Certified Implicit Hitting Set Solving for Pseudo-Boolean Optimization

Benjamin Bogø   Xiamin Chen   Wietze Koops   Pinyan Lu   **Jakob Nordström**
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Dagstuhl Workshop 25371
Interactions in Constraint Optimization
September 11, 2025

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(*) *Thanks for the slides!*

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- This talk: 0-1 linear objective and inequalities (pseudo-Boolean in SAT-speak)
- IHS solving: Benders decomposition in OR-speak

Certified Solving using Proof Logging

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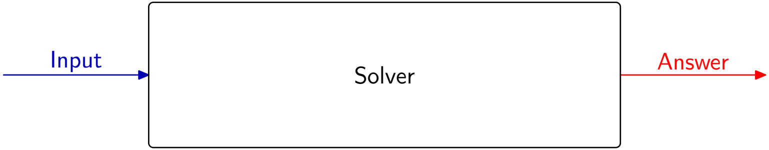
Certified Solving using Proof Logging

- Modern combinatorial solvers very fast, but sometimes wrong [BLB10, AGJ+18, GSD19]
- Only currently feasible way of addressing this: Proof logging
  - Make solver certifying [ABM+11, MMNS11] by adding code so that it outputs
  - not only answer but also
  - simple, machine-verifiable proof that answer is correct

Proof Logging with Certifying Solvers: Workflow



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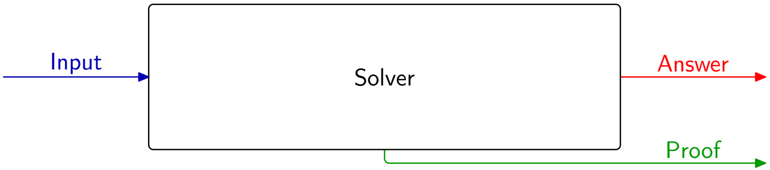

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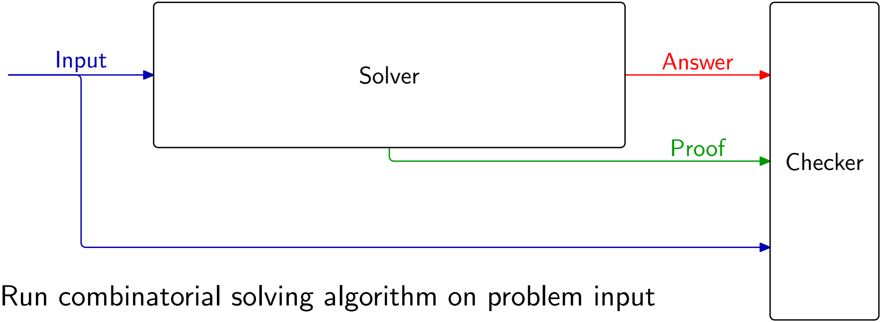

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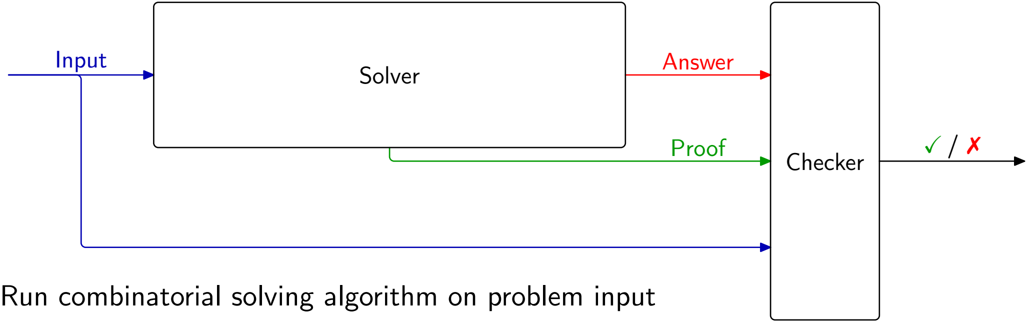

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3. Feed input + answer + proof to proof checker
4. Verify that proof checker says answer is correct

