Understanding Conflict-Driven SAT Solving Through the Lens of Proof Complexity

Jakob Nordström

KTH Royal Institute of Technology
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Theoretical Foundations of SAT Solving
Fields Institute, Toronto, Canada
August 15–19, 2016

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SAT Solving in Theory and Practice

The unreasonable effectiveness of SAT solvers

- The Boolean satisfiability problem (SAT) is NP-complete and so should be exponentially hard
- Yet current state-of-the-art conflict-driven clause learning (CDCL) SAT solvers can deal with formulas containing millions of variables
- How can they work so well? What are their limits?

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- Community structure
- Parameterized complexity
- This talk: **proof complexity**
  Rigorous analysis of underlying method of reasoning

Purpose of This Presentation

- Survey some of the research in the area (including some ongoing work)
- Show some theoretical "benchmark formulas" used to understand potential and limitations of SAT solvers
- Discuss some (of the many) remaining open problems

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Caveats:

- By necessity, selective and somewhat subjective coverage
- Won't do too much name-dropping — full references at end of slides

Some More Caveats and Clarifications

Only basic propositional logic proof search

- No SMT or first-order logic or anything in this talk
- No discussion of preprocessing techniques

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Limitations of proof complexity

- Asking for rigorous analysis is asking a lot. . .
- In addition, proof complexity considers optimal algorithms (so restrict focus to unsatisfiable formulas)
- Still possible to prove some highly nontrivial theorems
- Separate question how to interpret these theoretical theorems

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Why theory benchmarks?

- See what SAT solvers can do (sometimes very neat things)
- See what SAT solvers cannot do (provably hard instances)
- See what SAT solvers "should be able" to do (formulas easy for proof system but hard for corresponding SAT solvers)

Outline

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$ (or $\neg x$)
- Clause $C = a_1 \lor \cdots \lor a_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- CNF formula $F = C_1 \land \cdots \land C_m$: conjunction of clauses
- k-CNF formula: CNF formula with clauses of size $\leq k$
  (where k is some constant)
- N denotes size of formula (# literals counted with repetitions)
- $\mathcal{O}(f(N))$ grows at most as quickly as $f(N)$ asymptotically
  $\Omega(g(N))$ grows at least as quickly as $g(N)$ asymptotically
  $\Theta(h(N))$ grows equally quickly as $h(N)$ asymptotically

The Resolution Proof System Underlying CDCL

Goal: refute **unsatisfiable** CNF

Start with clauses of formula (axioms)

Derive new clauses by resolution rule

$$\frac{C \vee x \qquad D \vee \overline{x}}{C \vee D}$$

Done when empty clause $\perp$ derived

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- annotated list or
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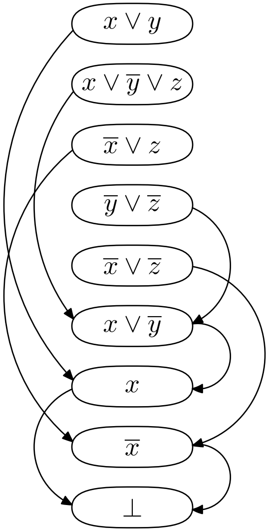

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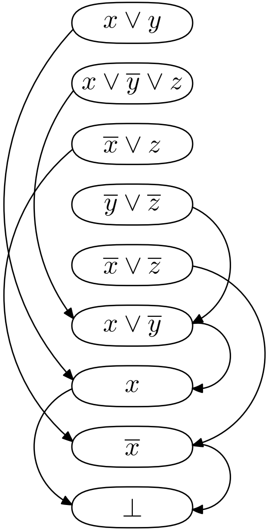
Tree-like resolution if DAG is tree



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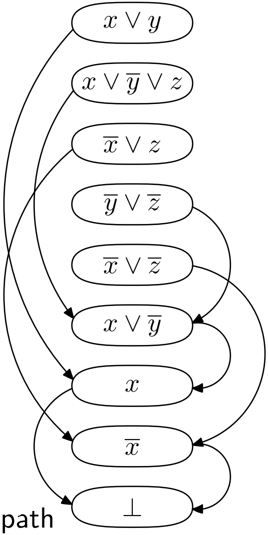
Regular if resolved variables don't repeat on path



Making the Connection to DPLL

Basis of best modern SAT solvers still DPLL method [DP60, DLL62]

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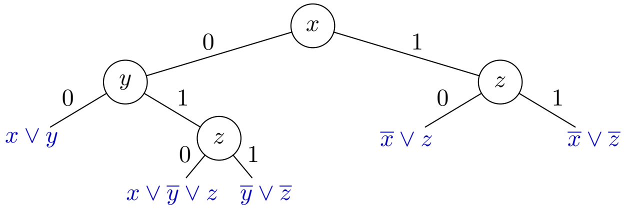
Visualize execution of DPLL algorithm as search tree

- Branch on variable assignments in internal nodes
- Stop in leaves when falsfied clause found



DPLL Execution as Resolution Proof

A DPLL execution is essentially a resolution proof

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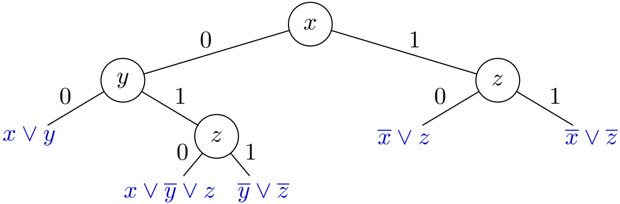
Look at our example again:



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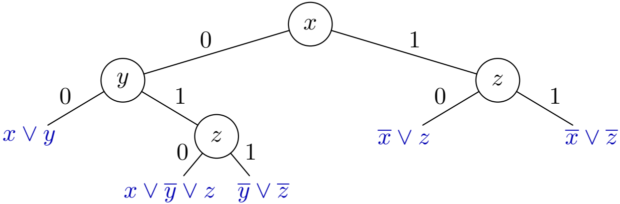

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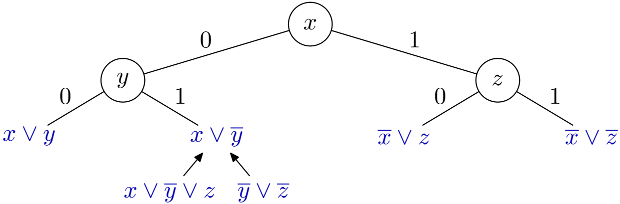

