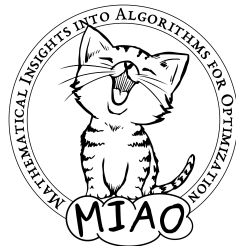


A One-Size-Fits-All Proof Logging System?

Jakob Nordström

University of Copenhagen and Lund University

1st International Workshop on Organizing
and Optimizing Proof-logging Systems
May 23–24, 2024



Based on joint work with Jeremias Berg, Bart Bogaerts, Jan Elffers, Ambros Gleixner, Stephan Gocht, Alexander Hoen, Hannes Ihalainen, Matti Järvisalo, Ciaran McCreesh, Matthew McIlree, Magnus O. Myreen, Andy Oertel, Yong Kiam Tan, and Dieter Vandersande

Combinatorial Solving and Optimization

- Astounding progress last couple of decades on **combinatorial solvers** for, e.g.:
 - Boolean satisfiability (SAT) solving and optimization [BHvMW21]
 - Constraint programming [RvBW06]
 - Mixed integer linear programming [AW13, BR07]
 - Satisfiability modulo theories (SMT) solving [BHvMW21]
- Solvers very fast, but **sometimes wrong** (even best commercial ones) [BLB10, CKSW13, AGJ⁺18, GSD19, GS19, BMN22, BBN⁺23]
- Even get feasibility of solutions wrong (though this should be straightforward!)
- And how to check the absence of solutions?
- Or that a solution is optimal? (Even off-by-one mistakes can snowball into large errors if solver used as subroutine)

What Can Be Done About Solver Bugs?

- **Software testing**

Hard to get good test coverage for sophisticated solvers

Progress using fuzzing and delta debugging [BB09, BLB10, KB22, NPB22, PB23]

But inherently can only detect presence of bugs, not absence

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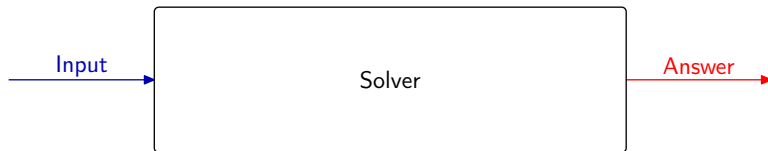
Current techniques cannot scale to level of complexity in modern solvers

- **Proof logging**

Make solver **certifying** [ABM⁺11, MMNS11] by adding code so that it outputs

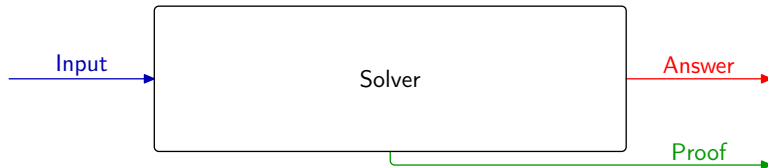
- ① not only **answer** but also
- ② simple, machine-verifiable **proof** that answer is correct

Proof Logging with Certifying Solvers: Workflow



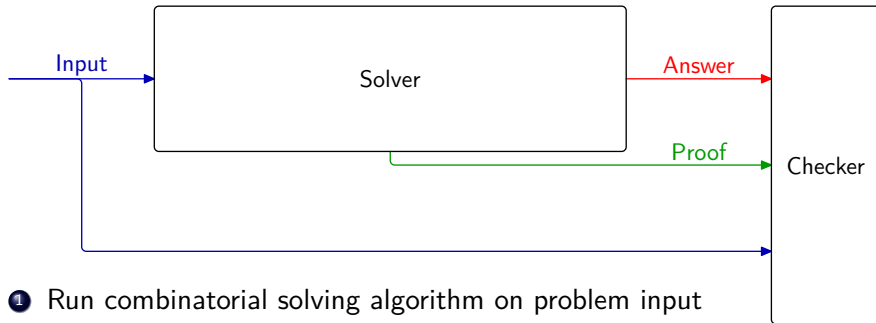
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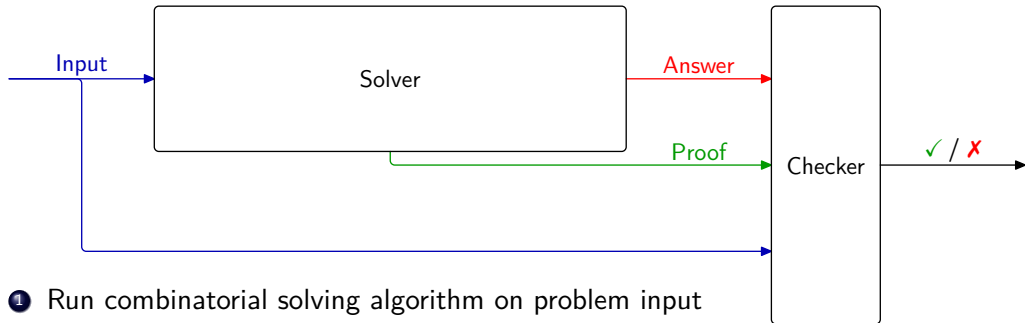
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Proof Logging with Certifying Solvers: Workflow



