Proof Complexity as a Computational Lens: Lecture 2

Theory Basics, Resolution, and the Pigeonhole Principle

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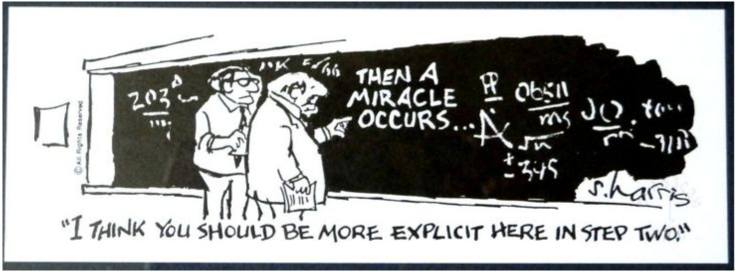
November 4, 2025

What is a Proof?



The Subject Matter of This Course

- What is a proof?
- Which (logical) statements have efficient proofs?
- How can we find such proofs? (Is it even possible?)
- What are good methods of reasoning about logical statements?
- What are natural notions of "efficiency" of proofs? (size, complexity, et cetera)
- How are these notions related?

Today's Lecture

- More "theory-oriented" introduction to proof complexity
- Some "teasers" for what to expect in coming lectures
- Recap of resolution proof system
- Proof that resolution cannot reason efficiently about the pigeonhole principle (on the board)
- Introductory slides might go slightly fast, but
  - everything will be online to allow recap
  - we will repeat everything more carefully when we need it later

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Claim: 25957 is the product of two primes

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- $$\begin{array}{ll} 25957 \equiv 1 \pmod{2} & 25957 \equiv 0 \pmod{101} \\ 25957 \equiv 1 \pmod{3} & 25957 \equiv 1 \pmod{103} \\ 25957 \equiv 2 \pmod{5} & \vdots \\ \vdots & 25957 \equiv 0 \pmod{257} \\ 25957 \equiv 19 \pmod{99} & \vdots \end{array}$$
  OK, but maybe even a bit of overkill

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  OK, but maybe even a bit of overkill
- "$25957 = 101 \cdot 257$; check yourself that these are primes"

Key demand: A proof should be efficiently verifiable

Proof system

Proof system for a language L (adapted from Cook & Reckhow [CR79]):

Deterministic algorithm $\mathcal{P}(x, \pi)$ that runs in time polynomial in $|x|$ and $|\pi|$ such that

- for all $x \in L$ there is a string π (a proof) for which $\mathcal{P}(x, \pi) = 1$
- for all $x \notin L$ it holds for all strings π that $\mathcal{P}(x, \pi) = 0$

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Propositional proof system: proof system for the language TAUT of all valid propositional logic formulas (or tautologies)

Propositional Logic: Syntax

Set $Vars$ of Boolean variables ranging over $\{0, 1\}$ (false and true)

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- conjunction $\wedge$
- disjunction $\vee$
- implication $\rightarrow$
- equivalence $\leftrightarrow$

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Set PROP of propositional logic formulas is smallest set X such that

- $x \in X$ for all propositional logic variables $x \in \mathit{Vars}$
- if $F, G \in X$ then $(F \wedge G), (F \vee G), (F \rightarrow G), (F \leftrightarrow G) \in X$
- if $F \in X$ then $(\neg F) \in X$

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Extend α from variables to formulas by:

- $\alpha(\neg F) = 1$ if $\alpha(F) = 0$
- $\alpha(F \lor G) = 1$ unless $\alpha(F) = \alpha(G) = 0$
- $\alpha(F \land G) = 1$ if $\alpha(F) = \alpha(G) = 1$
- $\alpha(F \to G) = 1$ unless $\alpha(F) = 1$ and $\alpha(G) = 0$
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We say that F is

- satisfiable if there is an assignment α with $\alpha(F) = 1$
- valid or tautological if all assignments satisfy F
- falsifiable if there is an assignment α with $\alpha(F) = 0$
- unsatisfiable or contradictory if all assignments falsify F

Example Propositional Proof System

Example (Truth table)

p	q	r	$(p \land (q \lor r)) \leftrightarrow ((p \land q) \lor (p \land r))$
0	0	0	1
0	0	1	1
0	1	0	1
0	1	1	1
1	0	0	1
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Certainly polynomial-time checkable measured in “proof” size
Why does this not make us happy?

Proof System Complexity

Complexity $cplx(\mathcal{P})$ of a proof system $\mathcal{P}$:

Smallest $g : \mathbb{N} \rightarrow \mathbb{N}$ such that $x \in L$ if and only if there is a proof π of size $|\pi| \leq g(|x|)$ such that $\mathcal{P}(x, \pi) = 1$

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Example (Truth table continued)

Truth table is a propositional proof system, but of exponential complexity!