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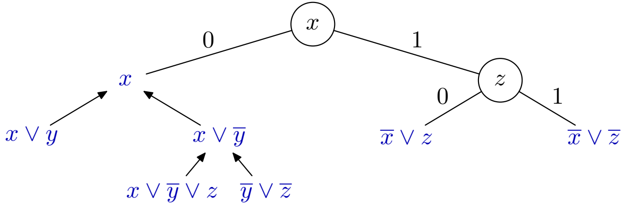

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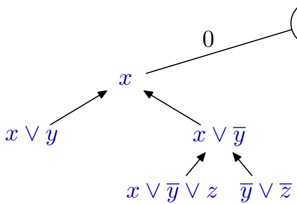

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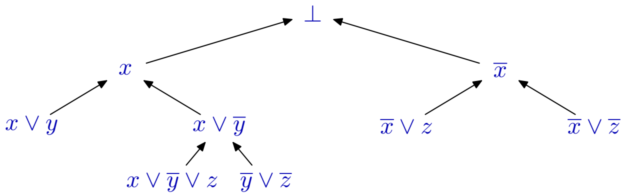

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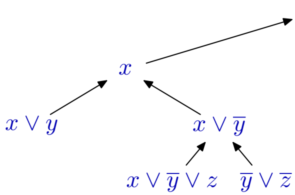

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(Slightly more needed to turn this into formal theorem, but this is essentially it)

CDCL Execution as Resolution Proof

Many more ingredients in modern CDCL SAT solvers [BS97, MS99, MMZ$^+$01], for instance:

- Choice of branching variables crucial
- In leaf, compute & add reason for failure (clause learning)
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Will talk more about this later in the presentation

Resolution Size/Length

Size/length of proof $=$ # clauses (9 in our example)
Length of refuting F $=$ min over all proofs for F

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Most fundamental measure in proof complexity

Lower bound on CDCL running time
(can extract resolution proof from execution trace)

Never worse than $\exp(\mathcal{O}(N))$

Matching $\exp(\Omega(N))$ lower bounds known

Some Examples of Hard Formulas w.r.t. Length (1/3)

Pigeonhole principle (PHP) [Hak85]
"$n+1$ pigeons don't fit into n holes"

Variables $p_{i,j} =$ "pigeon i goes into hole j"

$p_{i,1} \lor p_{i,2} \lor \cdots \lor p_{i,n}$	every pigeon i gets a hole
$\overline{p}_{i,j} \lor \overline{p}_{i',j}$	no hole j gets two pigeons $i \neq i'$

Can also add "functionality" and "onto" axioms

$\overline{p}_{i,j} \lor \overline{p}_{i,j'}$	no pigeon i gets two holes $j \neq j'$
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Even onto functional PHP formula is hard for resolution
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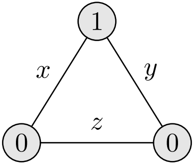
But only length lower bound $\exp(\Omega(\sqrt[3]{N}))$ in terms of formula size

Some Examples of Hard Formulas w.r.t. Length (2/3)

Tseitin formulas [Urq87]
"Sum of degrees of vertices in graph is even"

Variables = edges (in undirected graph of bounded degree)

- Label every vertex 0/1 so that sum of labels odd
- Write CNF requiring parity of # true incident edges = label



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\begin{aligned}
&(x \lor y) && \land\ (\overline{x} \lor z) \\
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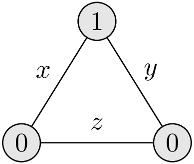

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Requires length $\exp(\Omega(N))$ on well-connected so-called expanders
"Resolution cannot count $\mathrm{mod}\ 2$"

Some Examples of Hard Formulas w.r.t. Length (3/3)

Subset cardinality formulas [Spe10, VS10, MN14]

Variables = 1s in matrix with four 1s per row/column + extra 1
Row ⇒ majority of variables true; column ⇒ majority false

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Lower bound $\exp(\Omega(N))$ on expanding matrices (well spread-out)

Resolution Space

Space = max # clauses in memory when performing refutation

Motivated by solver memory usage (but also of intrinsical theory interest)

Can be measured in different ways — makes most sense here to focus on clause space

Space at step t = # clauses at steps $\leq t$ used at steps $\geq t$

1.	$x \lor y$	Axiom
2.	$x \lor \overline{y} \lor z$	Axiom
3.	$\overline{x} \lor z$	Axiom
4.	$\overline{y} \lor \overline{z}$	Axiom
5.	$\overline{x} \lor \overline{z}$	Axiom
6.	$x \lor \overline{y}$	Res$(2, 4)$
7.	x	Res$(1, 6)$
8.	$\overline{x}$	Res$(3, 5)$
9.	$\perp$	Res$(7, 8)$

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Example: Space at step 7 …

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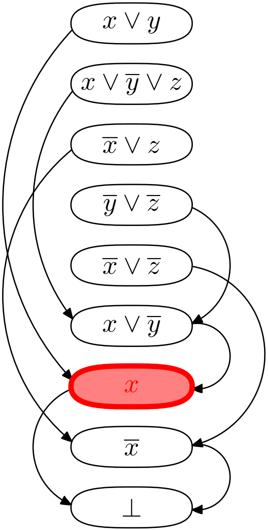

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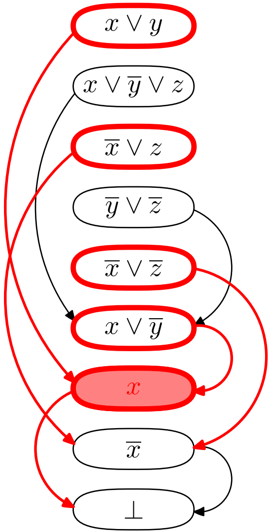

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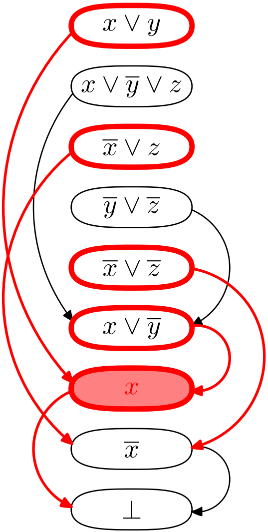

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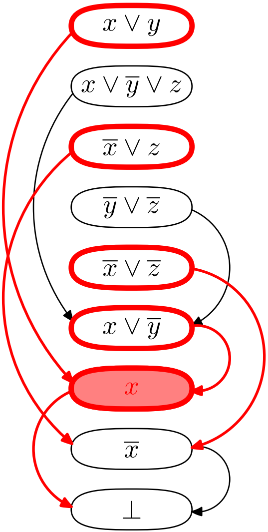
Space of proof = max over all steps
Space of refuting F = min over all proofs



Bounds on Resolution Space

Space always at most $N + \mathcal{O}(1)$ (!) [ET01]

Matching $\Omega(N)$ lower bounds known [ABRW02, BG03, ET01]

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Linear space lower bounds might not seem so impressive...