- ① Run combinatorial solving algorithm on problem input
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- ③ Feed input + answer + proof to proof checker

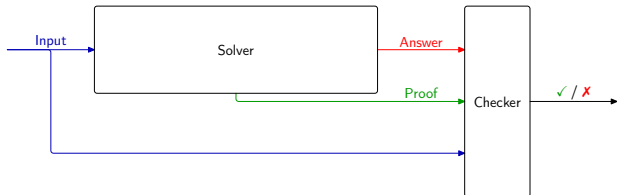
Proof Logging with Certifying Solvers: Workflow



- ① Run combinatorial solving algorithm on problem input
- ② Get as output not only answer but also proof
- ③ Feed input + answer + proof to proof checker
- ④ Verify that proof checker says answer is correct

Proof Logging Desiderata

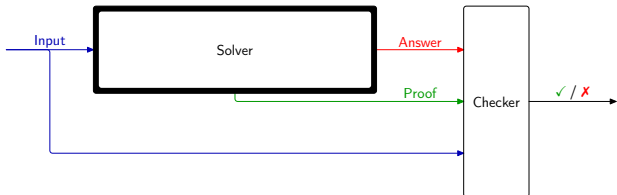
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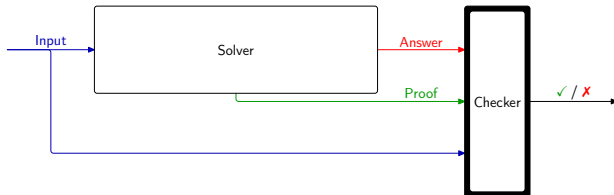
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Proof Logging Desiderata

Proof format for certifying solver should be

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- **dead simple:** checking correctness of proofs should be (almost) trivial

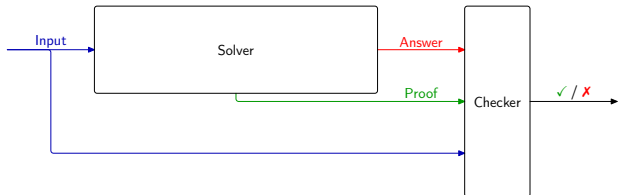


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Clear conflict expressivity vs. simplicity!



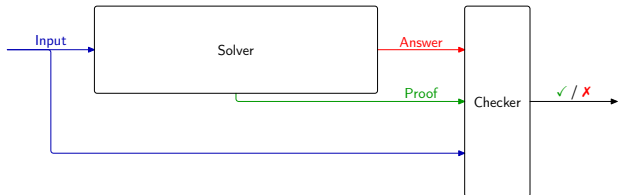
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Asking for both perhaps a little bit too good to be true?



Message of This Talk (and This Workshop)

Proof logging for combinatorial optimization is possible with **single, unified method!**

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- Build on successes in SAT solving with proof formats such as DRAT [HHW13a, HHW13b, WHH14], GRIT [CMS17], LRAT [CHH⁺17], ...
- But represent constraints as **0–1 integer linear inequalities**
- Formalize reasoning using **cutting planes** [CCT87] proof system
- Add well-chosen **strengthening rules** [Goc22, GN21, BGMN23]
- Implemented in **VERiPB** (<https://gitlab.com/MIA0research/software/VeriPB>)

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Purpose of this talk:

- ① Marketing pitch ☺
- ② Overview of proof system behind VERIPB
- ③ Sample of applications and future challenges

Outline of This Talk

- 1 Proof Logging Principles
 - Pseudo-Boolean Basics
 - Proof Logging Goals
 - Workflow
- 2 Proof System
 - Cutting Planes Proof System and VERIPB Proof Format
 - Strengthening Rules and Deletion
 - Proofs for Decision and Optimization Problems
- 3 Example Applications and Future Directions
 - Advanced SAT Solving Techniques
 - Subgraph Isomorphism Solving
 - Further Challenges

The Sales Pitch For Proof Logging

- ① Certifies correctness of computed results
- ② Detects errors even if due to compiler bugs, hardware failures, or cosmic rays
- ③ Provides debugging support during software development
[EG21, GMM⁺20, KM21, BBN⁺23]
- ④ Facilitates performance analysis
- ⑤ Helps identify potential for further improvements
- ⑥ Enables auditability
- ⑦ Serves as stepping stone towards explainability

