >= 1 ;
  ConstraintId 063: >= 1
line 030: output NONE
line 031: conclusion BOUNDS -4 -4
line 032: end pseudo-Boolean proof
== end trace ==

Verification succeeded.
```

# Different Clique Algorithms

Different search orders?

- ✓ Irrelevant for proof logging.

Using local search to initialise?

- ✓ Just log the incumbent.

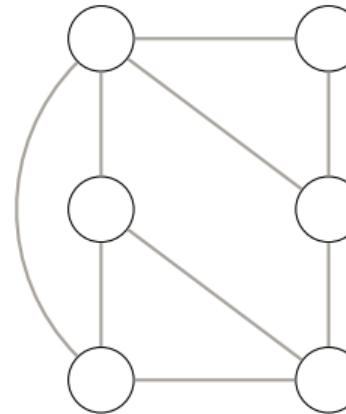
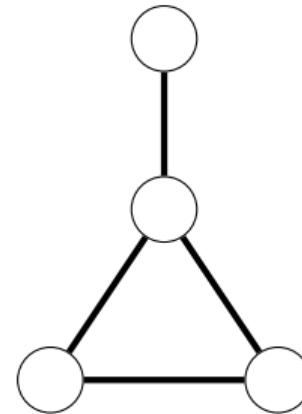
Different bound functions?

- Is cutting planes strong enough to justify every useful bound function ever invented?
- So far, seems like it...

Weighted cliques?

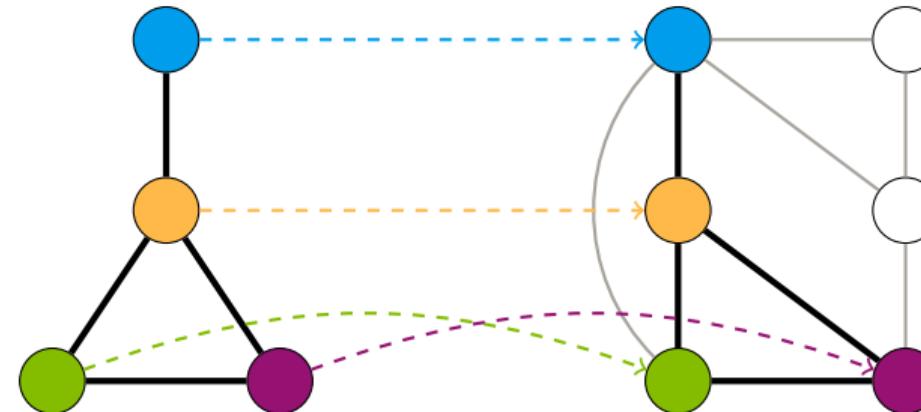
- ✓ Multiply a colour class by its largest weight.
- ✓ Also works for vertices “split between colour classes”.

# Subgraph Isomorphism



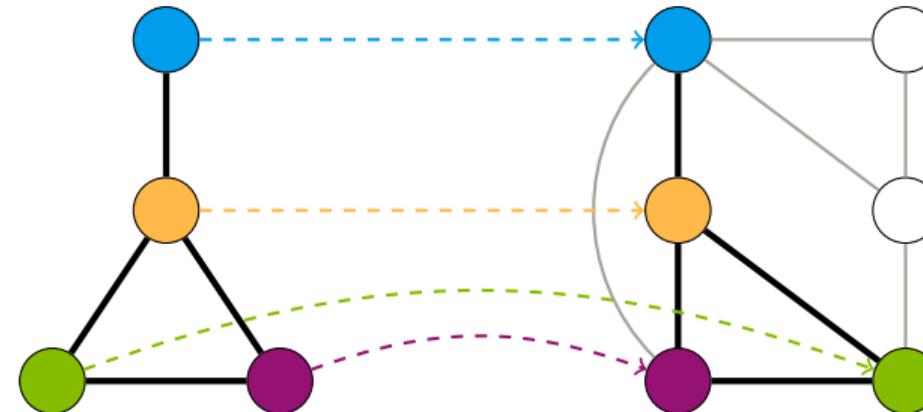
- Find the *pattern* inside the *target*.
- Applications in compilers, biochemistry, model checking, pattern recognition, ...
- Often want to find *all* matches.

# Subgraph Isomorphism



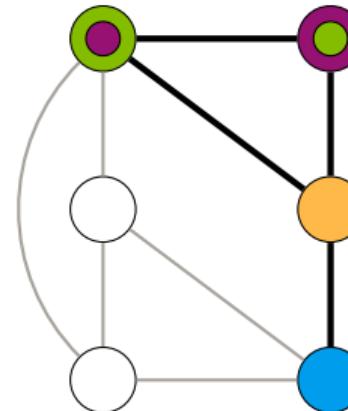
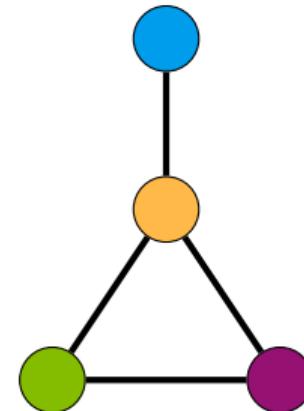
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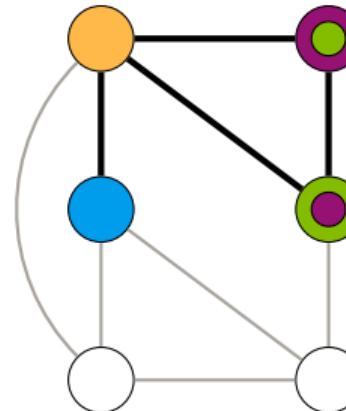
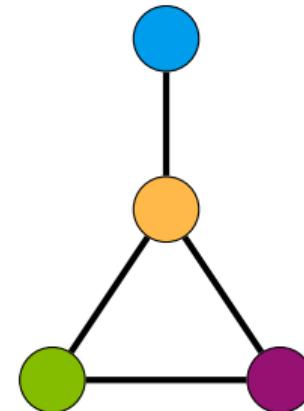
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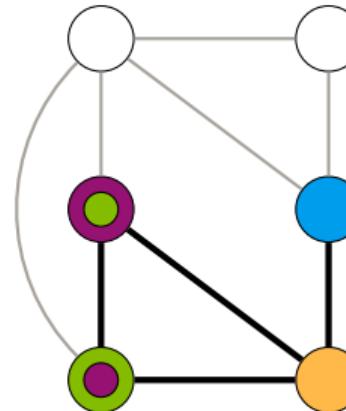
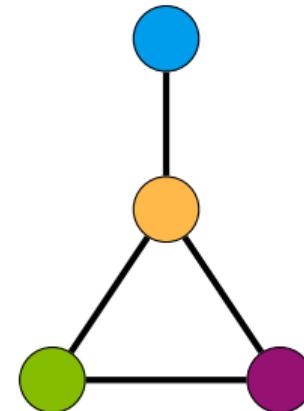
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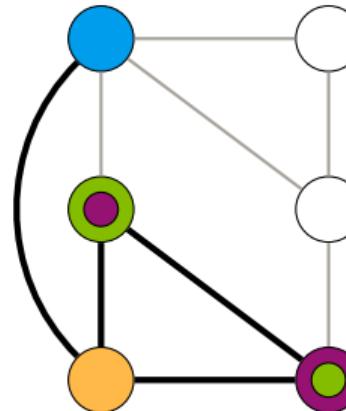
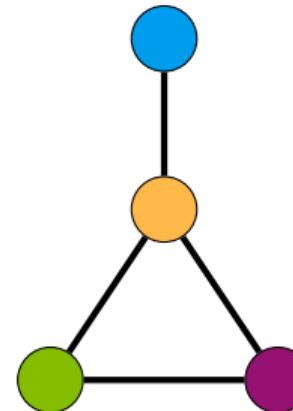
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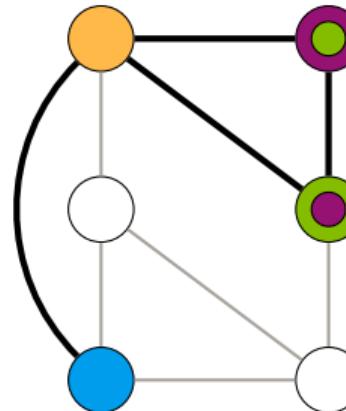
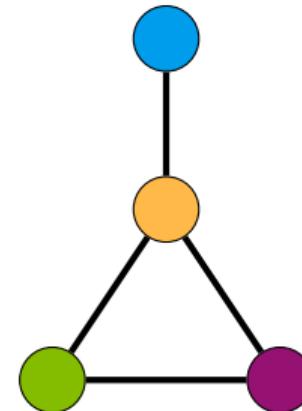
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# Subgraph Isomorphism in Pseudo-Boolean Form

Each pattern vertex gets a target vertex:

$$\sum_{t \in V(T)} x_{p,t} = 1 \quad p \in V(P)$$

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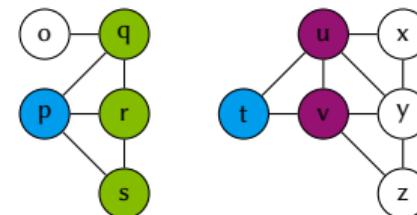
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$$\sum_{p \in V(P)} -x_{p,t} \geq -1 \quad t \in V(T)$$

Adjacency constraints, if  $p$  is mapped to  $t$ , then  $p$ 's neighbours must be mapped to  $t$ 's neighbours:

$$\bar{x}_{p,t} + \sum_{u \in N(t)} x_{q,u} \geq 1 \quad p \in V(P), q \in N(p), t \in V(T)$$

# Degree Reasoning in Cutting Planes



A pattern vertex  $p$  of degree  $\deg(p)$  can never be mapped to a target vertex  $t$  of degree  $\deg(p) - 1$  or lower in any subgraph isomorphism.

Observe  $N(p) = \{q, r, s\}$  and  $N(t) = \{u, v\}$ .

We wish to derive  $\bar{x}_{p,t} \geq 1$ .

# Degree Reasoning in Cutting Planes

Adjacency:

$$\bar{x}_{p,t} + x_{q,u} + x_{q,v} \geq 1$$

$$\bar{x}_{p,t} + x_{r,u} + x_{r,v} \geq 1$$

$$\bar{x}_{p,t} + x_{s,u} + x_{s,v} \geq 1$$

Injectivity:

$$-x_{o,u} + -x_{p,u} + -x_{q,u} + -x_{r,u} + -x_{s,u} \geq -1$$

$$-x_{o,v} + -x_{p,v} + -x_{q,v} + -x_{r,v} + -x_{s,v} \geq -1$$

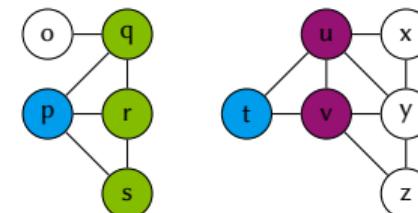
Literal axioms:

$$x_{o,u} \geq 0$$

$$x_{o,v} \geq 0$$

$$x_{p,u} \geq 0$$

$$x_{p,v} \geq 0$$



Add these together and divide by 3 to get

$$\bar{x}_{p,t} \geq 1$$

# Degree Reasoning in VERIPB

