Certified CNF Translations for Pseudo-Boolean Solving

Jakob Nordström

University of Copenhagen
and Lund University

Swedish Operations Research Conference (SOAK 2022)

October 24, 2022

Joint work with Stephan Gocht, Ruben Martins, and Andy Oertel

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Thanks for the slides!

The Pseudo-Boolean (PB) Problem

pseudo-Boolean formula:

$$x_1 + x_2 + x_3 \geq 2$$
$$x_2 + x_3 + x_4 \geq 2$$
$$\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 + \overline{x}_4 \geq 3$$

$\rightarrow$ Pseudo-Boolean Solver $\rightarrow$

Result: SAT/UNSAT

- Input: Pseudo-Boolean formula (a.k.a. 0-1 integer linear program)
  - Collection of 0-1 integer linear constraints

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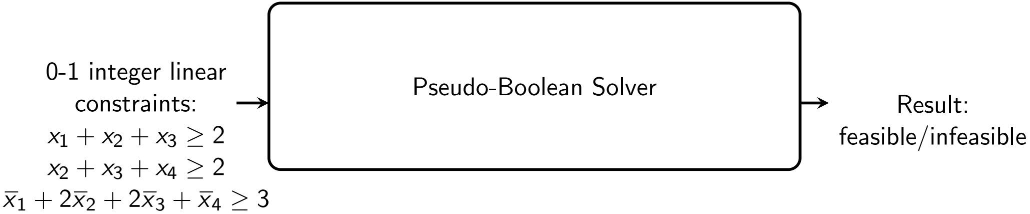

- Input: Pseudo-Boolean formula (a.k.a. 0-1 integer linear program)
  - Collection of 0-1 integer linear constraints
- Pseudo-Boolean solvers:
  - Native: Sat4j [LP10], RoundingSAT [EN18]
  - SAT-based: MiniSAT+ [ES06], Open-WBO [MML14], NaPS [SN15]

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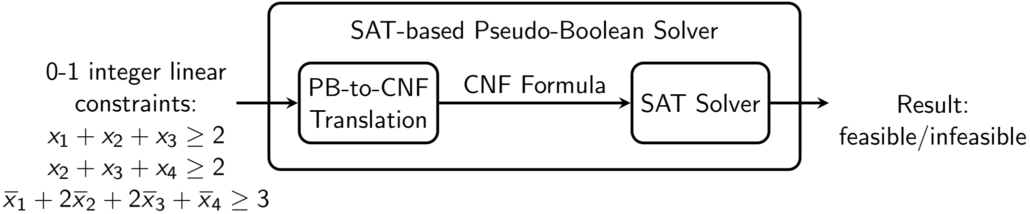

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Certification with Proof Logging

Solver

Certification with Proof Logging

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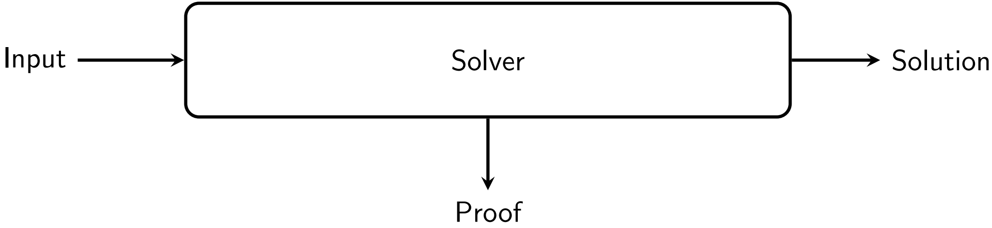

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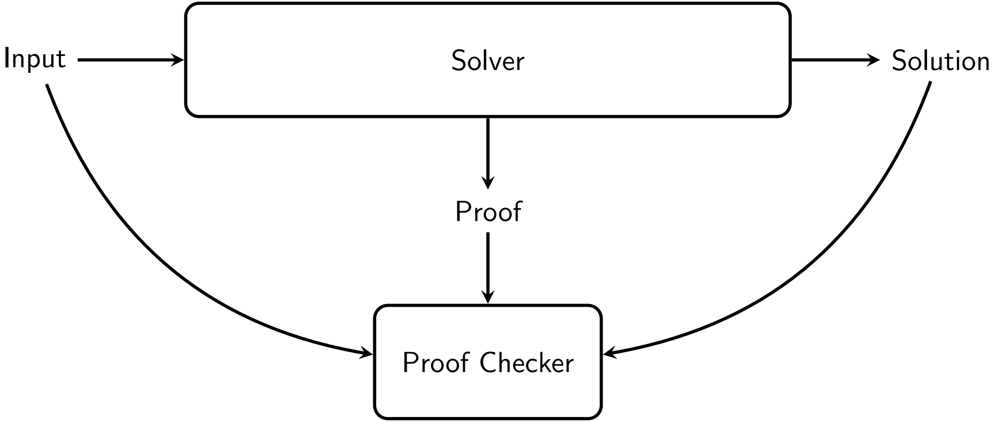