Proof Language: Pseudo-Boolean Constraints

Proof consists of **0-1 integer linear inequalities** or **pseudo-Boolean constraints**:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- **literals** ℓ_i : x_i or \bar{x}_i (where $x_i + \bar{x}_i = 1$)
- variables x_i take values **0 = false** or **1 = true**

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Sometimes convenient to use **normalized form** [Bar95] with **all a_i, A positive** (without loss of generality)

Some Types of Pseudo-Boolean Constraints

① Clauses

$$x \vee \bar{y} \vee z \quad \Leftrightarrow \quad x + \bar{y} + z \geq 1$$

② Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

③ General pseudo-Boolean constraints

$$x_1 + 2\bar{x}_2 + 3x_3 + 4\bar{x}_4 + 5x_5 \geq 7$$

Pseudo-Boolean Proof Logging Wishlist

Paradigms

- SAT solving
- (linear) pseudo-Boolean solving
- graph solving
- constraint programming
- automated planning
- mixed integer linear programming
- SMT solving

Problem types

- decision / feasibility
- optimization
- multi-objective optimization
- projected model enumeration
- projected model counting
- preprocessing / problem reformulation

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Design Principles for Proof Logging

Proof logging implementation

- Don't change solver
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Proof system

- Keep proof language maximally simple
- Reason about XOR constraints, CP propagators, symmetries, etc within language
- Combine proof logging with formally verified proof checker

Pseudo-Boolean Proof Logging — How and Why?

If problem is (special case of) 0-1 integer linear program

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- ① 0-1 ILP **expressive formalism** for combinatorial problems (including objective)
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- ③ Efficient **reification** of constraints

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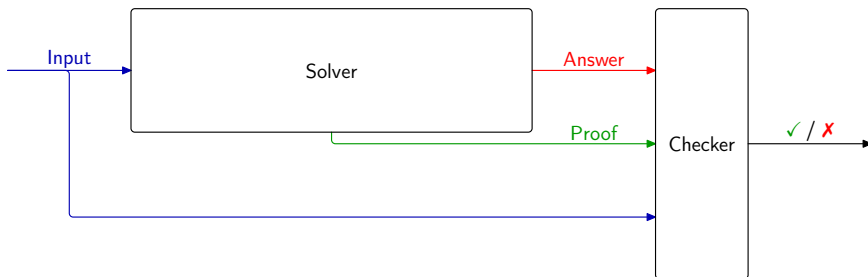
$$r \Rightarrow x_1 + 2\bar{x}_2 + 3x_3 + 4\bar{x}_4 + 5x_5 \geq 7$$

$$7\bar{r} + x_1 + 2\bar{x}_2 + 3x_3 + 4\bar{x}_4 + 5x_5 \geq 7$$

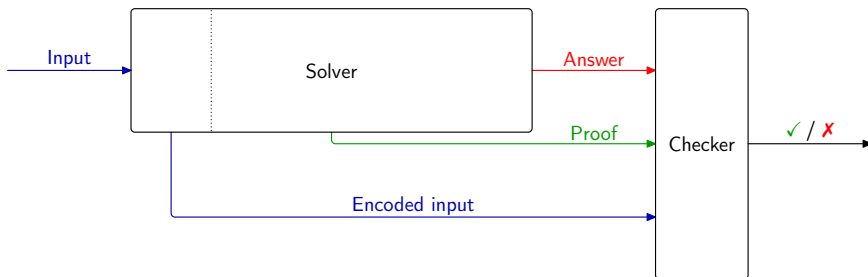
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$$9r + \bar{x}_1 + 2x_2 + 3\bar{x}_3 + 4x_4 + 5\bar{x}_5 \geq 9$$

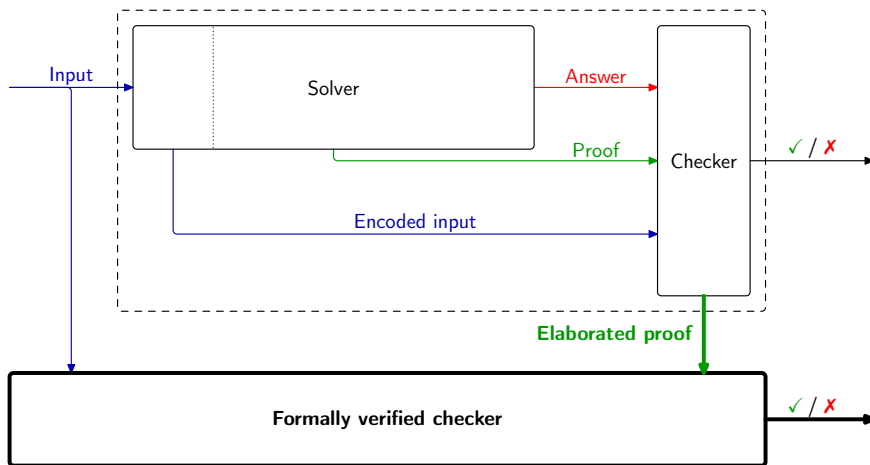
Proof Logging with Formally Verified Checking: Full Workflow



