Certified Symmetry and Dominance Breaking for Combinatorial Optimisation

Jakob Nordström

University of Copenhagen
and Lund University

SAT Encodings and Beyond
Dagstuhl Seminar 23261
June 25–30, 2023

Joint *AAAI '22* paper with Bart Bogaerts, Stephan Gocht, and Ciaran McCreesh

Combinatorial Solving and Optimisation

- Revolution last couple of decades in combinatorial solvers for
  - Boolean satisfiability (SAT) solving [BHvMW21]
  - Satisfiability modulo theories (SMT) solving [BHvMW21]
  - Constraint programming (CP) [RvBW06]
  - Mixed integer linear programming (MIP) [AW13, BR07]
- Solve NP problems (or worse) very successfully in practice!
- Except solvers are sometimes wrong... (Even best commercial ones) [BLB10, CKSW13, AGJ+18, GSD19, GS19]
- Software testing doesn't suffice to resolve this problem
- Formal verification techniques cannot deal with level of complexity of modern solvers

Certified Results with Proof Logging

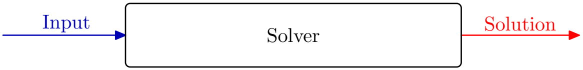
Design certifying algorithms [ABM+11, MMNS11] that

- not only solve problem but also
- do proof logging to certify that solution is correct

Certified Results with Proof Logging

Design certifying algorithms [ABM+11, MMNS11] that

- not only solve problem but also
- do proof logging to certify that solution is correct



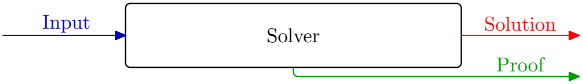
Workflow:

1. Run solver on problem input

Certified Results with Proof Logging

Design certifying algorithms [ABM+11, MMNS11] that

- not only solve problem but also
- do proof logging to certify that solution is correct



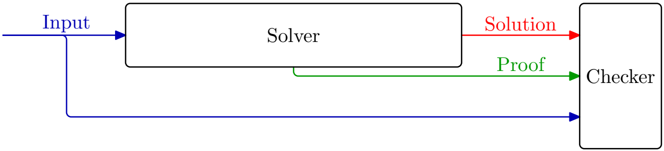
Workflow:

1. Run solver on problem input
2. Get as output not only solution but also proof

Certified Results with Proof Logging

Design certifying algorithms [ABM+11, MMNS11] that

- not only solve problem but also
- do proof logging to certify that solution is correct



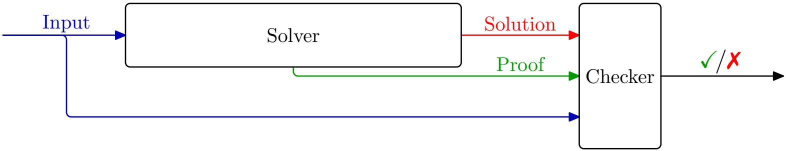
Workflow:

1. Run solver on problem input
2. Get as output not only solution but also proof
3. Feed input + solution + proof to proof checker

Certified Results with Proof Logging

Design certifying algorithms [ABM+11, MMNS11] that

- not only solve problem but also
- do proof logging to certify that solution is correct



Workflow:

1. Run solver on problem input
2. Get as output not only solution but also proof
3. Feed input + solution + proof to proof checker
4. Verify that proof checker says solution is correct

Yet Another SAT Success Story

Many proof logging formats for SAT solving using CNF clausal format:

- DRAT [HHW13a, HHW13b, WHH14]
- GRIT [CMS17]
- LRAT [CHH+17]
- . . .

Well established — required in main track of SAT competitions

Yet Another SAT Success Story

Many proof logging formats for SAT solving using CNF clausal format:

- DRAT [HHW13a, HHW13b, WHH14]
- GRIT [CMS17]
- LRAT [CHH+17]
- ...

Well established — required in main track of SAT competitions

But efficient proof logging has remained out of reach for stronger paradigms

Yet Another SAT Success Story(?)

Many proof logging formats for SAT solving using CNF clausal format:

- DRAT [HHW13a, HHW13b, WHH14]
- GRIT [CMS17]
- LRAT [CHH+17]
- ...

Well established — required in main track of SAT competitions

But efficient proof logging has remained out of reach for stronger paradigms

And, in fact, even for some advanced SAT solving techniques:

- cardinality reasoning
- Gaussian elimination
- symmetry handling

Clausal Proof Logging Approaches

Cardinality and pseudo-Boolean reasoning [SB06, BBH22]
Evaluated on fairly specific crafted benchmarks
More challenging and/or real-world benchmarks would be valuable

Clausal Proof Logging Approaches

Cardinality and pseudo-Boolean reasoning [SB06, BBH22]
Evaluated on fairly specific crafted benchmarks
More challenging and/or real-world benchmarks would be valuable

Gaussian elimination [PR16, Bry22, SB22]
Problems with proof logging overhead and proof file size

Clausal Proof Logging Approaches

Cardinality and pseudo-Boolean reasoning [SB06, BBH22]
Evaluated on fairly specific crafted benchmarks
More challenging and/or real-world benchmarks would be valuable

Gaussian elimination [PR16, Bry22, SB22]
Problems with proof logging overhead and proof file size

Symmetry handling [HHW15, TD20]
No fully general method for symmetry breaking (i.e., adding constraints to remove symmetric solutions)
Method for symmetric learning (i.e., adding symmetric versions of derived constraints) not compatible with SAT preprocessing

Our Work: Efficient Proof Logging for Symmetry Breaking

Paper *Certified Symmetry and Dominance Breaking for Combinatorial Optimisation* at *AAAI '22* [BGMN22]:

Implementation in proof checker VeriPB [Ver]

- First general & efficient proof logging method for symmetry breaking
- Supports also pseudo-Boolean reasoning and Gaussian elimination
- Based on 0-1 integer linear constraints instead of clauses
- Uses cutting planes method [CCT87] with additional rules

Outline of Presentation

What I hope to cover in the rest of this presentation:

- Basics of proof logging with 0-1 linear constraints
- New rule for symmetry and dominance breaking
- Application to symmetry breaking for SAT (and some other problems)
- Some future research directions

0-1 Integer Linear (a.k.a. Pseudo-Boolean) Constraints

Pseudo-Boolean (PB) constraints are 0-1 integer linear constraints

$$C \doteq \sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
- variables x_i take values $0 = \mathit{false}$ or $1 = \mathit{true}$

0-1 Integer Linear (a.k.a. Pseudo-Boolean) Constraints

Pseudo-Boolean (PB) constraints are 0-1 integer linear constraints

$$C \doteq \sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
- variables x_i take values $0 = \mathit{false}$ or $1 = \mathit{true}$

Pseudo-Boolean formulas $F \doteq \bigwedge_{i=1}^{m} C_i$ are conjunctions of pseudo-Boolean constraints

Some Types of Pseudo-Boolean Constraints

1. Clauses

$$x \vee \overline{y} \vee z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

Some Types of Pseudo-Boolean Constraints

1. Clauses

$$x \vee \overline{y} \vee z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

Some Types of Pseudo-Boolean Constraints

1. Clauses

$$x \vee \overline{y} \vee z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Linear combination $$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i) \ell_i \geq c_A A + c_B B} \quad [c_A, c_B \in \mathbb{N}]$$

