Tutorial on Mixed Integer Linear Programming (MIP) and Pseudo-Boolean Optimization

Jakob Nordström

University of Copenhagen and Lund University

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Outline of Lecture on MIP Solving and PB Optimization

1. Mixed Integer Linear Programming (MIP) and Integer Linear Programming (ILP)
   - MIP Preliminaries
   - Branch-and-Bound and Branch-and-Cut
   - Additional Techniques

2. Combining PB and MIP Techniques
   - Some Challenges When Integrating PB and LP Solving
   - A Proof-of-Concept Hybrid PB-LP Solver
   - Evaluation and Conclusions

An Acknowledgement and an Apology

The MIP material relies heavily on the presentation *Computational Mixed-Integer Programming* by Ambros Gleixner at the Casa Matemática Oaxaca (CMO) workshop *Theory and Practice of Satisfiability Solving* in 2018 (`https://tinyurl.com/MIPtutorial`)

A bit too many references are still missing — see Gleixner' slides for full details

Mixed Integer Linear Programming

Mixed integer linear program

- Minimize $\sum_j a_j x_j$
- Subject to $\sum_j a_{i,j} x_j \leq A_i$, $i = 1, \ldots, m$
- $x_j \in \mathbb{N}$ for $j = 1, \ldots, n$
- $x_j \in \mathbb{R}_{\geq 0}$ for $j = n+1, \ldots, N$

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- Integer-valued variables
- Real-valued variables
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- No real-valued variables: integer linear program (ILP)
- $0 \leq x_j \leq 1$ for all j: 0-1 ILP
- Vacuous objective $\sum_j 0 \cdot x_j$: decision problem
- But MIP best for optimization

Two Differences Compared to SAT/PB

Academia vs. industry

- Best solvers are commercial and closed-source
- E.g., CPLEX [CPL], GUROBI [Gur], and XPRESS [Xpr]
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Search vs. backtracking

- SAT/PB: Fast decisions; careful, slow(er) conflict analysis
- MIP: Lots of time & effort on decisions; backtracking not so advanced

MIP Solving at a High Level

1. Preprocessing (called presolving)
2. Linear programming + branch-and-bound
3. Add cutting planes ruling out infeasible LP-solutions (branch-and-cut method going back to [Gom58])
4. Heuristics for quickly finding good feasible solutions

Linear Programming Relaxation

Linear Programming Relaxation (LPR)

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- $x_j \in \mathbb{R}_{\geq 0}$ for $j = n + 1, \ldots, N$

- Fast to solve (just linear programming)
- LP solution x^* yields lower bound
- Or, if x^* "accidentally" feasible, have optimal solution
- Use simplex algorithm — will have many LP calls for same problem with different variable bounds; need efficient hot restarts

LP-Based Branch-and-Bound

Branch-and-bound

Choose integer-valued x_j and $B \in \mathbb{N}$

- Solve MIP plus constraint $x_j \geq B$
- Solve MIP plus constraint $x_j \leq B - 1$

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Prune subproblem/node when

- LP is infeasible
- LP bound $>$ incumbent (current best solution)

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Branch on

- Variables
- General linear constraints (powerful but difficult)
  Corresponds to stabbing planes proof system [BFI+18]

Branch-and-Cut

General cutting plane method

1. Solve LP relaxation
2. If solution x^* feasible for MIP $\Rightarrow$ found optimum
3. Otherwise generate and add constraint $\sum_j b_j x_j \leq B$ that is
    - valid for MIP
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4. Repeat from the top

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Branch-and-cut

- Run branch-and-bound
- But in each subproblem, use cutting plane method to repeatedly
    - solve LP relaxation
    - add cut

Example Cut 1: Knapsack Cover Cut

Given constraint

$$\sum_{j \in I} a_j x_j \leq A$$

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(In cutting planes, weaken & divide $\sum_{j \in I} a_j \overline{x}_j \geq -A + \sum_{j \in I} a_j$ to get disjunctive clause $\sum_{j \in C} \overline{x}_j \geq 1$)

Example Cut 2: Mixed Integer Rounding (MIR) Cut

Mixed integer rounding (MIR) cut [MW01] applied to (normalized) pseudo-Boolean constraint

$$\sum_i a_i \ell_i \geq A$$

with divisor $d \in \mathbb{N}^+$ produces constraint

$$\sum_i \Big(\min(a_i \bmod d, A \bmod d) + \left\lfloor \frac{a_i}{d} \right\rfloor (A \bmod d) \Big) \ell_i \geq \left\lceil \frac{A}{d} \right\rceil (A \bmod d)$$

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For comparison, division by 3 and multiplication by 2 produces

$$2x + 2y + 2z + 4w + 4u \geq 4$$

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Some simple (but efficient) techniques:

- Substitution of fixed variables
- Normalization of constraints: divide integer constraints by $\gcd$ on left-hand side and round on right-hand side
- Probing: tentatively assign binary variables and propagate
- Dominance test: remove constraints implied by other constraints

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For more details, see talk by Gleixner `https://tinyurl.com/MIPtutorial`

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MIP conflict analysis [Ach07] analogous to CDCL, but

- operate on clausal reasons extracted from constraints
- **not** on constraints themselves

Exponential loss in power!

