

# Combinatorial Solving with Provably Correct Results

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# Combinatorial Solving and Optimisation

- Revolution last couple of decades in **combinatorial solvers** for
    - Boolean satisfiability (SAT) solving [BHvMW21]<sup>1</sup>
    - Constraint programming (CP) [RvBW06]
    - Mixed integer linear programming (MIP) [AW13, BR07]
  - Solve NP-complete problems (or worse) very successfully in practice!
  - Except solvers are sometimes wrong... (Even best commercial ones)  
[BLB10, CKSW13, AGJ<sup>+</sup>18, GSD19, GS19, BMN22, BBN<sup>+</sup>23]
  - Even get feasibility of solutions wrong (though this should be straightforward!)
  - And how to check the absence of solutions?
  - Or that a solution is optimal? (Even off-by-one mistakes can snowball into large errors if solver used as subroutine)

<sup>1</sup>See end of slides for all references with bibliographic details

# What Can Be Done About Solver Bugs?

## ■ Software testing

Hard to get good test coverage for sophisticated solvers

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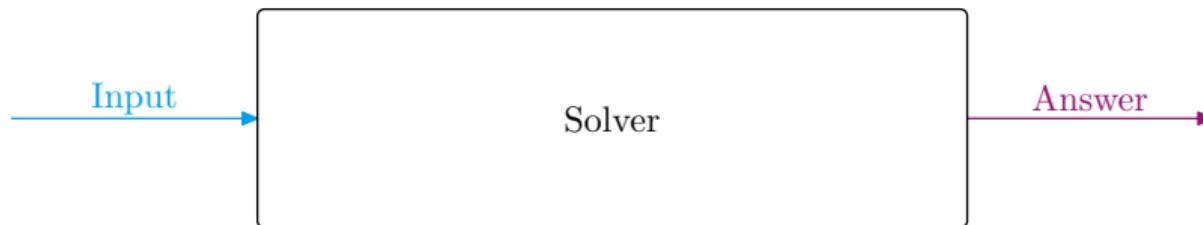
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## ■ Proof logging

Make solver certifying [ABM<sup>+</sup>11, MMNS11] by outputting

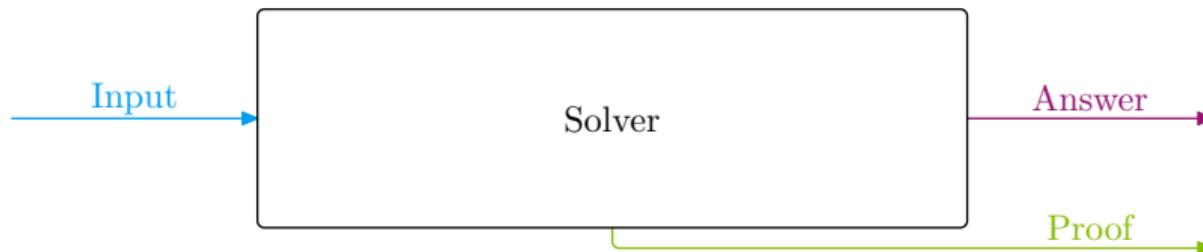
- 1 not only **answer** but also
  - 2 simple, machine-verifiable **proof** that answer is correct

# Proof Logging with Certifying Solvers: Workflow



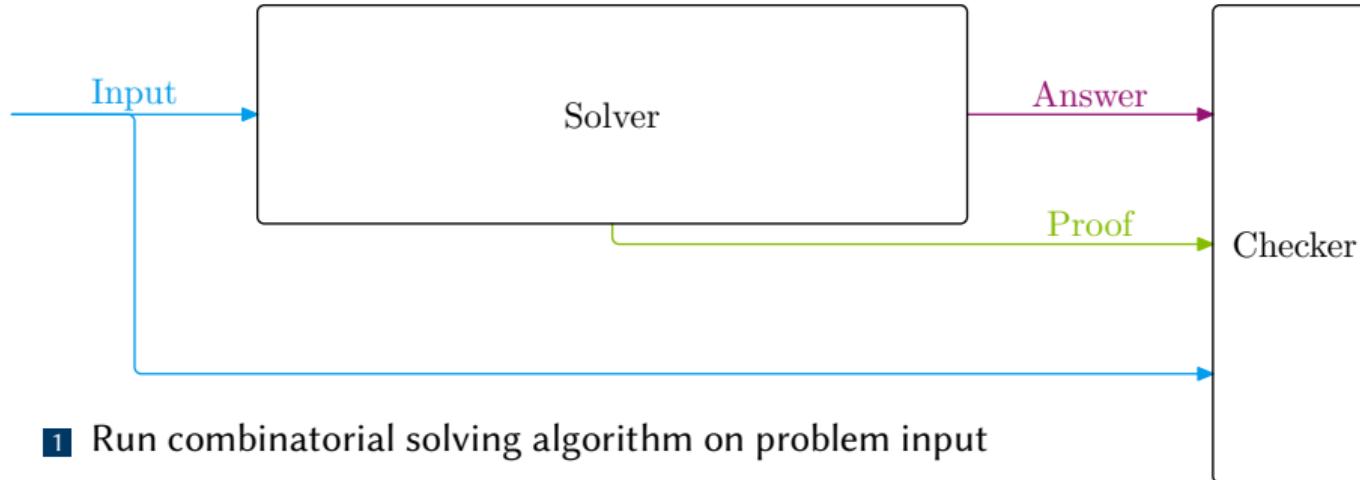
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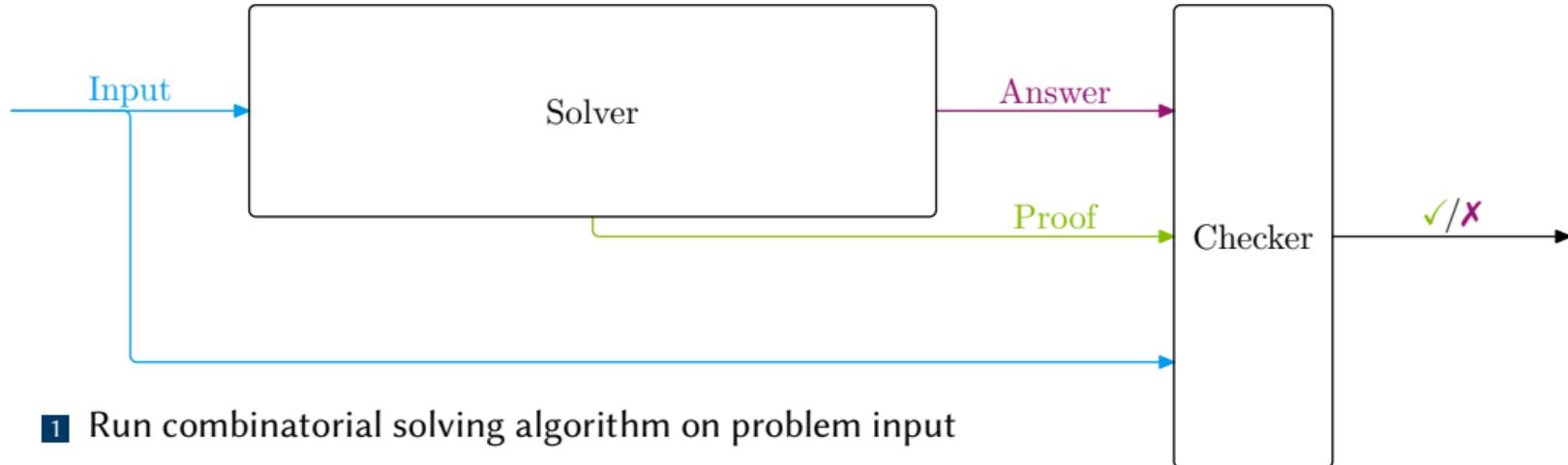
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  - 3 Feed input + answer + proof to proof checker

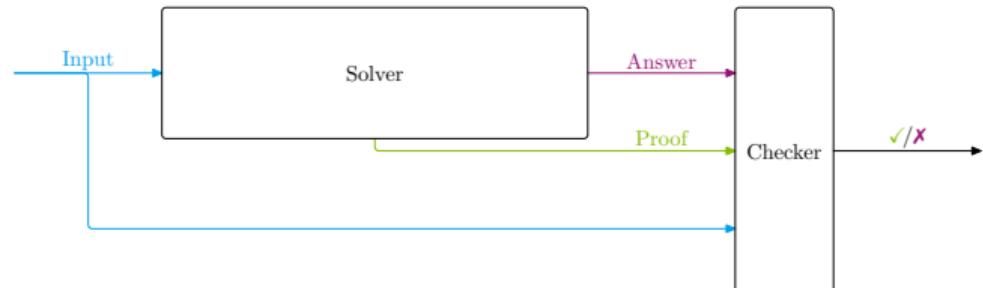
# Proof Logging with Certifying Solvers: Workflow



- 1 Run combinatorial solving algorithm on problem input
  - 2 Get as output not only answer but also proof
  - 3 Feed input + answer + proof to proof checker
  - 4 Verify that proof checker says answer is correct

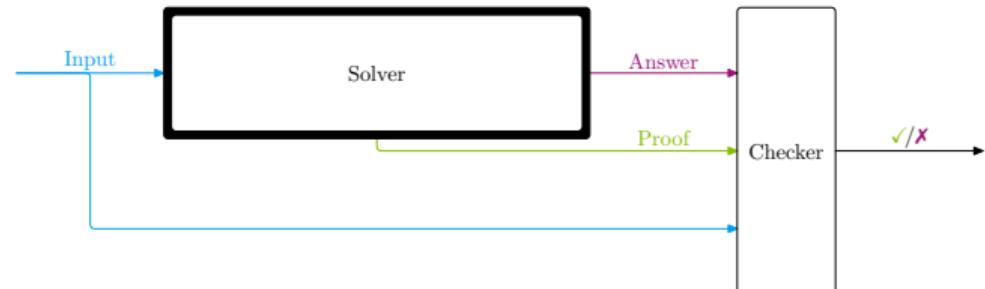
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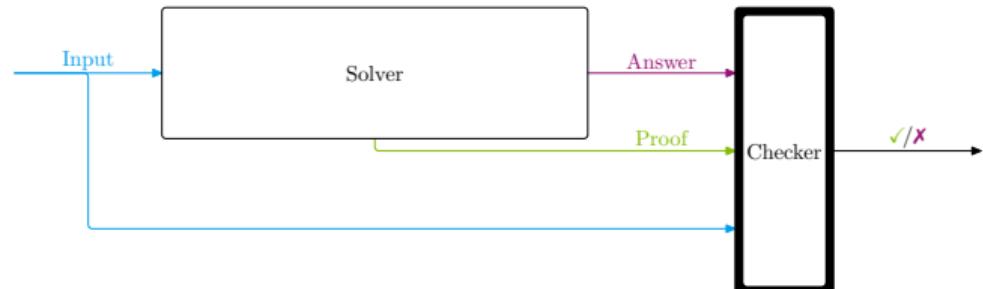
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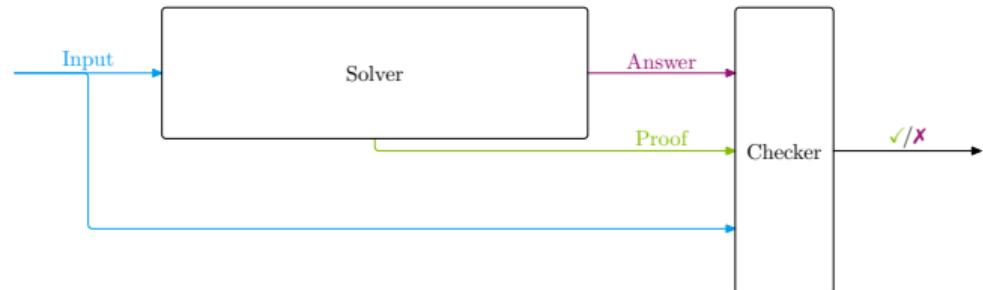
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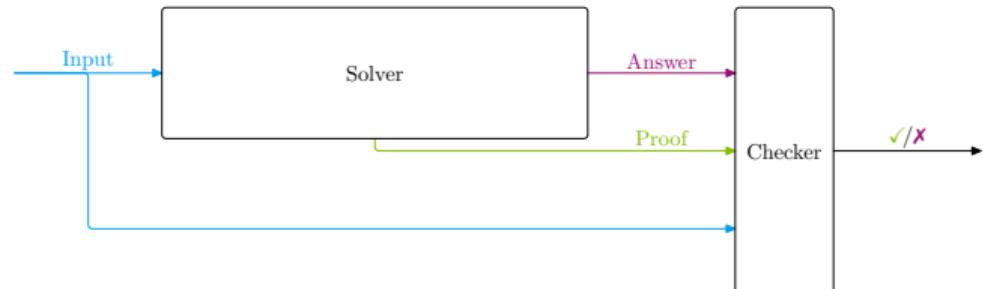


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Asking for both perhaps a little bit too good to be true?

## Take-Away Message from This Tutorial

Proof logging for combinatorial optimisation is possible with **single, unified method!**

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Proof logging for combinatorial optimisation is possible with single, unified method!

- Build on successes in proof logging for SAT solvers with proof formats such as DRAT [HHW13a, HHW13b, WHH14], GRIT [CMS17], LRAT [CHH<sup>+</sup>17], ...
  - But represent constraints as 0–1 integer linear inequalities
  - Formalize reasoning using cutting planes [CCT87] proof system
  - Add well-chosen strengthening rules [Goc22, GN21, BGMN23]
  - Implemented in **VERIPB** (<https://gitlab.com/MIA0research/software/VeriPB>)

## The Sales Pitch For Proof Logging

- 1 Certifies correctness of computed results
  - 2 Detects errors even if due to compiler bugs, hardware failures, or cosmic rays
  - 3 Provides debugging support during development [EG21, GMM<sup>+</sup>20, KM21, BBN<sup>+</sup>23]
  - 4 Facilitates performance analysis
  - 5 Helps identify potential for further improvements
  - 6 Enables auditability
  - 7 Serves as stepping stone towards explainability

# The Rest of This Tutorial

Explain how to use **VERIPB** to do proof logging for

- SAT solving (including advanced techniques)
- SAT-based optimisation (MaxSAT)
- Subgraph algorithms
- Constraint programming
- Symmetry and dominance reasoning

in a unified way

## The SAT Problem

- Variable  $x$ : takes value **true** (=1) or **false** (=0)
  - Literal  $\ell$ : variable  $x$  or its negation  $\bar{x}$
  - Clause  $C = \ell_1 \vee \cdots \vee \ell_k$ : disjunction of literals  
(Consider as sets, so no repetitions and order irrelevant)
  - Conjunctive normal form (**CNF**) formula  $F = C_1 \wedge \cdots \wedge C_m$ : conjunction of clauses

## The SAT Problem

Given a CNF formula  $F$ , is it satisfiable?

For instance, what about:

$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge \\ (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$

## Proofs for SAT

For satisfiable instances: just specify satisfying assignment

For unsatisfiability: a sequence of clauses (CNF constraints)

- Each clause follows “obviously” from everything we know so far
  - Final clause is empty, meaning contradiction (written  $\perp$ )
  - Means original formula must be inconsistent

# What Is Obvious? Unit Propagation

## Unit Propagation

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Proof checker should know how to unit propagate until saturation

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DPLL [DP60, DLL62]: Assign variables and propagate; backtrack when clause violated

“Proof trace”: when backtracking, write negation of guesses made

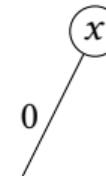
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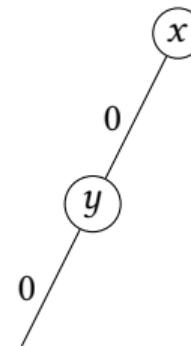


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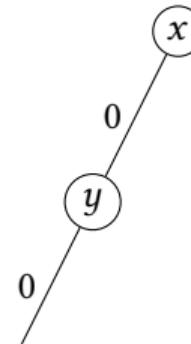


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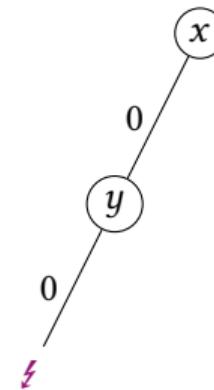
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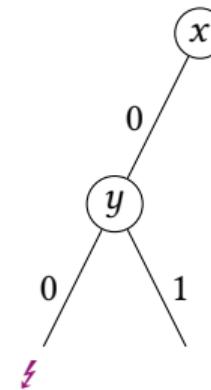
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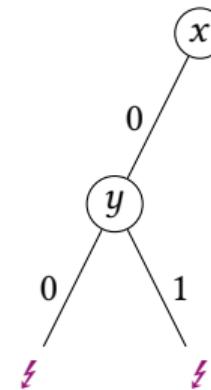
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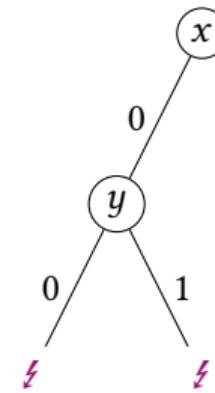
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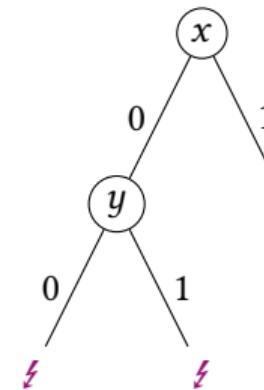
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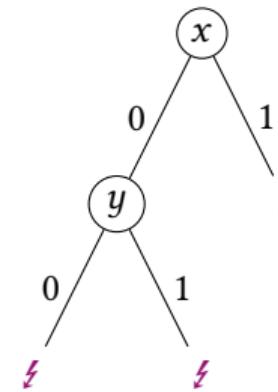
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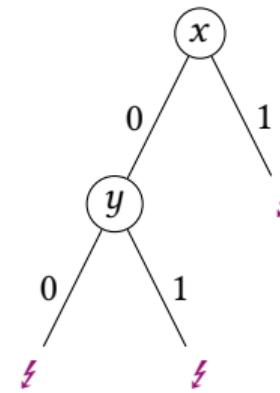
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  - 5**  $\perp$



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## Fact

Backtrack clauses from DPLL solver generate a RUP proof

# What About Conflict-Driven Clause Learning (CDCL)?

Run CDCL [BS97, MS99, MMZ<sup>+</sup>01] on our favourite CNF formula:

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## Decision

Free choice to assign value to variable

Notation  $p \stackrel{d}{=} 0$

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$$p \stackrel{d}{=} 0$$

## Decision

Free choice to assign value to variable

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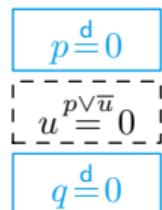
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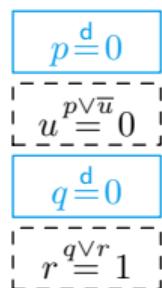
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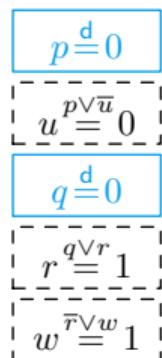
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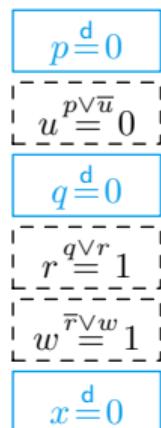
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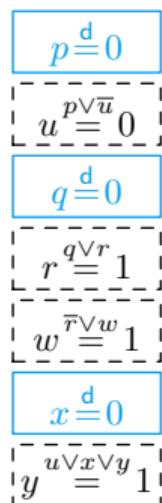
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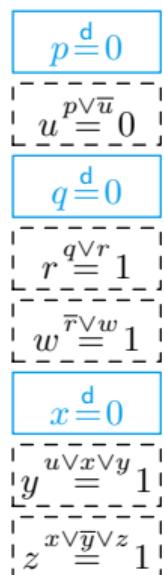
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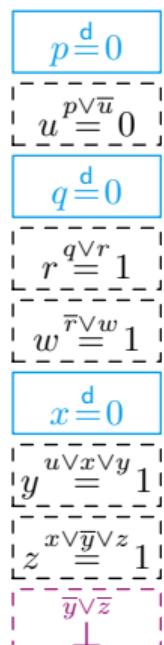
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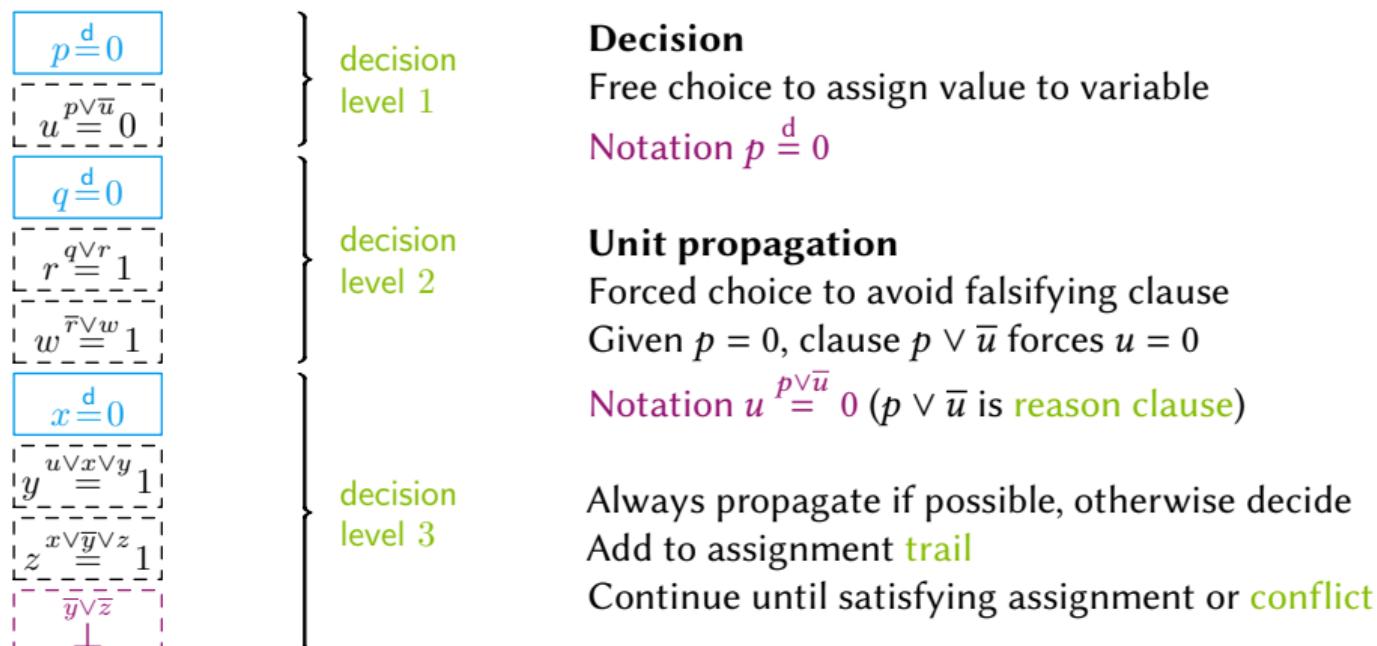
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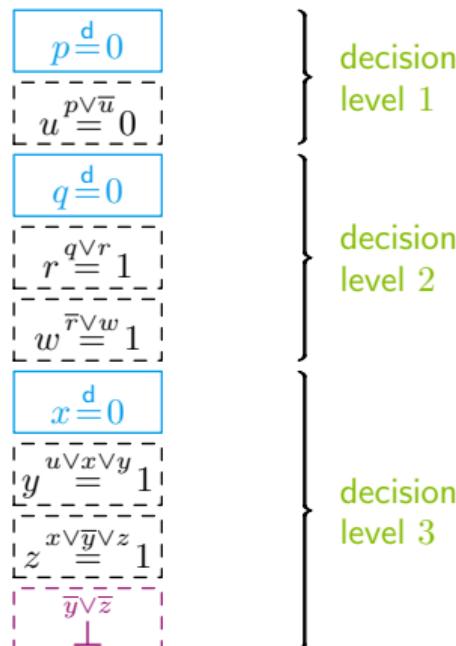
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## Conflict Analysis

Time to analyse this conflict and learn from it

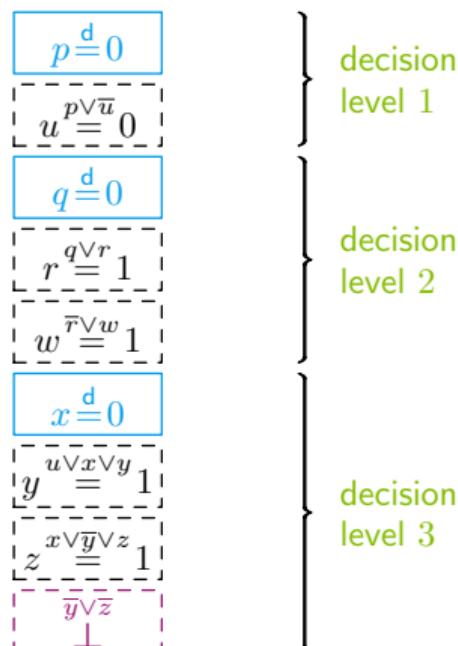
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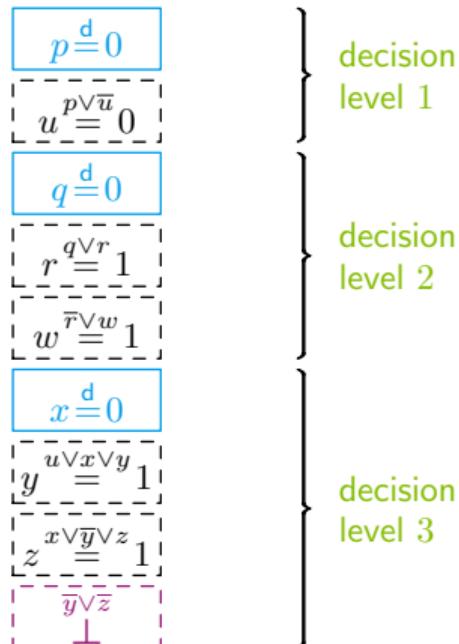


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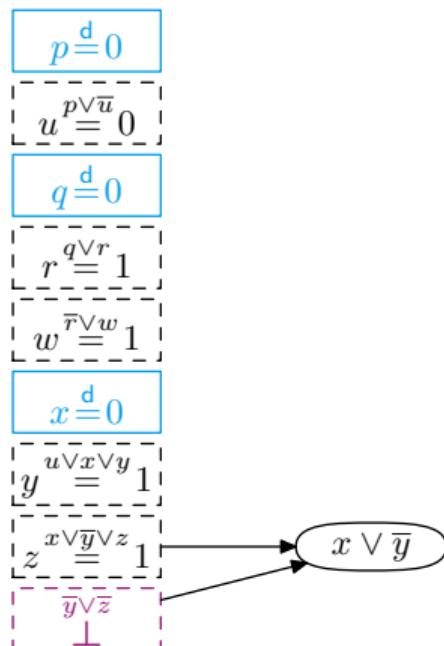
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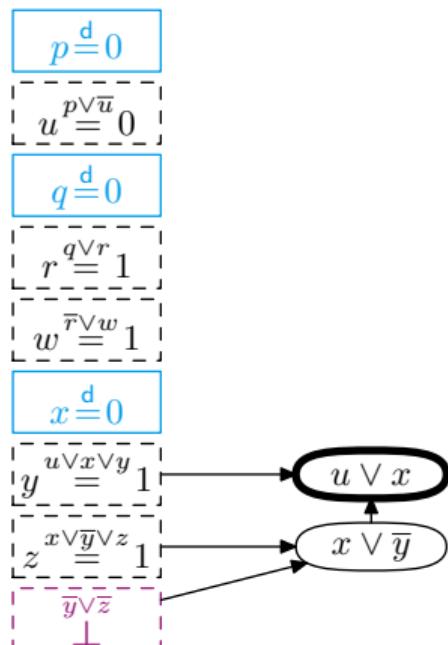
Case analysis over  $z$  for last two clauses:

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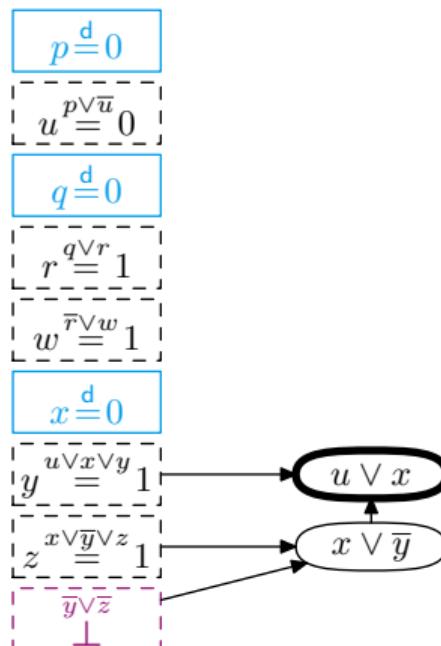
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Repeat until **UIP clause** with only 1 variable at conflict level after last decision — **learn** and **backjump**

# Complete Example of CDCL Execution

**Backjump:** undo max #decisions while learned clause propagates

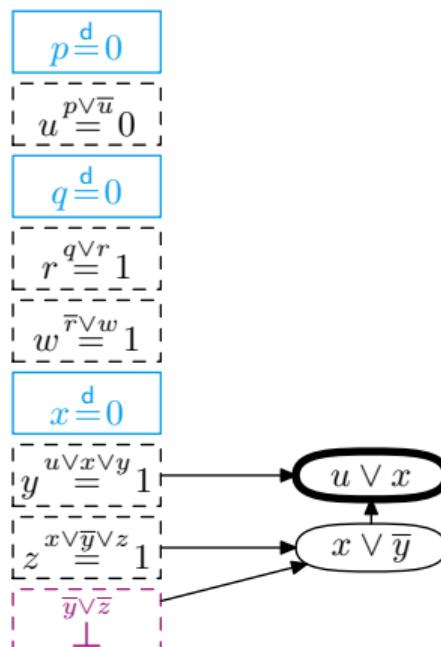
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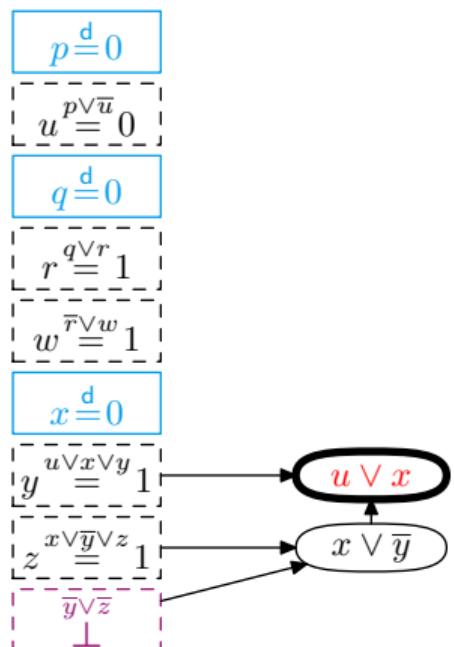


