①

$R \subseteq [m]^n \times (\{0,1\}^m)^n$ is ρ-like iff $G(R) = C_\rho^{-1}(1)$

$\iff \forall z \in \{0,1\}^n$ consistent with ρ: $\exists\, x \in [m]^n;\ y \in (\{0,1\}^m)^n$:

$$G(x,y) = \mathrm{Ind}_m^n(x,y) = z.$$

A random variable $\bar{X} \in [m]^J$ is h-dense if for every $I \neq \emptyset \subseteq J$:

X_I has min-entropy $H_\infty(X_I) \geq h \cdot |I|$.

$\hookrightarrow \min_x \log\left(\frac{1}{\Pr[X=x]}\right)$

A rectangle $R = X \times Y$ is ρ-structured if

1) $X_{\mathrm{dom}(\rho)}$ is fixed, and every $z \in G(R)$: $z \in C_\rho^{-1}(1)$

$\Leftarrow y_i$ is chosen appropriately.

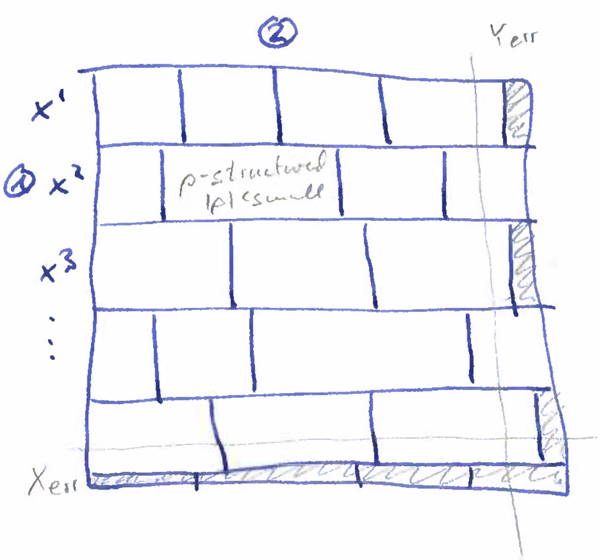
2) $X_{\mathrm{free}(\rho)}$ $\rho^{-1}(*)$ is $0.95 \log m$-dense

3) Y is large: $H_\infty(Y) \geq m \cdot |\rho^{-1}(*)| - n \cdot \log m$.

Full range lemma:

If $X \times Y$ is ρ-structured, then there is an $x \in X$ such that $\{x\} \times Y$ is ρ-like.

How to go from a rectangle $R := X \times Y \in \Pi$ to structured rectangles.



① Let $I_i \subseteq [n]$ be maximal such that X_{I_i} has min-entropy $\leq 0.95 \log m\, |I_i|$.

Let $\alpha_i \in [m]^{I_i}$ witness this:

$$\Pr[X_{I_i} = \alpha_i] > m^{-0.95|I_i|}.$$

$X^i := \{x : x_{I_i} = \alpha_i\}$

$X = X \setminus X^i$.

② For each X^i; $\gamma \in \{0,1\}^{I_i}$:

$$Y^{i,\gamma} := \{y : g^{I_i}(\alpha_i, y) = \gamma\}$$

output $\{R^{i,\gamma} : X^i \times \underbrace{Y^{i,\gamma}}_{\neq \emptyset}\}$

Rectangle Lemma

Let $R = X \times Y$ and $d < n$; let $R = \cup R^i_\epsilon$ be the rectangles from the above partition. Then, there are error sets $X_{err} \subseteq X$; $Y_{err} \subseteq Y$ with density $\leq 2^{-2d \log m}$ in $[m]^n$ and $(\{0,1\}^m)^n$ respectively such that either

- R^i is ρ^i structured for ρ^i of size $\leq O(d)$.
- R^i is covered by error rows/cols;

$$R^i \subseteq X_{err} \times (\{0,1\}^m)^n \cup Y_{err} \times [m]^n.$$

Finally: for $x \in [m]^n \setminus X_{err}$ there is an $I_x \subseteq [n]$: $|I_x| \leq O(d)$ and every structured R^i intersecting row x has $\mathrm{dom}(\rho^i) \subseteq I_x$.

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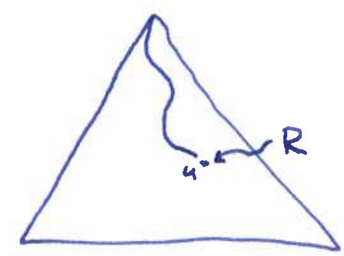
given a rectangle-dag π solving $S \circ G$ of size $|\pi| = m^d$, then $w(S) \leq O(d)$.