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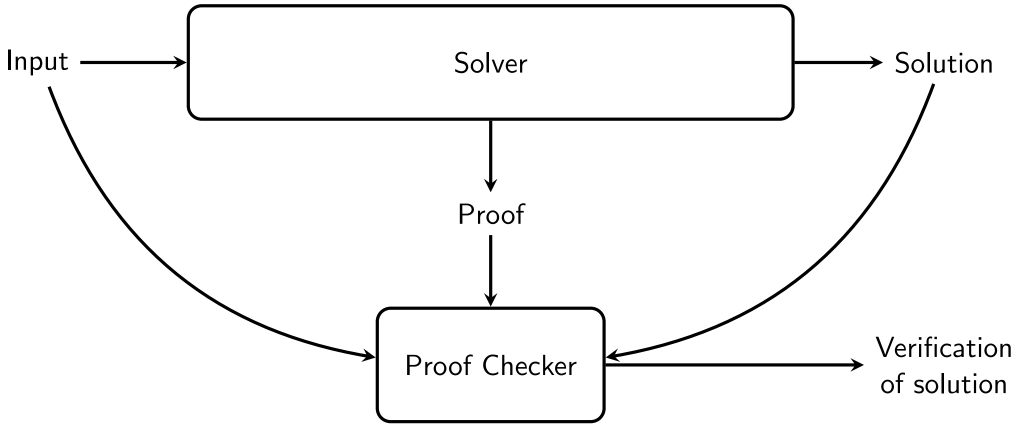

Certifying Correctness

0-1 integer-linear constraints:

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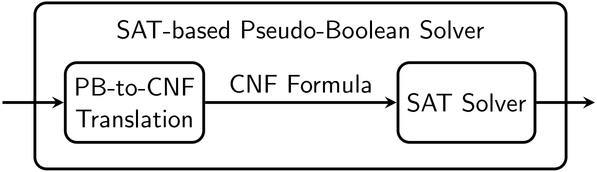

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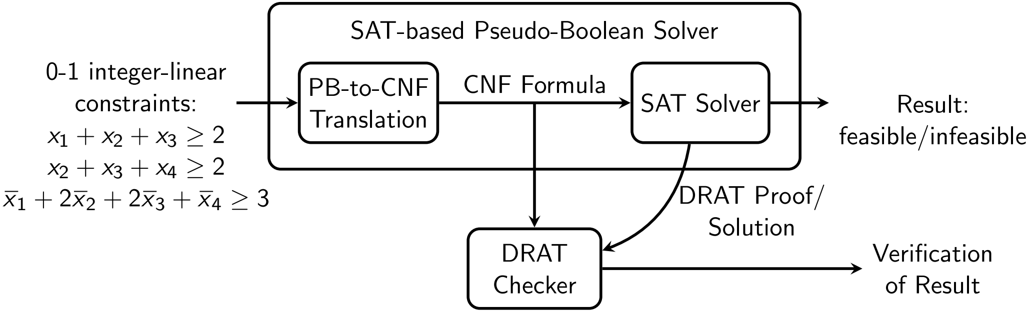

- Correctness of SAT solver solution can be certified [HHW13a, HHW13b, WHH14]

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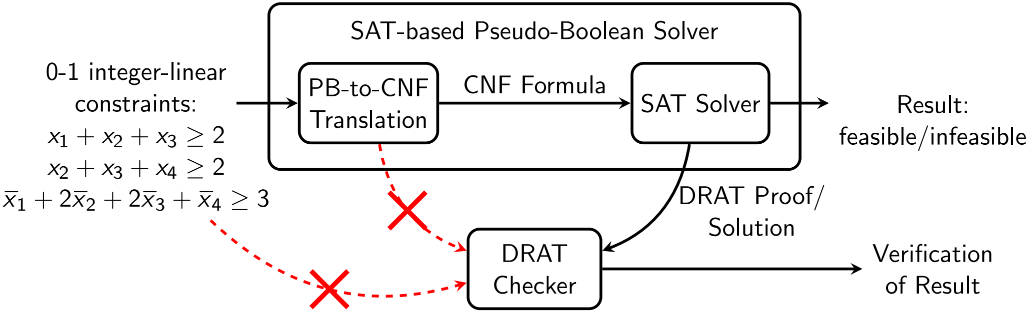

