

Proof complexity and SAT solving

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SAT/SMT/AR Summer School
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Three Simple Problems. . .

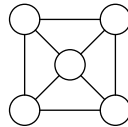
COLOURING

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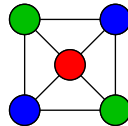


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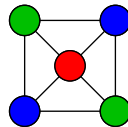


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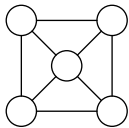
3-colouring? Yes, but no 2-colouring

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CLIQUE

Is there a **clique** in the graph $G = (V, E)$ with k vertices that are all pairwise connected by edges in E ?

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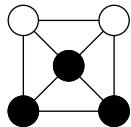


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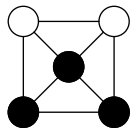


3-clique? Yes

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3-clique? Yes, but no 4-clique

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Given propositional logic formula, is there a **satisfying assignment**?

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- Variables should be set to **true** or **false**
- Constraint $(x \vee \neg y \vee z)$: means x or z should be true or y false
- \wedge means all constraints should hold simultaneously
- Is there a truth value assignment satisfying all constraints?

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COLOURING: frequency allocation for mobile base stations

CLIQUE: bioinformatics, computational chemistry

SAT: easily models these and many other problems

...with Huge Practical Implications

- Some more examples of problems that can be encoded as propositional logic formulas:
 - computer hardware verification
 - computer software testing
 - artificial intelligence
 - operations research
 - cryptography
 - bioinformatics
 - et cetera...
- Leads to **humongous formulas** (100,000s or even 1,000,000s of variables)
- Can we use computers to solve these problems efficiently?

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- **NP-complete**, so probably very hard [Coo71, Lev73]
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 - COLOURING [Kho01, Zuc07]
 - CLIQUE [Hås99]
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Solving NP in Theory and Practice

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 - COLOURING [Kho01, Zuc07]
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 - SAT [Hås01]
 - Except that in practice, there are good algorithms for
 - COLOURING [DLMM08, DLMO09, DLMM11]
 - CLIQUE [Pro12, McC17]
- and amazing **conflict-driven clause learning (CDCL)** solvers [BS97, MS99, MMZ⁺01] that solve huge SAT formulas

How can we understand real-world algorithms for NP-hard problems?

This talk: Use proof complexity (not only conceivable answer)

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- ① Is there a short proof using rules in this proof system?
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Study **infeasible problems** — proofs of feasibility are trivial

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Question 2: Topic for separate lecture(s) — lots of recent exciting progress; mostly negative (worst-case) results that proof search is hard, e.g., [AM20, GKMP20, dRGN⁺21]

Applications of Proof Complexity

Three applied reasons for proof complexity:

- ① Understand real-world applied algorithmic paradigms [**this lecture**]
- ② Get ideas for algorithmic improvements
[EN18, EN20, DGD⁺21, DGN21, KBBN22]
(See, e.g., tutorials <https://www.youtube.com/watch?v=LZ8VztiplaQ> and https://www.youtube.com/watch?v=wD_2tx1rTaw about **ROUNDINGSAT**)
- ③ Enhance algorithms to write machine-verifiable certificates of correctness
[EGMN20, GMN20, GMM⁺20, GN21, GMN22, GMNO22, VDB22, BGMN23, BBN⁺23, MM23, GMM⁺24, HOGN24, BBN⁺24, DMM⁺24, IOT⁺24, MMN24]
(See tutorial https://www.youtube.com/watch?v=s_5BIi4I22w about **VERIPB**)

Outline

- 1 DPLL, CDCL, and Resolution
 - Davis-Putnam-Logemann-Loveland (DPLL) Method
 - Conflict-Driven Clause Learning (CDCL)
 - Resolution Proof System
- 2 Algebraic and Semi-algebraic Approaches
 - Nullstellensatz
 - Gröbner Bases and Polynomial Calculus
 - Pseudo-Boolean Solving and Cutting Planes
- 3 Some More Advanced Proof Systems We Might Not Have Time for
 - Sherali-Adams and Sums of Squares
 - Stabbing Planes
 - Extended Resolution

Formal Description of SAT Problem

- **Variable** x : takes value **true** ($= 1$) or **false** ($= 0$)
- **Literal** ℓ : variable x or its negation \bar{x} (write \bar{x} instead of $\neg x$)
- **Clause** $C = \ell_1 \vee \dots \vee \ell_k$: disjunction of literals
(Consider as sets, so no repetitions and order irrelevant)
- **Conjunctive normal form (CNF) formula** $F = C_1 \wedge \dots \wedge C_m$: conjunction of clauses

The SATISFIABILITY (or just SAT) Problem

Given a CNF formula F , is it satisfiable?

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The SATISFIABILITY (or just SAT) Problem

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Here is our example formula again:

$$(x \vee z) \wedge (y \vee \bar{z}) \wedge (x \vee \bar{y} \vee u) \wedge (\bar{y} \vee \bar{u}) \\ \wedge (u \vee v) \wedge (\bar{x} \vee \bar{v}) \wedge (\bar{u} \vee w) \wedge (\bar{x} \vee \bar{u} \vee \bar{w})$$

The Same Problem in Three Different Shapes

$$\begin{aligned} & (x \vee z) \wedge (y \vee \bar{z}) \wedge (x \vee \bar{y} \vee u) \wedge (\bar{y} \vee \bar{u}) \\ & \wedge (u \vee v) \wedge (\bar{x} \vee \bar{v}) \wedge (\bar{u} \vee w) \wedge (\bar{x} \vee \bar{u} \vee \bar{w}) \end{aligned}$$

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$$(1 - x)(1 - z) = 0$$

$$(1 - y)z = 0$$

$$(1 - x)y(1 - u) = 0$$

$$yu = 0$$

$$(1 - u)(1 - v) = 0$$

$$xv = 0$$

$$u(1 - w) = 0$$

$$xuw = 0$$

For **true** = 1 and **false** = 0, is there a $\{0, 1\}$ -valued solution?

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$$1 - x - z + xz = 0$$

$$z - yz = 0$$

$$y - xy - yu + xyu = 0$$

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$$1 - x - z + xz = 0$$

$$x + z \geq 1$$

$$z - yz = 0$$

$$y + (1 - z) \geq 1$$

$$y - xy - yu + xyu = 0$$

$$x + (1 - y) + u \geq 1$$

$$yu = 0$$

$$(1 - y) + (1 - u) \geq 1$$

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$$x - y + u \geq 0$$

$$yu = 0$$

$$-y - u \geq -1$$

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$$u + v \geq 1$$

$$xv = 0$$

$$-x - v \geq -1$$

$$u - uw = 0$$

$$-u + w \geq 0$$

$$xuw = 0$$

$$-x - u - w \geq -2$$

For **true** = 1 and **false** = 0, is there a $\{0, 1\}$ -valued solution?

DPLL: Attempting Smart Case Analysis

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- 4 Set $x = 0$, simplify F and **make recursive call**
- 5 Set $x = 1$, simplify F and **make recursive call**
- 6 If result in both cases “**unsatisfiable**”, then report “**unsatisfiable**” and return

A DPLL Toy Example

$$\begin{aligned} F = & (x \vee z) \wedge (y \vee \bar{z}) \wedge (x \vee \bar{y} \vee u) \wedge (\bar{y} \vee \bar{u}) \\ & \wedge (u \vee v) \wedge (\bar{x} \vee \bar{v}) \wedge (\bar{u} \vee w) \wedge (\bar{x} \vee \bar{u} \vee \bar{w}) \end{aligned}$$

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Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