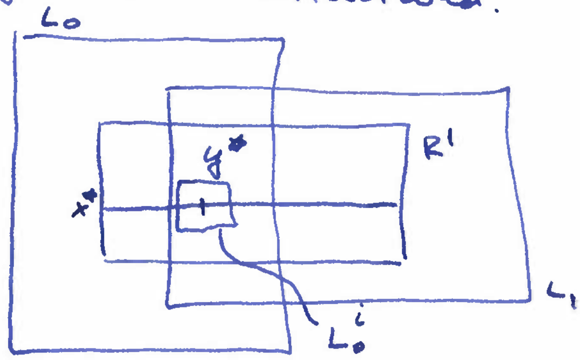
[ignoring error sets].

- maintain a $\overset{\leq O(d)}{\rho}$-structured $R' \subseteq R$.



(1) Root: the rectangle is $*^n$-structured.

(2) Step:



$X' \times Y' =: R'$ is ρ-structured $\Rightarrow \exists\, x^* \in X'$: $x^* \times Y'$ is ρ-like

- Consider partition of L_0; L_1.

$\rightsquigarrow$ the rectangles intersecting row x^*:

$\exists\, I_0; I_1$: $\forall L_b^i$ intersecting x^* : $\mathrm{dom}(\rho_b^i) \subseteq I_b$.
$\hookrightarrow \rho_b^i$-like

$\rightsquigarrow$ Query $I_0 \cup I_1$ $\rightsquigarrow \rho^*$ (small; $O(d)$).

- $x^* \times Y'$ is ρ-like $\rightarrow \exists\, y^* \in Y'$: $g(x^*, y^*) = z$ is consistent with ρ^*.

$\rightarrow$ Consider $\underset{\rho_b^i\text{-structured}}{L_b^i}$: $(x^*, y^*) \in L_b^i$. $\rightsquigarrow$ forget everything except $\mathrm{dom}(\rho_b^i)$.

(3) Leaf case: game state ρ; R': ρ-struct.

leaf labelled by $o \in O$:

$$R' \subseteq (S \circ G)^{-1}(o)$$

$$\Longleftrightarrow C_\rho^{-1}(1) = G(R') \subseteq S^{-1}(o)$$

<u>Error</u>: traverse π in topological order from leaves to root;

$R_1, \dots, R_{m^d}$.

$$X^*_{err} ; Y^*_{err} = \emptyset.$$

Consider R_i:

- update $R_i \leftarrow R_i \setminus \left(X^*_{err} \times (\{0,1\}^n)^n \cup [m]^n \times Y^*_{err}\right)$
- apply partition scheme; keep the structured rects.
- $X^*_{err} \leftarrow X^*_{err} \cup X_{err}$; $Y^*_{err} \leftarrow Y^*_{err} \cup Y_{err}$.

$\rightsquigarrow$ same proof as before on $(X \setminus X_{err}) \times (Y \setminus Y_{err})$.

(1) <u>Root</u>: the density of the error sets on X $< m^d \cdot m^{-2d} \ll 1\%$.

on Y less than m^{-d} fraction

$\to$ the remaining rectangle is $*^n$-structured.

(2) <u>Step</u>: Error sets shrink as we walk down the proof π.

$\to$ cover property is maintained.

Proof of the rectangle lemma:

- X_{err}: while there is $R^i = X \times Y$ such that $|I^i| > 40d$ ($I^i = \mathrm{dom}(\rho^i)$)
  update $X_{err} \leftarrow X_{err} \cup X$.
- Y_{err}: while there is $R^i = X \times Y$ such that $|Y \setminus Y_{err}| < 2^{m \cdot |\bar{I}^i| - 5d\log m}$ ($-5d\log m$) needed? don't think so?
  update $Y_{err} \leftarrow Y_{err} \cup Y$.

<u>Claim 1</u>: if R^i is not covered by $X_{err}; Y_{err}$, then R^i is ρ^i-struct ~~R^i is fixed on I~~; with $|\mathrm{dom}(\rho^i)| \le O(d)$.

- P1: obvious;
- P2: min-entropy holds by maximality.
- P3: by construction.

$\rightsquigarrow$ error set density?

$$|X_{err}| \le m^n \cdot 2^{-2d\log m}$$

unless X_{err} is empty $\exists j$: (min)

x^j added to X_{err}.

$\rightarrow |I_j| > 40d$.

