Combinatorial Solving with Provably Correct Results

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Based on joint work with Jeremias Berg, Bart Bogaerts, Emir Demirović, Jan Elffers, Ambros Gleixner, Stephan Gocht, Alexander Hoen, Hannes Ihalainen, Matti Järvisalo, Ciaran McCreesh, Matthew McIlree, Magnus O. Myreen, Andy Oertel, Tobias Paxian, Konstantin Sidorov, Yong Kiam Tan, and Dieter Vandesande

The Success of Combinatorial Solving (and the Dirty Little Secret)

- Astounding progress last couple of decades on combinatorial solvers for, e.g.:
  - Boolean satisfiability (SAT) solving and optimization [BHvMW21]
  - Constraint programming [RvBW06]
  - Mixed integer linear programming [AW13, BR07]
  - Satisfiability modulo theories (SMT) solving [BHvMW21]
- Solvers very fast, but sometimes wrong (even best commercial ones) [BLB10, CKSW13, AGJ+18, GSD19, GS19, BMN22, BBN+23]
- Solvers can propose infeasible "solutions" (which should be possible to avoid!)
- How can we get reliable claims of infeasibility?
- Or of optimality? (Even off-by-one mistakes can snowball into large errors if solver used as subroutine)

What Can Be Done About Solver Bugs?

- **Software testing**
  Hard to get good test coverage for sophisticated solvers
  Progress using fuzzing and delta debugging [BB09, BLB10, KB22, NPB22, PB23]
  But testing inherently can only detect presence of bugs, not absence

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  Prove that solver implementation adheres to formal specification
  Current techniques cannot scale to level of complexity in modern solvers

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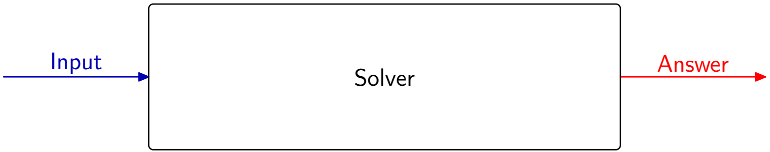
- **Proof logging**
  Make solver certifying [ABM+11, MMNS11] by adding code so that it outputs
  1. not only answer but also
  2. simple, machine-verifiable proof that answer is correct

Proof Logging with Certifying Solvers: Workflow



1. Run combinatorial solving algorithm on problem input

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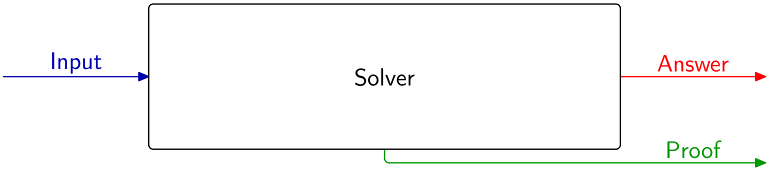

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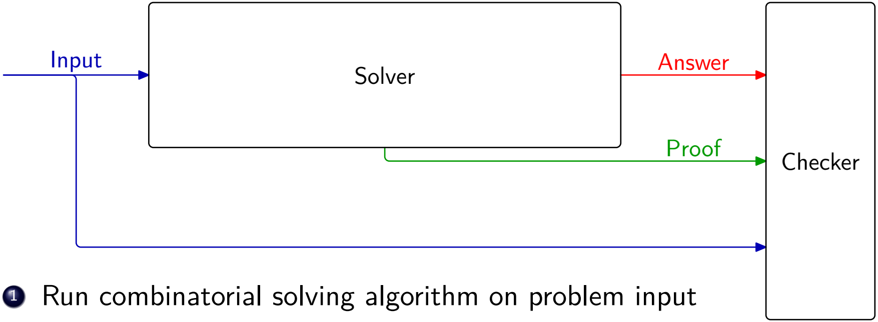

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3. Feed input + answer + proof to proof checker