IHS Proof Logging

- Proof logging implemented for state-of-the-art solvers for other optimization paradigms
  - Solution-improving search [BBN+24]
  - Core-guided search [VDB22, BBN+23]

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- Successful IHS implementations for both MaxSAT [DB11] and pseudo-Boolean optimization [SBJ21, SBJ22], but so far no proof logging for IHS
  - Mixed integer programming (MIP) solver used for IHS problem
  - Closed source — cannot add proof logging to code
  - Also, not known how to do proof logging for all MIP solving techniques

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- Possible approaches to get certified IHS solving:
  1. Use pseudo-Boolean solver with proof logging for IHS problem
  2. Use local search to find solutions for IHS problem
  3. Find optimal solution with MIP, then let other certifying solver prove lower bound

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- Study if and why MIP technique crucial for implicit hitting set solving
- Compare pros and cons from point of view certified solving
- Explore ways of integrating IHS in "hybrid methods" using also other optimization paradigms (cf. [DGD+21, DGN21])

Pseudo-Boolean Optimization (PBO) Problem

- Pseudo-Boolean formula $\mathcal{F}$: collection of 0-1 integer linear inequalities

Example

$$
\begin{aligned}
x_1 + x_2 + 2\,\overline{x_4} &\geq 2 \\
x_1 + 2\,x_3 + \overline{x_5} &\geq 2 \\
x_4 + x_5 &\geq 1
\end{aligned}
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- 0-1 linear objective function $\mathcal{O}$ to minimize

Example

$$\text{min: } x_1 + x_2 + 3\,x_3$$

Implicit Hitting Set Solving in More Detail

- Split PBO problem $(\mathcal{F}, \mathcal{O})$ into two subproblems

PBO formula

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\text{min: } x_1 + x_2 + 3\,x_3 \\
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- Split PBO problem $(\mathcal{F}, \mathcal{O})$ into two subproblems
  - Decision subproblem $\mathcal{F}$ (all constraints)

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Decision subproblem

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Implicit Hitting Set Solving in More Detail

- Split PBO problem $(\mathcal{F}, \mathcal{O})$ into two subproblems
  - Decision subproblem $\mathcal{F}$ (all constraints)
  - IHS subproblem (core constraints over objective variables only)

PBO formula

$$\begin{aligned} \text{min: } & x_1 + x_2 + 3\,x_3 \\ x_1 + x_2 + 2\,\overline{x_4} &\geq 2 \\ x_1 + 2\,x_3 + \overline{x_5} &\geq 2 \\ x_4 + x_5 &\geq 1 \end{aligned}$$

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IHS subproblem

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(core constraints over x_1, x_2, x_3)

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Example

$$\begin{aligned} \{\overline{x_1}, \overline{x_2}, \overline{x_3}\} &\rightarrow x_1 + x_3 \geq 1 \\ \{x_1, \overline{x_2}, \overline{x_3}\} &\rightarrow x_2 + x_3 \geq 1 \\ \{x_1, x_2, \overline{x_3}\} & \end{aligned}$$

Decision subproblem

$$\begin{aligned} x_1 + x_2 + 2\,\overline{x_4} \geq 2 \\ x_1 + 2\,x_3 + \overline{x_5} \geq 2 \\ x_4 + x_5 \geq 1 \end{aligned}$$

Implicit Hitting Set Solving

1. Find optimal solution α to current core constraints
2. Try to extend α to solution for all constraints in $\mathcal{F}$
   a. Solution extended: Optimum found!
   b. Otherwise: Extract new core & go to 1

IHS subproblem

$$\text{min: } x_1 + x_2 + 3\,x_3$$
$$x_1 + x_3 \geq 1$$
$$x_2 + x_3 \geq 1$$

Example

$$\{\overline{x_1}, \overline{x_2}, \overline{x_3}\} \rightarrow x_1 + x_3 \geq 1$$
$$\{x_1, \overline{x_2}, \overline{x_3}\} \rightarrow x_2 + x_3 \geq 1$$
$$\{x_1, x_2, \overline{x_3}\} \rightarrow \{x_1, x_2, \overline{x_3}, x_4, \overline{x_5}\}$$