① $|X^j| \le |X^{\geq j}| \cdot 2^{-0.95\,|I_j| \log m}$

$$|X^j| = |X^{\geq j}| \cdot \Pr_{x \sim X^{\geq j}}[x_{I_j} = \alpha_j] \le |X^{\geq j}| \cdot 2^{-0.95\,|I_j|\log m}.$$

$$\rightarrow H_\infty(X^j) \geq H_\infty(X^{\geq j}) - .95|I_j| \log m$$

$$(n - |I_j|)\log m \geq H_\infty(X^j).$$

$$\rightsquigarrow H_\infty(X^{\geq j}) \le (n - 0.05\,|I_j|)\log m.$$

$$|X_{err}| \le |X^{\geq j}| < 2^{(n - 0.05 \cdot 40d)\log m}$$
$$\le m^n \cdot 2^{-2d\log m}.$$

Y_{err}: each $Y^{i,\delta}$ is defined by

$$(I_i, \alpha_i, \delta)$$

for $k \in [40d]$: # of such $Y^{i,\delta} \leq \binom{n}{k} m^k 2^k < 2^{3k \log m}$.

$\rightarrow$ by a union bound:

$$|Y_{err}| \leq \sum_{k=1}^{40d} 2^{3k \log m} \cdot 2^{m(n-k) - 5d \log m}$$

$$\leq 40d \cdot 2^{m(n-1) - 2d \log m} \ll 2^{mn} 2^{-2d \log m}.$$

Full range lemma. $R := X \times Y$; ρ-~~like~~ structured

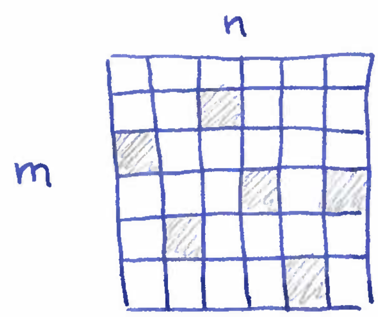
want to argue that there is a row x^* such that

$$\mathrm{Ind}_m^n(x^*, Y) = \underbrace{C_\rho^{-1}(1)}_{\text{all assignments "compatible" with } \rho}$$

By contradiction: For every row $x \in [m]^n$, there is a $z \in \{0,1\}^n$:

$$\forall y \in Y: \underbrace{(y_{x_1}, \dots, y_{x_n})}_{\{0,1\}^{mn}} \neq z$$

$$\Longleftrightarrow \quad z \notin \mathrm{Ind}(x, Y).$$



x picks a box per column.

$\rightarrow$ no matter what value $y \in Y$ you choose, you will never see the assignment z in the boxes.

We want to argue that since these constraints have high ~~min entropy~~; h-density, there cannot be many y satisfying these constraints.

But first, let us think of what the "worst case" is w.r.t. the constraints; when do they rule out the <u>fewest</u> $y \in (\{0,1\}^m)^n$ (with respect to the choice of z).

<u>Claim:</u> setting all $z = \vec{1}$ is the worst-case; max y will satisfy the constraints.

← all assignments with $x = y = z = w = 1$ are ruled out.

w.l.o.g. $w' = 1$

if $x = 1 = y = z \Rightarrow$ max overlap of ruled out subcubes; assignments

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want to analyze the event that for $y \sim_{u.a.r.} \{0,1\}^{mn}$ the boxes chosen by x are all different from $\vec{1}$.

$\leadsto$ Apply Janson's inequality:

$$\Pr_y\left[\forall x \in X : x \not\subseteq y\right] \leq e^{(-\mu^2/\Lambda)}$$

set indicator

$$\mu := \mathbb{E}[\#\text{of contained sets}] = |X| \cdot 2^{-n}$$

$$\Lambda := \sum_{\substack{(i,j):\\ X_i \cap X_j \neq \emptyset}} \mathbb{E}\left[\mathbb{1}_{\{x_i \cup x_j \subseteq y\}}\right]$$

Remains to bound Λ.

1) Fix the set $x \in X$

2) Fix the size of the intersection a.

Use denseness to argue that there are <u>few</u> sets that intersect in a given choice of a points;

$$|X| \cdot m^{-0.95 \cdot a}.$$

$$\Rightarrow \Lambda \leq |X| \cdot \sum_{a=1}^{n} \binom{n}{a} |X| \cdot m^{-0.95 \cdot a} \cdot 2^{-2n+a}$$

$$\leq \mu^2 \cdot \left(\left(1 + \frac{2}{m^{0.95}}\right)^n - 1 \right)$$

$$\leq \mu^2 \cdot \frac{4n}{m^{0.95}}$$

$$\Rightarrow \Pr_y [\forall x \in X: x \not\subseteq y] \leq \exp\left(-\frac{m^{0.95}}{4n}\right) \leq \exp(-n \cdot \log m)$$

$$\Rightarrow |Y| \leq 2^{nm - n \cdot \log m} \quad ; \quad \text{contradiction} \qquad \square.$$

If we want to optimize m, need to be more careful ~~with~~ with the used bounds; see [Rao20, Lemma 4].

$\rightsquigarrow$ get $m \sim n^{1+\varepsilon}$.