LICS 2021 Lecture 11: Boolean Satisfiability (SAT) Solving

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Three Simple Problems. . .

Colouring

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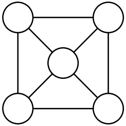

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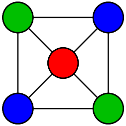

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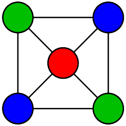

3-colouring? Yes, but no 2-colouring

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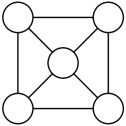

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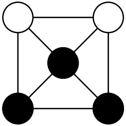

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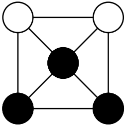

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- Variables should be set to **true** or **false**
- Constraint $(x \lor \neg y \lor z)$: means x or z should be true or y false
- $\land$ means all constraints should hold simultaneously
- Is there a truth value assignment satisfying all constraints?

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COLOURING:	frequency allocation for mobile base stations
CLIQUE:	bioinformatics, computational chemistry
SAT:	easily models these and many other problems

. . . with Huge Practical Implications

- Some more examples of problems that can be encoded as propositional logic formulas:
  - computer hardware verification
  - computer software testing
  - artificial intelligence
  - cryptography
  - bioinformatics
  - et cetera. . .

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- Question mentioned already in Gödel's famous letter in 1956 to von Neumann (the "father of computer science")
- Topic of intense research in computer science ever since 1960s

Solving Logic Formulas in Practice

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  - It's 2022 now — can we go beyond techniques from 1960s?

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. . . And in the process also touch on some of the research being done in the Mathematical Insights into Algorithms for Optimization (MIAO) group

Formal Description of SAT Problem

- Variable x: takes value 1 (**true**) or 0 (**false**)
- Literal ℓ: variable x or its negation $\overline{x}$ (write $\overline{x}$ instead of $\neg x$)
- Clause $C = \ell_1 \vee \cdots \vee \ell_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
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For instance, what about our example formula?

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To understand how large this number is, consider that even if every atom in the known universe was a modern supercomputer that had been running at full speed ever since the beginning of time some 13.7 billion years ago, all of them together would only have covered a completely negligible fraction of these cases by now. So we really would not have time to wait for them to finish...

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- The family of problems for which solutions are easy to check have a name: NP

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- Widely believe to be impossible to solve efficiently on computer in the worst case, but we really don't know

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- Set $x = 0$, simplify F and make recursive call
- Set $x = 1$, simplify F and make recursive call
- If result in both cases "unsatisfiable", then report "unsatisfiable" and return

A DPLL Toy Example

$$\begin{aligned} F = \quad & (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \\ & \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w}) \end{aligned}$$

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Pick variables in internal nodes; terminate in leaves when falsified clause found (i.e., when conflict reached)

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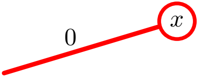

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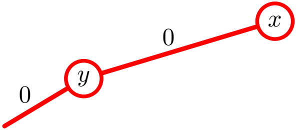

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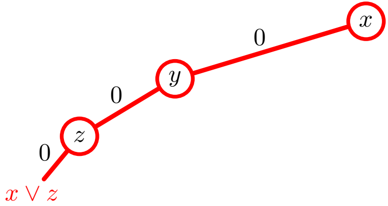

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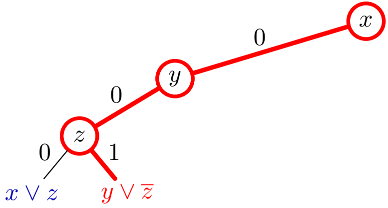

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$$F = (\quad z) \land (y \lor \overline{z}) \land (\quad u) \land (\quad \overline{u}) \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when falsified clause found (i.e., when conflict reached)

"Simplify formula" by (mentally) removing

- satisfied clauses
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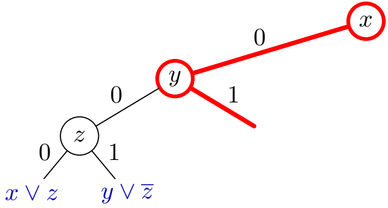

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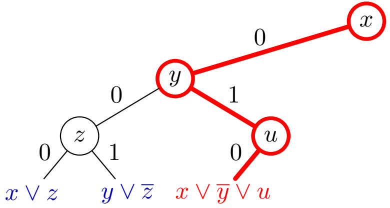

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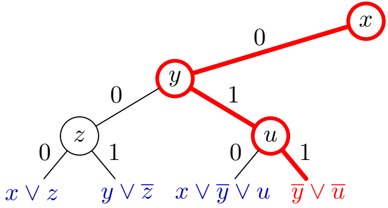

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$$\begin{aligned} F = \quad & (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \\ & \land (u \lor v) \land (\overline{v}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w}) \end{aligned}$$

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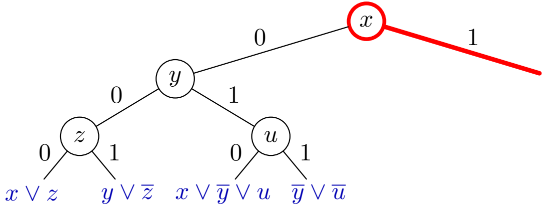

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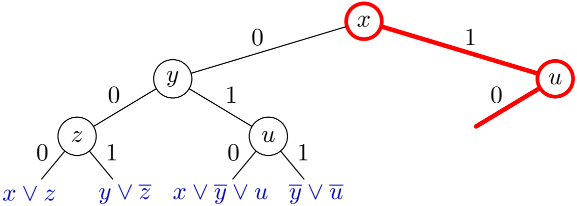