Proof Logging with Formally Verified Checking: Full Workflow



Proof Logging with Formally Verified Checking: Full Workflow



VERIPB Proof Structure

❶ Preamble

Load input formula
Specify settings

❷ Derivation section

Derivations of new constraints
Logging of solutions

❸ Output section

Listing of constraints currently in database
Input to next stage (or for debugging)

❹ Conclusions section

Specification of what was established

- satisfiability / unsatisfiability
- optimality (or upper and lower bounds)
- other types of conclusions to be added

VERIPB Proof Configuration

Core set \mathcal{C}

- Contains input formula at the start
- Maintains “equivalence” with input formula

Derived set \mathcal{D}

- All constraints derived during search
- Also intermediate constraints used in proof logging [but not used by solver]

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Objective $f = \sum_i w_i l_i + k$

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Order \mathcal{O}

- Pseudo-Boolean formula encoding pre-order (reflexive and transitive)
- Syntactic proof of properties required
- Applied to specified variable set \mathcal{Z}

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input axioms

From the input

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input axioms

Literal axioms

From the input

$$\overline{l_i \geq 0}$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input axioms

Literal axioms

Addition

From the input

$$\frac{\overline{\ell_i \geq 0} \quad \sum_i a_i \ell_i \geq A \quad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i) \ell_i \geq A + B}$$

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Multiplication for any $c \in \mathbb{N}^+$

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$$\frac{\sum_i a_i l_i \geq A}{\sum_i c a_i l_i \geq cA}$$

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Multiplication for any $c \in \mathbb{N}^+$

Division for any $c \in \mathbb{N}^+$
(constraint in normalized form)

From the input

$$\begin{array}{c}
 \overline{\ell_i \geq 0} \\
 \hline
 \sum_i a_i \ell_i \geq A \quad \sum_i b_i \ell_i \geq B \\
 \hline
 \sum_i (a_i + b_i) \ell_i \geq A + B \\
 \\
 \frac{\sum_i a_i \ell_i \geq A}{\sum_i c a_i \ell_i \geq c A} \\
 \\
 \frac{\sum_i a_i \ell_i \geq A}{\sum_i \lceil \frac{a_i}{c} \rceil \ell_i \geq \lceil \frac{A}{c} \rceil}
 \end{array}$$

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Literal axioms

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Division for any $c \in \mathbb{N}^+$
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Saturation
(constraint in normalized form)

From the input

$$\begin{array}{c}
 \overline{\ell_i \geq 0} \\
 \hline
 \frac{\sum_i a_i \ell_i \geq A \quad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i) \ell_i \geq A + B} \\
 \hline
 \frac{\sum_i a_i \ell_i \geq A}{\sum_i c a_i \ell_i \geq c A} \\
 \hline
 \frac{\sum_i a_i \ell_i \geq A}{\sum_i \lceil \frac{a_i}{c} \rceil \ell_i \geq \lceil \frac{A}{c} \rceil} \\
 \hline
 \frac{\sum_i a_i \ell_i \geq A}{\sum_i \min(a_i, A) \cdot \ell_i \geq A}
 \end{array}$$

Cutting Planes Toy Example

$$w + 2x + y \geq 2$$

Cutting Planes Toy Example

Multiply by 2 $\frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$

Cutting Planes Toy Example

Multiply by 2 $\frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \quad w + 2x + 4y + 2z \geq 5$

Cutting Planes Toy Example

$$\begin{array}{l} \text{Multiply by 2} \quad \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \\ \text{Add} \quad \frac{2w + 4x + 2y \geq 4 \quad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} \end{array}$$

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 \text{Add} & \frac{3w + 6x + 6y}{3w + 6x + 6y + 2z} \geq 7 & \\
 \text{Divide by 3} & \frac{3w + 6x + 6y}{w + 2x + 2y} \geq 7 & \\
 & w + 2x + 2y \geq 2\frac{1}{3} &
 \end{array}$$

Cutting Planes Toy Example

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 \text{Divide by 3} & \frac{3w + 6x + 6y}{w + 2x + 2y} & \geq 3
 \end{array}$$

