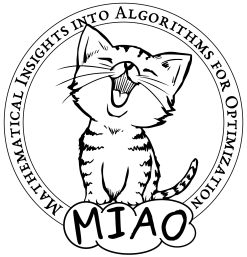
An Introduction to Pseudo-Boolean Proof Logging

Jakob Nordström

University of Copenhagen and Lund University



2nd International Workshop on Highlights in Organizing and Optimizing Proof-logging Systems
Paris, France
September 13–14, 2025

Based on Joint Work With…

- Markus Anders
- Jeremias Berg
- Bart Bogaerts
- Benjamin Bogø
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- Emir Demirović
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Combinatorial Solving and Optimisation

- Revolution last couple of decades in combinatorial solvers for
  - Boolean satisfiability (SAT) solving [BHvMW21][1]
  - Constraint programming (CP) [RvBW06]
  - Mixed integer linear programming (MIP) [AW13, BR07]
- Solve NP-complete problems (or worse) very successfully in practice!
- Except solvers are sometimes wrong… (Even best commercial ones) [BLB10, CKSW13, AGJ+18, GSD19, BMN22, GCS23]
- Solvers can propose infeasible "solutions" (but such errors can be detected)
- More challenging: How to achieve reliable claims of infeasibility?
- Or that a solution is optimal? (Even off-by-one mistakes can snowball into large errors if solver used as subroutine)

[1] See end of slides for all references with bibliographic details

What Can Be Done About Solver Bugs?

- **Software testing**
  Very useful, but bugs slip through even with careful domain-specific testing
  Progress using fuzzing and delta debugging [BB09, BLB10, KB22, NPB22, PB23]
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  Prove that solver implementation adheres to formal specification
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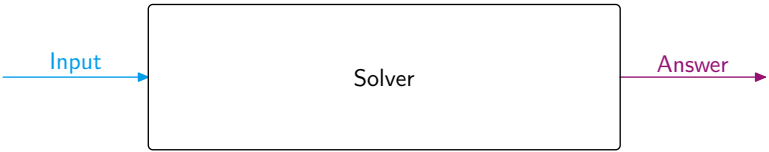
  (Despite valiant efforts in, e.g., [Fle20])
- **Proof logging**
  Make solver certifying [ABM+11, MMNS11] by adding code so that it outputs
  1. not only answer but also
  2. simple, machine-verifiable proof that answer is correct

Proof Logging with Certifying Solvers: Workflow



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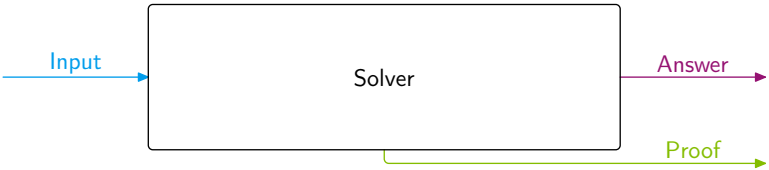

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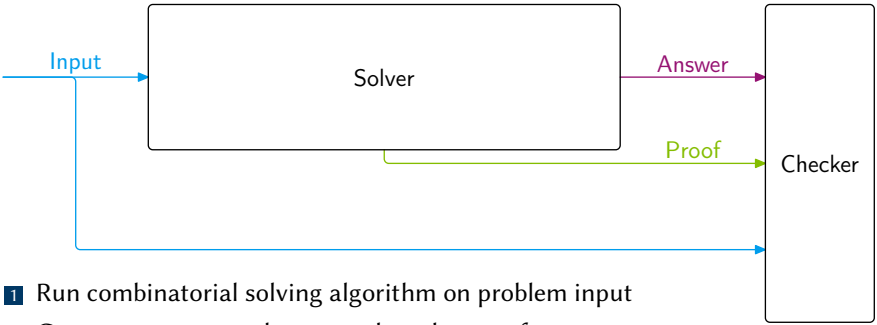

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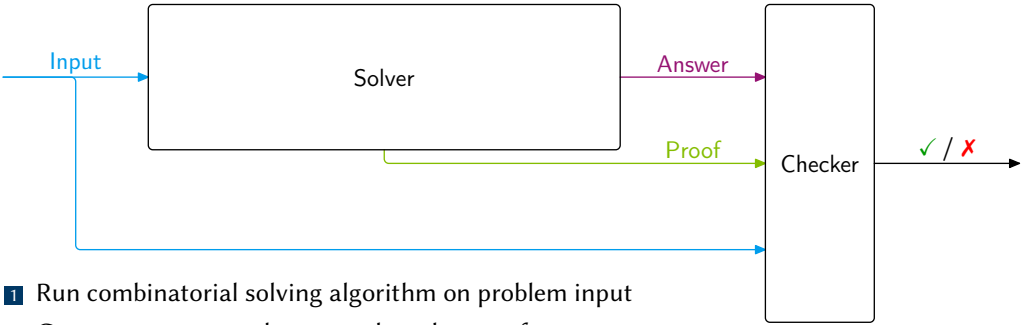

1. Run combinatorial solving algorithm on problem input
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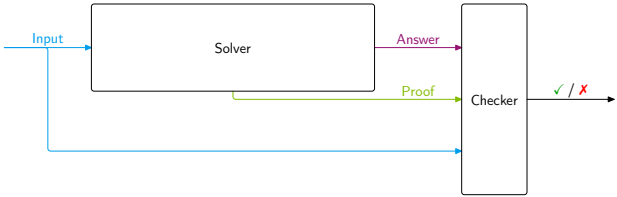
3. Feed input + answer + proof to proof checker
4. Verify that proof checker says answer is correct

Proof Logging Desiderata

Proof format for certifying solver should be



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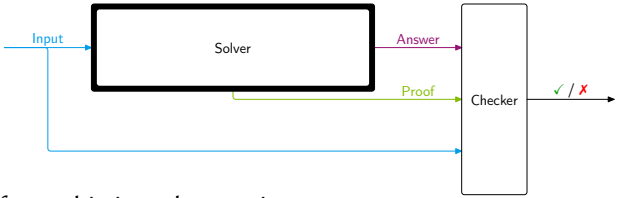

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- **very powerful:** minimal overhead for sophisticated reasoning

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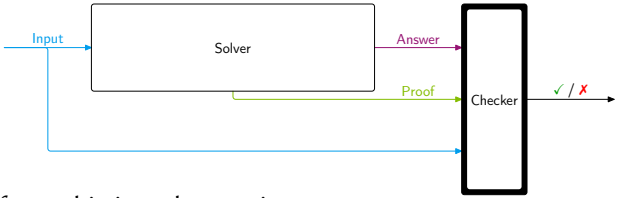

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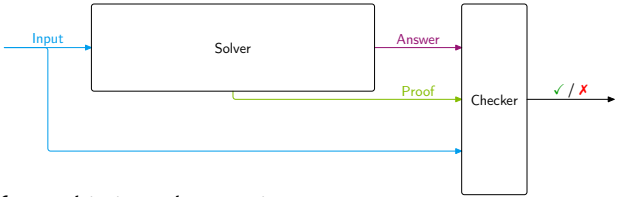

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Clear conflict expressivity vs. simplicity!