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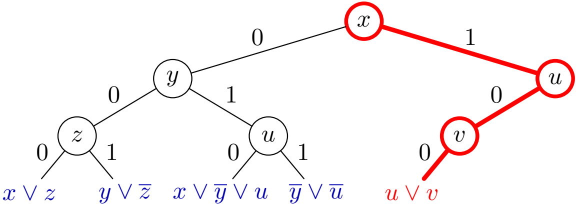

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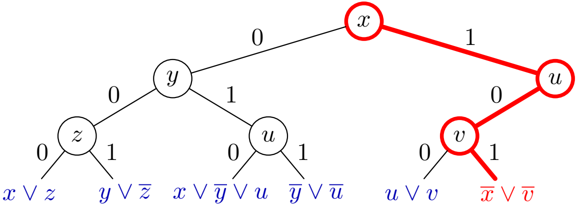

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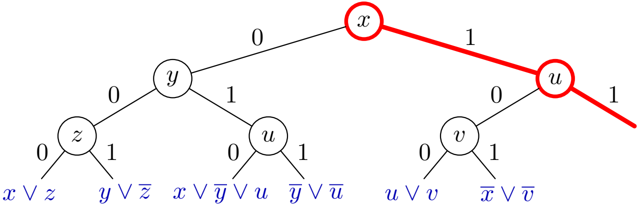

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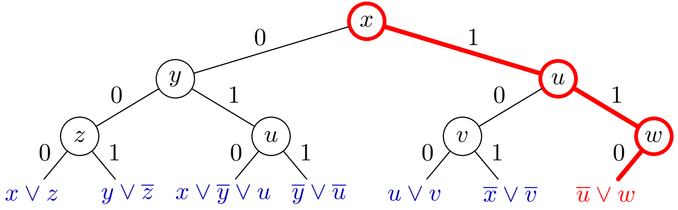

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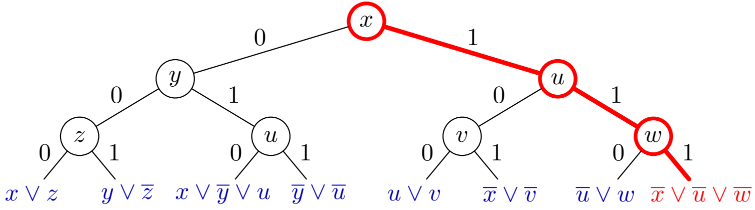

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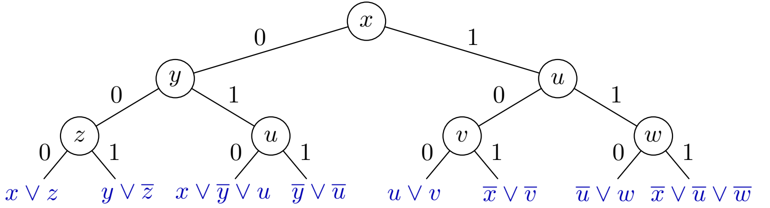

State-of-the-art SAT solvers: Ingredients

Many more ingredients in modern conflict-driven clause learning (CDCL) SAT solvers (as pioneered in [MS99, MMZ^{+}01]), e.g.:

- Branching or decision heuristic (choice of pivot variables crucial)
- When reaching leaf, compute explanation for conflict and add to formula as new clause (clause learning)
- Every once in a while, restart from beginning (but save computed info)

Let us discuss these ingredients

Variable Assignment Heuristics

Unit propagation

- Suppose current assignment ρ falsifies all literals in $C = \ell_1 \lor \ell_2 \lor \cdots \lor \ell_k$ except one (say ℓ_k) — C is unit under ρ
- Then ℓ_k has to be true, so set it to true
- Known as unit progagation or Boolean constraint progagation
- Always propagate if possible — in modern solvers aim for 99% of assignments being unit propagations

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VSIDS (Variable state independent decaying sum)

- When backtracking, score $+1$ for variables "causing conflict"
- Also multiply all scores with factor $\kappa < 1$ — exponential filter rewarding variables involved in recent conflicts
- When no propagations, decide on variable with highest score

Clause Learning

- At conflict, want to add clause avoiding same part of search tree being explored again

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- In practice, more advanced learning schemes

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- Suppose we can compute that decisions $x = 1$, $y = 0$, $z = 1$ responsible for conflict
- Then can add $\overline{x} \vee y \vee \overline{z}$ to avoid these decisions being made again — decision learning scheme
- In practice, more advanced learning schemes
- Derive new clause from clauses unit propagating on the way to conflict

Decisions, Unit Propagations, and Conflict

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

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Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

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Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

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Add to assignment trail

Until satisfying assignment or conflict

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$r \stackrel{q \vee r}{=} 1$

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$y \stackrel{u \lor x \lor y}{=} 1$

Decision

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$y \overset{u \lor x \lor y}{=} 1$

$z \overset{x \lor \overline{y} \lor z}{=} 1$

$\overset{\overline{y} \lor \overline{z}}{\bot}$

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$\boxed{p \stackrel{\mathsf{d}}{=} 0}$
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decision level 1

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$
$r \stackrel{q \vee r}{=} 1$
$w \stackrel{\overline{r} \vee w}{=} 1$
decision level 2

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$
$y \stackrel{u \vee x \vee y}{=} 1$
$z \stackrel{x \vee \overline{y} \vee z}{=} 1$
$\stackrel{\overline{y} \vee \overline{z}}{\perp}$
decision level 3