But:

- Apply for space on top of storing formula
- Hold even for optimal algorithms that magically know exactly which clauses to throw away or keep
- So significantly more space might be needed in practice
- And linear space upper bound obtained for proofs of exponential size

Length and Space

Exist space-efficient proofs $\Rightarrow$ exist short proofs [AD08]
(for k-CNF formulas, to be precise)

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Pebbling formulas [Nor09, NH13, BN08]

- Can be refuted in length $\mathcal{O}(N)$
- May require space $\Omega(N/\log N)$

Pebbling Formulas

Encode so-called pebble games on DAGs [BW01]

1. $u_1 \oplus u_2$
2. $v_1 \oplus v_2$
3. $w_1 \oplus w_2$
4. $(u_1 \oplus u_2) \wedge (v_1 \oplus v_2) \rightarrow (x_1 \oplus x_2)$
5. $(v_1 \oplus v_2) \wedge (w_1 \oplus w_2) \rightarrow (y_1 \oplus y_2)$
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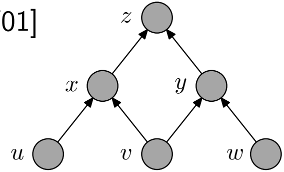
7. $\neg(z_1 \oplus z_2)$



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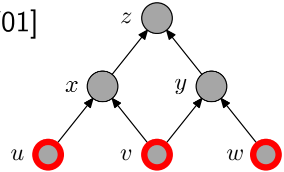

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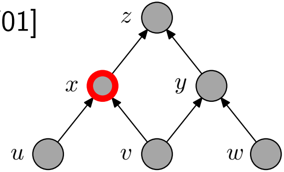

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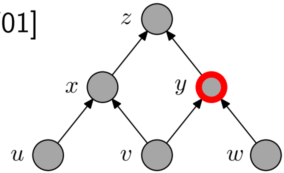

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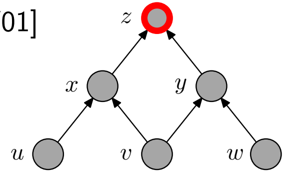

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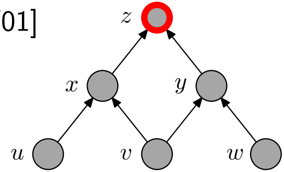
4. $(u_1 \oplus u_2) \wedge (v_1 \oplus v_2) \rightarrow (x_1 \oplus x_2)$
5. $(v_1 \oplus v_2) \wedge (w_1 \oplus w_2) \rightarrow (y_1 \oplus y_2)$
6. $(x_1 \oplus x_2) \wedge (y_1 \oplus y_2) \rightarrow (z_1 \oplus z_2)$
7. $\neg(z_1 \oplus z_2)$



- sources are true
- truth propagates upwards
- but sink is false

Pebbling Formulas

Encode so-called pebble games on DAGs [BW01]

1. $u_1 \oplus u_2$
2. $v_1 \oplus v_2$
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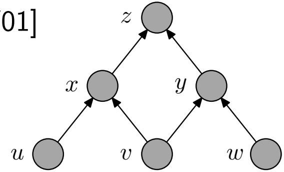

- sources are true
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Write in CNF
E.g., $(x_1 \oplus x_2) \rightarrow (y_1 \oplus y_2)$ becomes

$$\begin{aligned} &(x_1 \vee \overline{x}_2 \vee y_1 \vee y_2) \wedge (x_1 \vee \overline{x}_2 \vee \overline{y}_1 \vee \overline{y}_2) \\ \wedge\ &(\overline{x}_1 \vee x_2 \vee y_1 \vee y_2) \wedge (\overline{x}_1 \vee x_2 \vee \overline{y}_1 \vee \overline{y}_2) \end{aligned}$$

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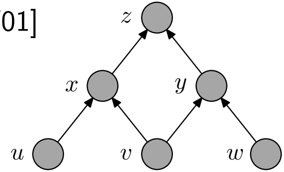

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Pebbling space lower bounds $\Rightarrow$ resolution space lower bounds

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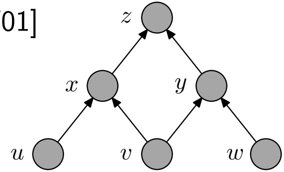

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$$\wedge\ (\overline{x}_1 \vee x_2 \vee y_1 \vee y_2) \wedge (\overline{x}_1 \vee x_2 \vee \overline{y}_1 \vee \overline{y}_2)$$

Pebbling space lower bounds $\Rightarrow$ resolution space lower bounds
(Works also for other functions than $\oplus$)

Length-Space Trade-offs

Length $\approx$ running time; space $\approx$ memory consumption

SAT solvers aggressively try to minimize both — is this possible?

Length-Space Trade-offs

Length ≈ running time; space ≈ memory consumption

SAT solvers aggressively try to minimize both — is this possible?