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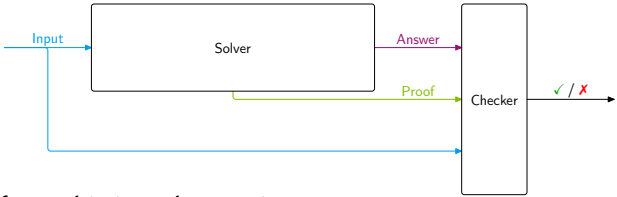

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Asking for both perhaps a little bit too good to be true?

Take-Away Message from This Tutorial Day

Proof logging for combinatorial optimisation is possible with single, unified method!

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Proof logging for combinatorial optimisation is possible with single, unified method!

- Build on successes in proof logging for SAT solvers with proof formats such as DRAT [HHW13a, HHW13b, WHH14], GRIT [CMS17], LRAT [CHH+17], ...
- But represent constraints as 0–1 linear inequalities
- Formalize reasoning using cutting planes [CCT87] proof system
- Add well-chosen strengthening rules [Goc22, GN21, BGMN23]
- Implemented in VeriPB (`https://gitlab.com/MIAOresearch/software/VeriPB`)

The Sales Pitch For Proof Logging

1. Certifies correctness of computed results
2. Detects errors even if due to compiler bugs, hardware failures, or cosmic rays
3. Provides debugging support during software development [GMM+20, KM21, BBN+23, EG23, KLM+25]
4. Facilitates performance analysis
5. Helps identify potential for further improvements
6. Enables auditability
7. Serves as stepping stone towards explainability

Design Principles for Proof Logging

Proof logging implementation

- Don't change solver
- Just add proof logging print statements (plus some book-keeping) to solver code

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Performance goals

- Proof logging overhead small constant fraction of running time ($\lessapprox 10\%$)
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Proof system

- Keep language simple — no XOR constraints, CP propagators, symmetries, ...
- But reason efficiently about such notions using power of proof system
- Combine proof logging with formally verified proof checker

Program for This Tutorial Day

Explain how to use VeriPB as a unified proof logging method for

- **09:00** SAT solving (Jakob Nordström)
- **10:00** Subgraph solving (Ciaran McCreesh)
- **11:30** Constraint programming (Matthew McIlree)
- **14:00** Pseudo-Boolean optimisation (Wietze Koops)
- **15:30** Preprocessing/presolving in MaxSAT and 0–1 linear programming (Andy Oertel)
- **16:30** Symmetry breaking (Markus Anders)

But Let Us Start from the Beginning…

Review of some basic concepts:

- Satisfiability (SAT) problem
- Unit propagation
- DPLL and CDCL algorithms
- Resolution proof system

The Satisfiability (SAT) Problem

- Variable x: takes value **true** (=1) or **false** (=0)
- Literal ℓ: variable x or its negation $\overline{x}$
- Clause $C = \ell_1 \vee \cdots \vee \ell_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- Conjunctive normal form (CNF) formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses

The SAT Problem

Given a CNF formula F, is it satisfiable?

For instance, what about:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge$$
$$(x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

Proofs for SAT

For satisfiable instances: just specify satisfying assignment

For unsatisfiability: a sequence of clauses (CNF constraints)

- Each clause follows “obviously” from everything we know so far
- Final clause is empty, meaning contradiction (written $\bot$)
- Means original formula must be inconsistent

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Unit Propagation

Clause C unit propagates ℓ under partial assignment ρ if ρ falsifies all literals in C except ℓ

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Proof checker should know how to unit propagate until saturation

Davis-Putman-Logemann-Loveland (DPLL)

DPLL [DP60, DLL62]: Assign variables and propagate; backtrack when clause violated

"Proof trace": when backtracking, write negation of guesses made

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

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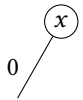

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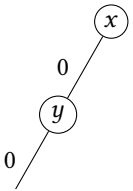

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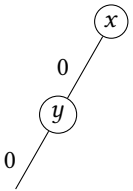

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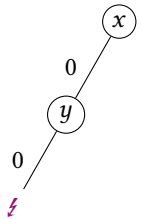

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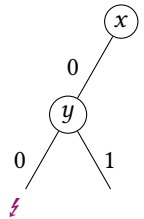

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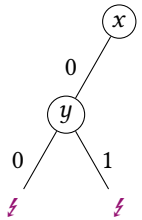
2. $x \vee \overline{y}$



Davis-Putman-Logemann-Loveland (DPLL)

DPLL [DP60, DLL62]: Assign variables and propagate; backtrack when clause violated

"Proof trace": when backtracking, write negation of guesses made

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

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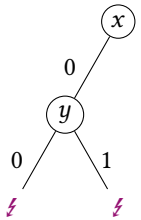

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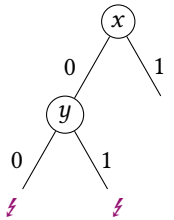

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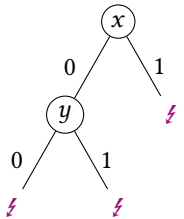

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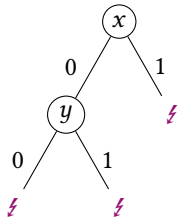
5. $\bot$



Reverse Unit Propagation (RUP)

To make this a proof, need backtrack clauses to be easily verifiable

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Fact

Backtrack clauses from DPLL solver generate a RUP proof

What About Conflict-Driven Clause Learning (CDCL)?

Run CDCL [BS97, MS99, MMZ+01] on our favourite CNF formula:

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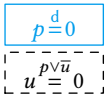

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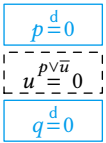

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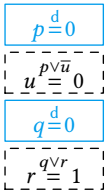

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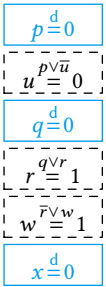

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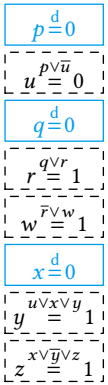

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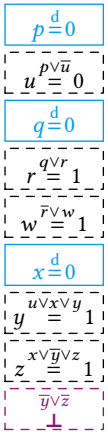

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decision level 1

$q \overset{\mathrm{d}}{=} 0$

$r \overset{q \vee r}{=} 1$

$w \overset{\overline{r} \vee w}{=} 1$

decision level 2

$x \overset{\mathrm{d}}{=} 0$

$y \overset{u \vee x \vee y}{=} 1$

$z \overset{x \vee \overline{y} \vee z}{=} 1$

$\overline{y} \vee \overline{z}$

$\bot$

decision level 3

Decision

Free choice to assign value to variable

Notation $p \overset{\mathrm{d}}{=} 0$

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Conflict Analysis

Time to analyse this conflict and learn from it!