“Simplify formula” by (mentally) removing

- satisfied clauses
- falsified literals

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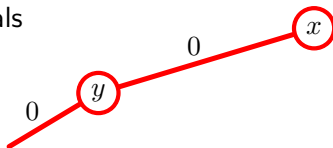
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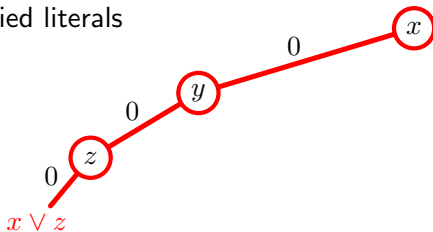
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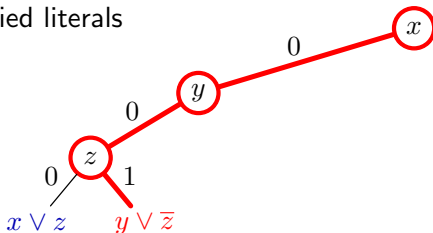
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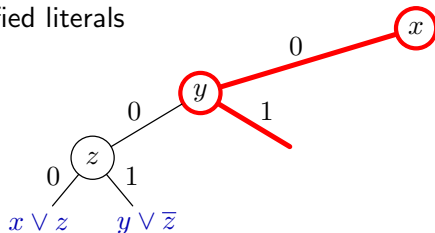
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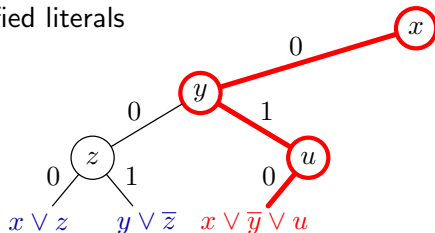
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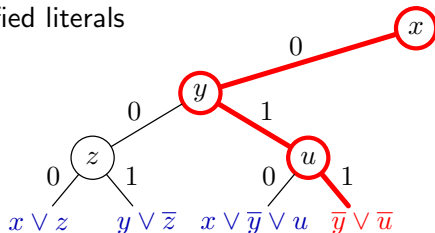
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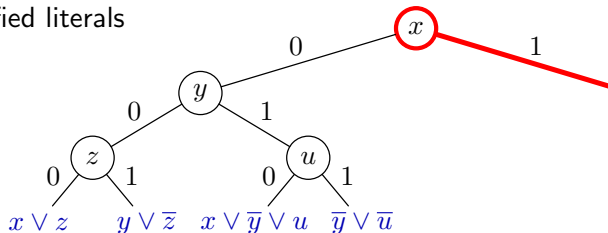
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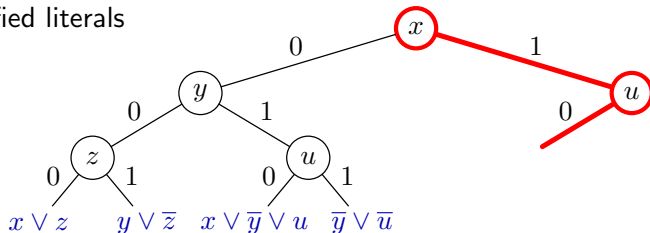
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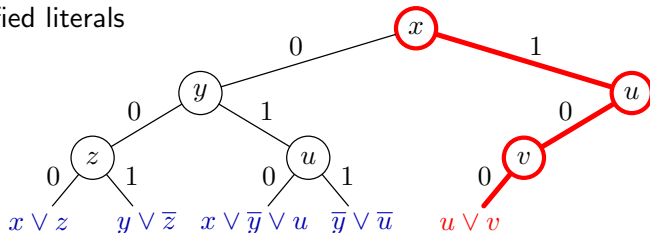
$$F = (x \vee z) \wedge (y \vee \bar{z}) \wedge (x \vee \bar{y}) \wedge (\bar{y} \vee \bar{u}) \\ \wedge (u \vee v) \wedge (\bar{v}) \wedge (\bar{u} \vee w) \wedge (\bar{u} \vee \bar{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

“Simplify formula” by (mentally) removing

- satisfied clauses
- falsified literals



A DPLL Toy Example

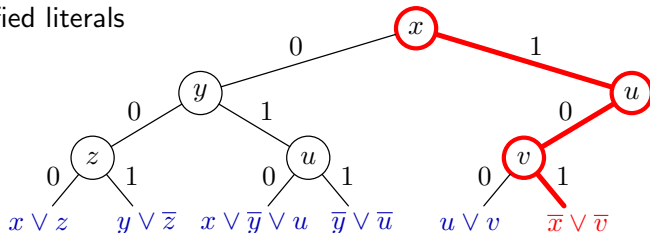
$$F = (x \vee z) \wedge (y \vee \bar{z}) \wedge (x \vee \bar{y}) \wedge (\bar{y} \vee \bar{u}) \\ \wedge (v) \wedge (\bar{x} \vee \bar{v}) \wedge (\bar{u} \vee w) \wedge (\bar{u} \vee \bar{w})$$

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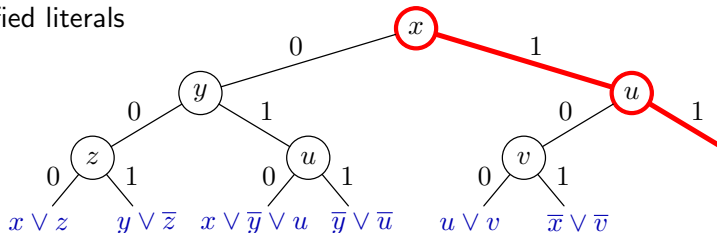
$$F = (x \vee z) \wedge (y \vee \bar{z}) \wedge (x \vee \bar{y} \vee u) \wedge (\bar{y} \vee \bar{u}) \\ \wedge (u \vee v) \wedge (\bar{v}) \wedge (w) \wedge (\bar{w})$$

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A DPLL Toy Example

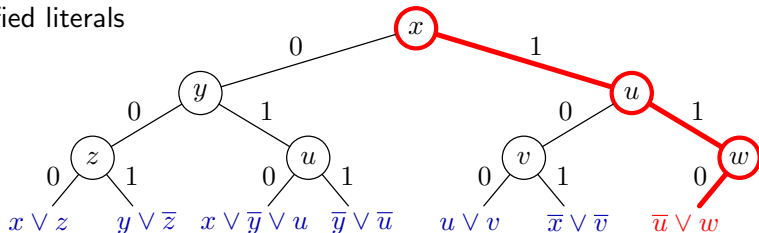
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A DPLL Toy Example

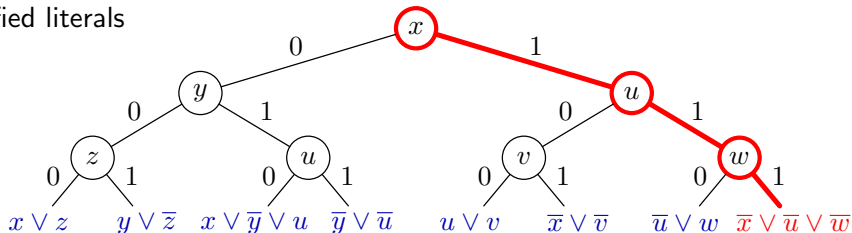
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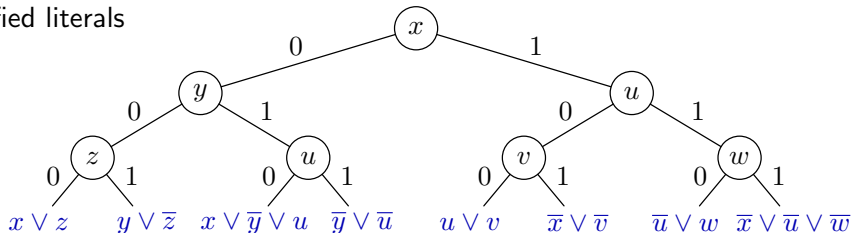
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Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

“Simplify formula” by (mentally) removing

- satisfied clauses
- falsified literals



State-of-the-Art SAT Solving in One Slide

High-level description of modern **conflict-driven clause learning (CDCL)** SAT solving (as pioneered in [BS97, MS99, MMZ⁺01]):

- Try to build satisfying assignment for formula (**branching** or **decision heuristic** crucial)
- When partial assignment violates formula, **compute explanation for conflict** and **add to formula** as new clause (**clause learning**)
- Every once in a while, **restart** from beginning (but save computed info)

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$

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Decision

Free choice to assign value to variable

Notation $p \stackrel{d}{=} 0$

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$

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$$p \stackrel{d}{=} 0$$

Decision

Free choice to assign value to variable

Notation $p \stackrel{d}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \bar{u}$ forces $u = 0$

Notation $u \stackrel{p \vee \bar{u}}{=} 0$ ($p \vee \bar{u}$ is **reason clause**)

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$$p \stackrel{d}{=} 0$$

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$$p \stackrel{d}{=} 0$$

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Always propagate if possible, otherwise decide

Add to assignment **trail**

Continue until satisfying assignment or **conflict**

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$

$$p \stackrel{d}{=} 0$$

$$u \stackrel{p \vee \bar{u}}{=} 0$$

$$q \stackrel{d}{=} 0$$

$$r \stackrel{q \vee r}{=} 1$$

Decision

Free choice to assign value to variable

Notation $p \stackrel{d}{=} 0$

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$$p \stackrel{d}{=} 0$$

$$u \stackrel{p \vee \bar{u}}{=} 0$$

$$q \stackrel{d}{=} 0$$

$$r \stackrel{q \vee r}{=} 1$$

$$w \stackrel{\bar{r} \vee w}{=} 1$$

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$$p \stackrel{d}{=} 0$$

$$u \stackrel{p \vee \bar{u}}{=} 0$$

$$q \stackrel{d}{=} 0$$

$$r \stackrel{q \vee r}{=} 1$$

$$w \stackrel{\bar{r} \vee w}{=} 1$$

$$x \stackrel{d}{=} 0$$

Decision

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$$p \stackrel{d}{=} 0$$

$$u \stackrel{p \vee \bar{u}}{=} 0$$

$$q \stackrel{d}{=} 0$$

$$r \stackrel{q \vee r}{=} 1$$

$$w \stackrel{\bar{r} \vee w}{=} 1$$

$$x \stackrel{d}{=} 0$$

$$y \stackrel{u \vee x \vee y}{=} 1$$

Decision

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Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