Theorem ([BN11, BBI12, BNT13])

There are formulas for which

- *exist refutations in short length*
- *exist refutations in small space*
- *optimization of one measure causes dramatic blow-up for other measure*

Holds for

- Pebbling formulas on the right graphs
- Tseitin formulas on long, narrow rectangular grids

So simultaneous optimization not possible [at least in theory]

Abstract Description of CDCL (1/2)

Trail: a stack of decisions $x_i \stackrel{\mathsf{d}}{=} b$ and unit propagations $x_i \stackrel{C}{=} b$

$$\Big(\underbrace{x_7 \stackrel{\mathsf{d}}{=} 0}_{\text{dec. level 1}}, \underbrace{x_2 \stackrel{\mathsf{d}}{=} 1, x_{12} \stackrel{C_1}{=} 0}_{\text{decision level 2}}, \underbrace{x_6 \stackrel{\mathsf{d}}{=} 1, x_4 \stackrel{C_2}{=} 1, x_1 \stackrel{C_3}{=} 0}_{\text{decision level 3}}, \underbrace{x_{11} \stackrel{\mathsf{d}}{=} 0, x_{59} \stackrel{C_4}{=} 1}_{\text{decision level 4}} \Big)$$

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Clause database $\mathcal{D}$: original formula + learned clauses

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else if some $C \in \mathcal{D}$ unit w.r.t. trail, move to **Unit**;

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else if some $C \in \mathcal{D}$ unit w.r.t. trail, move to **Unit**;
else if restart, set trail to $()$ and move to **Default**;

Abstract Description of CDCL (1/2)

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Clause database $\mathcal{D}$: original formula + learned clauses

Start in **Default** mode; transit to **Conflict**, **Unit**, or **Decision**

Default **If** trail falsifies clause $C \in \mathcal{D}$, move to **Conflict**;
else if all variables assigned, output `SAT`;
else if some $C \in \mathcal{D}$ unit w.r.t. trail, move to **Unit**;
else if restart, set trail to () and move to **Default**;
else

1. decide if to apply database reduction to $\mathcal{D}$;
2. move to **Decision**

Abstract Description of CDCL (2/2)

Unit Pick clause $C \in \mathcal{D}$ that is unit w.r.t. trail
Add propagated assignment $x \stackrel{C}{=} b$ to trail
Move to **Default**

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Conflict **If** trail contains no decisions, output `UNSAT`;
else

- apply learning scheme to derive asserting clause C;
- backjump, i.e., remove decision levels $>$ assertion level of C from trail;
- move to **Unit**

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Add propagated assignment $x \overset{C}{=} b$ to trail
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Decision Use decision scheme to add decision $x \overset{\mathsf{d}}{=} b$ to trail
Move to **Default**

Description from [EJL$^+$16] drawing heavily on [AFT11, BHJ08, PD11]

CDCL Execution Example

Too small formula for interesting example. . .

$(x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z})$

CDCL Execution Example

Too small formula for interesting example... So expand slightly:

$$(u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$$

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$w \stackrel{\mathsf{d}}{=} 0$

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$\boxed{w \stackrel{\mathsf{d}}{=} 0}$

$\boxed{u \stackrel{\overline{u} \lor w}{=} 0}$

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$w \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{\overline{u} \lor w}{=} 0$

$x \stackrel{\mathsf{d}}{=} 0$

CDCL Execution Example

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$\boxed{w \stackrel{\mathsf{d}}{=} 0}$

$\boxed{u \stackrel{\overline{u} \lor w}{=} 0}$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$\boxed{y \stackrel{u \lor x \lor y}{=} 1}$

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$$(u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$$

$\boxed{w \stackrel{\mathsf{d}}{=} 0}$

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$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$\boxed{y \stackrel{u \lor x \lor y}{=} 1}$

$\boxed{z \stackrel{x \lor \overline{y} \lor z}{=} 1}$

CDCL Execution Example

Too small formula for interesting example... So expand slightly:

$$(u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{u} \vee w) \wedge (\overline{u} \vee \overline{w})$$

$\boxed{w \stackrel{\mathsf{d}}{=} 0}$

$\boxed{u \stackrel{\overline{u} \vee w}{=} 0}$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$\boxed{y \stackrel{u \vee x \vee y}{=} 1}$

$\boxed{z \stackrel{x \vee \overline{y} \vee z}{=} 1}$

$\boxed{\stackrel{\overline{y} \vee \overline{z}}{\perp}}$

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$$(u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$$

$w \overset{\mathsf{d}}{=} 0$

$u \overset{\overline{u} \lor w}{=} 0$

$x \overset{\mathsf{d}}{=} 0$

$y \overset{u \lor x \lor y}{=} 1$

$z \overset{x \lor \overline{y} \lor z}{=} 1$ → $x \lor \overline{y}$

$\overset{\overline{y} \lor \overline{z}}{\bot}$ → $x \lor \overline{y}$

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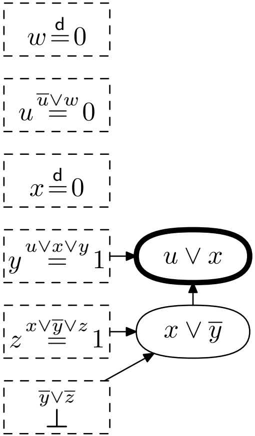

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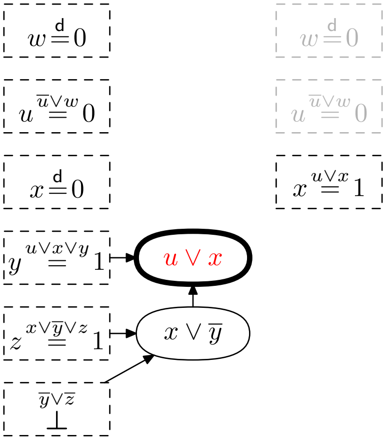

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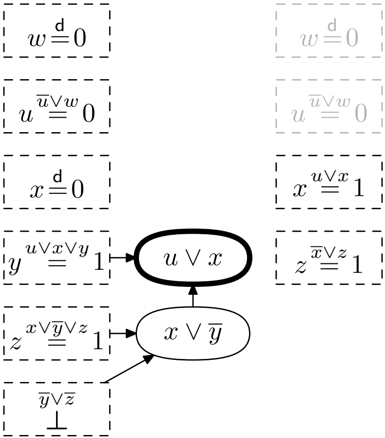

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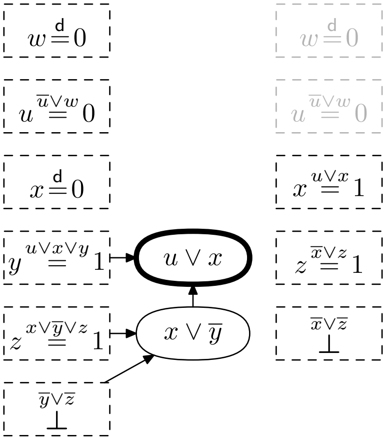