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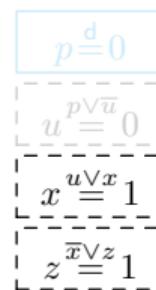
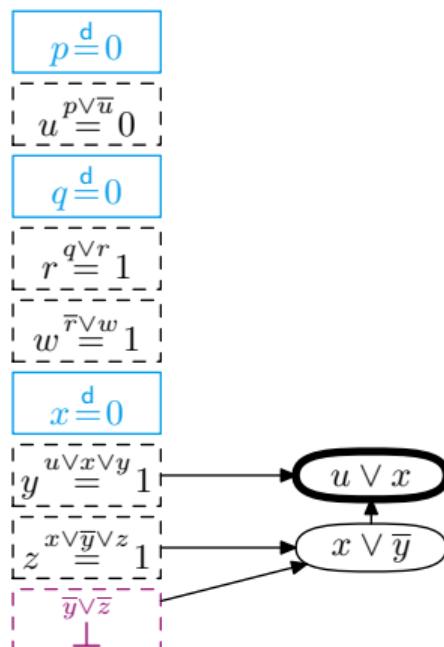
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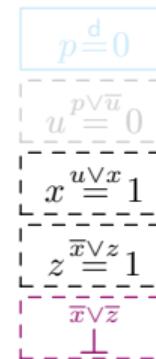
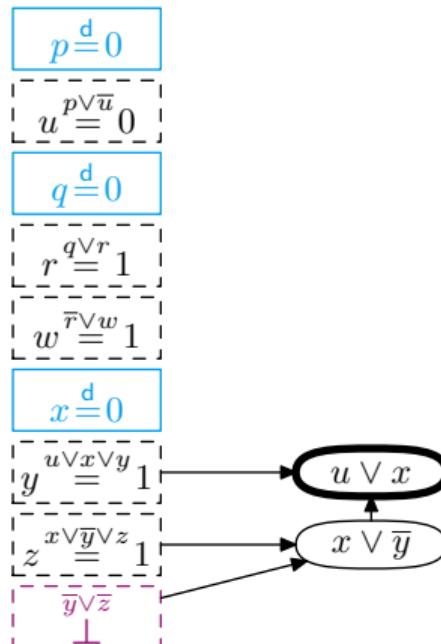
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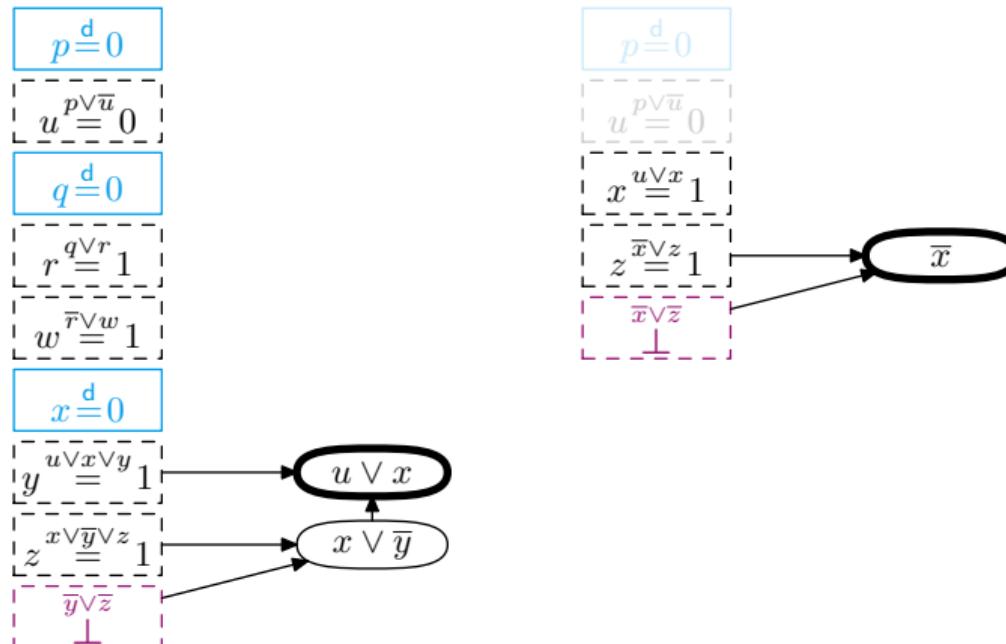
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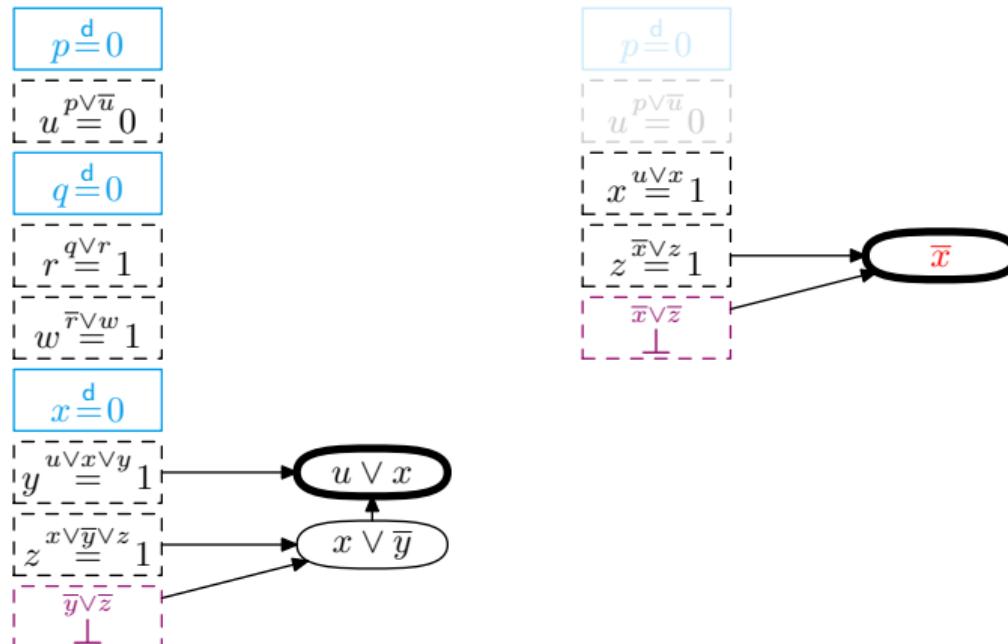
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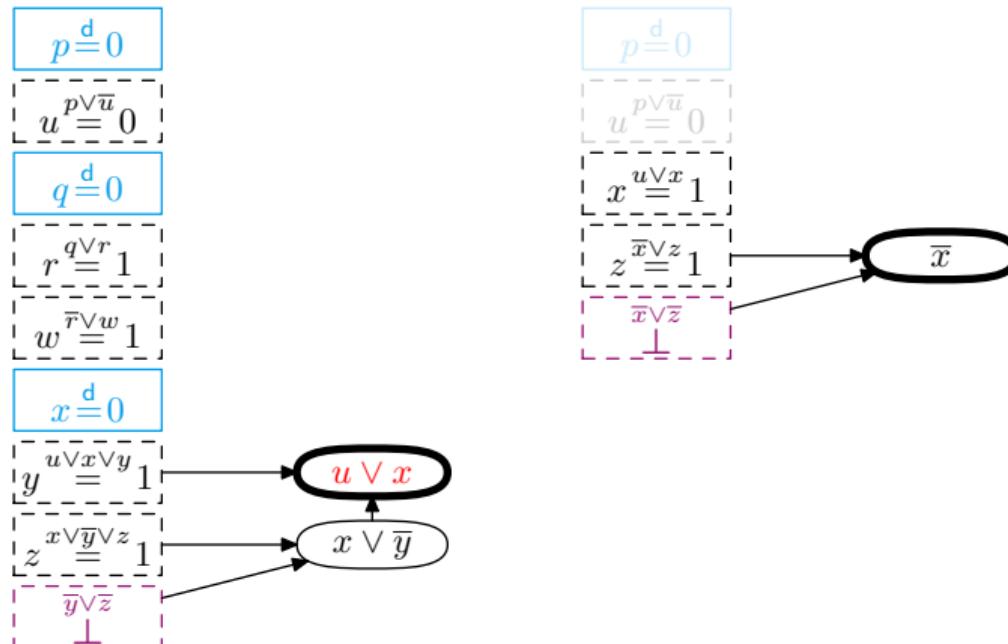
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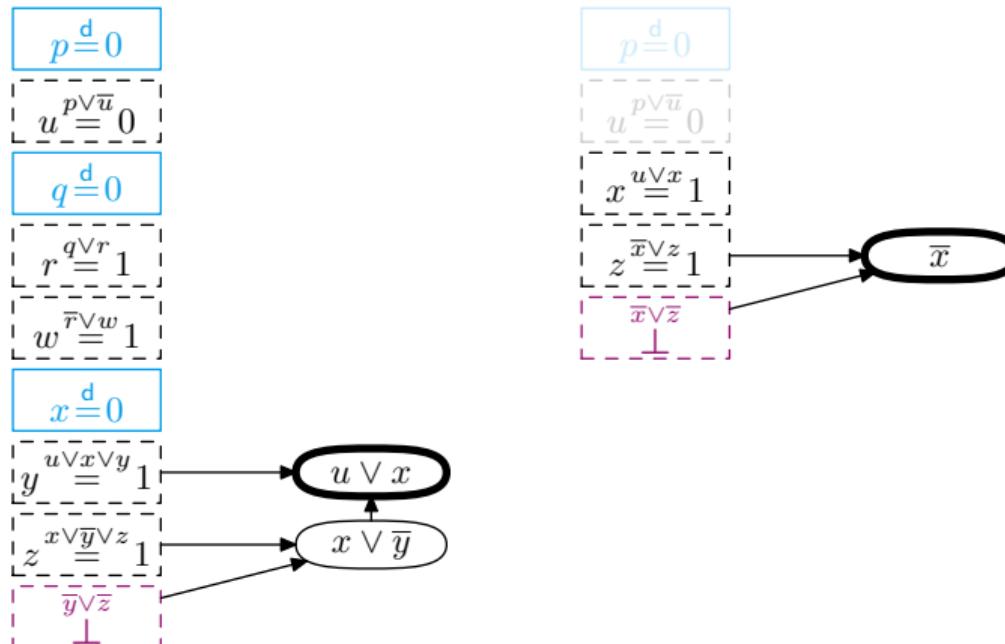
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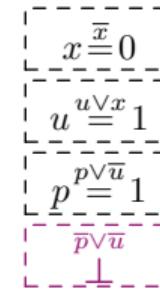
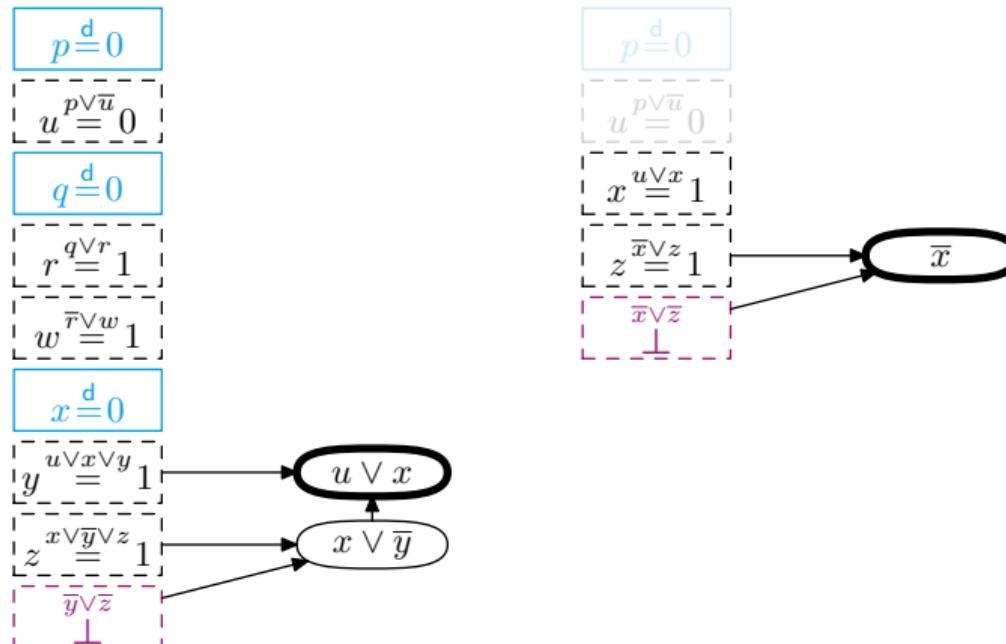
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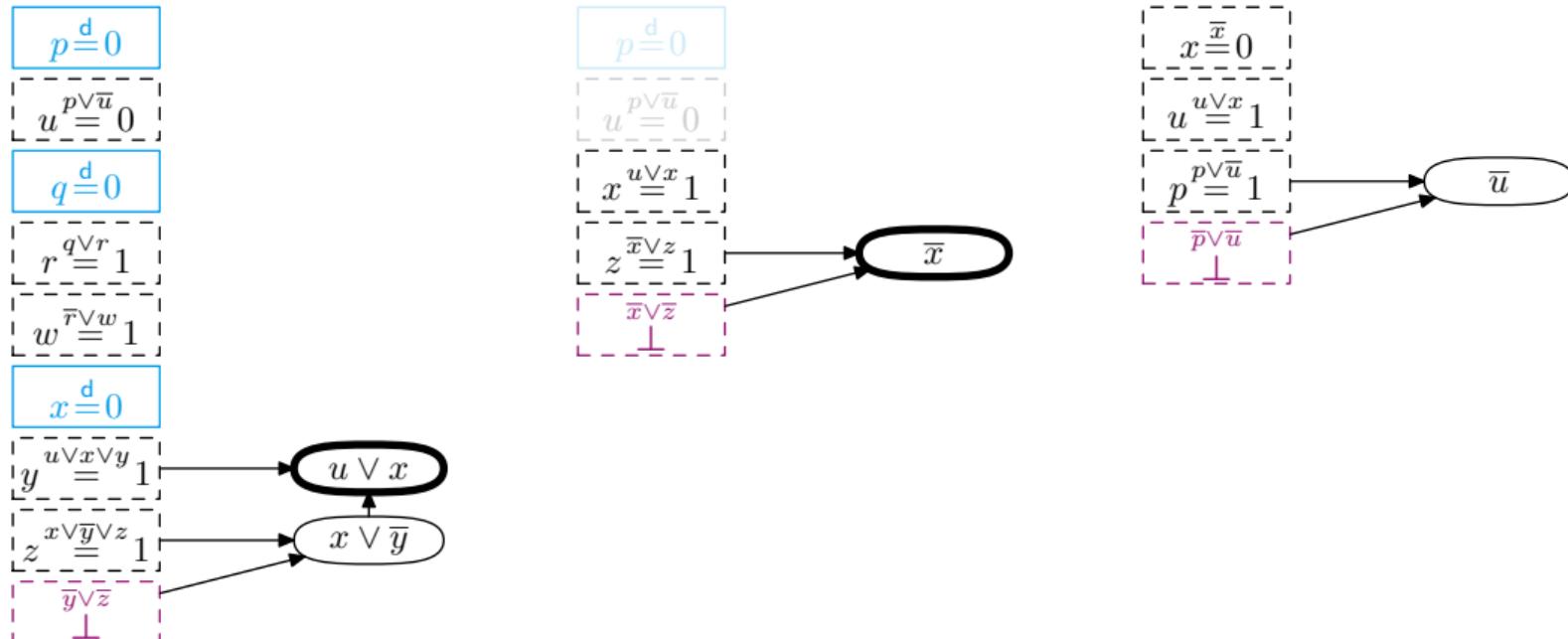
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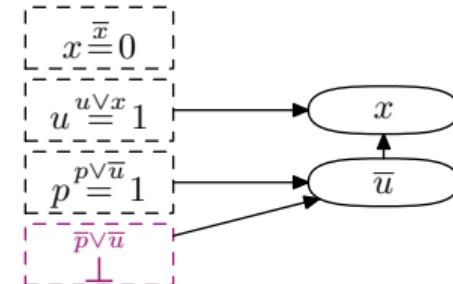
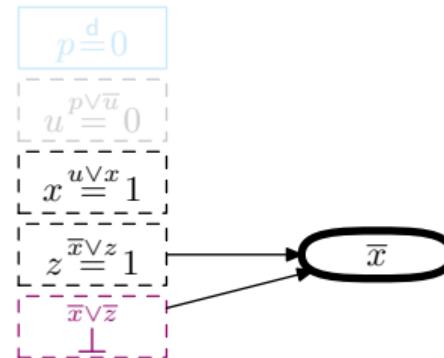
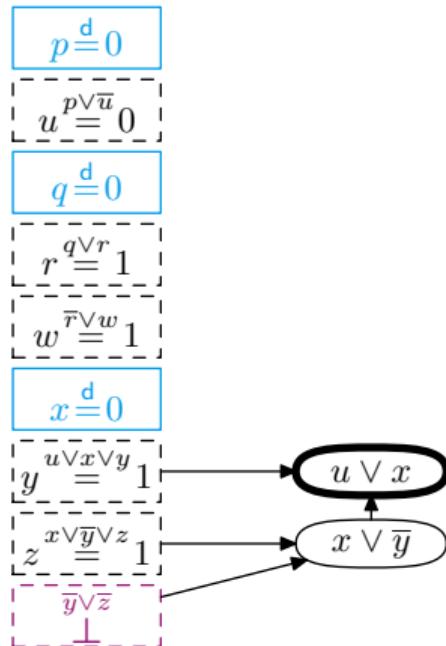
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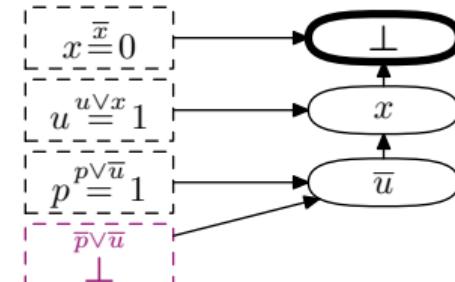
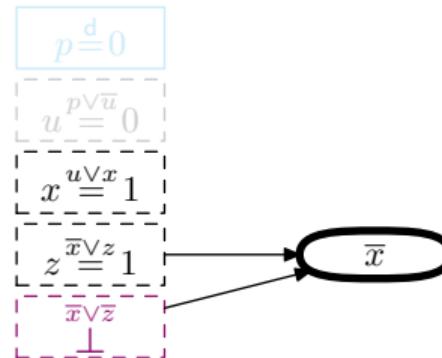
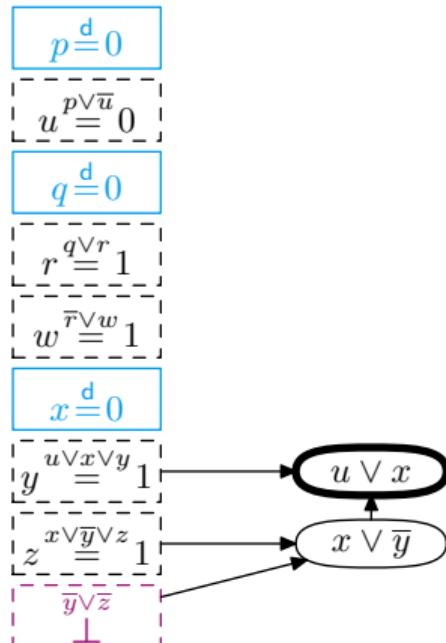
$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$



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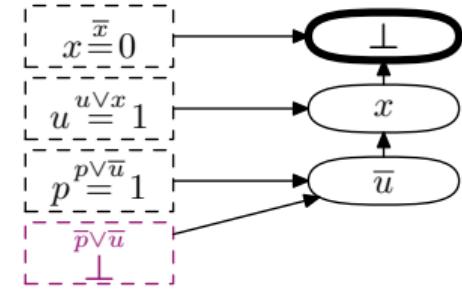
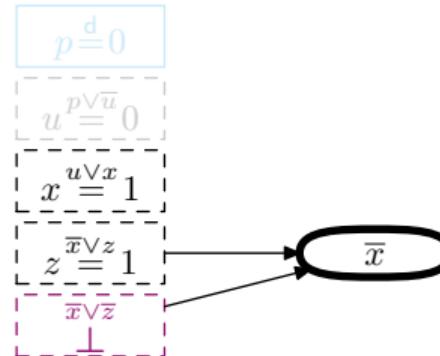
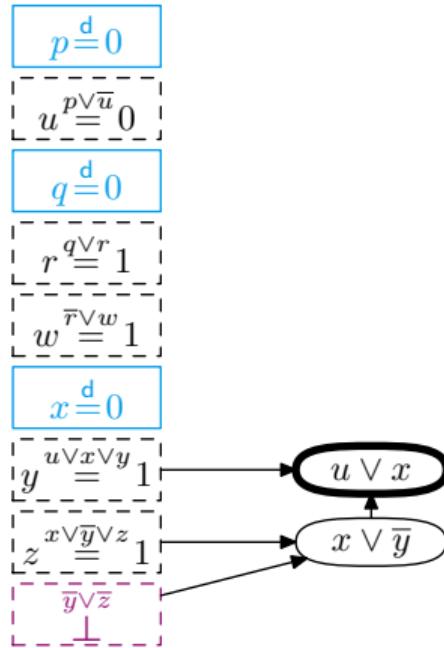
(\*) Ignores pre- and inprocessing, but we will get there...

# Resolution Proofs from CDCL Executions

Obtain resolution proof...

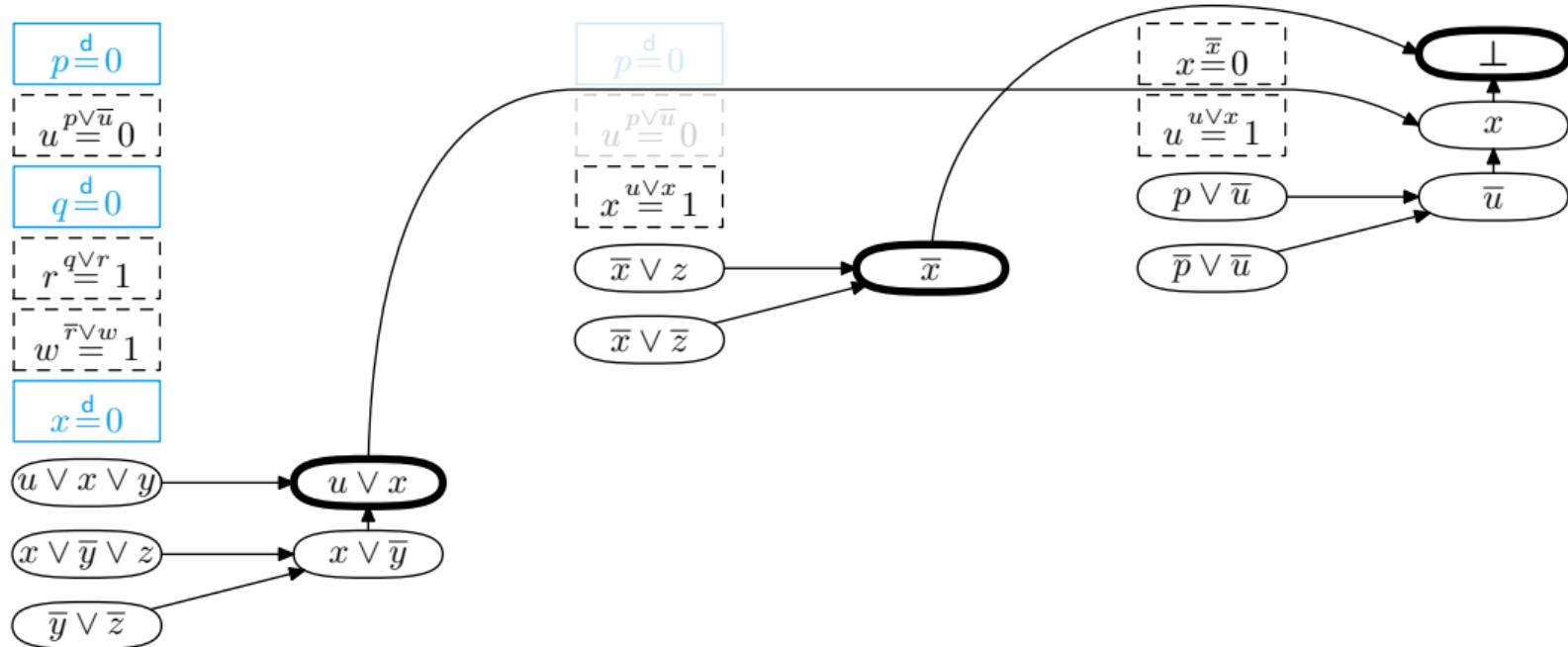
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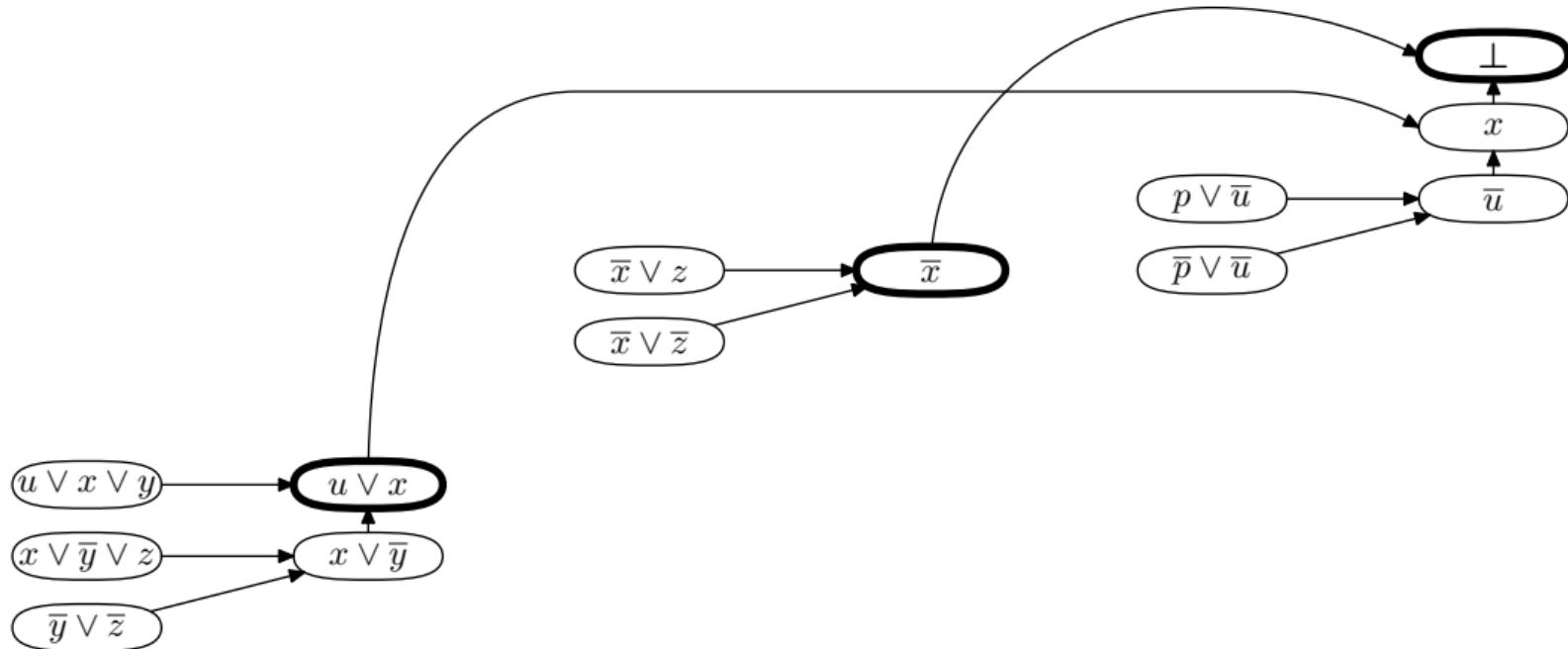
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is sequence of reverse unit propagation (RUP) clauses

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is sequence of reverse unit propagation (RUP) clauses

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is sequence of reverse unit propagation (RUP) clauses

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# More Ingredients in Proof Logging for SAT

## Fact

RUP proofs can be viewed as shorthand for resolution proofs

See [BN21] for more on this and connections to SAT solving

But RUP and resolution are not enough for preprocessing, inprocessing, and some other kinds of reasoning

# Extension Variables, Part 1

Suppose we want a variable  $a$  encoding

$$a \Leftrightarrow (x \wedge y)$$

Extended resolution [Tse68]

Resolution rule plus **extension rule** introducing clauses

$$a \vee \bar{x} \vee \bar{y} \quad \bar{a} \vee x \quad \bar{a} \vee y$$

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## Fact

Extended resolution (RUP + definition of new variables) is essentially equivalent to the DRAT proof logging system most commonly used for SAT solving

# Why Aren't We Done?

Practical limitations of current SAT proof logging technology:

- Difficulties dealing with stronger reasoning efficiently (even for SAT solving)
- Clausal proofs can't easily reflect what algorithms for other problems do

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Surprising claim: a slight change to **0-1 integer linear inequalities** does the job!