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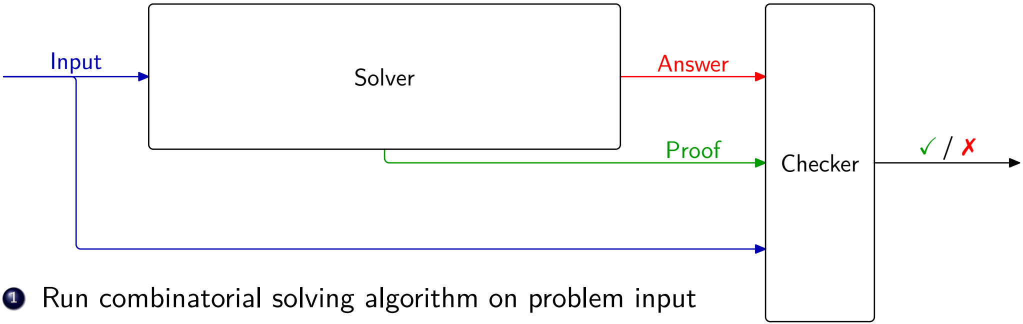
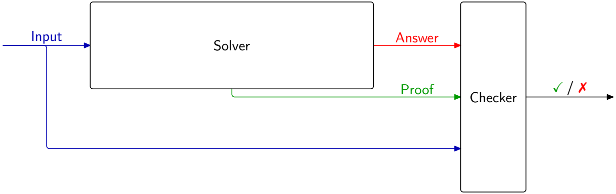


1. Run combinatorial solving algorithm on problem input
2. Get as output not only answer but also proof
3. Feed input + answer + proof to proof checker
4. Verify that proof checker says answer is correct

Proof Logging Desiderata

Proof format for certifying solver should be



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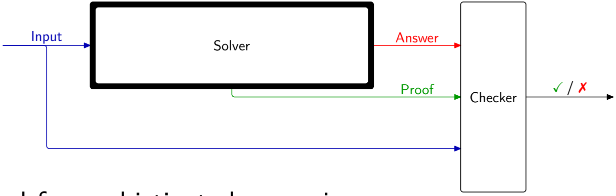

Proof format for certifying solver should be

- **very powerful:** minimal overhead for sophisticated reasoning

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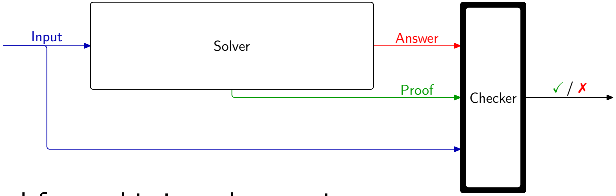

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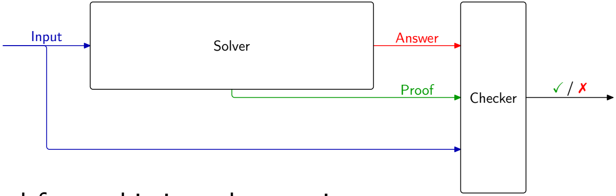

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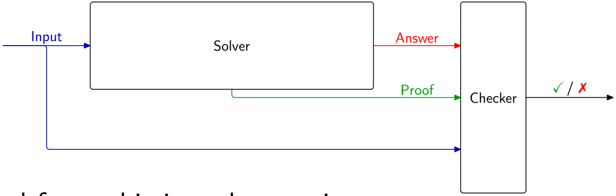

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Clear conflict expressivity vs. simplicity!

Asking for both perhaps a little bit too good to be true?

Some Previous Proof Logging Work

Boolean satisfiability (SAT) solving

- Well established since over decade with several proof formats such as
  - DRAT [HHW13a, HHW13b, WHH14]
  - GRIT [CMS17]
  - LRAT [CHH+17]
- But no efficient support for most advanced techniques such as
  - Gaussian elimination
  - symmetry breaking

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Mixed integer linear programming

- Work on proof format VIPR [CGS17, EG23]
- But only for exact solving and without support for advanced techniques

Message of This Talk

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- Build on successes in proof logging for SAT solving
- But represent constraints as 0–1 integer linear inequalities
- Formalize reasoning using cutting planes [CCT87] proof system
- Add well-chosen strengthening rules [Goc22, GN21, BGMN23]
- Implemented in VeriPB (`https://gitlab.com/MIAOresearch/software/VeriPB`)

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Purpose of this talk:

1. Marketing pitch ☺

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2. Describe foundations of proof logging method
3. Discuss future challenges and directions