$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$

$p \overset{\mathrm{d}}{=} 0$

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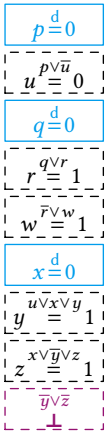

decision level 1

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Could backtrack by erasing conflict level & flipping last decision

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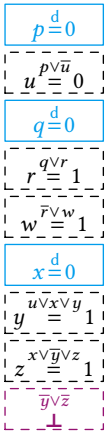

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But want to learn from conflict and cut away as much of search space as possible

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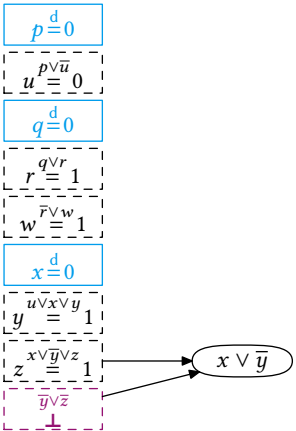

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Case analysis over z for last two clauses:

- $x \vee \overline{y} \vee z$ wants $z = 1$
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- Resolve clauses by merging them & removing z — must satisfy $x \vee \overline{y}$

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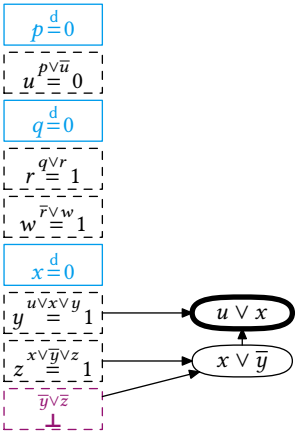

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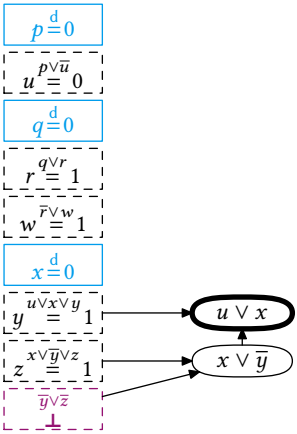
Repeat until UIP clause with only 1 variable at conflict level after last decision — learn and backjump

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$



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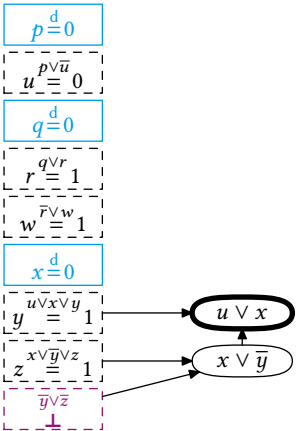

Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

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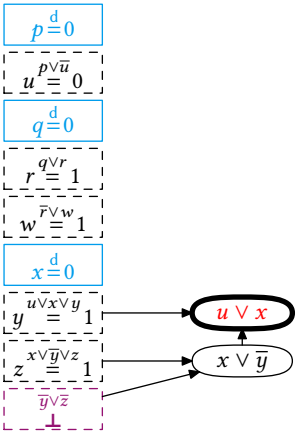

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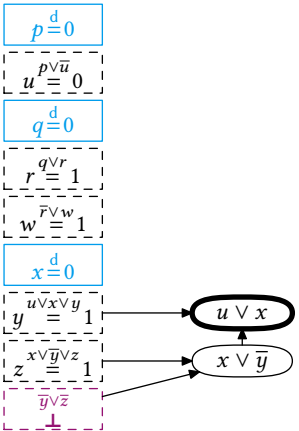

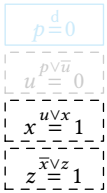

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Then continue as before…

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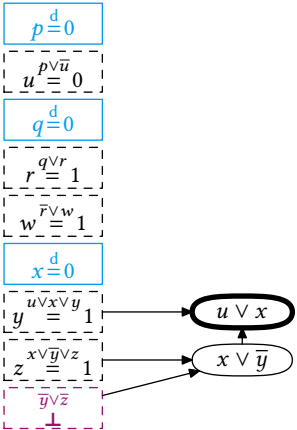

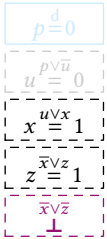

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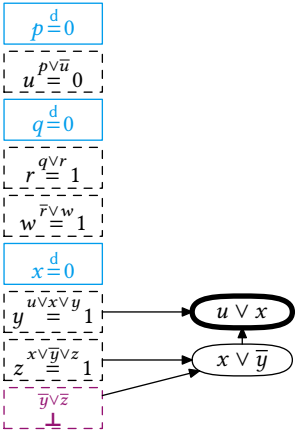

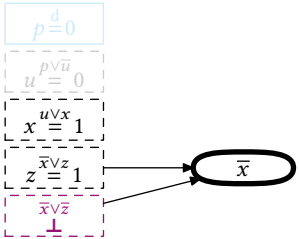

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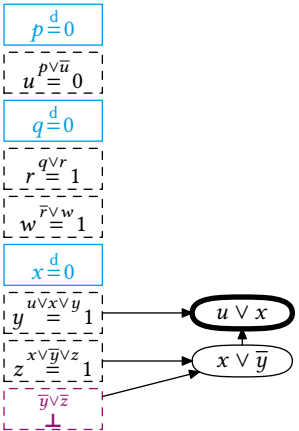

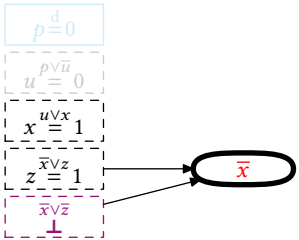

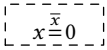

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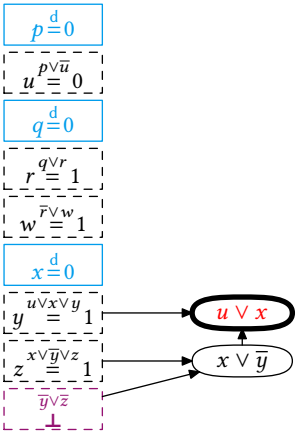

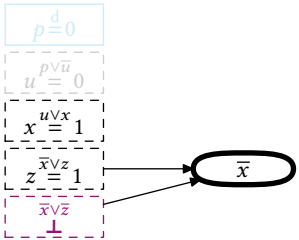

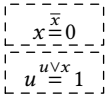

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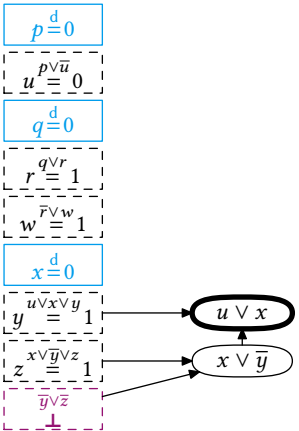

