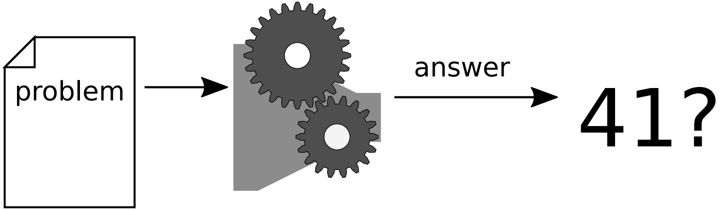
Certifying Correctness for Combinatorial Algorithms by Using Pseudo-Boolean Reasoning

Stephan Gocht

1st June 2022

Do you trust your computer?



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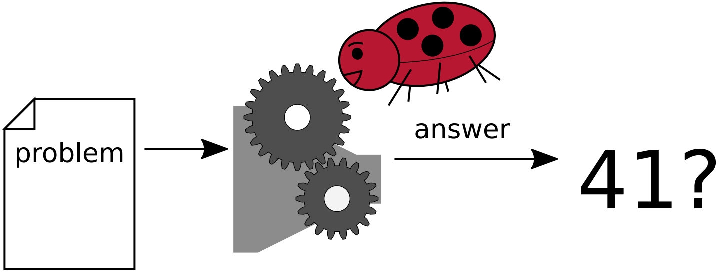

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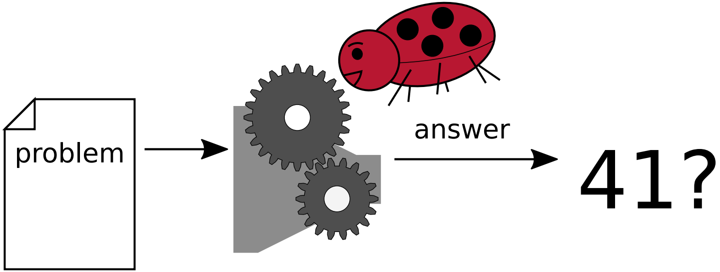

- what if answer is used for high-stakes decision?
  - e.g., combinatorial auction, kidney exchange program

Software Verification — How to ensure software behaves as intended?

- Software testing
  - run collection of test cases to check if software behaves as intended
    - depends on quality of test cases, likely to miss non-trivial defects
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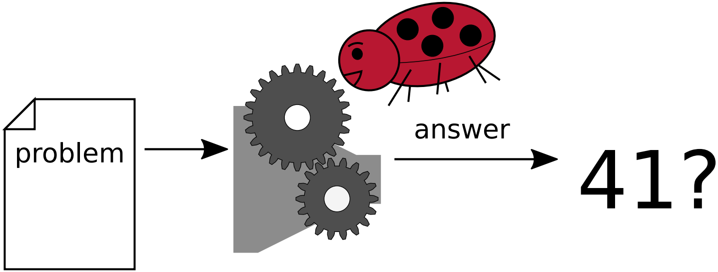
- Certifying algorithms, also known as proof logging (this talk)
  - let algorithm output answer and *proof* that answer is correct
  - proof: sequence of simple, efficiently machine-verifiable steps

Detecting Bugs with Certifying Algorithms



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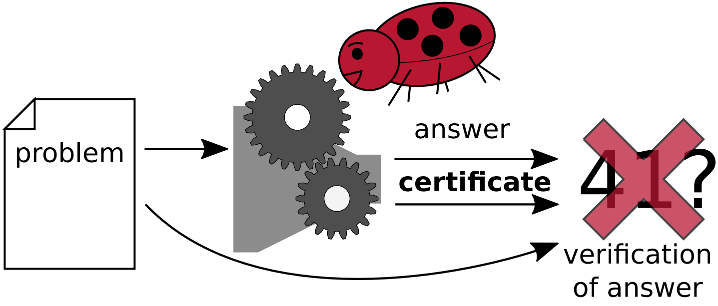
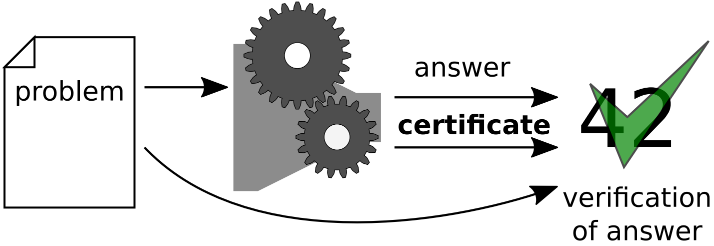