```
pol 18p~t:q 19p~t:r + 20p~t:s +      * sum adjacency constraints
  12inj(u) + 13inj(v) +                  * sum injectivity constraints
  xo_u + xo_v +                            * cancel stray xo_*
  xp_u + xp_v +                            * cancel stray xp_*
  3 d                                      * divide, and we're done
```

Or we can ask VERIPB to do the last bit of simplification automatically:

```
pol 18p~t:q 19p~t:r + 20p~t:s +      * sum adjacency constraints
  12inj(u) + 13inj(v) +                  * sum injectivity constraints
  ia -1 : 1 ~xp_t >= 1 ;                  * desired conclusion is implied
```

# Other Forms of Reasoning

We can also log all of the other things state of the art subgraph solvers do:

- Injectivity reasoning and filtering.
- Distance filtering.
- Neighbourhood degree sequences.
- Path filtering.
- Supplemental graphs.

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- Injectivity reasoning and filtering.
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- Supplemental graphs.

Proof steps are “efficient” using cutting planes.

- The length of the proof steps are no worse than the time complexity of the reasoning algorithms.
- Most proof steps require only trivial additional computations.

# Limitations

Why trust the encoding?

- Here we can formally verify the correctness of the encoding! Work in progress...

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- Unit propagation is much slower than bit-parallel algorithms.

Works up to moderately-sized hard instances

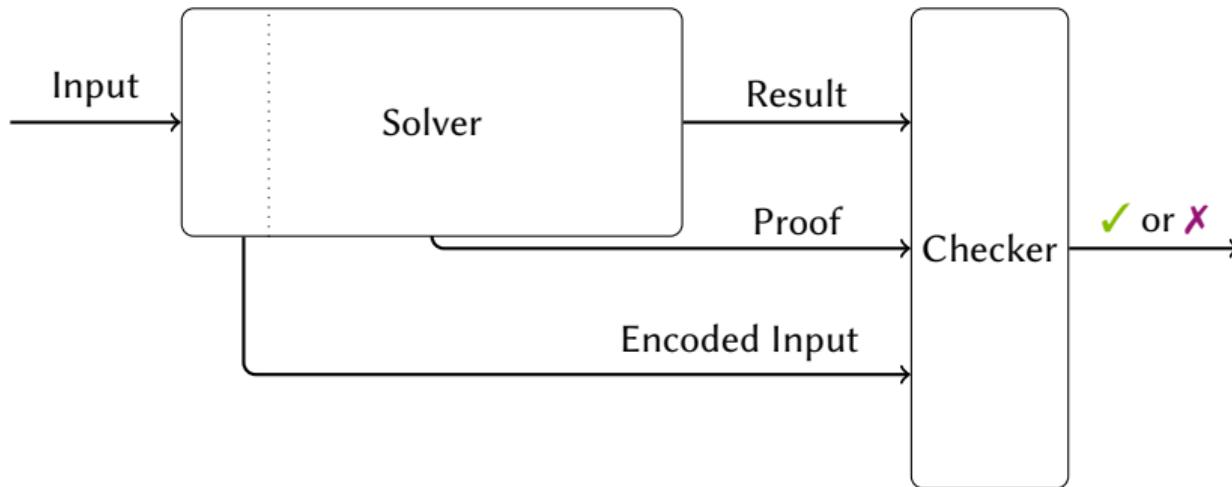
- Even an  $O(n^3)$  encoding is painful.
- Particularly bad when the pseudo-Boolean encoding talks about “non-edges” but large sparse graphs are “easy”.

# Code

<https://github.com/ciaranm/glasgow-subgraph-solver>

Released under MIT Licence.

# Recap (1 / 2)



## Recap (2 / 2)

### Proof logging implementation

- Don't change solver
- Just add proof logging statements (plus some book-keeping)

### Performance goals

Want linear(ish) scaling in terms of **solver running time** for

- **proof size**
- **proof checking time**

# What About Constraint Programming?

Non-Boolean variables?

Constraints?

- Encoding constraints as Pseudo-Boolean constraints?
- Justifying inference?

Reformulation?

# Compiling CP Variables

Given  $A \in \{-3 \dots 9\}$ , the direct encoding is:

$$\begin{aligned} a_{-3} + a_{-2} + a_{-1} + a_0 + a_1 + a_2 + a_3 \\ + a_4 + a_5 + a_6 + a_7 + a_8 + a_9 = 1 \end{aligned}$$

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This doesn't work for large domains.

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This doesn't work for large domains.

We could use a binary encoding:

$$\begin{aligned} -16a_{\text{neg}} + 1a_{b0} + 2a_{b1} + 4a_{b2} + 8a_{b3} \geq -3 \text{ and} \\ 16a_{\text{neg}} + -1a_{b0} + -2a_{b1} + -4a_{b2} + -8a_{b3} \geq -9 \end{aligned}$$

This doesn't propagate much, but that isn't a problem for proof logging.

# Compiling CP Variables

We can mix binary and an order encoding. Where needed, define:

$$a_{\geq 4} \Leftrightarrow -16a_{\text{neg}} + 1a_{b0} + 2a_{b1} + 4a_{b2} + 8a_{b3} \geq 4$$

$$a_{\geq 5} \Leftrightarrow -16a_{\text{neg}} + 1a_{b0} + 2a_{b1} + 4a_{b2} + 8a_{b3} \geq 5$$

$$a_4 \Leftrightarrow a_{\geq 4} \wedge \bar{a}_{\geq 5}$$

When creating  $a_{\geq i}$  also introduce pseudo-Boolean constraints encoding

$$a_{\geq i} \Rightarrow a_{\geq j} \quad \text{and} \quad a_{\geq h} \Rightarrow a_{\geq i}$$

for the closest values  $j < i < h$  that already exist.

We can do this:

- Inside the pseudo-Boolean model, where needed.
- Otherwise lazily during proof logging.

# Compiling Constraints

- Also need to compile every constraint to pseudo-Boolean form.
- Doesn't need to be a propagating encoding.
- Can use additional variables.

# Compiling Constraints

Given  $2A + 3B + 4C \geq 42$ , where  $A, B, C \in \{-3 \dots 9\}$ ,

$$\begin{aligned} & -32a_{\text{neg}} + 2a_{b0} + 4a_{b1} + 8a_{b2} + 16a_{b3} \\ & + -48b_{\text{neg}} + 3b_{b0} + 6b_{b1} + 12b_{b2} + 24b_{b3} \\ & + -64c_{\text{neg}} + 4c_{b0} + 8c_{b1} + 16c_{b2} + 32c_{b3} \geq 42 \end{aligned}$$

# Compiling Constraints

Constraints can be specified *extensionally* as list of feasible tuples, called a *table*. We have to pick one of the tuples from the table, and give it to the associated variables.

Given a table constraint  $(A, B, C) \in [(1, 2, 3), (1, 3, 4), (2, 2, 5)]$ , define

$$3\bar{t}_0 + a_{=1} + b_{=2} + c_{=3} \geq 3 \quad \text{i.e.} \quad t_0 \Rightarrow (a_{=1} \wedge b_{=2} \wedge c_{=3})$$

$$3\bar{t}_1 + a_{=1} + b_{=4} + c_{=4} \geq 3 \quad \text{i.e.} \quad t_1 \Rightarrow (a_{=1} \wedge b_{=4} \wedge c_{=4})$$

$$3\bar{t}_2 + a_{=2} + b_{=2} + c_{=5} \geq 3 \quad \text{i.e.} \quad t_2 \Rightarrow (a_{=2} \wedge b_{=2} \wedge c_{=5})$$

using a tuple selector variable

$$t_0 + t_1 + t_2 = 1$$

# Encoding Constraint Definitions

We already know how to do it for any constraint that has a sane encoding using some combination of

- CNF,
- Integer linear inequalities,
- Table constraints,
- Auxiliary variables.

Simplicity is important, propagation strength isn't.

# Justifying Search

Mostly this works as in earlier examples.

Restarts are easy.

No need to justify guesses or decisions. We only justify backtracking.

# Justifying Inference

Key idea: anything the constraint programming solver knows must follow from unit propagation of guessed assignments in the proof.

If it follows from unit propagation from the encoding, nothing needed.

Some propagators and encodings need RUP steps for inferences.

- A lot of propagators are effectively “doing a little bit of lookahead” but in an efficient way.

A few need explicit cutting planes justifications.

- Linear inequalities just need to multiply and add.
- All-different needs a bit more.

# Justifying All-Different Failures

$$V \in \{ 1 \quad 4 \quad 5 \}$$

$$W \in \{ 1 \ 2 \ 3 \ }$$

$$X \in \{ \quad 2 \ 3 \ }$$

$$Y \in \{ 1 \quad 3 \ }$$

$$Z \in \{ 1 \quad 3 \ }$$

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# Justifying All-Different Failures

$$V \in \{ 1 \quad 4 \quad 5 \}$$

$$W \in \{ 1 \ 2 \ 3 \} \quad w_{=1} + w_{=2} + w_{=3} \geq 1$$

$$X \in \{ 2 \ 3 \}$$

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$$Z \in \{ 1 \ 3 \}$$

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$$X \in \{ 2 \ 3 \} \quad x_{=2} + x_{=3} \geq 1$$

$$Y \in \{ 1 \ 3 \} \quad y_{=1} + y_{=3} \geq 1$$

$$Z \in \{ 1 \ 3 \} \quad z_{=1} + z_{=3} \geq 1$$

$$\rightarrow -v_{=1} + -w_{=1} + -y_{=1} + -z_{=1} \geq -1$$

$$\rightarrow -w_{=2} + -x_{=2} \geq -1$$

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# Justifying All-Different Failures

$$V \in \{ 1 \quad 4 \quad 5 \}$$

$$W \in \{ 1 \ 2 \ 3 \} \quad w_{=1} + w_{=2} + w_{=3} \geq 1$$

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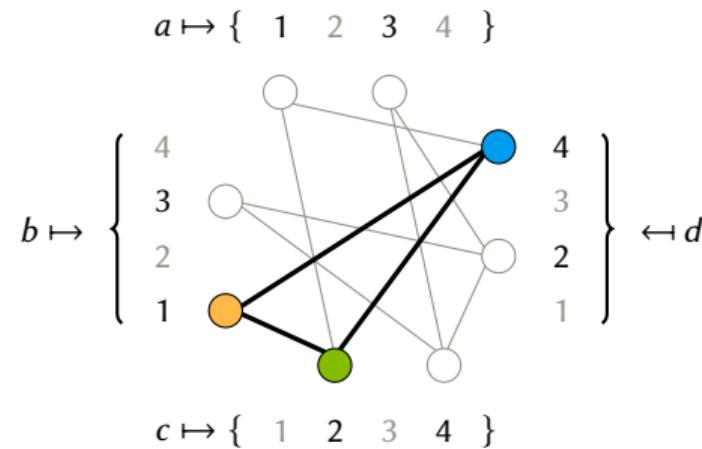
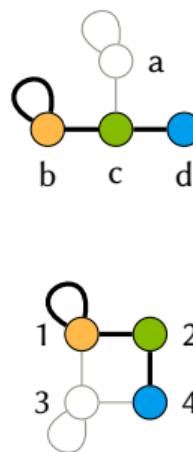
$$0 \geq 1$$

## Reformulation

Auto-tabulation is possible.

- Heavy use of extension variables.

Can re-encode maximum common subgraph as a clique problem, without changing the pseudo-Boolean model.



# High Level Modelling Languages?

High level modelling languages like MiniZinc and Essence have complicated compilers.

How do we know we're giving a proof for the problem the user actually specified?

Future research...

# Code

<https://github.com/ciaranm/glasgow-constraint-solver>

Released under MIT Licence.

Supports proof logging for global constraints including:

- All-different.
- Integer linear inequality (including for very large domains).
- Smart table and regular.
- Minimum / maximum of an array.
- Element.
- Absolute value.
- Circuit.

Details in [GMN22, MM23].

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Sometimes weaker criterion needed. Recall: we wanted new, fresh variable  $a$  encoding

$$a \Leftrightarrow (x \wedge y)$$

Introduce pseudo-Boolean constraints

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More generally: **without-loss-of-generality** argument.

Now: rules that derive (possibly) non-implied constraints

# Redundance-Based Strengthening

$C$  is redundant with respect to  $F$  if  $F$  and  $F \wedge C$  are equisatisfiable

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$C$  is redundant with respect to  $F$  iff there is a substitution  $\omega$  (mapping variables to truth values or literals), called a witness, for which

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Witness  $\omega$  should be specified, and implication be efficiently verifiable (which is the case, e.g., if all constraints in  $(F \wedge C) \upharpoonright_\omega$  are RUP)

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Choose binary encoding of two integers in  $[0, 15]$   
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$$\begin{aligned}1 \cdot x_0 + 2 \cdot x_1 + 4 \cdot x_2 + 8 \cdot x_3 \\+ 1 \cdot y_0 + 2 \cdot y_1 + 4 \cdot y_2 + 8 \cdot y_3 = 25 \\x_0 = y_0\end{aligned}$$

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To derive without loss of generality  $x \leq y$   
(argument: we can always swap them)

$$1 \cdot x_0 + 2 \cdot x_1 + 4 \cdot x_2 + 8 \cdot x_3 \leq 1 \cdot y_0 + 2 \cdot y_1 + 4 \cdot y_2 + 8 \cdot y_3$$

pseudo-Boolean proof version 2.0