Division $$\frac{\sum_i c a_i \ell_i \geq A}{\sum_i a_i \ell_i \geq \lceil A/c \rceil} \quad [c \in \mathbb{N}^+]$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Linear combination $$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i)\ell_i \geq c_A A + c_B B} \quad [c_A, c_B \in \mathbb{N}]$$

Division $$\frac{\sum_i c a_i \ell_i \geq A}{\sum_i a_i \ell_i \geq \lceil A/c \rceil} \quad [c \in \mathbb{N}^+]$$

Toy example:

Lin comb $$\frac{2x + 4y + 2z + w \geq 5 \qquad 2x + y + w \geq 2}{}$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Literal axioms $\dfrac{}{\ell_i \geq 0}$

Linear combination $\dfrac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i)\ell_i \geq c_A A + c_B B} \quad [c_A, c_B \in \mathbb{N}]$

Division $\dfrac{\sum_i c a_i \ell_i \geq A}{\sum_i a_i \ell_i \geq \lceil A/c \rceil} \quad [c \in \mathbb{N}^+]$

Toy example:

Lin comb $\dfrac{2x + 4y + 2z + w \geq 5 \qquad 2x + y + w \geq 2}{(2x + 4y + 2z + w) + 2 \cdot (2x + y + w) \geq 5 + 2 \cdot 2}$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Linear combination $$\frac{\sum_i a_i\ell_i \geq A \qquad \sum_i b_i\ell_i \geq B}{\sum_i (c_A a_i + c_B b_i)\ell_i \geq c_A A + c_B B} \quad [c_A, c_B \in \mathbb{N}]$$

Division $$\frac{\sum_i c a_i\ell_i \geq A}{\sum_i a_i\ell_i \geq \lceil A/c \rceil} \quad [c \in \mathbb{N}^+]$$

Toy example:

Lin comb $$\frac{2x + 4y + 2z + w \geq 5 \qquad 2x + y + w \geq 2}{6x + 6y + 2z + 3w \geq 9}$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Linear combination $$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i)\ell_i \geq c_A A + c_B B} \quad [c_A, c_B \in \mathbb{N}]$$

Division $$\frac{\sum_i c a_i \ell_i \geq A}{\sum_i a_i \ell_i \geq \lceil A/c \rceil} \quad [c \in \mathbb{N}^+]$$

Toy example:

$$\dfrac{\dfrac{2x + 4y + 2z + w \geq 5 \qquad 2x + y + w \geq 2}{6x + 6y + 2z + 3w \geq 9}\text{Lin comb} \qquad \overline{z} \geq 0}{}\text{Lin comb}$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Literal axioms $\dfrac{}{\ell_i \geq 0}$

Linear combination $\dfrac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i)\ell_i \geq c_A A + c_B B}$ $[c_A, c_B \in \mathbb{N}]$

Division $\dfrac{\sum_i c a_i \ell_i \geq A}{\sum_i a_i \ell_i \geq \lceil A/c \rceil}$ $[c \in \mathbb{N}^+]$

Toy example:

$$\dfrac{\text{Lin comb}\ \dfrac{2x + 4y + 2z + w \geq 5 \qquad 2x + y + w \geq 2}{6x + 6y + 2z + 3w \geq 9} \qquad \overline{z} \geq 0}{6x + 6y + 2z + 3w + 2 \cdot \overline{z} \geq 9 + 2 \cdot 0}\ \text{Lin comb}$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Literal axioms $\dfrac{}{\ell_i \geq 0}$

Linear combination $\dfrac{\sum_i a_i\ell_i \geq A \qquad \sum_i b_i\ell_i \geq B}{\sum_i (c_A a_i + c_B b_i)\ell_i \geq c_A A + c_B B}$ $[c_A, c_B \in \mathbb{N}]$

Division $\dfrac{\sum_i c a_i\ell_i \geq A}{\sum_i a_i\ell_i \geq \lceil A/c \rceil}$ $[c \in \mathbb{N}^+]$

Toy example:

$$\text{Lin comb}\ \dfrac{\text{Lin comb}\ \dfrac{2x + 4y + 2z + w \geq 5 \qquad 2x + y + w \geq 2}{6x + 6y + 2z + 3w \geq 9} \qquad \overline{z} \geq 0}{6x + 6y + 3w + 2 \geq 9}$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Linear combination $$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i)\ell_i \geq c_A A + c_B B} \quad [c_A, c_B \in \mathbb{N}]$$

Division $$\frac{\sum_i c a_i \ell_i \geq A}{\sum_i a_i \ell_i \geq \lceil A/c \rceil} \quad [c \in \mathbb{N}^+]$$

Toy example:

$$\text{Lin comb}\ \dfrac{\text{Lin comb}\ \dfrac{2x + 4y + 2z + w \geq 5 \qquad 2x + y + w \geq 2}{6x + 6y + 2z + 3w \geq 9} \qquad \overline{z} \geq 0}{6x + 6y + 3w \geq 7}$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Linear combination $$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i)\ell_i \geq c_A A + c_B B} \quad [c_A, c_B \in \mathbb{N}]$$

Division $$\frac{\sum_i c a_i \ell_i \geq A}{\sum_i a_i \ell_i \geq \lceil A/c \rceil} \quad [c \in \mathbb{N}^+]$$

Toy example:

$$\dfrac{\text{Lin comb}\ \dfrac{\text{Lin comb}\ \dfrac{2x + 4y + 2z + w \geq 5 \qquad 2x + y + w \geq 2}{6x + 6y + 2z + 3w \geq 9} \qquad \overline{z} \geq 0}{6x + 6y + 3w \geq 7}}{2x + 2y + w \geq 2\frac{1}{3}}\ \text{Div}$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Linear combination $$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (c_A a_i + c_B b_i)\ell_i \geq c_A A + c_B B} \quad [c_A, c_B \in \mathbb{N}]$$

Division $$\frac{\sum_i c a_i \ell_i \geq A}{\sum_i a_i \ell_i \geq \lceil A/c \rceil} \quad [c \in \mathbb{N}^+]$$

Toy example:

$$\dfrac{\dfrac{\dfrac{2x + 4y + 2z + w \geq 5 \qquad 2x + y + w \geq 2}{6x + 6y + 2z + 3w \geq 9}\text{Lin comb} \qquad \overline{z} \geq 0}{6x + 6y + 3w \geq 7}\text{Lin comb}}{2x + 2y + w \geq 3}\text{Div}$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Literal axioms $$\frac{}{\ell_i \geq 0}$$

Linear combination $$\frac{\sum_i a_i\ell_i \geq A \qquad \sum_i b_i\ell_i \geq B}{\sum_i (c_A a_i + c_B b_i)\ell_i \geq c_A A + c_B B} \quad [c_A, c_B \in \mathbb{N}]$$

Division $$\frac{\sum_i c a_i\ell_i \geq A}{\sum_i a_i\ell_i \geq \lceil A/c \rceil} \quad [c \in \mathbb{N}^+]$$

Toy example:

$$\dfrac{\dfrac{\dfrac{2x+4y+2z+w \geq 5 \qquad 2x+y+w \geq 2}{6x+6y+2z+3w \geq 9}\text{Lin comb} \qquad \overline{z} \geq 0}{6x+6y+3w \geq 7}\text{Lin comb}}{2x+2y+w \geq 3}\text{Div}$$

(See [BN21] for more details about cutting planes)