Proof systems and P vs. NP

Theorem (Cook & Reckhow [CR79])

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NP is *exactly* the set of languages with p-bounded proof systems. $\square$

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If NP $=$ coNP, then $\mathrm{TAUT} \in$ NP has a p-bounded proof system by definition.

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($\Leftarrow$) Suppose there exists a p-bounded proof system. Then $\mathrm{TAUT} \in$ NP, and since TAUT is complete for coNP it follows that NP $=$ coNP. $\square$

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Definition (p-simulation)

$\mathcal{P}_1$ polynomially simulates, or p-simulates, $\mathcal{P}_2$ if there exists a polynomial-time computable function f such that for all $F \in \mathrm{TAUT}$ it holds that $\mathcal{P}_2(F, \pi) = 1$ iff $\mathcal{P}_1(F, f(\pi)) = 1$

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Weak p-simulation: $cplx(\mathcal{P}_1) = (cplx(\mathcal{P}_2))^{\mathcal{O}(1)}$ but we do not know explicit translation function f from $\mathcal{P}_2$-proofs to $\mathcal{P}_1$-proofs

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Lots of results proven relating strength of different proof systems
Will see some examples in this course

A Fundamental Theoretical Problem. . .

The constructive version of the problem:

Problem

Given a propositional logic formula F, can we decide efficiently whether is it true no matter how we assign values to its variables?

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These days recognized as one of the main challenges for all of mathematics — one of the million dollar "Millennium Problems" of the Clay Mathematics Institute [Mil00]

. . . with Huge Practical Implications

- All known algorithms run in exponential time in worst case
- But enormous progress on applied computer programs last 30 years (see, e.g., [BS97, MS99, MMZ$^+$01, ES04, AS09, Bie10] or [BHvMW21] for more comprehensive references)
- These so-called SAT solvers are routinely deployed to solve large-scale real-world problems with 100 000s or even 1 000 000s of variables
- Used in, e.g., hardware verification, software testing, software package management, artificial intelligence, cryptography, bioinformatics, operations research, railway signalling systems, et cetera (and even in pure mathematics)
- But we also know small example formulas with only hundreds of variables that trip up even state-of-the-art SAT solvers

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Approach:

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Interesting and (arguably) important questions
But messy reality is hard to model with clean mathematics...

Proof Search Algorithms and Automatability

Proof search algorithm $A_{\mathcal{P}}$ for propositional proof system $\mathcal{P}$:
Deterministic algorithm with

- input: formula F
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Definition (Automatizability)

$\mathcal{P}$ is automatizable if there exists a proof search algorithm $A_{\mathcal{P}}$ such that if $F \in \mathrm{TAUT}$ then $A_{\mathcal{P}}$ on input F outputs a $\mathcal{P}$-proof of F in time polynomial in size of F plus size of a smallest $\mathcal{P}$-proof of F

Short Proofs Seem Hard to Find (at Least in Theory)

Example (Truth table continued)

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We want proof systems that are both

- strong (i.e., have short proofs for all tautologies) and
- automatizable (i.e., we can find these short proofs efficiently)

Seems that this is not possible unless P = NP [AM20]

But can find proof search algorithms that work really well "in practice"

Potential and Limitations of Mathematical Reasoning

Reason 3 for proof complexity: understand how deep / hard various mathematical truths are

- Look at logic encoding of various mathematical theorems (e.g., combinatorial principles such as pigeonhole principle, least number principle, handshaking lemma, et cetera)
- Determine how strong proof systems are needed to provide efficient proofs
- Tells us how powerful mathematical tools are needed for establishing such statements

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Fascinating questions that are systematically explored in bounded arithmetic
Some of the results we will cover are tangentially related, but this is not our main focus

Transforming Tautologies to Unsatisfiable CNF Formulas

Any propositional logic formula F can be converted to formula F' in conjunctive normal form (CNF) such that

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Approach by Tseitin [Tse68]:

- Introduce new variable x_G for each subformula $G \doteq H_1 \circ H_2$ in F, $\circ \in \{\land, \lor, \rightarrow, \leftrightarrow\}$
- Translate G to set of disjunctive clauses $Cl(G)$ which enforces that truth value of x_G is computed correctly given x_{H_1} and x_{H_2}