```
f 4
red 1 y0 2 y1 4 y2 8 y3
→ -1 x0 -2 x1 -4 x2 -8 x3 >= 0 ;
→ y0 → x0 x0 → y0 y1 → x1 x1 → y1
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Why does this work? Need to show

$$F \wedge \neg C \models (F \wedge C) \upharpoonright_{\omega}$$

- $F \upharpoonright_{\omega}$  equals  $F$  (swaps last two constraints)
- $C \upharpoonright_{\omega}$  says  $y \leq x$  while  $\neg C$  says  $y < x$

# Deriving $a \Leftrightarrow (x \wedge y)$ Using the Redundance Rule

Want to derive

$$a + \bar{x} + \bar{y} \geq 1 \quad 2\bar{a} + x + y \geq 2$$

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Choose  $\omega = \{a \mapsto 0\}$  —  $F$  untouched; new constraint satisfied

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2  $F \wedge (2\bar{a} + x + y \geq 2) \wedge \neg(a + \bar{x} + \bar{y} \geq 1) \models (F \wedge (2\bar{a} + x + y \geq 2) \wedge (a + \bar{x} + \bar{y} \geq 1)) \upharpoonright_{\omega}$

Choose  $\omega = \{a \mapsto 1\}$  —  $F$  untouched; new constraint satisfied

$\neg(a + \bar{x} + \bar{y} \geq 1)$  forces  $x \mapsto 1$  and  $y \mapsto 1$ , hence  $2\bar{a} + x + y \geq 2$  remains satisfied after forcing  $a$  to be true

# Redundance and Dominance Rules for Optimisation

## Redundance-based strengthening, optimisation version

Add constraint  $C$  to formula  $F$  if exists witness substitution  $\omega$  s.t.

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- Applying  $\omega$  should **strictly decrease**  $f$ .
- If so, don't need to show that  $C \upharpoonright_{\omega}$  holds!

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- 7 ...
- 8 Can't go on forever, so finally reach  $\alpha'$  satisfying  $F \wedge C$ .

# Strength of Dominance Rule

Dominance-based strengthening (stronger, still simplified)

If  $C_1, C_2, \dots, C_{m-1}$  have been derived from  $F$  (maybe using dominance), then can derive  $C_m$  if exists witness substitution  $\omega$  s.t.

$$F \wedge \bigwedge_{i=1}^{m-1} C_i \wedge \neg C_m \models F \upharpoonright \omega \wedge f \upharpoonright \omega < f$$

Only consider  $F$  – no need to show that any  $C_i \upharpoonright \omega$  implied!

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Further extensions:

- Define dominance rule w.r.t. order independent of objective.
- Switch between different orders in same proof.
- See [BGMN23] for details.

# Using the Dominance rule for Symmetry Handling

Dominance rule **very powerful**; can be used for symmetry and dominance breaking

# Using the Dominance rule for Symmetry Handling

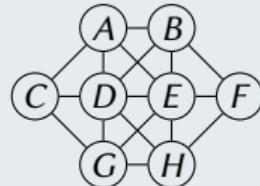
Dominance rule **very powerful**; can be used for symmetry and dominance breaking

## Examples

- Symmetries in CP (manual symmetry breaking)
- Symmetries in clique solving (automatic dominance breaking during search)
- Symmetries in SAT solving (automatic symmetry breaking in preprocessing)

# Symmetry Elimination (CP)

## The Crystal Maze Puzzle



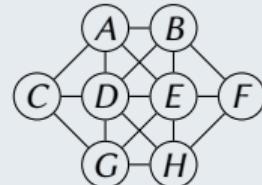
Place numbers 1 to 8 without repetition, adjacent circles cannot have consecutive numbers.

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Human modellers might add:

- $A < G$  (mirror vertically)
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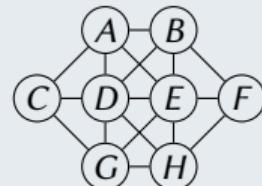
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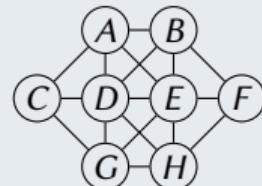
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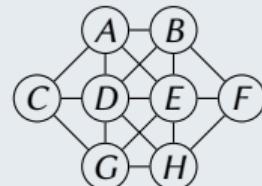
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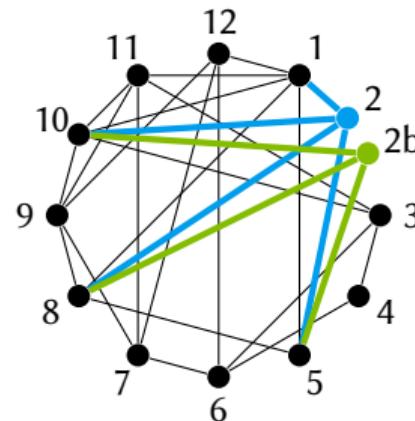
**Research challenge:** a CP toolchain supporting this.

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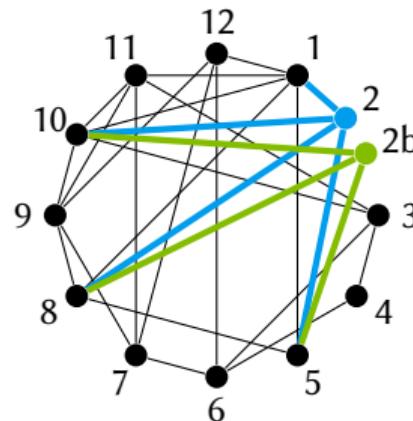
# Lazy Global Domination For Maximum Clique [MP16]



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Dominance rule can justify this.

- Even when detected dynamically during search.

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Order: “Pigeon 1 preferred in the smallest hole; next pigeon 2, ...”

$$f \doteq 2^{11} \cdot p_{13} + 2^{10} \cdot p_{12} + 2^9 \cdot p_{11} + 2^8 \cdot p_{23} + \dots + 1 \cdot p_{41}$$

# Breaking a Single Simple Symmetry (Example)

- $F$  is a formula expressing PHP constraints with  $F \upharpoonright_{\sigma_{(12)}} = F$
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Similar to DRAT symmetry breaking [HHW15]

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Can “restore” its truth by applying  $\sigma_{(1234)}$  **once, twice, or thrice**.