By naming constraints by integers and literal axioms by the literal involved as

$$\text{Constraint 1} \doteq 2x + y + w \geq 2$$

$$\text{Constraint 2} \doteq 2x + 4y + 2z + w \geq 5$$

$$\sim z \doteq \bar{z} \geq 0$$

Cutting Planes Toy Example

$$\begin{array}{rcl}
 \text{Multiply by 2} & \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} & \\
 \text{Add} & \frac{w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} & \frac{\bar{z} \geq 0}{2\bar{z} \geq 0} \quad \text{Multiply by 2} \\
 \text{Add} & \frac{3w + 6x + 6y + 2z \geq 9}{3w + 6x + 6y} & \\
 \text{Divide by 3} & \frac{3w + 6x + 6y}{w + 2x + 2y} \geq 7 & \\
 & w + 2x + 2y \geq 3 &
 \end{array}$$

By naming constraints by integers and literal axioms by the literal involved as

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such a calculation is written in the proof log in reverse Polish notation as

pol 1 2 * 2 + ~z 2 * + 3 d

More About VERIPB Proofs

Variables

- start with a letter in A-Z or a-z
- continue with characters in A-Z, a-z, 0-9, or `[]{}-_^`
(square and curly brackets, hyphen, underscore, and caret)
- contain at least two characters

Constraints

Are referred to by positive integers (constraint IDs)

Derivation rules and requirements

Come in two flavours

- ① **kernel format** for formally verified proof checker
- ② **augmented format** with convenience rules such as **reverse unit propagation (RUP)**

Open Problem: Division Versus Saturation

$$\text{Division} \frac{\sum_i a_i \ell_i \geq A}{\sum_i \lceil \frac{a_i}{c} \rceil \ell_i \geq \lceil \frac{A}{c} \rceil}$$

$$\text{Saturation} \frac{\sum_i a_i \ell_i \geq A}{\sum_i \min(a_i, A) \cdot \ell_i \geq A}$$

How do division and saturation rules compare?

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How do division and saturation rules compare?

- Strengths of rules as such incomparable [GNY19]
- Cutting planes with division can be exponentially stronger than cutting planes with saturation
- Unknown whether cutting planes with saturation can be stronger than cutting planes with division

Redundance-Based Strengthening

C is **redundant** with respect to F if F and $F \cup \{C\}$ are **equisatisfiable**

Want to allow adding such “redundant” constraints

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C is redundant with respect to F if and only if there is a **substitution** ω (mapping variables to truth values or literals), called a **witness**, for which

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- Proof sketch for interesting direction: If α satisfies F but falsifies C , then $\alpha \circ \omega$ satisfies $F \cup \{C\}$
- In a proof, the implication needs to be **efficiently verifiable** — every $D \in (F \cup \{C\})|_{\omega}$ should follow from $F \cup \{\neg C\}$ either
 - ① “obviously” (e.g., by so-called weakening or unit propagation) or
 - ② by explicitly presented derivation

Example: Deriving $a \leftrightarrow (x \wedge y)$ Using the Redundance Rule

Want to derive

$$2\bar{a} + x + y \geq 2$$

$$a + \bar{x} + \bar{y} \geq 1$$

using condition $F \cup \{\neg C\} \models (F \cup \{C\}) \upharpoonright_{\omega}$

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Choose $\omega = \{a \mapsto 1\}$ — F untouched; new constraint satisfied

$\neg(a + \bar{x} + \bar{y} \geq 1)$ forces $x \mapsto 1$ and $y \mapsto 1$, hence $2\bar{a} + x + y \geq 2$ remains satisfied after forcing a to be true

Redundance and Dominance Rules in VERIPB (Slightly Simplified)

Redundance-based strengthening, optimization version [BGMN23]

Add constraint C to derived set \mathcal{D} if exists witness substitution ω such that

$$C \cup \mathcal{D} \cup \{\neg C\} \models (C \cup \mathcal{D} \cup \{C\}) \upharpoonright_{\omega} \cup \{f \upharpoonright_{\omega} \leq f\}$$

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- Applying ω should **strictly decrease** f
- If so, don't need to show that $(\mathcal{D} \cup \{C\})|_{\omega}$ implied!