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$$p \stackrel{d}{=} 0$$

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$$q \stackrel{d}{=} 0$$

$$r \stackrel{q \vee r}{=} 1$$

$$w \stackrel{\bar{r} \vee w}{=} 1$$

$$x \stackrel{d}{=} 0$$

$$y \stackrel{u \vee x \vee y}{=} 1$$

$$z \stackrel{x \vee \bar{y} \vee z}{=} 1$$

Decision

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Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$

$$p \stackrel{d}{=} 0$$

$$u \stackrel{p \vee \bar{u}}{=} 0$$

$$q \stackrel{d}{=} 0$$

$$r \stackrel{q \vee r}{=} 1$$

$$w \stackrel{\bar{r} \vee w}{=} 1$$

$$x \stackrel{d}{=} 0$$

$$y \stackrel{u \vee x \vee y}{=} 1$$

$$z \stackrel{x \vee \bar{y} \vee z}{=} 1$$

$$\bar{y} \vee \bar{z} \stackrel{\perp}{=}$$

Decision

Free choice to assign value to variable

Notation $p \stackrel{d}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \bar{u}$ forces $u = 0$

Notation $u \stackrel{p \vee \bar{u}}{=} 0$ ($p \vee \bar{u}$ is **reason clause**)

Always propagate if possible, otherwise decide

Add to assignment **trail**

Continue until satisfying assignment or **conflict**

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$

$$p \stackrel{d}{=} 0$$

decision
level 1

Decision

Free choice to assign value to variable

Notation $p \stackrel{d}{=} 0$

$$u \stackrel{p \vee \bar{u}}{=} 0$$

$$q \stackrel{d}{=} 0$$

decision
level 2

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \bar{u}$ forces $u = 0$

Notation $u \stackrel{p \vee \bar{u}}{=} 0$ ($p \vee \bar{u}$ is **reason clause**)

$$r \stackrel{q \vee r}{=} 1$$

$$w \stackrel{\bar{r} \vee w}{=} 1$$

$$x \stackrel{d}{=} 0$$

decision
level 3

Always propagate if possible, otherwise decide

Add to assignment **trail**

Continue until satisfying assignment or **conflict**

$$y \stackrel{u \vee x \vee y}{=} 1$$

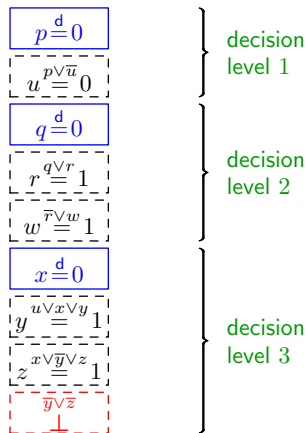
$$z \stackrel{x \vee \bar{y} \vee z}{=} 1$$

$$\bar{y} \vee \bar{z}$$

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$



Conflict Analysis

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$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$

$$p \stackrel{d}{=} 0$$

$$u \stackrel{p \vee \bar{u}}{=} 0$$

$$q \stackrel{d}{=} 0$$

$$r \stackrel{q \vee r}{=} 1$$

$$w \stackrel{\bar{r} \vee w}{=} 1$$

$$x \stackrel{d}{=} 0$$

$$y \stackrel{u \vee x \vee y}{=} 1$$

$$z \stackrel{x \vee \bar{y} \vee z}{=} 1$$

$$\bar{y} \vee \bar{z} \quad \perp$$

decision
level 1

decision
level 2

decision
level 3

Could backtrack by erasing **conflict level** & flipping last decision

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$

$$p \stackrel{d}{=} 0$$

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$$w \stackrel{\bar{r} \vee w}{=} 1$$

$$x \stackrel{d}{=} 0$$

$$y \stackrel{u \vee x \vee y}{=} 1$$

$$z \stackrel{x \vee \bar{y} \vee z}{=} 1$$

$$\bar{y} \vee \bar{z} \quad \perp$$

decision
level 1

Could backtrack by erasing **conflict level** & flipping last decision

decision
level 2

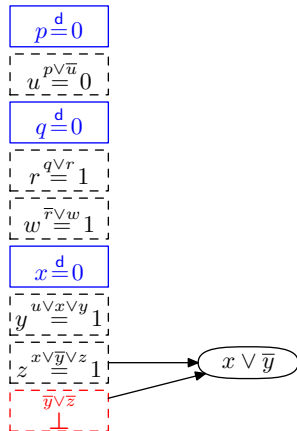
But want to **learn** from conflict and cut away as much of search space as possible

decision
level 3

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$



Could backtrack by erasing **conflict level** & flipping last decision

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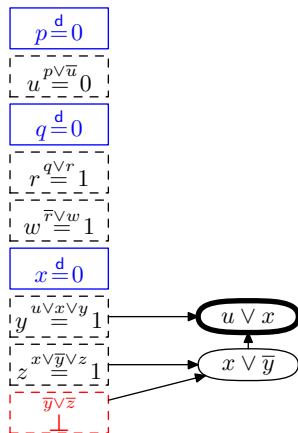
Case analysis over z for last two clauses:

- $x \vee \bar{y} \vee z$ wants $z = 1$
- $\bar{y} \vee \bar{z}$ wants $z = 0$
- Merge clauses & remove z — must satisfy $x \vee \bar{y}$

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$



Could backtrack by erasing **conflict level** & flipping last decision

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Case analysis over z for last two clauses:

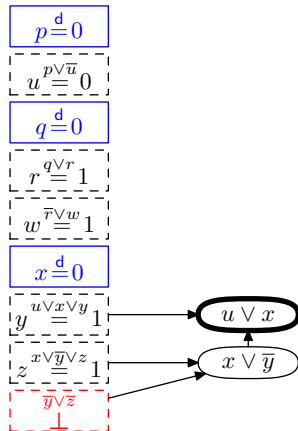
- $x \vee \bar{y} \vee z$ wants $z = 1$
- $\bar{y} \vee \bar{z}$ wants $z = 0$
- Merge clauses & remove z — must satisfy $x \vee \bar{y}$

Repeat until **UIP clause** with only 1 variable at conflict level after last decision — **learn** and **backjump**

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$



Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$

$$p \stackrel{d}{=} 0$$

$$u \stackrel{p \vee \bar{u}}{=} 0$$

$$q \stackrel{d}{=} 0$$

$$r \stackrel{q \vee r}{=} 1$$

$$w \stackrel{\bar{r} \vee w}{=} 1$$

$$x \stackrel{d}{=} 0$$

$$y \stackrel{u \vee x \vee y}{=} 1$$

$$z \stackrel{x \vee \bar{y} \vee z}{=} 1$$

$$\bar{y} \vee \bar{z} \quad \perp$$

$$p \stackrel{d}{=} 0$$

$$u \stackrel{p \vee \bar{u}}{=} 0$$

Assertion level 1 (2nd largest level in learned clause) —
trim trail to that level

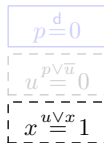
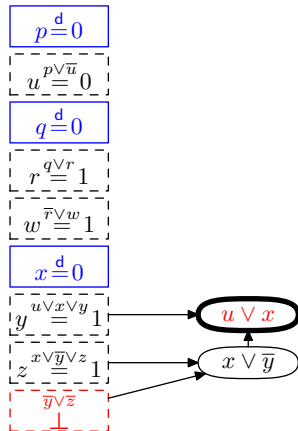
$$u \vee x$$

$$x \vee \bar{y}$$

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$



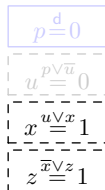
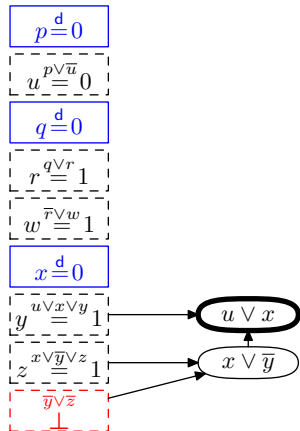
Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

Now UIP literal guaranteed to flip (**assert**) — but this is a **propagation**, not a decision

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \vee \bar{u}) \wedge (q \vee r) \wedge (\bar{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \bar{y} \vee z) \wedge (\bar{x} \vee z) \wedge (\bar{y} \vee \bar{z}) \wedge (\bar{x} \vee \bar{z}) \wedge (\bar{p} \vee \bar{u})$$



Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

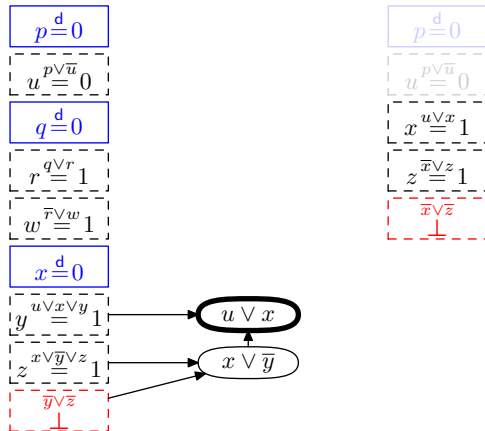
Now UIP literal guaranteed to flip (**assert**) — but this is a **propagation**, not a decision

Then continue as before. . .