The Sales Pitch For Proof Logging

1. Certifies correctness of computed results
2. Detects errors even if due to compiler bugs, hardware failures, or cosmic rays
3. Provides debugging support during software development [GMM+20, KM21, BBN+23, EG23]
4. Facilitates performance analysis
5. Helps identify potential for further improvements
6. Enables auditability
7. Serves as stepping stone towards explainability

Design Principles for Proof Logging

Proof logging implementation

- Don't change solver
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Performance goals

- Proof logging overhead small constant fraction of running time ($\lessapprox 10\%$)
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Proof system

- Keep language simple — no XOR constraints, CP propagators, symmetries, ...
- But reason efficiently about such notions using power of proof system
- Combine proof logging with formally verified proof checker

Proof Language: Pseudo-Boolean Constraints

Proof consists of 0-1 integer linear inequalities or pseudo-Boolean constraints:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
- variables x_i take values $0 = \mathit{false}$ or $1 = \mathit{true}$

Sometimes convenient to use normalized form [Bar95] with all a_i, A positive (without loss of generality)

Some Types of Pseudo-Boolean Constraints

1. Disjunctive clauses

$$x \lor \overline{y} \lor z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Pseudo-Boolean Proof Logging Wishlist

Paradigms

- Boolean satisfiability (SAT) solving
- (linear) pseudo-Boolean solving
- subgraph solving
- constraint programming
- automated planning
- mixed integer linear programming
- SMT solving

Problem types

- decision / feasibility
- optimization
- multi-objective optimization
- projected model enumeration
- projected model counting
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Pseudo-Boolean Proof Logging — How and Why?

If problem is (special case of) 0-1 integer linear program

- just do proof logging [basically: add print statements to solver code]

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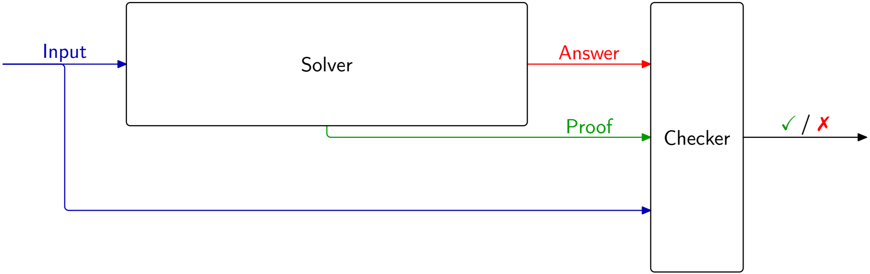
Goldilocks compromise between expressivity and simplicity:

1. 0-1 ILP expressive formalism for combinatorial problems (including objective)
2. Powerful reasoning capturing many combinatorial arguments

Proof Logging with Formally Verified Checking: Full Workflow



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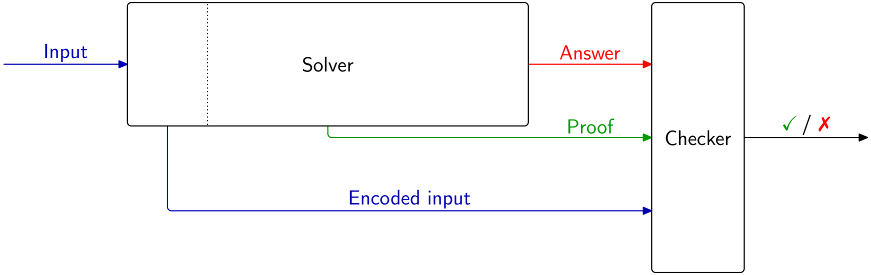