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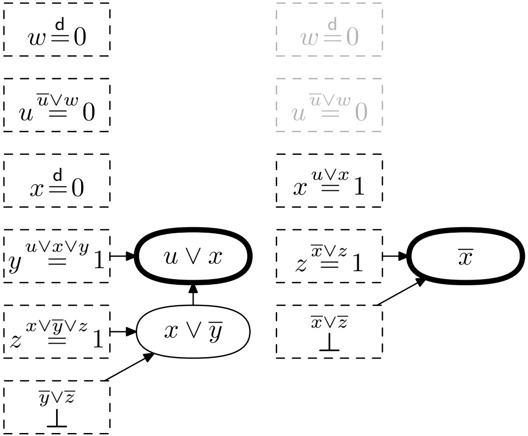

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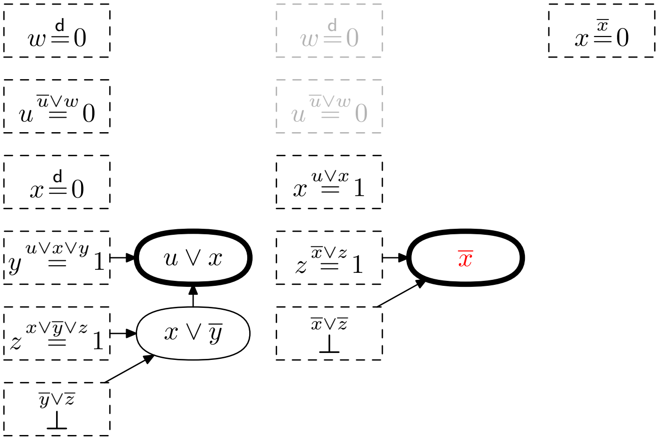

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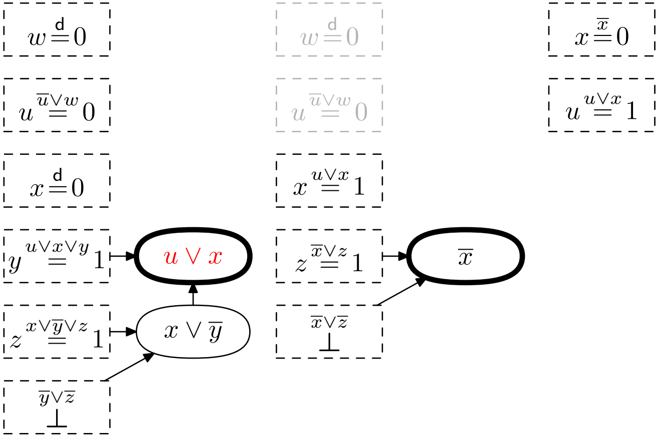

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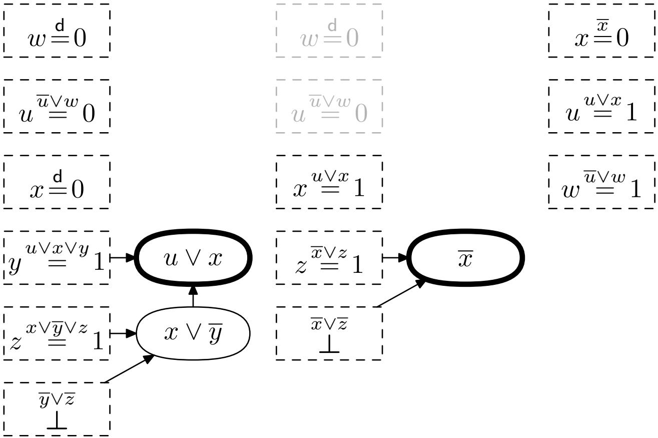

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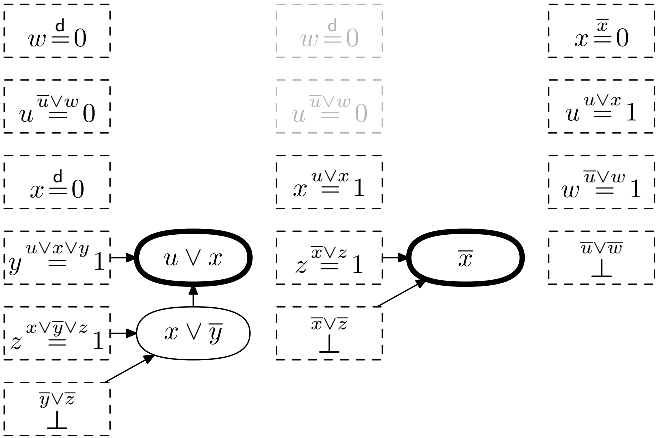

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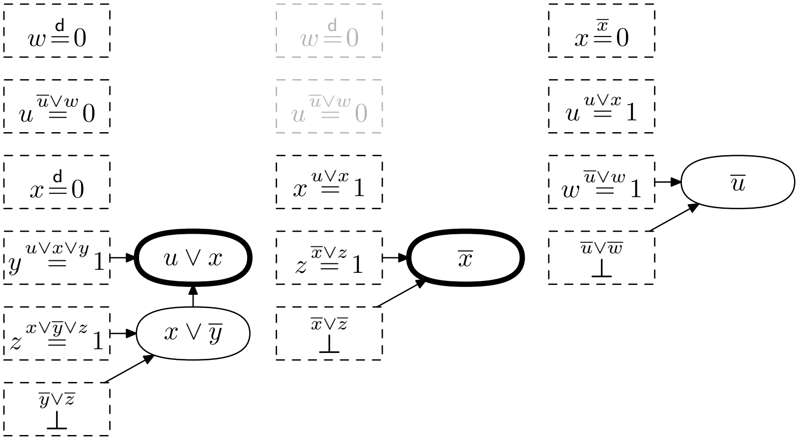

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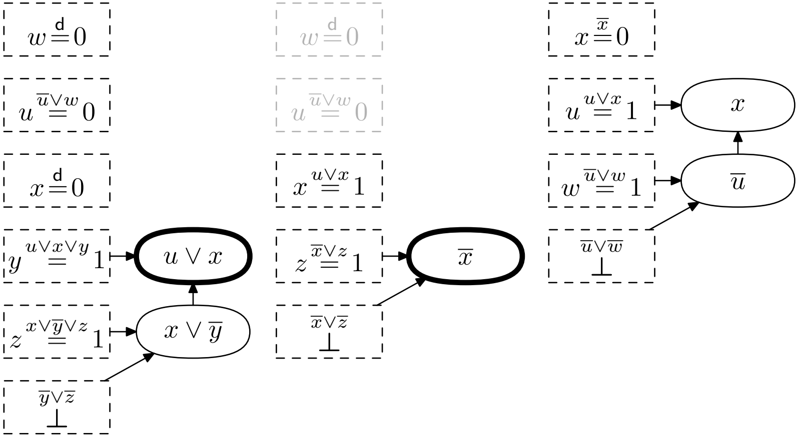