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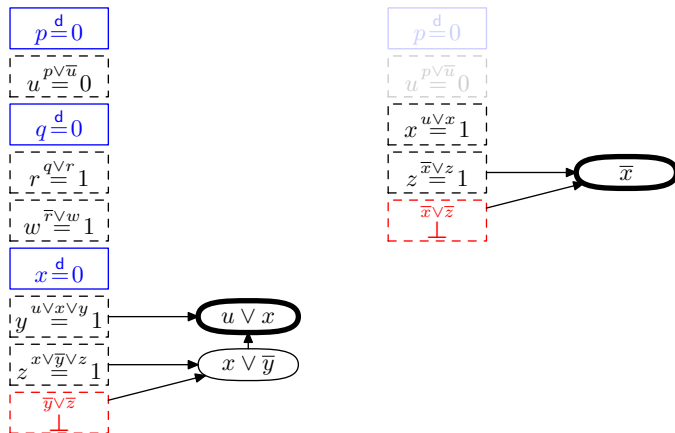
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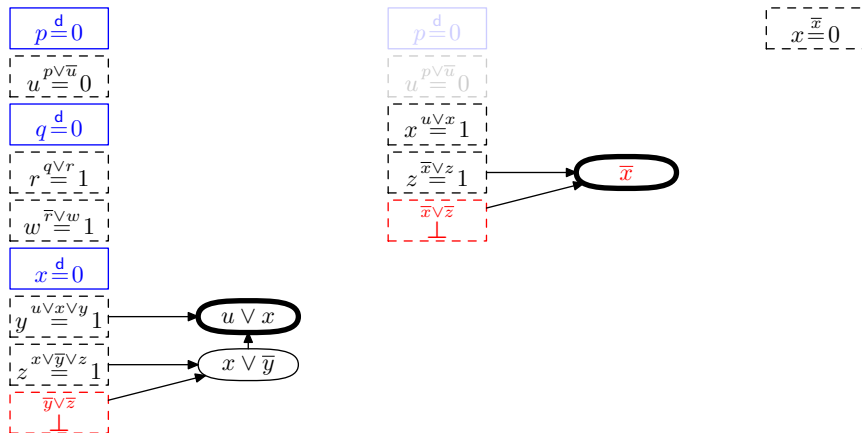
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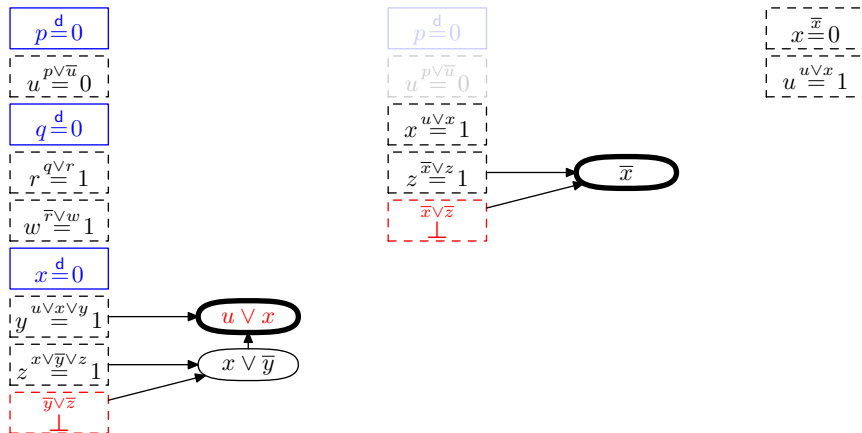
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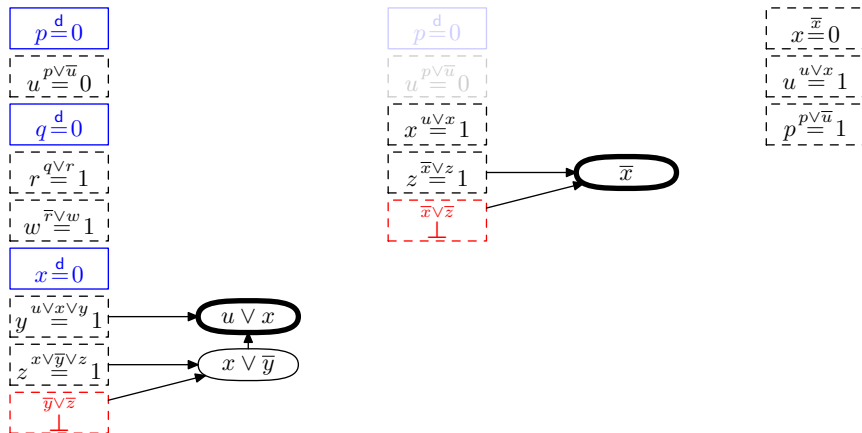
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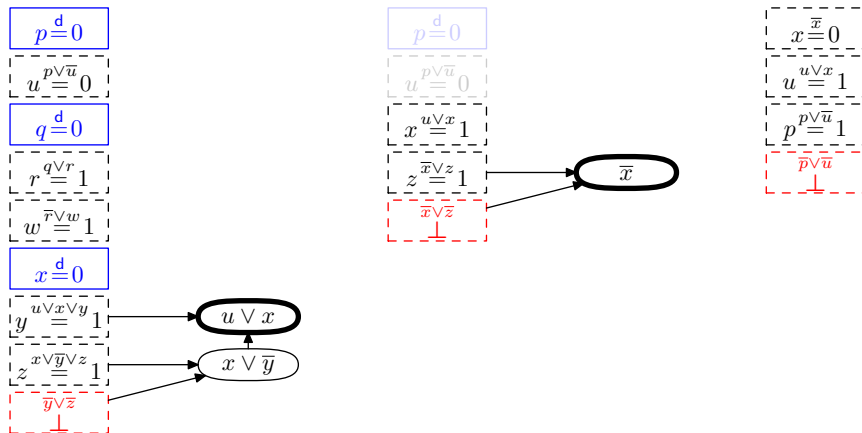
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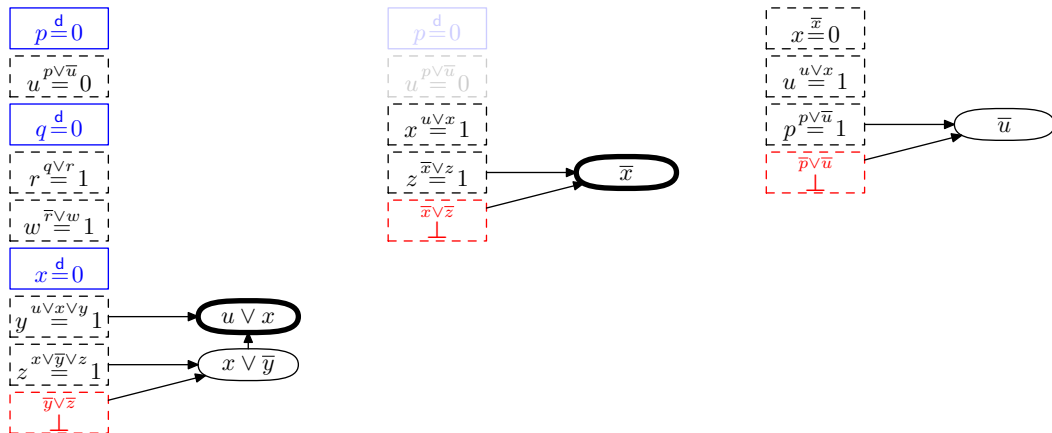
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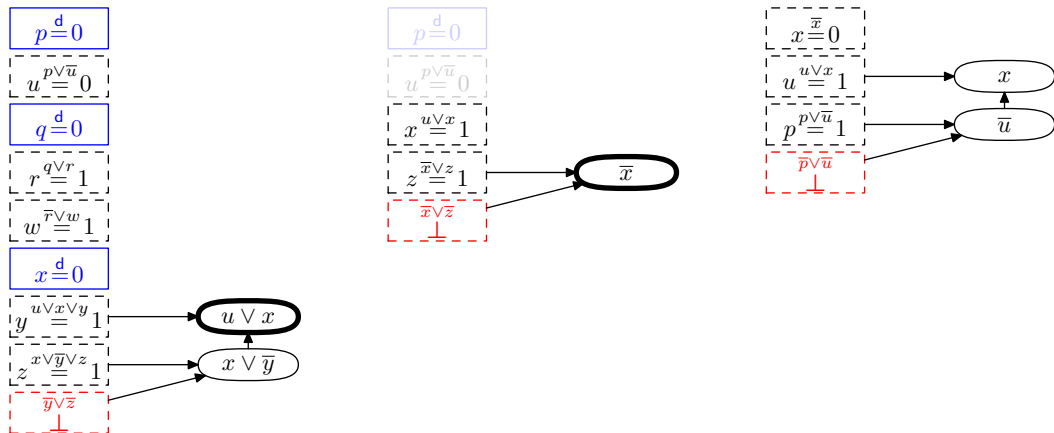
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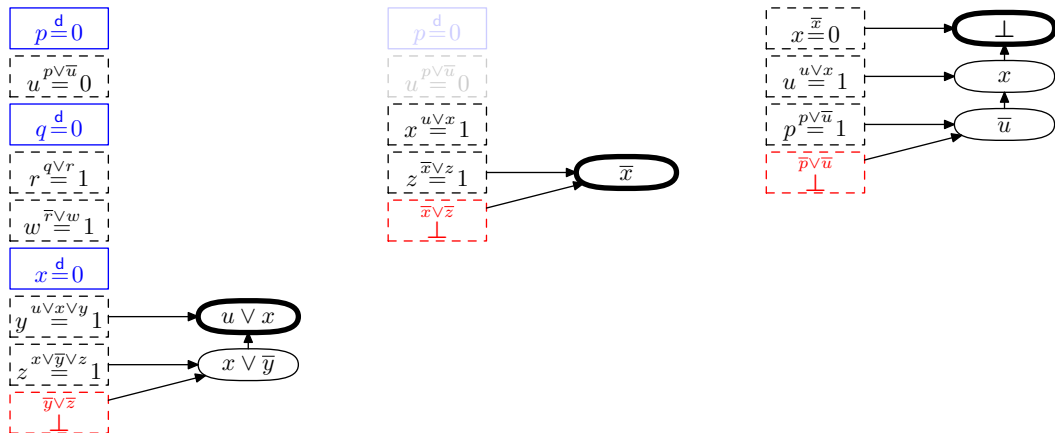
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SAT Solver Analysis and the Resolution Proof System