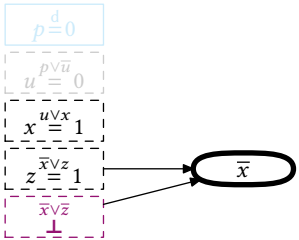

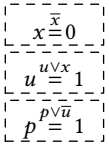

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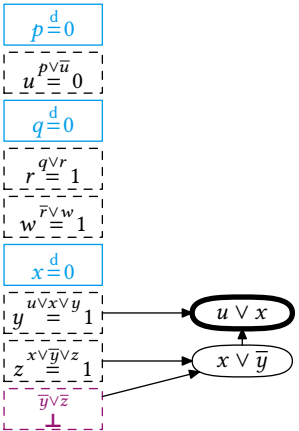

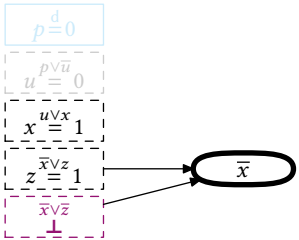

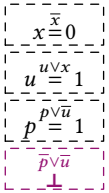

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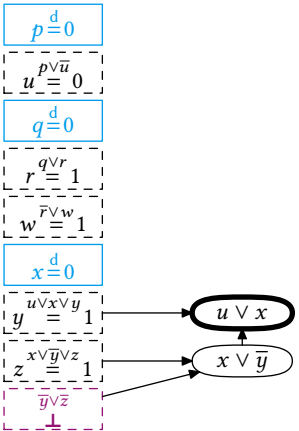

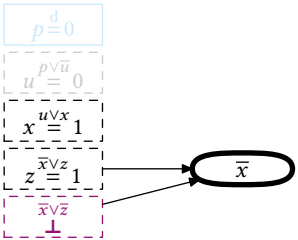

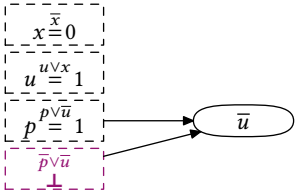

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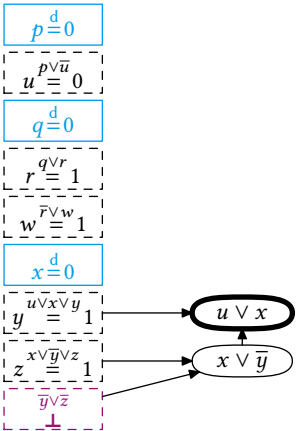

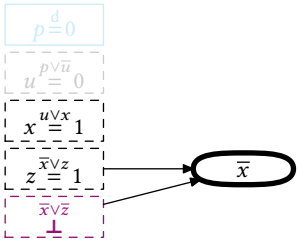

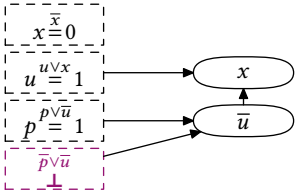

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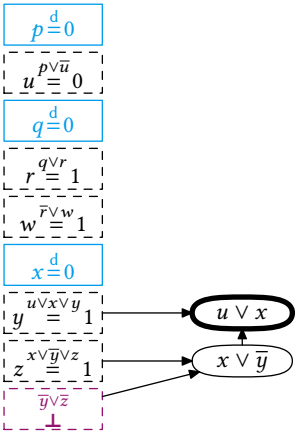

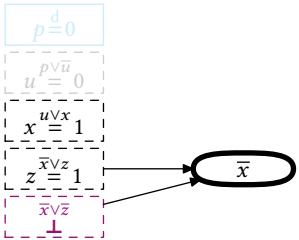

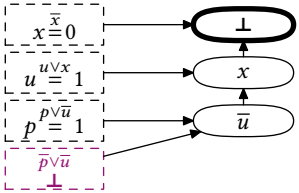

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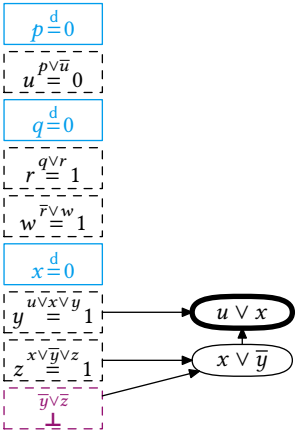

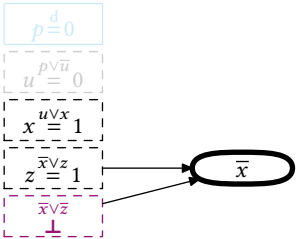

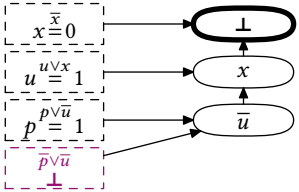

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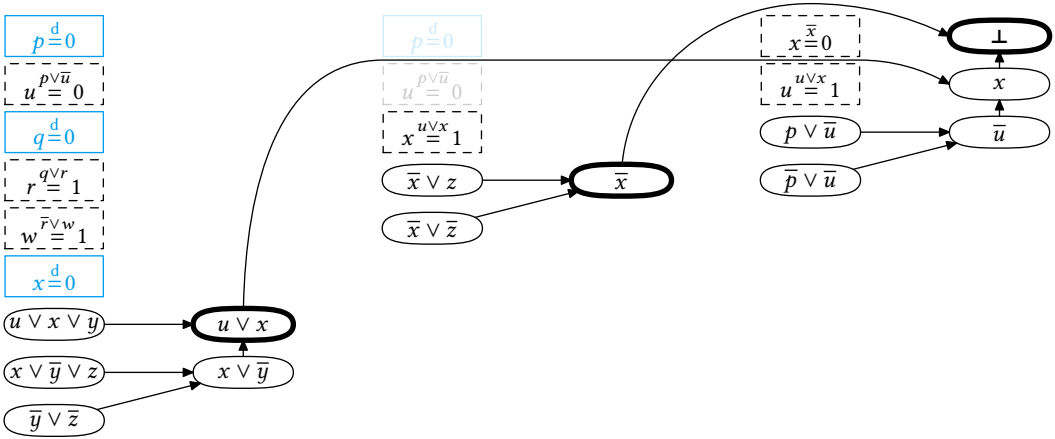