- Enables proof logging for **advanced SAT techniques** so far beyond reach for efficient DRAT proof logging:
  - Cardinality reasoning
  - Gaussian elimination
  - Symmetry breaking
- Supports use of SAT solvers for **optimisation problems (MaxSAT)**
- Can justify **graph reasoning** without knowing what a graph is
- Can justify **constraint programming** inference without knowing what an integer variable is

## Pseudo-Boolean Constraints

0-1 integer linear inequalities or (linear) pseudo-Boolean constraints:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
  - literals  $\ell_i$ :  $x_i$  or  $\bar{x}_i$  (where  $x_i + \bar{x}_i = 1$ )

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Sometimes convenient to use normalized form [Bar95] with all  $a_i, A$  positive (without loss of generality)

## Some Types of Pseudo-Boolean Constraints

1 Clauses

$$x_1 \vee \bar{x}_2 \vee x_3 \Leftrightarrow x_1 + \bar{x}_2 + x_3 \geq 1$$

## 2 Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

### 3 General pseudo-Boolean constraints

$$x_1 + 2\bar{x}_2 + 3x_3 + 4\bar{x}_4 + 5x_5 \geq 7$$

# Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

**Input/model axioms**

From the input

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**Division** for any  $c \in \mathbb{N}^+$   
(assumes normalized form)

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \lceil \frac{a_i}{c} \rceil \ell_i \geq \lceil \frac{A}{c} \rceil}$$

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Naming constraints by integers and literal axioms by the literal involved (with  $\sim$  for negation) as

$$\begin{aligned} \text{Constraint 1} &\doteq 2x + y + w \geq 2 \\ \text{Constraint 2} &\doteq 2x + 4y + 2z + w \geq 5 \\ \sim z &\doteq \bar{z} \geq 0 \end{aligned}$$

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such a calculation is written in the proof log in reverse Polish notation as

pol 1 2 \* 2 + ~z 2 \* + 3 d

# Resolution and Cutting Planes

To simulate resolution step such as

$$\frac{\bar{y} \vee \bar{z} \quad x \vee \bar{y} \vee z}{x \vee \bar{y}}$$

we can perform the cutting planes steps

$$\text{Add } \frac{\bar{y} + \bar{z} \geq 1 \quad x + \bar{y} + z \geq 1}{x + 2\bar{y} \geq 1}$$

Divide by 2  $\frac{x + 2\bar{y} \geq 1}{x + \bar{y} \geq 1}$

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Given that the premises are clauses 7 and 5 in our example CNF formula, using references

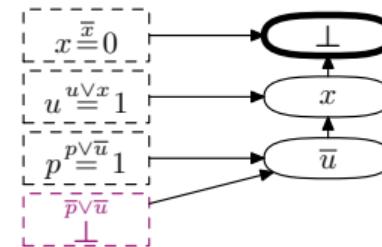
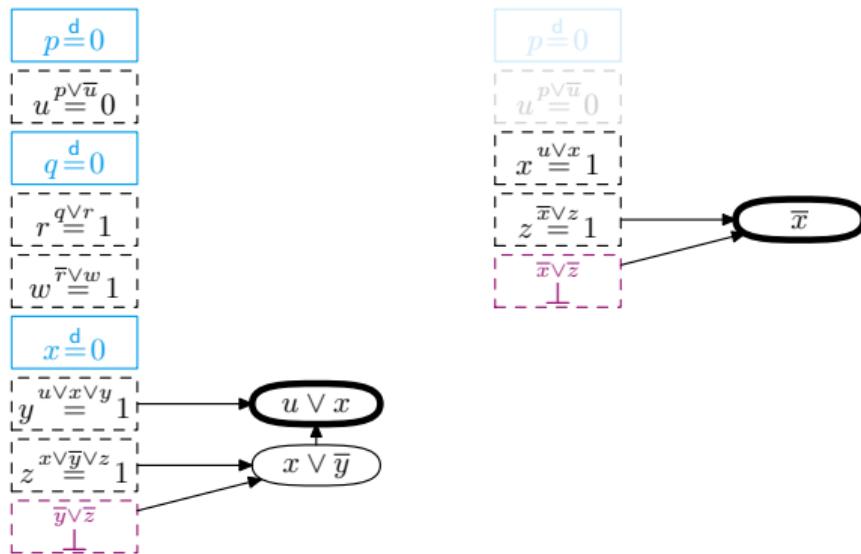
$$\text{Constraint 7} \doteq \bar{y} + \bar{z} \geq 1$$

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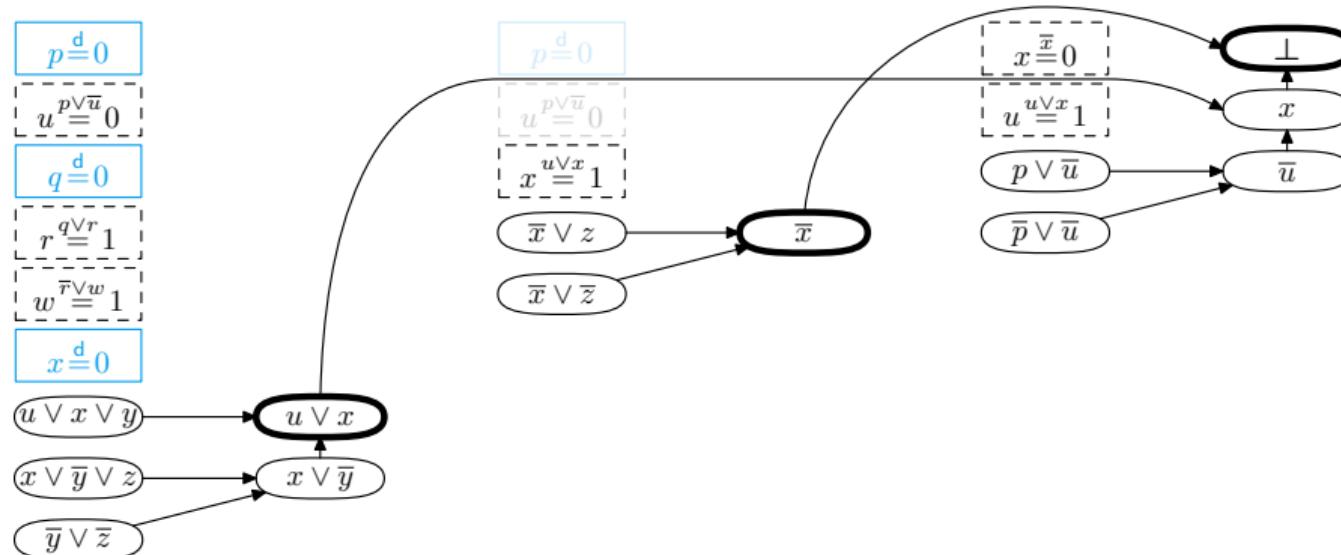
we can write this in the proof log as

pol 7 5 + 2 d

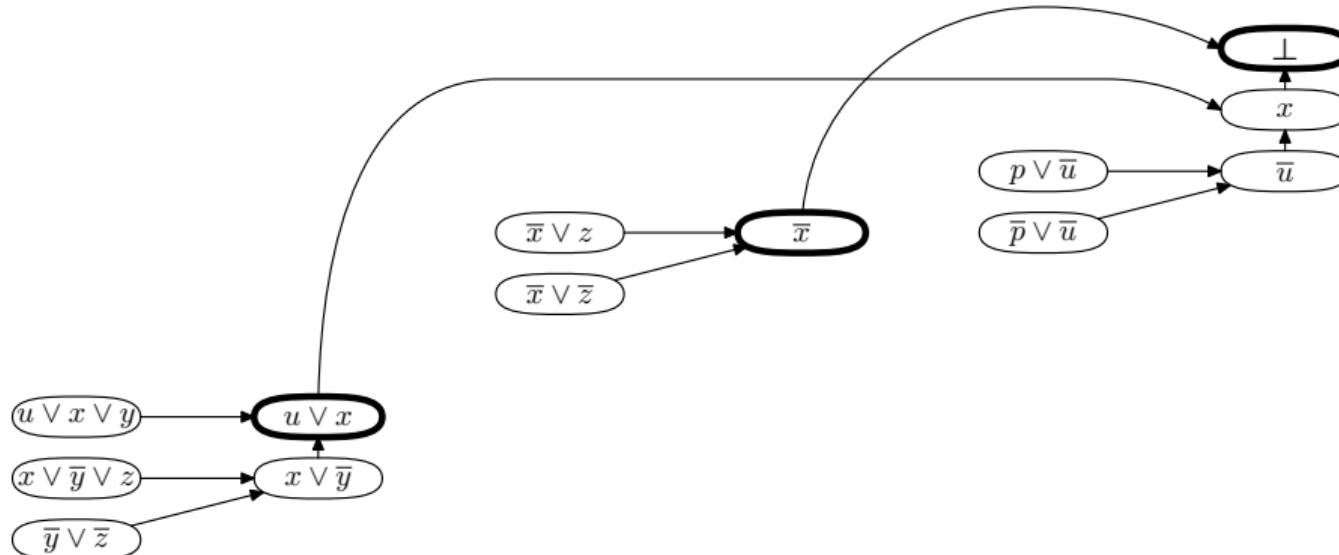
Pseudo-Boolean Proof Logging for Example CDCL Conflict Analyses



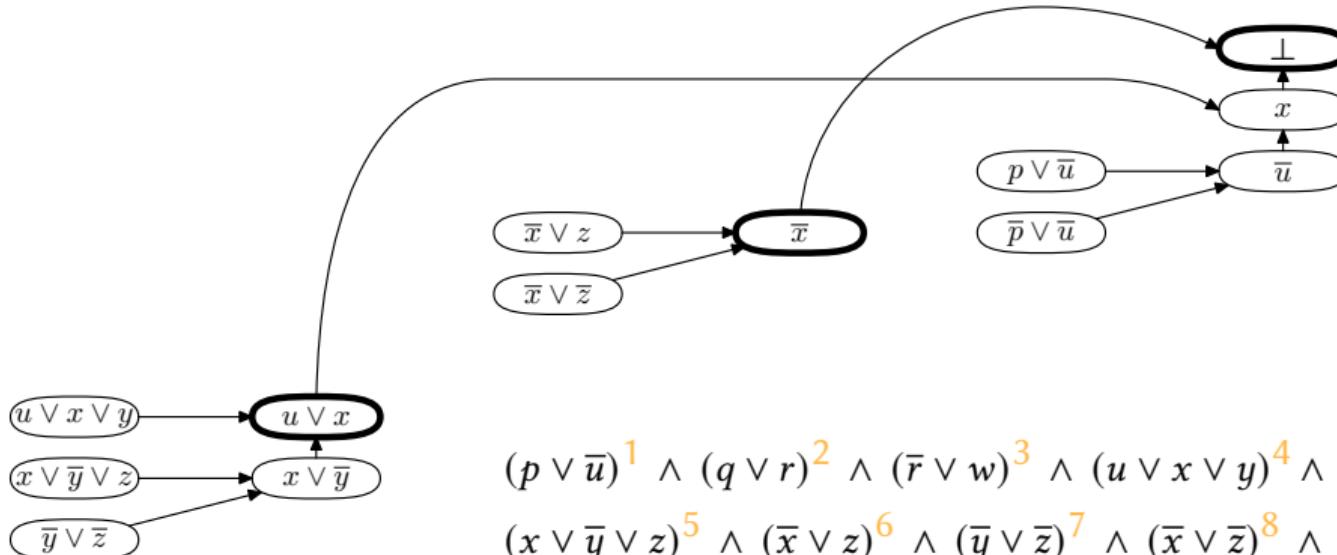
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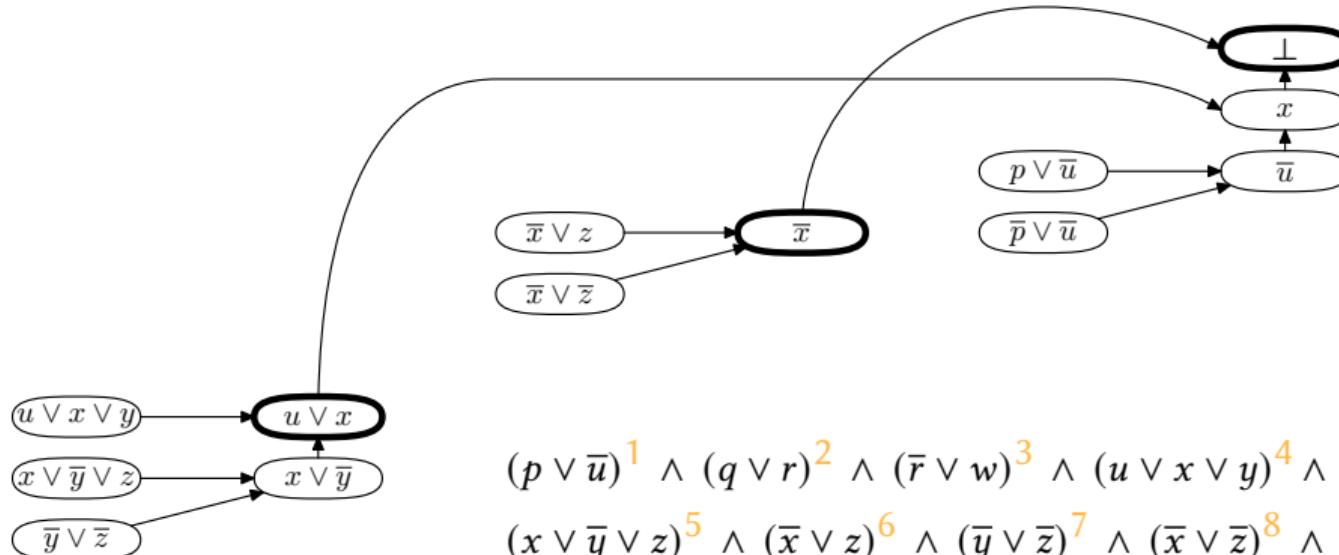


## Pseudo-Boolean Proof Logging for Example CDCL Conflict Analyses



$$(p \vee \bar{u})^1 \wedge (q \vee r)^2 \wedge (\bar{r} \vee w)^3 \wedge (u \vee x \vee y)^4 \wedge \\ (x \vee \bar{y} \vee z)^5 \wedge (\bar{x} \vee z)^6 \wedge (\bar{y} \vee \bar{z})^7 \wedge (\bar{x} \vee \bar{z})^8 \wedge (\bar{p} \vee \bar{u})^9$$

# Pseudo-Boolean Proof Logging for Example CDCL Conflict Analyses



# RUP Revisited

Can define (reverse) unit propagation in a pseudo-Boolean setting

Constraint  $C$  propagates variable  $x$  if setting  $x$  to “wrong value” would make  $C$  unsatisfiable

# RUP Revisited

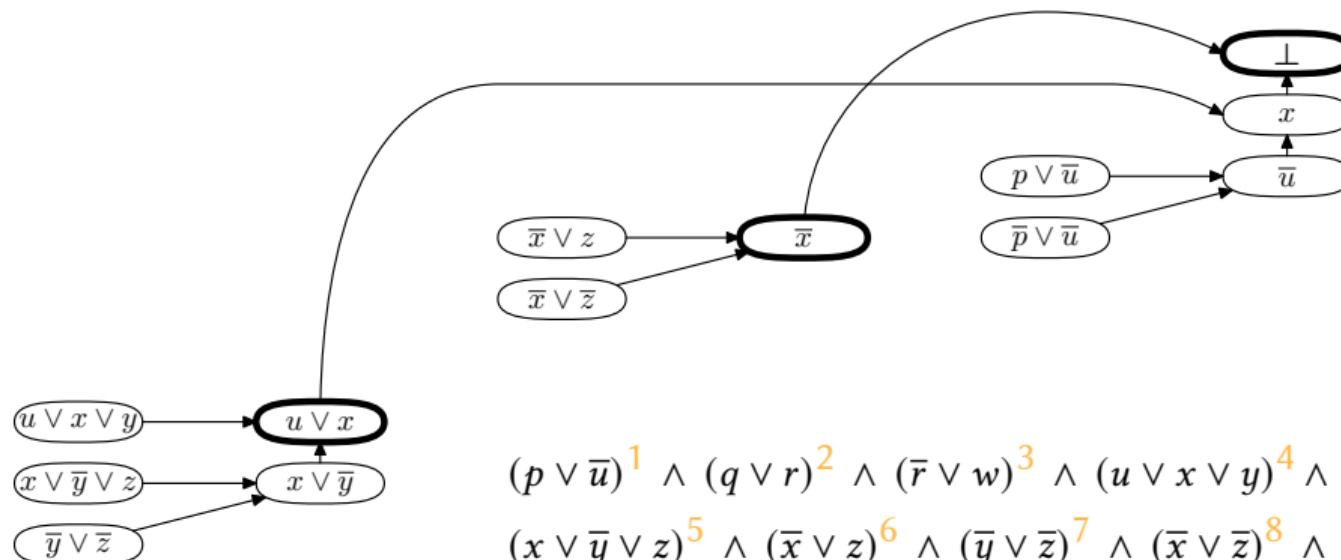
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Risk for confusion:

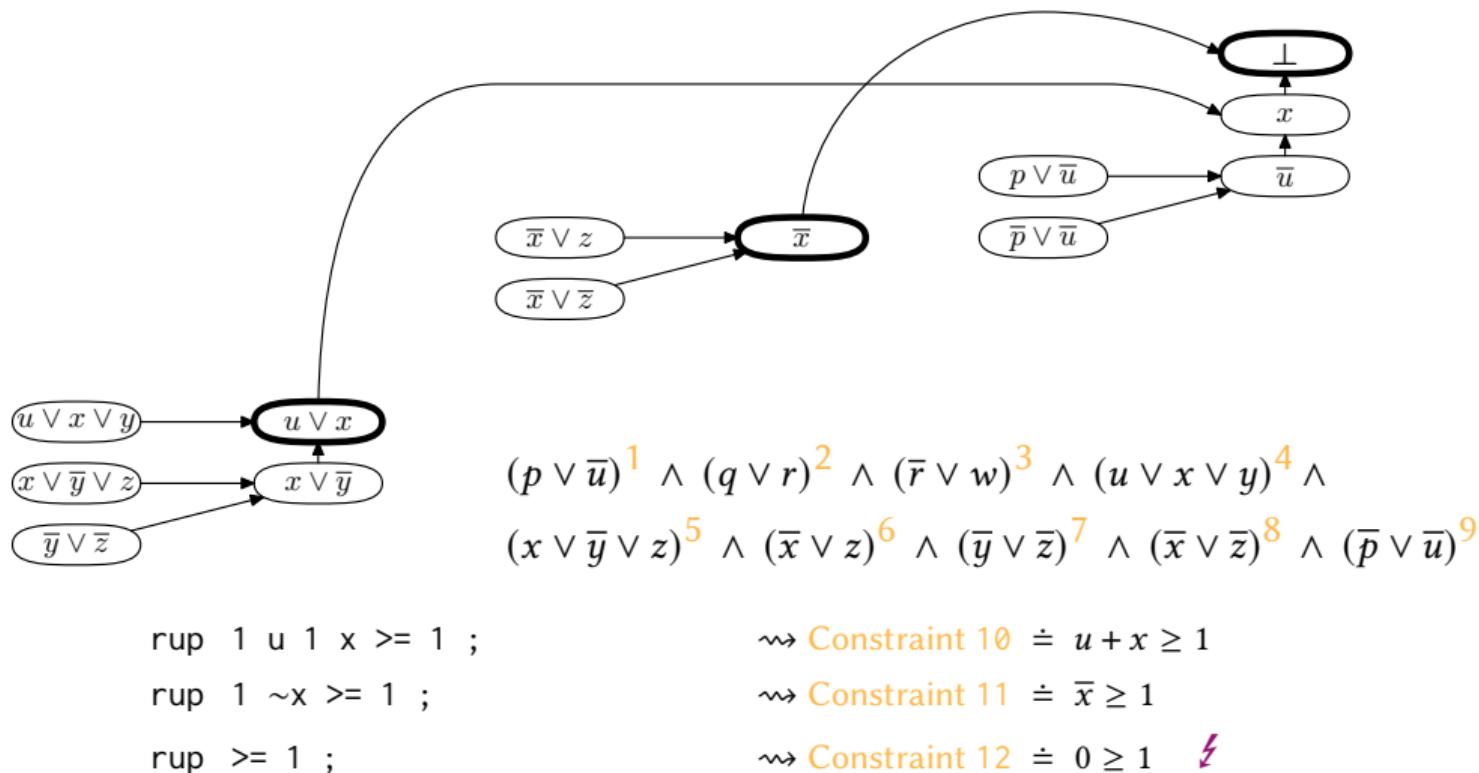
- Constraint programming people might call this (reverse) integer bounds consistency
  - Does the same thing if we’re working with clauses
  - More interesting for general pseudo-Boolean constraints
- SAT people beware: constraints can propagate multiple times and multiple variables

## Pseudo-Boolean Proof Logging for Example CDCL Execution with RUP



$$(p \vee \bar{u})^1 \wedge (q \vee r)^2 \wedge (\bar{r} \vee w)^3 \wedge (u \vee x \vee y)^4 \wedge \\ (x \vee \bar{y} \vee z)^5 \wedge (\bar{x} \vee z)^6 \wedge (\bar{y} \vee \bar{z})^7 \wedge (\bar{x} \vee \bar{z})^8 \wedge (\bar{p} \vee \bar{u})^9$$

# Pseudo-Boolean Proof Logging for Example CDCL Execution with RUP



## Extension Variables, Part 2

Suppose we want new, fresh variable  $a$  encoding

$$a \Leftrightarrow (3x + 2y + z + w \geq 3)$$

This time, introduce constraints

$$3\bar{a} + 3x + 2y + z + w \geq 3 \quad 5a + 3\bar{x} + 2\bar{y} + \bar{z} + \bar{w} \geq 5$$

Again, needs support from the proof system

# Proof Logs for “Extended Cutting Planes”

For satisfiable instances: just specify a satisfying assignment.

For unsatisfiability: a sequence of **pseudo-Boolean constraints** in (slight extension of) OPB format [RM16]

- Each constraint follows “obviously” from what is known so far
- Either implicitly, by RUP...
- Or by an explicit cutting planes derivation...
- Or as an extension variable reifying a new constraint\*
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(\*) Not actually implemented this way — details to come later...

# Deleting Constraints

In practice, important to erase constraints to save memory and time during verification

Fairly straightforward to deal with from the point of view of proof logging

So ignored in this tutorial for simplicity and clarity

# Enumeration and Optimisation Problems

Enumeration:

- When a solution is found, can log it
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- So the proof semantics is “infeasible, except for all the solutions I told you about”

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For optimisation:

- Define an objective  $f = \sum_i w_i \ell_i$ ,  $w_i \in \mathbb{Z}$ , to minimise subject to the constraints in the formula
- To maximise, negate objective
- Log a solution  $\alpha$ ; get an objective-improving constraint  $\sum_i w_i \ell_i \leq -1 + \sum_i w_i \alpha(\ell_i)$
- Semantics for proof of optimality: “infeasible to find better solution than best so far”

# Pseudo-Boolean Proof Logging – How and Why?

If problem is (special case of) 0–1 integer linear program (ILP)

- just do proof logging

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- 1 0–1 ILP **expressive formalism** for combinatorial problems (including objective)
- 2 **Powerful reasoning** capturing many combinatorial arguments (even for SAT)
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$$r \Rightarrow x_1 + 2\bar{x}_2 + 3x_3 + 4\bar{x}_4 + 5x_5 \geq 7$$

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$$7\bar{r} + x_1 + 2\bar{x}_2 + 3x_3 + 4\bar{x}_4 + 5x_5 \geq 7$$

$$9r + \bar{x}_1 + 2x_2 + 3\bar{x}_3 + 4x_4 + 5\bar{x}_5 \geq 9$$

# The VERIPB Format and Tool

<https://gitlab.com/MIA0research/software/VeriPB>



Released under MIT Licence

Various features to help development:

- Extended variable name syntax allowing human-readable names
- Proof tracing
- “Trust me” assertions for incremental proof logging

Documentation:

- Description of VERIPB checker [BMM<sup>+</sup>23] used in SAT 2023 competition (<https://satcompetition.github.io/2023/checkers.html>)
- Specific details on different proof logging techniques covered in research papers [EGMN20, GMN20, GMM<sup>+</sup>20, GN21, GMN22, GMNO22, VDB22, BBN<sup>+</sup>23, BGPN23, MM23]
- Lots of concrete example files at <https://gitlab.com/MIA0research/software/VeriPB>

# Parity (XOR) Reasoning

Given clauses

$$x \vee y \vee z$$

$$x \vee \bar{y} \vee \bar{z}$$

$$\bar{x} \vee y \vee \bar{z}$$

$$\bar{x} \vee \bar{y} \vee z$$

and

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$$x + y + z = 1 \pmod{2}$$

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But used in *CryptoMiniSat* [Cry]

DRAT proof logging like [PR16] too inefficient in practice!

Could add XORs to language, but prefer to keep things super-simple

# Pseudo-Boolean Proof Logging for XOR Reasoning

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Introduce extension variables  $a, b$  and derive

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(“=” syntactic sugar for “ $\geq$ ” plus “ $\leq$ ”)

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Add to get

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From this can extract

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VERIPB can certify XOR reasoning [GN21]

# CDCL Solvers on Pseudo-Boolean Inputs

Can re-encode to CNF and run CDCL:

- *MiniSat+* [ES06]
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E.g., encode pseudo-Boolean constraint

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

to clauses with extension variables

$$s_{i,k} \Leftrightarrow \sum_{j=1}^i x_j \geq k$$

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$$k \cdot \bar{s}_{i,k} + \sum_{j=1}^i x_j \geq k$$

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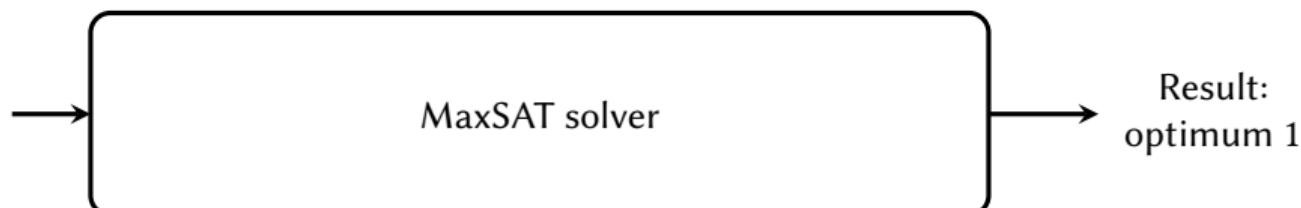
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VERIPB can certify **pseudo-Boolean-to-CNF rewriting** [GMNO22, VDB22]

# Certified Maximum Satisfiability (MaxSAT) Solving

Minimize linear objective subject to satisfying formula in conjunctive normal form (CNF)

$$\begin{aligned} \min & 2x_1 + x_2 \\ \text{s.t. } & x_1 \vee \bar{z} \\ & z \vee x_2 \end{aligned}$$

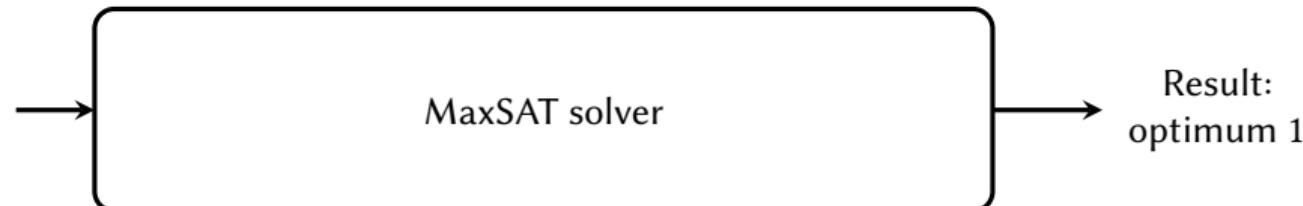


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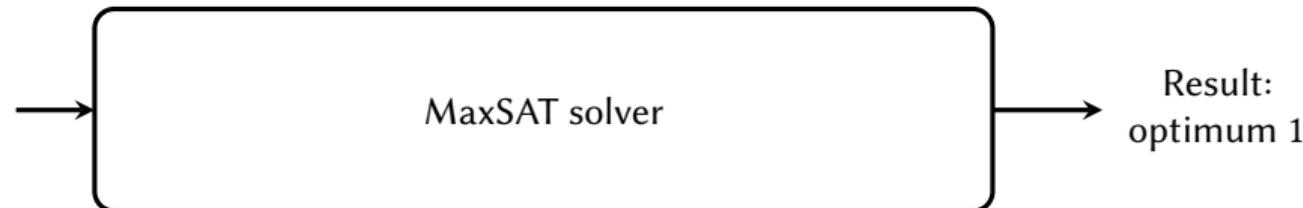
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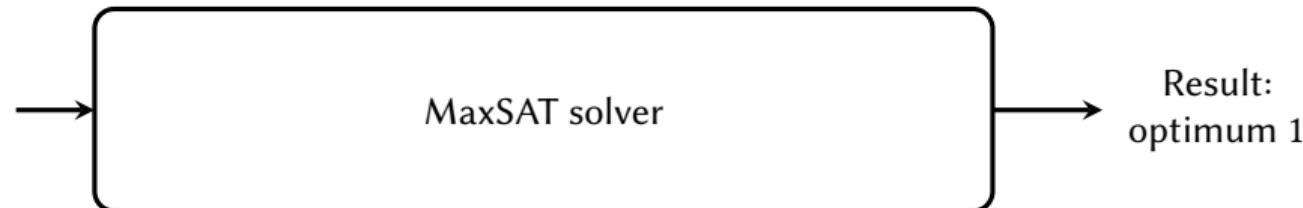
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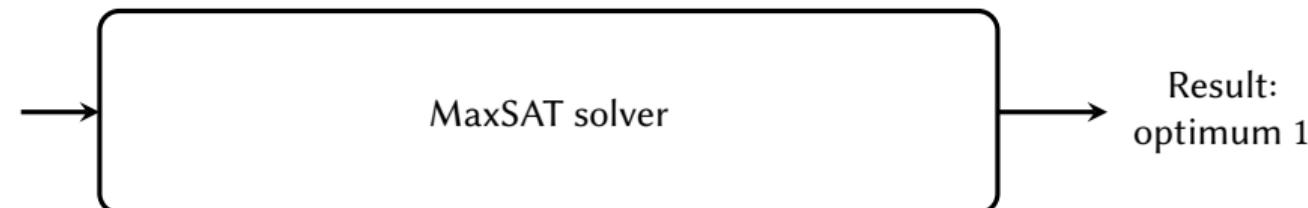
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**Does not work** Only proves answer correct, not reasoning within solver!