- verification of answer with external tool can detect bugs

Guaranteeing Correctness with Certifying Algorithms



- successful verification of answer with external tool guarantees correct answer

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  - increase trust in solution
  - detect hardware errors

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- while solving
  - increase trust in solution
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- after solving
  - analyse certificate to understand and improve solving process
  - could use certificate to audit solution afterwards
- during development
  - simplifies testing: not necessary to know correct answer a priory
  - find bugs even if result is correct
  - locate first unsound step

Requirements for Certifying Algorithms

- certificate verification
  - should be efficiently machine-verifiable
  - ideally so simple that proof checker can be formally verified
  - want: simple, easy to verify steps / rules

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  - want: expressive rules for concise reasoning

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But how?

SAT Solving — A Success Story for Certifying Algorithms . . .

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- SAT competition requires solver to produce certificate (aka proof logging)
- proof formats such as RUP [GN03], TraceCheck [Bie06], GRIT [CMS17], LRAT [CHH$^+$17]; DRAT [WHH14] has become standard

. . . But Need for Further Research

- some SAT techniques don't have efficient DRAT proof logging
  - parity reasoning
  - symmetry breaking
  - symmetric explanation learning

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  - MaxSAT solving
  - constraint programming (CP)
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  - algebraic reasoning / Gröbner basis computations
  - pseudo-Boolean satisfiability and optimization

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Need to look beyond DRAT!

New Proof Systems are Being Developed

many new proof systems

- propagation redundancy (PR) [HKB17]
- branch and bound in integer programming [CGS17, EG21]
- practical polynomial calculus (PAC) [RBK18, KFB20, KFBK22]
- extensible RAT (FRAT) [BCH21]
- propagation redundancy for BDDs [BB21]
- Max-SAT resolution [PCH21]
- **pseudo-Boolean proofs** [EGMN20, GN21, BGMN22]

High Level Idea of Pseudo-Boolean Proofs

- use pseudo-Boolean constraints (0-1 linear inequalities) to describe problem
  - e.g., $x_1 + x_2 + x_3 \geq 1$ or $2z + x_1 + x_2 + x_3 \geq 2$
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- proof constructs sequence of constraints $D_1, D_2, D_3, \ldots, D_L$
  - each constraint is derived by rule in proof system
  - annotation can contain additional information necessary for efficient verification
  - proves there is no solution if D_L is $0 \geq 1$
  - proves optimality if D_L is bound on objective matching known solution
- rest of this talk will explain and refine these concepts

Our Approach

- use pseudo-Boolean proofs (PBP)
- reference implementation of verifier: VeriPB[1]
- **multi-purpose** format: proof logging for wide range of problems / algorithms
  - reasoning with 0-1 linear inequalities (by design)

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  - pseudo-Boolean solving via translation to CNF [GMNO22]

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Running Example — Matching

- bipartite graph $G = (U \cup V, E)$
- find maximum matching $M \subseteq E$
- such that no node is incident to two edges in M

Basics — Pseudo-Boolean Problems

- Boolean variable x is 0 (false) or 1 (true)
- Literal: x or its negation $\overline{x} = 1 - x$
- pseudo-Boolean constraint: linear inequality over literals
  e.g., $\overline{x_1} + \overline{x_2} \geq 1$ or $x_1 + 2x_2 + \overline{x_3} \geq 2$
- formula F: set of constraints
- objective function f to be minimized
- Clause: at-least-one constraint, e.g., $\overline{x_1} + \overline{x_2} \geq 1$
- Contradiction: $\perp$ or $0 \geq 1$ is constraint that can't be satisfied

Goal: find assignment minimizing objective and satisfying all constraints

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Literal Axioms

$$\frac{}{x \geq 0} \qquad \frac{}{\overline{x} \geq 0}$$

- can add variable bound
- rule is annotated by literal

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Addition Rule

$$\frac{\overline{x}_1 + \overline{x}_2 \geq 1 \qquad \overline{x}_2 + \overline{x}_3 \geq 1}{\overline{x}_1 + 2\overline{x}_2 + \overline{x}_3 \geq 2}$$