# Breaking Symmetries With the Dominance Rule (1/2)

## Definition

Given a symmetry  $\sigma$ , the (pseudo-Boolean) breaking constraint of  $\sigma$  is

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## Theorem

$C_\sigma$  can be derived from  $F$  using dominance **with witness**  $\sigma$

$$F \wedge \neg C_\sigma \models F \upharpoonright_\sigma \wedge f \upharpoonright_\sigma < f$$

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Why does it work?

- Witness need not satisfy all derived constraints
- Sufficient to just produce “better” assignment

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If  $y_{k-1}$  is true,  $x_k$  is at most  $\sigma(x_k)$

(derivable from the PB breaking constraint)

# Back to our pigeons

Start the proof and load  
input formula

```
pseudo-Boolean proof version 2.0
f 22
pre_order exp
  vars
    left u1 u2 u3 u4 u5 u6 u7 u8 u9 u10 u11 u12
    right v1 v2 v3 v4 v5 v6 v7 v8 v9 v10 v11 v12
    aux
  end
  def
    -1 u12 1 v12 -2 u11 2 v11 [...] -1024 u2 1024 v2 -2048 u1 2048 v1 >= 0;
  end
  transitivity
  vars
    fresh_right w1 w2 w3 w4 w5 w6 w7 w8 w9 w10 w11 w12
  end
  proof
    proofgoal #1
    pol 1 2 + 3 +
    qed -1
  qed
  end
end
load_order exp p13 p12 p11 p23 p22 p21 p31 p32 p33 p41 p42 p43
```

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Start the proof and load input formula

- 1 Pretend to solve optimisation problem

minimizing  
 $f \doteq 2^{11} \cdot p_{13} + 2^{10} \cdot p_{12} + 2^9 \cdot p_{11} + 2^8 \cdot p_{23} + \dots + 1 \cdot p_{41}$

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Derived constraints ( $D$ ):

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23 >= 0 ; p11 -> p21 p12 -> p22 p13 -> p23 p21 -> p22
proofgoal #2
  pol -1 -2 +
  qed -1
end
red 1 y0 >= 1 ; y0 -> 1
rup 1 ~y0 1 ~p13 1 p23 >= 1 ;
red 1 ~y1 1 y0 >= 1 ; y1 -> 0
red 1 ~y1 1 ~p23 1 p13 >= 1 ; y1 -> 0
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pol 26 32 2048 * +
del id 26
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```

Pseudo-Boolean breaking constraint

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Pseudo-Boolean breaking constraint

Use dominance with witness  $\sigma = (p_{11}p_{21})(p_{12}p_{22})(p_{13}p_{23})$

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red 1 ~y1 1 y0 >= 1 ; y1 -> 0
red 1 ~y1 1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1 p23 1 ~y0 1 y1 >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0 1 y1 >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1 1 ~p12 1 p22 >= 1 ;
```

Pseudo-Boolean breaking constraint

Use dominance with witness  $\sigma = (p_{11}p_{21})(p_{12}p_{22})(p_{13}p_{23})$

$$F \wedge \neg C \models F \upharpoonright_{\omega} \wedge f \upharpoonright_{\omega} < f$$

Everything autoproven except that  $\neg C \wedge C \models \perp$ .

# Back to our pigeons – Deriving the Constraints

Derived constraints ( $D$ ):

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 p12 -> p22 p13 -> p23 p21 -> p20
proofgoal #2
  pol -1 -2 +
  qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0 1 ~p13 1 p23 >= 1 ;
red 1 ~y1 1 y0  >= 1 ; y1 -> 0
red 1 ~y1 1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1 p23 1 ~y0 1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0 1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1 1 ~p12 1 p22 >= 1 ;
```

Derivable by **redundance** with witness  $\omega : y_0 \mapsto 1$

$$F \wedge D \wedge \{\bar{y}_0 \geq 1\} \models (F \wedge D) \upharpoonright_\omega \wedge \{y_0 \geq 1\} \upharpoonright_\omega$$

# Back to our pigeons – Deriving the Constraints

**Derived constraints ( $D$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 p12 -> p22 p13 -> p23 p21 -> p22
proofgoal #2
  pol -1 -2 +
  qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0 1 ~p13 1 p23 >= 1 ;
red 1 ~y1 1 y0  >= 1 ; y1 -> 0
red 1 ~y1 1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1 p23 1 ~y0 1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0 1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1 1 ~p12 1 p22 >= 1 ;
```

Derivable by **redundance** with witness  $\omega : y_0 \mapsto 1$

$$F \wedge D \wedge \{\bar{y}_0 \geq 1\} \models (F \wedge D) \upharpoonright_\omega \wedge \{y_0 \geq 1\} \upharpoonright_\omega$$

$$F \wedge D \wedge \{\bar{y}_0 \geq 1\} \models (F \wedge D) \upharpoonright_\omega \wedge \{1 \geq 1\}$$

# Back to our pigeons – Deriving the Constraints

**Derived constraints ( $D$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 p12 -> p22 p13 -> p23 p21 -> p20
proofgoal #2
  pol -1 -2 +
  qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0 1 ~p13 1 p23 >= 1 ;
red 1 ~y1 1 y0  >= 1 ; y1 -> 0
red 1 ~y1 1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1 p23 1 ~y0 1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0 1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1 1 ~p12 1 p22 >= 1 ;
```

Derivable by RUP

$$F \wedge D \wedge \neg(\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1)$$

# Back to our pigeons – Deriving the Constraints

**Derived constraints ( $D$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 p12 -> p22 p13 -> p23 p21 -> p22
proofgoal #2
  pol -1 -2 +
  qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0 1 ~p13 1 p23 >= 1 ;
red 1 ~y1 1 y0  >= 1 ; y1 -> 0
red 1 ~y1 1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1 p23 1 ~y0 1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0 1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1 1 ~p12 1 p22 >= 1 ;
```

Derivable by RUP

$$F \wedge D \wedge \neg(\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1) = F \wedge D \wedge y_0 \geq 1 \wedge p_{13} \geq 1 \wedge \bar{p}_{23} \geq 1$$

# Back to our pigeons – Deriving the Constraints

**Derived constraints ( $D$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 p12 -> p22 p13 -> p23 p21 -> p24
proofgoal #2
  pol -1 -2 +
  qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0 1 ~p13 1 p23 >= 1 ;
red 1 ~y1 1 y0  >= 1 ; y1 -> 0
red 1 ~y1 1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1 p23 1 ~y0 1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0 1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1 1 ~p12 1 p22 >= 1 ;
```

Derivable by RUP

$$F \wedge D \wedge \neg(\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1) = F \wedge D \wedge y_0 \geq 1 \wedge p_{13} \geq 1 \wedge \bar{p}_{23} \geq 1$$

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \dots \geq 0$$

# Back to our pigeons – Deriving the Constraints

**Derived constraints ( $D$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 p12 -> p22 p13 -> p23 p21 -> p24
proofgoal #2
  pol -1 -2 +
  qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0 1 ~p13 1 p23 >= 1 ;
red 1 ~y1 1 y0  >= 1 ; y1 -> 0
red 1 ~y1 1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1 p23 1 ~y0 1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0 1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1 1 ~p12 1 p22 >= 1 ;
```

Derivable by RUP

$$F \wedge D \wedge \neg(\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1) = F \wedge D \wedge y_0 \geq 1 \wedge p_{13} \geq 1 \wedge \bar{p}_{23} \geq 1$$

$$2^{11} \cdot (-1) + 2^{10} \cdot (p_{22} - p_{12}) + \dots \geq 0$$

# Back to our pigeons – Deriving the Constraints

**Derived constraints ( $D$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 p12 -> p22 p13 -> p23 p21 -> p24
proofgoal #2
  pol -1 -2 +
  qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0 1 ~p13 1 p23 >= 1 ;
red 1 ~y1 1 y0  >= 1 ; y1 -> 0
red 1 ~y1 1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1 p23 1 ~y0 1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0 1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1 1 ~p12 1 p22 >= 1 ;
```

Derivable by RUP

$$F \wedge D \wedge \neg(\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1) = F \wedge D \wedge y_0 \geq 1 \wedge p_{13} \geq 1 \wedge \bar{p}_{23} \geq 1$$

$$2^{11} \cdot (-1) + 2^{10} \cdot (p_{22} - p_{12}) + \dots \geq 0$$

$$\text{with } \sum_{i=1}^{10} 2^i \leq 2^{11} - 1$$

# Back to our pigeons – Deriving the Constraints

**Derived constraints ( $D$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\bar{y}_1 + y_0 \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 p12 -> p22 p13 -> p23 p21 -> p22
proofgoal #2
  pol -1 -2 +
  qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0 1 ~p13 1 p23 >= 1 ;
red 1 ~y1 1 y0  >= 1 ; y1 -> 0
red 1 ~y1 1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1 p23 1 ~y0 1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0 1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1 1 ~p12 1 p22 >= 1 ;
```

Derivable by **redundance** with witness  $\omega : y_1 \mapsto 0$

$$F \wedge D \wedge \neg(\bar{y}_1 + y_0 \geq 1)$$

$$\models (F \wedge D) \upharpoonright_{\omega} \wedge \{\bar{y}_1 + y_0 \geq 1\} \upharpoonright_{\omega}$$

# Back to our pigeons – Deriving the Constraints

**Derived constraints ( $D$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\bar{y}_1 + y_0 \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 p12 -> p22 p13 -> p23 p21 -> p22
proofgoal #2
  pol -1 -2 +
  qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0 1 ~p13 1 p23 >= 1 ;
red 1 ~y1 1 y0  >= 1 ; y1 -> 0
red 1 ~y1 1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1 p23 1 ~y0 1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0 1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1 1 ~p12 1 p22 >= 1 ;
```

Derivable by **redundance** with witness  $\omega : y_1 \mapsto 0$

$$F \wedge D \wedge \neg(\bar{y}_1 + y_0 \geq 1)$$

$$\models (F \wedge D) \upharpoonright_\omega \wedge \{\bar{y}_1 + y_0 \geq 1\} \upharpoonright_\omega$$

$$F \wedge D \wedge \neg(\bar{y}_1 + y_0 \geq 1)$$

$$\models (F \wedge D) \upharpoonright_\omega \wedge \{1 + y_0 \geq 1\}$$

# Back to our pigeons – Deriving the Constraints

Derived constraints ( $D$ ):