Proof Logging for SAT with Pseudo-Boolean Reasoning

- View clauses as pseudo-Boolean constraints

Proof Logging for SAT with Pseudo-Boolean Reasoning

- View clauses as pseudo-Boolean constraints
- Operate on constraints with cutting planes rules

Proof Logging for SAT with Pseudo-Boolean Reasoning

- View clauses as pseudo-Boolean constraints
- Operate on constraints with cutting planes rules
- Prove unsatisfiability by deriving $0 \geq 1$

Proof Logging for SAT with Pseudo-Boolean Reasoning

- View clauses as pseudo-Boolean constraints
- Operate on constraints with cutting planes rules
- Prove unsatisfiability by deriving $0 \geq 1$
- Generalize **reverse unit propagation (RUP)** rule [GN03, Van08] to PB constraints — just convenient shorthand for derivation

Proof Logging for SAT with Pseudo-Boolean Reasoning

- View clauses as pseudo-Boolean constraints
- Operate on constraints with cutting planes rules
- Prove unsatisfiability by deriving $0 \geq 1$
- Generalize **reverse unit propagation (RUP)** rule [GN03, Van08] to PB constraints — just convenient shorthand for derivation
- Also need **extension** rule (analogue of RAT [JHB12]) to deal with, e.g., preprocessing

Extension Rule: Redundance-Based Strengthening

C is redundant with respect to F if F and $F \wedge C$ are equisatisfiable

Want to allow adding redundant constraints

Extension Rule: Redundance-Based Strengthening

C is redundant with respect to F if F and $F \wedge C$ are equisatisfiable
Want to allow adding redundant constraints

Redundance-based strengthening [BT19, GN21]

C is redundant with respect to F if and only if there is a substitution ω (mapping variables to truth values or literals), called a witness, for which

$$F \wedge \neg C \models (F \wedge C)\restriction_{\omega}$$

Extension Rule: Redundance-Based Strengthening

C is redundant with respect to F if F and $F \wedge C$ are equisatisfiable
Want to allow adding redundant constraints

Redundance-based strengthening [BT19, GN21]

C is redundant with respect to F if and only if there is a substitution ω (mapping variables to truth values or literals), called a witness, for which

$$F \wedge \neg C \models (F \wedge C)\restriction_{\omega}$$

- Proof sketch for interesting direction: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \wedge C$

Extension Rule: Redundance-Based Strengthening

C is redundant with respect to F if F and $F \land C$ are equisatisfiable
Want to allow adding redundant constraints

Redundance-based strengthening [BT19, GN21]

C is redundant with respect to F if and only if there is a substitution ω (mapping variables to truth values or literals), called a witness, for which

$$F \land \neg C \models (F \land C)\restriction_{\omega}$$

- Proof sketch for interesting direction: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \land C$
- Implication should be efficiently verifiable — every $D \in (F \land C)\restriction_{\omega}$ should follow from $F \land \neg C$ by, e.g.,
  1. weakening (addition of literal axioms $\ell_i \geq 0$)
  2. reverse unit propagation (RUP)
  3. explicit derivation presented in proof log

The Power of Proof Logging with Extended Cutting Planes

0-1 linear inequalities convenient to capture SAT reasoning (with clauses)

The Power of Proof Logging with Extended Cutting Planes

0-1 linear inequalities convenient to capture SAT reasoning (with clauses)

And yields efficient proof logging for wider range of problems/algorithms:

- Pre- and inprocessing [GN21] (since redundance rule subsumes RAT)
- Pseudo-Boolean reasoning (by design)
- Gaussian elimination [GN21]
- Subgraph problems [GMN20, GMM$^+$20]
- Pseudo-Boolean solving via translation to CNF [GMNO22]
- Core-guided MaxSAT solving [VDB22, BBN$^+$23]
- Constraint programming [EGMN20, GMN22]
- **This talk:** Symmetry and dominance breaking [BGMN22]

The Power of Proof Logging with Extended Cutting Planes

0-1 linear inequalities convenient to capture SAT reasoning (with clauses)

And yields efficient proof logging for wider range of problems/algorithms:

- Pre- and inprocessing [GN21] (since redundance rule subsumes RAT)
- Pseudo-Boolean reasoning (by design)
- Gaussian elimination [GN21]
- Subgraph problems [GMN20, GMM$^+$20]
- Pseudo-Boolean solving via translation to CNF [GMNO22]
- Core-guided MaxSAT solving [VDB22, BBN$^+$23]
- Constraint programming [EGMN20, GMN22]
- **This talk:** Symmetry and dominance breaking [BGMN22]

More info in tutorial Combinatorial solving with provably correct results (`https://youtu.be/s_5BIi4I22w`) on pseudo-Boolean proof logging

The Challenge of Symmetries

(Syntactic) symmetry: substitution σ preserving F ($F\restriction_\sigma \doteq F$)

- Show up in some hard SAT benchmarks
- Can play crucial role in CP and MIP problems [AW13, GSVW14]

The Challenge of Symmetries

(Syntactic) symmetry: substitution σ preserving F ($F\restriction_\sigma \doteq F$)

- Show up in some hard SAT benchmarks
- Can play crucial role in CP and MIP problems [AW13, GSVW14]

Symmetry breaking in SAT
Add constraints filtering out symmetric solutions [ASM06, DBBD16]

Symmetric learning in SAT
Allow to add all symmetric versions of learned constraint [DBB17]

The Challenge of Symmetries

(Syntactic) symmetry: substitution σ preserving F ($F{\upharpoonright}_\sigma \doteq F$)

- Show up in some hard SAT benchmarks
- Can play crucial role in CP and MIP problems [AW13, GSVW14]

Symmetry breaking in SAT
Add constraints filtering out symmetric solutions [ASM06, DBBD16]

Symmetric learning in SAT
Allow to add all symmetric versions of learned constraint [DBB17]

Not supported by standard SAT proof logging!

Optimisation Problems

Deal with symmetry breaking by switching focus to optimisation
(which the title of the talk kind of promised anyway)

Optimisation Problems

Deal with symmetry breaking by switching focus to optimisation (which the title of the talk kind of promised anyway)

Pseudo-Boolean optimisation

Minimize $f = \sum_i w_i \ell_i$ (for $w_i \in \mathbb{N}$) subject to constraints in F

Optimisation Problems

Deal with symmetry breaking by switching focus to optimisation
(which the title of the talk kind of promised anyway)

Pseudo-Boolean optimisation

Minimize $f = \sum_i w_i \ell_i$ (for $w_i \in \mathbb{N}$) subject to constraints in F

Proof of optimality:

- F satisfied by α
- $F \wedge \left(\sum_i w_i \ell_i < \sum_i w_i \cdot \alpha(\ell_i)\right)$ is infeasible

Optimisation Problems

Deal with symmetry breaking by switching focus to optimisation (which the title of the talk kind of promised anyway)

Pseudo-Boolean optimisation

Minimize $f = \sum_i w_i \ell_i$ (for $w_i \in \mathbb{N}$) subject to constraints in F

Proof of optimality:

- F satisfied by α
- $F \wedge \left(\sum_i w_i \ell_i < \sum_i w_i \cdot \alpha(\ell_i)\right)$ is infeasible

Note that $\sum_i w_i \ell_i < \sum_i w_i \cdot \alpha(\ell_i)$ means $\sum_i w_i \ell_i \leq -1 + \sum_i w_i \cdot \alpha(\ell_i)$

Proof Logging for Optimisation Problems

How does proof system change?