Soundness of Dominance Rule

Dominance-based strengthening

Add constraint C to derived set \mathcal{D} if exists witness substitution ω such that

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Why is this sound? Let $\mathcal{D} = \emptyset$ for simplicity

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- ⑦ ...
- ⑧ Can't go on forever, so finally reach α' satisfying $\mathcal{C} \cup \{C\}$

Soundness of Dominance Rule (Continued)

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- Same inductive proof as before, but also nested forward induction over derivation
- Or pick α satisfying $C \cup \mathcal{D}$ and minimizing f and argue by contradiction

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Further extensions:

- Define dominance rule with respect to order \mathcal{O} independent of objective function
- Switch between different orders in same proof
- See [BGMN23] for details

Strengthening Rules in Their (Almost) Full Formal Glory

Witness ω : substitution mapping variables to truth values or literals

Redundance-based strengthening (witness ω show how to “patch assignment”)

Derive constraint C from $\mathcal{C} \cup \mathcal{D}$ if exists witness ω such that

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Dominance-based strengthening (witness ω “drives down potential”)

Derive constraint C from $\mathcal{C} \cup \mathcal{D}$ if exists witness ω such that

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Strengthening Rules: Proof Format

```
red  $\langle \text{Constraint } C \rangle$  ;  $\langle var1 \rangle \rightarrow \langle val1 \rangle \dots \langle varN \rangle \rightarrow \langle valN \rangle$  ; begin  
  subproofs for proof goals  
end
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- Witness ω should be explicitly specified in proof log
- Subproofs of proof goals should also be explicit
- But can be skipped for proof goals “obvious” to proof checker (like by weakening or unit propagation)

The Problem of Deleting Constraints

Important to allow deletions of constraints from database

- Improves practical performance
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- Satisfiable formulas can turn unsatisfiable(!)

Solution: distinguish between deletion from core set \mathcal{C} and derived set \mathcal{D}

Deletion, Core Transfer, and Order Change

Deletion

- ① Deletion of constraint C **always OK** from derived set \mathcal{D}
- ② **OK from core set \mathcal{C} only if C can be rederived from $\mathcal{C} \setminus \{C\}$** with redundancy rule (otherwise **unchecked deletion** — special conditions apply)

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Constraints from \mathcal{D} can be moved to \mathcal{C}

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Change of order

Possible to change order only if $\mathcal{D} = \emptyset$

Conclusions for Decision Problems

NONE

Status is undetermined

SAT [: $\langle assignment \rangle$]

Propagate given assignment w.r.t. database, then check against original formula

If no assignment given, then

- solution should have been logged
- no unchecked deletion must have occurred

UNSAT [: $\langle constraint ID \rangle$]

Only valid if no solution has been logged

Check that specified constraint is contradictory (technically: negative slack)

If no constraint given, check that database unit propagates to contradiction

Optimization Problems

Any solution α found is logged with `sol` “log solution and improve” command

- provided solution α checked against current core set \mathcal{C}
- **Objective-improving constraint** $\sum_i w_i \ell_i \leq -1 + \sum_i w_i \cdot \alpha(\ell_i)$ added to core set (forces search for better solutions)

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Proof of optimality: Contradiction derived from objective-improving constraint

Proof format supports not just optimality, but also non-tight upper and lower bounds

Conclusions for Optimization Problems

NONE

No solution or lower bound found

BOUNDS $\langle LB \rangle$ [: $\langle constraint\ ID \rangle$] $\langle UB \rangle$ [: $\langle assignment \rangle$]

$\langle LB \rangle$ and $\langle UB \rangle$ are integers or `inf`; optimality if $\langle LB \rangle = \langle UB \rangle$

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Constraint $\langle constraint\ ID \rangle$, if specified, should imply lower bound

Otherwise, $f \geq \langle LB \rangle$ should be “obvious” to proof checker from current database

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Otherwise, $f \geq \langle LB \rangle$ should be “obvious” to proof checker from current database

Upper bound

Propagate given assignment w.r.t. database, then check against original formula