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But can find other, more interesting benchmarks where MIP conflict analysis seems to really suffer from this problem [DGN21]

Branching Heuristics

Dual gain

Given LP solution x^*, branch on x_j such that $x_j \geq \lceil x_j^* \rceil$ and $x_j \leq \lfloor x_j^* \rfloor$ both provide good lower bound increase

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Look ahead (strong branching)

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Keep also other statistics about variables to guide search

Node Selection

How to grow search tree?

- Depth-first search (DFS): keeps cost for simplex calls small *[corresponds to what SAT and PB solvers **always** do]*
- Best bound search (BBS): Focus on improving lower bound (dual bound)
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Combine BBS and BES with DFS plunges to exploit simplex hot restarts

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Example: Relaxation-enforced neighbourhood search

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Example of "fix-and-MIP" local neighbourhood search heuristic
(Note that, interestingly, this turns ILP into 0-1 ILP subproblem)

And More. . .

1. Decomposition
   - Branch-and-price / column generation
   - Bender's decomposition
     [Core-guided and IHS search similar in spirit to logic-based Benders decomposition [HO03]]
2. Symmetry handling
   - Via graph automorphism
   - Or dedicated symmetry detection (commercial solvers)
3. Extended formulations (with new variables and constraints)
4. Parallelization
5. Restarts

Numerics and Correctness

Numerics

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- Can lead to rounding errors
- Exact MIP solvers like [CKSW13, EG21]
  - are significantly slower
  - don't support the full range of state-of-the-art techniques

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Proof logging / certification

- Currently not available for state-of-the-art MIP solvers
- Though known that even best commercial solvers sometimes give wrong results
- Some work on proof logging in [CGS17, EG21] — challenges:
  - How to capture wide diversity of techniques?
  - What is a convenient format?
  - How to generate proofs efficiently on-the-fly?

Some Interesting MIP Questions

1. Develop better heuristics to branch on general linear constraints (cf. stabbing planes [BFI+18])
2. Design stronger conflict analysis operating directly on linear constraints (borrow ideas from native pseudo-Boolean solvers?)
3. Provide rigorous understanding of MIP solver performance
4. Develop families of theory benchmarks and computational complexity results for them (cf. SAT solving and proof complexity [BN21])
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Combining PB Solving and Mixed Integer Programming

Pseudo-Boolean solvers

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Why not merge the two to get the best of both worlds of SAT-style conflict-driven search and MIP-style branch-and-cut?

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Need to carefully balance time allocation for PB solver and LP solver

Backtracking from LP Infeasibility?

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- Obviously, PB solver should backtrack
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- And PB solver blissfully unaware of any conflict...

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More subtle issue:

- Efficient LP solvers use inexact floating-point arithmetic
- How to incorporate into Boolean solver that must maintain perfectly sound reasoning?

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Cut constraints from LP solver

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Cut constraints from PB solver

- PB solvers learns new constraints at high rate from conflict analysis
- These learned constraints can also be viewed as cuts
- Should such constraints be passed from PB solver to LP solver?

Report on Proof-of-Concept PB-LP Integration [DGN21]

1. Interleave LP solving within conflict-driven PB search
   - Limit LP time by enforcing total #LP pivots $\leq$ #PB conflicts
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4. Also explore letting PB solver pass learned constraints to LP solver

(What We Need from) Farkas Lemma [Far02]

Pseudo-Boolean Farkas Lemma

Given

- Pseudo-Boolean formula $F = \{C_1, \ldots, C_m\}$,
- partial assignment ρ,

such that LP relaxation of residual formula $F{\restriction_\rho}$ infeasible
Then $\exists$ coefficients $k_i \in \mathbb{N}$ such that linear combination

$$\sum_{i=1}^{m} k_i \cdot C_i$$

is violated by ρ, i.e.,

$$slack\left(\sum_{i=1}^{m} k_i \cdot C_i; \rho\right) < 0$$

Observed in [MM04] that $\sum_{i=1}^{m} k_i \cdot C_i$ is valid starting point for pseudo-Boolean conflict analysis

Relation to MIP Solvers with Conflict Analysis?