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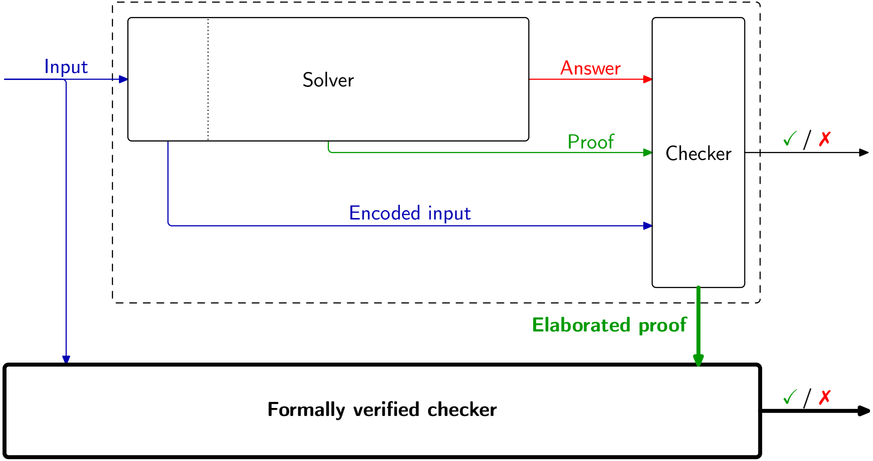

VeriPB Proof Structure

1. **Preamble**
   Load input formula
   Specify settings
2. **Derivation section**
   Derivations of new constraints
   Logging of solutions
3. **Output section**
   Listing of constraints currently in database
   Input to next stage (or for debugging)
4. **Conclusions section**
   Specification of what was established
   - satisfiability / unsatisfiability
   - optimality (or upper and lower bounds)
   - other types of conclusions

VeriPB Proof Configuration (Slightly Simplified)

Core set $\mathcal{C}$

- Contains input formula at the start
- Maintains "equivalence" with input formula

Derived set $\mathcal{D}$

- All constraints derived during search
- Also intermediate constraints used in proof logging [but not used by solver]

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- Contains input formula at the start
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Objective $f = \sum_i w_i \ell_i + k$

- 0–1 linear function to minimize
- Or $f = 0$ for decision problem
- Keep track of best known bound; initialize to ∞

Derived set $\mathcal{D}$

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Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input axioms From the input

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Literal axioms $$\overline{\ell_i \geq 0}$$

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$$\frac{}{\ell_i \geq 0}$$

Addition

$$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i) \ell_i \geq A + B}$$

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Multiplication for any $c \in \mathbb{N}^+$

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i c a_i \ell_i \geq cA}$$

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Division for any $c \in \mathbb{N}^+$
(constraint in normalized form)

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \left\lceil \frac{a_i}{c} \right\rceil \ell_i \geq \left\lceil \frac{A}{c} \right\rceil}$$

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Saturation
(constraint in normalized form)

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \min(a_i, A) \cdot \ell_i \geq A}$$

Cutting Planes Toy Example

$$w + 2x + y \geq 2$$

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Multiply by 2 $$\frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

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Multiply by 2 $\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$ $\qquad w + 2x + 4y + 2z \geq 5$

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$$\dfrac{\text{Multiply by } 2\ \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}\ \text{Add}$$

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$$\text{Add}\ \dfrac{\text{Multiply by } 2\ \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5 \qquad \overline{z} \geq 0}{3w + 6x + 6y + 2z \geq 9}$$

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$$
\dfrac{3w + 6x + 6y + 2z \geq 9 \qquad 2\overline{z} \geq 0}{3w + 6x + 6y \qquad\qquad \geq 7}\ \text{Add}
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By naming constraints by integers and literal axioms by the literal involved as

$$
\begin{aligned}
\text{Constraint 1} &\doteq 2x + y + w \geq 2 \\
\text{Constraint 2} &\doteq 2x + 4y + 2z + w \geq 5 \\
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$$

such a calculation is written in the proof log in reverse Polish notation as

```
pol  1  2  *  2  +  ~z  2  *  +  3  d
```

Deriving Non-implied Constraints by Redundance-Based Strengthening

C is said to be "redundant" with respect to F if F and $F \cup \{C\}$ are equisatisfiable

Need to allow adding such "redundant" non-implied constraints

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$$F \cup \{\neg C\} \models (F \cup \{C\})\restriction_{\omega}$$

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- Proof sketch for interesting direction: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \cup \{C\}$

Deriving Non-implied Constraints by Redundance-Based Strengthening

C is said to be "redundant" with respect to F if F and $F \cup \{C\}$ are equisatisfiable
Need to allow adding such "redundant" non-implied constraints

Redundance-based strengthening ([BT19, GN21], inspired by [JHB12])