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\bar{y}_1 + y_0 \geq 1$$

$$\bar{y}_1 + \overline{\sigma(p_{13})} + p_{13} \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 p12 -> p22 p13 -> p23 p21 -> p20
proofgoal #2
  pol -1 -2 +
  qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0 1 ~p13 1 p23 >= 1 ;
red 1 ~y1 1 y0  >= 1 ; y1 -> 0
red 1 ~y1 1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1 p23 1 ~y0 1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0 1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1 1 ~p12 1 p22 >= 1 ;
```

Derivable by **redundance** with witness  $\omega : y_1 \mapsto 0$   
(same argument)

# Back to our pigeons – Deriving the Constraints

**Derived constraints ( $D$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\bar{y}_1 + y_0 \geq 1$$

$$\bar{y}_1 + \overline{\sigma(p_{13})} + p_{13} \geq 1$$

$$y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 p12 -> p22 p13 -> p23 p21 -> p22
proofgoal #2
  pol -1 -2 +
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end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0 1 ~p13 1 p23 >= 1 ;
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red 1 ~y1 1 ~p23 1 p13 >= 1 ; y1 -> 0
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red 1 ~p13 1 ~y0 1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1 1 ~p12 1 p22 >= 1 ;
```

Derivable by **redundance** with witness  $\omega : y_1 \mapsto 1$

$$F \wedge D \wedge \neg(y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1)$$

$$\models (F \wedge D) \upharpoonright_{\omega} \wedge \{y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1\} \upharpoonright_{\omega}$$

# Back to our pigeons – Deriving the Constraints

**Derived constraints ( $D$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\bar{y}_1 + y_0 \geq 1$$

$$\bar{y}_1 + \overline{\sigma(p_{13})} + p_{13} \geq 1$$

$$y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1$$

```

dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 p12 -> p22 p13 -> p23 p21 -> p
proofgoal #2
  pol -1 -2 +
  qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0 1 ~p13 1 p23 >= 1 ;
red 1 ~y1 1 y0  >= 1 ; y1 -> 0
red 1 ~y1 1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1 p23 1 ~y0 1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0 1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1 1 ~p12 1 p22 >= 1 ;

```

Derivable by **redundance** with witness  $\omega : y_1 \mapsto 1$

$$F \wedge D \wedge \neg(y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1)$$

$$\models (F \wedge D) \upharpoonright_{\omega} \wedge \{y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1\} \upharpoonright_{\omega}$$

$$F \wedge D \wedge \{\bar{y}_1 \geq 1 \wedge y_0 \geq 1 \wedge p_{13} \geq 1\}$$

$$\models \dots \wedge D \upharpoonright_{\omega} \wedge \dots$$

# Back to our pigeons – Deriving the Constraints

**Derived constraints ( $D$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\bar{y}_1 + y_0 \geq 1$$

$$\bar{y}_1 + \overline{\sigma(p_{13})} + p_{13} \geq 1$$

$$y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1$$

```

dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 p12 -> p22 p13 -> p23 p21 -> p
proofgoal #2
  pol -1 -2 +
  qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0 1 ~p13 1 p23 >= 1 ;
red 1 ~y1 1 y0  >= 1 ; y1 -> 0
red 1 ~y1 1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1 p23 1 ~y0 1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0 1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1 1 ~p12 1 p22 >= 1 ;

```

Derivable by **redundance** with witness  $\omega : y_1 \mapsto 1$

$$F \wedge D \wedge \neg(y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1)$$

$$\models (F \wedge D) \upharpoonright_{\omega} \wedge \{y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1\} \upharpoonright_{\omega}$$

$$F \wedge D \wedge \{\bar{y}_1 \geq 1 \wedge y_0 \geq 1 \wedge p_{13} \geq 1\}$$

$$\models \dots \wedge D \upharpoonright_{\omega} \wedge \dots$$

# Back to our pigeons – Deriving the Constraints

**Derived constraints ( $D$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\bar{y}_1 + y_0 \geq 1$$

$$\bar{y}_1 + \overline{\sigma(p_{13})} + p_{13} \geq 1$$

$$y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1$$

```

dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 p12 -> p22 p13 -> p23 p21 -> p
proofgoal #2
  pol -1 -2 +
  qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0 1 ~p13 1 p23 >= 1 ;
red 1 ~y1 1 y0  >= 1 ; y1 -> 0
red 1 ~y1 1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1 p23 1 ~y0 1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0 1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1 1 ~p12 1 p22 >= 1 ;

```

Derivable by **redundance** with witness  $\omega : y_1 \mapsto 1$

$$F \wedge D \wedge \neg(y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1)$$

$$\models (F \wedge D) \upharpoonright_{\omega} \wedge \{y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1\} \upharpoonright_{\omega}$$

$$F \wedge D \wedge \{\bar{y}_1 \geq 1 \wedge y_0 \geq 1 \wedge p_{13} \geq 1\}$$

$$\models \dots \wedge D \upharpoonright_{\omega} \wedge \dots$$

# Back to our pigeons – Deriving the Constraints

Derived constraints ( $D$ ):

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

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$$\bar{y}_1 + \overline{\sigma(p_{13})} + p_{13} \geq 1$$

$$y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1$$

$$y_1 + \bar{y}_0 + \sigma(p_{13}) \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 p12 -> p22 p13 -> p23 p21 -> p20
proofgoal #2
  pol -1 -2 +
  qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0 1 ~p13 1 p23 >= 1 ;
red 1 ~y1 1 y0  >= 1 ; y1 -> 0
red 1 ~y1 1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1 p23 1 ~y0 1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0 1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1 1 ~p12 1 p22 >= 1 ;
```

Derivable by **redundance** with witness  $\omega : y_1 \mapsto 1$   
(same argument)

# Back to our pigeons – Deriving the Constraints

Derived constraints ( $D$ ):

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\bar{y}_1 + y_0 \geq 1$$

$$\bar{y}_1 + \overline{\sigma(p_{13})} + p_{13} \geq 1$$

$$y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1$$

$$y_1 + \bar{y}_0 + \sigma(p_{13}) \geq 1$$

$$2^{11}\bar{y}_1 + 2^{10} \cdot (p_{22} - p_{12}) \dots \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 p12 -> p22 p13 -> p23 p21 -> p22
proofgoal #2
  pol -1 -2 +
  qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0 1 ~p13 1 p23 >= 1 ;
red 1 ~y1 1 y0  >= 1 ; y1 -> 0
red 1 ~y1 1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1 p23 1 ~y0 1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0 1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1 1 ~p12 1 p22 >= 1 ;
```

Simplify the PB breaking constraint and get rid of original

# Back to our pigeons – Deriving the Constraints

**Derived constraints ( $D$ ):**

$$\begin{aligned} 2^{11} \cdot (p_{23} - p_{13}) + \\ 2^{10} \cdot (p_{22} - p_{12}) + \\ \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\bar{y}_1 + y_0 \geq 1$$