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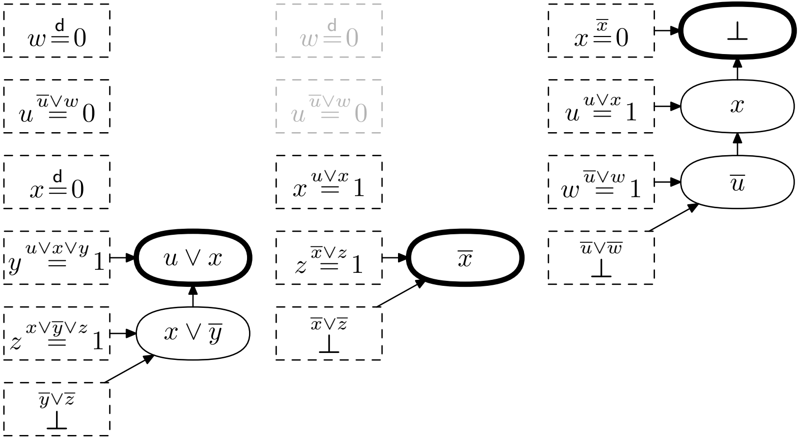

CDCL Execution Example as Resolution Refutation

Obtain resolution refutation...

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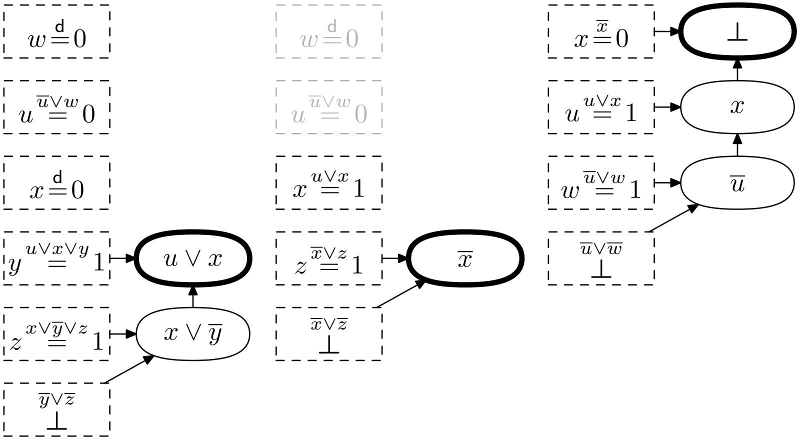
Obtain resolution refutation from CDCL execution...



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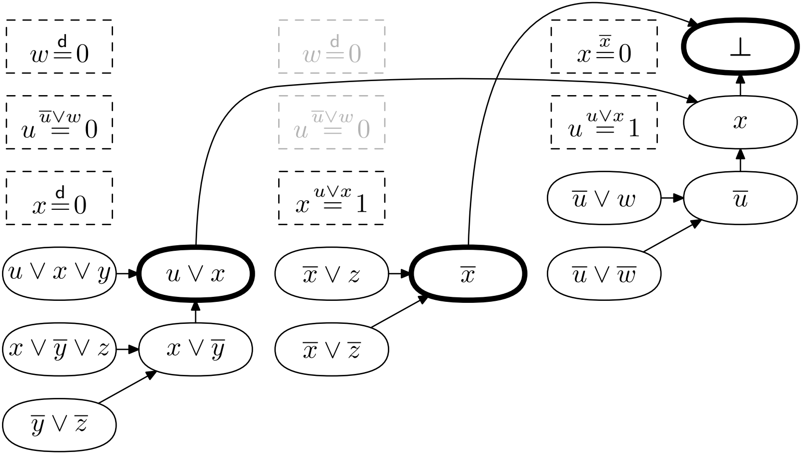

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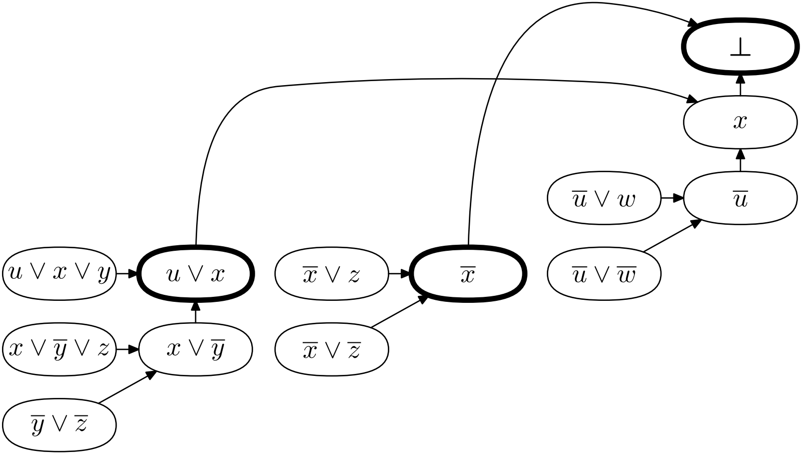

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- Long line of work aimed at proving that CDCL explores resolution search space efficiently, e.g., [BKS04, Van05, BHJ08, HBPV08]
- Challenging problem — progress only by making assumptions such as
  - artificial preprocessing
  - decisions past conflicts
  - non-standard learning scheme
  - no unit propagation(!)

Proof Plan for CDCL Simulation of Resolution

General idea is obvious:

- Given resolution proof $(C_1, C_2, \ldots, C_\tau)$
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Not as easy as it seems...

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- Suppose have $C \lor x$ and $D \lor \overline{x}$ and now want to learn $C \lor D$
- Why not just decide to make $C \lor D$ false $\Rightarrow$ conflict on x?!
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Non-standard assumptions needed precisely for these reasons

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- Good, so then we're done understanding CDCL?
  Not quite...

Room for Further Improvement of [AFT11, PD11]? (1/2)

Learning scheme

- Learned clause assertive but otherwise adversarially chosen
- Very strong aspect of result
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Restart policy

- Restarts are *not too frequent* (unless Luby is too frequent)
- But no progress at all in between restarts
- Restarts seem important, but not quite *that* important?!