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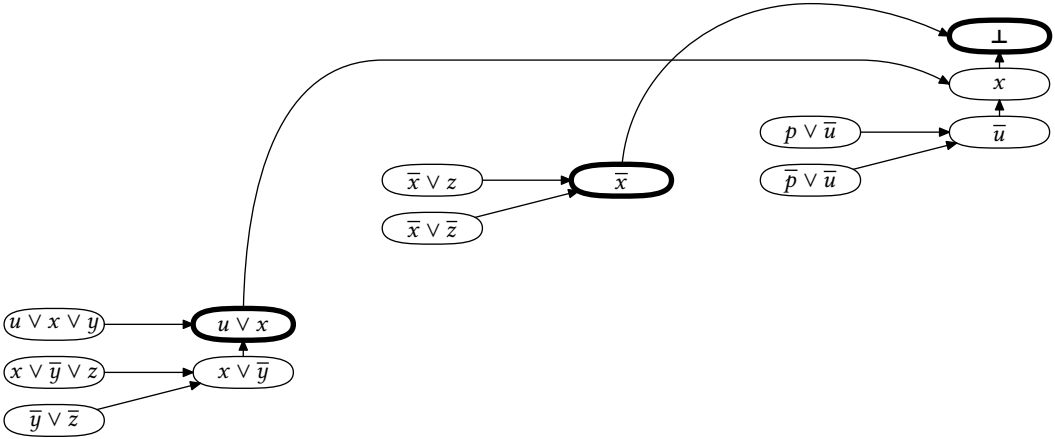

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is sequence of reverse unit propagation (RUP) clauses

1. $u \vee x$
2. $\overline{x}$
3. $\bot$

RUP Proofs and CDCL

But it turns out we can be lazier…

Fact

All learned clauses generated by CDCL solver are RUP clauses

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More Ingredients in Proof Logging for SAT

Fact

RUP proofs can be viewed as shorthand for resolution proofs

See proof complexity and SAT solving survey [BN21] for more on this

But RUP and resolution are not enough for preprocessing, inprocessing, and some other kinds of reasoning

Extension Variables, Part 1

Suppose we want a variable a encoding

$$a \Leftrightarrow (x \wedge y)$$

Extended resolution [Tse68]

Resolution rule plus extension rule introducing clauses

$$a \vee \overline{x} \vee \overline{y} \qquad \overline{a} \vee x \qquad \overline{a} \vee y$$

for fresh variable a (this is fine since a doesn't appear anywhere previously)

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Fact

Extended resolution (RUP + definition of new variables) is essentially equivalent to the DRAT proof logging system most commonly used for SAT solving

Why Aren't We Done?

Practical limitations of current SAT proof logging technology:

- Difficulties dealing with stronger reasoning efficiently (even for SAT solving)
- Clausal proofs can't easily reflect what algorithms for other problems do

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- Can justify graph reasoning without knowing what a graph is
- Can justify constraint programming inference without knowing what an integer variable is

Pseudo-Boolean Constraints

0–1 integer linear inequalities or (linear) pseudo-Boolean constraints:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)

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Sometimes convenient to use normalised form [Bar95] with all a_i, A positive
(without loss of generality)

Some Types of Pseudo-Boolean Constraints

1. Clauses

$$x_1 \vee \overline{x}_2 \vee x_3 \quad \Leftrightarrow \quad x_1 + \overline{x}_2 + x_3 \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input/model axioms From the input

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input/model axioms From the input

Literal axioms $\overline{\ell_i \geq 0}$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input/model axioms From the input

Literal axioms

$$\frac{}{\ell_i \geq 0}$$

Addition

$$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i)\ell_i \geq A + B}$$

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Multiplication for any $c \in \mathbb{N}^+$

$$\frac{\sum_i a_i\ell_i \geq A}{\sum_i ca_i\ell_i \geq cA}$$

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Division for any $c \in \mathbb{N}^+$

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \left\lceil \frac{a_i}{c} \right\rceil \ell_i \geq \left\lceil \frac{A}{c} \right\rceil}$$

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Saturation for any $c \in \mathbb{N}^+$
(assumes normalised form)

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \min(a_i, A) \cdot \ell_i \geq A}$$

Cutting Planes Toy Example

$$w + 2x + y \geq 2$$

Cutting Planes Toy Example

$$\text{Multiply by } 2 \quad \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

Cutting Planes Toy Example

$$\text{Multiply by 2}\ \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5$$

Cutting Planes Toy Example

$$\dfrac{\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}\ \text{Multiply by 2} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}\ \text{Add}$$

Cutting Planes Toy Example

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$$\dfrac{3w + 6x + 6y \geq 7}{w + 2x + 2y \geq 2\frac{1}{3}}\ \text{Divide by 3}$$

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Naming constraints by integers and literal axioms by the literal involved (with $\sim$ for negation) as

$$\text{Constraint 1} \doteq 2x + y + w \geq 2$$
$$\text{Constraint 2} \doteq 2x + 4y + 2z + w \geq 5$$
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such a calculation is written in the proof log in reverse Polish notation as*

```
pol 1 2 * 2 + ~z 2 * + 3 d ;
```

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(*) Except VeriPB variables have to be at least two characters long, but we abuse syntax slightly here for simplicity

Resolution and Cutting Planes

To simulate resolution step such as

$$\frac{\overline{y} \vee \overline{z} \qquad x \vee \overline{y} \vee z}{x \vee \overline{y}}$$

we can perform the cutting planes steps

$$\text{Add}\ \frac{\overline{y} + \overline{z} \geq 1 \qquad x + \overline{y} + z \geq 1}{\text{Divide by 2}\ \dfrac{x + 2\overline{y} \geq 1}{x + \overline{y} \geq 1}}$$

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$$\cfrac{\cfrac{\overline{y} + \overline{z} \geq 1 \qquad x + \overline{y} + z \geq 1}{x + 2\overline{y} \geq 1}\ \text{Add}}{x + \overline{y} \geq 1}\ \text{Divide by 2}$$

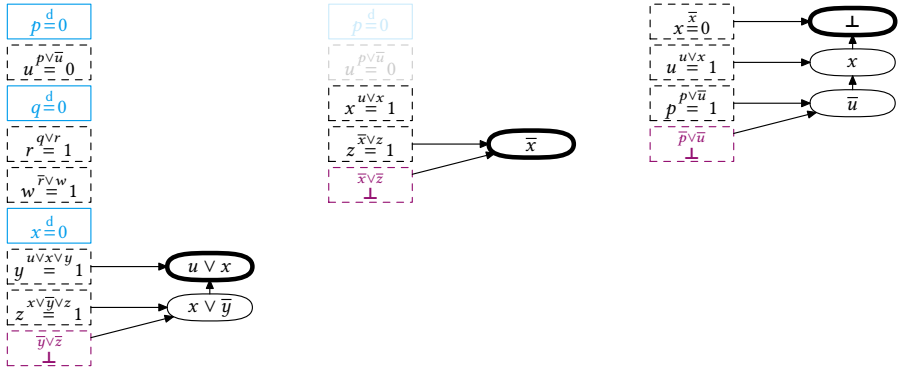
Given that the premises are clauses 7 and 5 in our example CNF formula, using references