Proof Logging for Optimisation Problems

How does proof system change?

Rules must preserve (at least one) optimal solution

Proof Logging for Optimisation Problems

How does proof system change?
Rules must preserve (at least one) optimal solution

1. Standard cutting planes rules OK — derive constraints that must hold for any satisfying assignment

Proof Logging for Optimisation Problems

How does proof system change?
Rules must preserve (at least one) optimal solution

1. Standard cutting planes rules OK — derive constraints that must hold for any satisfying assignment
2. Once solution α has been found, allow constraint $\sum_i w_i\ell_i < \sum_i w_i \cdot \alpha(\ell_i)$ to force search for better solutions

Proof Logging for Optimisation Problems

How does proof system change?
Rules must preserve (at least one) optimal solution

1. Standard cutting planes rules OK — derive constraints that must hold for any satisfying assignment
2. Once solution α has been found, allow constraint $\sum_i w_i \ell_i < \sum_i w_i \cdot \alpha(\ell_i)$ to force search for better solutions
3. Redundance rule must not destroy good solutions

Proof Logging for Optimisation Problems

How does proof system change?
Rules must preserve (at least one) optimal solution

1. Standard cutting planes rules OK — derive constraints that must hold for any satisfying assignment

2. Once solution α has been found, allow constraint $\sum_i w_i \ell_i < \sum_i w_i \cdot \alpha(\ell_i)$ to force search for better solutions

3. Redundance rule must not destroy good solutions

Redundance-based strengthening, optimisation version

Add constraint C to formula F if exists witness substitution ω such that

$$F \wedge \neg C \models (F \wedge C)\restriction_\omega \;\wedge\; f\restriction_\omega \leq f$$

Redundance and Dominance Rules

Redundance-based strengthening, optimisation version

Add constraint C to formula F if exists witness substitution ω such that

$$F \land \neg C \models (F \land C)\restriction_\omega \land f\restriction_\omega \leq f$$

Redundance and Dominance Rules

Redundance-based strengthening, optimisation version

Add constraint C to formula F if exists witness substitution ω such that

$$F \land \neg C \models (F \land C)\restriction_\omega \land f\restriction_\omega \leq f$$

Can be more aggressive if witness ω strictly improves solution

Redundance and Dominance Rules

Redundance-based strengthening, optimisation version

Add constraint C to formula F if exists witness substitution ω such that

$$F \wedge \neg C \models (F \wedge C)\restriction_\omega \wedge f\restriction_\omega \leq f$$

Can be more aggressive if witness ω strictly improves solution

Dominance-based strengthening (simplified)

Add constraint C to formula F if exists witness substitution ω such that

$$F \wedge \neg C \models F\restriction_\omega \wedge f\restriction_\omega < f$$

Redundance and Dominance Rules

Redundance-based strengthening, optimisation version

Add constraint C to formula F if exists witness substitution ω such that

$$F \wedge \neg C \models (F \wedge C)\restriction_\omega \ \wedge \ f\restriction_\omega \leq f$$

Can be more aggressive if witness ω strictly improves solution

Dominance-based strengthening (simplified)

Add constraint C to formula F if exists witness substitution ω such that

$$F \wedge \neg C \models F\restriction_\omega \ \wedge \ f\restriction_\omega < f$$

- Applying ω should strictly decrease f
- If so, don't need to show that $C\restriction_\omega$ implied!

Soundness of Dominance Rule

Dominance-based strengthening (simplified)

Add constraint C to formula F if exists witness substitution ω such that

$$F \wedge \neg C \models F\!\restriction_\omega \wedge\ f\!\restriction_\omega < f$$

Why is this sound?

Soundness of Dominance Rule

Dominance-based strengthening (simplified)

Add constraint C to formula F if exists witness substitution ω such that

$$F \wedge \neg C \models F\restriction_\omega \wedge f\restriction_\omega < f$$

Why is this sound?

1. Suppose α satisfies F but falsifies C (i.e., satisfies $\neg C$)

Soundness of Dominance Rule

Dominance-based strengthening (simplified)

Add constraint C to formula F if exists witness substitution ω such that

$$F \wedge \neg C \models F\restriction_\omega \wedge f\restriction_\omega < f$$

Why is this sound?

1. Suppose α satisfies F but falsifies C (i.e., satisfies $\neg C$)
2. Then $\alpha \circ \omega$ satisfies F and $f(\alpha \circ \omega) < f(\alpha)$

Soundness of Dominance Rule

Dominance-based strengthening (simplified)

Add constraint C to formula F if exists witness substitution ω such that

$$F \wedge \neg C \models F\restriction_{\omega} \wedge f\restriction_{\omega} < f$$

Why is this sound?

1. Suppose α satisfies F but falsifies C (i.e., satisfies $\neg C$)
2. Then $\alpha \circ \omega$ satisfies F and $f(\alpha \circ \omega) < f(\alpha)$
3. If $\alpha \circ \omega$ satisfies C, we're done

Soundness of Dominance Rule

Dominance-based strengthening (simplified)

Add constraint C to formula F if exists witness substitution ω such that

$$F \wedge \neg C \models F\restriction_\omega \wedge f\restriction_\omega < f$$

Why is this sound?

1. Suppose α satisfies F but falsifies C (i.e., satisfies $\neg C$)
2. Then $\alpha \circ \omega$ satisfies F and $f(\alpha \circ \omega) < f(\alpha)$
3. If $\alpha \circ \omega$ satisfies C, we're done
4. Otherwise $(\alpha \circ \omega) \circ \omega$ satisfies F and $f((\alpha \circ \omega) \circ \omega) < f(\alpha \circ \omega)$

Soundness of Dominance Rule

Dominance-based strengthening (simplified)

Add constraint C to formula F if exists witness substitution ω such that

$$F \wedge \neg C \models F\restriction_\omega \wedge f\restriction_\omega < f$$

Why is this sound?

1. Suppose α satisfies F but falsifies C (i.e., satisfies $\neg C$)
2. Then $\alpha \circ \omega$ satisfies F and $f(\alpha \circ \omega) < f(\alpha)$
3. If $\alpha \circ \omega$ satisfies C, we're done
4. Otherwise $(\alpha \circ \omega) \circ \omega$ satisfies F and $f((\alpha \circ \omega) \circ \omega) < f(\alpha \circ \omega)$
5. If $(\alpha \circ \omega) \circ \omega$ satisfies C, we're done

Soundness of Dominance Rule

Dominance-based strengthening (simplified)

Add constraint C to formula F if exists witness substitution ω such that

$$F \land \neg C \models F\restriction_\omega \;\land\; f\restriction_\omega < f$$

Why is this sound?

1. Suppose α satisfies F but falsifies C (i.e., satisfies $\neg C$)
2. Then $\alpha \circ \omega$ satisfies F and $f(\alpha \circ \omega) < f(\alpha)$
3. If $\alpha \circ \omega$ satisfies C, we're done
4. Otherwise $(\alpha \circ \omega) \circ \omega$ satisfies F and $f((\alpha \circ \omega) \circ \omega) < f(\alpha \circ \omega)$
5. If $(\alpha \circ \omega) \circ \omega$ satisfies C, we're done
6. Otherwise $((\alpha \circ \omega) \circ \omega) \circ \omega$ satisfies F and $f(((\alpha \circ \omega) \circ \omega) \circ \omega) < f((\alpha \circ \omega) \circ \omega)$

Soundness of Dominance Rule

Dominance-based strengthening (simplified)

Add constraint C to formula F if exists witness substitution ω such that

$$F \wedge \neg C \models F{\restriction_\omega} \ \wedge \ f{\restriction_\omega} < f$$

Why is this sound?