If no assignment given, then

- solution with value $\langle UB \rangle$ should have been logged
- no unchecked deletion must have occurred

Parity (XOR) Reasoning in SAT Solving

Given clauses

$$x \vee y \vee z$$

$$x \vee \bar{y} \vee \bar{z}$$

$$\bar{x} \vee y \vee \bar{z}$$

$$\bar{x} \vee \bar{y} \vee z$$

and

$$y \vee z \vee w$$

$$y \vee \bar{z} \vee \bar{w}$$

$$\bar{y} \vee z \vee \bar{w}$$

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want to derive

$$x \vee \bar{w}$$

$$\bar{x} \vee w$$

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want to derive

$$x \vee \bar{w}$$

$$\bar{x} \vee w$$

This is just parity reasoning:

Parity (XOR) Reasoning in SAT Solving

Given clauses

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But used in, e.g., CRYPTOMINISAT [Cry]

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But used in, e.g., CRYPTOMINISAT [Cry]

DRAT proof logging like [PR16] too inefficient in practice!

Could add XORs to language, but prefer to keep things super-simple

Pseudo-Boolean Proof Logging for XOR Reasoning

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Use redundance rule with fresh variables a, b to derive

$$x + y + z + 2a = 3$$

$$y + z + w + 2b = 3$$

(“=” syntactic sugar for “ \geq ” plus “ \leq ”)

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From this can extract

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VERIPB can certify **XOR reasoning** [GN21]

Symmetry Breaking in SAT Solving

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- ❸ Derive **symmetry breaking clauses** from this PB constraint:

y_0	$\bar{y}_j \vee \overline{\sigma(x_j)} \vee x_j$
$\bar{y}_{j-1} \vee \bar{x}_j \vee \sigma(x_j)$	$y_j \vee \bar{y}_{j-1} \vee \bar{x}_j$
$\bar{y}_j \vee y_{j-1}$	$y_j \vee \bar{y}_{j-1} \vee \sigma(x_j)$

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$$\begin{array}{ll}
 y_0 \geq 1 & \bar{y}_j + \overline{\sigma(x_j)} + x_j \geq 1 \\
 \bar{y}_{j-1} + \bar{x}_j + \sigma(x_j) \geq 1 & y_j + \bar{y}_{j-1} + \bar{x}_j \geq 1 \\
 \bar{y}_j + y_{j-1} \geq 1 & y_j + \bar{y}_{j-1} + \sigma(x_j) \geq 1
 \end{array}$$

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VERIPB can certify fully general **SAT symmetry breaking** [BGMN23]

Open Problem: Symmetry Breaking with Redundance Rule?

Is the dominance rule really needed for fully general symmetry breaking?

Or could the redundance rule be enough?

Weaker DRAT strengthening rule sufficient for “pigeonhole-style” symmetries [HHW15]

Open Problem: Efficient Substitution Proofs?

Can cutting planes with redundance and dominance support **proofs with lemmas/substitution** efficiently?

Special case: **symmetric learning** in SAT solving [DBB17]

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Can cutting planes with redundance and dominance support **proofs with lemmas/substitution** efficiently?

Special case: **symmetric learning** in SAT solving [DBB17]

Can be done in principle, but seems very finicky. . .

Extension and substitution proof systems don't mix well

The Subgraph Isomorphism Problem

Input

- **Pattern** graph \mathcal{P} with vertices $V(\mathcal{P}) = \{a, b, c, \dots\}$
- **Target** graph \mathcal{T} with vertices $V(\mathcal{T}) = \{u, v, w, \dots\}$

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Task

- Find all **subgraph isomorphisms** $\varphi : V(\mathcal{P}) \rightarrow V(\mathcal{T})$
- I.e., if
 - ① $\varphi(a) = u$
 - ② $\varphi(b) = v$
 - ③ $(a, b) \in E(\mathcal{P})$

then must have $(u, v) \in E(\mathcal{T})$

Pseudo-Boolean Proof Logging for Subgraph Isomorphism Solving

All reasoning steps in Glasgow Subgraph Solver [ADH⁺19, GSS] can be formalized efficiently in the cutting planes proof system [GMN20]

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Means that

- ① Solver can justify each step by writing local formal derivation
- ② Local derivations can be chained into global correctness proof
- ③ Proof checkable by stand-alone verifier that knows nothing about graphs

Subgraph Isomorphism as a Pseudo-Boolean Formula

- **Pattern** graph \mathcal{P} with $V(\mathcal{P}) = \{a, b, c, \dots\}$
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- No loops (for simplicity)

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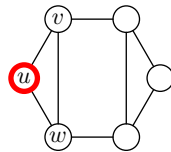
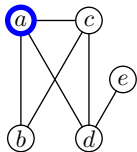
Pseudo-Boolean encoding

$$\sum_{v \in V(\mathcal{T})} x_{a \mapsto v} = 1 \quad [\text{every } a \text{ maps somewhere}]$$

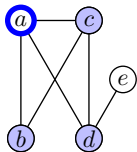
$$\sum_{b \in V(\mathcal{P})} \bar{x}_{b \mapsto u} \geq |V(\mathcal{P})| - 1 \quad [\text{mapping is one-to-one}]$$

$$\bar{x}_{a \mapsto u} + \sum_{v \in N(u)} x_{b \mapsto v} \geq 1 \quad [\text{edge } (a, b) \text{ maps to edge } (u, v)]$$