# MaxSAT Solvers

Three main categories:

- Linear SAT-UNSAT search
  - 1 Call SAT solver to find some solution
  - 2 Add clauses encoding “I want a better solution”
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No proof logging available **yet**

# MaxSAT example (LSU search)

Objective:  $\min \sum_i r_i$

VERIPB proof:

derived

justification

$$\bar{x}_1 \vee x_2$$

$$x_1 \vee \bar{x}_2$$

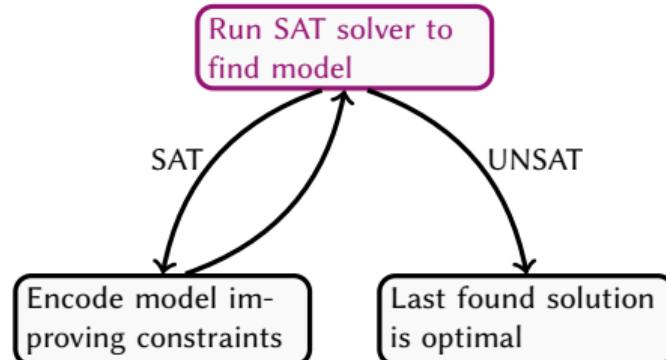
$$\bar{x}_2 \vee x_3$$

$$\bar{x}_3 \vee x_4$$

$$\bar{x}_1 \vee \bar{x}_2 \vee r_1$$

$$x_1 \vee x_2 \vee r_2$$

$$x_2 \vee x_4 \vee r_3$$



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$$\bar{x}_2 \vee x_3$$

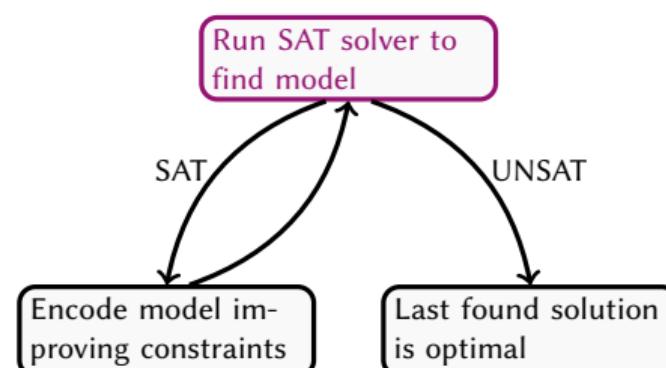
$$\bar{x}_3 \vee x_4$$

$$\bar{x}_1 \vee \bar{x}_2 \vee r_1$$

$$x_1 \vee x_2 \vee r_2$$

$$x_2 \vee x_4 \vee r_3$$

$$x_2 \vee r_2$$



# MaxSAT example (LSU search)

Objective:  $\min \sum_i r_i$

VERIPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation

$$\bar{x}_1 \vee x_2$$

$$x_1 \vee \bar{x}_2$$

$$\bar{x}_2 \vee x_3$$

$$\bar{x}_3 \vee x_4$$

$$\bar{x}_1 \vee \bar{x}_2 \vee r_1$$

$$x_1 \vee x_2 \vee r_2$$

$$x_2 \vee x_4 \vee r_3$$

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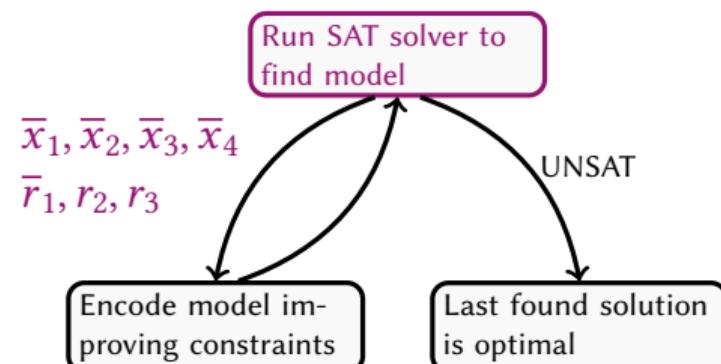
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$$\begin{array}{ll}\overline{x}_1 \vee x_2 & \overline{x}_1 \vee \overline{x}_2 \vee r_1 \\x_1 \vee \overline{x}_2 & x_1 \vee x_2 \vee r_2 \\\overline{x}_2 \vee x_3 & x_2 \vee x_4 \vee r_3 \\\overline{x}_3 \vee x_4 & \textcolor{violet}{x}_2 \vee \textcolor{violet}{r}_2\end{array}$$



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Objective:  $\min \sum_i r_i$

VERIPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\bar{x}_1, \dots, \bar{x}_4, \bar{r}_1, r_2, r_3\}$	Incumbent solution

$$\bar{x}_1 \vee x_2$$

$$x_1 \vee \bar{x}_2$$

$$\bar{x}_2 \vee x_3$$

$$\bar{x}_3 \vee x_4$$

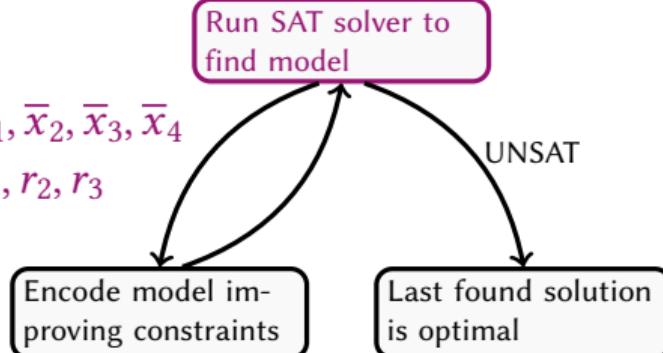
$$\bar{x}_1 \vee \bar{x}_2 \vee r_1$$

$$x_1 \vee x_2 \vee r_2$$

$$x_2 \vee x_4 \vee r_3$$

$$x_2 \vee r_2$$

$\bar{x}_1, \bar{x}_2, \bar{x}_3, \bar{x}_4$   
 $\bar{r}_1, r_2, r_3$



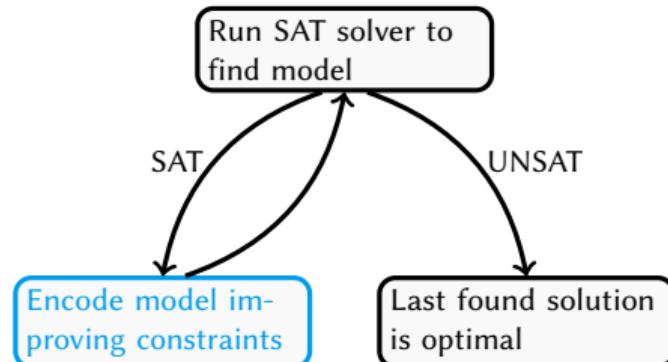
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$x_2 + r_2 \geq 1$	Reverse Unit Propagation
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$\sum_i r_i \leq 1$	Objective Improvement Rule

$$\begin{array}{ll}\bar{x}_1 \vee x_2 & \bar{x}_1 \vee \bar{x}_2 \vee r_1 \\x_1 \vee \bar{x}_2 & x_1 \vee x_2 \vee r_2 \\\bar{x}_2 \vee x_3 & x_2 \vee x_4 \vee r_3 \\\bar{x}_3 \vee x_4 & \textcolor{violet}{x_2 \vee r_2}\end{array}$$



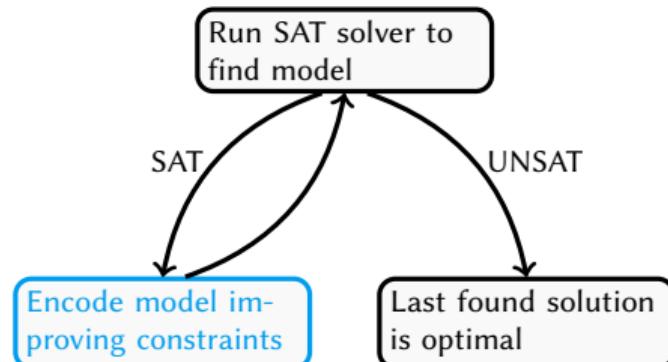
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$\text{PB}(p_1 \Leftrightarrow (\sum_i r_i \geq 1))$	Fresh variable (RBS)
$\text{PB}(p_2 \Leftrightarrow (\sum_i r_i \geq 2))$	

$$\begin{array}{ll}\bar{x}_1 \vee x_2 & \bar{x}_1 \vee \bar{x}_2 \vee r_1 \\x_1 \vee \bar{x}_2 & x_1 \vee x_2 \vee r_2 \\\bar{x}_2 \vee x_3 & x_2 \vee x_4 \vee r_3 \\\bar{x}_3 \vee x_4 & \textcolor{violet}{x_2 \vee r_2}\end{array}$$



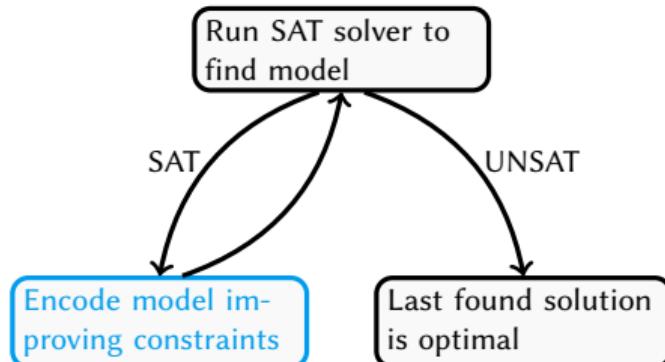
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$j \cdot \bar{p}_j + \sum_i r_i \geq j$	Fresh variable (RBS)
$(4 - j) \cdot p_j + \sum_i \bar{r}_i \geq 4 - j$	

$$\begin{array}{ll}\bar{x}_1 \vee x_2 & \bar{x}_1 \vee \bar{x}_2 \vee r_1 \\x_1 \vee \bar{x}_2 & x_1 \vee x_2 \vee r_2 \\\bar{x}_2 \vee x_3 & x_2 \vee x_4 \vee r_3 \\\bar{x}_3 \vee x_4 & \textcolor{violet}{x}_2 \vee \textcolor{violet}{r}_2\end{array}$$



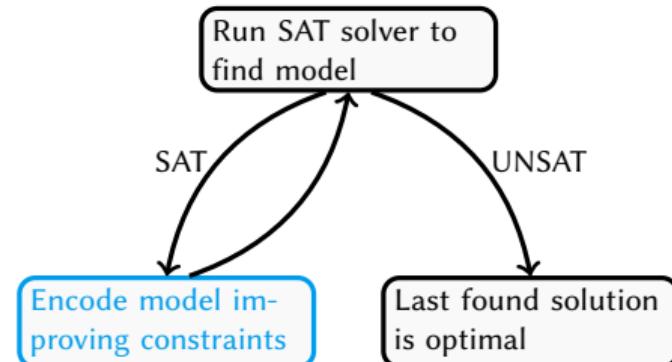
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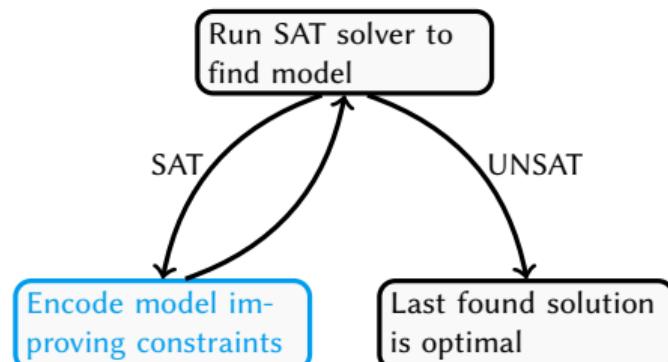
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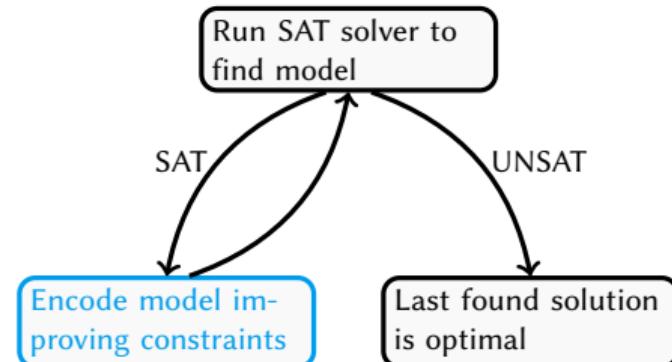
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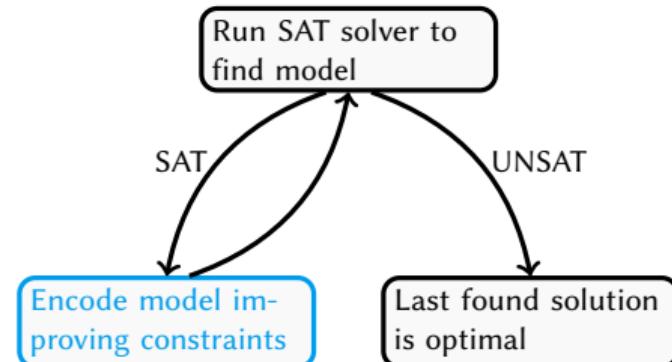
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$(4 - j) \cdot p_j + \sum_i \bar{r}_i \geq 4 - j$	Explicit CP derivation
$\text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
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 \bar{x}_3 \vee x_4 & x_2 \vee r_2 \\
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 \bar{p}_2 &
 \end{array}$$



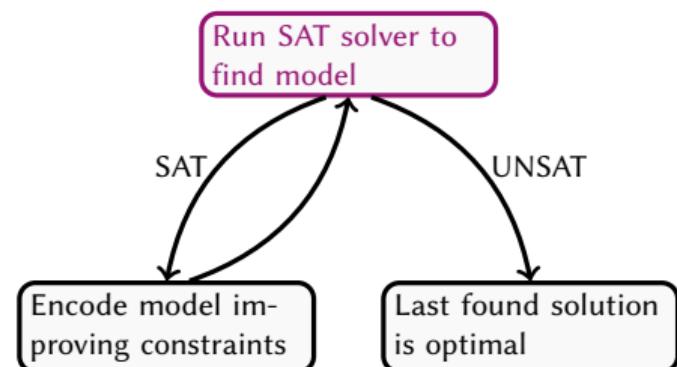
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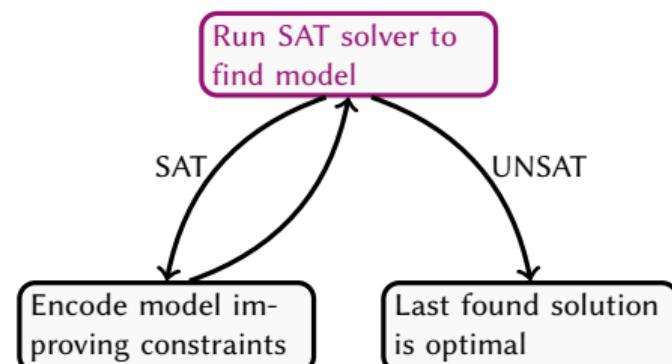
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$\text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\bar{p}_2 \geq 1$	Reverse Unit Propagation
$x_4 \geq 1$	

$$\begin{array}{ll}
 \bar{x}_1 \vee x_2 & \bar{x}_1 \vee \bar{x}_2 \vee r_1 \\
 x_1 \vee \bar{x}_2 & x_1 \vee x_2 \vee r_2 \\
 \bar{x}_2 \vee x_3 & x_2 \vee x_4 \vee r_3 \\
 \bar{x}_3 \vee x_4 & x_2 \vee r_2 \\
 \text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j)) & \\
 \bar{p}_2 & x_4
 \end{array}$$



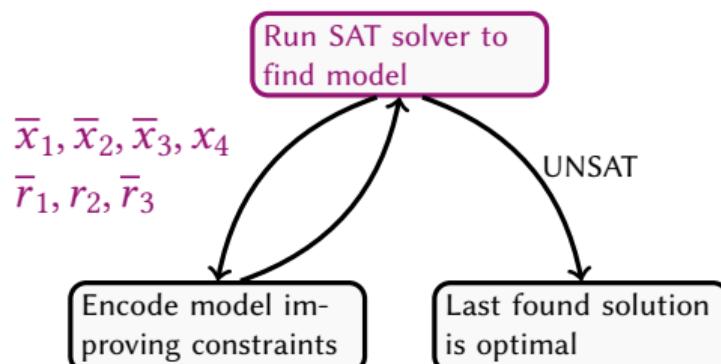
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Objective:  $\min \sum_i r_i$

VERIPB proof:

derived	justification
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$\{\bar{x}_1, \bar{x}_2, \bar{x}_3, x_4, \bar{r}_1, r_2, \bar{r}_3\}$	

$$\begin{array}{ll}
 \bar{x}_1 \vee x_2 & \bar{x}_1 \vee \bar{x}_2 \vee r_1 \\
 x_1 \vee \bar{x}_2 & x_1 \vee x_2 \vee r_2 \\
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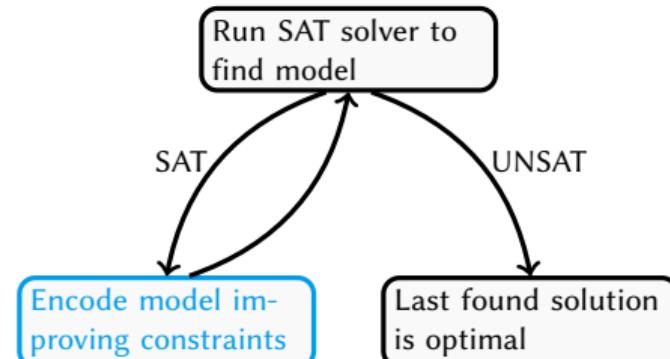
# MaxSAT example (LSU search)

Objective:  $\min \sum_i r_i$

VERIPB proof:

derived	justification
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$\{\bar{x}_1, \dots, \bar{x}_4, \bar{r}_1, r_2, r_3\}$	Incumbent solution
$\sum_i r_i \leq 1$	Objective Improvement Rule
$j \cdot \bar{p}_j + \sum_i r_i \geq j$	Fresh variable (RBS)
$(4 - j) \cdot p_j + \sum_i \bar{r}_i \geq 4 - j$	Explicit CP derivation
$\text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\bar{p}_2 \geq 1$	Reverse Unit Propagation
$x_4 \geq 1$	Incumbent solution
$\{\bar{x}_1, \bar{x}_2, \bar{x}_3, x_4, \bar{r}_1, r_2, \bar{r}_3\}$	Objective Improvement Rule
$\sum_i r_i \leq 0$	

$$\begin{array}{ll}
 \bar{x}_1 \vee x_2 & \bar{x}_1 \vee \bar{x}_2 \vee r_1 \\
 x_1 \vee \bar{x}_2 & x_1 \vee x_2 \vee r_2 \\
 \bar{x}_2 \vee x_3 & x_2 \vee x_4 \vee r_3 \\
 \bar{x}_3 \vee x_4 & x_2 \vee r_2 \\
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 \end{array}$$



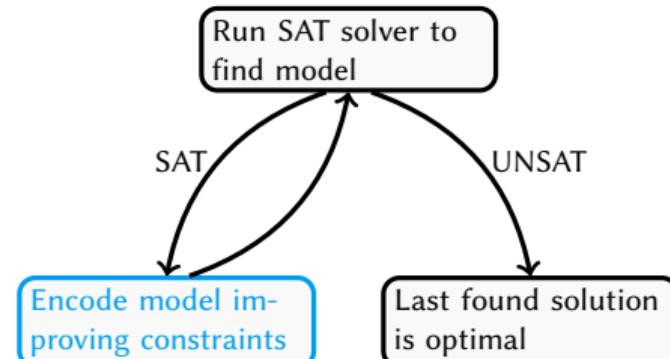
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Objective:  $\min \sum_i r_i$

VERIPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
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$(4 - j) \cdot p_j + \sum_i \bar{r}_i \geq 4 - j$	Explicit CP derivation
$\text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\bar{p}_2 \geq 1$	Reverse Unit Propagation
$x_4 \geq 1$	Incumbent solution
$\{\bar{x}_1, \bar{x}_2, \bar{x}_3, x_4, \bar{r}_1, r_2, \bar{r}_3\}$	Objective Improvement Rule
$\sum_i r_i \leq 0$	Explicit CP derivation
$\bar{p}_1 \geq 1$	

$$\begin{array}{ll}
 \bar{x}_1 \vee x_2 & \bar{x}_1 \vee \bar{x}_2 \vee r_1 \\
 x_1 \vee \bar{x}_2 & x_1 \vee x_2 \vee r_2 \\
 \bar{x}_2 \vee x_3 & x_2 \vee x_4 \vee r_3 \\
 \bar{x}_3 \vee x_4 & x_2 \vee r_2 \\
 \text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j)) & \\
 \bar{p}_2 & x_4 \\
 \bar{p}_1 &
 \end{array}$$



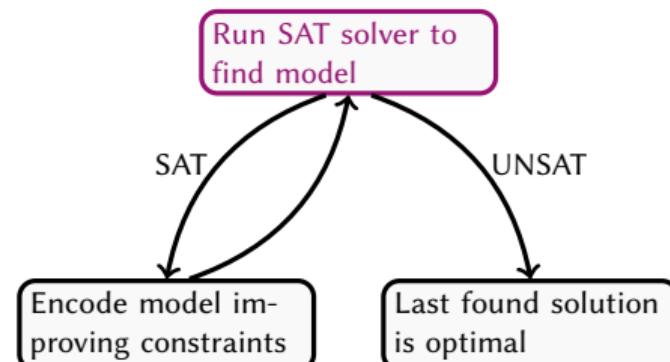
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VERIPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
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$\{\bar{x}_1, \bar{x}_2, \bar{x}_3, x_4, \bar{r}_1, r_2, \bar{r}_3\}$	Objective Improvement Rule
$\sum_i r_i \leq 0$	Explicit CP derivation
$\bar{p}_1 \geq 1$	

$$\begin{array}{ll}
 \bar{x}_1 \vee x_2 & \bar{x}_1 \vee \bar{x}_2 \vee r_1 \\
 x_1 \vee \bar{x}_2 & x_1 \vee x_2 \vee r_2 \\
 \bar{x}_2 \vee x_3 & x_2 \vee x_4 \vee r_3 \\
 \bar{x}_3 \vee x_4 & x_2 \vee r_2 \\
 \text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j)) & \\
 \bar{p}_2 & x_4 \\
 \bar{p}_1 &
 \end{array}$$



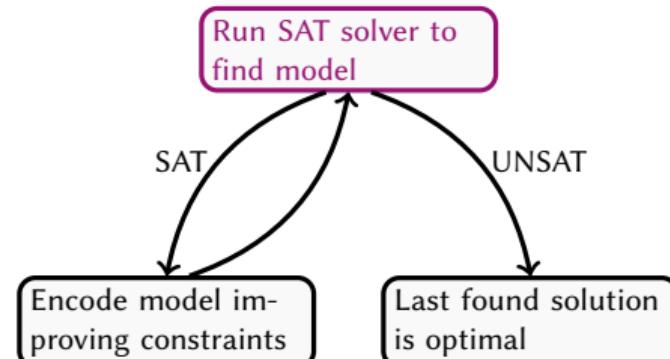
# MaxSAT example (LSU search)

Objective:  $\min \sum_i r_i$

VERIPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\bar{x}_1, \dots, \bar{x}_4, \bar{r}_1, r_2, r_3\}$	Incumbent solution
$\sum_i r_i \leq 1$	Objective Improvement Rule
$j \cdot \bar{p}_j + \sum_i r_i \geq j$	Fresh variable (RBS)
$(4 - j) \cdot p_j + \sum_i \bar{r}_i \geq 4 - j$	Explicit CP derivation
$\text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j))$	Explicit CP derivation
$\bar{p}_2 \geq 1$	Reverse Unit Propagation
$x_4 \geq 1$	Incumbent solution
$\{\bar{x}_1, \bar{x}_2, \bar{x}_3, x_4, \bar{r}_1, r_2, \bar{r}_3\}$	Objective Improvement Rule
$\sum_i r_i \leq 0$	Explicit CP derivation
$\bar{p}_1 \geq 1$	Reverse Unit Propagation
$0 \geq 1$	

$$\begin{array}{ll}
 \bar{x}_1 \vee x_2 & \bar{x}_1 \vee \bar{x}_2 \vee r_1 \\
 x_1 \vee \bar{x}_2 & x_1 \vee x_2 \vee r_2 \\
 \bar{x}_2 \vee x_3 & x_2 \vee x_4 \vee r_3 \\
 \bar{x}_3 \vee x_4 & x_2 \vee r_2 \\
 \text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j)) & \\
 \bar{p}_2 & x_4 \\
 \bar{p}_1 & \perp
 \end{array}$$



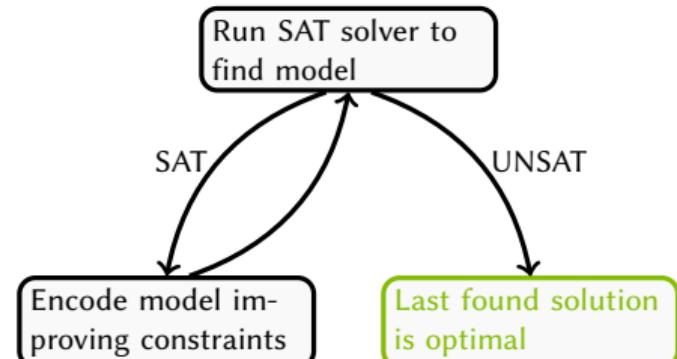
# MaxSAT example (LSU search)

Objective:  $\min \sum_i r_i$

VERIPB proof:

derived	justification
$x_2 + r_2 \geq 1$	Reverse Unit Propagation
$\{\bar{x}_1, \dots, \bar{x}_4, \bar{r}_1, r_2, r_3\}$	Incumbent solution
$\sum_i r_i \leq 1$	Objective Improvement Rule
$j \cdot \bar{p}_j + \sum_i r_i \geq j$	Fresh variable (RBS)
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$\bar{p}_2 \geq 1$	Reverse Unit Propagation
$x_4 \geq 1$	Incumbent solution
$\{\bar{x}_1, \bar{x}_2, \bar{x}_3, x_4, \bar{r}_1, r_2, \bar{r}_3\}$	Objective Improvement Rule
$\sum_i r_i \leq 0$	Explicit CP derivation
$\bar{p}_1 \geq 1$	Reverse Unit Propagation
$0 \geq 1$	

$$\begin{array}{ll}
 \bar{x}_1 \vee x_2 & \bar{x}_1 \vee \bar{x}_2 \vee r_1 \\
 x_1 \vee \bar{x}_2 & x_1 \vee x_2 \vee r_2 \\
 \bar{x}_2 \vee x_3 & x_2 \vee x_4 \vee r_3 \\
 \bar{x}_3 \vee x_4 & x_2 \vee r_2 \\
 \text{CNF}(p_j \Leftrightarrow (\sum_i r_i \geq j)) & \\
 \bar{p}_2 & x_4 \\
 \bar{p}_1 & \perp
 \end{array}$$



# Progress So Far

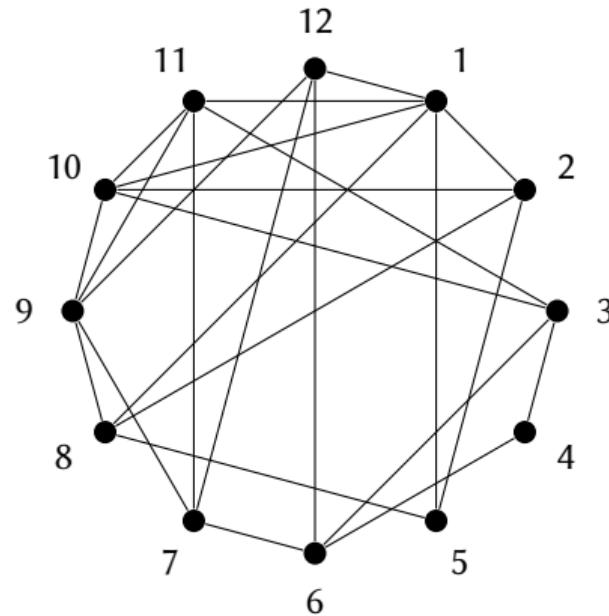
We've seen proof logging, and how it works for SAT

We've learned about

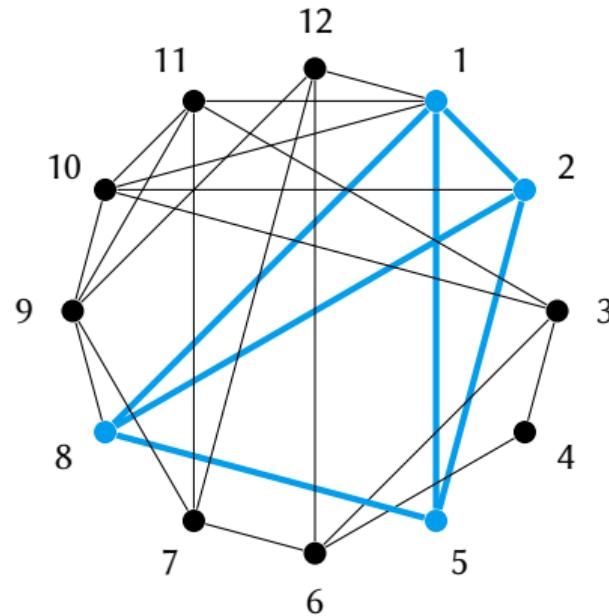
- pseudo-Boolean constraints (0–1 linear inequalities)
- cutting planes reasoning
- VERIPB

Coming next, some worked examples from dedicated graph solvers

# The Maximum Clique Problem



# The Maximum Clique Problem



# Maximum Clique Solvers

There are a lot of dedicated solvers for clique problems

But there are issues:

- “State-of-the-art” solvers have been buggy.
- Often undetected: error rate of around 0.1 [MPP19]