$$\bar{y}_1 + \overline{\sigma(p_{13})} + p_{13} \geq 1$$

$$y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1$$

$$y_1 + \bar{y}_0 + \sigma(p_{13}) \geq 1$$

$$2^{11}\bar{y}_1 + 2^{10} \cdot (p_{22} - p_{12}) \dots \geq 1$$

$$\bar{y}_1 + \bar{p}_{12} + \sigma(p_{22}) \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23 >= 0 ; p11 -> p21 p12 -> p22 p13 -> p23 p21 -> p22
proofgoal #2
pol -1 -2 +
qed -1
end
red 1 y0 >= 1 ; y0 -> 1
rup 1 ~y0 1 ~p13 1 p23 >= 1 ;
red 1 ~y1 1 y0 >= 1 ; y1 -> 0
red 1 ~y1 1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1 p23 1 ~y0 1 y1 >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0 1 y1 >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1 1 ~p12 1 p22 >= 1 ;
```

Same story with next variable

...

# Future Research Directions

## Performance and reliability of pseudo-Boolean proof logging

- Trim proof while verifying (as in *DRAT-Trim* [HHW13a])
- Compress proof file using binary format
- Design formally verified proof checker (*work in progress* [BMM<sup>+</sup>23])

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## Proof logging for other combinatorial problems and techniques

- Symmetric learning and recycling (substitution) of subproofs
- Mixed integer linear programming (*work on SCIP* in [CGS17, EG21])
- Satisfiability modulo theories (SMT) solving (*work by Bjørner and others*)
- High-level modelling languages

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## And more...

- Use proof logs for algorithm analysis or explainability purposes
- Lots of other challenging problems and interesting ideas

# Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity

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Come talk to us. We're hiring and open to collaboration!

The end. Or rather, the beginning!

# The VERIPB Format and Tool

<https://gitlab.com/MIA0research/software/VeriPB>



Released under MIT Licence

Various features to help development:

- Extended variable name syntax allowing human-readable names
- Proof tracing
- “Trust me” assertions for incremental proof logging

Documentation:

- Description of VERIPB checker [BMM<sup>+</sup>23] used in SAT 2023 competition (<https://satcompetition.github.io/2023/checkers.html>)
- Specific details on different proof logging techniques covered in research papers [EGMN20, GMN20, GMM<sup>+</sup>20, GN21, GMN22, GMNO22, VDB22, BBN<sup>+</sup>23, BGPN23, MM23]
- Lots of concrete example files at <https://gitlab.com/MIA0research/software/VeriPB>

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<https://gitlab.com/MIA0research/software/VeriPB>

<https://github.com/ciaranm/glasgow-constraint-solver>

<https://github.com/ciaranm/glasgow-subgraph-solver>

<https://bitbucket.org/krr/breakid>



# Parity Reasoning: Experiments

Implemented parity reasoning and PB proof logging engine<sup>2</sup>