Decision
Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation
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Notation $u \stackrel{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

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Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$$\left.\begin{array}{l} \boxed{p \stackrel{\mathsf{d}}{=} 0} \\ u \stackrel{p \vee \overline{u}}{=} 0 \end{array}\right\} \text{decision level } 1$$

$$\left.\begin{array}{l} \boxed{q \stackrel{\mathsf{d}}{=} 0} \\ r \stackrel{q \vee r}{=} 1 \\ w \stackrel{\overline{r} \vee w}{=} 1 \end{array}\right\} \text{decision level } 2$$

$$\left.\begin{array}{l} \boxed{x \stackrel{\mathsf{d}}{=} 0} \\ y \stackrel{u \vee x \vee y}{=} 1 \\ z \stackrel{x \vee \overline{y} \vee z}{=} 1 \\ \stackrel{\overline{y} \vee \overline{z}}{\bot} \end{array}\right\} \text{decision level } 3$$

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$p \overset{\mathsf{d}}{=} 0$

$u \overset{p \lor \overline{u}}{=} 0$

decision level 1

$q \overset{\mathsf{d}}{=} 0$

$r \overset{q \lor r}{=} 1$

$w \overset{\overline{r} \lor w}{=} 1$

decision level 2

$x \overset{\mathsf{d}}{=} 0$

$y \overset{u \lor x \lor y}{=} 1$

$z \overset{x \lor \overline{y} \lor z}{=} 1$

$\overset{\overline{y} \lor \overline{z}}{\bot}$

decision level 3

Could backtrack by removing last decision level & flipping last decision

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$p \stackrel{\mathsf{d}}{=} 0$
$u \stackrel{p \lor \overline{u}}{=} 0$
} decision level 1

$q \stackrel{\mathsf{d}}{=} 0$
$r \stackrel{q \lor r}{=} 1$
$w \stackrel{\overline{r} \lor w}{=} 1$
} decision level 2

$x \stackrel{\mathsf{d}}{=} 0$
$y \stackrel{u \lor x \lor y}{=} 1$
$z \stackrel{x \lor \overline{y} \lor z}{=} 1$
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But want to learn from conflict and cut away as much of search space as possible

Conflict Analysis

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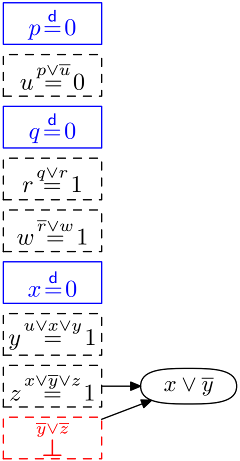

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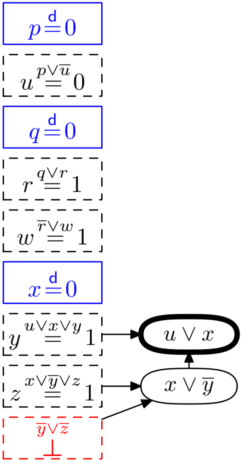
Case analysis over z for last two clauses:

- $x \vee \overline{y} \vee z$ wants $z = 1$
- $\overline{y} \vee \overline{z}$ wants $z = 0$
- Resolve clauses by merging them & removing z — must satisfy $x \vee \overline{y}$

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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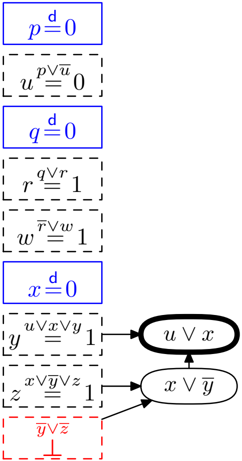
- $x \lor \overline{y} \lor z$ wants $z = 1$
- $\overline{y} \lor \overline{z}$ wants $z = 0$
- Resolve clauses by merging them & removing z — must satisfy $x \lor \overline{y}$

Repeat until UIP clause with only 1 variable after last decision — learn and backjump

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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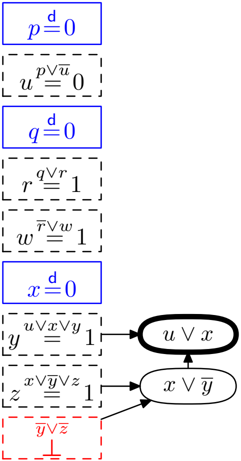
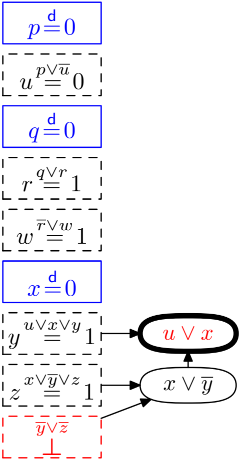


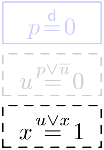
Assertion level 1 (max for non-UIP literal in learned clause) — keep trail to that level

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$





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Now UIP literal guaranteed to flip (assert) — but this is a propagation, not a decision

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$w \stackrel{\overline{r} \lor w}{=} 1$

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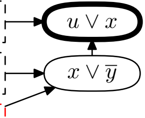
$\stackrel{\overline{y} \lor \overline{z}}{\bot}$



$p \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{p \lor \overline{u}}{=} 0$

$x \stackrel{u \lor x}{=} 1$

$z \stackrel{\overline{x} \lor z}{=} 1$

Assertion level 1 (max for non-UIP literal in learned clause) — keep trail to that level

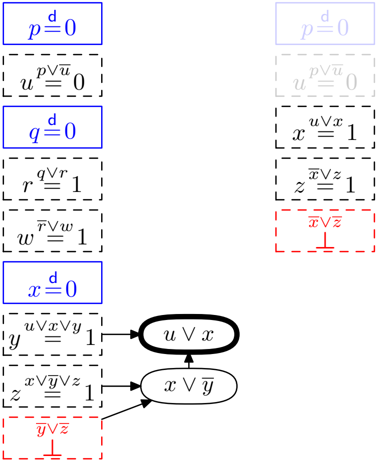
Now UIP literal guaranteed to flip (assert) — but this is a propagation, not a decision

Then continue as before...