- Correctness of SAT solver solution can be certified [HHW13a, HHW13b, WHH14]
- **PB-to-CNF translation uncertified!**

Pseudo-Boolean Proof Logging

- **Multi-purpose** proof format
- Allows easy proof logging for
  - Reasoning with pseudo-Boolean constraints (by design)
  - SAT solving (including advanced techniques) [GN21, BGMN22]
  - Constraint programming [EGMN20, GMN22]
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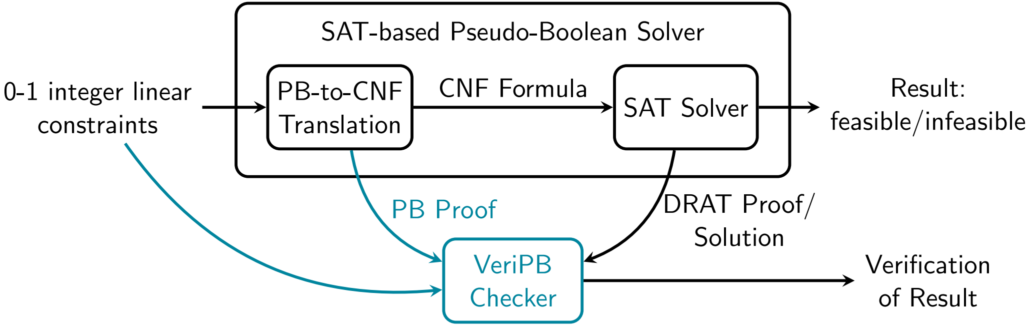
This Work

- Proof logging for translating pseudo-Boolean constraints to CNF
- **General framework** to certify many different encodings
- Promising foundation for certifying MaxSAT solving and PB optimization

Workflow



SAT-based Pseudo-Boolean Solver
0-1 integer linear constraints
PB-to-CNF Translation
CNF Formula
SAT Solver
Result: feasible/infeasible
PB Proof
DRAT Proof/ Solution
VeriPB Checker
Verification of Result

Basic Notation

- Boolean variable x: with domain 0 (false) and 1 (true)
- Literal ℓ: x or negation $\overline{x} = 1 - x$
- 0-1 integer linear constraint: integer linear inequality over literals

$$3x_1 + 2x_2 + 5\overline{x}_3 \geq 5$$

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- Equality constraint: syntactic sugar for 2 inequalities

$$3x_1 + 2x_2 + 5\overline{x}_3 = 5 \longrightarrow \begin{array}{l} 3x_1 + 2x_2 + 5\overline{x}_3 \geq 5 \\ 3x_1 + 2x_2 + 5\overline{x}_3 \leq 5 \end{array}$$

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- Clause: disjunction of literals / at-least-one constraint

$$x_1 \vee \overline{x}_2 \vee \overline{x}_3 \iff x_1 + \overline{x}_2 + \overline{x}_3 \geq 1$$

Cutting Planes Proof System [CCT87]

Rules:

- Literal axiom

$$\frac{}{\ell_i \geq 0}$$

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$$\frac{x_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3 \qquad \overline{x}_2 + 3x_3 \geq 3}{x_1 + 3\overline{x}_2 + x_3 \geq 4} \text{Add}$$

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$$\frac{x_1 + 2\overline{x}_2 \geq 3}{2x_1 + 4\overline{x}_2 \geq 6} \text{ Multiply by 2}$$

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$$\frac{x_1 + 2\overline{x}_2 \geq 3}{2x_1 + 4\overline{x}_2 \geq 6} \text{ Multiply by 2}$$

- Division

$$\frac{2x_1 + 2\overline{x}_2 + 4x_3 \geq 5}{x_1 + \overline{x}_2 + 2x_3 \geq 3} \text{ Divide by 2}$$

Extended Cutting Planes: Reification

Extension rule to introduce fresh variables:

- Reification (special case of redundance rule in [GN21, BGMN22])

$$a \Leftrightarrow x_1 + \overline{x}_2 + 2x_3 \geq 2 \longrightarrow \begin{array}{ll} 2\overline{a} + x_1 + \overline{x}_2 + 2x_3 \geq 2 & \qquad (a \Rightarrow x_1 + \overline{x}_2 + 2x_3 \geq 2) \\ 3a + \overline{x}_1 + x_2 + 2\overline{x}_3 \geq 3 & \qquad (a \Leftarrow x_1 + \overline{x}_2 + 2x_3 \geq 2) \end{array}$$