Pseudo-Boolean Proof Logging Example: Degree Preprocessing



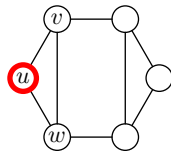
Pseudo-Boolean Proof Logging Example: Degree Preprocessing



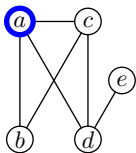
$$\bar{x}_{a \mapsto u} + x_{b \mapsto v} + x_{b \mapsto w} \geq 1$$

$$\bar{x}_{a \mapsto u} + x_{c \mapsto v} + x_{c \mapsto w} \geq 1$$

$$\bar{x}_{a \mapsto u} + x_{d \mapsto v} + x_{d \mapsto w} \geq 1$$



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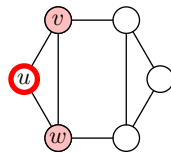
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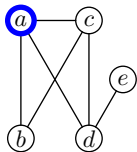
$$\bar{x}_{a \mapsto u} + x_{d \mapsto v} + x_{d \mapsto w} \geq 1$$

$$\bar{x}_{a \mapsto v} + \bar{x}_{b \mapsto v} + \bar{x}_{c \mapsto v} + \bar{x}_{d \mapsto v} + \bar{x}_{e \mapsto v} \geq 4$$

$$\bar{x}_{a \mapsto w} + \bar{x}_{b \mapsto w} + \bar{x}_{c \mapsto w} + \bar{x}_{d \mapsto w} + \bar{x}_{e \mapsto w} \geq 4$$



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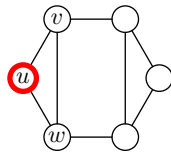
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$$x_{a \mapsto v} \geq 0$$

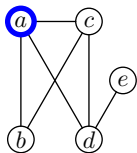
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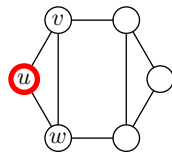
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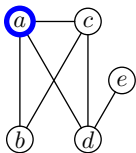
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Sum up all constraints & divide by 3 to obtain

Pseudo-Boolean Proof Logging Example: Degree Preprocessing



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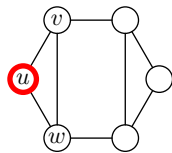
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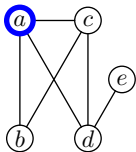
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Sum up all constraints & divide by 3 to obtain

$$3\bar{x}_{a \mapsto u} + 10 \geq 11$$

Pseudo-Boolean Proof Logging Example: Degree Preprocessing



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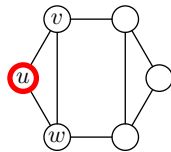
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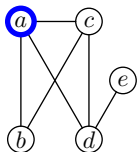
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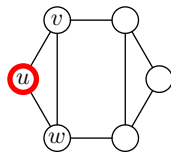
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Future Research Directions

Performance and reliability of pseudo-Boolean proof logging

- Trim proof while verifying (as in DRAT-TRIM [HHW13a])
- Compress proof file using binary format
- Design formally verified proof checker (*work in progress* [BMM⁺23])

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Proof logging for other combinatorial problems and techniques

- Model enumeration and counting
- Mixed integer linear programming (*work on SCIP in* [CGS17, EG21, DEGH23])
- Satisfiability modulo theories (SMT) solving (*work on* cvc5, Z3, ... [BBC⁺23])

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- Compress proof file using binary format
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Proof logging for other combinatorial problems and techniques

- Model enumeration and counting
- Mixed integer linear programming (*work on SCIP in* [CGS17, EG21, DEGH23])
- Satisfiability modulo theories (SMT) solving (*work on* cvc5, Z3, ... [BBC⁺23])

And more...

- Use proof logs for algorithm analysis or explainability purposes
- Lots of other challenging problems and interesting ideas

Future Research Directions

Performance and reliability of pseudo-Boolean proof logging

- Trim proof while verifying (as in DRAT-TRIM [HHW13a])
- Compress proof file using binary format
- Design formally verified proof checker (*work in progress* [BMM⁺23])

Proof logging for other combinatorial problems and techniques

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And more...

- Use proof logs for algorithm analysis or explainability purposes
- Lots of other challenging problems and interesting ideas
- **We're hiring!** Talk to me to join the pseudo-Boolean proof logging revolution! 😊

VERIPB Documentation

VERIPB tutorial at *CP* '22 [BMN22]

- video at youtu.be/s_5BIi4I22w
- updated slides for *IJCAI* '23 tutorial [BMN23]



Description of VERIPB and CAKEPB [BMM⁺23] for SAT 2023 competition

- Available at satcompetition.github.io/2023/checkers.html

Specific details on different proof logging techniques covered in research papers [EGMN20, GMN20, GMM⁺20, GN21, GMN22, GMNO22, VDB22, BBN⁺23, BGMN23, MM23, GMM⁺24, HOGN24, IOT⁺24, MMN24]

Lots of concrete example files at gitlab.com/MIA0research/software/VeriPB

Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity
- **Action point:** What problems can VERIPB solve for you? 😊



Summing up

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Thank you for your attention!



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