1. Suppose α satisfies F but falsifies C (i.e., satisfies $\neg C$)
2. Then $\alpha \circ \omega$ satisfies F and $f(\alpha \circ \omega) < f(\alpha)$
3. If $\alpha \circ \omega$ satisfies C, we're done
4. Otherwise $(\alpha \circ \omega) \circ \omega$ satisfies F and $f((\alpha \circ \omega) \circ \omega) < f(\alpha \circ \omega)$
5. If $(\alpha \circ \omega) \circ \omega$ satisfies C, we're done
6. Otherwise $((\alpha \circ \omega) \circ \omega) \circ \omega$ satisfies F and $f(((\alpha \circ \omega) \circ \omega) \circ \omega) < f((\alpha \circ \omega) \circ \omega)$
7. ...

Soundness of Dominance Rule

Dominance-based strengthening (simplified)

Add constraint C to formula F if exists witness substitution ω such that

$$F \wedge \neg C \models F\restriction_\omega \wedge f\restriction_\omega < f$$

Why is this sound?

1. Suppose α satisfies F but falsifies C (i.e., satisfies $\neg C$)
2. Then $\alpha \circ \omega$ satisfies F and $f(\alpha \circ \omega) < f(\alpha)$
3. If $\alpha \circ \omega$ satisfies C, we're done
4. Otherwise $(\alpha \circ \omega) \circ \omega$ satisfies F and $f((\alpha \circ \omega) \circ \omega) < f(\alpha \circ \omega)$
5. If $(\alpha \circ \omega) \circ \omega$ satisfies C, we're done
6. Otherwise $((\alpha \circ \omega) \circ \omega) \circ \omega$ satisfies F and $f(((\alpha \circ \omega) \circ \omega) \circ \omega) < f((\alpha \circ \omega) \circ \omega)$
7. ...
8. Can't go on forever, so finally reach α' satisfying $F \wedge C$

Strength of Dominance Rule

Dominance-based strengthening (stronger, still simplified)

If $C_1, C_2, \ldots, C_{m-1}$ have been derived from F (maybe using dominance), then can derive also C_m if exists witness substitution ω such that

$$F \wedge \bigwedge_{i=1}^{m-1} C_i \wedge \neg C_m \models F\restriction_\omega \wedge f\restriction_\omega < f$$

Only consider original formula — no need to show that any $C_i\restriction_\omega$ implied!

Strength of Dominance Rule

Dominance-based strengthening (stronger, still simplified)

If $C_1, C_2, \ldots, C_{m-1}$ have been derived from F (maybe using dominance), then can derive also C_m if exists witness substitution ω such that

$$F \wedge \bigwedge_{i=1}^{m-1} C_i \wedge \neg C_m \models F{\restriction_\omega} \wedge f{\restriction_\omega} < f$$

Only consider original formula — no need to show that any $C_i{\restriction_\omega}$ implied!

Now why is **this** sound?

- Same inductive proof as before, but nested
- Or pick α satisfying F and minimizing f and argue by contradiction

Strength of Dominance Rule

Dominance-based strengthening (stronger, still simplified)

If $C_1, C_2, \ldots, C_{m-1}$ have been derived from F (maybe using dominance), then can derive also C_m if exists witness substitution ω such that

$$F \wedge \bigwedge_{i=1}^{m-1} C_i \wedge \neg C_m \models F\restriction_\omega \wedge f\restriction_\omega < f$$

Only consider original formula — no need to show that any $C_i\restriction_\omega$ implied!

Now why is **this** sound?

- Same inductive proof as before, but nested
- Or pick α satisfying F and minimizing f and argue by contradiction

Further extensions:

- Define dominance rule w.r.t. order independent of objective function
- Switch between different orders in same proof
- See [BGMN22] for details

Strategy for SAT Symmetry Breaking

1. Pretend to solve optimisation problem minimizing $f \doteq \sum_{i=1}^{n} 2^{n-i} \cdot x_i$ (searching lexicographically smallest assignment satisfying formula)

Strategy for SAT Symmetry Breaking

1. Pretend to solve optimisation problem minimizing $f \doteq \sum_{i=1}^{n} 2^{n-i} \cdot x_i$ (searching lexicographically smallest assignment satisfying formula)

2. Derive pseudo-Boolean lex-leader constraint

$$\begin{aligned} C_\sigma &\doteq f \leq f\restriction_\sigma \\ &\doteq \sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0 \end{aligned}$$

Strategy for SAT Symmetry Breaking

1. Pretend to solve optimisation problem minimizing $f \doteq \sum_{i=1}^{n} 2^{n-i} \cdot x_i$ (searching lexicographically smallest assignment satisfying formula)
2. Derive pseudo-Boolean lex-leader constraint

$$
\begin{aligned}
C_\sigma &\doteq f \leq f\restriction_\sigma \\
&\doteq \sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0
\end{aligned}
$$

3. Derive CNF encoding of lex-leader constraints from PB constraint (in same spirit as [GMNO22])

$$
\begin{array}{ll}
y_0 & \overline{y}_j \vee \overline{\sigma(x_j)} \vee x_j \\
\overline{y}_{j-1} \vee \overline{x}_j \vee \sigma(x_j) & y_j \vee \overline{y}_{j-1} \vee \overline{x}_j \\
\overline{y}_j \vee y_{j-1} & y_j \vee \overline{y}_{j-1} \vee \sigma(x_j)
\end{array}
$$

Breaking Symmetries With the Dominance Rule (1/2)

Theorem

$C_\sigma \doteq f \leq f\restriction_\sigma$ can be derived from F using dominance with witness σ

$$F \wedge \neg C_\sigma \models F\restriction_\sigma \wedge f\restriction_\sigma < f$$

Breaking Symmetries With the Dominance Rule (1/2)

Theorem

$C_\sigma \doteq f \leq f\restriction_\sigma$ *can be derived from* F *using dominance with witness* σ

$$F \wedge \neg C_\sigma \models F\restriction_\sigma \wedge f\restriction_\sigma < f$$

Comparison to DRAT-style proofs

Redundance-based strengthening can be used analogously to [HHW15]

Breaking Symmetries With the Dominance Rule (1/2)

Theorem

$C_\sigma \doteq f \leq f\restriction_\sigma$ can be derived from F using dominance with witness σ

$$F \wedge \neg C_\sigma \models F\restriction_\sigma \wedge f\restriction_\sigma < f$$

Comparison to DRAT-style proofs

Redundance-based strengthening can be used analogously to [HHW15]

- but only guaranteed to work for breaking single symmetry σ

Breaking Symmetries With the Dominance Rule (1/2)

Theorem

$C_\sigma \doteq f \leq f\restriction_\sigma$ can be derived from F using dominance with witness σ

$$F \wedge \neg C_\sigma \models F\restriction_\sigma \wedge f\restriction_\sigma < f$$

Comparison to DRAT-style proofs

Redundance-based strengthening can be used analogously to [HHW15]

- but only guaranteed to work for breaking single symmetry σ
- if σ is involution (i.e., its own inverse)

Breaking Symmetries With the Dominance Rule (1/2)