Translating 0-1 ILP to CNF: Outline

1. Construct circuit evaluating left-hand side of 0-1 integer linear constraint
2. Encode circuit to CNF using so-called Tseitin translation
3. Enforce constraint

Translating 0-1 ILP to CNF: Step 1

1. **Construct circuit evaluating left-hand side of 0-1 integer linear constraint**

- Several approaches to construct logical circuit evaluating PB constraint
  - Sequential counter [Sin05], totalizer [BB03], adder network [ES06], ...

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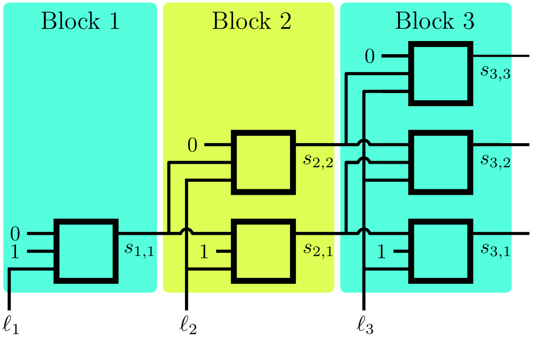
Example: $\ell_1 + \ell_2 + \ell_3 \geq 2$

Meaning of $s_{i,j}$ variable:
$s_{i,j}$ true if and only if
$\ell_1 + \ldots + \ell_i \geq j$



Translating 0-1 ILP to CNF: Step 2

2. **Encode circuit to CNF using so-called Tseitin translations**

- Introduce fresh variable for each wire
- Encode using clauses describing behavior of each component

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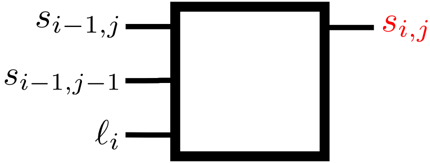
Example: Sequential counter component



Specification of $s_{i,j}$

$$s_{i,j} \leftrightarrow (\ell_i \wedge s_{i-1,j-1}) \vee s_{i-1,j}$$

Clausal encoding

$$\overline{\ell}_i \vee \overline{s}_{i-1,j-1} \vee s_{i,j}$$
$$\overline{s}_{i-1,j} \vee s_{i,j}$$
$$\ell_i \vee s_{i-1,j} \vee \overline{s}_{i,j}$$
$$s_{i-1,j-1} \vee \overline{s}_{i,j}$$

Translating 0-1 ILP to CNF: Step 3

3. **Enforce constraint**

- Add clauses enforcing comparison with right-hand side

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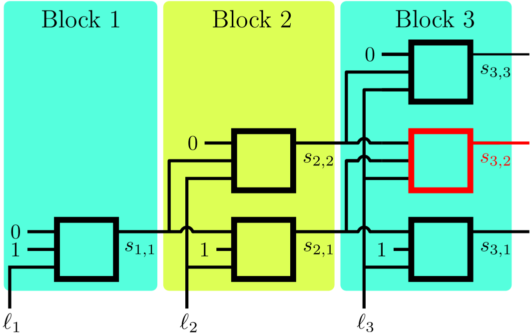
Example: $\ell_1 + \ell_2 + \ell_3 \geq 2$



At least 2 true literals if $s_{3,2}$ true.

Add unary clause

$s_{3,2}$

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Is the translation correct?

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- Proof logging!
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This means

- 0-1 ILP has feasible solutions $\Longrightarrow$ CNF translation satisfiable
- Solver finds no solution to CNF translation $\Longrightarrow$ 0-1 ILP is infeasible

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End-to-end verification of SAT-based pseudo-Boolean solvers!

Rest of This Talk: Some technical details?