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

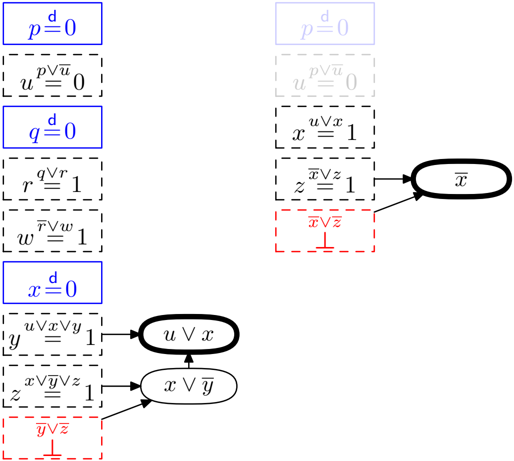
$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$



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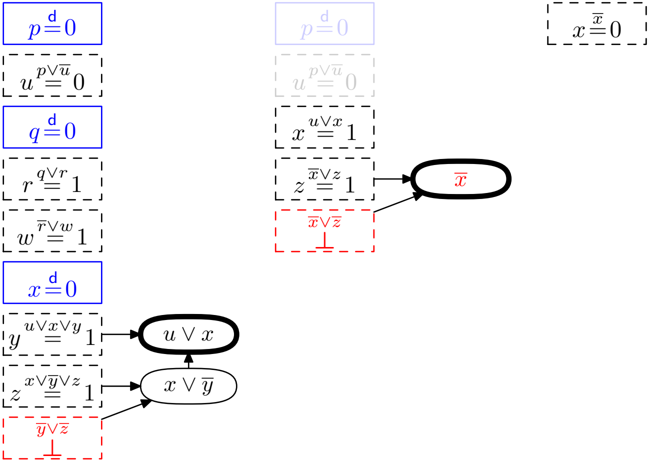

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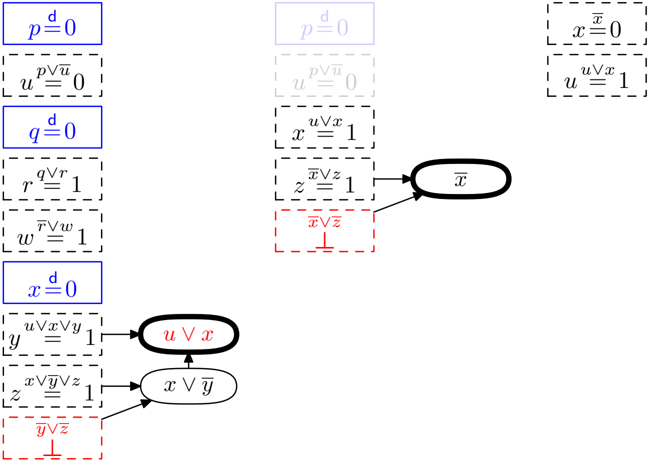

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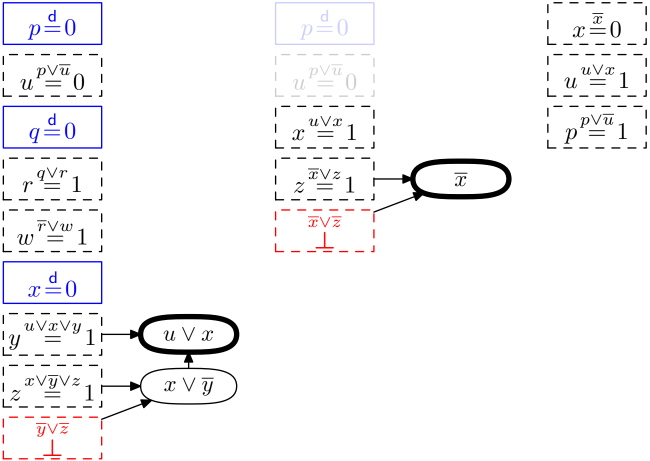

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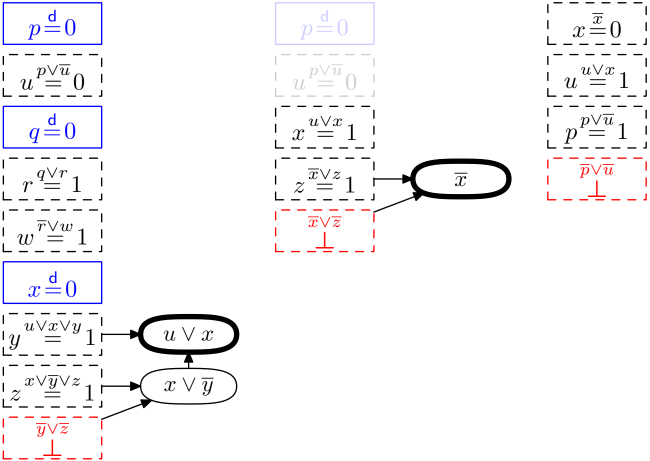