MIP solvers also combine constraint propagation and SAT-style clause learning with LP solving

- Implemented in SCIP [ABKW08]
- And also in closed-source solvers (see [AW13])

Important to understand similarities and differences — let's give high-level description of PB search and conflict analysis phrased in MIP language

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Pseudo-Boolean search

1. Make decision to assign free variable to 0 or 1
2. Propagate all assignments implied by some linear constraint until saturation
3. If no contradiction, go to step 1
4. Otherwise some constraint C violated $\Rightarrow$ trigger conflict analysis

PB Conflict Analysis "in MIP Language"

Pseudo-Boolean conflict analysis (simplified description)

1. Find reason constraint R responsible for propagating last variable x in C to "wrong value"

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Pseudo-Boolean conflict analysis (simplified description)

1. Find reason constraint R responsible for propagating last variable x in C to "wrong value"
2. Apply division/saturation to generate (globally valid) cut R_{cut} propagating x to $\{0, 1\}$-value (over the reals)

PB Conflict Analysis "in MIP Language"

Pseudo-Boolean conflict analysis (simplified description)

1. Find reason constraint R responsible for propagating last variable x in C to "wrong value"
2. Apply division/saturation to generate (globally valid) cut R_{cut} propagating x to $\{0, 1\}$-value (over the reals)
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7. Switch back to search phase

Comparison to MIP Propagation and Conflict Analysis

Propagation in SCIP

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- Instead use disjunctive clauses extracted from reason constraints
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Arithmetic

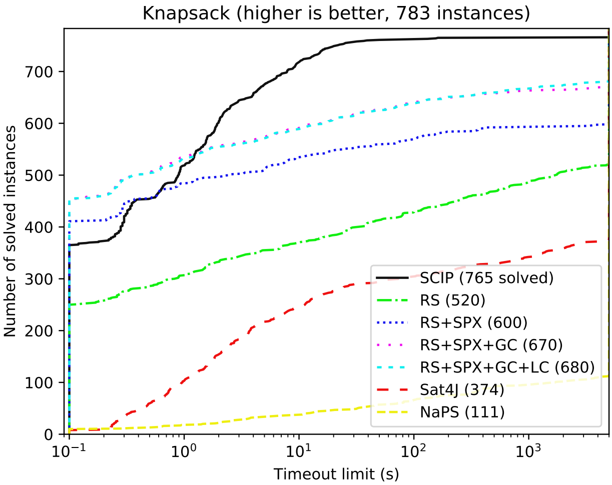
- SCIP uses floating point
- Reasoning steps in PB solver computed with exact integer arithmetic
- No issues with possible rounding errors

Experimental Results for Knapsack Benchmarks [Pis05]

RoundingSat (RS) enhanced with

- LP solver SoPlex (SPX) (from SCIP)
- Gomory cuts (GC)
- shared learned PB cuts (LC)

compared to other solvers



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	SCIP	RS	+SPX	+GC	+LC	Sat4j	NaPS
PB16dec (1783)	1123	**1472**	**1453**	**1452**	1451	1432	1400
PB16opt (1600)	**1057**	862	**988**	986	**993**	776	896
MIPdec (556)	**264**	203	**263**	**261**	259	169	170
MIPopt (291)	**125**	78	101	**102**	**102**	62	65

Performance of Integrated PB-LP Solver

1. Best of both worlds?
    - At least well-rounded performance
    - Hybrid PB-LP solver always competitive with best solver
    - Pretty dramatic improvements for optimization problems compared to pseudo-Boolean state of the art
    - SCIP is hard to beat, but also pulls quite a few extra tricks that we haven't implemented

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3. Sharing Gomory cuts and learned cuts not so helpful
    - Except for knapsack benchmarks, where they help a lot
    - And maybe we could/should fine-tune how sharing is done?

Usefulness/Usage of Constraints

Estimate usefulness of different types of constraints

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Constraints learned after Farkas-based conflicts

- Less useful than regular learned constraints
- But big spread in usage measurements

PB Solver Performance: Balancing the Picture

Actually, RoundingSat can also outperform commercial MIP solvers by 1-2 orders of magnitude for, e.g.,

- matching of children with adoptive families [DGG+19]
- automated planning using binarized neural networks [SS18]

as reported by authors of these papers

(See also our paper [SDNS20])

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RoundingSat seems particularly good for "big-M constraints" like

$$A\overline{z} + \sum_i a_i \ell_i \geq A$$

encoding $z \Rightarrow \sum_i a_i \ell_i \geq A$

LP relaxations are quite uninformative for such constraints

Future Research Directions for PB-LP Integration (1/2)

1. Fine-tune heuristics
   - Improved LP-based cut generation?
   - Smarter sharing of PB constraints with LP solver?
   - Dynamic allocation of PB and LP solving time based on contributions?

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4. Use MIP presolving in pseudo-Boolean solvers
5. Use MIR cuts and/or other MIP cut rules to improve pseudo-Boolean conflict analysis

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8. Export pseudo-Boolean conflict analysis to MIP
9. Use hybrid PB-LP solver to solve 0-1 MIP problems à la Bender
   - PB solver decides on Boolean variables and propagates
   - LP solver takes care of real-valued variables

Summing up

- Revolution in performance last two decades in
  - Boolean satisfiability (SAT) solving
  - Mixed integer linear programming (MIP)
- More recent addition Cutting-planes-based conflict-driven search
- Quite different approaches
  - Complementary strengths
  - Lots of room for synergies?
- Lots of exciting research waiting to be done ☺

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Thanks for sticking till the end!

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