We develop general framework certifying PB-to-CNF translations

- But let us stay with our example:

Sequential counter encoding of $\ell_1 + \ell_2 + \ell_3 \geq 2$

Circuit Specification in Pseudo-Boolean Form

Using Cutting Planes + reification, do syntactic derivation of circuit specification:

- Specification of $s_{i,j}$ variables

$$s_{i,j} \Leftrightarrow \sum_{k=1}^{i-1} s_{i-1,k} + \ell_i \geq j$$

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- Ordering of $s_{i,j}$ variables

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- Ordering of $s_{i,j}$ variables

$$s_{i,j} \geq s_{i,j+1}$$

- Preservation of sum

$$\sum_{k=1}^{i} s_{i,k} = \sum_{k=1}^{i-1} s_{i-1,k} + \ell_i$$

Deriving the CNF Translation

We now have 0-1 integer linear constraints:

$$s_{1,1} = \ell_1 \qquad s_{2,1} + s_{2,2} = s_{1,1} + \ell_2 \qquad s_{3,1} + s_{3,2} + s_{3,3} = s_{2,1} + s_{2,2} + \ell_3$$
$$s_{2,1} \geq s_{2,2} \qquad s_{3,1} \geq s_{3,2} \qquad s_{3,2} \geq s_{3,3} \qquad s_{3,1} + s_{3,2} + s_{3,3} \geq 2$$

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But we want clauses:

$$\begin{array}{rrrr}
\overline{\ell}_1 \lor s_{1,1} & \overline{\ell}_2 \lor \overline{s}_{1,1} \lor s_{2,2} & \ell_3 \lor s_{2,1} \lor \overline{s}_{3,1} & \overline{\ell}_3 \lor \overline{s}_{2,2} \lor s_{3,3} \\
\ell_1 \lor \overline{s}_{1,1} & \ell_2 \lor \overline{s}_{2,2} & \overline{\ell}_3 \lor \overline{s}_{2,1} \lor s_{3,2} & \ell_3 \lor \overline{s}_{3,3} \\
\overline{\ell}_2 \lor s_{2,1} & s_{1,1} \lor \overline{s}_{2,2} & \overline{s}_{2,2} \lor s_{3,2} & s_{2,2} \lor \overline{s}_{3,3} \\
\overline{s}_{1,1} \lor s_{2,1} & \overline{\ell}_3 \lor s_{3,1} & \ell_3 \lor s_{2,2} \lor \overline{s}_{3,2} & s_{3,2} \\
\ell_2 \lor s_{1,1} \lor \overline{s}_{2,1} & \overline{s}_{2,1} \lor s_{3,1} & s_{2,1} \lor \overline{s}_{3,2} &
\end{array}$$

Deriving the CNF Translation

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\end{array}$$

- Follow easily from PB specification by so-called reverse unit propagation [GN03, Van08]
- See SAT'22 paper for details [GMNO22]

Experiments

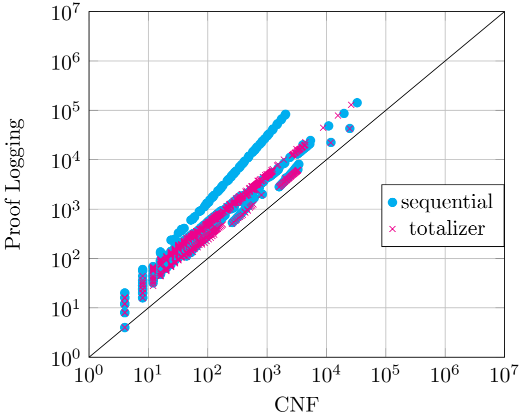
- Certified translations for the following CNF encodings:[1]
  - Sequential counter [Sin05]
  - Totalizer [BB03]
  - Generalized totalizer [JMM15]
  - Adder network [ES06]

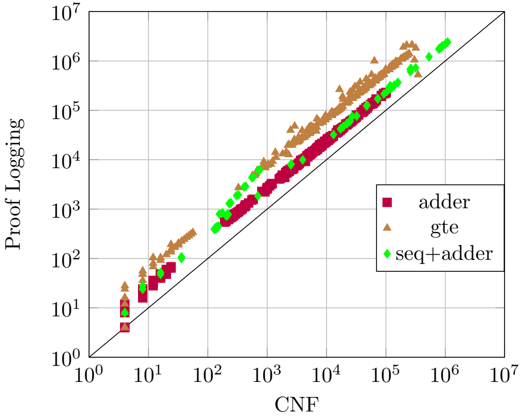
- Proof verified by proof checker VeriPB[2]

- Benchmarks from PB 2016 Evaluation:[3]
  - `SMALLINT` decision benchmarks without purely clausal formulas
  - 3 subclasses of benchmarks:
    - Only cardinality constraints (sequential counter, totalizer)
    - Only general 0-1 ILP constraints (generalized totalizer, adder network)
    - Mixed cardinality & general 0-1 ILP constraints (sequential counter + adder network)