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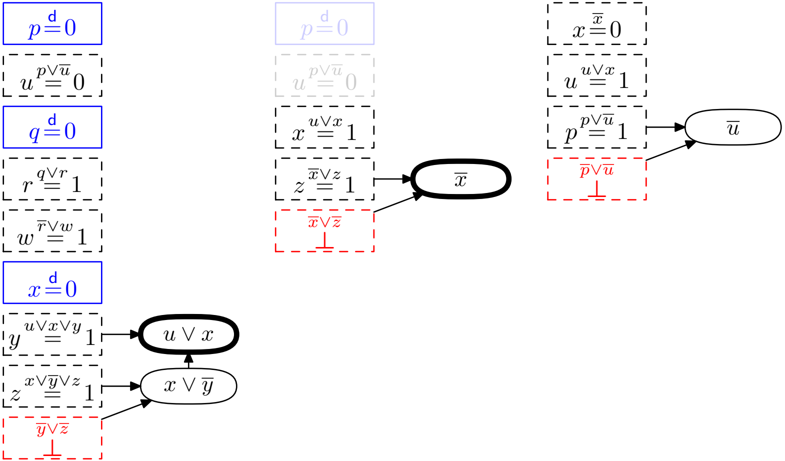

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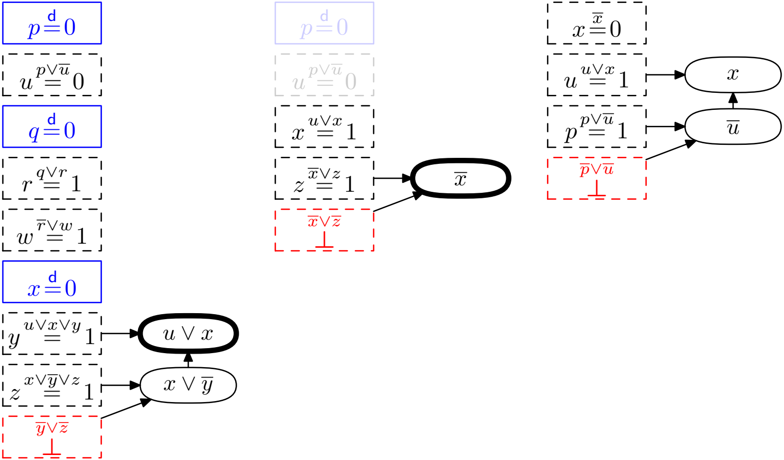

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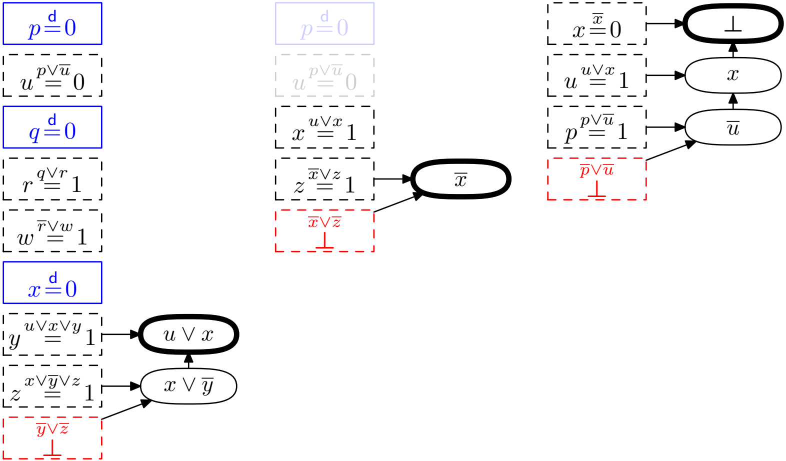

Clause Database Reduction

- In addition to learning clauses, also erase learned clauses that don't seem useful
- Modern solvers do this very aggressively
- Speeds up CDCL search (in particular, unit propagation, which dominates running time)
- But erasing too aggressively can throw away clauses that would have made solver terminate faster [EGG+18]
- So trade-off between search speed and search quality
- Except sometimes getting rid of clauses improves search quality too! [KN20]

Restarts

- Fairly frequently, start search all over (but keep learned clauses)
- Original intuition: stuck in bad part of search tree — go somewhere else
- Not the reason this is done now
- Popular variables with high VSIDS scores get set again [MMZ$^+$01]
- Are even set to same values (phase saving) [PD07]
- Current intution: improves the search by focusing on important variables
- Restart at fixed intervals or (better) make adaptive restarts depending on "quality" of learned clauses [AS09, AS12]

CDCL Main Loop Pseudocode

Algorithm 1: CDCL(F)

$\mathcal{D} \leftarrow F$; `// initialize clause database to contain formula`
$\rho \leftarrow \emptyset$; `// initialize assignment trail to empty`
forever do
  if *ρ falsifies some clause $C \in \mathcal{D}$* **then**
    $A \leftarrow$ ConflictAnalysis($\mathcal{D}, \rho, C$) ;
    if $A = \bot$ **then** output `UNSATISFIABLE` and exit;
    else
      add A to $\mathcal{D}$ and backjump by shrinking ρ ;
  else if *exists clause $C \in \mathcal{D}$ unit propagating x to $b \in \{0, 1\}$ under ρ* **then**
    add propagated assignment $x \stackrel{D}{=} b$ to ρ ;
  else if *time to restart* **then** $\rho \leftarrow \emptyset$;
  else if *time for clause database reduction* **then**
    erase (roughly) half of learned clauses in $\mathcal{D} \setminus F$ from $\mathcal{D}$
  else if *all variables assigned* **then** output `SATISFIABLE` and exit;
  else
    use decision scheme to choose assignment $x \stackrel{\mathsf{d}}{=} b$ to add to ρ ;