Theorem

$C_\sigma \doteq f \leq f\!\restriction_\sigma$ can be derived from F using dominance with witness σ

$$F \land \neg C_\sigma \models F\!\restriction_\sigma \land f\!\restriction_\sigma < f$$

Comparison to DRAT-style proofs

Redundance-based strengthening can be used analogously to [HHW15]

- but only guaranteed to work for breaking single symmetry σ
- if σ is involution (i.e., its own inverse)
- not known how to deal with symmetries that are complex or interact

Breaking Symmetries With the Dominance Rule (2/2)

Breaking symmetries with the dominance rule

- Surprisingly simple

Breaking Symmetries With the Dominance Rule (2/2)

Breaking symmetries with the dominance rule

- Surprisingly simple
- Generalizes well (compared to redundance-based symmetry breaking)

Breaking Symmetries With the Dominance Rule (2/2)

Breaking symmetries with the dominance rule

- Surprisingly simple
- Generalizes well (compared to redundance-based symmetry breaking)
  - Works for arbitrary symmetries

Breaking Symmetries With the Dominance Rule (2/2)

Breaking symmetries with the dominance rule

- Surprisingly simple
- Generalizes well (compared to redundance-based symmetry breaking)
  - Works for arbitrary symmetries
  - Works for multiple symmetries (ignore previously derived constraints)

$$F \wedge C_\sigma \wedge \neg C_\tau \models F{\restriction_\tau} \wedge f{\restriction_\tau} < f$$

Breaking Symmetries With the Dominance Rule (2/2)

Breaking symmetries with the dominance rule

- Surprisingly simple
- Generalizes well (compared to redundance-based symmetry breaking)
    - Works for arbitrary symmetries
    - Works for multiple symmetries (ignore previously derived constraints)

$$F \wedge C_\sigma \wedge \neg C_\tau \models F\restriction_\tau \wedge f\restriction_\tau < f$$

Why does it work?

- Witness need not satisfy all derived constraints
- Sufficient to just produce “better” assignment

Applied Symmetry Breaking for SAT Solving

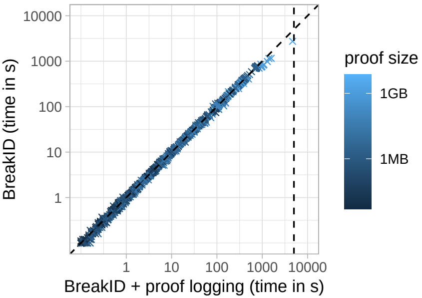
1. Break symmetries with BreakID and get proofs for symmetry breaking clauses
2. Concatenate with CDCL solver proof (DRAT rewritten in VeriPB format)

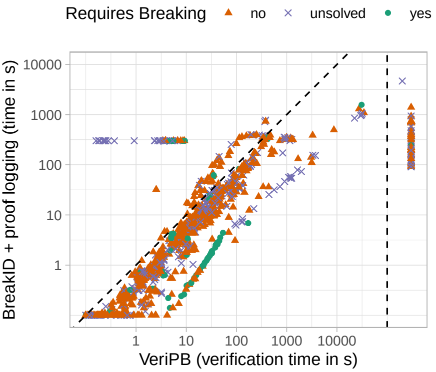
Short dictionary for DRAT-to-VeriPB translations

DRAT	VeriPB
`1`	`x1`
`-2`	`~x2`
`1 -2 3 0`	`+1 x1 +1 ~x2 + 1 x3 >= 1 ;`
`1 -2 3 0` is RUP	`rup +1 x1 +1 ~x2 + 1 x3 >= 1 ;`
`1 -2 3 0` is RAT	`red +1 x1 +1 ~x2 + 1 x3 >= 1 ; x1 ->1`

Experimental Evaluation

- Evaluated on SAT competition benchmarks
- BreakID [DBBD16, Bre] used to find and break symmetries





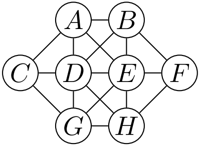
- proof logging overhead negligible
- verification at most 20 times slower than solving for 95% of instances

Symmetry Breaking for Constraint Programming

Crystal Maze puzzle

Place numbers 1 to 8 without repetition

Adjacent circles mustn't have consecutive numbers

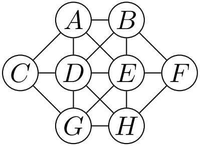


Symmetry Breaking for Constraint Programming

Crystal Maze puzzle

Place numbers 1 to 8 without repetition

Adjacent circles mustn't have consecutive numbers

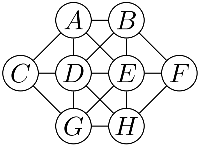


Without loss of generality:

- $A < G$ (horizontal mirror symmetry)
- $A < B$ (vertical mirror symmetry)
- $A \leq 4$ (value symmetry)

Symmetry Breaking for Constraint Programming

Crystal Maze puzzle
Place numbers 1 to 8 without repetition
Adjacent circles mustn't have consecutive numbers



Without loss of generality:

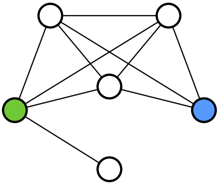
- $A < G$ (horizontal mirror symmetry)
- $A < B$ (vertical mirror symmetry)
- $A \leq 4$ (value symmetry)

Technical challenge: integer-valued variables
See [GMN22] for more detailed discussion

Dominance Breaking for Maximum Clique Solving

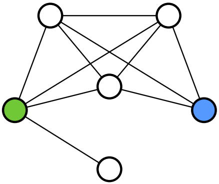
Maximum clique solving

Find largest fully connected component



Dominance Breaking for Maximum Clique Solving

Maximum clique solving
Find largest fully connected component



Lazy global domination [MP16]
Only consider green and not blue vertex
(since every neighbour of blue is also neighbour of green)

Technical challenge: vertex domination detected only lazily during search
Dominance rule (rather than redundance rule) really helpful here

Future Research Directions

Performance and reliability of pseudo-Boolean proof logging

- Trim proof while verifying (as in DRAT-TRIM [HHW13a])
- Compress proof file using binary format
- Design formally verified proof checker *(work in progress [BMM+23])*

Future Research Directions

Performance and reliability of pseudo-Boolean proof logging

- Trim proof while verifying (as in DRAT-TRIM [HHW13a])
- Compress proof file using binary format
- Design formally verified proof checker *(work in progress [BMM+23])*

Proof logging for other combinatorial problems and techniques

- Symmetric learning and recycling (substitution) of subproofs
- MaxSAT solving and PB optimization *(work in progress [VDB22, BBN+23])*
- Mixed integer linear programming *(work on SCIP in [CGS17, EG21])*
- Satisfiability modulo theories (SMT) solving *(work by Bjørner and others)*
- Planning *(work in progress)*

Future Research Directions

Performance and reliability of pseudo-Boolean proof logging

- Trim proof while verifying (as in DRAT-TRIM [HHW13a])
- Compress proof file using binary format
- Design formally verified proof checker *(work in progress [BMM+23])*

Proof logging for other combinatorial problems and techniques

- Symmetric learning and recycling (substitution) of subproofs
- MaxSAT solving and PB optimization *(work in progress [VDB22, BBN+23])*
- Mixed integer linear programming *(work on SCIP in [CGS17, EG21])*
- Satisfiability modulo theories (SMT) solving *(work by Bjørner and others)*
- Planning *(work in progress)*

And more...