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Proof Logging for IHS Solving in More Detail

- Reasoning for decision subproblem
  - Conflict-driven search — use pseudo-Boolean proof logging [KLBM+25]
  - Core extraction — just special case of conflict analysis
    (so-called decision learning scheme)

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  - Conflict-driven search — use pseudo-Boolean proof logging [KLBM+25]
  - Core extraction — just special case of conflict analysis (so-called decision learning scheme)
- Reasoning for IHS subproblem
  - More challenging
  - Incremental problem — new core constraints keep getting added

Proof Logging for Implicit Hitting Set Subproblem

- Optimization solvers use found solutions to trim search space
  - Infer new constraints from requirement to improve solution further
  - Solution with value $v \Rightarrow$ add objective-improving constraint $\mathcal{O} \leq v - 1$

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  3. Automatic book-keeping via reified constraints

Book-keeping for Invalidated Constraints

IHS subproblem

$$\text{min: } x_1 + x_2 + 3\,x_3$$

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$$x_1 + x_2 + 3\,x_3 \leq 4 \qquad \text{Solution: } 5 \qquad (1)$$

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$$x_1 + x_3 \geq 1 \qquad \text{Add core constraint} \qquad (3)$$

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$x_1 + x_2 + 3\,x_3 \leq 4$	Solution: 5	(1)
$x_1 + x_2 + 3\,x_3 \leq -1$	Optimum: 0	(2)
$x_1 + x_3 \geq 1$	Add core constraint	(3)
$x_1 + x_2 + 3\,x_3 \leq 3$	Solution: 4	(4)

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$x_1 + x_2 + 3\,x_3 \leq 1$	Optimum: 2	(8)

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Book-keeping for Invalidated Constraints

IHS subproblem

$$\text{min: } x_1 + x_2 + 3\,x_3$$

$-\overline{s_5} + x_1 + x_2 + 3\,x_3 \leq 4$	Solution: 5	(1)
$x_1 + x_2 + 3\,x_3 \leq -1$	Optimum: 0	(2)
$x_1 + x_3 \geq 1$	Add core constraint	(3)
$x_1 + x_2 + 3\,x_3 \leq 3$	Solution: 4	(4)
$x_1 + \overline{x_2} \geq 1$	Infer by (3) and (4)	(5)
$x_1 + x_2 + 3\,x_3 \leq 0$	Optimum: 1	(6)
$x_2 + x_3 \geq 1$	Add core constraint	(7)
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$$
\begin{aligned}
-\,\overline{s_5} + x_1 + x_2 + 3\,x_3 &\leq 4 && \text{Solution: } 5 && (1)\\
-6\,\overline{s_0} + x_1 + x_2 + 3\,x_3 &\leq -1 && \text{Optimum: } 0 && (2)\\
x_1 + x_3 &\geq 1 && \text{Add core constraint} && (3)\\
-2\,\overline{s_4} + x_1 + x_2 + 3\,x_3 &\leq 3 && \text{Solution: } 4 && (4)\\
x_1 + \overline{x_2} &\geq 1 && \text{Infer by (3) and (4)} && (5)\\
-5\,\overline{s_1} + x_1 + x_2 + 3\,x_3 &\leq 0 && \text{Optimum: } 1 && (6)\\
x_2 + x_3 &\geq 1 && \text{Add core constraint} && (7)\\
-4\,\overline{s_2} + x_1 + x_2 + 3\,x_3 &\leq 1 && \text{Optimum: } 2 && (8)
\end{aligned}
$$

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$-5\,\overline{s_1} + x_1 + x_2 + 3\,x_3 \le 0$	Optimum: 1	(6)
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What About Performance?