Sketch of Transformation

Two examples for $\lor$ and $\rightarrow$ ($\land$ and $\leftrightarrow$ are analogous):

$$G \equiv H_1 \lor H_2 : \qquad Cl(G) := \begin{aligned}&(\neg x_G \lor x_{H_1} \lor x_{H_2})\\ \land\ &(x_G \lor \neg x_{H_1})\\ \land\ &(x_G \lor \neg x_{H_2})\end{aligned}$$

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- Finally, add clause $\neg x_F$

Proof Systems for Refuting Unsatisfiable CNFs

- Easy to verify that constructed CNF formula F' is unsatisfiable iff F is a tautology
- So any sound and complete proof system which produces refutations of formulas in CNF can be used as a propositional proof system
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Warning:

- Because of this duality, proof complexity terminology is slightly schizophrenic
- Unsatisfiable formulas sometimes referred to as "tautologies" in the literature
- We won't go quite that far...
- But throughout the course "proof" and "refutation" will be synonyms

Sequential Proof Systems

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More formally, a proof system $\mathcal{P}$ is sequential if a proof π in $\mathcal{P}$ is a

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We will mostly study sequential proof systems in this course

The Resolution Proof System

Resolution:

- Most well-studied proof system in all of proof complexity
- Originally described by Blake [Bla37]
- Used in the context of SAT solving [DP60, DLL62, Rob65]
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Just one inference rule, the resolution rule:

$$\frac{B \vee x \quad C \vee \overline{x}}{B \vee C}$$

$B \vee C$ is the resolvent of $B \vee x$ and $C \vee \overline{x}$

Soundness and Completeness of Resolution

Resolution derivation π from CNF formula F:

- Start with clauses in F
- Interatively derive new clauses by resolution rule and add
- Final clause in π is $A \Leftrightarrow \pi$ is derivation of A (notation: $\pi : F \vdash A$)

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Resolution is:

Sound If there is a resolution derivation $\pi : F \vdash A$ then $F \vDash A$ (easy to show)

Complete If $F \vDash A$ then there is a resolution derivation $\pi : F \vdash A'$ for some $A' \subseteq A$ (not hard to prove, but we will skip this)

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In particular:

$$F \text{ is unsatisfiable}$$
$$\Updownarrow$$
$$\exists \text{ resolution refutation of } F = \text{derivation of unsatisfiable empty clause } \bot$$

Example Resolution Refutation

Recap of set-up:

- Goal: refute **unsatisfiable** CNF
- Start with clauses of formula (axioms)
- Derive new clauses by resolution rule

$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

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1. $x \lor y$
2. $x \lor \overline{y} \lor z$
3. $\overline{x} \lor z$
4. $\overline{y} \lor \overline{z}$
5. $\overline{x} \lor \overline{z}$

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Can represent refutation as

- annotated list or
- directed acyclic graph

1.	$x \vee y$	Axiom
2.	$x \vee \overline{y} \vee z$	Axiom
3.	$\overline{x} \vee z$	Axiom
4.	$\overline{y} \vee \overline{z}$	Axiom
5.	$\overline{x} \vee \overline{z}$	Axiom
6.	$x \vee \overline{y}$	Res$(2,4)$
7.	x	Res$(1,6)$
8.	$\overline{x}$	Res$(3,5)$
9.	$\perp$	Res$(7,8)$

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3.	$\overline{x} \lor z$	Axiom
4.	$\boldsymbol{\overline{y} \lor \overline{z}}$	Axiom
5.	$\overline{x} \lor \overline{z}$	Axiom
6.	$x \lor \overline{y}$	Res$(2, 4)$
7.	x	Res$(1, 6)$
8.	$\overline{x}$	Res$(3, 5)$
9.	$\bot$	Res$(7, 8)$

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Example Resolution Refutation

Recap of set-up:

- Goal: refute **unsatisfiable** CNF
- Start with clauses of formula (axioms)
- Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

- Refutation/proof ends when empty clause $\perp$ derived

Can represent refutation as

- annotated list or
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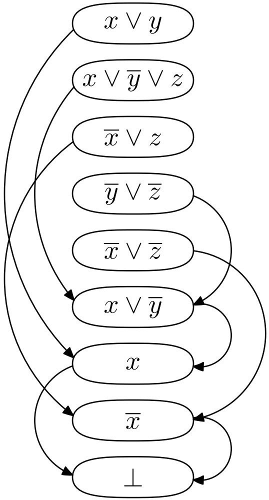

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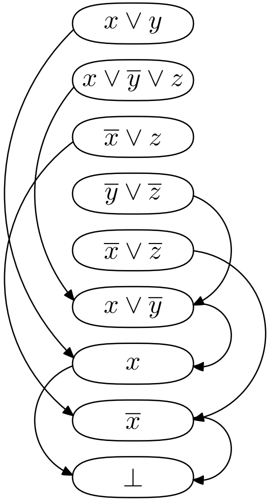
Tree-like resolution if DAG is tree



