Supercritical Space-Width Trade-offs for Resolution

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Joint work with Christoph Berkholz

The Resolution Proof System

Goal: refute **unsatisfiable** CNF

- Start with axiom clauses in formula
- Derive new clauses by resolution rule

$$\frac{C \lor x \qquad D \lor \overline{x}}{C \lor D}$$

- Done when empty clause $\bot$ derived

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Can represent refutation/proof as

- annotated list or
- directed acyclic graph (DAG)

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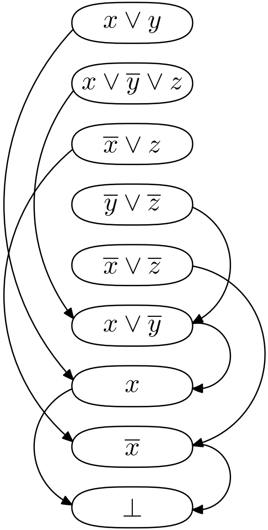

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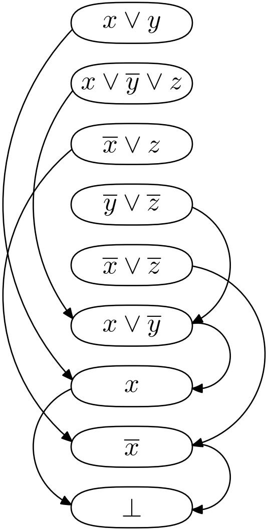
Tree-like resolution if DAG is tree



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Width of proof = # literals in largest clause (3 in our example)
Width of refuting F = min width over all proofs for F

Width at most linear, so here obviously care about linear factors

Resolution Space

Space = amount of memory needed
when performing refutation

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Example: Clause space at step 7

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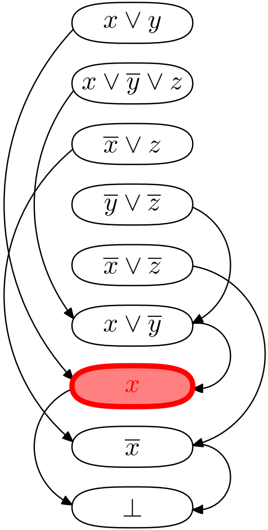

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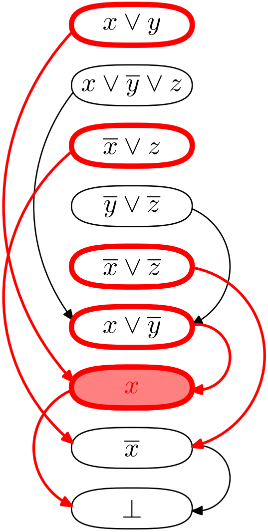

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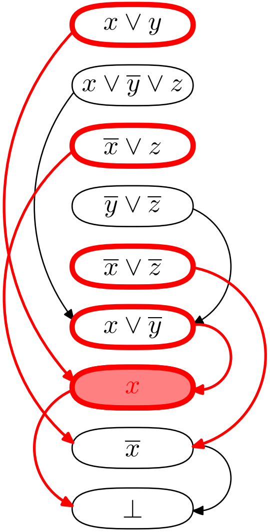

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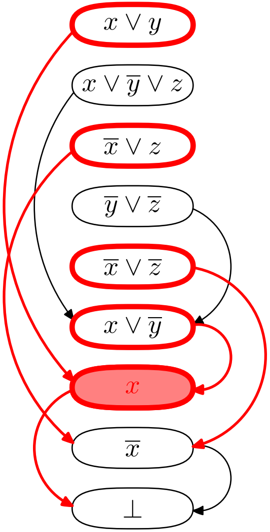
Space of proof = max over all steps
Space of refuting F = min over all proofs



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This talk: focus on width and clause space
But results translate to total space by:

$$\text{clause space} \leq \text{total space} \leq \text{clause space} \cdot \text{width}$$

Lower Bounds via Resolution Width

For n-variable k-CNFs (k constant) it holds that:

width $\leq$ Ω(clause space) [Atserias & Dalmau '03]

Lower Bounds via Resolution Width

For n-variable k-CNFs (k constant) it holds that:

$$\begin{array}{lll} \text{width} & \leq & \Omega\big(\text{clause space}\big) \\ \text{width}^2 & \leq & \Omega\big(\text{total space}\big) \end{array}$$