- can add two pseudo-Boolean constraints
- rule is annotated by (reference to) the constraints to be added

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Multiplication Rule

$$\frac{\overline{x}_1 + \overline{x}_2 \geq 1}{2\overline{x}_1 + 2\overline{x}_2 \geq 2}$$

- can multiply constraint by positive number
- rule is annotated by (reference to) the constraint and used factor

Division Rule

$$\frac{2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_2 \geq 3}{\overline{x}_1 + \overline{x}_2 + \overline{x}_2 \geq 2}$$

- can divide constraint by positive number and round up
- rule is annotated by (reference to) the constraint and used divisor

- rules so far are known as the cutting planes proof system [CCT87]

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Saturation Rule

$$\frac{4\overline{x}_1 + 3\overline{x}_2 + 2\overline{x}_2 \geq 3}{3\overline{x}_1 + 3\overline{x}_2 + 2\overline{x}_2 \geq 3}$$

- can reduce too large coefficients (assuming all coefficients are positive)
- rule is annotated by (reference to) the constraint

Basics — Manipulating Constraints

- can negate constraint

$$C: \quad x + y + z \geq 2$$
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- Implication: $F \models C$ if every assignment satisfying F also satisfies C

Constraints that Remove Solutions

- so far, any solution satisfying F also satisfies added constraints (rules are implicational)
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  - but there is assignment α satisfying F but falsifying C
  - if we find assignment α' satisfying F and $f_{\upharpoonright\alpha'} < f_{\upharpoonright\alpha}$
  - then α wasn't optimal $\rightarrow$ OK that α falsifies C
  - save to add C if for all α falsifying C we find such an α'

Verifying Constraints that Remove Solutions

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  - assume α satisfies $F \cup \{\, \neg C \,\}$ (i.e., α satisfies F and falsifies C)
  - by implication above, α satisfies $F_{\restriction \omega} \cup \{\, f_{\restriction \omega} < f \,\}$
  - hence $\alpha' = \alpha \circ \omega$ satisfies F and has better objective value:

$$\begin{aligned} & (F_{\restriction \omega} \cup \{f_{\restriction \omega} < f\})_{\restriction \alpha} \\ = \; & F_{\restriction \alpha \circ \omega} \cup \{f_{\restriction \alpha \circ \omega} < f_{\restriction \alpha}\} \\ = \; & F_{\restriction \alpha'} \cup \{f_{\restriction \alpha'} < f_{\restriction \alpha}\} \end{aligned}$$

Verifying the Condition

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- problem: implication hard to check in general
- solution: provide proof (using previous rules), showing

$$F \cup \{ \neg C, \neg D \} \models \bot \text{ for } D \in F_{\restriction\omega} \cup \{ f_{\restriction\omega} < f \}$$

Dominance Rule (simplified)

$$\frac{F \cup \{\neg C\} \models F{\restriction_\omega} \cup \{f{\restriction_\omega} < f\}}{C}$$

- rule is annotated by:
  - used substitution ω
  - for each $D \in F{\restriction_\omega} \cup \{f{\restriction_\omega} < f\}$ a proof showing $F \cup \{\neg C, \neg D\} \models \bot$

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Redundance Rule (simplified)

- idea: (generalize redundancy from SAT [HKB17, BT19] to PB and optimization)
  - don't need to improve objective strictly
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  - sufficient if one optimal solution remains
  - let G_i, be set of constraints added so far ($G_i = F \cup \{ D_1, \ldots, D_{i-1} \}$)

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$$\frac{G_i \cup \{ \neg D_i \} \models (G_i \cup D_i)_{\restriction\omega} \cup \{ f_{\restriction\omega} \leq f \}}{D_i}$$

- rule is annotated by:
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  - for each $C \in (G_i \cup D_i)_{\restriction\omega} \cup \{ f_{\restriction\omega} \leq f \}$ a proof showing $G_i \cup \{ \neg D_i, \neg C \} \models \bot$

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- then there is assignment ρ' satisfying $F \cup \{ D_1, \ldots, D_i \}$,
- and $f_{\upharpoonright\rho'} = f_{\upharpoonright\rho}$

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note:

- can terminate all proofs with contradiction $(0 \geq 1)$

Deletion Rule

deleting constraints . . .