$$\text{Constraint 7} \doteq \overline{y} + \overline{z} \geq 1$$
$$\text{Constraint 5} \doteq x + \overline{y} + z \geq 1$$

we can write this in the proof log as

```
pol 7 5 + 2 d ;
```

Pseudo-Boolean Proof Logging for Example CDCL Conflict Analyses



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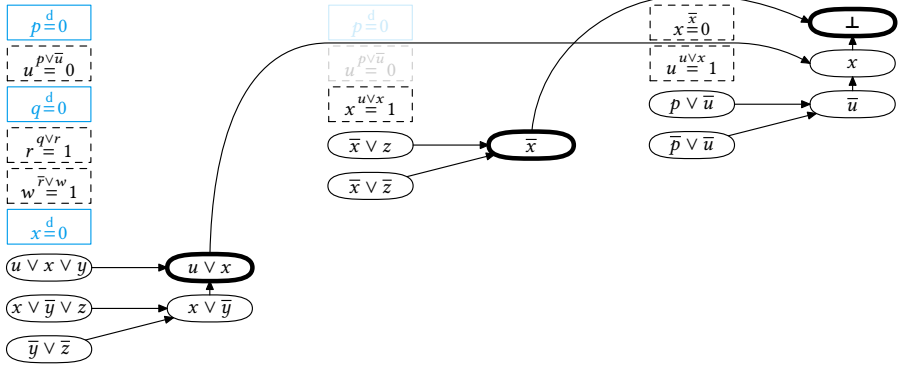

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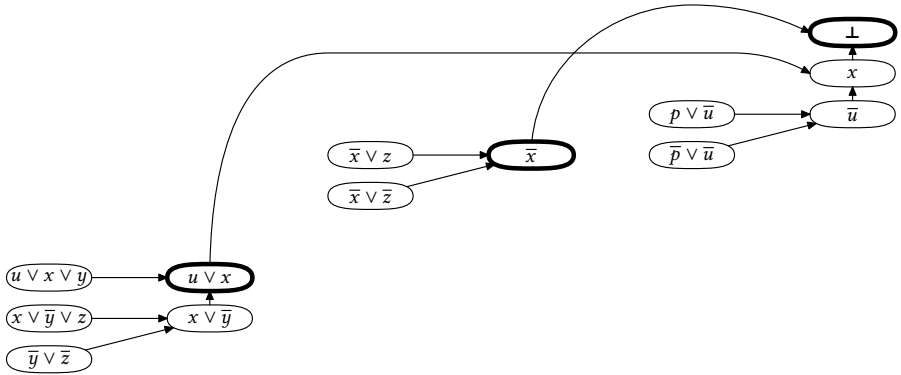

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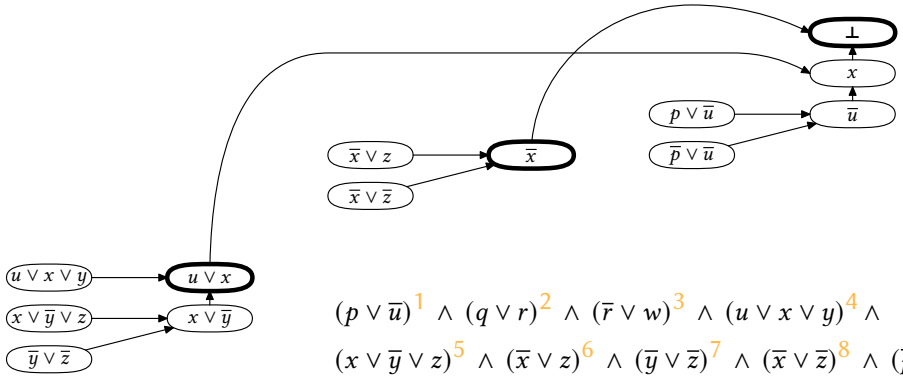

$$(p \vee \overline{u})^{1} \wedge (q \vee r)^{2} \wedge (\overline{r} \vee w)^{3} \wedge (u \vee x \vee y)^{4} \wedge$$
$$(x \vee \overline{y} \vee z)^{5} \wedge (\overline{x} \vee z)^{6} \wedge (\overline{y} \vee \overline{z})^{7} \wedge (\overline{x} \vee \overline{z})^{8} \wedge (\overline{p} \vee \overline{u})^{9}$$

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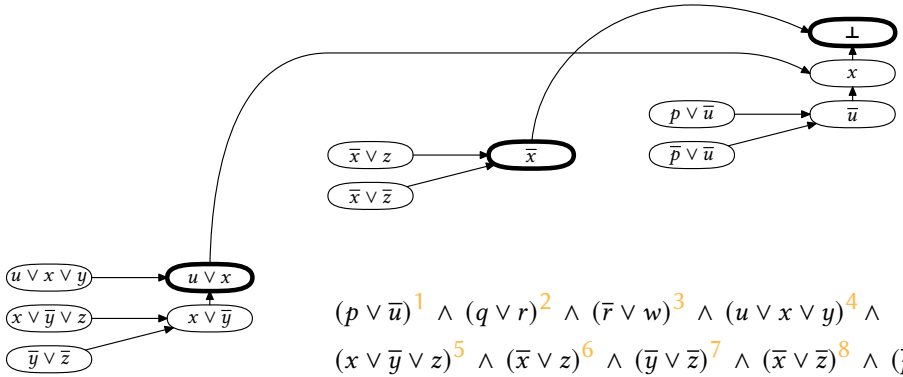