Room for Further Improvement of [AFT11, PD11]? (2/2)

Decision strategy

- In [PD11], crucially relies on (unknown) resolution proof
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Simulation efficiency

- Solvers typically have to run in (close to) linear time $\mathcal{O}(n)$
- But simulation will yield something like $\mathcal{O}(n^5)$ running time

What We Would Want

Want a more fine-grained and realistic CDCL model...

- Capture restarts, clause learning, memory management, etc.
- Modular design to allow study of different features
- Theoretical analogue of projects in [KSM11, SM11, ENSS16]

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... Leading to improved theoretical insights

- Can CDCL proof search be time *and space* efficient?
- And can it be *really* efficient? (No large polynomial blow-ups)
- How does memory management affect proof search quality?
- Do restarts increase reasoning power?
- How do other heuristics help or hinder proof search?

What We Have So Far (1/2)

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- **Time/Size:** # decisions + propagations + conflict analysis steps
  Space: (Size of clause database) − (size of formula)

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- **Cannot locally verify proof** but need global view (doubleplusunnice)

Sanity Check: CDCL Cannot Do Better than Resolution

Theorem ([EJL+16])

If *CDCL with "standard" learning scheme (e.g., 1UIP) decides* F *in time* τ *and space* s
then F *has resolution proof in size* $\leq \tau$ *and space* $\leq s + \mathcal{O}(1)$

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So lower bounds in resolution trade-offs automatically carry over
But can CDCL find time-efficient and space-efficient proofs?

Time-Space Trade-Offs for CDCL (in Math Notation)

We obtain CDCL analogues of (almost all) trade-off results in [BN11, BBI12, BNT13] — here is one sample:

Theorem ([EJL$^+$16], slightly informal)

For your favourite $k \in \mathbb{N}^+$ $\exists$ explicit formulas F_N of size $\approx N$ such that

- *CDCL with 1UIP learning and no restarts can decide F_N in time $\mathcal{O}(N^k)$ and space $\mathcal{O}(N^k)$*
- *CDCL with 1UIP learning and no restarts can decide F_N in space $\mathcal{O}(\log^2 N)$ and time $N^{\mathcal{O}(\log N)}$*
- *For any CDCL run in time τ and space s using any learning scheme and restart policy it holds that $\tau \gtrapprox (N^k/s)^{\Omega(\log\log N/\log\log\log N)}$*

Time-Space Trade-Offs for CDCL (in English)

Very informal statement of theorem to convey high-level message:

- Somewhat tricky formulas F_N (require superlinear time)
- CDCL can solve them efficiently for generous memory management (even without restarts)
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Interpretation:

- This is only a mathematical theorem about asymptotic properties of theoretical benchmarks
- But have some indications of similar behaviour for scaled-down versions in practical experiments [ENSS16]

Cutting Planes

Introduced in [CCT87] based on integer LP in [Gom63, Chv73]

Clauses interpreted as linear inequalities over the reals with integer coefficients (identifying $1 \equiv \mathit{true}$ and $0 \equiv \mathit{false}$)

Example: $x \lor y \lor \overline{z}$ gets translated to $x + y + (1 - z) \geq 1$

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Derivation rules

Variable axioms $\dfrac{}{0 \leq x \leq 1}$

Multiplication $\dfrac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA}$

Addition $\dfrac{\sum a_i x_i \geq A \quad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$

Division $\dfrac{\sum c a_i x_i \geq A}{\sum a_i x_i \geq \lceil A/c \rceil}$

Goal: Derive $0 \geq 1 \Leftrightarrow$ formula unsatisfiable

Size, Length and Space

Length = total # lines/inequalities in refutation

Size = sum also size of coefficients

Space = max # lines in memory during refutation

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Hard Formulas w.r.t. Cutting Planes Length

Clique-coclique formulas [Pud97]
"A graph with an m-clique is not $(m-1)$-colourable"

$p_{i,j}$ = indicator variables for edges in an n-vertex graph
$q_{k,i}$ = identifiers for members of m-clique in graph
$r_{i,\ell}$ = encoding of legal $(m-1)$-colouring of vertices

$q_{k,1} \lor q_{k,2} \lor \cdots \lor q_{k,n}$	some vertex is kth member of clique
$\overline{q}_{k,i} \lor \overline{q}_{k,j}$	kth clique member is uniquely defined
$p_{i,j} \lor \overline{q}_{k,i} \lor \overline{q}_{k',j}$	clique members are connected by edges
$r_{i,1} \lor r_{i,2} \lor \cdots \lor r_{i,m-1}$	every vertex i has a colour
$\overline{p}_{i,j} \lor \overline{r}_{i,\ell} \lor \overline{r}_{j,\ell}$	neighbours have distinct colours

Exponential lower bound via interpolation and circuit complexity
Technique very specifically tied to structure of formula

Open Problems for Cutting Planes Length and Space

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Prove length lower bounds for cutting planes

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Open Problems

Prove space lower bounds for cutting planes

- *with constant-size coefficients (very weak bounds in [GPT15])*
- *with polynomial-size coefficients (nothing known)*

Size-Space Trade-offs for Cutting Planes?

- Short cutting planes refutations of (lifted) Tseitin formulas on expanders need large space [GP14] (but probably don't exist)
- Short cutting planes refutations of (some) pebbling formulas need large space [HN12, GP14] (and such refutations exist)

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Open Problem

Are there trade-offs where the space-efficient CP refutations have small coefficients? (Say, of polynomial or even constant size)

Size-Space Trade-offs for Cutting Planes!

Recent news: Yes, there are such trade-offs!

Theorem ([dRNV16])

There exist flavours of pebbling formulas such that

- $\exists$ *small-size refutations with constant-size coefficients*
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- Again uses communication complexity (+ several other twists)
- Downside: Parameters worse than in previous results

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Several challenges:

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- Linear constraints more complicated than clauses — and integer arithmetic can become expensive
- Not obvious how to do conflict analysis
  - Can sometimes skip "resolution steps" in conflict analysis with propagating constraints on reason side — good or bad?
  - Can happen that "resolvent" is not conflicting — can be fixed in several ways, but what way is best?