- important for performance and memory efficiency
- only makes problem more satisfiable
  (except in connection with dominance — explanation in second part)

Dealing with Lazy Programmers ;-)

- goal: proof system should be easy to use
- problem: often "obvious" that adding constraint is OK, but tedious to write down
- solution: let verifier take care of "obvious" cases

Omitting Obvious Steps

from dominance rule:

- for each $D \in F_{\upharpoonright\omega} \cup \{ f_{\upharpoonright\omega} < f \}$ a proof showing $F \cup \{ \neg C, \neg D \} \models \bot$

can omitt proof if

- $\neg D = \bot$
- $D \in F$ (because $\neg D + D = \bot$)
- $\neg C + \neg D = \bot$

Reverse Unit Propagation [GN03, Van08]

- assume we have

$$C_1 : x + y >= 1$$
$$C_2 : \overline{y} + z >= 1$$

- and want to add

$$C_3 : x + z >= 1$$

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  - but now C_2 falsified by ρ
  - $\Rightarrow$ no assignment ρ satisfies C_1, C_2 and $\neg C_3$

Future Work

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proof logging for more algorithms and problems:

- MaxSAT (optimization for SAT)
- more propagators in constraint programming
- symmetric explanation learning
- integer programming

Conclusion

- proof logging is well-established standard for SAT solving
- so far not usable for
  - some techniques in SAT (e.g. symmetry breaking)
  - richer problem formalisms including optimization

2

Conclusion

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  - some techniques in SAT (e.g. symmetry breaking)
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our work: proof logging via pseudo-Boolean proofs + verification (VeriPB[2])

- simple to implement + efficient proof checking
- applicable to wide range of combinatorial problems / algorithms
- resolve open problems for proof logging in SAT

[2] `https://gitlab.com/MIAOresearch/VeriPB`

Menu for second part

- live demo
- translating DRAT proofs
- full form of dominance / redundance
- deletion and dominance

Translating DRAT Proof to Pseudo-Boolean Proof

- step in DRAT proof is clausal form of redundance rule
- witness ω implicitly set by first literal
- literals represented by numbers

```
c DRAT PROOF
c loads formula implicitly
1 2 -3 0
```

```
pseudo-Boolean proof version 1.3
* load formula explicitly
f
red 1 x1 1 x2 1 ~x3 >= 1 ; x1 -> 1
```

Dominance Rule (explanation in second part)

in [BGMN22] rule is more general:

- allow to improve arbitrary preorder $\preceq$ instead of objective
- define preorder as set of constraints $\mathcal{O}_{\preceq}(\vec{u}, \vec{v})$
- proof file shows that $\mathcal{O}_{\preceq}(\vec{u}, \vec{v})$ defines preorder
- $\mathscr{C}$ is set of constraints last time preorder was changed
- $\mathscr{D}$ is set of constraints added after last time preorder was changed
- allows to use constraints in $\mathscr{D}$
- check $\alpha' \preceq \alpha$

$$\mathscr{C} \cup \mathscr{D} \cup \{\neg C\} \models \mathscr{C}{\upharpoonright_\omega} \cup \mathcal{O}_{\preceq}(\vec{x}{\upharpoonright_\omega}, \vec{x}) \cup \{f{\upharpoonright_\omega} \leq f\}$$

- check $\alpha \not\preceq \alpha'$

$$\mathscr{C} \cup \mathscr{D} \cup \{\neg C\} \cup \mathcal{O}_{\preceq}(\vec{x}, \vec{x}{\upharpoonright_\omega}) \models \bot$$

Redundance Rule (explanation in second part)

in [BGMN22] rule is more general:

- preorder defined through $\mathcal{O}_{\preceq}(\vec{u}, \vec{v})$
- $\mathscr{C}$ is set of constraints last time preorder was changed
- $\mathscr{D}$ is set of constraints added after last time preorder was changed

$$\frac{\mathscr{C} \cup \mathscr{D} \cup \{\, \neg C \,\} \models (\mathscr{C} \cup \mathscr{D} \cup \{\, C \,\})_{\restriction\omega} \cup \mathcal{O}_{\preceq}(\vec{x}_{\restriction\omega}, \vec{x}) \cup \{\, f_{\restriction\omega} \leq f \,\}}{C}$$

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