Often used inside other solvers

- An off-by-one result can cause much larger errors

# A Brief and Incomplete Guide to Clique Solving (1/4)

Recursive maximum clique algorithm:

- Pick a vertex  $v$
- Either  $v$  is in the clique...
  - Throw away every vertex not adjacent to  $v$
  - If vertices remain, recurse
- ...or  $v$  is not in the clique
  - Throw  $v$  away and pick another vertex

# A Brief and Incomplete Guide to Clique Solving (2/4)

Key data structures:

- Growing clique  $C$
- Set of potential vertices  $P$ 
  - All the vertices we haven't thrown away yet
  - Every  $v \in P$  is adjacent to every  $w \in C$

# A Brief and Incomplete Guide to Clique Solving (2/4)

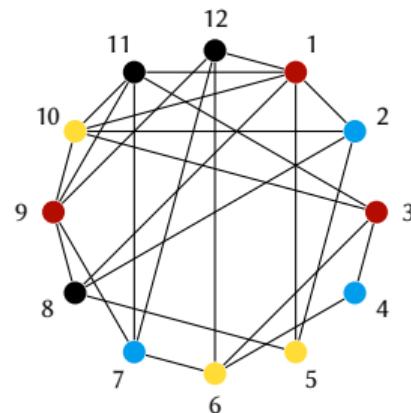
Key data structures:

- Growing clique  $C$
- Set of potential vertices  $P$ 
  - All the vertices we haven't thrown away yet
  - Every  $v \in P$  is adjacent to every  $w \in C$

Branch and bound:

- Remember the biggest clique  $C^*$  found so far
- If  $|C| + |P| \leq |C^*|$ , no need to keep going

# A Brief and Incomplete Guide to Clique Solving (3/4)



Given a  $k$ -colouring of a subgraph, that subgraph cannot have a clique of more than  $k$  vertices  
We can use  $|C| + \#colours(P)$  as a bound, for any colouring

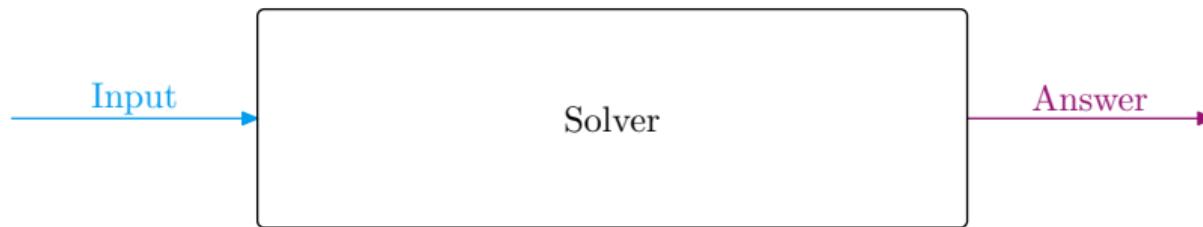
# A Brief and Incomplete Guide to Clique Solving (4/4)

- This brings us to 1997
- Many improvements since then
  - better bound functions
  - clever vertex selection heuristics
  - efficient data structures
  - local search
  - ...
- But key ideas for proof logging can be explained without worrying about such things

# Making a Proof Logging Clique Solver

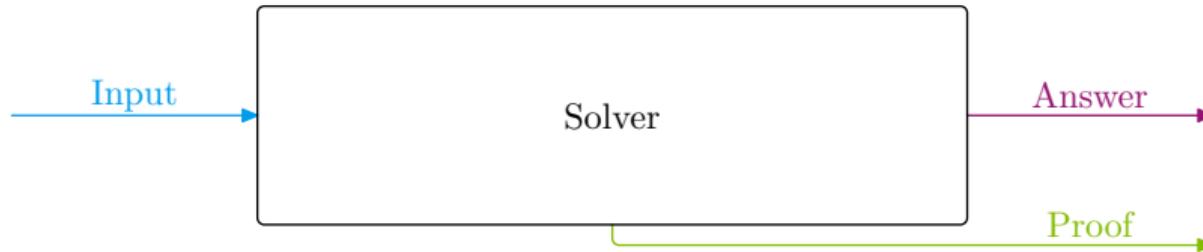
- 1** Output a pseudo-Boolean encoding of the problem
  - Clique problems have several standard file formats
- 2** Make the solver log its search tree
  - Output a small header
  - Output something on every backtrack
  - Output something every time a solution is found
  - Output a small footer
- 3** Figure out how to log the bound function

# A Slightly Different Proof Logging Workflow



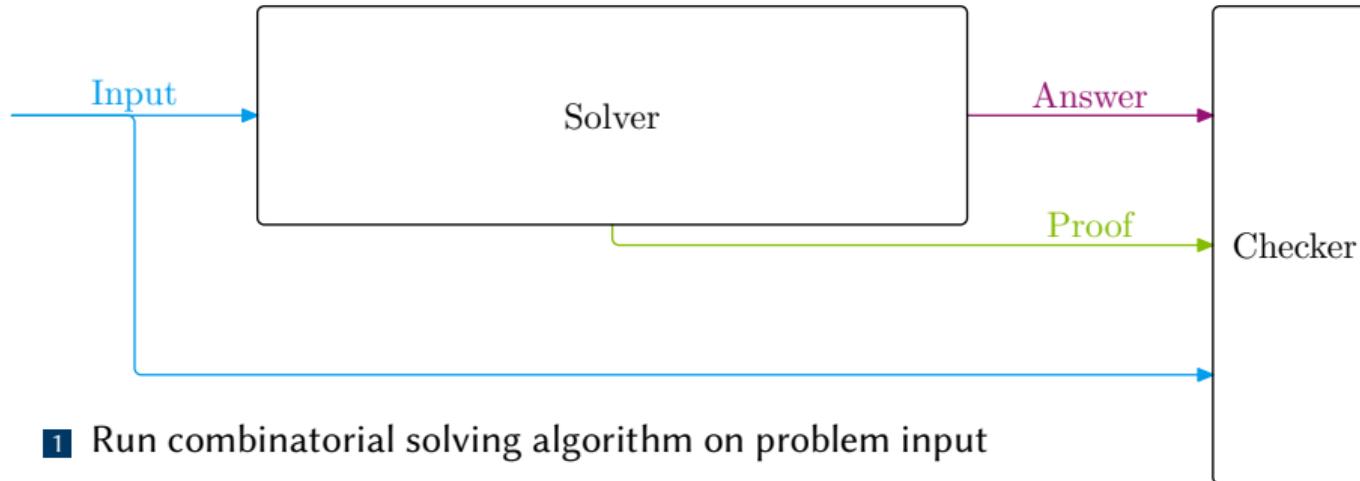
- 1 Run combinatorial solving algorithm on problem input

# A Slightly Different Proof Logging Workflow



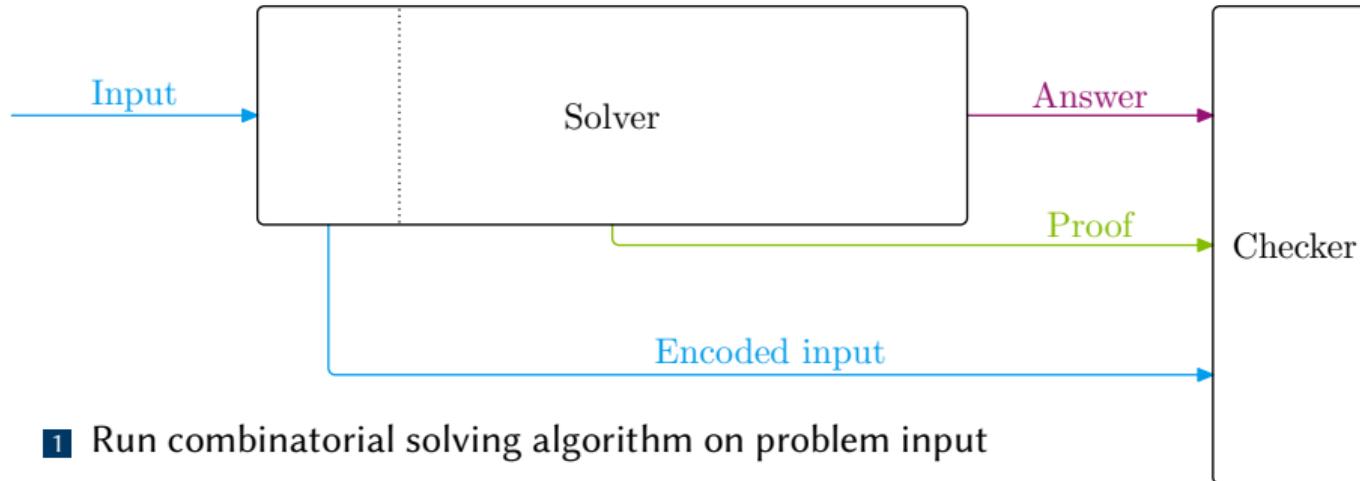
- 1 Run combinatorial solving algorithm on problem input
- 2 Get as output not only answer but also proof

# A Slightly Different Proof Logging Workflow



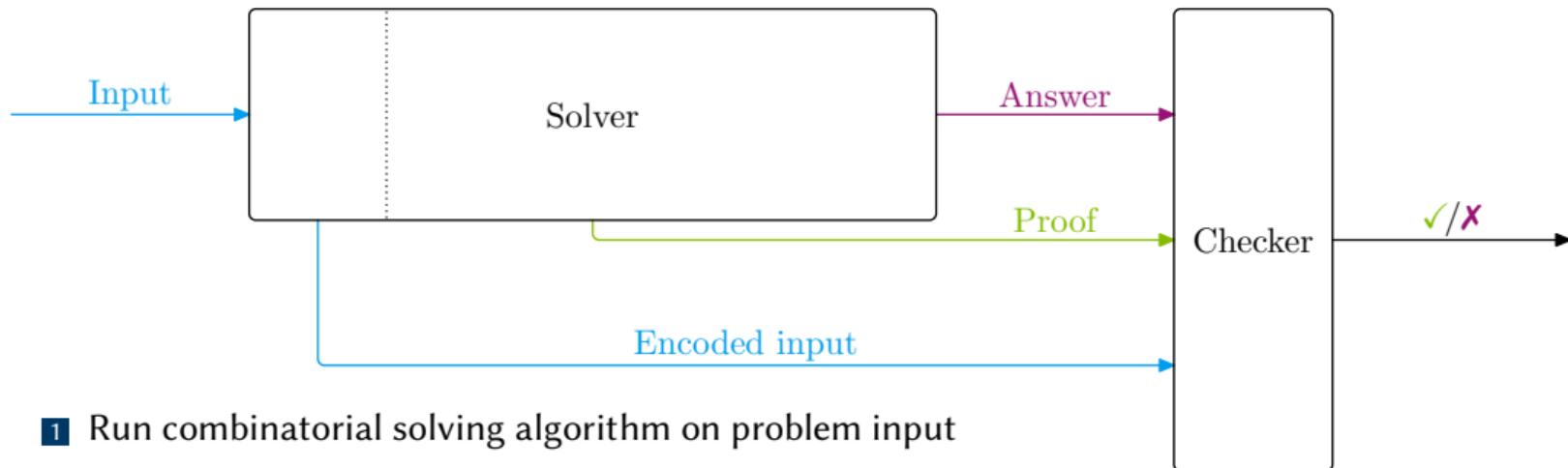
- 1 Run combinatorial solving algorithm on problem input
- 2 Get as output not only answer but also proof
- 3 Feed answer + proof to proof checker together with input

# A Slightly Different Proof Logging Workflow



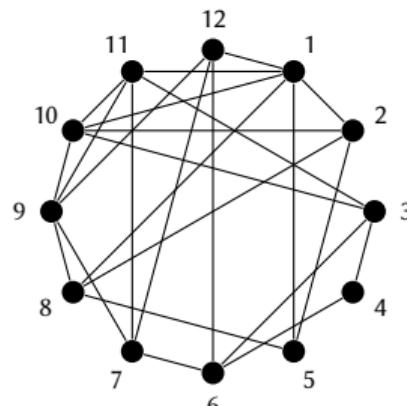
- 1 Run combinatorial solving algorithm on problem input
- 2 Get as output not only answer but also proof
- 3 Feed answer + proof to proof checker together with encoded input

# A Slightly Different Proof Logging Workflow



- 1 Run combinatorial solving algorithm on problem input
- 2 Get as output not only answer but also proof
- 3 Feed answer + proof to proof checker together with **encoded input**
- 4 Verify that proof checker says answer is correct

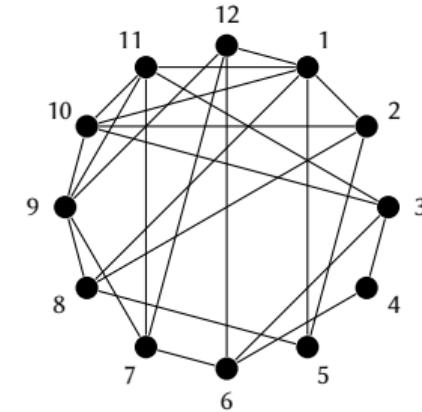
# A Pseudo-Boolean Encoding for Clique (in OPB Format)



```
* #variable= 12 #constraint= 41
min: -1 x1 -1 x2 -1 x3 -1 x4 . . . and so on. . . -1 x11 -1 x12 ;
1 ~x3 1 ~x1 >= 1 ;
1 ~x3 1 ~x2 >= 1 ;
1 ~x4 1 ~x1 >= 1 ;
* . . . and a further 38 similar lines for the remaining non-edges
```

# First Attempt at a Proof

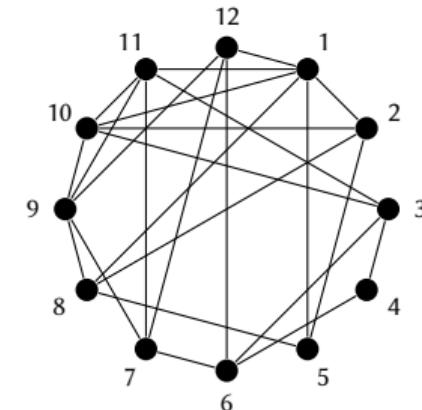
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



# First Attempt at a Proof

```
pseudo-Boolean proof version 2.0  
f 41
```

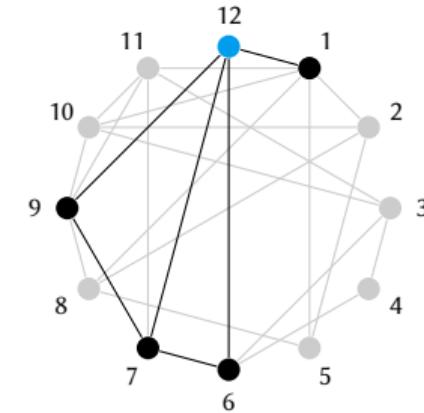
```
soli x7 x9 x12  
rup 1 ~x12 1 ~x7 >= 1 ;  
rup 1 ~x12 >= 1 ;  
rup 1 ~x11 1 ~x10 >= 1 ;  
rup 1 ~x11 >= 1 ;  
soli x1 x2 x5 x8  
rup 1 ~x8 1 ~x5 >= 1 ;  
rup 1 ~x8 >= 1 ;  
rup >= 1 ;  
output NONE  
conclusion BOUNDS -4 -4  
end pseudo-Boolean proof
```



```
Start with a header  
Load the 41 problem axioms
```

# First Attempt at a Proof

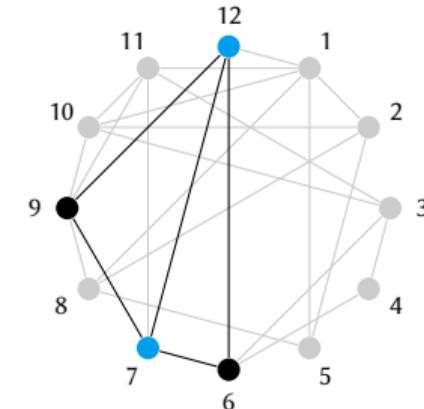
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
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rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Branch accepting 12  
Throw away non-adjacent vertices

# First Attempt at a Proof

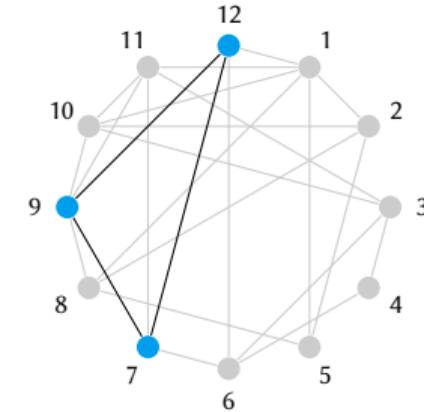
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Branch also accepting 7  
Throw away non-adjacent vertices

# First Attempt at a Proof

```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Branch also accepting 9  
Throw away non-adjacent vertices

# First Attempt at a Proof

pseudo-Boolean proof version 2.0

f 41

soli x7 x9 x12

rup 1 ~x12 1 ~x7 >= 1 ;

rup 1 ~x12 >= 1 ;

rup 1 ~x11 1 ~x10 >= 1 ;

rup 1 ~x11 >= 1 ;

soli x1 x2 x5 x8

rup 1 ~x8 1 ~x5 >= 1 ;

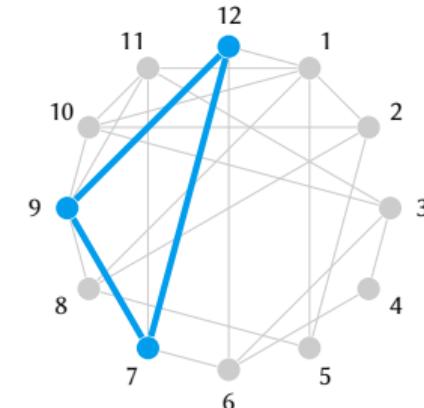
rup 1 ~x8 >= 1 ;

rup >= 1 ;

output NONE

conclusion BOUNDS -4 -4

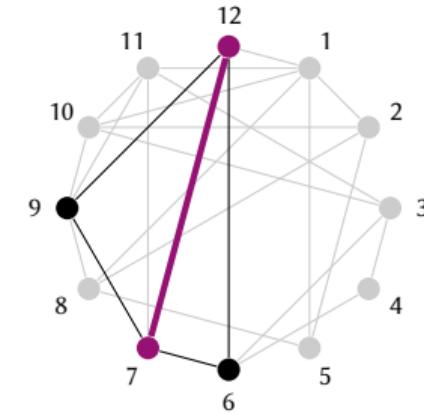
end pseudo-Boolean proof



We branched on 12, 7, 9  
Found a new incumbent  
Now looking for a  $\geq 4$  vertex clique

# First Attempt at a Proof

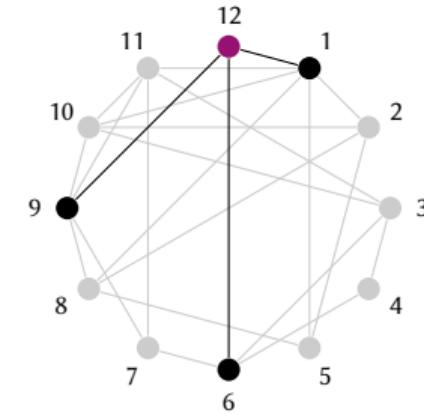
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Backtrack from 12, 7  
9 explored already, only 6 feasible  
No  $\geq 4$  vertex clique possible  
Effectively this deletes the 7–12 edge

# First Attempt at a Proof

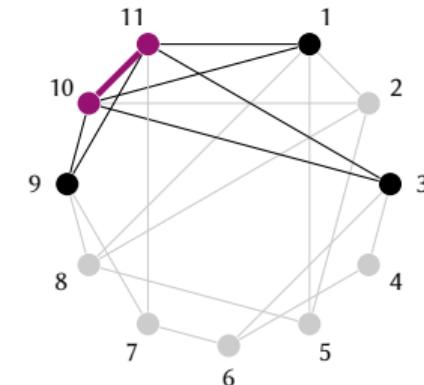
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Backtrack from 12  
Only 1, 6 and 9 feasible (1-colourable)  
No  $\geq 4$  vertex clique possible  
Effectively this deletes vertex 12

# First Attempt at a Proof

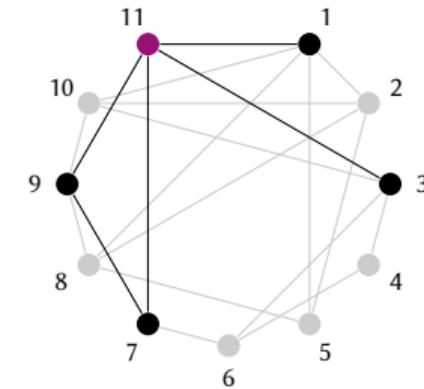
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Branch on 11 then 10  
Only 1, 3 and 9 feasible (1-colourable)  
No  $\geq 4$  vertex clique possible  
Backtrack, deleting the edge

# First Attempt at a Proof

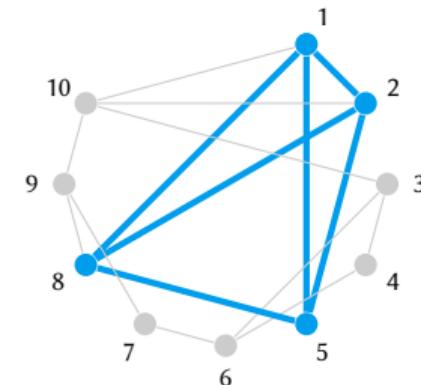
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Backtrack from 11  
2-colourable, so no  $\geq 4$  clique  
Delete the vertex

## First Attempt at a Proof

```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```

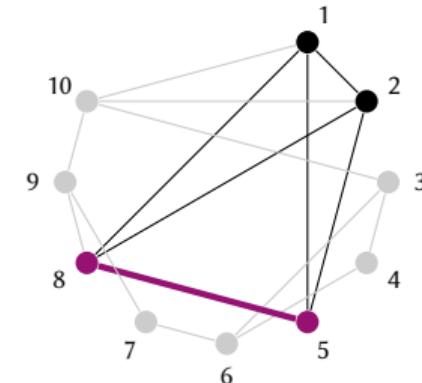


Branch on 8, 5, 1, 2  
Find a new incumbent  
Now looking for a  $\geq 5$  vertex clique

# First Attempt at a Proof

pseudo-Boolean proof version 2.0

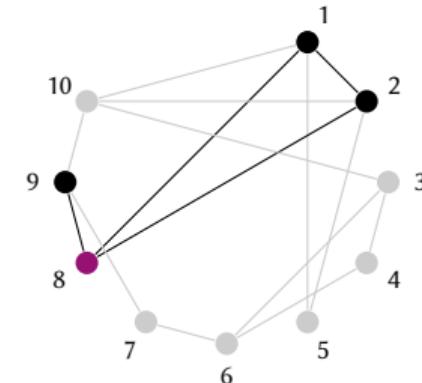
```
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Backtrack from 8, 5  
Only 4 vertices; can't have a  $\geq 5$  clique  
Delete the edge

# First Attempt at a Proof

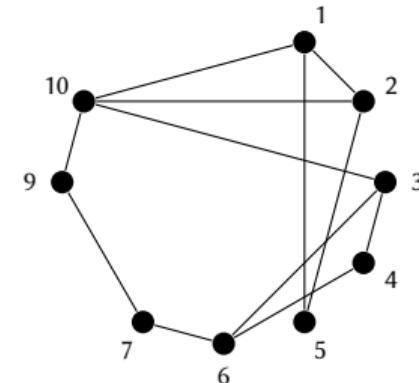
```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Backtrack from 8  
Still not enough vertices  
Delete the vertex

# First Attempt at a Proof

```
pseudo-Boolean proof version 2.0
f 41
soli x7 x9 x12
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
soli x1 x2 x5 x8
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```



Remaining graph is 3-colourable  
Backtrack from root node

# First Attempt at a Proof

pseudo-Boolean proof version 2.0

f 41

soli x7 x9 x12

rup 1 ~x12 1 ~x7 >= 1 ;

rup 1 ~x12 >= 1 ;

rup 1 ~x11 1 ~x10 >= 1 ;

rup 1 ~x11 >= 1 ;

soli x1 x2 x5 x8

rup 1 ~x8 1 ~x5 >= 1 ;

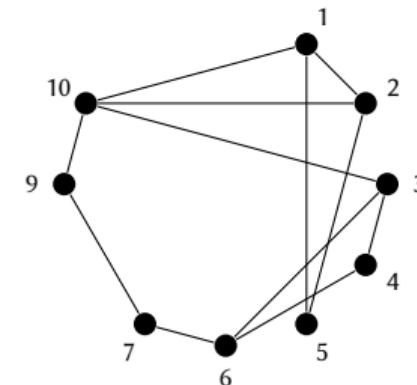
rup 1 ~x8 >= 1 ;

rup >= 1 ;

output NONE

conclusion BOUNDS -4 -4

end pseudo-Boolean proof



Finish with what we've concluded

We specify a lower and an upper bound

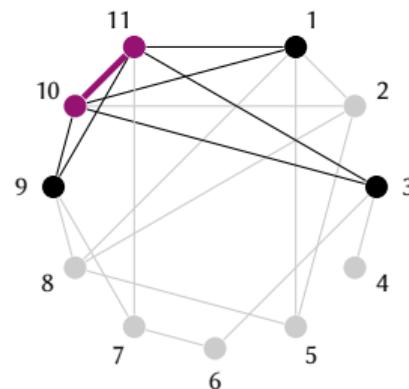
Remember we're minimising  $\sum_v -1 \times v$ , so a 4-clique has an objective value of -4

# Verifying This Proof (Or Not...)

```
$ veripb clique.opb clique-attempt-one.veripb
Verification failed.
Failed in proof file line 6.
Hint: Failed to show '1 ~x10 1 ~x11 >= 1' by reverse unit propagation.
```

## Verifying This Proof (Or Not...)

```
$ veripb clique.opb clique-attempt-one.veripb
Verification failed.
Failed in proof file line 6.
Hint: Failed to show ' $1 \simx 10 1 \simx 11 \geq 1$ ' by reverse unit propagation.
```



# Verifying This Proof (Or Not...)