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Conflict Analysis Pseudocode

Algorithm 2: ConflictAnalysis$(\mathcal{D}, \rho, C_{\text{confl}})$

$C_{\text{learn}} \leftarrow C_{\text{confl}}$;
while C_{learn} *not UIP clause* **and** $C_{\text{learn}} \neq \bot$ **do**
    $\ell \leftarrow$ literal assigned last on trail ρ;
    if ℓ *propagated* **and** $\overline{\ell}$ *occurs in* C_{learn} **then**
        $C_{\text{reason}} \leftarrow$ reason$(\ell, \rho, \mathcal{D})$;
        $C_{\text{learn}} \leftarrow$ resolve$(C_{\text{learn}}, C_{\text{reason}})$;
    $\rho \leftarrow \rho \setminus \{\ell\}$;
return C_{learn};

State-of-the-art SAT solvers: What About the Recipe?

List of ingredients again (not exhaustive):

- Variable decisions & propagations
- Clause learning
- Restarts
- Clause database reduction

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Why SAT solvers actually work so well
is a poorly understood question

Lots of research to comprehend this better
(Among other places in the MIAO group)

SAT Solver Analysis and the Resolution Proof System

How to make rigorous analysis of SAT solver performance?
Many intricate, hard-to-understand heuristics
So focus instead on underlying method of reasoning

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Resolution proof system

- Start with clauses of CNF formula (axioms)
- Derive new clauses by resolution rule

$$\frac{C_1 \lor x \qquad C_2 \lor \overline{x}}{C_1 \lor C_2}$$

Resolution Proofs by Contradction

Resolution rule:

$$\frac{C_1 \lor x \qquad C_2 \lor \overline{x}}{C_1 \lor C_2}$$

Observation

If F is a satisfiable CNF formula and D is derived from clauses $D_1, D_2 \in F$ by the resolution rule, then $F \land D$ is satisfiable.

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Such proof by contradiction also called resolution refutation

DPLL and Resolution Proofs

A DPLL execution is essentially a resolution proof

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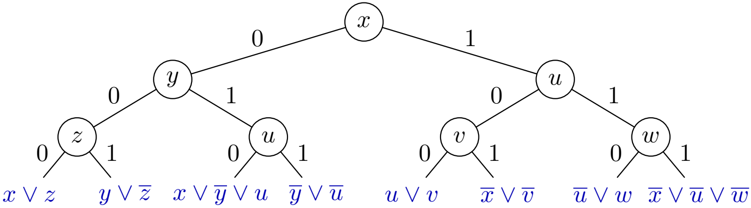
Look at our example again



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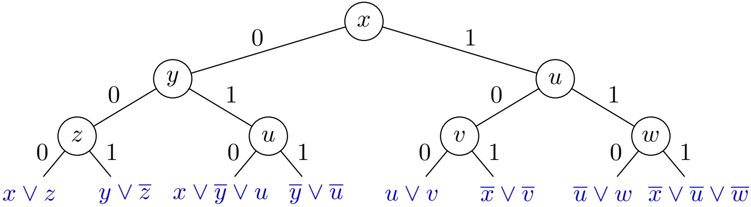

and apply resolution rule $\frac{C_1 \lor x \quad C_2 \lor \overline{x}}{C_1 \lor C_2}$ bottom-up

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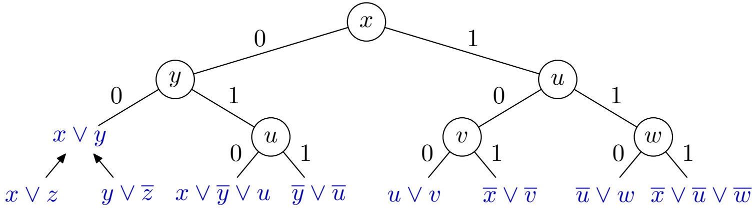

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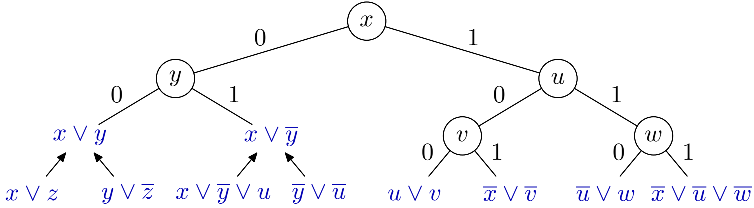

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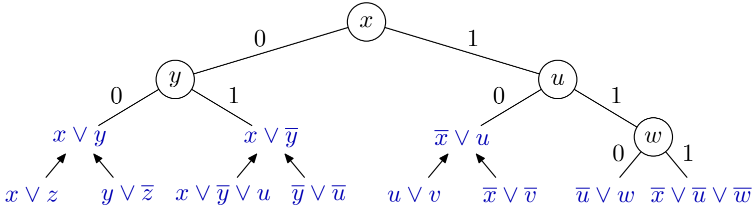