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Many intricate, hard-to-understand heuristics

So focus instead on **underlying method of reasoning**

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Resolution proof system [Bla37, Rob65]

- Start with clauses of CNF formula (**axioms**)
- Derive new clauses by **resolution rule**

$$\frac{C_1 \vee x \quad C_2 \vee \bar{x}}{C_1 \vee C_2}$$

Resolution Proofs by Contradiction

Resolution rule:

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Observation

If F is a satisfiable CNF formula and D is derived from clauses $D_1, D_2 \in F$ by the resolution rule, then $F \wedge D$ is satisfiable.

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So can prove F **unsatisfiable** by deriving the unsatisfiable empty clause (denoted \perp) from F by resolution

Such proof by contradiction also called **resolution refutation**

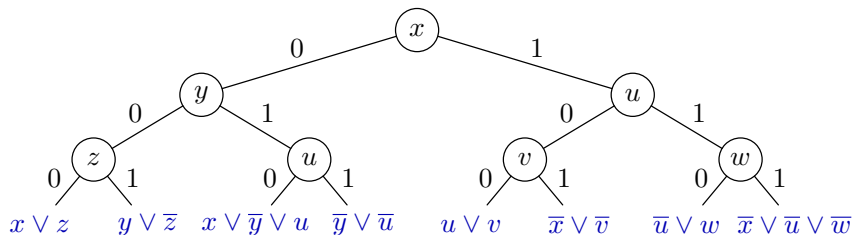
DPLL and Resolution Proofs

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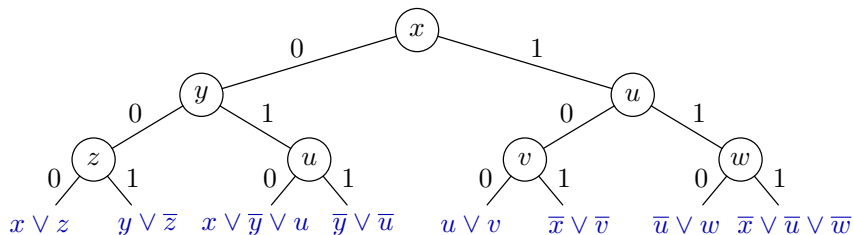
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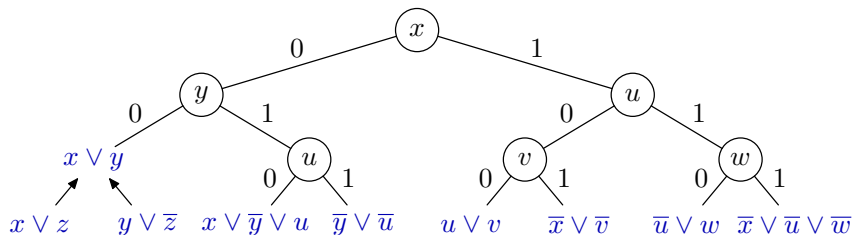


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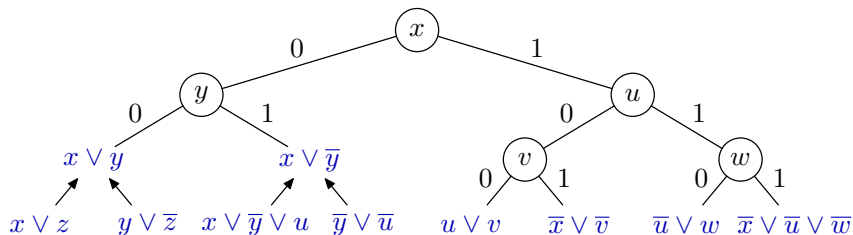


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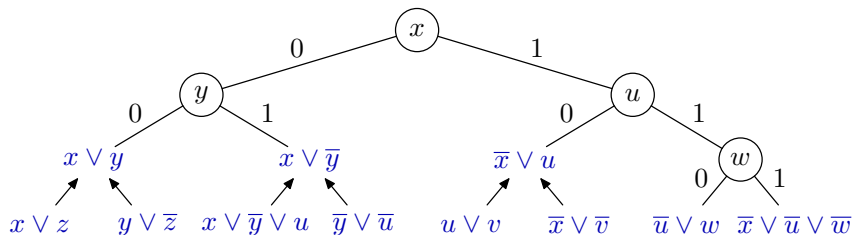


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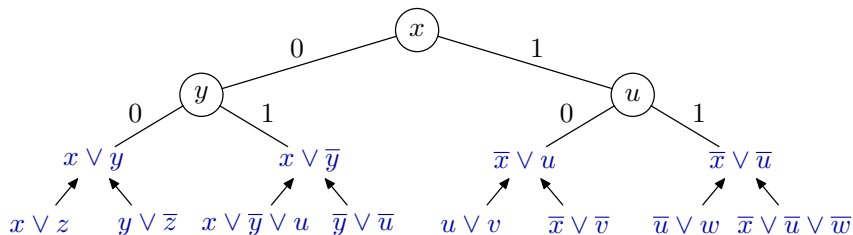


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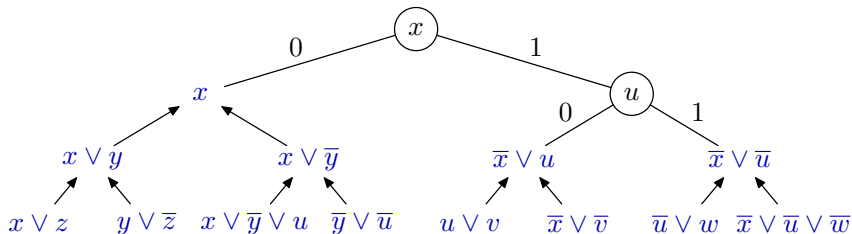