```
$ veripb --trace clique.opb clique-attempt-one.veripb
line 002: f 41
  ConstraintId 001: 1 ~x1 1 ~x3 >= 1
  ConstraintId 002: 1 ~x2 1 ~x3 >= 1
...
  ConstraintId 041: 1 ~x11 1 ~x12 >= 1
line 003: soli x7 x9 x12 ~x1 ~x2 ~x3 ~x4 ~x5 ~x6 ~x8 ~x10 ~x11
  ConstraintId 042: 1 x1 1 x2 1 x3 1 x4 1 x5 1 x6 1 x7 1 x8 1 x9 1 x10 1 x11 1 x12 >= 4
line 004: rup 1 ~x12 1 ~x7 >= 1 ;
  ConstraintId 043: 1 ~x7 1 ~x12 >= 1
line 005: rup 1 ~x12 >= 1 ;
  ConstraintId 044: 1 ~x12 >= 1
line 006: rup 1 ~x11 1 ~x10 >= 1 ;
Verification failed.
Failed in proof file line 6.
Hint: Failed to show '1 ~x10 1 ~x11 >= 1' by reverse unit propagation.
```

# Dealing With Colourings

The colour bound doesn't follow by RUP...

But we can lazily recover at-most-one constraints for each colour class!

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The colour bound doesn't follow by RUP...

But we can lazily recover at-most-one constraints for each colour class!

$$\begin{aligned} & (\bar{x}_1 + \bar{x}_6 \geq 1) \\ & + (\bar{x}_1 + \bar{x}_9 \geq 1) & = 2\bar{x}_1 + \bar{x}_6 + \bar{x}_9 \geq 2 \\ & + (\bar{x}_6 + \bar{x}_9 \geq 1) & = 2\bar{x}_1 + 2\bar{x}_6 + 2\bar{x}_9 \geq 3 \\ & / 2 & = \bar{x}_1 + \bar{x}_6 + \bar{x}_9 \geq 2 \\ & & \text{i.e. } x_1 + x_6 + x_9 \leq 1 \end{aligned}$$

# Dealing With Colourings

The colour bound doesn't follow by RUP...

But we can lazily recover at-most-one constraints for each colour class!

$$\begin{aligned} & (\bar{x}_1 + \bar{x}_6 \geq 1) \\ & + (\bar{x}_1 + \bar{x}_9 \geq 1) & = 2\bar{x}_1 + \bar{x}_6 + \bar{x}_9 \geq 2 \\ & + (\bar{x}_6 + \bar{x}_9 \geq 1) & = 2\bar{x}_1 + 2\bar{x}_6 + 2\bar{x}_9 \geq 3 \\ & \quad / 2 & = \bar{x}_1 + \bar{x}_6 + \bar{x}_9 \geq 2 \\ & & \text{i.e. } x_1 + x_6 + x_9 \leq 1 \end{aligned}$$

This generalises to colour classes of any size  $v$

- Each non-edge is used exactly once,  $v(v - 1)$  additions
- $v - 3$  multiplications and  $v - 2$  divisions

Solvers don't need to “understand” cutting planes to write this derivation to proof log

# What This Looks Like in the Proof Log

```
pseudo-Boolean proof version 2.0
f 41
soli x12 x7 x9
rup 1 ~x12 1 ~x7 >= 1 ;
* bound, colour classes [ x1 x6 x9 ]
pol 71~6 191~9 + 246~9 + 2 d
pol 42obj -1 +
rup 1 ~x12 >= 1 ;
* bound, colour classes [ x1 x3 x9 ]
pol 11~3 191~9 + 213~9 + 2 d
pol 42obj -1 +
rup 1 ~x11 1 ~x10 >= 1 ;
* bound, colour classes [ x1 x3 x7 ]
* [ x9 ]
pol 11~3 101~7 + 123~7 + 2 d
pol 42obj -1 +
rup 1 ~x11 >= 1 ;
```

```
soli x8 x5 x2 x1
rup 1 ~x8 1 ~x5 >= 1 ;
* bound, colour classes [ x1 x9 ] [ x2 ]
pol 53obj 191~9 +
rup 1 ~x8 >= 1 ;
* bound, colour classes [ x1 x3 x7 ]
* [ x2 x4 x9 ] [ x5 x6 x10 ]
pol 11~3 101~7 + 123~7 + 2 d
pol 53obj -1 +
pol 42~4 202~9 + 224~9 + 2 d
pol 53obj -3 + -1 +
pol 95~6 265~10 + 276~10 + 2 d
pol 53obj -5 + -3 + -1 +
rup >= 1 ;
output NONE
conclusion BOUNDS -4 -4
end pseudo-Boolean proof
```

# Verifying This Proof (For Real, This Time)

```
$ veripb --trace clique.opb clique-attempt-two.veripb
== begin trace ==
line 002: f 41
  ConstraintId 001: 1 ~x1 1 ~x3 >= 1
  ConstraintId 002: 1 ~x2 1 ~x3 >= 1
...
  ConstraintId 041: 1 ~x11 1 ~x12 >= 1
line 003: soli x7 x9 x12 ~x1 ~x2 ~x3 ~x4 ~x5 ~x6 ~x8 ~x10 ~x11
  ConstraintId 042: 1 x1 1 x2 1 x3 1 x4 1 x5 1 x6 1 x7 1 x8 1 x9 1 x10 1 x11 1 x12 >= 4
line 004: rup 1 ~x12 1 ~x7 >= 1 ;
  ConstraintId 043: 1 ~x7 1 ~x12 >= 1
line 005: * bound, colour classes [ x1 x6 x9 ]
line 006: pol 7 19 + 24 + 2 d
  ConstraintId 044: 1 ~x1 1 ~x6 1 ~x9 >= 2
line 007: pol 42 43 +
  ConstraintId 045: 1 x1 1 x2 1 x3 1 x4 1 x5 1 x6 1 x8 1 x9 1 x10 1 x11 >= 3
...
  ConstraintId 061: 1 ~x5 1 ~x6 1 ~x10 >= 2
line 028: pol 53 57 + 59 + 61 +
  ConstraintId 062: 1 x8 1 x11 1 x12 >= 2
line 029: rup >= 1 ;
  ConstraintId 063: >= 1
line 030: output NONE
line 031: conclusion BOUNDS -4 -4
line 032: end pseudo-Boolean proof
== end trace ==

Verification succeeded.
```

# Different Clique Algorithms

Different search orders?

- ✓ Irrelevant for proof logging

Using local search to initialise?

- ✓ Just log the incumbent

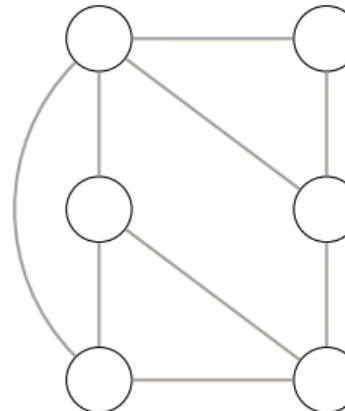
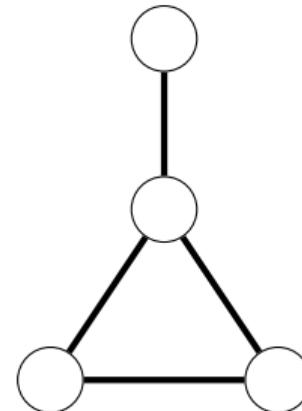
Different bound functions?

- Is cutting planes strong enough to justify every useful bound function ever invented?
- So far, seems like it...

Weighted cliques?

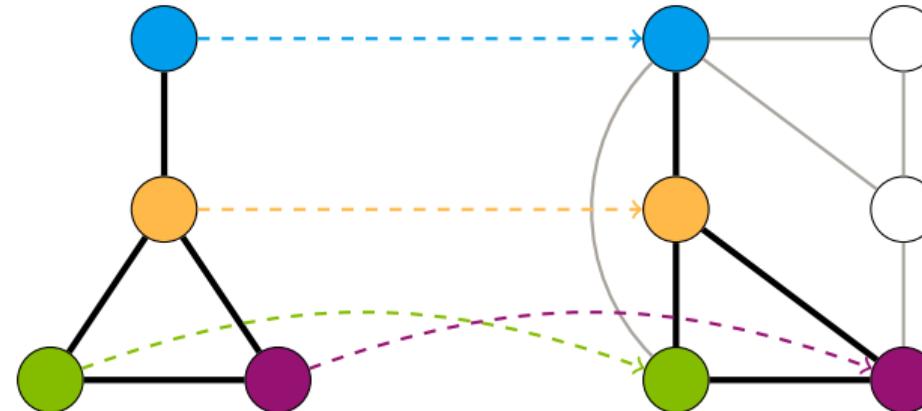
- ✓ Multiply a colour class by its largest weight
- ✓ Also works for vertices “split between colour classes”

# Subgraph Isomorphism



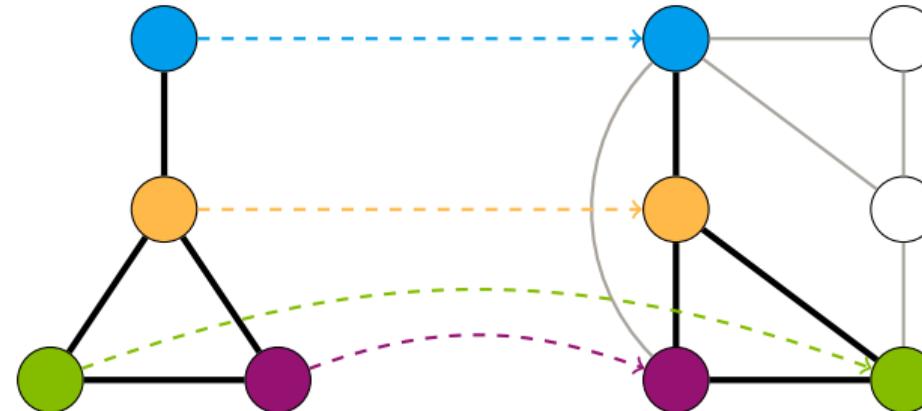
- Find the **pattern** inside the **target**
- Applications in compilers, biochemistry, model checking, pattern recognition, ...
- Often want to find **all** matches

# Subgraph Isomorphism



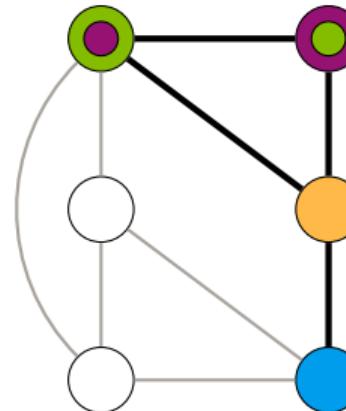
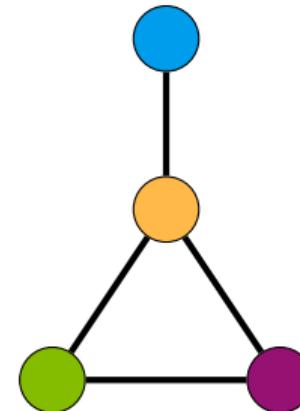
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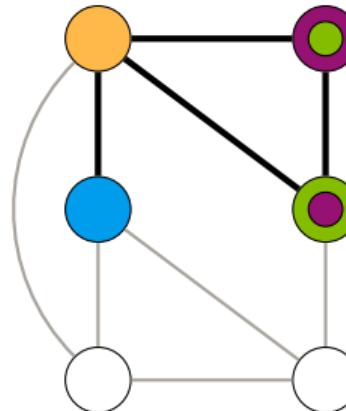
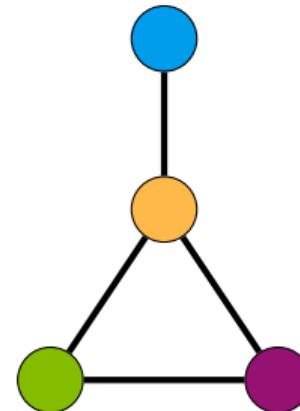
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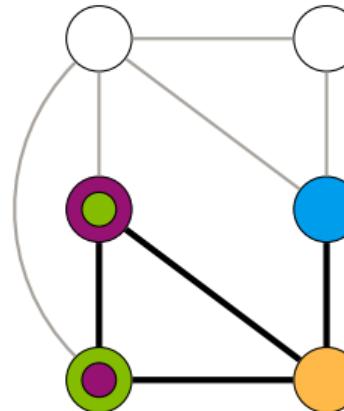
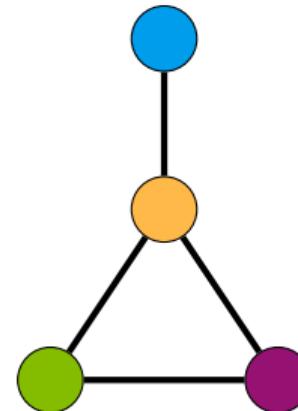
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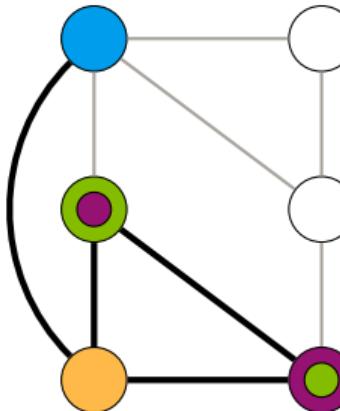
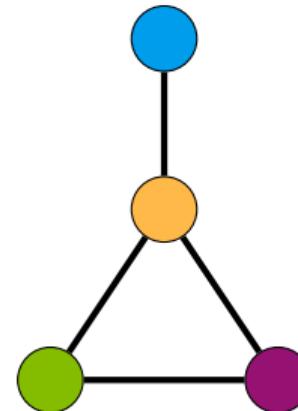
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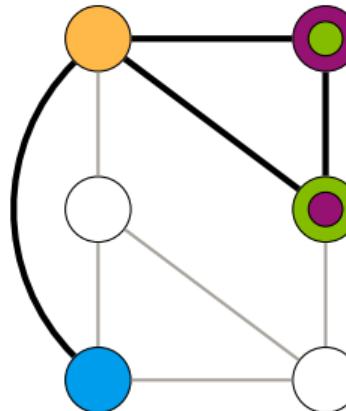
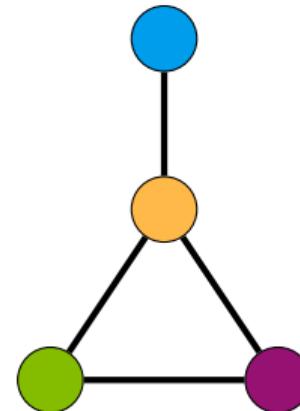
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# Subgraph Isomorphism in Pseudo-Boolean Form

Each pattern vertex gets a target vertex:

$$\sum_{t \in V(T)} x_{p,t} = 1 \quad p \in V(P)$$

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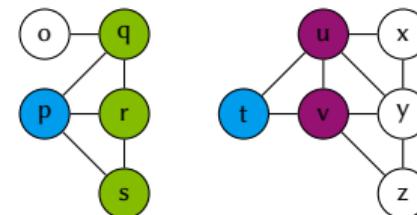
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Adjacency constraints, if  $p$  is mapped to  $t$ , then  $p$ 's neighbours must be mapped to  $t$ 's neighbours:

$$\bar{x}_{p,t} + \sum_{u \in N(t)} x_{q,u} \geq 1 \quad p \in V(P), q \in N(p), t \in V(T)$$

# Degree Reasoning in Cutting Planes



Pattern vertex  $p$  of degree  $\deg(p)$  can never be mapped to target vertex  $t$  of degree  $< \deg(p)$  in any subgraph isomorphism

Observe  $N(p) = \{q, r, s\}$  and  $N(t) = \{u, v\}$

We wish to derive  $\bar{x}_{p,t} \geq 1$

# Degree Reasoning in Cutting Planes

Adjacency:

$$\bar{x}_{p,t} + x_{q,u} + x_{q,v} \geq 1$$

$$\bar{x}_{p,t} + x_{r,u} + x_{r,v} \geq 1$$

$$\bar{x}_{p,t} + x_{s,u} + x_{s,v} \geq 1$$

Injectivity:

$$-x_{o,u} + -x_{p,u} + -x_{q,u} + -x_{r,u} + -x_{s,u} \geq -1$$

$$-x_{o,v} + -x_{p,v} + -x_{q,v} + -x_{r,v} + -x_{s,v} \geq -1$$

Literal axioms:

$$x_{o,u} \geq 0$$

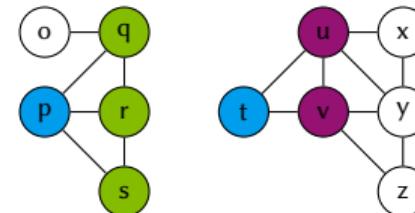
$$x_{o,v} \geq 0$$

$$x_{p,u} \geq 0$$

$$x_{p,v} \geq 0$$

Add these together and divide by 3 to get

$$\bar{x}_{p,t} \geq 1$$



# Degree Reasoning in VERIPB

```
pol 18p~t:q 19p~t:r + 20p~t:s + * sum adjacency constraints
  12inj(u) + 13inj(v) + * sum injectivity constraints
    xo_u + xo_v + * cancel stray xo_*
    xp_u + xp_v + * cancel stray xp_*
  3 d * divide, and we're done
```

Or we can ask VERIPB to do the last bit of simplification automatically:

```
pol 18p~t:q 19p~t:r + 20p~t:s + * sum adjacency constraints
  12inj(u) + 13inj(v) + * sum injectivity constraints
ia -1 : 1 ~xp_t >= 1 ; * desired conclusion is implied
```

# Other Forms of Reasoning

We can also log all of the other things state of the art subgraph solvers do:

- Injectivity reasoning and filtering
- Distance filtering
- Neighbourhood degree sequences
- Path filtering
- Supplemental graphs

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Proof steps are “efficient” using cutting planes

- Length of proof  $\approx$  time complexity of the reasoning algorithms
- Most proof steps require only trivial additional computations

# Limitations

Why trust the encoding?

- Correctness of encoding can be formally verified! Work in progress...

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Works up to moderately-sized hard instances

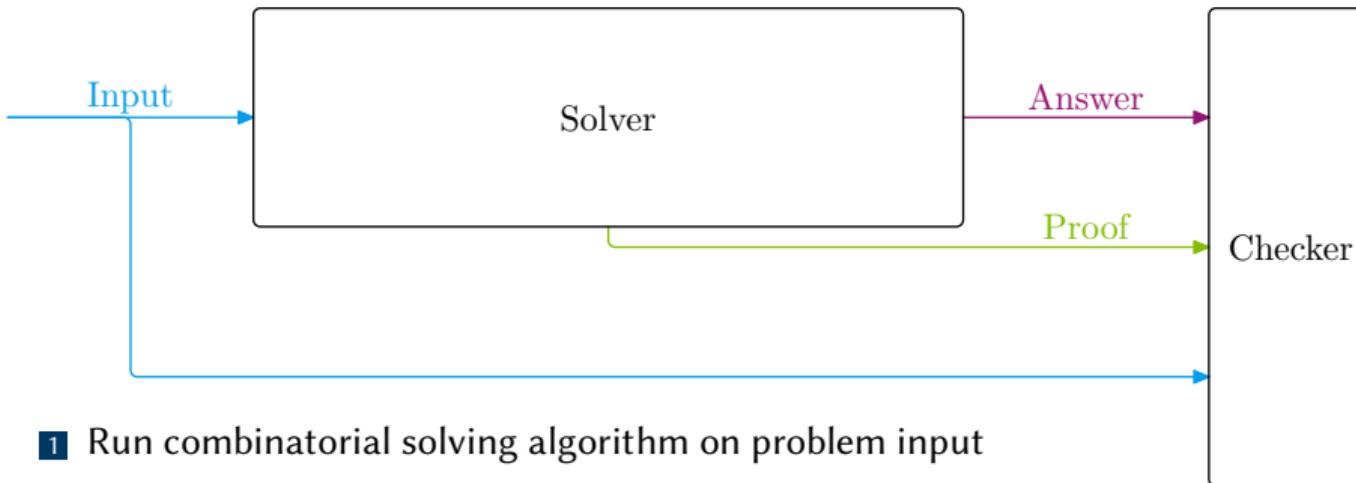
- Even an  $O(n^3)$  encoding is painful
- Particularly bad when the pseudo-Boolean encoding talks about “non-edges” but large sparse graphs are “easy”

# Code for Proof Logging Subgraph Solver

<https://github.com/ciaranm/glasgow-subgraph-solver>

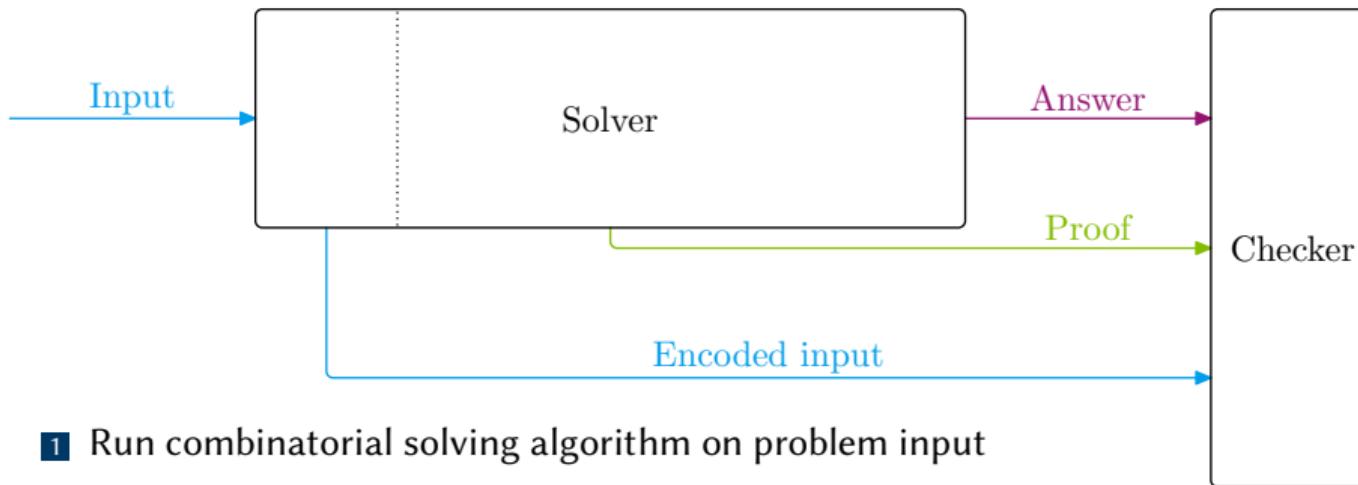
Released under MIT Licence

# Recap (1/2)



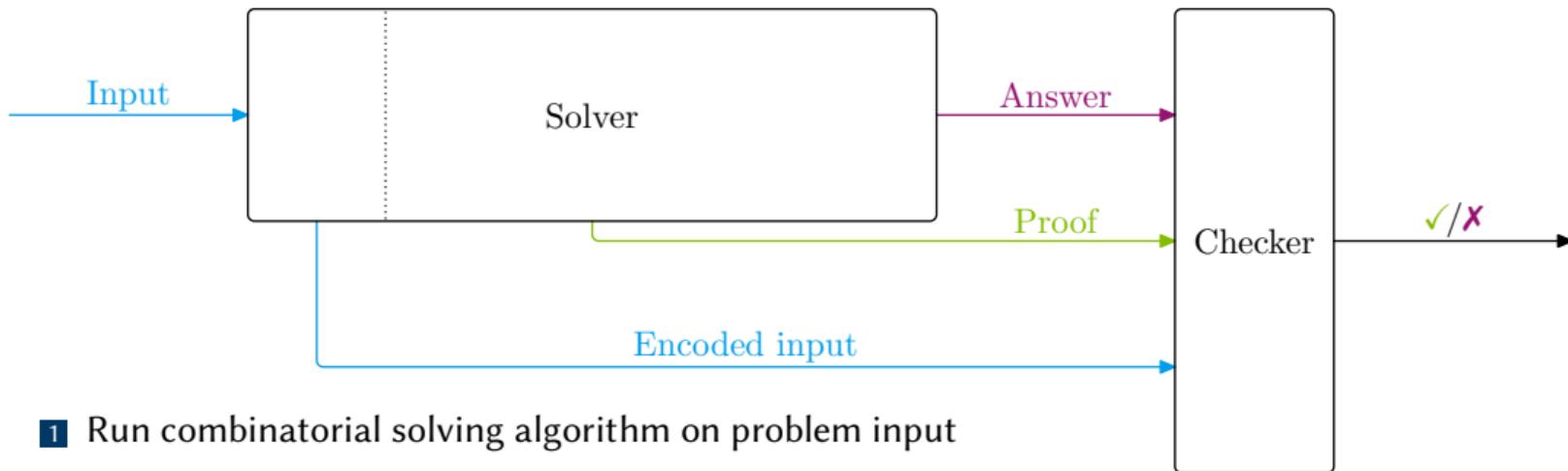
- 1 Run combinatorial solving algorithm on problem input
- 2 Get as output not only answer but also proof
- 3 Feed answer + proof to proof checker together with input

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## Recap (1/2)



- 1 Run combinatorial solving algorithm on problem input
- 2 Get as output not only answer but also proof
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- 4 Verify that proof checker says answer is correct

## Recap (2/2)

### Proof logging implementation

- Don't change solver
- Just add proof logging statements (plus some book-keeping)

### Performance goals

Want linear(ish) scaling in terms of **solver running time** for

- **proof size**
- **proof checking time**

# What About Constraint Programming?

Non-Boolean variables?

Constraints?

- Encoding constraints in pseudo-Boolean form?
- Justifying inferences?

Reformulations?

# Compiling CP Variables (1/2)

Given  $A \in \{-3 \dots 9\}$ , the direct encoding is:

$$\begin{aligned} & a_{=-3} + a_{=-2} + a_{=-1} + a_{=0} + a_{=1} + a_{=2} + a_{=3} \\ & + a_{=4} + a_{=5} + a_{=6} + a_{=7} + a_{=8} + a_{=9} = 1 \end{aligned}$$

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This doesn't work for large domains...

We could use a binary encoding:

$$\begin{aligned} & -16a_{\text{neg}} + 1a_{b0} + 2a_{b1} + 4a_{b2} + 8a_{b3} \geq -3 \text{ and} \\ & 16a_{\text{neg}} + -1a_{b0} + -2a_{b1} + -4a_{b2} + -8a_{b3} \geq -9 \end{aligned}$$

This doesn't propagate much, but that isn't a problem for proof logging

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**Convention** in what follows:

- Upper-case  $A, B, C$  are CP variables;
- Lower-case  $a, b, c$  are corresponding Boolean variables in PB encoding

## Compiling CP Variables (2/2)

We can mix binary and an order encoding! Where needed, define:

$$a_{\geq 4} \Leftrightarrow -16a_{\text{neg}} + 1a_{b0} + 2a_{b1} + 4a_{b2} + 8a_{b3} \geq 4$$

$$a_{\geq 5} \Leftrightarrow -16a_{\text{neg}} + 1a_{b0} + 2a_{b1} + 4a_{b2} + 8a_{b3} \geq 5$$

$$a_{=4} \Leftrightarrow a_{\geq 4} \wedge \bar{a}_{\geq 5}$$

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When creating  $a_{\geq i}$ , also introduce pseudo-Boolean constraints encoding

$$a_{\geq i} \Rightarrow a_{\geq j} \quad \text{and} \quad a_{\geq h} \Rightarrow a_{\geq i}$$

for the closest values  $j < i < h$  that already exist

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We can do this:

- Inside the pseudo-Boolean model, where needed
- Otherwise lazily during proof logging

# Compiling Constraints

- Also need to compile every constraint to pseudo-Boolean form
- Doesn't need to be a propagating encoding
- Can use additional variables

# Compiling Linear Inequalities

Given inequality

$$2A + 3B + 4C \geq 42$$

where  $A, B, C \in \{-3 \dots 9\}$

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Encode in pseudo-Boolean form as

$$\begin{aligned} & -32a_{\text{neg}} + 2a_{b0} + 4a_{b1} + 8a_{b2} + 16a_{b3} \\ & + -48b_{\text{neg}} + 3b_{b0} + 6b_{b1} + 12b_{b2} + 24b_{b3} \\ & + -64c_{\text{neg}} + 4c_{b0} + 8c_{b1} + 16c_{b2} + 32c_{b3} \geq 42 \end{aligned}$$

# Compiling Table Constraints

Constraints can be specified **extensionally** as list of feasible tuples, called a **table**

Variable assignments must match some row in table

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Variable assignments must match some row in table

Given table constraint

$$(A, B, C) \in [(1, 2, 3), (1, 3, 4), (2, 2, 5)]$$

define

$$3\bar{t}_0 + a_{=1} + b_{=2} + c_{=3} \geq 3$$

$$\text{i.e., } t_0 \Rightarrow (a_{=1} \wedge b_{=2} \wedge c_{=3})$$

$$3\bar{t}_1 + a_{=1} + b_{=4} + c_{=4} \geq 3$$

$$\text{i.e., } t_1 \Rightarrow (a_{=1} \wedge b_{=4} \wedge c_{=4})$$

$$3\bar{t}_2 + a_{=2} + b_{=2} + c_{=5} \geq 3$$

$$\text{i.e., } t_2 \Rightarrow (a_{=2} \wedge b_{=2} \wedge c_{=5})$$

using tuple selector variables

$$t_0 + t_1 + t_2 = 1$$

# Encoding Constraint Definitions

Already know how to do it for any constraint with a sane encoding using some combination of

- CNF
- Integer linear inequalities
- Table constraints
- Auxiliary variables

Simplicity is important, propagation strength isn't

# Justifying Search

Mostly this works as in earlier examples

Restarts are easy

No need to justify guesses or decisions — only justify backtracking

# Justifying Inference

## Key idea

Anything the constraint programming solver knows must follow from **unit propagation** of guessed assignments on **constraints in proof log**

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Some propagators and encodings need RUP steps for inferences

- A lot of propagators are effectively “doing a little bit of lookahead” but in an efficient way

A few need explicit cutting planes justifications written to the proof log

- **Linear inequalities** just need to multiply and add
- **All-different** needs a bit more

# Justifying All-Different Failures

$$V \in \{ 1 \quad 4 \quad 5 \}$$

$$W \in \{ 1 \ 2 \ 3 \ }$$

$$X \in \{ \quad 2 \ 3 \ }$$

$$Y \in \{ 1 \quad 3 \ }$$

$$Z \in \{ 1 \quad 3 \ }$$

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$$W \in \{ 1 \ 2 \ 3 \} \quad w_{=1} + w_{=2} + w_{=3} \geq 1 \quad [W \text{ takes some value}]$$

$$X \in \{ 2 \ 3 \}$$

$$Y \in \{ 1 \ 3 \}$$

$$Z \in \{ 1 \ 3 \}$$

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$$X \in \{ 2 \ 3 \} \quad x_{=2} + x_{=3} \geq 1 \quad [ X \text{ takes some value} ]$$

$$Y \in \{ 1 \ 3 \} \quad y_{=1} + y_{=3} \geq 1 \quad [ Y \text{ takes some value} ]$$

$$Z \in \{ 1 \ 3 \} \quad z_{=1} + z_{=3} \geq 1 \quad [ Z \text{ takes some value} ]$$

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$$Z \in \{ 1 \ 3 \} \quad z_{=1} + z_{=3} \geq 1 \quad [ Z \text{ takes some value} ]$$

$$\rightarrow -v_{=1} + -w_{=1} + -y_{=1} + -z_{=1} \geq -1 \quad [ \text{At most one variable} = 1 ]$$

$$\rightarrow -w_{=2} + -x_{=2} \geq -1 \quad [ \text{At most one variable} = 2 ]$$

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# Justifying All-Different Failures

$$V \in \{ 1 \quad 4 \quad 5 \}$$

$$W \in \{ 1 \ 2 \ 3 \} \quad w_{=1} + w_{=2} + w_{=3} \geq 1 \quad [ W \text{ takes some value} ]$$

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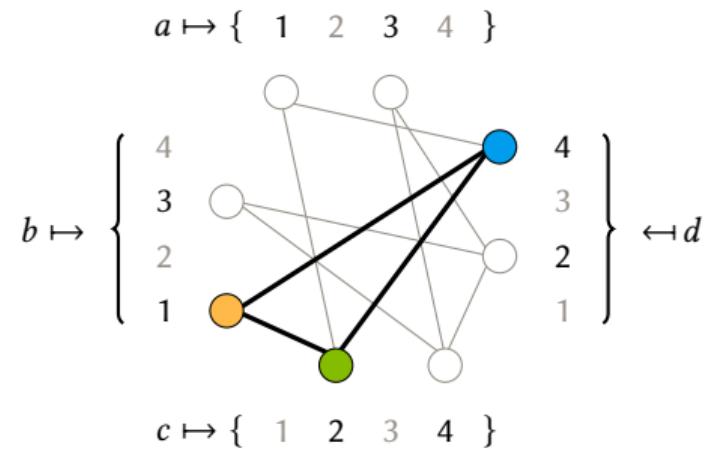
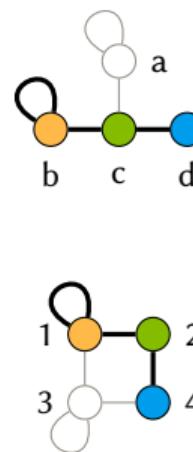
$$0 \geq 1 \quad [ \text{Sum above two constraints} ]$$

# Reformulation

Auto-tabulation is possible

- Heavy use of extension variables

Can re-encode maximum common subgraph as a clique problem, without changing pseudo-Boolean encoding



# High Level Modelling Languages?

High level modelling languages like MINIZINC and ESSENCE have complicated compilers

How do we know we're giving a proof for the problem the user actually specified?

Future research...

# Code

<https://github.com/ciaranm/glasgow-constraint-solver>

Released under MIT Licence

Supports proof logging for global constraints including:

- All-different
- Integer linear inequality (including for very large domains)
- Smart table and regular
- Minimum / maximum of an array
- Element
- Absolute value
- (Hamiltonian) Circuit

Details in [EGMN20, GMN22, MM23]

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we introduced pseudo-Boolean constraints

$$3\bar{a} + 3x + 2y + z + w \geq 3 \quad 5a + 3\bar{x} + 2\bar{y} + \bar{z} + \bar{w} \geq 5$$

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Wish to allow without-loss-of-generality arguments that can derive non-implied constraints

# Redundance-Based Strengthening

$C$  is **redundant** with respect to  $F$  if  $F$  and  $F \wedge C$  are **equisatisfiable**

Adding redundant constraints should be OK

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Witness  $\omega$  should be specified, and implication should be efficiently verifiable, which is the case for constraints in  $(F \wedge C) \upharpoonright \omega$  that are, e.g.,

- Reverse unit propagation (RUP) constraints w.r.t.  $F \wedge \neg C$
- Obviously implied by a single constraint among  $F \wedge \neg C$

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$$\begin{aligned}1 \cdot x_0 + 2 \cdot x_1 + 4 \cdot x_2 + 8 \cdot x_3 \\+ 1 \cdot y_0 + 2 \cdot y_1 + 4 \cdot y_2 + 8 \cdot y_3 = 25 \\x_0 = y_0\end{aligned}$$

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$$-1 \cdot x_0 - 2 \cdot x_1 - 4 \cdot x_2 - 8 \cdot x_3$$

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To derive without loss of generality  $x \leq y$   
(argument: we can always swap them)

$$1 \cdot x_0 + 2 \cdot x_1 + 4 \cdot x_2 + 8 \cdot x_3 \leq 1 \cdot y_0 + 2 \cdot y_1 + 4 \cdot y_2 + 8 \cdot y_3$$

pseudo-Boolean proof version 2.0

f 4

red 1 y0 2 y1 4 y2 8 y3

$\rightarrow -1 \cdot x_0 - 2 \cdot x_1 - 4 \cdot x_2 - 8 \cdot x_3 \geq 0 ;$

$\rightarrow y_0 \rightarrow x_0 x_0 \rightarrow y_0 y_1 \rightarrow x_1 x_1 \rightarrow y_1$

$\rightarrow y_2 \rightarrow x_2 x_2 \rightarrow y_2 y_3 \rightarrow x_3 x_3 \rightarrow y_3$

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pseudo-Boolean proof version 2.0