- Lots of challenging problems and interesting ideas

Future Research Directions

Performance and reliability of pseudo-Boolean proof logging

- Trim proof while verifying (as in DRAT-TRIM [HHW13a])
- Compress proof file using binary format
- Design formally verified proof checker *(work in progress [BMM+ 23])*

Proof logging for other combinatorial problems and techniques

- Symmetric learning and recycling (substitution) of subproofs
- MaxSAT solving and PB optimization *(work in progress [VDB22, BBN+ 23])*
- Mixed integer linear programming *(work on SCIP in [CGS17, EG21])*
- Satisfiability modulo theories (SMT) solving *(work by Bjørner and others)*
- Planning *(work in progress)*

And more...

- Lots of challenging problems and interesting ideas
- We're hiring! Talk to me to join the proof logging revolution! ☺

Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity
- **This work:** Efficient proof logging for symmetry and dominance breaking using cutting planes with extensions

Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity
- **This work:** Efficient proof logging for symmetry and dominance breaking using cutting planes with extensions

Thank you for your attention!

References I

[ABM+11] Eyad Alkassar, Sascha Böhme, Kurt Mehlhorn, Christine Rizkallah, and Pascal Schweitzer. An introduction to certifying algorithms. *it - Information Technology Methoden und innovative Anwendungen der Informatik und Informationstechnik*, 53(6):287–293, December 2011.

[AGJ+18] Özgür Akgün, Ian P. Gent, Christopher Jefferson, Ian Miguel, and Peter Nightingale. Metamorphic testing of constraint solvers. In *Proceedings of the 24th International Conference on Principles and Practice of Constraint Programming (CP '18)*, volume 11008 of *Lecture Notes in Computer Science*, pages 727–736. Springer, August 2018.

[ASM06] Fadi A. Aloul, Karem A. Sakallah, and Igor L. Markov. Efficient symmetry breaking for Boolean satisfiability. *IEEE Transactions on Computers*, 55(5):549–558, May 2006. Preliminary version in *IJCAI '03*.

[AW13] Tobias Achterberg and Roland Wunderling. Mixed integer programming: Analyzing 12 years of progress. In Michael Jünger and Gerhard Reinelt, editors, *Facets of Combinatorial Optimization*, pages 449–481. Springer, 2013.

References II

[BBH22] Randal E. Bryant, Armin Biere, and Marijn J. H. Heule. Clausal proofs for pseudo-Boolean reasoning. In *Proceedings of the 28th International Conference on Tools and Algorithms for the Construction and Analysis of Systems (TACAS '22)*, volume 13243 of *Lecture Notes in Computer Science*, pages 443–461. Springer, April 2022.

[BBN+23] Jeremias Berg, Bart Bogaerts, Jakob Nordström, Andy Oertel, and Dieter Vandesande. Certified core-guided MaxSAT solving. In *Proceedings of the 29th International Conference on Automated Deduction (CADE-29)*, July 2023. To appear.

[BGMN22] Bart Bogaerts, Stephan Gocht, Ciaran McCreesh, and Jakob Nordström. Certified symmetry and dominance breaking for combinatorial optimisation. In *Proceedings of the 36th AAAI Conference on Artificial Intelligence (AAAI '22)*, pages 3698–3707, February 2022.

[BHvMW21] Armin Biere, Marijn J. H. Heule, Hans van Maaren, and Toby Walsh, editors. *Handbook of Satisfiability*, volume 336 of *Frontiers in Artificial Intelligence and Applications*. IOS Press, 2nd edition, February 2021.

References III

[BLB10] Robert Brummayer, Florian Lonsing, and Armin Biere. Automated testing and debugging of SAT and QBF solvers. In *Proceedings of the 13th International Conference on Theory and Applications of Satisfiability Testing (SAT '10)*, volume 6175 of *Lecture Notes in Computer Science*, pages 44–57. Springer, July 2010.

[BMM+23] Bart Bogaerts, Ciaran McCreesh, Magnus O. Myreen, Jakob Nordström, Andy Oertel, and Yong Kiam Tan. Documentation of VeriPB and CakePB for the SAT competition 2023. Available at `https://satcompetition.github.io/2023/checkers.html`, March 2023.

[BN21] Samuel R. Buss and Jakob Nordström. Proof complexity and SAT solving. In Biere et al. [BHvMW21], chapter 7, pages 233–350.

[BR07] Robert Bixby and Edward Rothberg. Progress in computational mixed integer programming—A look back from the other side of the tipping point. *Annals of Operations Research*, 149(1):37–41, February 2007.

[Bre] Breakid. `https://bitbucket.org/krr/breakid`.

[Bry22] Randal E. Bryant. TBUDDY: a proof-generating BDD package. EasyChair Preprint 8471, July 2022. Available at `https://easychair.org/publications/preprint/DbRN`.

References IV

[BT19] Samuel R. Buss and Neil Thapen. DRAT proofs, propagation redundancy, and extended resolution. In *Proceedings of the 22nd International Conference on Theory and Applications of Satisfiability Testing (SAT '19)*, volume 11628 of *Lecture Notes in Computer Science*, pages 71–89. Springer, July 2019.

[CCT87] William Cook, Collette Rene Coullard, and György Turán. On the complexity of cutting-plane proofs. *Discrete Applied Mathematics*, 18(1):25–38, November 1987.

[CGS17] Kevin K. H. Cheung, Ambros M. Gleixner, and Daniel E. Steffy. Verifying integer programming results. In *Proceedings of the 19th International Conference on Integer Programming and Combinatorial Optimization (IPCO '17)*, volume 10328 of *Lecture Notes in Computer Science*, pages 148–160. Springer, June 2017.

[CHH+17] Luís Cruz-Filipe, Marijn J. H. Heule, Warren A. Hunt Jr., Matt Kaufmann, and Peter Schneider-Kamp. Efficient certified RAT verification. In *Proceedings of the 26th International Conference on Automated Deduction (CADE-26)*, volume 10395 of *Lecture Notes in Computer Science*, pages 220–236. Springer, August 2017.

[CKSW13] William Cook, Thorsten Koch, Daniel E. Steffy, and Kati Wolter. A hybrid branch-and-bound approach for exact rational mixed-integer programming. *Mathematical Programming Computation*, 5(3):305–344, September 2013.

References V

[CMS17] Luís Cruz-Filipe, João P. Marques-Silva, and Peter Schneider-Kamp. Efficient certified resolution proof checking. In *Proceedings of the 23rd International Conference on Tools and Algorithms for the Construction and Analysis of Systems (TACAS '17)*, volume 10205 of *Lecture Notes in Computer Science*, pages 118–135. Springer, April 2017.

[DBB17] Jo Devriendt, Bart Bogaerts, and Maurice Bruynooghe. Symmetric explanation learning: Effective dynamic symmetry handling for SAT. In *Proceedings of the 20th International Conference on Theory and Applications of Satisfiability Testing (SAT '17)*, volume 10491 of *Lecture Notes in Computer Science*, pages 83–100. Springer, August 2017.

[DBBD16] Jo Devriendt, Bart Bogaerts, Maurice Bruynooghe, and Marc Denecker. Improved static symmetry breaking for SAT. In *Proceedings of the 19th International Conference on Theory and Applications of Satisfiability Testing (SAT '16)*, volume 9710 of *Lecture Notes in Computer Science*, pages 104–122. Springer, July 2016.