- Work in progress — so far, so crappy...
- Book-keeping with reified objective-improving constraints involves serious challenges
- But the solver works!
- First certifying IHS solver with proofs that can be checked (somewhat) efficiently
- Submitted to standard and certified tracks of Pseudo-Boolean Competition 2025 [Pse25]
- Not great competition results, but not the worst solver either
  (which is a bit of a miracle given how many features are missing)

Limited Experimental Evaluation

Set-up:

- Benchmarks: Pseudo-Boolean Competition 2024 OPT-LIN optimization instances [Pse24]
- Memory: 16 GB
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  - RoundingSat both for decision subproblem and IHS subproblem (two different solvers)

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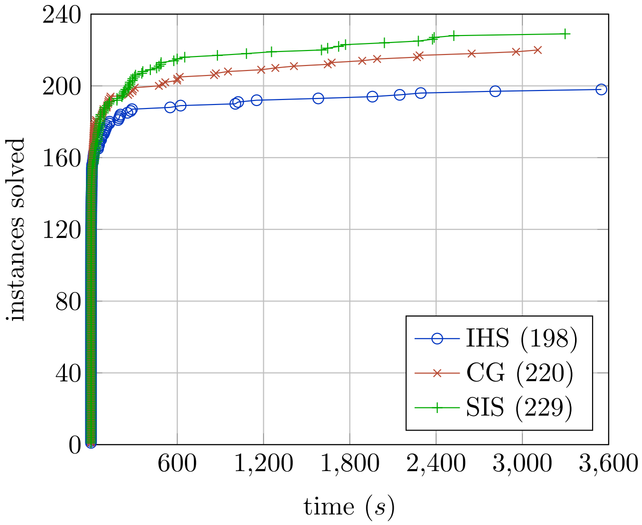
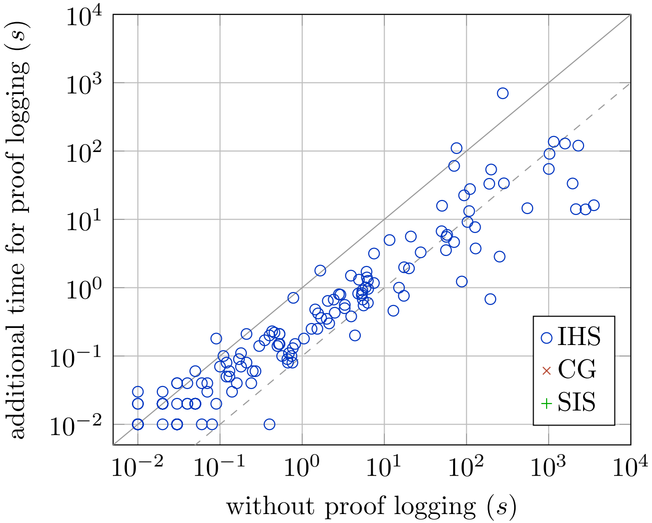
Evaluate

- Pure implicit hitting set (IHS) solving
  - RoundingSat both for decision subproblem and IHS subproblem (two different solvers)
- Compared to core-guided (CG) and solution-improving search (SIS) [KLBM+25]
  - Both as implemented in RoundingSat
  - ... Which uses LP solver SoPlex as important subroutine

Time vs Solved Instances



Solving Time vs Proof Logging Time



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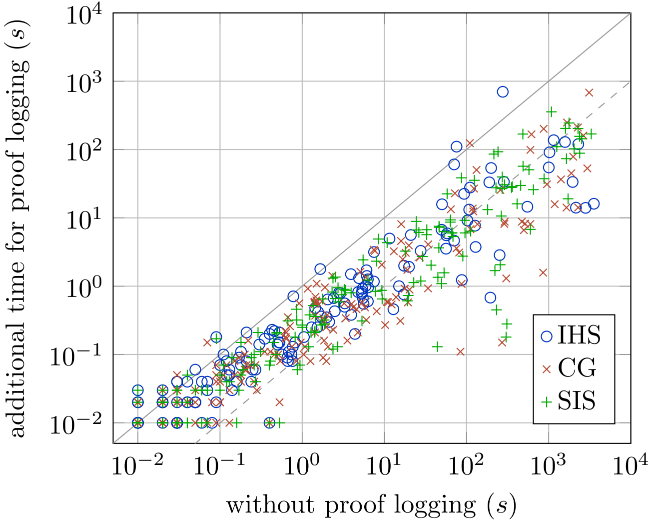
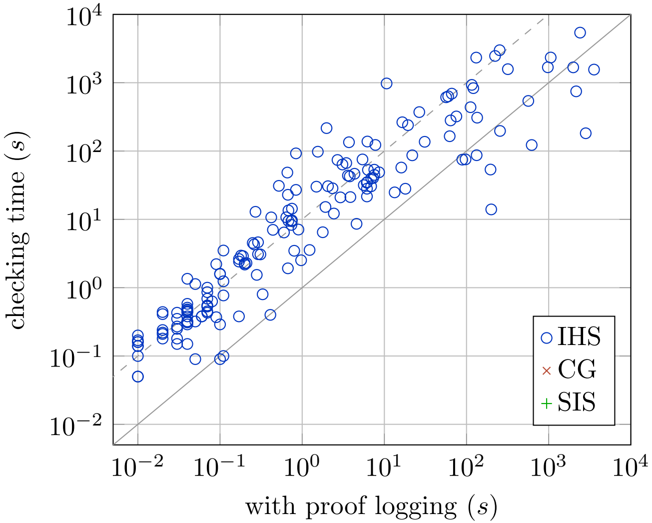