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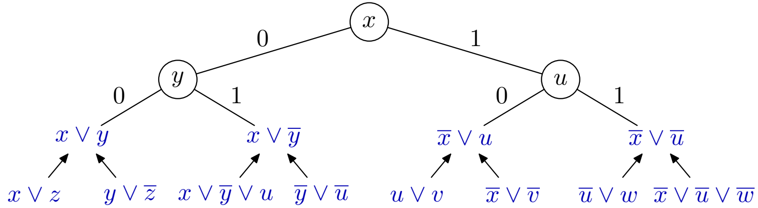

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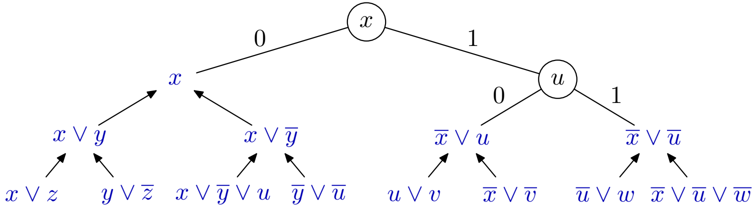

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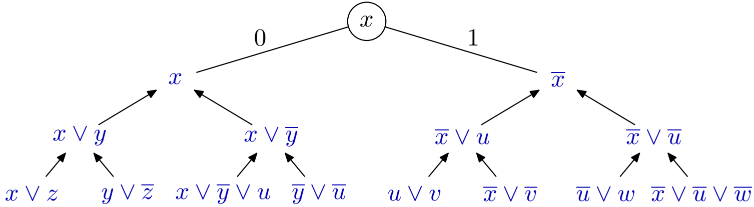
A DPLL execution is essentially a resolution proof

Look at our example again



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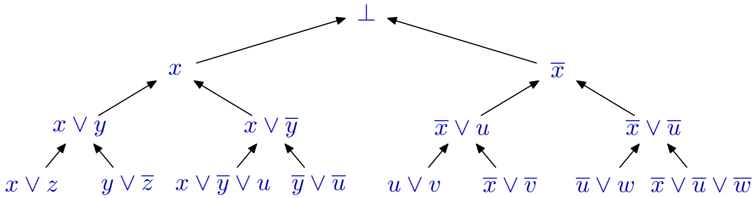

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- Conflict-driven clause learning adds "shortcut edges" in tree, but still yields resolution proof

CDCL and Resolution Proofs

Obtain resolution proof...

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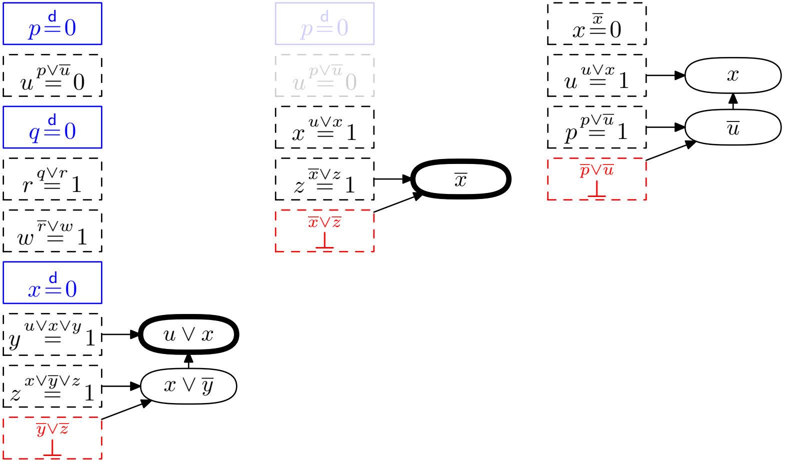

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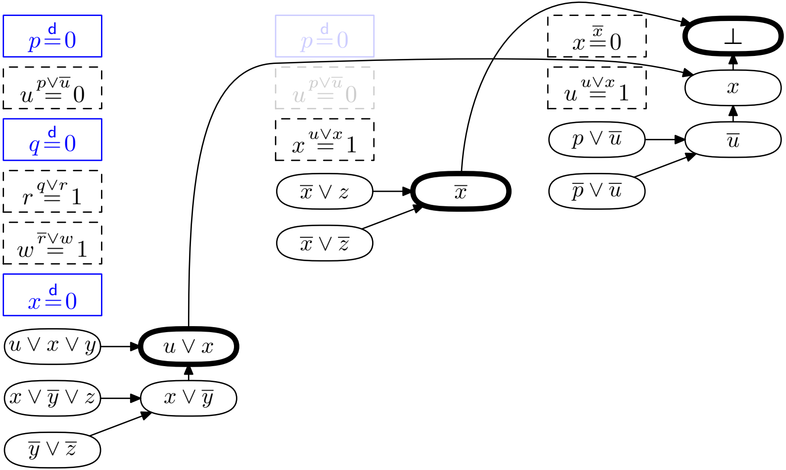

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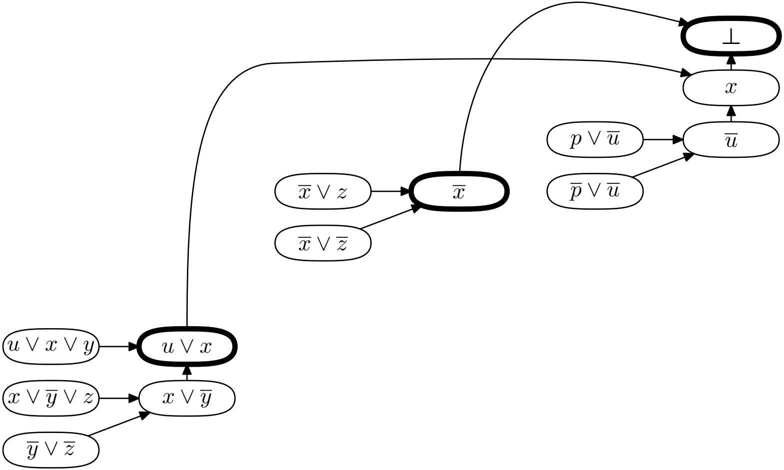

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(*) Except for some preprocessing techniques, which is an important omission, but this gets complicated and we don't have time to go into details...