Also DRAT proof logging as described in [PR16]

Experiments with MINISAT<sup>3</sup>

Set-up:<sup>4</sup>

- Intel Core i5-1145G7 @2.60GHz × 4
- Memory limit 8GiB
- Disk write speed roughly 200 MiB/s
- Read speed of 2 GiB/s

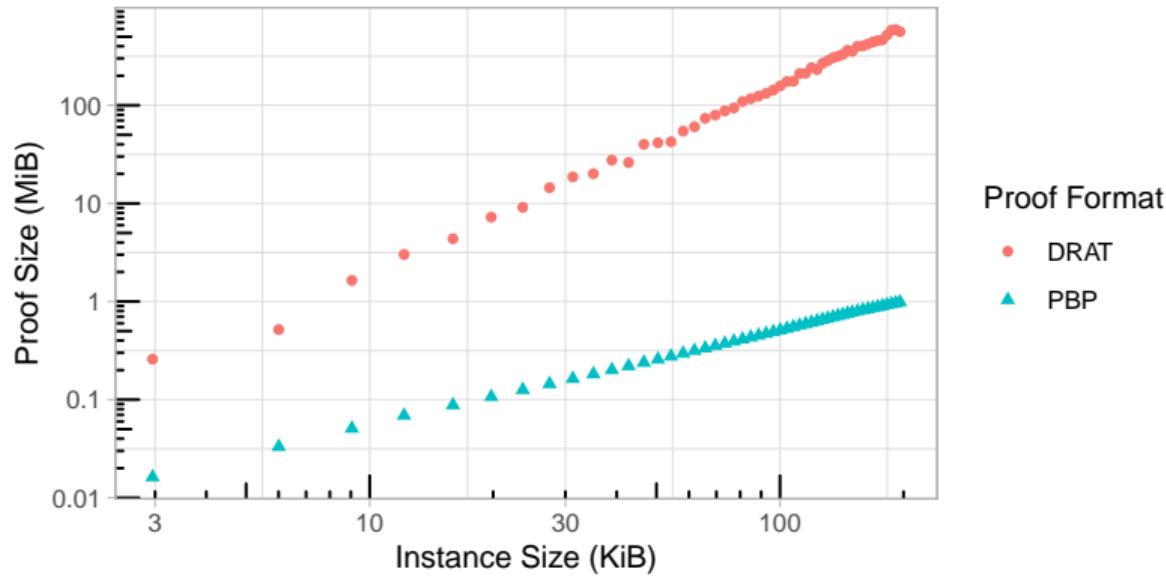
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<sup>2</sup><https://gitlab.com/MIA0research/tools-and-utilities/xorengine>

<sup>3</sup><http://minisat.se/>

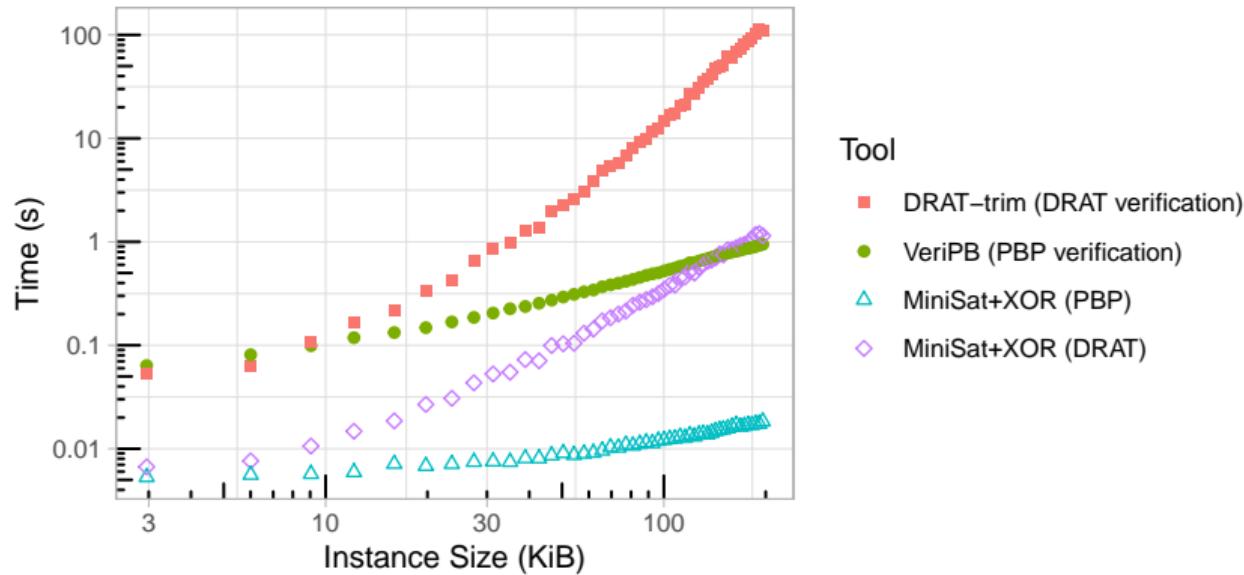
<sup>4</sup>Tools, benchmarks, data and evaluation scripts available at <https://doi.org/10.5281/zenodo.7083485>

# Parity Reasoning: Proof Size



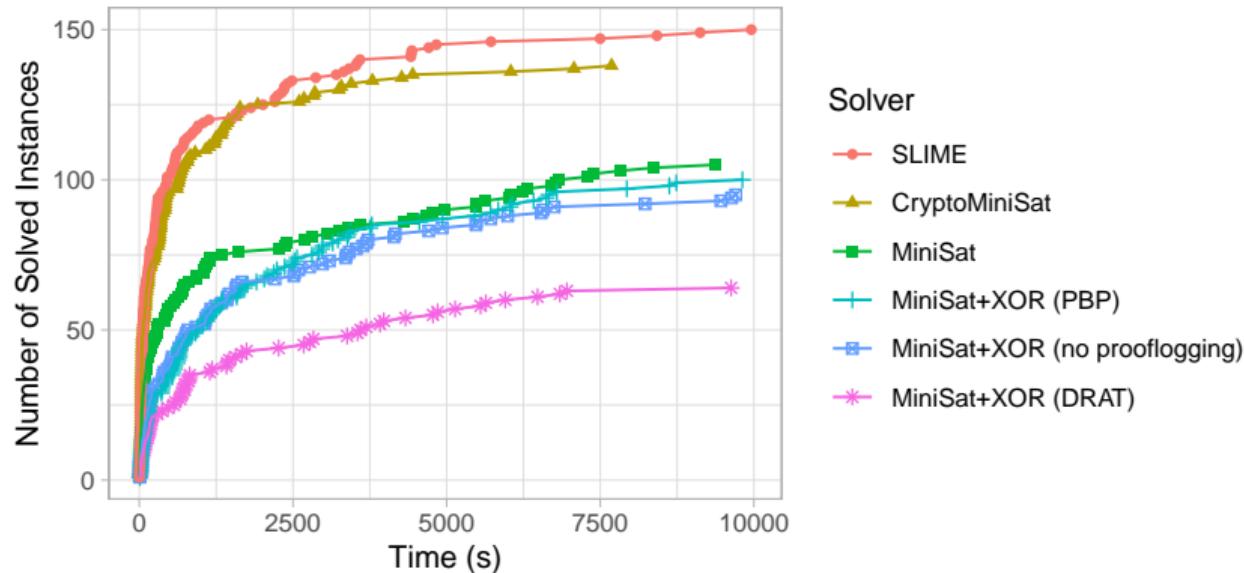
Proof sizes for Tseitin formulas using DRAT and PB proof logging

## Parity Reasoning: Solving and Verification Time



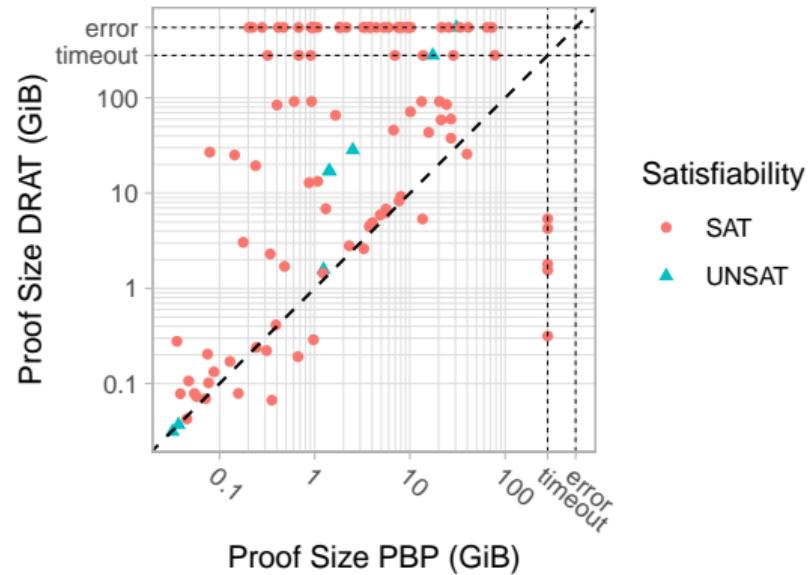
## Solving and verification time for Tseitin formulas

# Parity Reasoning: Crypto Track of SAT 2021 Competition



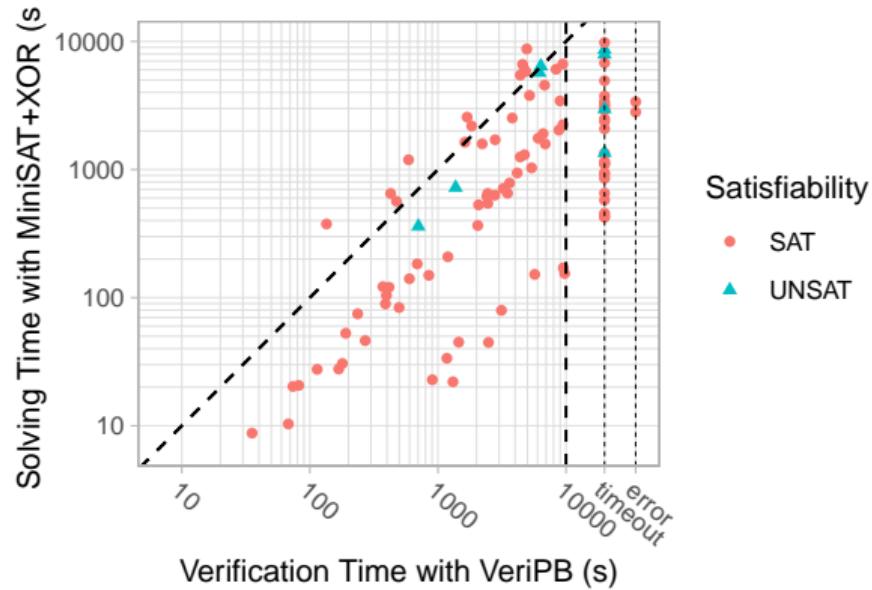
Cumulative plot for the crypto track of the SAT Competition 2021

# Parity Reasoning: Crypto Track Proof Size



DRAT and PB proof sizes for crypto track of SAT Competition 2021

# Parity Reasoning: Crypto Track Verification Time



Time required for solving and verifying crypto instances

# PB-to-CNF Translation: Experiments

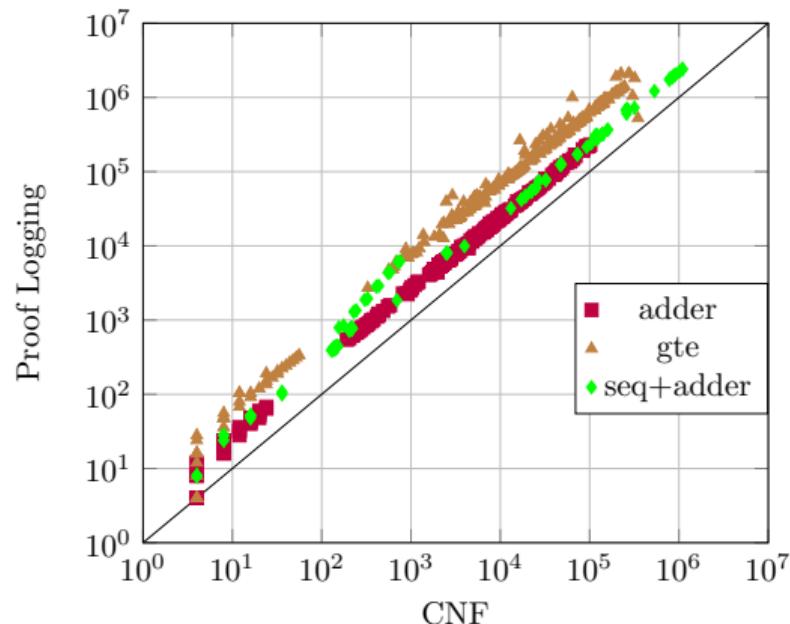
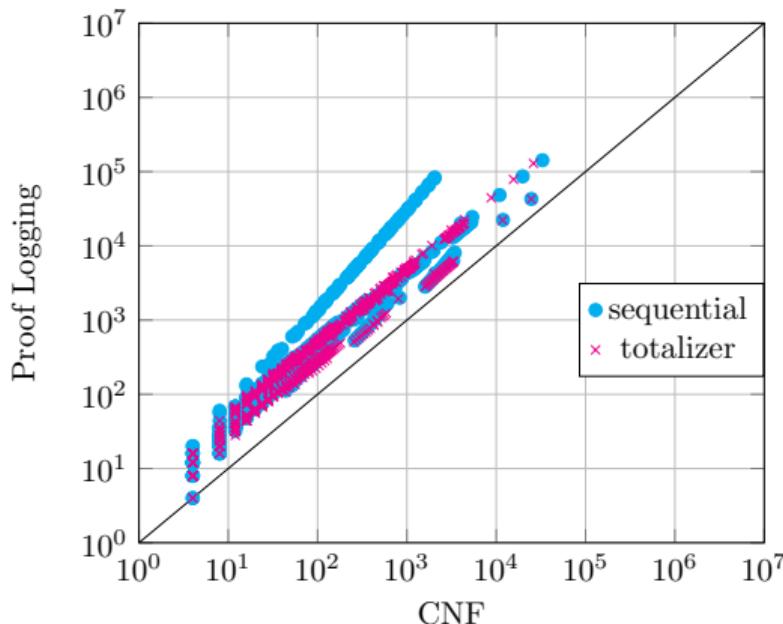
- Certified translations for the following CNF encodings:<sup>5</sup>
  - Sequential counter [Sin05]
  - Totalizer [BB03]
  - Generalized totalizer [JMM15]
  - Adder network [ES06]
- Proof verified by proof checker VERIPB
- Benchmarks from PB 2016 Evaluation:<sup>6</sup>
  - SMALLINT decision benchmarks without purely clausal formulas
  - 3 subclasses of benchmarks:
    - Only cardinality constraints (sequential counter, totalizer)
    - Only general 0-1 ILP constraints (generalized totalizer, adder network)
    - Mixed cardinality & general 0-1 ILP constraints (sequential counter + adder network)

---

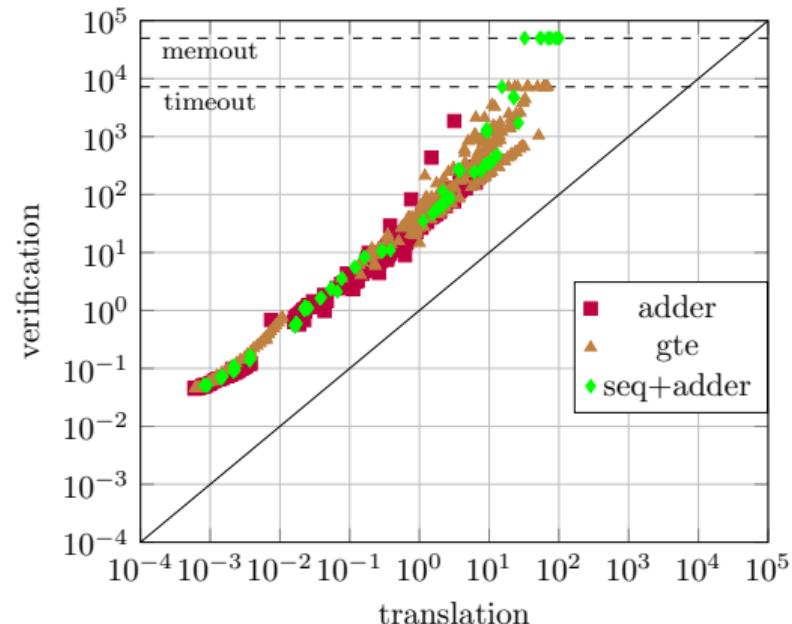
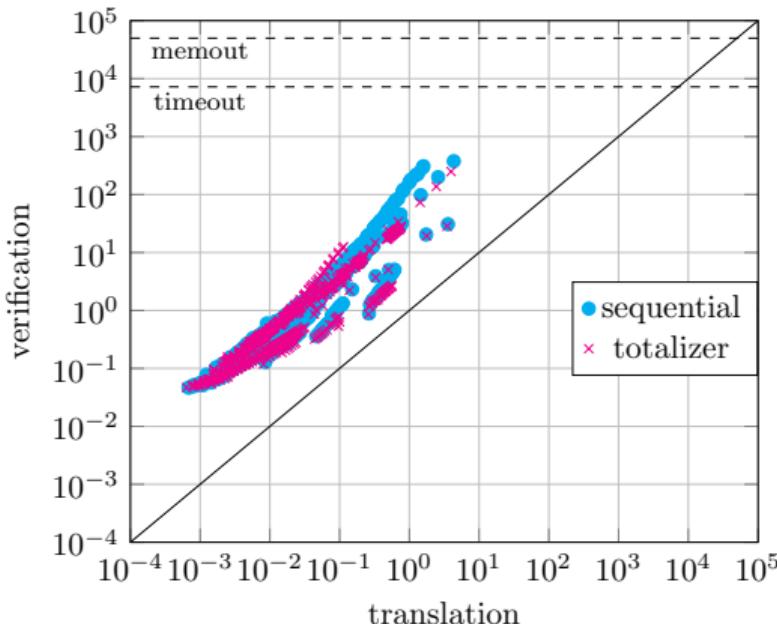
<sup>5</sup><https://github.com/forge-lab/VeritasPBLib>

<sup>6</sup><http://www.cril.univ-artois.fr/PB16/>

# PB-to-CNF: CNF Size vs Proof Size in KiB

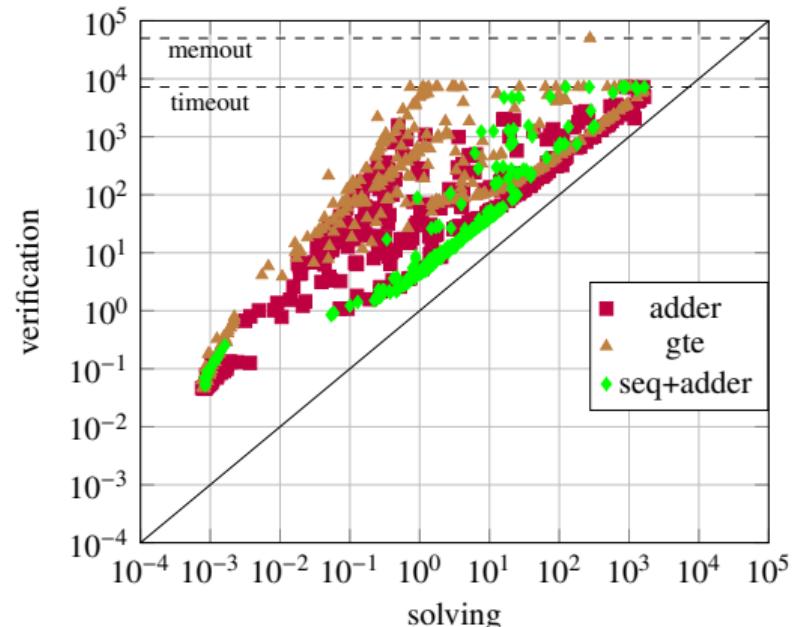
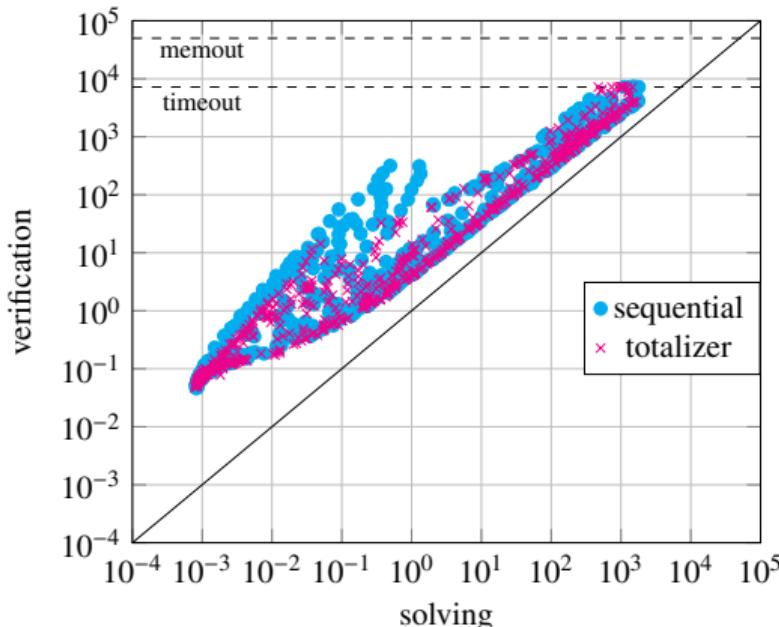


# PB-to-CNF: Translation vs Verification Time in Seconds



- Translation just generates clauses and proof
- Verification slower, as reasoning has to be performed

# PB-to-CNF: Solving Time vs Verification Time in Seconds



- Solved with fork of Kissat<sup>7</sup> syntactically modified to output pseudo-Boolean proofs
- Room for improvement, but clearly shows approach is viable

# PB-to-CNF: Future Work

Improving performance:

- Cutting Planes derivations instead of reverse unit propagations [VDB22]