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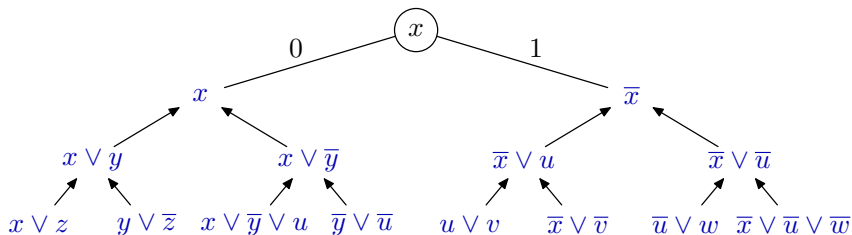


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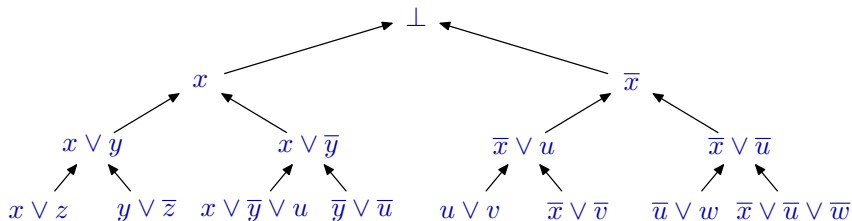


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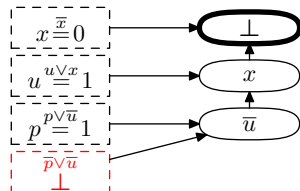
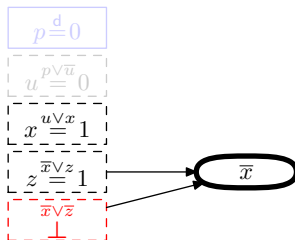
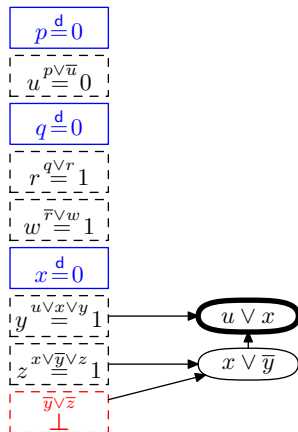
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- Conflict-driven clause learning adds “shortcut edges” in tree, but still yields resolution proof

CDCL and Resolution Proofs

Obtain resolution proof. . .

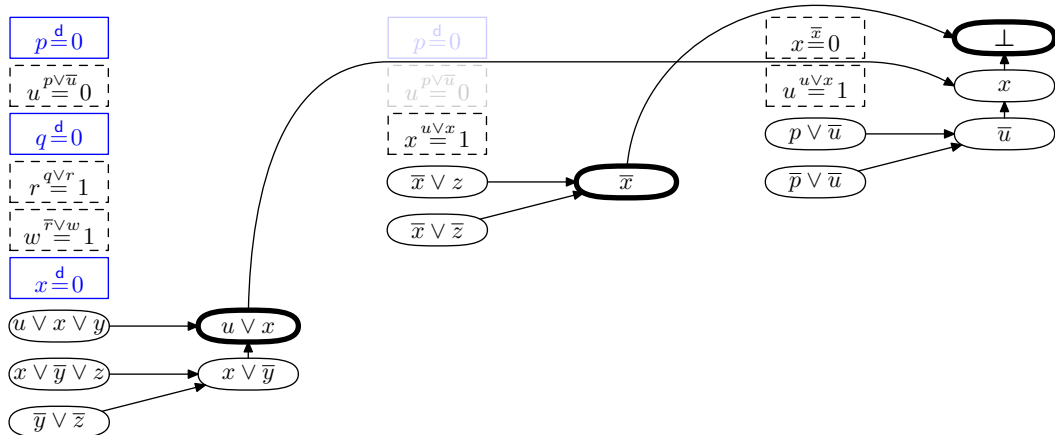
CDCL and Resolution Proofs

Obtain resolution proof from our example CDCL execution...



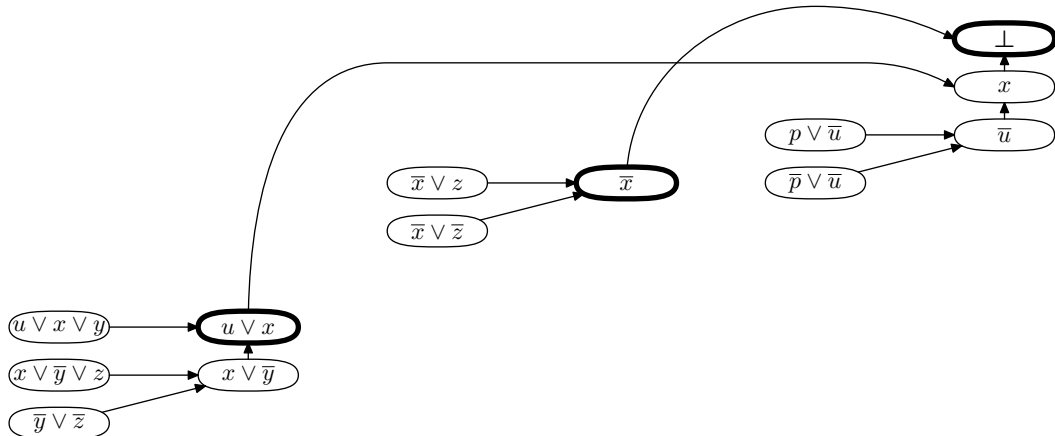
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(*) Except for some **preprocessing techniques**, which is an important omission, but this gets complicated and we don't have time to go into details. . .

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- Paradox: resolution quite weak proof system; many strong proof complexity lower bounds for (seemingly) “obvious” formulas

Examples of Hard Formulas For Resolution (1/3)

Pigeonhole principle (PHP) formulas [Hak85]

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$$p_{i,1} \vee p_{i,2} \vee \cdots \vee p_{i,n}$$

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Even onto functional PHP hard — **“resolution cannot count”**

Resolution proof requires $\exp(\Omega(n)) = \exp(\Omega(\sqrt[3]{N}))$ clauses
(measured in terms of formula size N)

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Tseitin formulas [Urq87]

“Sum of degrees of vertices in graph is even”

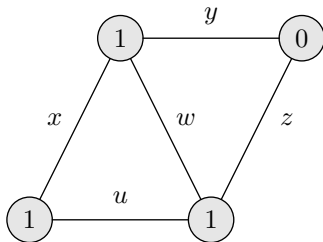
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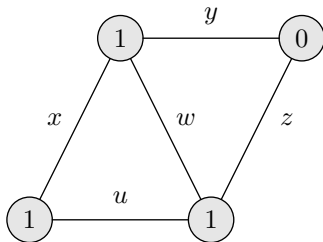
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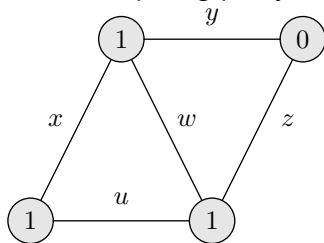
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Requires **proof size** $\exp(\Omega(N))$ on well-connected so-called **expander graphs** —

“**resolution cannot count mod 2**”

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Random k -CNF formulas [CS88]

Δn randomly sampled k -clauses over n variables

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But no such strong lower bounds known for CLIQUE!

- Refuting existence of k -clique should require proof size $n^{\Omega(k)}$
- Only known for restricted so-called regular resolution [ABdR⁺21]

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- Add Boolean axioms

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Hilbert's Nullstellensatz

Consider any system of polynomial equations

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Hilbert's Nullstellensatz

System infeasible \Leftrightarrow exist $q_i, r_j \in \mathbb{F}[x_1, \dots, x_n]$ such that

$$\sum_{i=1}^m q_i(x_1, \dots, x_n) \cdot p_i(x_1, \dots, x_n) + \sum_{j=1}^n r_j(x_1, \dots, x_n) \cdot (x_j^2 - x_j) = 1$$

Nullstellensatz Proof System [BIK⁺94]

Nullstellensatz refutation of

$$\begin{array}{ll} p_i(x_1, \dots, x_n) = 0 & i \in [m] \\ x_j^2 - x_j = 0 & j \in [n] \end{array}$$

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Complexity measures of refutations:

- **Size**: number of monomials (when all polynomials expanded out)
- **Degree**: highest total degree of any polynomial

Nullstellensatz Example (Not Expanded out)

$$(x \vee z) \wedge (y \vee \neg z) \wedge (x \vee \neg y \vee u) \wedge (\neg y \vee \neg u) \\ \wedge (u \vee v) \wedge (\neg x \vee \neg v) \wedge (\neg u \vee w) \wedge (\neg x \vee \neg u \vee \neg w)$$