Resolution Length and Size

Length $=$ # clauses in resolution refutation (9 in our example)

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Size $=$ total # literals in refutation, strictly speaking

In practice, ignore linear factor and set size $=$ length for resolution

Proof size/length is the most fundamental measure in proof complexity
Main complexity measure of interest in this course

Resolution Space

Space = amount of memory needed when performing refutation

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Can be measured in different ways:

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Clause space at step t: $\#$ clauses at steps $\leq t$ used at steps $\geq t$

Total space at step t: Count also literals

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Clause space at step t: # clauses at steps $\leq t$ used at steps $\geq t$
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Example: Line space at step 7

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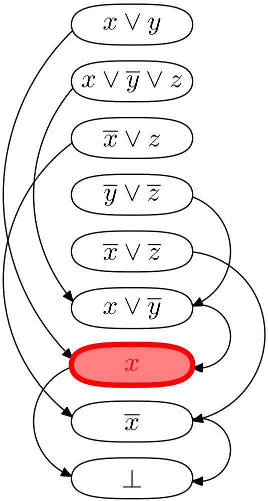

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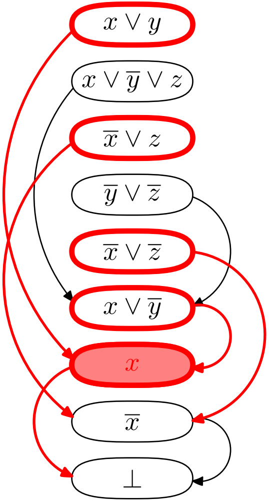
Example: Line space at step 7 is 5



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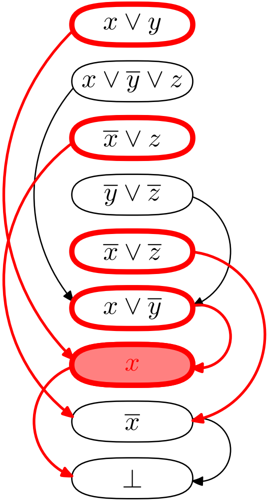
Example: Line space at step 7 is 5
Total space at step 7 is 9



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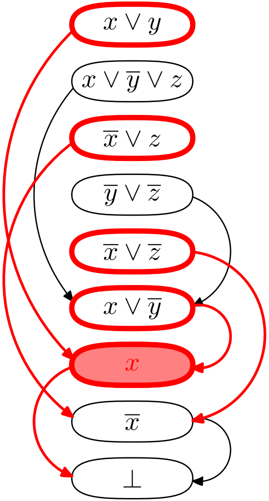
Space of refutation: Max over all steps



Refutation Size and Space

For any unsatisfiable CNF formula F and any proof system $\mathcal{P}$:

Size of refuting F = size of smallest $\mathcal{P}$-refutation of F

Clause space of refuting F = max # lines in memory in most space-efficient $\mathcal{P}$-refutation of F

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Interesting to study:

- size bounds (≈ SAT solver running time)
- space bounds (≈ SAT solver memory usage)
- size-space trade-offs (because solvers aggressively minimize both)

How to Prove Size/Length Lower Bounds

- Find suitable family of unsatisfiable CNF formulas with size scaling polynomially
- Show that smallest possible refutations in proof system $\mathcal{P}$ of these formulas scale superpolynomially or even exponentially
- How to prove this? Have to establish that no short proofs exist, even totally crazy ones!
- In order to do so, need to understand formulas really well
- So the formulas we know how to prove lower bounds for are mostly formulas that look very easy to humans
- A bit of a paradox... Let's now turn to the most famous formula family

Pigeonhole Principle (PHP) Formulas

"$n + 1$ pigeons don't fit into n holes"

Variables $p_{i,j}$ = "pigeon i goes into hole j", $i \in [n+1]$, $j \in [n]$

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$p_{i,1} \lor p_{i,2} \lor \cdots \lor p_{i,n}$	[every pigeon i gets a hole]
$\overline{p}_{i,j} \lor \overline{p}_{i',j}$	[no hole j gets two pigeons $i \neq i'$]

Can also add "functionality" and/or "onto" axioms

$\overline{p}_{i,j} \lor \overline{p}_{i,j'}$	[no pigeon i gets two holes $j \neq j'$]
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All versions are hard for resolution [Hak85]

We will give a proof for the simplest PHP version following the exposition in [Pud00]

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