Conflict-Driven CP Solvers: Two Concrete Obstacles

- **Roadblock 1:** Given CNF input, solvers cannot discover and use cardinality constraints (too limited form of addition)

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- **Roadblock 2(?):** Solvers seem inefficient for systems of inequalities that have rational but not integral solutions (too limited form of division?)
- Fail on, e.g., even colouring formulas [Mar06] for no obvious good reason
- Not well understood at all — work in progress

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- Study effect of different CDCL heuristics on performance

Theoretically Easy Combinatorial Benchmarks

- Study tweaked versions of well-studied formulas with:
  - short resolution proofs that can in principle be found by CDCL
  - without any preprocessing
  - often even without any restarts
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- Test theoretical results in [AFT11, PD11]: Does CDCL search for proofs efficiently?
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- Practical note: many (though not quite all) formulas generated using the tool **CNFgen** [CNF, LENV16]

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To run experiments, add "knobs" to Glucose [AS09, Glu] and vary settings for:

- restart policy
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Yields huge number of potential combinations

- Not all combinations make sense, but many do
- Test also settings where "convential wisdom" knows answer

Some Preliminary Conclusions (1/2)

Importance of restarts

- Sometimes very frequent restarts very important
- Crucial in [AFT11, PD11] for CDCL to simulate resolution efficiently
- Also seems to matter in practice for some formulas which are hard for subsystems of resolution such as regular resolution (stone formulas [AJPU07])

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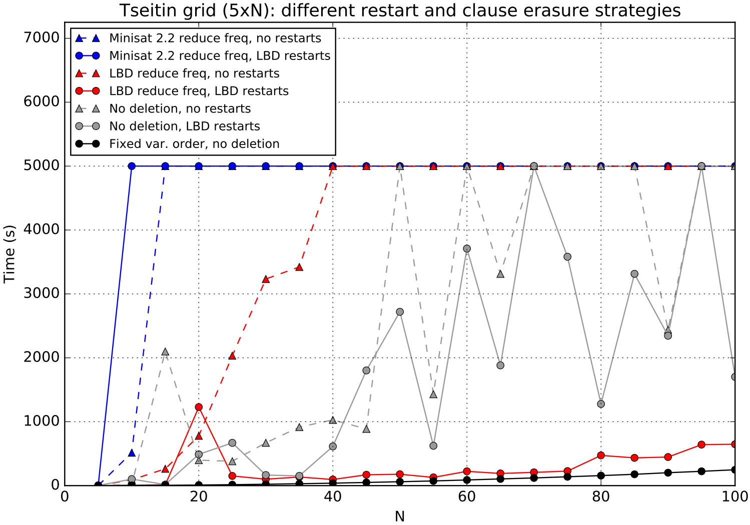
Clause erasure

- Theory says very aggressive clause removal could hurt badly
- Seem to see this on scaled-down versions of time-space trade-off formulas in [BBI12, BNT13] (Tseitin formulas)
- Even no erasure at all can be competitive for these formulas for frequent enough restarts

Plot 1: Tseitin Formulas on Grids



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- Can LBD (literal block distance) heuristic balance aggressive erasures by identifying important clauses? Maybe...
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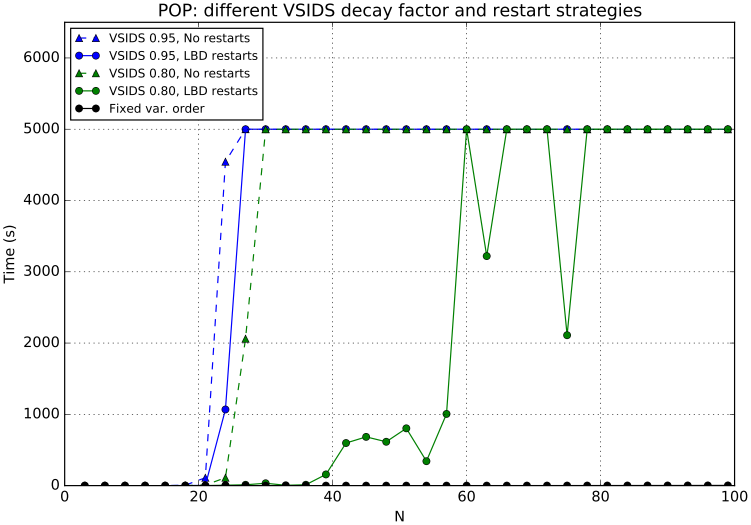
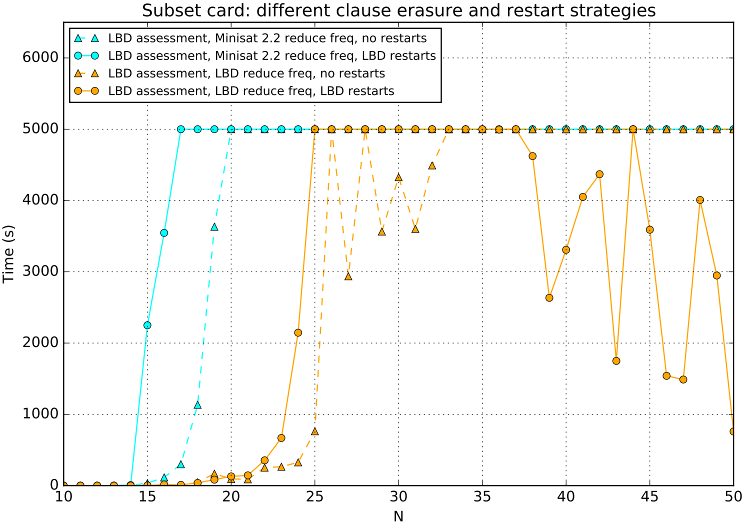
CDCL vs. resolution

- Sometimes CDCL fails miserably on easy formulas (Tseitin, even colouring) — VSIDS just goes dead wrong
- Sometimes strange easy-hard-easy patterns (subset cardinality)

Plot 2: Ordering Principle Formulas



Plot 3: Subset Cardinality Formulas



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This presentation:

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- Can CDCL simulate resolution time- and space-efficiently?
- Is standard CDCL without restarts weaker than resolution?
- Are there formulas for which VSIDS goes *provably* wrong?
- Can study of subsystems of cutting planes explain power and limitations of pseudo-Boolean solvers?
- Is it possible to build SAT solvers based on stronger proof systems than resolution that beat CDCL solvers?

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Thank you for your attention!

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