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```
pol 7 5 + 2 d 4 + 2 d ;
pol 8 6 + 2 d ;
pol 9 1 + 2 d 10 + 2 d 11 + 2 d ;
```

$\rightsquigarrow$ Constraint 10 $\doteq u + x \geq 1$

$\rightsquigarrow$ Constraint 11 $\doteq \overline{x} \geq 1$

$\rightsquigarrow$ Constraint 12 $\doteq 0 \geq 1$ ↯

RUP Revisited

Can define (reverse) unit propagation in a pseudo-Boolean setting [EGMN20]

Constraint C propagates variable x if setting x to "wrong value" would make C unsatisfiable

E.g., if x_5 is false,

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

would propagate $\overline{x}_4$ (since other coefficients do not add up to 7)

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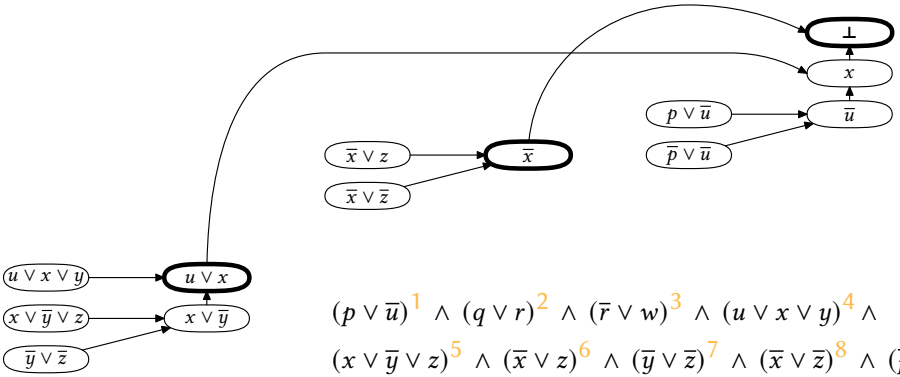
Risk for confusion!

- Constraint programming people might call this (reverse) integer bounds consistency
  - Does the same thing if we're working with clauses
  - More interesting for general pseudo-Boolean constraints
- SAT people beware: constraints can propagate multiple times and multiple variables

Pseudo-Boolean Proof Logging for Example CDCL Execution with RUP



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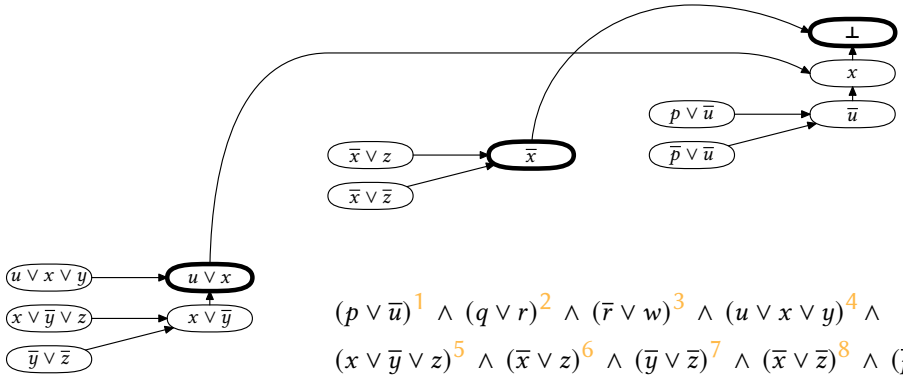

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```
rup 1 u 1 x >= 1 ;
```
$\rightsquigarrow$ Constraint 10 $\doteq u + x \geq 1$

```
rup 1 ~x >= 1 ;
```
$\rightsquigarrow$ Constraint 11 $\doteq \overline{x} \geq 1$

```
rup >= 1 ;
```
$\rightsquigarrow$ Constraint 12 $\doteq 0 \geq 1$ ↯

Extension Variables, Part 2

Suppose we want new, fresh variable a encoding

$$a \Leftrightarrow (3x + 2y + z + w \geq 3)$$

This time, introduce constraints

$$3\overline{a} + 3x + 2y + z + w \geq 3 \qquad 5a + 3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \geq 5$$

Again, needs support from the proof system in the form of strengthening rules

Proof Logs for "Cutting Planes with Strengthening"

For satisfiable instances: just specify a satisfying assignment.

For unsatisfiability: a sequence of pseudo-Boolean constraints in (slight extension of) OPB format [RM16]

- Each constraint follows "obviously" from what is known so far
- Either implicitly, by RUP...
- Or by an explicit cutting planes derivation...
- Or as an extension variable reifying a new constraint*
- Final constraint is $0 \geq 1$

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- Final constraint is $0 \geq 1$

(*) Not actually implemented this way — more details in a few slides …

Deleting Constraints

In practice, important to erase constraints to save memory and time during verification

Unsatisfiability proofs: fairly straightforward to deal with from point of view of proof logging

Optimisation proofs: significantly more delicate

We will mostly ignore deletions during this tutorial day for simplicity and clarity

Enumeration and Optimisation Problems

Enumeration:

- When a solution is found, can log it with `solx` rule
- Introduces a new constraint saying “not this solution”
- So the proof semantics is “infeasible, except for all the solutions I told you about”

Enumeration and Optimisation Problems

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- When a solution is found, can log it with `solx` rule
- Introduces a new constraint saying "not this solution"
- So the proof semantics is "infeasible, except for all the solutions I told you about"

Optimisation:

- Define an objective $f = \sum_i w_i \ell_i$, $w_i \in \mathbb{Z}$, to minimise subject to the contraints in the formula
- To maximise, negate objective
- Log solution α with `soli` rule $\Rightarrow$ objective-improving constraint $\sum_i w_i \ell_i \leq -1 + \sum_i w_i \alpha(\ell_i)$
- Semantics for proof of optimality: "infeasible to find better solution than best so far"
- Can also derive (potentially non-tight) lower bound $\sum_i w_i \ell_i \geq LB$

Pseudo-Boolean Proof Logging — How and Why?

If problem is (special case of) 0-1 integer linear program

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$$r \implies x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

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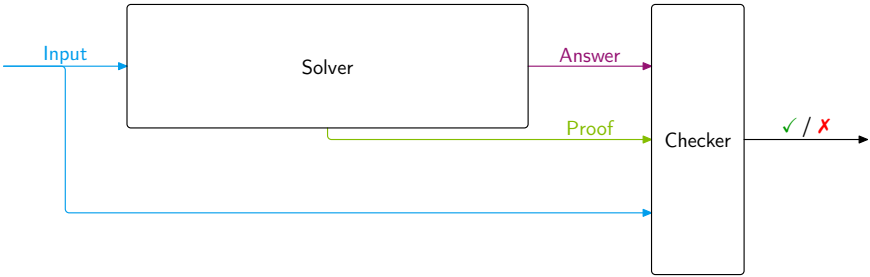
$r \Rightarrow x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$ $\quad$ $7\overline{r} + x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$

$r \Leftarrow x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$ $\quad$ $9r + \overline{x}_1 + 2x_2 + 3\overline{x}_3 + 4x_4 + 5\overline{x}_5 \geq 9$

Proof Logging with Formally Verified Checking: Full Workflow



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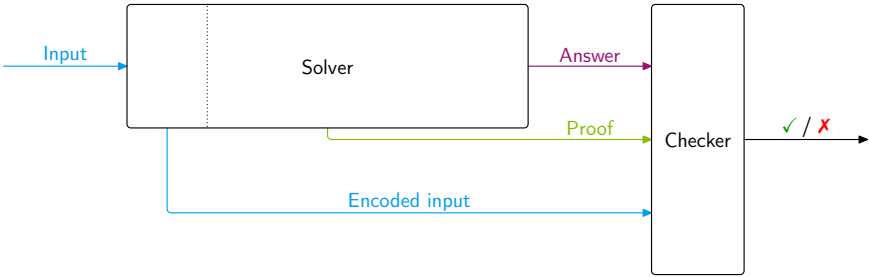