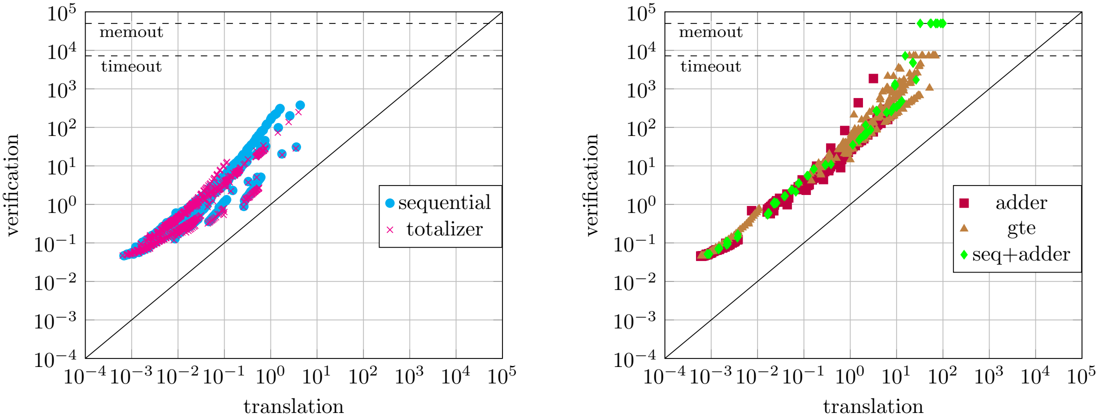
[1] `https://github.com/forge-lab/VeritasPBLib`
[2] `https://gitlab.com/MIAOresearch/software/VeriPB`
[3] `http://www.cril.univ-artois.fr/PB16/`

CNF Size vs Proof Size in KiB



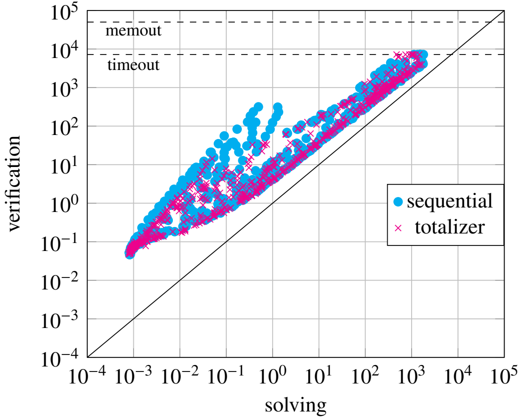


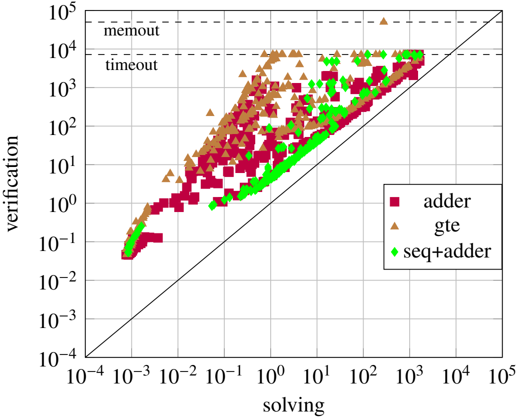
Translation Time vs Verification Time in Seconds



- Translation just generates clauses and proof
- Verification slower, as reasoning has to be performed

Solving Time vs Verification Time in Seconds





- Solved with fork of Kissat[4] syntactically modified to output pseudo-Boolean proofs
- Room for improvement, but this clearly shows that our approach is viable

[4] `https://gitlab.com/MIAOresearch/tools-and-utilities/kissat_fork`

Future Work

Improving performance:

- Cutting Planes derivations instead of reverse unit propagations [VDB22]
- Backwards checking/trimming for verification (as in DRAT-trim [HHW13a])

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Extend proof logging further:

- Sorting networks like odd-even mergesort, bitonic sorter [Bat68]
- MaxSAT solving and pseudo-Boolean optimization
- Mixed integer linear programming

Conclusion

This work:

- General approach for certifying different PB-to-CNF translations
- End-to-end verification of SAT-based pseudo-Boolean solving

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Pseudo-Boolean reasoning provides unified proof logging method for:

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- Subgraph problems [GMN20, GMM$^+$20]
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- **Future goal(?):** Mixed integer linear programming

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- **This work:** SAT-based pseudo-Boolean solving
- **Next up:** MaxSAT solving and pseudo-Boolean optimization
- **Future goal(?):** Mixed integer linear programming

Thank you for your attention!

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