C is redundant with respect to F if and only if there is a substitution ω (mapping variables to truth values or literals), called a witness, for which

$$F \cup \{\neg C\} \models (F \cup \{C\})\restriction_\omega$$

- Proof sketch for interesting direction: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \cup \{C\}$
- In a proof, the implication needs to be efficiently verifiable — every $D \in (F \cup \{C\})\restriction_\omega$ should follow from $F \cup \{\neg C\}$ either
  1. "obviously" or
  2. by explicitly presented derivation

Example: Deriving $r \leftrightarrow (x \wedge y)$ Using the Redundance Rule

Want to derive

$$2\overline{r} + x + y \geq 2 \qquad\qquad r + \overline{x} + \overline{y} \geq 1$$

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   Choose $\omega = \{r \mapsto 1\}$ — F untouched; new constraint satisfied
   $\neg(r + \overline{x} + \overline{y} \geq 1)$ forces $x \mapsto 1$ and $y \mapsto 1$, hence $2\overline{r} + x + y \geq 2$ remains satisfied after forcing r to be true

Redundance and Dominance Rules in VeriPB (Slightly Simplified)

Redundance-based strengthening, optimization version [BGMN23]

Add constraint C to derived set $\mathcal{D}$ if exists witness substitution ω such that

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- Applying ω should strictly decrease f
- If so, don't need to show that $(\mathcal{D} \cup \{C\})\restriction_\omega$ implied!

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7. ...
8. Can't go on forever, so finally reach α' satisfying $\mathcal{C} \cup \{C\}$

Strengthening Rules: Proof Format

red ⟨Constraint C⟩ ; ⟨*var1*⟩ -> ⟨*val1*⟩ ... ⟨*varN*⟩ -> ⟨*valN*⟩ ; begin
    subproofs for proof goals
end

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- Witness ω should be explicitly specified in proof log
- Subproofs of proof goals should also be explicit
- But can be skipped for proof goals “obvious” to proof checker

Successful Applications of VeriPB Proof Logging

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1. Boolean satisfiability (SAT) solving including advanced techniques such as
   - Gaussian elimination [GN21]
   - symmetry breaking [BGMN23]
2. SAT-based optimization (MaxSAT) [VDB22, BBN⁺23, BBN⁺24, IOT⁺24]
3. (Linear) Pseudo-Boolean solving [GMNO22]
4. Subgraph solving (max clique, subgraph isomorphism, max common connected subgraph) [GMN20, GMM⁺20, GMM⁺24]
5. Dynamic programming and decision diagrams [DMM⁺24]
6. Presolving in 0–1 integer linear programming [HOGN24]
7. Constraint programming [EGMN20, GMN22, MM23, MMN24]

Future Research Directions

Performance of pseudo-BoolEean proof logging and checking

- Trim proof while verifying (as in DRAT-TRIM [HHW13a])
- Compress proof file using binary format
- More careful software engineering in proof checker (such as faster propagation)

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- Model enumeration and counting
- SMT solving *(work on solvers* CVC5, SMTINTERPOL, Z3, . . . *[BBC+23, HS22])*
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And more. . .

- Use proof logs for algorithm analysis or explainability purposes
- Lots of other challenging problems and interesting ideas

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And more...

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- Lots of other challenging problems and interesting ideas
- We're hiring! Talk to me to join the pseudo-Boolean proof logging revolution! ☺

VeriPB Documentation

VeriPB tutorial at *CP '22* [BMN22]

- video at `youtu.be/s_5BIi4I22w`
- updated slides for *IJCAI '23* tutorial [BMN23]

Description of VeriPB and CakePB [BMM+23] for SAT 2023 competition

- Available at `satcompetition.github.io/2023/checkers.html`

Specific details on different proof logging techniques covered in research papers [EGMN20, GMN20, GMM+20, GN21, GMN22, GMNO22, VDB22, BBN+23, BGMN23, MM23, BBN+24, DMM+24, GMM+24, HOGN24, IOT+24, MMN24]

Lots of concrete example files at `gitlab.com/MIAOresearch/software/VeriPB`

Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity
- **Action point:** What problems can VeriPB solve for you? ☺

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Thank you for your attention!

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