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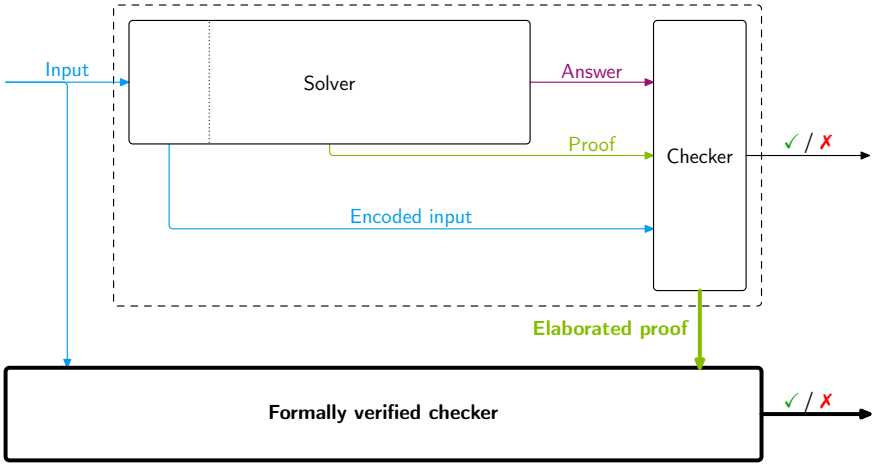

Strengthening Rules (And the Truth About Extension Variables)

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Sometimes weaker criterion needed — recall that to get variable a encoding

$$a \Leftrightarrow (3x + 2y + z + w \geq 3)$$

we introduced pseudo-Boolean constraints

$$3\overline{a} + 3x + 2y + z + w \geq 3 \qquad 5a + 3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \geq 5$$

Cutting planes method inherently cannot certify such constraints — they are not implied!

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Wish to allow without-loss-of-generality arguments that can derive non-implied constraints

Redundance-Based Strengthening

C is "redundant" with respect to F if F and $F \cup \{C\}$ are equisatisfiable

[apologies for the terminology — this is inherited from SAT proof logging]

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Redundance-based strengthening [BT19, GN21] (extending RAT rule of SAT proof logging)

C is redundant with respect to F if and only if there is a substitution ω (mapping variables to truth values or literals), called a witness, for which

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Witness ω should be specified, and implication needs to be efficiently verifiable — every $D \in (F \cup \{C\})\restriction_\omega$ should follow from $F \cup \{\neg C\}$ either

- "obviously" (e.g., by reverse unit propagation) or
- by explicitly presented derivation

Deriving $a \Leftrightarrow (3x + 2y + z + w \geq 3)$ Using the Redundance Rule

Want to derive

$$3\overline{a} + 3x + 2y + z + w \geq 3 \qquad 5a + 3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \geq 5$$

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   $\big(F \cup \{3\overline{a} + 3x + 2y + z + w \geq 3,\ 5a + 3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \geq 5\}\big)\restriction_\omega$
   Choose $\omega = \{a \mapsto 1\}$ — F untouched; new constraint satisfied
   $\neg(5a + 3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \geq 5)$ forces $3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \leq 4$
   This is the same constraint as $3x + 2y + z + w \geq 3$
   And VeriPB can automatically detect this

Redundance and Dominance Rules for Optimisation

Redundance-based strengthening, optimisation version

Add constraint C to formula F if exists witness substitution ω s.t.

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- Applying ω should strictly decrease f
- If so, don't need to show that $C\restriction_\omega$ holds!

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7. ...
8. Can't go on forever, so finally reach α' satisfying $F \wedge C$

Strength of Dominance Rule

Dominance-based strengthening (stronger, still simplified)

If $C_1, C_2, \ldots, C_{m-1}$ have been derived from F (maybe using dominance), then can derive C_m if exists witness substitution ω s.t.

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Further extensions:

- Define dominance rule w.r.t. order independent of objective
- Switch between different orders in same proof
- See [BGMN23] for details

Strengthening Rules: Proof Format

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    subproofs for proof goals
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- Witness ω should be explicitly specified in proof log
- Subproofs of proof goals should also be explicit
- Except can be skipped for proof goals that "obviously" follow, e.g.,
  - by reverse unit propagation (RUP)
  - by simple syntactic implication from other constraint
  - since the proof goal is not affected by the witness substitution ω

Successful Applications of VeriPB Proof Logging

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Pseudo-Boolean reasoning with strengthening rules sufficient to certify surprisingly wide range of combinatorial solving paradigms:

1. Boolean satisfiability (SAT) solving including advanced techniques such as
   - Gaussian elimination [GN21]
   - symmetry breaking [BGMN23]
2. SAT-based optimization (MaxSAT) [VDB22, BBN+23, BBN+24, IOT+24]
3. (Linear) Pseudo-Boolean solving [GMNO22, KLM+25]
4. Subgraph solving (max clique, subgraph isomorphism, max common connected subgraph) [GMN20, GMM+20, GMM+24]
5. Dynamic programming and decision diagrams [DMM+24]
6. Presolving in 0–1 integer linear programming [HOGN24]
7. Constraint programming [EGMN20, GMN22, MM23, MMN24, MM25]
8. Automated planning [DHN+25]

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Lots of concrete example files at `gitlab.com/MIAOresearch/software/VeriPB`

Future Research Directions

Performance and reliability of pseudo-Boolean proof logging and checking

- Trim proof while checking (as in *DRAT-Trim* [HHW13a])
- Compress proof file using binary format
- Formally verifed end-to-end checking with CakePB (as in [GMM+24, IOT+24, KLM+25])
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And more…

- Use proof logs for algorithm analysis or explainability purposes
- Lots of other challenging problems and interesting ideas

Future Research Directions

Performance and reliability of pseudo-Boolean proof logging and checking

- Trim proof while checking (as in *DRAT-Trim* [HHW13a])
- Compress proof file using binary format
- Formally verifed end-to-end checking with CakePB (as in [GMM+24, IOT+24, KLM+25])
- Faster proof logging and checking!

Proof logging for other combinatorial problems and techniques

- Model counting
- Mixed integer linear programming *(suggested extension of VeriPB in [DEGH23])*
- Satisfiability modulo theories (SMT) solving *(work on solvers cvc5, SMTInterpol, Z3, … [BBC+23, HS22])*

And more…

- Use proof logs for algorithm analysis or explainability purposes
- Lots of other challenging problems and interesting ideas
- Talk to us if you want to join the pseudo-Boolean proof logging revolution! ☺
  We're happy to collaborate, and we're hiring

Summing up

- Combinatorial solving and optimisation is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity

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- **Action point:** What problems can VeriPB solve for you?

  Come talk to us. We're hiring and open to collaboration!

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Thank you for your attention!

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