```
f 4
red 1 y0 2 y1 4 y2 8 y3
→ -1 x0 -2 x1 -4 x2 -8 x3 >= 0 ;
→ y0 → x0 x0 → y0 y1 → x1 x1 → y1
→ y2 → x2 x2 → y2 y3 → x3 x3 → y3
```

Why does this work? Need to show

$$F \wedge \neg C \models (F \wedge C) \upharpoonright_{\omega}$$

- $F \upharpoonright_{\omega}$  equals  $F$  (swaps last two constraints)
- $C \upharpoonright_{\omega}$  says  $y \leq x$  while  $\neg C$  says  $y < x$

# Deriving $a \Leftrightarrow (3x + 2y + z + w \geq 3)$ Using the Redundance Rule

Want to derive

$$3\bar{a} + 3x + 2y + z + w \geq 3 \quad 5a + 3\bar{x} + 2\bar{y} + \bar{z} + \bar{w} \geq 5$$

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Choose  $\omega = \{a \mapsto 0\}$  —  $F$  untouched; new constraint satisfied

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2  $F \wedge (3\bar{a} + 3x + 2y + z + w \geq 3) \wedge \neg(5a + 3\bar{x} + 2\bar{y} + \bar{z} + \bar{w} \geq 5) \models (F \wedge (3\bar{a} + 3x + 2y + z + w \geq 3) \wedge (5a + 3\bar{x} + 2\bar{y} + \bar{z} + \bar{w} \geq 5)) \upharpoonright_{\omega}$

Choose  $\omega = \{a \mapsto 1\}$  —  $F$  untouched; new constraint satisfied

$\neg(5a + 3\bar{x} + 2\bar{y} + \bar{z} + \bar{w} \geq 5)$  forces  $3\bar{x} + 2\bar{y} + \bar{z} + \bar{w} \leq 4$

This is the same constraint as  $3\bar{a} + 3x + 2y + z + w \geq 3$

And VERIPB can automatically detect this implication

# Redundance and Dominance Rules for Optimisation

## Redundance-based strengthening, optimisation version

Add constraint  $C$  to formula  $F$  if exists witness substitution  $\omega$  s.t.

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- Applying  $\omega$  should **strictly decrease**  $f$
- If so, don't need to show that  $C \upharpoonright_{\omega}$  holds!

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- 7 ...
- 8 Can't go on forever, so finally reach  $\alpha'$  satisfying  $F \wedge C$

# Strength of Dominance Rule

Dominance-based strengthening (stronger, still simplified)

If  $C_1, C_2, \dots, C_{m-1}$  have been derived from  $F$  (maybe using dominance), then can derive  $C_m$  if exists witness substitution  $\omega$  s.t.

$$F \wedge \bigwedge_{i=1}^{m-1} C_i \wedge \neg C_m \models F \upharpoonright \omega \wedge f \upharpoonright \omega < f$$

Only consider  $F$  – no need to show that any  $C_i \upharpoonright \omega$  implied!

# Strength of Dominance Rule

Dominance-based strengthening (stronger, still simplified)

If  $C_1, C_2, \dots, C_{m-1}$  have been derived from  $F$  (maybe using dominance), then can derive  $C_m$  if exists witness substitution  $\omega$  s.t.

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Further extensions:

- Define dominance rule w.r.t. order independent of objective
- Switch between different orders in same proof
- See [BGMN23] for details

# Using the Dominance Rule for Symmetry Handling

Dominance rule **very powerful**; can be used for symmetry and dominance breaking

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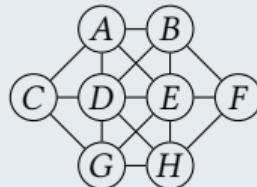
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Examples:

- 1 Symmetries in constraint programming (manual symmetry breaking)
- 2 Vertex dominance in clique solving (automatic dominance breaking during search)
- 3 Symmetries in SAT solving (automatic symmetry breaking in preprocessing)

# Symmetry Elimination (CP)

## The Crystal Maze Puzzle



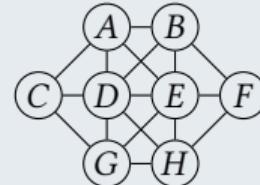
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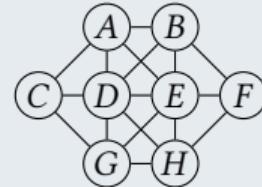
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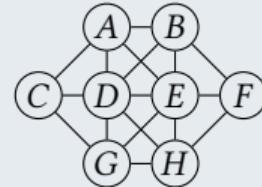
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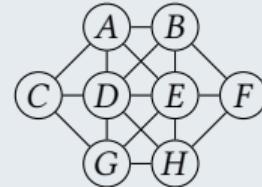
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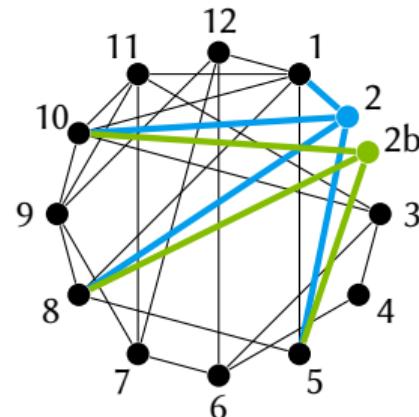
**Research challenge:** Constraint programming toolchain supporting this

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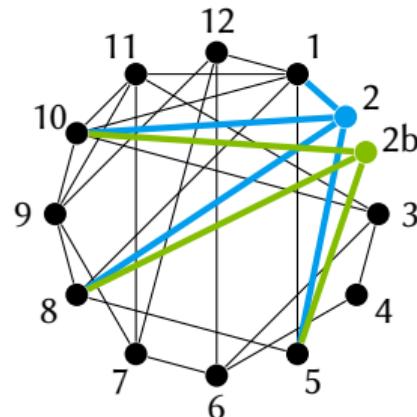
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# Strategy for SAT Symmetry Breaking in SAT Solving

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Order: “Pick smallest hole for pigeon 1, then smallest for pigeon 2, ...”

$$f \doteq 2^{11} \cdot p_{13} + 2^{10} \cdot p_{12} + 2^9 \cdot p_{11} + 2^8 \cdot p_{23} + \cdots + 1 \cdot p_{41}$$

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Similar to DRAT symmetry breaking [HHW15]

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Can satisfy this constraint by applying  $\sigma_{(1234)}$  **once, twice, or thrice**

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## Definition

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## Theorem ([BGMN23])

$C_\sigma$  can be derived from  $F$  using dominance *with witness*  $\sigma$

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Why does it work?

- Witness need not satisfy all derived constraints
- Sufficient to just produce “better” assignment

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If  $y_{k-1}$  is true,  $x_k$  is at most  $\sigma(x_k)$   
(derivable from the PB breaking constraint)

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# Back to Our Pigeons — Setting up the Pretend Optimisation Problem

Start the proof and load  
input formula

```
pseudo-Boolean proof version 2.0
f 22
pre_order exp
  vars
    left u1 u2 u3 u4 u5 u6 u7 u8 u9 u10 u11 u12
    right v1 v2 v3 v4 v5 v6 v7 v8 v9 v10 v11 v12
    aux
  end
  def
    -1 u12 1 v12 -2 u11 2 v11 [...] -1024 u2 1024 v2 -2048 u1 2048 v1 >= 0;
  end
  transitivity
    vars
      fresh_right w1 w2 w3 w4 w5 w6 w7 w8 w9 w10 w11 w12
    end
  proof
    proofgoal #1
      pol 1 2 + 3 +
      qed -1
    qed
  end
end
load_order exp p13 p12 p11 p23 p22 p21 p31 p32 p33 p41 p42 p43
```

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1 Pretend to solve  
optimisation  
problem

$$\begin{aligned} \text{minimizing } f \doteq \\ 2^{11} \cdot p_{13} + 2^{10} \cdot p_{12} + \\ 2^9 \cdot p_{11} + 2^8 \cdot p_{23} + \\ \cdots + 2 \cdot p_{42} + 1 \cdot p_{41} \end{aligned}$$

```
pseudo-Boolean proof version 2.0
f 22
pre_order exp
  vars
    left u1 u2 u3 u4 u5 u6 u7 u8 u9 u10 u11 u12
    right v1 v2 v3 v4 v5 v6 v7 v8 v9 v10 v11 v12
    aux
  end
  def
    -1 u12 1 v12 -2 u11 2 v11 [...] -1024 u2 1024 v2 -2048 u1 2048 v1 >= 0;
  end
  transitivity
    vars
      fresh_right w1 w2 w3 w4 w5 w6 w7 w8 w9 w10 w11 w12
    end
  proof
    proofgoal #1
      pol 1 2 + 3 +
      qed -1
    qed
  end
end
load_order exp p13 p12 p11 p23 p22 p21 p31 p32 p33 p41 p42 p43
```

# Back to Our Pigeons — Setting up the Pretend Optimisation Problem

Start the proof and load  
input formula

1 Pretend to solve  
optimisation  
problem

$$\begin{aligned} \text{minimizing } f \doteq \\ 2^{11} \cdot p_{13} + 2^{10} \cdot p_{12} + \\ 2^9 \cdot p_{11} + 2^8 \cdot p_{23} + \\ \cdots + 2 \cdot p_{42} + 1 \cdot p_{41} \end{aligned}$$

(Actually defining an  
order — see [BGMN23]  
for details)

```
pseudo-Boolean proof version 2.0
f 22
pre_order exp
  vars
    left u1 u2 u3 u4 u5 u6 u7 u8 u9 u10 u11 u12
    right v1 v2 v3 v4 v5 v6 v7 v8 v9 v10 v11 v12
    aux
  end
  def
    -1 u12 1 v12 -2 u11 2 v11 [...] -1024 u2 1024 v2 -2048 u1 2048 v1 >= 0;
  end
  transitivity
    vars
      fresh_right w1 w2 w3 w4 w5 w6 w7 w8 w9 w10 w11 w12
    end
    proof
      proofgoal #1
        pol 1 2 + 3 +
        qed -1
      qed
    end
  end
load_order exp p13 p12 p11 p23 p22 p21 p31 p32 p33 p41 p42 p43
```

# Back to Our Pigeons — Deriving the Constraints

Derived constraints ( $\mathcal{D}$ ):

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23 >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
    proofgoal #2
        pol -1 -2 +
        qed -1
    end
    red 1 y0 >= 1 ; y0 -> 1
    rup 1 ~y0 1 ~p13 1 p23 >= 1 ;
    red 1 ~y1 1 y0 >= 1 ; y1 -> 0
    red 1 ~y1 1 ~p23 1 p13 >= 1 ; y1 -> 0
    red 1 p23 1 ~y0 1 y1 >= 1 ; y1 -> 1
    red 1 ~p13 1 ~y0 1 y1 >= 1 ; y1 -> 1
    pol 26 32 2048 * +
    del id 26
    rup 1 ~y1 1 ~p12 1 p22 >= 1 ;
```

Pseudo-Boolean breaking constraint

# Back to Our Pigeons — Deriving the Constraints

Derived constraints ( $\mathcal{D}$ ):

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
    qed -1
  end
  red 1 y0  >= 1 ; y0 -> 1
  rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
  red 1 ~y1  1 y0      >= 1 ; y1 -> 0
  red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
  red 1 p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
  red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
  pol 26 32 2048 * +
  del id 26
  rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

## Pseudo-Boolean breaking constraint

Use **dominance** with witness  $\sigma = (p_{11}p_{21})(p_{12}p_{22})(p_{13}p_{23})$

# Back to Our Pigeons — Deriving the Constraints

**Derived constraints ( $\mathcal{D}$ ):**

$$2^{11} \cdot (p_{23} - p_{13}) +$$

$$2^{10} \cdot (p_{22} - p_{12}) +$$

$$\dots \geq 0$$

```

dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
    proofgoal #2
        pol -1 -2 +
        qed -1
end
red 1 y0  >= 1 ; y0 -> 1
rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
red 1 ~y1  1 y0      >= 1 ; y1 -> 0
red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
red 1 p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
pol 26 32 2048 * +
del id 26
rup 1 ~y1  1 ~p12 1 p22 >= 1 ;

```

## Pseudo-Boolean breaking constraint

Use **dominance** with witness  $\sigma = (p_{11}p_{21})(p_{12}p_{22})(p_{13}p_{23})$

$$F \wedge \neg C \models F \upharpoonright_\omega \wedge (f \upharpoonright_\omega < f)$$

VERIPB fills in all missing subproofs except for  $\neg C \wedge C \models \perp$

# Back to Our Pigeons — Deriving the Constraints

**Derived constraints ( $\mathcal{D}$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

```

dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
    qed -1
  end
  red 1 y0  >= 1 ; y0 -> 1
  rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
  red 1 ~y1  1 y0      >= 1 ; y1 -> 0
  red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
  red 1 p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
  red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
  pol 26 32 2048 * +
  del id 26
  rup 1 ~y1  1 ~p12 1 p22 >= 1 ;

```

Derivable by **redundance** with witness  $\omega = \{y_0 \mapsto 1\}$

$$F \wedge \mathcal{D} \wedge \neg(y_0 \geq 1) \models (F \wedge \mathcal{D}) \upharpoonright_\omega \wedge (y_0 \geq 1) \upharpoonright_\omega$$

# Back to Our Pigeons — Deriving the Constraints

**Derived constraints ( $\mathcal{D}$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
    qed -1
  end
  red 1 y0  >= 1 ; y0 -> 1
  rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
  red 1 ~y1  1 y0      >= 1 ; y1 -> 0
  red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
  red 1 p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
  red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
  pol 26 32 2048 * +
  del id 26
  rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by **redundance** with witness  $\omega = \{y_0 \mapsto 1\}$

$$F \wedge \mathcal{D} \wedge \neg(y_0 \geq 1) \models (F \wedge \mathcal{D}) \upharpoonright_\omega \wedge (y_0 \geq 1) \upharpoonright_\omega$$

$$F \wedge \mathcal{D} \wedge (\bar{y}_0 \geq 1) \models (F \wedge \mathcal{D}) \wedge (1 \geq 1)$$

# Back to Our Pigeons — Deriving the Constraints

**Derived constraints ( $\mathcal{D}$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
    qed -1
  end
  red 1 y0  >= 1 ; y0 -> 1
  rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
  red 1 ~y1  1 y0      >= 1 ; y1 -> 0
  red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
  red 1 p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
  red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
  pol 26 32 2048 * +
  del id 26
  rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by RUP

$$F \wedge \mathcal{D} \wedge \neg(\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1)$$

# Back to Our Pigeons — Deriving the Constraints

**Derived constraints ( $\mathcal{D}$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
    qed -1
  end
  red 1 y0  >= 1 ; y0 -> 1
  rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
  red 1 ~y1  1 y0      >= 1 ; y1 -> 0
  red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
  red 1 p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
  red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
  pol 26 32 2048 * +
  del id 26
  rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by RUP

$$F \wedge \mathcal{D} \wedge \neg(\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1) \models F \wedge \mathcal{D} \wedge (y_0 \geq 1) \wedge (p_{13} \geq 1) \wedge (\bar{p}_{23} \geq 1)$$

# Back to Our Pigeons — Deriving the Constraints

**Derived constraints ( $\mathcal{D}$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
    qed -1
  end
  red 1 y0  >= 1 ; y0 -> 1
  rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
  red 1 ~y1  1 y0  >= 1 ; y1 -> 0
  red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
  red 1 p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
  red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
  pol 26 32 2048 * +
  del id 26
  rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by RUP

$$F \wedge \mathcal{D} \wedge \neg(\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1) \models F \wedge \mathcal{D} \wedge (y_0 \geq 1) \wedge (p_{13} \geq 1) \wedge (\bar{p}_{23} \geq 1)$$

$$2^{11} \cdot (p_{23} - p_{13}) + 2^{10} \cdot (p_{22} - p_{12}) + \dots \geq 0$$

# Back to Our Pigeons — Deriving the Constraints

**Derived constraints ( $\mathcal{D}$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
    qed -1
  end
  red 1 y0  >= 1 ; y0 -> 1
  rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
  red 1 ~y1  1 y0  >= 1 ; y1 -> 0
  red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
  red 1 p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
  red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
  pol 26 32 2048 * +
  del id 26
  rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by RUP

$$F \wedge \mathcal{D} \wedge \neg(\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1) \models F \wedge \mathcal{D} \wedge (y_0 \geq 1) \wedge (p_{13} \geq 1) \wedge (\bar{p}_{23} \geq 1)$$

$$2^{11} \cdot (-1) + 2^{10} \cdot (p_{22} - p_{12}) + \dots \geq 0$$

# Back to Our Pigeons — Deriving the Constraints

**Derived constraints ( $\mathcal{D}$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
    qed -1
  end
  red 1 y0  >= 1 ; y0 -> 1
  rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
  red 1 ~y1  1 y0      >= 1 ; y1 -> 0
  red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
  red 1 p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
  red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
  pol 26 32 2048 * +
  del id 26
  rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by RUP

$$F \wedge \mathcal{D} \wedge \neg(\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1) \models F \wedge \mathcal{D} \wedge (y_0 \geq 1) \wedge (p_{13} \geq 1) \wedge (\bar{p}_{23} \geq 1)$$

$$2^{11} \cdot (-1) + 2^{10} \cdot (p_{22} - p_{12}) + \dots \geq 0$$

$$\text{where } \sum_{i=1}^{10} 2^i < 2^{11}$$

# Back to Our Pigeons — Deriving the Constraints

**Derived constraints ( $\mathcal{D}$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\bar{y}_1 + y_0 \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
    qed -1
  end
  red 1 y0  >= 1 ; y0 -> 1
  rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
  red 1 ~y1  1 y0      >= 1 ; y1 -> 0
  red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
  red 1 p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
  red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
  pol 26 32 2048 * +
  del id 26
  rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by **redundance** with witness  $\omega = \{y_1 \mapsto 0\}$

$$F \wedge \mathcal{D} \wedge \neg(\bar{y}_1 + y_0 \geq 1)$$

$$\models (F \wedge \mathcal{D}) \upharpoonright_{\omega} \wedge (\bar{y}_1 + y_0 \geq 1) \upharpoonright_{\omega}$$

# Back to Our Pigeons — Deriving the Constraints

**Derived constraints ( $\mathcal{D}$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\bar{y}_1 + y_0 \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
    qed -1
  end
  red 1 y0  >= 1 ; y0 -> 1
  rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
  red 1 ~y1  1 y0      >= 1 ; y1 -> 0
  red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
  red 1 p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
  red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
  pol 26 32 2048 * +
  del id 26
  rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by **redundance** with witness  $\omega = \{y_1 \mapsto 0\}$

$$F \wedge \mathcal{D} \wedge \neg(\bar{y}_1 + y_0 \geq 1)$$

$$\models (F \wedge \mathcal{D}) \upharpoonright_\omega \wedge (\bar{y}_1 + y_0 \geq 1) \upharpoonright_\omega$$

$$F \wedge \mathcal{D} \wedge (y_1 + \bar{y}_0 \geq 2)$$

$$\models (F \wedge \mathcal{D}) \quad \wedge (1 + y_0 \geq 1)$$

# Back to Our Pigeons — Deriving the Constraints

Derived constraints ( $\mathcal{D}$ ):

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\bar{y}_1 + y_0 \geq 1$$

$$\bar{y}_1 + \overline{\sigma(p_{13})} + p_{13} \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
    qed -1
  end
  red 1 y0  >= 1 ; y0 -> 1
  rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
  red 1 ~y1  1 y0      >= 1 ; y1 -> 0
  red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
  red 1  p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
  red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
  pol 26 32 2048 * +
  del id 26
  rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by **redundance** with witness  $\omega = \{y_1 \mapsto 0\}$   
(essentially same argument)

# Back to Our Pigeons — Deriving the Constraints

**Derived constraints ( $\mathcal{D}$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\bar{y}_1 + y_0 \geq 1$$

$$\bar{y}_1 + \overline{\sigma(p_{13})} + p_{13} \geq 1$$

$$y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1$$

```
dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
    qed -1
  end
  red 1 y0  >= 1 ; y0 -> 1
  rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
  red 1 ~y1  1 y0      >= 1 ; y1 -> 0
  red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
  red 1 p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
  red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
  pol 26 32 2048 * +
  del id 26
  rup 1 ~y1  1 ~p12 1 p22 >= 1 ;
```

Derivable by **redundance** with witness  $\omega = \{y_1 \mapsto 1\}$

$$F \wedge \mathcal{D} \wedge \neg(y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1)$$

$$\models (F \wedge \mathcal{D}) \upharpoonright_{\omega} \wedge (y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1) \upharpoonright_{\omega}$$

# Back to Our Pigeons — Deriving the Constraints

**Derived constraints ( $\mathcal{D}$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\bar{y}_1 + y_0 \geq 1$$

$$\bar{y}_1 + \overline{\sigma(p_{13})} + p_{13} \geq 1$$

$$y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1$$

```

dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
    qed -1
  end
  red 1 y0  >= 1 ; y0 -> 1
  rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
  red 1 ~y1  1 y0      >= 1 ; y1 -> 0
  red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
  red 1 p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
  red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
  pol 26 32 2048 * +
  del id 26
  rup 1 ~y1  1 ~p12 1 p22 >= 1 ;

```

Derivable by **redundance** with witness  $\omega = \{y_1 \mapsto 1\}$

$$F \wedge \mathcal{D} \wedge \neg(y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1)$$

$$\models (F \wedge \mathcal{D}) \upharpoonright_{\omega} \wedge (y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1) \upharpoonright_{\omega}$$

$$F \wedge \mathcal{D} \wedge (\bar{y}_1 + y_0 + p_{13} \geq 3)$$

$$\models \dots \wedge \mathcal{D} \upharpoonright_{\omega} \wedge \dots$$

# Back to Our Pigeons — Deriving the Constraints

**Derived constraints ( $\mathcal{D}$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\bar{y}_1 + y_0 \geq 1$$

$$\bar{y}_1 + \overline{\sigma(p_{13})} + p_{13} \geq 1$$

$$y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1$$

```

dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
    qed -1
  end
  red 1 y0  >= 1 ; y0 -> 1
  rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
  red 1 ~y1  1 y0      >= 1 ; y1 -> 0
  red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
  red 1 p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
  red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
  pol 26 32 2048 * +
  del id 26
  rup 1 ~y1  1 ~p12 1 p22 >= 1 ;

```

Derivable by **redundance** with witness  $\omega = \{y_1 \mapsto 1\}$

$$F \wedge \mathcal{D} \wedge \neg(y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1)$$

$$\models (F \wedge \mathcal{D}) \upharpoonright_{\omega} \wedge (y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1) \upharpoonright_{\omega}$$

$$F \wedge \mathcal{D} \wedge (\bar{y}_1 + y_0 + p_{13} \geq 3)$$

$$\models \dots \wedge \mathcal{D} \upharpoonright_{\omega} \wedge \dots$$

# Back to Our Pigeons — Deriving the Constraints

**Derived constraints ( $\mathcal{D}$ ):**

$$\begin{aligned} & 2^{11} \cdot (p_{23} - p_{13}) + \\ & 2^{10} \cdot (p_{22} - p_{12}) + \\ & \dots \geq 0 \end{aligned}$$

$$y_0 \geq 1$$

$$\bar{y}_0 + \bar{p}_{13} + \sigma(p_{13}) \geq 1$$

$$\bar{y}_1 + y_0 \geq 1$$

$$\bar{y}_1 + \overline{\sigma(p_{13})} + p_{13} \geq 1$$

$$y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1$$

```

dom -64 p21 64 [...] -2048 p13 2048 p23  >= 0 ; p11 -> p21 [...] p23 -> p13 ; begin
  proofgoal #2
    pol -1 -2 +
    qed -1
  end
  red 1 y0  >= 1 ; y0 -> 1
  rup 1 ~y0  1 ~p13 1 p23 >= 1 ;
  red 1 ~y1  1 y0      >= 1 ; y1 -> 0
  red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
  red 1 p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
  red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
  pol 26 32 2048 * +
  del id 26
  rup 1 ~y1  1 ~p12 1 p22 >= 1 ;

```

Derivable by **redundance** with witness  $\omega = \{y_1 \mapsto 1\}$

$$F \wedge \mathcal{D} \wedge \neg(y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1)$$

$$\models (F \wedge \mathcal{D}) \upharpoonright_{\omega} \wedge (y_1 + \bar{y}_0 + \bar{p}_{13} \geq 1) \upharpoonright_{\omega}$$

$$F \wedge \mathcal{D} \wedge (\bar{y}_1 + y_0 + p_{13} \geq 3)$$

$$\models \dots \wedge \mathcal{D} \upharpoonright_{\omega} \wedge \dots$$

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Derivable by **redundance** with witness  $\omega = \{y_1 \mapsto 1\}$   
(same argument)

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Simplify the pseudo-Boolean breaking constraint and delete original constraint

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  red 1 ~y1  1 ~p23 1 p13 >= 1 ; y1 -> 0
  red 1 p23 1 ~y0  1 y1  >= 1 ; y1 -> 1
  red 1 ~p13 1 ~y0  1 y1  >= 1 ; y1 -> 1
  pol 26 32 2048 * +
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```

Continue in the same way for following  $y_i$ -variables

...

# Future Research Directions

## Performance and reliability of pseudo-Boolean proof logging

- Trim proof while verifying (as in *DRAT-Trim* [HHW13a])
- Compress proof file using binary format
- Design formally verified proof checker (*work in progress* [BMM<sup>+</sup>23])

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## Proof logging for other combinatorial problems and techniques

- Symmetric learning and recycling (substitution) of subproofs
- Mixed integer linear programming (*some work on SCIP* in [CGS17, EG21])
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## And more...

- Use proof logs for algorithm analysis or explainability purposes
- Lots of other challenging problems and interesting ideas

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## And more...

- Use proof logs for algorithm analysis or explainability purposes
- Lots of other challenging problems and interesting ideas
- Talk to us if you want to join the proof logging revolution! ☺  
We're happy to **collaborate**, and **we're hiring**

# Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity

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The end.

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- Action point: What problems can VERIPB solve for you?

The end. Or rather, the beginning!