Solving and Proof Logging Time vs Checking Time



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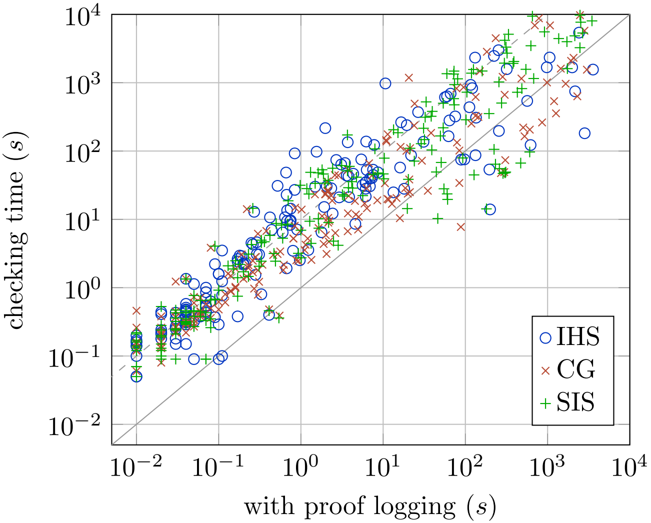

Future Work

- Pseudo-Boolean (PB) solving
  - More efficient book-keeping (with or without reified variables)

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- Investigate trade-offs between MIP usage and proof logging by comparing
  - MIP solver for IHS + PB decision solver generating proof for claimed optimal solution
  - PB IHS optimizer with book-keeping for objective-improving constraints

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- Ongoing and future work
  - Improve performance of book-keeping for objective-improving constraints
  - Evaluate also local search and independent proof generation for MIP claim
  - Make certified IHS solving competitive with other optimization approaches (by making it part of hybrid methods)

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Thank you for your attention!

References I

[ABM+11] Eyad Alkassar, Sascha Böhme, Kurt Mehlhorn, Christine Rizkallah, and Pascal Schweitzer. An introduction to certifying algorithms. *it - Information Technology Methoden und innovative Anwendungen der Informatik und Informationstechnik*, 53(6):287–293, December 2011.

[AGJ+18] Özgür Akgün, Ian P. Gent, Christopher Jefferson, Ian Miguel, and Peter Nightingale. Metamorphic testing of constraint solvers. In *Proceedings of the 24th International Conference on Principles and Practice of Constraint Programming (CP '18)*, volume 11008 of *Lecture Notes in Computer Science*, pages 727–736. Springer, August 2018.

[BBN+23] Jeremias Berg, Bart Bogaerts, Jakob Nordström, Andy Oertel, and Dieter Vandesande. Certified core-guided MaxSAT solving. In *Proceedings of the 29th International Conference on Automated Deduction (CADE-29)*, volume 14132 of *Lecture Notes in Computer Science*, pages 1–22. Springer, July 2023.