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$$(1 - x)(1 - z)$$

$$(1 - y)z$$

$$(1 - x)y(1 - u)$$

$$yu$$

$$(1 - u)(1 - v)$$

$$xv$$

$$u(1 - w)$$

$$xuw$$

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$$\begin{aligned} & (1 - y) \cdot (1 - x)(1 - z) \\ + & (1 - x) \cdot (1 - y)z \\ + & 1 \cdot (1 - x)y(1 - u) \\ + & (1 - x) \cdot yu \\ + & x \cdot (1 - u)(1 - v) \\ + & (1 - u) \cdot xv \\ + & x \cdot u(1 - w) \\ + & 1 \cdot xuw \end{aligned}$$

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Size 27

Degree 3

(No use of Boolean axioms)

Nullstellensatz Proof Search

- Solve linear system of equations with coefficients of polynomials q_i, r_j as unknowns
- Used successfully to solve, e.g., graph colouring problems [DLMM08, DLMO09, DLMM11]
- Running time grows exponentially with degree, though high-degree refutations can be very small [BCIP02, dRMNR21]

Dual Variables

- Annoying problem: $x_1 \vee x_2 \vee x_3$ translates to polynomial

$$(1 - x_1)(1 - x_2)(1 - x_3) = 1 - x_1 - x_2 - x_3 + x_1x_2 + x_1x_3 + x_2x_3 - x_1x_2x_3$$

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$$\prod_{i \in \mathcal{P}} x'_i \cdot \prod_{j \in \mathcal{N}} x_j = 0$$

- Doesn't affect degree (obviously), but can decrease size exponentially [dRLNS21]
(also for other algebraic proof systems)

Dynamic Construction of Nullstellensatz Certificates

Nullstellensatz again

Infeasibility of

$$p_i(x_1, \dots, x_n) = 0 \quad i \in [m]$$

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- **Ideal** \mathcal{I} :

- 1 $p, q \in \mathcal{I} \Rightarrow p + q \in \mathcal{I}$

- 2 $p \in \mathcal{I} \Rightarrow r \cdot p \in \mathcal{I}$ for any r

- Compute polynomials in this ideal \mathcal{I} step by step

- Use “multivariate division” to check whether 1 lies in ideal or not

Gröbner Bases: Admissible Orderings and Leading Terms

Admissible ordering \preceq on monomials m, m', t :

- ① $m \preceq m' \Rightarrow t \cdot m \preceq t \cdot m'$
- ② $m \preceq t \cdot m$

Examples:

- Lexicographic
- Degree-lexicographic

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Can write $p = \text{lt}(p) + p'$ for $\text{lt}(p)$ leading term (largest w.r.t. \preceq)

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Can write $p = \text{lt}(p) + p'$ for **lt(p) leading term** (largest w.r.t. \preceq)

If $\text{lt}(p) = t \cdot \text{lt}(q)$, can **reduce** $p \bmod q$ by computing $p - t \cdot q$

“Multivariate division”: Reduce p modulo all q in set of polynomials \mathcal{G} until no further reductions possible

\mathcal{G} is a **Gröbner basis** if final result uniquely determined

Gröbner Bases: Buchberger's Algorithm

Buchberger's algorithm for computing Gröbner bases (**very** rough)

- 1 Let $\mathcal{G} :=$ all axioms
- 2 Pick unprocessed pair $p, q \in \mathcal{G}$ or terminate if none exists
- 3 Compute $p' = t_p \cdot p$ and $q' = t_q \cdot q$ to make leading terms cancel
- 4 Set $S := p' - q'$; reduce $S \bmod \mathcal{G}$ with multivariate division; add result to \mathcal{G} if non-zero
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- ⑤ Go to 2

Facts:

- Buchberger's algorithm computes Gröbner basis
- At termination, $1 \in \mathcal{G} \Leftrightarrow$ polynomial equations infeasible

Polynomial Calculus [CEI96, ABRW02]

- Compute polynomials in ideal \mathcal{I} generated by p_i , $x_j^2 - x_j$, and $x_j + x'_j - 1$ step by step:
 - $p_i \in \mathcal{I}$, $x_j^2 - x_j \in \mathcal{I}$, and $x_j + x'_j - 1 \in \mathcal{I}$ (axioms)
 - If $p, q \in \mathcal{I}$, then $\alpha p + \beta q \in \mathcal{I}$ for any $\alpha, \beta \in \mathbb{F}$ (linear combination)
 - If $p \in \mathcal{I}$, then $m \cdot p \in \mathcal{I}$ for any monomial $m = \prod_j x_j$ (multiplication)

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- A refutation is a derivation ending with the polynomial 1
- Complexity measures:
 - **Size**: total number of monomials in all polynomials in derivation expanded out
 - **Degree**: highest total degree of any polynomial
- Polynomial calculus (much) stronger than Nullstellensatz w.r.t. both size and degree

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$$\frac{x \vee \bar{y} \vee z \quad \bar{y} \vee \bar{z}}{x \vee \bar{y}}$$

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$$\frac{x \vee \bar{y} \vee z \quad \bar{y} \vee \bar{z}}{x \vee \bar{y}}$$

simulated by polynomial calculus derivation

$$\frac{x'yz' \quad \frac{\frac{yz}{x'yz} \quad \frac{z + z' - 1}{x'yz + x'yz' - x'y}}{-x'yz' + x'y}}{x'y}$$

Polynomial Calculus is Strictly Stronger than Resolution

Polynomial calculus **can be exponentially stronger** than resolution

For instance:

- Tseitin formulas on expander graphs if $\mathbb{F} = \text{GF}(2)$
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Other hard formulas:

- Tseitin-like formulas for counting mod p if $p \neq$ field characteristic [BGIP01]
- Random k -CNF formulas
 - all characteristics except 2 [BI99]
 - all characteristics [AR03]

COLOURING and CLIQUE for Polynomial Calculus

COLOURING

- Exponential worst-case lower bounds in [LN17]
- Exponential **average-case** lower bounds in [CdRN⁺23]

CLIQUE

Essentially nothing known!

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- Use **dual variables!** [KBBN22]

Gröbner bases: Some Problems and Questions

- ① Buchberger not a great SAT solving algorithm
Slow and memory-intensive, and computes too much info
Possible to use conflict-driven paradigm?!
- ② Dual variables increase reasoning power exponentially [dRLNS21]
But are immediately eliminated by multivariate division
Possible to design dual-variable-aware Buchberger?!
- ③ Analysis of polynomial calculus uses degree-lexicographic ordering
In computational algebra, many other orderings used
Prove proof complexity separation results for different orderings?

SAT as System of 0–1 Integer Linear Inequalities

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to 0-1 integer linear inequalities

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$$\sum_{i \in \mathcal{P}} x_i + \sum_{j \in \mathcal{N}} (1 - x_j) \geq 1$$

- Add variable axioms

$$\begin{aligned} x_j &\geq 0 \\ -x_j &\geq -1 \end{aligned}$$

for all variables

Cutting Planes Proof System [CCT87]

Cutting planes introduced in [CCT87] to model integer linear programming algorithm in [Gom63, Chv73]

Can be applied to any system of 0-1 integer linear inequalities

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Cutting planes derivation rules

$$\text{Multiplication} \quad \frac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA} \quad c \in \mathbb{N}^+$$

$$\text{Addition} \quad \frac{\sum a_i x_i \geq A \quad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$$

$$\text{Division} \quad \frac{\sum a_i x_i \geq A}{\sum \lceil a_i / c \rceil x_i \geq \lceil A / c \rceil} \quad c \in \mathbb{N}^+$$

Cutting Planes Derivations and Refutations

- A cutting planes derivation is a sequence of 0-1 integer linear inequalities derived using
 - Axioms (clauses and variable bounds)
 - Multiplication $\sum a_i x_i \geq A \Rightarrow \sum c a_i x_i \geq cA$
 - Addition $\sum a_i x_i \geq A, \sum b_i x_i \geq B \Rightarrow \sum (a_i + b_i) x_i \geq A + B$
 - Division $\sum a_i x_i \geq A \Rightarrow \sum \lceil a_i / c \rceil x_i \geq \lceil A / c \rceil$
- A refutation ends with the inequality $0 \geq 1$
- Complexity measures:
 - **Length**: # inequalities
 - **Size**: Count also bit size of representing all coefficients

Cutting Planes vs. Resolution

- Cutting planes can simulate resolution reasoning efficiently and can be exponentially stronger (e.g., for PHP, just count and argue that $\# \text{pigeons} > \# \text{holes}$)