[EG21] Leon Eifler and Ambros Gleixner. A computational status update for exact rational mixed integer programming. In *Proceedings of the 22nd International Conference on Integer Programming and Combinatorial Optimization (IPCO '21)*, volume 12707 of *Lecture Notes in Computer Science*, pages 163–177. Springer, May 2021.

References VI

[EGMN20] Jan Elffers, Stephan Gocht, Ciaran McCreesh, and Jakob Nordström. Justifying all differences using pseudo-Boolean reasoning. In *Proceedings of the 34th AAAI Conference on Artificial Intelligence (AAAI '20)*, pages 1486–1494, February 2020.

[GMM+20] Stephan Gocht, Ross McBride, Ciaran McCreesh, Jakob Nordström, Patrick Prosser, and James Trimble. Certifying solvers for clique and maximum common (connected) subgraph problems. In *Proceedings of the 26th International Conference on Principles and Practice of Constraint Programming (CP '20)*, volume 12333 of *Lecture Notes in Computer Science*, pages 338–357. Springer, September 2020.

[GMN20] Stephan Gocht, Ciaran McCreesh, and Jakob Nordström. Subgraph isomorphism meets cutting planes: Solving with certified solutions. In *Proceedings of the 29th International Joint Conference on Artificial Intelligence (IJCAI '20)*, pages 1134–1140, July 2020.

[GMN22] Stephan Gocht, Ciaran McCreesh, and Jakob Nordström. An auditable constraint programming solver. In *Proceedings of the 28th International Conference on Principles and Practice of Constraint Programming (CP '22)*, volume 235 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 25:1–25:18, August 2022.

References VII

[GMNO22] Stephan Gocht, Ruben Martins, Jakob Nordström, and Andy Oertel. Certified CNF translations for pseudo-Boolean solving. In *Proceedings of the 25th International Conference on Theory and Applications of Satisfiability Testing (SAT '22)*, volume 236 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 16:1–16:25, August 2022.

[GN03] Evgueni Goldberg and Yakov Novikov. Verification of proofs of unsatisfiability for CNF formulas. In *Proceedings of the Conference on Design, Automation and Test in Europe (DATE '03)*, pages 886–891, March 2003.

[GN21] Stephan Gocht and Jakob Nordström. Certifying parity reasoning efficiently using pseudo-Boolean proofs. In *Proceedings of the 35th AAAI Conference on Artificial Intelligence (AAAI '21)*, pages 3768–3777, February 2021.

[GS19] Graeme Gange and Peter Stuckey. Certifying optimality in constraint programming. Presentation at KTH Royal Institute of Technology. Slides available at `https://www.kth.se/polopoly_fs/1.879851.1550484700!/CertifiedCP.pdf`, February 2019.

References VIII

[GSD19] Xavier Gillard, Pierre Schaus, and Yves Deville. SolverCheck: Declarative testing of constraints. In *Proceedings of the 25th International Conference on Principles and Practice of Constraint Programming (CP '19)*, volume 11802 of *Lecture Notes in Computer Science*, pages 565–582. Springer, October 2019.

[GSVW14] Maria Garcia de la Banda, Peter J. Stuckey, Pascal Van Hentenryck, and Mark Wallace. The future of optimization technology. *Constraints*, 19(2):126–138, April 2014.

[HHW13a] Marijn J. H. Heule, Warren A. Hunt Jr., and Nathan Wetzler. Trimming while checking clausal proofs. In *Proceedings of the 13th International Conference on Formal Methods in Computer-Aided Design (FMCAD '13)*, pages 181–188, October 2013.

[HHW13b] Marijn J. H. Heule, Warren A. Hunt Jr., and Nathan Wetzler. Verifying refutations with extended resolution. In *Proceedings of the 24th International Conference on Automated Deduction (CADE-24)*, volume 7898 of *Lecture Notes in Computer Science*, pages 345–359. Springer, June 2013.

References IX

[HHW15] Marijn J. H. Heule, Warren A. Hunt Jr., and Nathan Wetzler. Expressing symmetry breaking in DRAT proofs. In *Proceedings of the 25th International Conference on Automated Deduction (CADE-25)*, volume 9195 of *Lecture Notes in Computer Science*, pages 591–606. Springer, August 2015.

[JHB12] Matti Järvisalo, Marijn J. H. Heule, and Armin Biere. Inprocessing rules. In *Proceedings of the 6th International Joint Conference on Automated Reasoning (IJCAR '12)*, volume 7364 of *Lecture Notes in Computer Science*, pages 355–370. Springer, June 2012.

[MMNS11] Ross M. McConnell, Kurt Mehlhorn, Stefan Näher, and Pascal Schweitzer. Certifying algorithms. *Computer Science Review*, 5(2):119–161, May 2011.

[MP16] Ciaran McCreesh and Patrick Prosser. Finding maximum k-cliques faster using lazy global domination. In *Proceedings of the 9th Annual Symposium on Combinatorial Search (SOCS '16)*, pages 72–80, July 2016.

[PR16] Tobias Philipp and Adrián Rebola-Pardo. DRAT proofs for XOR reasoning. In *Proceedings of the 15th European Conference on Logics in Artificial Intelligence (JELIA '16)*, volume 10021 of *Lecture Notes in Computer Science*, pages 415–429. Springer, November 2016.

References X

[RvBW06] Francesca Rossi, Peter van Beek, and Toby Walsh, editors. *Handbook of Constraint Programming*, volume 2 of *Foundations of Artificial Intelligence*. Elsevier, 2006.

[SB06] Carsten Sinz and Armin Biere. Extended resolution proofs for conjoining BDDs. In *Proceedings of the 1st International Computer Science Symposium in Russia (CSR '06)*, volume 3967 of *Lecture Notes in Computer Science*, pages 600–611. Springer, June 2006.

[SB22] Mate Soos and Randal E. Bryant. Combining CDCL, Gauss-Jordan elimination, and proof generation. EasyChair Preprint 8497, July 2022. Available at `https://easychair.org/publications/preprint/4rGK`.

[TD20] Rodrigue Konan Tchinda and Clémentin Tayou Djamégni. On certifying the UNSAT result of dynamic symmetry-handling-based SAT solvers. *Constraints*, 25(3–4):251–279, December 2020.

[Van08] Allen Van Gelder. Verifying RUP proofs of propositional unsatisfiability. In *10th International Symposium on Artificial Intelligence and Mathematics (ISAIM '08)*, 2008. Available at `http://isaim2008.unl.edu/index.php?page=proceedings`.

References XI

[VDB22] Dieter Vandesande, Wolf De Wulf, and Bart Bogaerts. QMaxSATpb: A certified MaxSAT solver. In *Proceedings of the 16th International Conference on Logic Programming and Non-monotonic Reasoning (LPNMR '22)*, volume 13416 of *Lecture Notes in Computer Science*, pages 429–442. Springer, September 2022.

[Ver] VeriPB: Verifier for pseudo-Boolean proofs. `https://gitlab.com/MIAOresearch/software/VeriPB`.

[WHH14] Nathan Wetzler, Marijn J. H. Heule, and Warren A. Hunt Jr. DRAT-trim: Efficient checking and trimming using expressive clausal proofs. In *Proceedings of the 17th International Conference on Theory and Applications of Satisfiability Testing (SAT '14)*, volume 8561 of *Lecture Notes in Computer Science*, pages 422–429. Springer, July 2014.