- Backwards checking/trimming for verification (as in DRAT-trim [HHW13a])

# PB-to-CNF: Future Work

Improving performance:

- Cutting Planes derivations instead of reverse unit propagations [VDB22]
- Backwards checking/trimming for verification (as in DRAT-trim [HHW13a])

Extend proof logging further:

- Sorting networks like odd-even mergesort, bitonic sorter [Bat68]
- MaxSAT solving

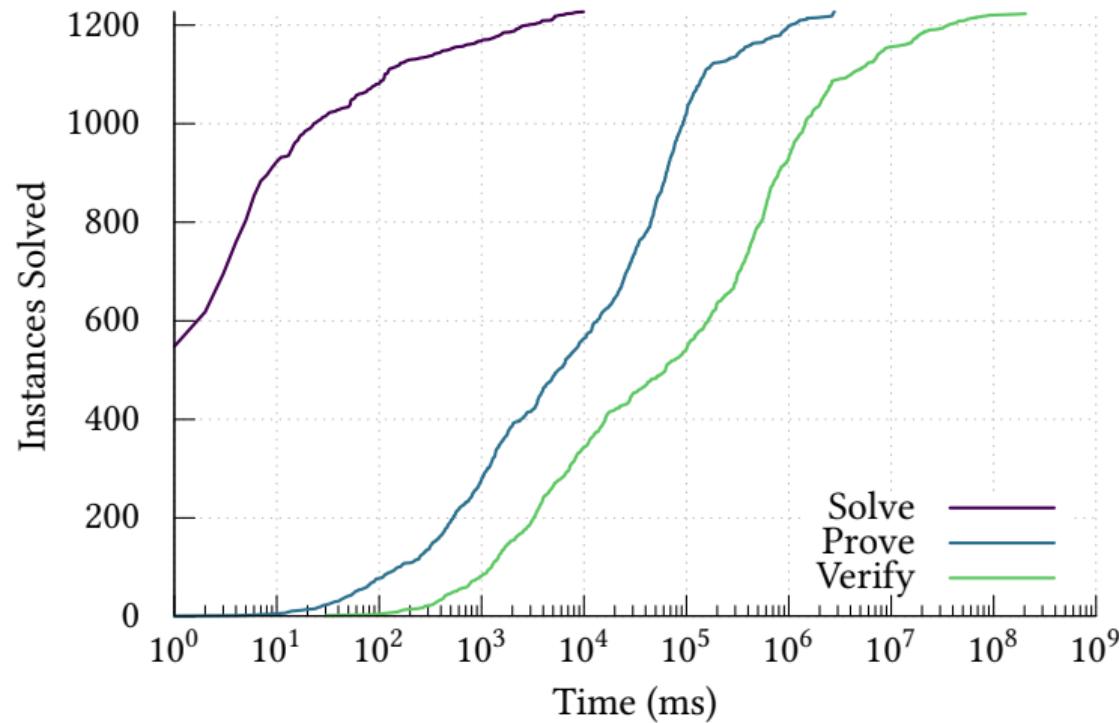
# Clique Results

- Implemented in the Glasgow Subgraph Solver.
  - Bit-parallel, can perform a colouring and recursive call in under a microsecond.
- 59 of the 80 DIMACS instances take under 1,000 seconds to solve without logging.
- Produced and verified proofs for 57 of these 59 instances (the other two reached 1TByte disk space).
- Mean slowdown from proof logging is 80.1 (due to disk I/O).
- Mean verification slowdown a further 10.1.
- Approximate implementation effort: one Masters student.

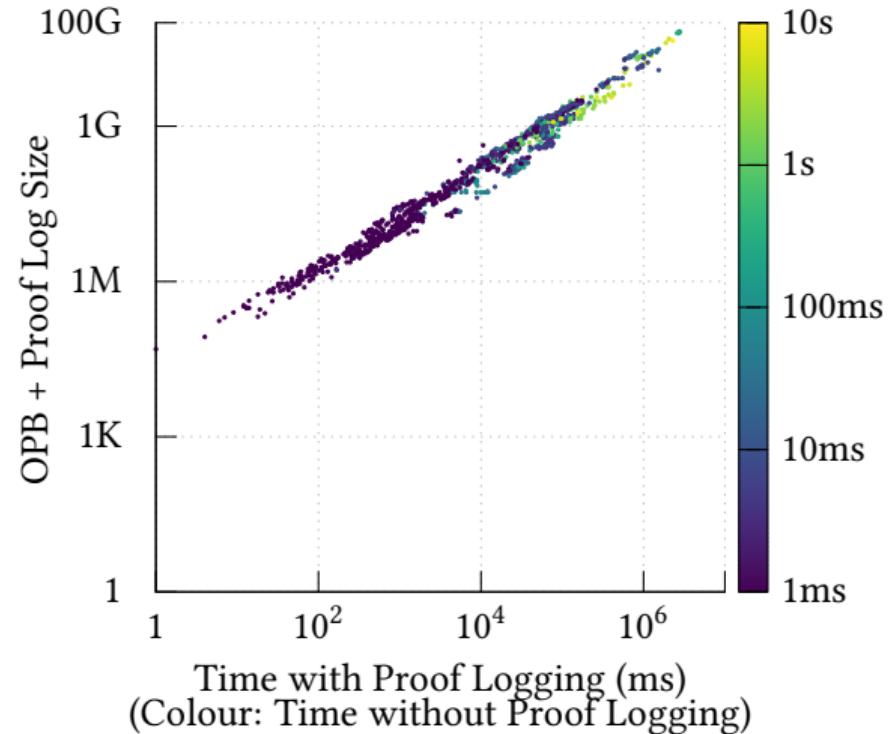
# Subgraph Isomorphism Results

- The Pseudo-Boolean models can be large: had to restrict to instances with no more than 260 vertices in the target graph.
- Took enumeration instances which could be solved without proof logging in under ten seconds.
- 1,227 instances from Solnon's benchmark collection:
  - 789 unsatisfiable, up to 50,635,140 solutions in the rest.
  - 498 instances solved without guessing.
  - Hardest solved satisfiable and unsatisfiable instances required 53,605,482 and 2,074,386 recursive calls.

# Subgraph Isomorphism Results



# Subgraph Isomorphism Results



# How Expensive is Proof Logging?

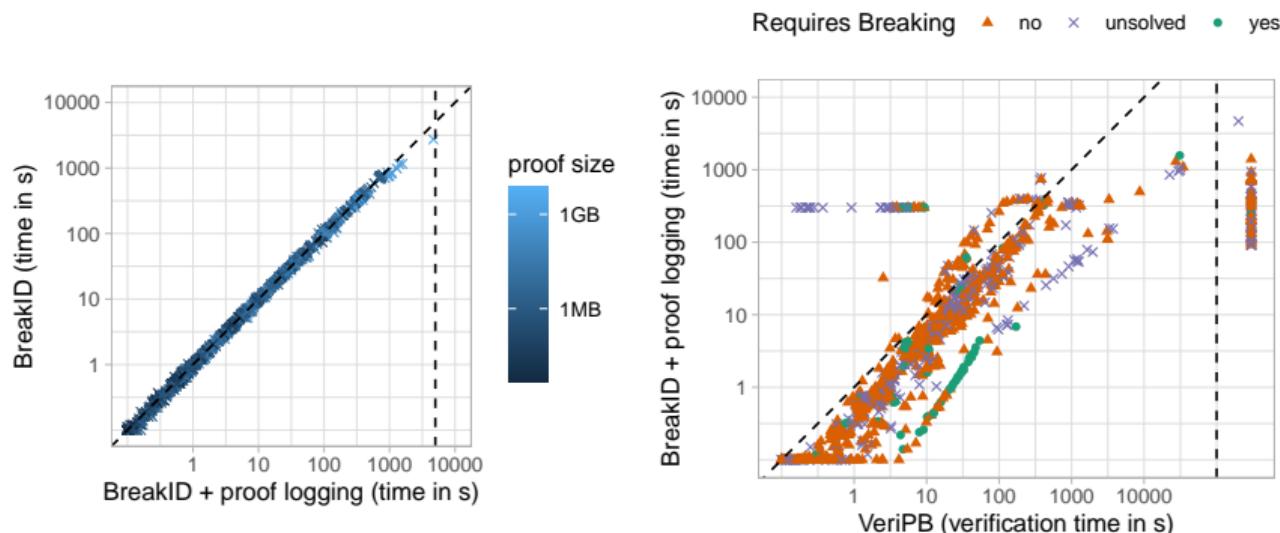
- Laurent D. Michel, Pierre Schaus, Pascal Van Hentenryck: MiniCP: a lightweight solver for constraint programming. *Math. Program. Comput.* 13(1) (2021).
- Five benchmark problems allowing comparison of solvers “doing the same thing”:
  - Simple models.
  - Fixed search order and well-defined propagation consistency levels.
  - Few global constraints (although we don’t have circuit yet).
- Probably close to the worst case for proof logging performance.
- Also: Crystal Maze and World’s Hardest Sudoku.

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- Our solver: faster than the fastest of MiniCP, OscaR, and Choco.
- Proof logging slowdown: between 8.4 to 61.1.
  - 800,000 to 3,000,000 inferences per second.
  - Proof logs can be hundreds of GBytes.
  - No effort put into making the proof-writing code run fast.
- Verification slowdown: a further 10 to 100.
  - Probably possible to reduce this substantially if we are prepared to put more care into writing proofs.

# Experimental Evaluation of SAT Symmetry Breaking

- Evaluated on SAT competition benchmarks
- BREAKID [DBBD16, Bre] used to find and break symmetries



- proof logging overhead negligible
- verification at most 20 times slower than solving for 95% of instances