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- Cutting planes can simulate resolution reasoning efficiently and can be exponentially stronger (e.g., for PHP, just count and argue that $\# \text{pigeons} > \# \text{holes}$)
- And 0-1 linear inequalities are similar to but much more concise than CNF

Compare

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 3$$

and

$$\begin{aligned} & (x_1 \vee x_2 \vee x_3 \vee x_4) \wedge (x_1 \vee x_2 \vee x_3 \vee x_5) \wedge (x_1 \vee x_2 \vee x_3 \vee x_6) \\ & \wedge (x_1 \vee x_2 \vee x_4 \vee x_5) \wedge (x_1 \vee x_2 \vee x_4 \vee x_6) \wedge (x_1 \vee x_2 \vee x_5 \vee x_6) \\ & \wedge (x_1 \vee x_3 \vee x_4 \vee x_5) \wedge (x_1 \vee x_3 \vee x_4 \vee x_6) \wedge (x_1 \vee x_3 \vee x_5 \vee x_6) \\ & \wedge (x_1 \vee x_4 \vee x_5 \vee x_6) \wedge (x_2 \vee x_3 \vee x_4 \vee x_5) \wedge (x_2 \vee x_3 \vee x_4 \vee x_6) \\ & \wedge (x_2 \vee x_3 \vee x_5 \vee x_6) \wedge (x_2 \vee x_4 \vee x_5 \vee x_6) \wedge (x_3 \vee x_4 \vee x_5 \vee x_6) \end{aligned}$$

Hard Formulas for Cutting Planes

Clique-colouring formulas [Pud97]

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Variables

- $p_{i,j}$ indicators of the edges in graph; $1 \leq i < j \leq n$
- $q_{k,i}$ identify members of m -clique; $1 \leq k \leq m$, $1 \leq i \leq n$
- $r_{i,\ell}$ specify colouring of vertices; $1 \leq \ell \leq m - 1$, $1 \leq i \leq n$

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$$q_{k,1} \vee q_{k,2} \vee \cdots \vee q_{k,n}$$

some vertex is the k th member of clique

$$\bar{q}_{k,i} \vee \bar{q}_{k',i}$$

clique members are uniquely defined ($k \neq k'$)

$$p_{i,j} \vee \bar{q}_{k,i} \vee \bar{q}_{k',j}$$

clique members are connected by edges

$$r_{i,1} \vee r_{i,2} \vee \cdots \vee r_{i,m-1}$$

every vertex i has a colour

$$\bar{p}_{i,j} \vee \bar{r}_{i,\ell} \vee \bar{r}_{j,\ell}$$

neighbours have distinct colours

More Hard Formulas for Cutting Planes?

Lower bound for clique-colouring formulas uses **interpolation** and **circuit complexity**

- From small cutting planes proof, build small circuit of special type that can decide whether graph has clique
- Prove separately that no such small circuits can exist
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Cutting planes not well understood at all

Clear need for development of new analysis methods

Some recent developments in [dRMN⁺20, HP17, FPPR22, GGKS20, Sok23]

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Nothing known for COLOURING or CLIQUE

Surprisingly, Tseitin formulas are at most quasi-polynomially hard for cutting planes [DT20]!

SAT Solvers Based on Cutting Planes?

So-called **pseudo-Boolean (PB) solvers** using (subset of) cutting planes reasoning developed in, e.g., [CK05, SS06, LP10, EN18]

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Challenge 1: Conjunctive normal form

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- Solvers can rewrite CNF to more helpful 0-1 linear inequalities [BLLM14, EN20], but this doesn't work so well in practice
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Challenge 2: Increased degrees of freedom(!?)

- Cutting planes much smarter method of reasoning
- But this makes it trickier to design smart search algorithms

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Challenge 1: Conjunctive normal form

- Pseudo-Boolean solvers terrible for CNF input
- Solvers can rewrite CNF to more helpful 0-1 linear inequalities [BLLM14, EN20], but this doesn't work so well in practice
- Better to encode problem with 0-1 integer linear inequalities from the start

Challenge 2: Increased degrees of freedom(!?)

- Cutting planes much smarter method of reasoning
- But this makes it trickier to design smart search algorithms

Is it truly harder to build good pseudo-Boolean solvers?

Or has just so much more work has been put into CDCL solvers?

Division Versus Saturation

Use negated literals as needed to get all a_i , A positive

Boolean derivation rules for 0–1 integer linear inequalities

$$\text{Division} \frac{\sum a_i \ell_i \geq A}{\sum \lceil a_i/c \rceil \ell_i \geq \lceil A/c \rceil} \quad c \in \mathbb{N}^+$$

$$\text{Saturation} \frac{\sum a_i \ell_i \geq A}{\sum \min\{a_i, A\} \cdot \ell_i \geq A}$$

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- Complexity literature of cutting planes uses division [CCT87]
- Pseudo-Boolean solvers instead adopted saturation [CK05, LP10]
- **Open how the two variants compare**, but clear that **division** can sometimes be better in theory [GNY19]

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- Complexity literature of cutting planes uses division [CCT87]
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- **Open how the two variants compare**, but clear that **division** can sometimes be better in theory [GNY19]
- ... And most often also in practice [EN18], though not always [LBD⁺20]

Sherali–Adams (SA) and Sums of Squares (SoS)

Refutation of $p_i \in \mathbb{R}[x_1, \dots, x_n]$, $i \in [m]$, and $x_j^2 - x_j$, $j \in [n]$

Nullstellensatz

$$\sum_{i=1}^m q_i \cdot p_i + \sum_{j=1}^n r_j \cdot (x_j^2 - x_j) = 1$$

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$$\sum_{i=1}^m q_i \cdot p_i + \sum_{j=1}^n r_j \cdot (x_j^2 - x_j) + \sum_{k=1}^t \alpha_k \prod_{i \in \mathcal{P}_t} (1 - x_i) \cdot \prod_{j \in \mathcal{N}_t} x_j = -1$$

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Sums of squares (SoS) ($s_k \in \mathbb{R}[x_1, \dots, x_n]$)

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Sherali–Adams, Sums of Squares, and Relations to Other Proof Systems

Sherali–Adams models linear programming (LP) hierarchies

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Strict hierarchy (over \mathbb{R}):

- Nullstellensatz
- Sherali–Adams
- Sums of squares

Sums of squares is strictly **stronger** than **polynomial calculus** (over \mathbb{R})

Sherali-Adams and **polynomial calculus** are **incomparable** [Ber18]

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Sums of squares very strong proof system (e.g., can reason about PHP)

But can't do parity reasoning efficiently [GV01, Gri01]

Survey [FKP19] recommended for more reading

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Intended to model modern 0-1 integer linear programming

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- ④ Recurse with $\mathcal{S} := \mathcal{S} \cup \{\sum_i a_i l_i \leq A - 1\}$

Complexity measures:

- **Length:** # branching nodes / sets \mathcal{S}
- **Size:** Count also bit size for representing all coefficients

Stabbing Planes and Cutting Planes

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Still possible that stabbing planes is exponentially more powerful than cutting planes, but hard to know what to believe

Extended Resolution [Tse68]

Resolution rule

$$\frac{C_1 \vee x \quad C_2 \vee \bar{x}}{C_1 \vee C_2}$$

Extension rule introducing clauses

$$a \vee \bar{x} \vee \bar{y} \quad \bar{a} \vee x \quad \bar{a} \vee y$$

for fresh variable a (encoding that $a \leftrightarrow (x \wedge y)$ must hold)

Extended Resolution and SAT Solving

- Closely related (and equivalent) to *DRAT* system used to justify correctness of some SAT preprocessing techniques [JHB12]
- *DRAT* also used for SAT solver proof logging
- Attempts to combine extended resolution with CDCL in, e.g., [AKS10, Hua10]
- Without restrictions, corresponds to extremely strong **extended Frege system** [CR79] — pretty much no lower bounds known
- To analyse solvers using extended resolution, would need to:
 - Describe heuristics/rules actually used
 - See if possible to reason about such restricted proof system

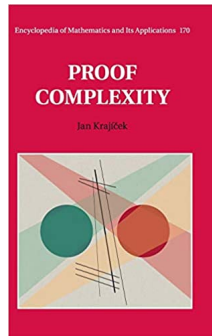
Some More References for Further Reading

Handbook of Satisfiability (Especially chapter 7 😊)



[BHvMW21]

Proof Complexity by Jan Krajíček



[Kra19]

Summing up This Presentation

Overview of some proof systems used in combinatorial solving:

- Resolution \longleftrightarrow conflict-driven clause learning (CDCL)
- Nullstellensatz and polynomial calculus \longleftrightarrow Gröbner bases
- Cutting planes \longleftrightarrow pseudo-Boolean solving

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Thank you for your attention!

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