[BBN+24] Jeremias Berg, Bart Bogaerts, Jakob Nordström, Andy Oertel, Tobias Paxian, and Dieter Vandesande. Certifying without loss of generality reasoning in solution-improving maximum satisfiability. In *Proceedings of the 30th International Conference on Principles and Practice of Constraint Programming (CP '24)*, volume 307 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 4:1–4:28, September 2024.

[BLB10] Robert Brummayer, Florian Lonsing, and Armin Biere. Automated testing and debugging of SAT and QBF solvers. In *Proceedings of the 13th International Conference on Theory and Applications of Satisfiability Testing (SAT '10)*, volume 6175 of *Lecture Notes in Computer Science*, pages 44–57. Springer, July 2010.

References II

[DB11] Jessica Davies and Fahiem Bacchus. Solving MAXSAT by solving a sequence of simpler SAT instances. In *Proceedings of the 17th International Conference on Principles and Practice of Constraint Programming (CP '11)*, volume 6876 of *Lecture Notes in Computer Science*, pages 225–239. Springer, September 2011.

[DEGH23] Jasper van Doornmalen, Leon Eifler, Ambros Gleixner, and Christopher Hojny. A proof system for certifying symmetry and optimality reasoning in integer programming. Technical Report 2311.03877, arXiv.org, November 2023.

[DGD+21] Jo Devriendt, Stephan Gocht, Emir Demirović, Jakob Nordström, and Peter Stuckey. Cutting to the core of pseudo-Boolean optimization: Combining core-guided search with cutting planes reasoning. In *Proceedings of the 35th AAAI Conference on Artificial Intelligence (AAAI '21)*, pages 3750–3758, February 2021.

[DGN21] Jo Devriendt, Ambros Gleixner, and Jakob Nordström. Learn to relax: Integrating 0-1 integer linear programming with pseudo-Boolean conflict-driven search. *Constraints*, 26(1–4):26–55, October 2021. Preliminary version in *CPAIOR '20*.

[GSD19] Xavier Gillard, Pierre Schaus, and Yves Deville. SolverCheck: Declarative testing of constraints. In *Proceedings of the 25th International Conference on Principles and Practice of Constraint Programming (CP '19)*, volume 11802 of *Lecture Notes in Computer Science*, pages 565–582. Springer, October 2019.

References III

[KLBM+25] Wietze Koops, Daniel Le Berre, Magnus O. Myreen, Jakob Nordström, Andy Oertel, Yong Kiam Tan, and Marc Vinyals. Practically Feasible Proof Logging for Pseudo-Boolean Optimization. In *31st International Conference on Principles and Practice of Constraint Programming (CP 2025)*, volume 340 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 21:1–21:27. Schloss Dagstuhl – Leibniz-Zentrum für Informatik, 2025.

[MMNS11] Ross M. McConnell, Kurt Mehlhorn, Stefan Näher, and Pascal Schweitzer. Certifying algorithms. *Computer Science Review*, 5(2):119–161, May 2011.

[Pse24] Pseudo-Boolean competition 2024. `https://www.cril.univ-artois.fr/PB24/`, August 2024.

[Pse25] Pseudo-Boolean competition 2025. `https://www.cril.univ-artois.fr/PB25/`, August 2025.

[SBJ21] Pavel Smirnov, Jeremias Berg, and Matti Järvisalo. Pseudo-Boolean optimization by implicit hitting sets. In *Proceedings of the 27th International Conference on Principles and Practice of Constraint Programming (CP '21)*, volume 210 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 51:1–51:20, October 2021.

[SBJ22] Pavel Smirnov, Jeremias Berg, and Matti Järvisalo. Improvements to the implicit hitting set approach to pseudo-Boolean optimization. In *Proceedings of the 25th International Conference on Theory and Applications of Satisfiability Testing (SAT '22)*, volume 236 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 13:1–13:18, August 2022.

References IV

[VDB22] Dieter Vandesande, Wolf De Wulf, and Bart Bogaerts. QMaxSATpb: A certified MaxSAT solver. In *Proceedings of the 16th International Conference on Logic Programming and Non-monotonic Reasoning (LPNMR '22)*, volume 13416 of *Lecture Notes in Computer Science*, pages 429–442. Springer, September 2022.