# References for Getting Started with VERIPB

<https://gitlab.com/MIA0research/software/VeriPB>

Released under MIT Licence



Various features to help development:

- Extended variable name syntax allowing human-readable names
- Proof tracing
- “Trust me” assertions for incremental proof logging

Documentation:

- Description of VERIPB checker [BMM<sup>+</sup>23] used in SAT 2023 competition (<https://satcompetition.github.io/2023/checkers.html>)
- Specific details on different proof logging techniques covered in research papers [EGMN20, GMN20, GMM<sup>+</sup>20, GN21, GMN22, GMNO22, VDB22, BBN<sup>+</sup>23, BGPN23, MM23]
- Lots of concrete example files at <https://gitlab.com/MIA0research/software/VeriPB>

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# Parity Reasoning: Experimental Evaluation

Implemented parity reasoning and PB proof logging engine<sup>2</sup>

Also DRAT proof logging for XOR constraints as described in [PR16]

Experiments with MINISAT<sup>3</sup>

Set-up:<sup>4</sup>

- Intel Core i5-1145G7 @2.60GHz × 4
- Memory limit 8GiB
- Disk write speed roughly 200 MiB/s
- Read speed of 2 GiB/s

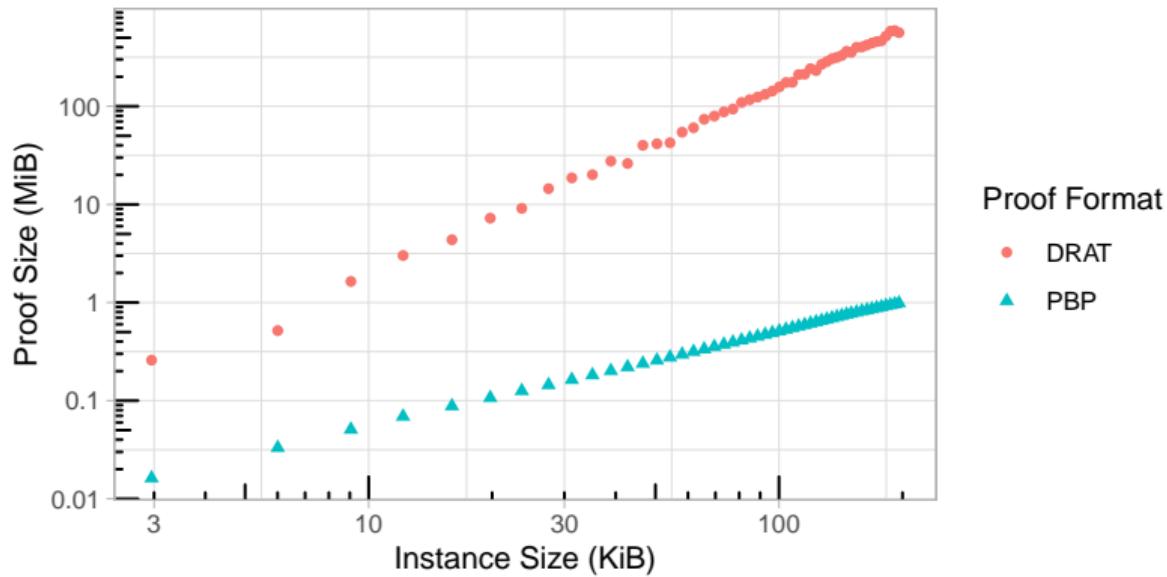
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<sup>2</sup><https://gitlab.com/MIA0research/tools-and-utilities/xorengine>

<sup>3</sup><http://minisat.se/>

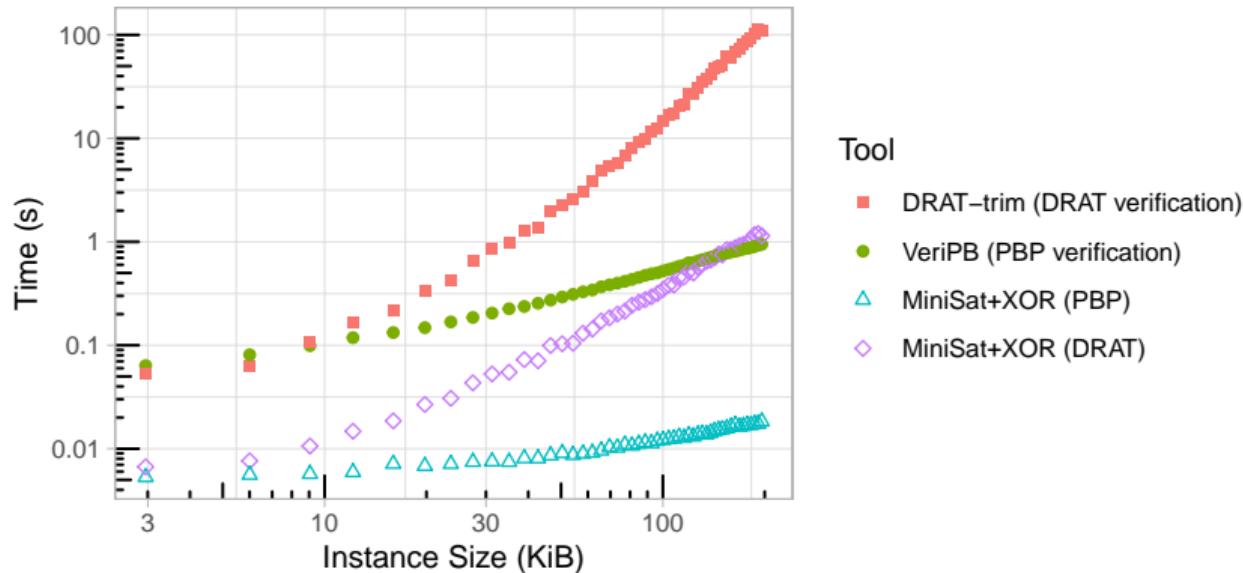
<sup>4</sup>Tools, benchmarks, data and evaluation scripts available at <https://doi.org/10.5281/zenodo.7083485>

# Parity Reasoning: Proof Size for DRAT and PB Proof Logging



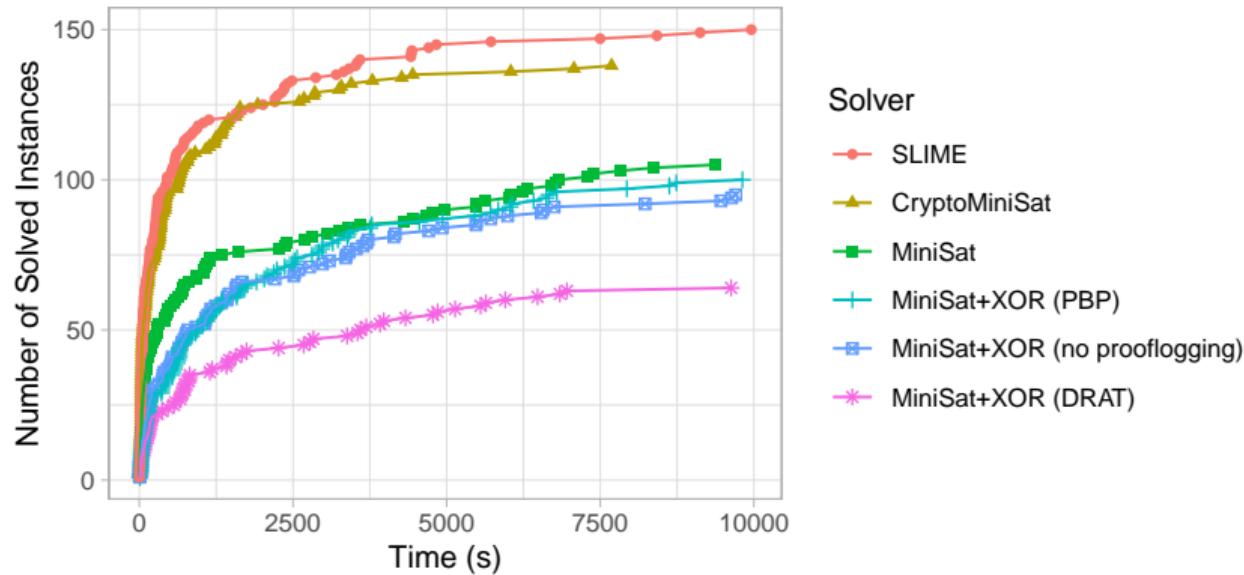
## Proof sizes for Tseitin formulas using DRAT and pseudo-Boolean proof logging

## Parity Reasoning: Solving and Proof Checking Time



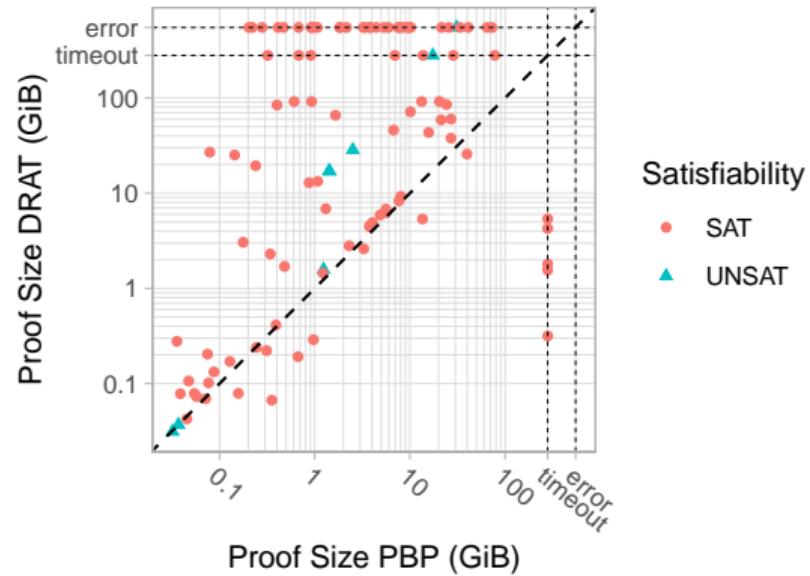
Solving and proof checking time for Tseitin formulas using DRAT and PB proof logging

# Parity Reasoning: Crypto Track of SAT 2021 Competition



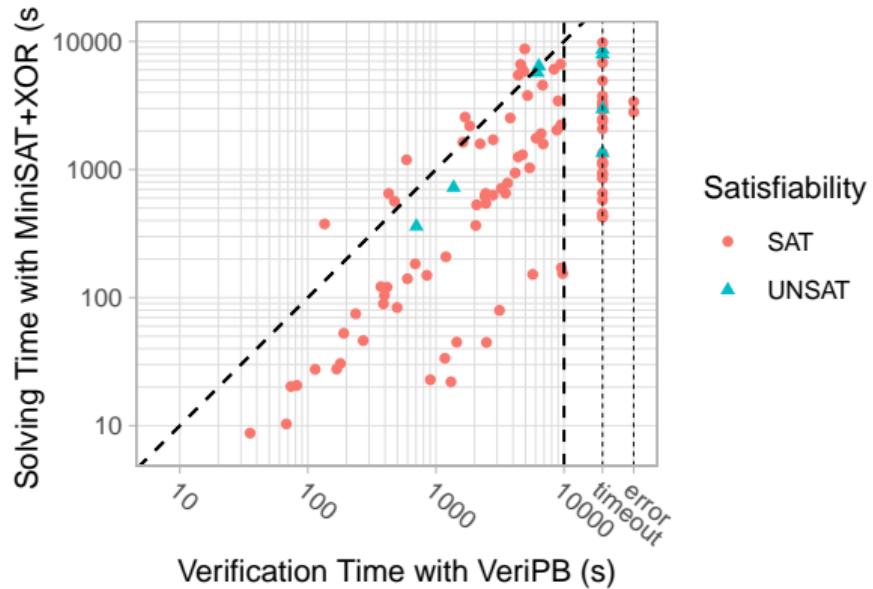
## Cumulative plot for the crypto track of the SAT Competition 2021

# Parity Reasoning: Crypto Track Proof Size



DRAT and PB proof sizes for crypto track of SAT Competition 2021

# Parity Reasoning: Crypto Track Solving & Proof Checking Time



Time required for solving and proof checking for cryptographic instances

# PB-to-CNF Translation: Experimental Evaluation

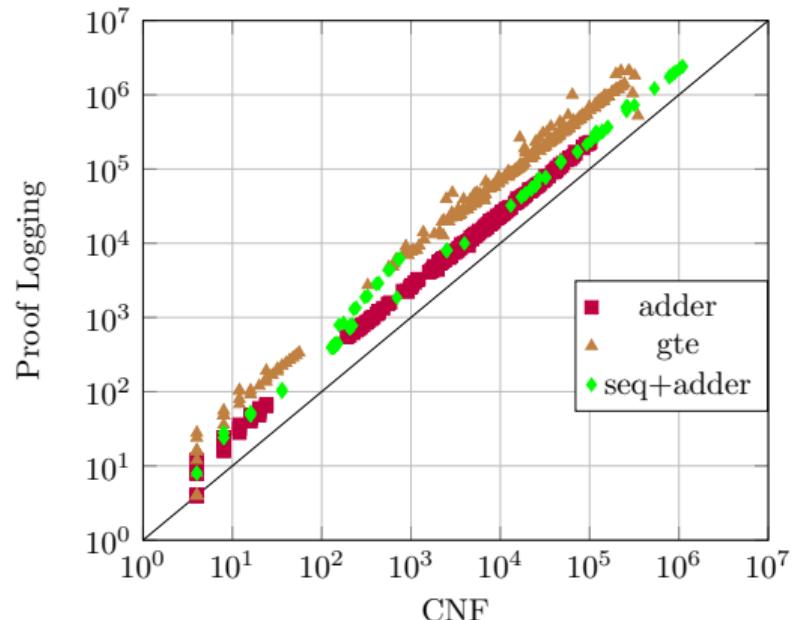
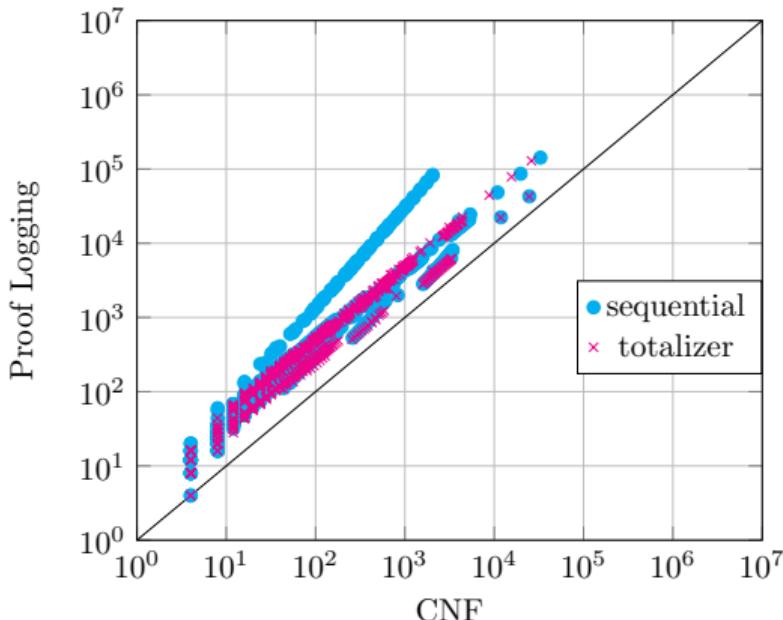
- Certified translations for CNF encodings with *VeritasPBLib*<sup>5</sup>
  - Sequential counter [Sin05]
  - Totalizer [BB03]
  - Generalized totalizer [JMM15]
  - Adder network [ES06]
- Proofs verified by proof checker VERIPB
- Formulas solved with fork of KISSAT<sup>6</sup> syntactically modified to output VERIPB proofs
- Benchmarks from PB 2016 Evaluation<sup>7</sup> in 3 categories
  - Only cardinality constraints (sequential counter, totalizer)
  - Only general 0-1 ILP constraints (generalized totalizer, adder network)
  - Mixed cardinality & general 0-1 ILP constraints (sequential counter + adder network)

<sup>5</sup><https://github.com/forge-lab/VeritasPBLib>

<sup>6</sup>[https://gitlab.com/MIA0research/tools-and-utilities/kissat\\_fork](https://gitlab.com/MIA0research/tools-and-utilities/kissat_fork)

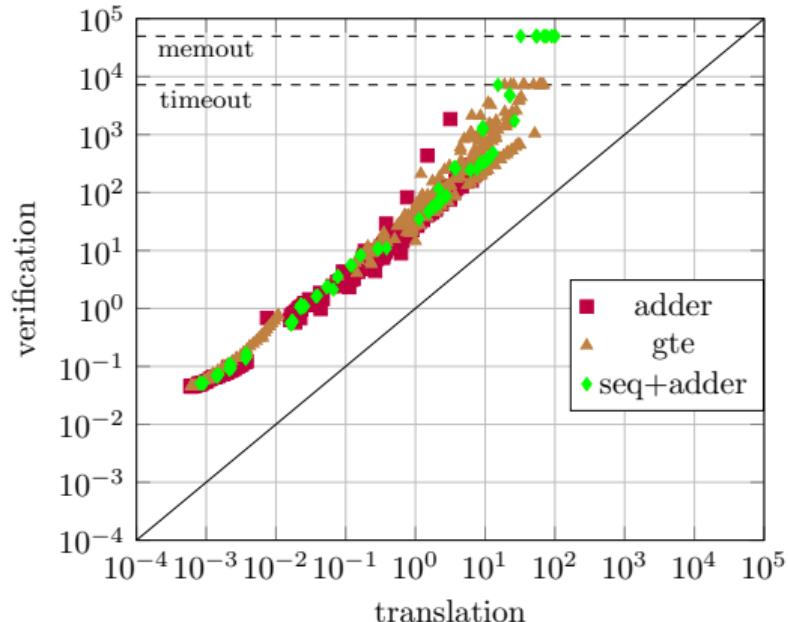
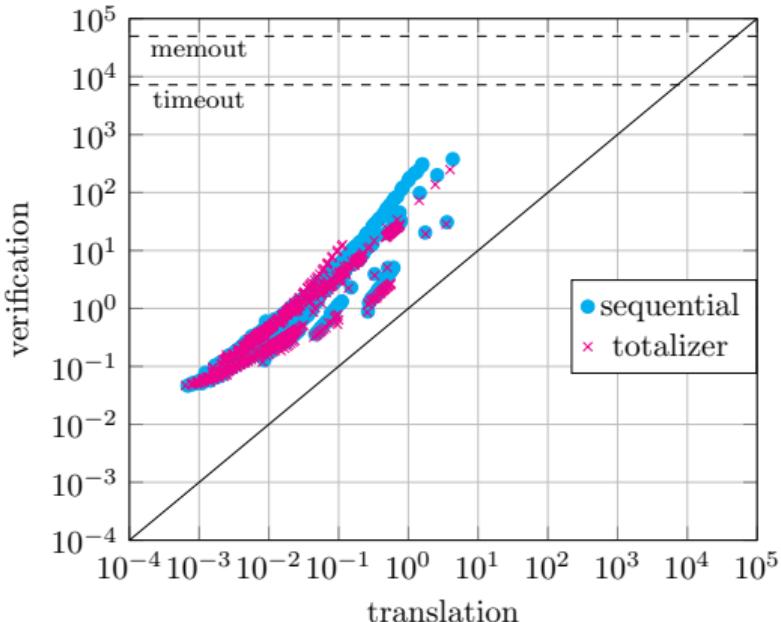
<sup>7</sup><http://www.cril.univ-artois.fr/PB16/>

# PB-to-CNF: CNF Size vs Proof Size in KiB



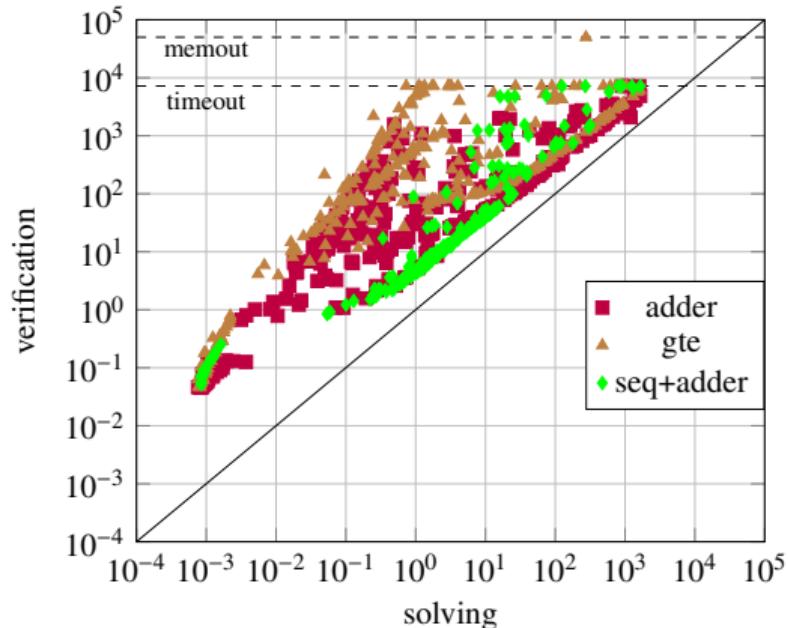
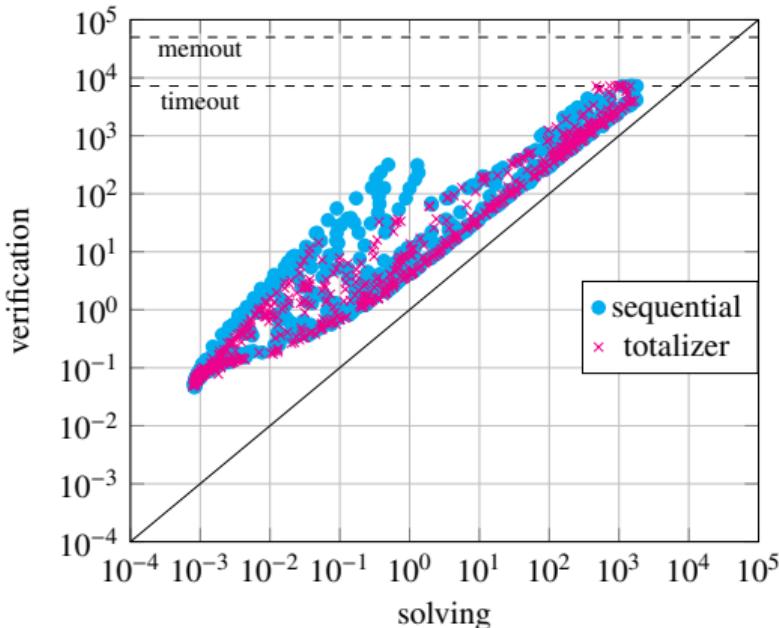
- Nice scaling for proof size in terms of original CNF formula size
- Except for some sequential encoding cases (which is not such a great encoding anyway)

# PB-to-CNF: Translation Time vs Proof Checking Time in Seconds



- Translation faster — only has to generate clauses and proof
- Proof checking slower — has to verify full proof

# PB-to-CNF: Solving Time vs Proof Checking Time in Seconds



- Room for improvement of end-to-end proof checking process
- But even first proof-of-concept implementation shows our approach is viable

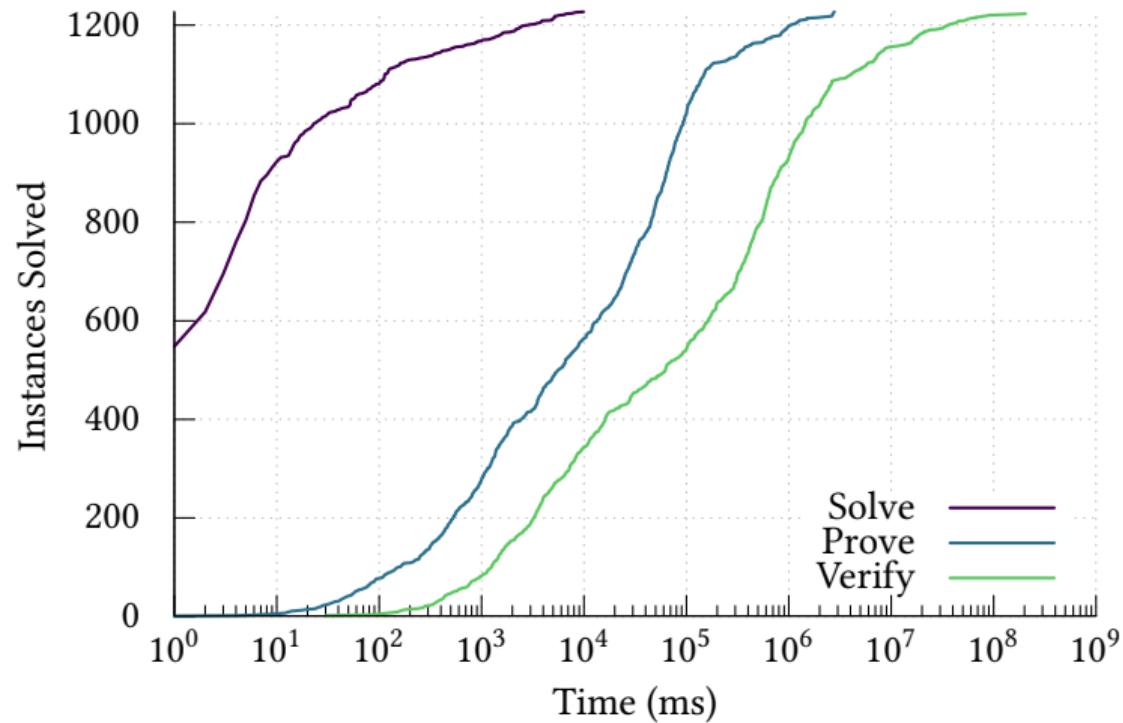
# Clique Solving: Experimental Evaluation

- Implemented in the *Glasgow Subgraph Solver*
  - Bit-parallel, can perform a colouring and recursive call in under a microsecond
- 59 of the 80 DIMACS instances take under 1,000 seconds to solve without logging
- Produced and verified proofs for 57 of these 59 instances (the other two reached 1TByte disk space)
- Mean slowdown from proof logging is 80.1 (due to disk I/O)
- Mean verification slowdown a further 10.1
- Approximate implementation effort: one Masters student

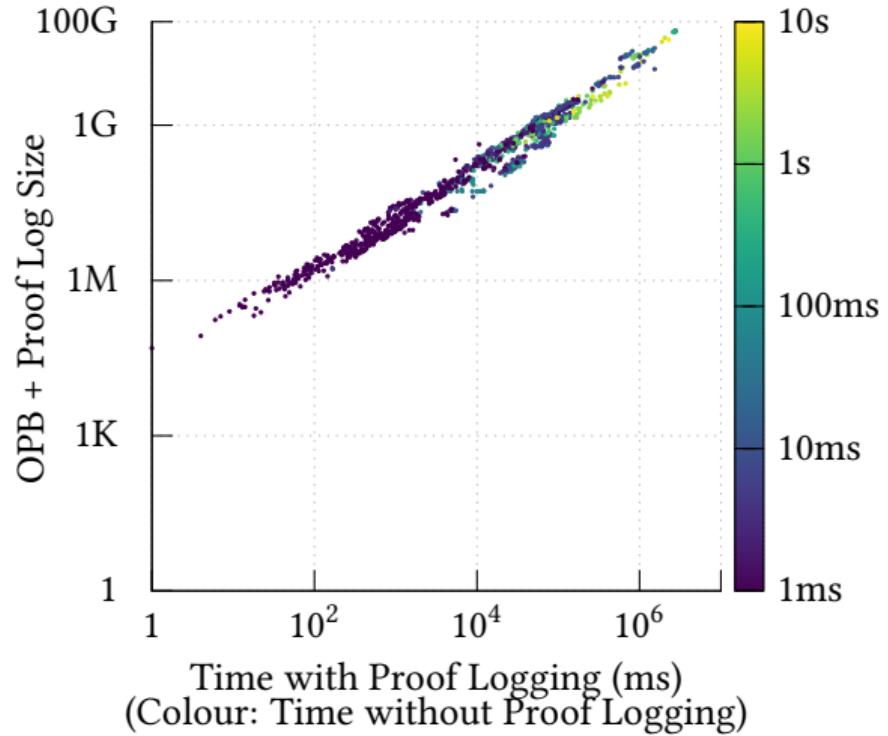
# Subgraph Isomorphism Solving: Experimental Evaluation (1/3)

- The Pseudo-Boolean models can be large: had to restrict to instances with no more than 260 vertices in the target graph
- Took enumeration instances which could be solved without proof logging in under ten seconds
- 1,227 instances from Solnon's benchmark collection:
  - 789 unsatisfiable, up to 50,635,140 solutions in the rest
  - 498 instances solved without guessing
  - Hardest solved satisfiable and unsatisfiable instances required 53,605,482 and 2,074,386 recursive calls

## Subgraph Isomorphism Solving: Experimental Evaluation (2/3)



# Subgraph Isomorphism Solving: Experimental Evaluation (3/3)



# Constraint Programming: How Expensive is Proof Logging? (1/2)

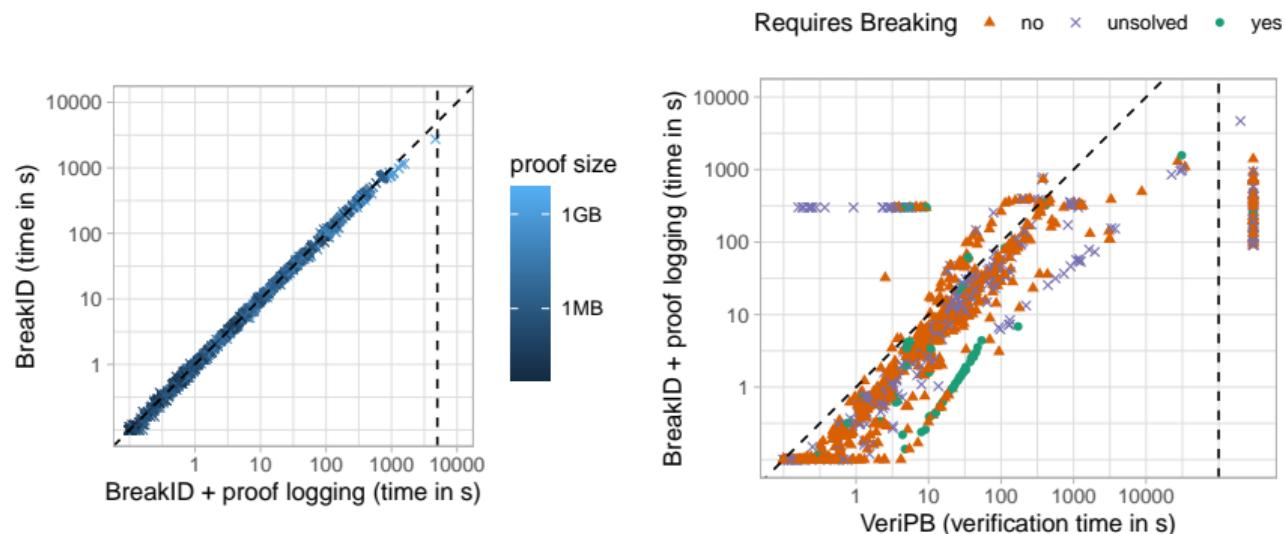
- Laurent D. Michel, Pierre Schaus, Pascal Van Hentenryck: *MiniCP: A Lightweight Solver for Constraint Programming* [MSH21]
- Five benchmark problems allowing comparison of solvers “doing the same thing”:
  - Simple models
  - Fixed search order and well-defined propagation consistency levels
  - Few global constraints
- Probably close to the worst case for proof logging performance
- Also: Crystal Maze and World’s Hardest Sudoku

# Constraint Programming: How Expensive is Proof Logging? (2/2)

- Our solver: faster than the fastest of *MiniCP*, *OscaR*, and *Choco*
- Proof logging slowdown: between 8.4 and 61.1 factor
  - 800,000 to 3,000,000 inferences per second
  - Proof logs can be hundreds of GBytes
  - No effort put into making the proof-writing code run fast
- Verification slowdown: a further factor 10 to 100
  - Probably possible to reduce this substantially if we are prepared to put more care into writing proofs

# SAT Symmetry Breaking: Experimental Evaluation

- Evaluated on SAT competition benchmarks
- *BreakID* [DBBD16, Bre] used to find and break symmetries



- Proof logging overhead negligible
- Proof checking at most 20 times slower than solving for 95% of instances