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- Very poor theoretical understanding:
  - Why do heuristics work?
  - Why are applied instances easy?
- Paradox: resolution quite weak proof system; many strong proof complexity lower bounds for (seemingly) "obvious" formulas

Examples of Hard Formulas For Resolution (1/3)

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Even onto functional PHP hard — **"resolution cannot count"**

Resolution proof requires $\exp(\Omega(n)) = \exp\big(\Omega\big(\sqrt[3]{N}\big)\big)$ clauses
(measured in terms of formula size N)

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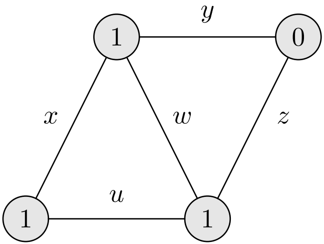

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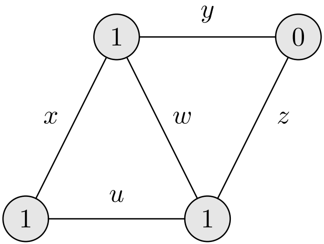

$$
\begin{array}{ll}
(u \lor x) & \land\ (y \lor \overline{z}) \\
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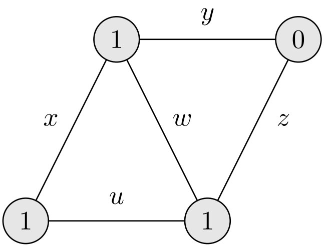

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Requires proof size $\exp(\Omega(N))$ on well-connected so-called expander graphs — **"resolution cannot count** $\bmod 2$**"**

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Δn randomly sampled k-clauses over n variables

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And more...

- Colouring [BCMM05]
- Clique and VertexCover [BIS07]
- Zero-one designs [Spe10, VS10, MN14]
- Et cetera... (See, e.g., [BN21] for overview)

Theoretical Lower Bounds and Practical Reality

- If resolution so weak, how can CDCL SAT solvers be so good?
- One answer: this kind of "tricky" formulas don't show up too often in practice
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- Explore stronger methods of reasoning!
- Algorithms based on such methods could potentially lead to exponential speed-ups

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Derivation rules

Variable axioms $\dfrac{}{0 \leq x \leq 1}$

Multiplication $\dfrac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA}$

Addition $\dfrac{\sum a_i x_i \geq A \quad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$

Division $\dfrac{\sum c a_i x_i \geq A}{\sum a_i x_i \geq \lceil A/c \rceil}$

Cutting Planes Refutation of CNF Formula

- Translate CNF formula to set of 0-1 linear inequalities
- Apply derivation rules
- Derive $0 \geq 1 \Leftrightarrow$ formula unsatisfiable
- Also makes sense for more general 0-1 linear inequalities (not just translations of CNF formulas)

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$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 3$$

and

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&(x_1 \lor x_2 \lor x_3 \lor x_4) \land (x_1 \lor x_2 \lor x_3 \lor x_5) \land (x_1 \lor x_2 \lor x_3 \lor x_6) \\
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- Construct more efficient SAT solvers using cutting planes?

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So-called pseudo-Boolean solvers using (subset of) cutting planes reasoning developed in, e.g., [CK05, LP10, EN18]

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Is it truly harder to build good pseudo-Boolean solvers?
Or has just so much more work has been put into CDCL solvers?

So... Is There a Smarter Way Than Brute-Force?

In theory, probably no...

- COLOURING, CLIQUE, SAT, and 1000s other problems are "all the same" — efficient algorithm for one can solve all (the problems are all NP-complete)
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Stark disconnect between theory and practice. . .

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Strengthen the mathematical analysis of algorithmic methods

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Construct stronger algorithms for combinatorial problems

- Use insights into stronger mathematical methods of reasoning to build algorithms for SAT and other combinatorial problems
- Aiming for exponential speed-ups over state of the art
- E.g., use cutting planes to build pseudo-Boolean solvers

Research Goals in the MIAO Group (2/2)

Improve understanding of efficient computation in practice

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Certify correctness for modern combinatorial solvers

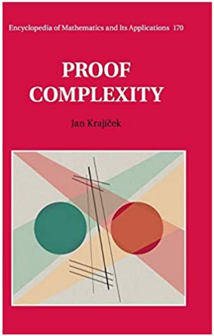
- In many combinatorial optimization paradigms, state-of-the-art solvers are known to be buggy
- Develop methods to make solvers output not just answer but machine-verifiable proof of correctness of this answer

Some References for Further Reading

Handbook of Satisfiability
(Especially chapter 7 ☺)

[BHvMW21]

Proof Complexity
by Jan Krajíček



[Kra19]

And survey papers, slides, and videos at `www.jakobnordstrom.se`

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Thanks for listening! See you again Tuesday Jan 18!

References I

[AS09] Gilles Audemard and Laurent Simon. Predicting learnt clauses quality in modern SAT solvers. In *Proceedings of the 21st International Joint Conference on Artificial Intelligence (IJCAI '09)*, pages 399–404, July 2009.

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