[Atserias & Dalmau '03]
[Bonacina '16]

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- Can have width $\Theta(\sqrt{n})$ and still size $\mathrm{poly}(n)$ [Bonet & Galesi '99]
- Can have width $\mathcal{O}(1)$ and still clause space $\Omega(n/\log n)$ [Ben-Sasson & Nordström '08]

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$$\textsf{size} \quad \leq \quad n^{\mathcal{O}(\textsf{width})}$$

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$$\begin{aligned} \text{size} &\leq n^{\mathcal{O}(\text{width})} \\ \text{time to find a refutation} &\leq n^{\mathcal{O}(\text{width})} \end{aligned}$$

for $w \leftarrow 3 \ldots n$ **do**
  Resolve all clauses & keep resolvents with at most w literals
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end for
Output `SAT`

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Algorithm (and resolution proof) requires $n^{\mathcal{O}(\text{width})}$ space!

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Recall: can always do clause space $\mathcal{O}(n)$

A Supercritical Space-Width Tradeoff

Theorem

For any $\varepsilon > 0$ *and* $6 \leq w \leq n^{\frac{1}{2}-\varepsilon}$ *exist* n*-variable* w-CNF*s* F_n *s.t.*

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Key components:

- Expander graphs
- XORification (substitution with exclusive or)

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- We feel "supercritical" is more descriptive

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Very well-connected so-called expander graphs play leading role in many proof complexity lower bounds

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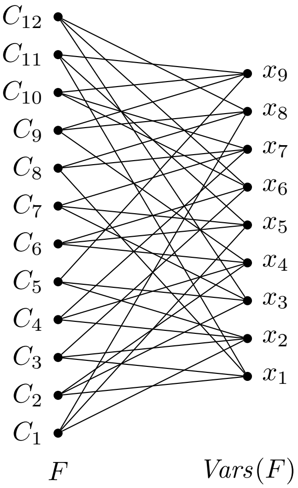

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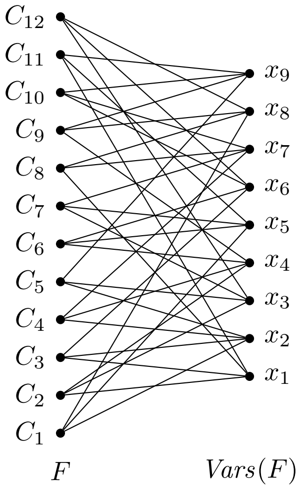

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If CVIG well-connected, then lower bounds for

- width, size, and space in resolution
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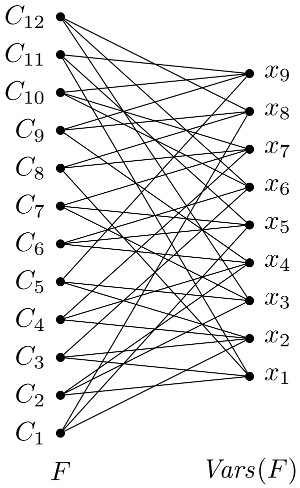

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Can also define more general graphs that capture "underlying combinatorial structure" and extend results [Mikša & Nordström '15]

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- # vars in memory $\geq s$ for $F \Longrightarrow$ clause space $\geq \Omega(s)$ for $F[\oplus_2]$ [Ben-Sasson & Nordström '08]

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Intuition behind proof

- Given resolution refutation π of $F[\oplus_2]$
- Extract the refutation π' of F that π is simulating
- Prove that extraction preserves complexity measures of interest

Pebbling Formulas

Encode pebble games on DAGs
[Ben-Sasson & Wigderson '99]

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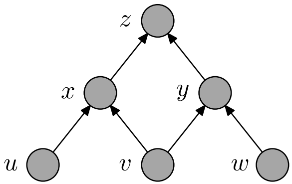

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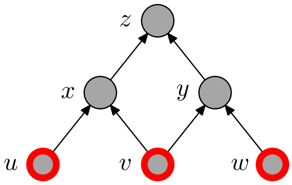

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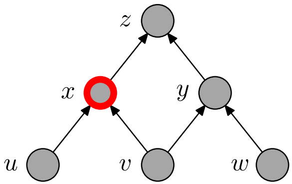

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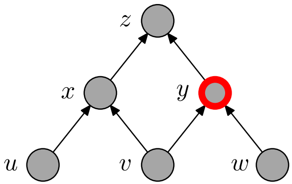

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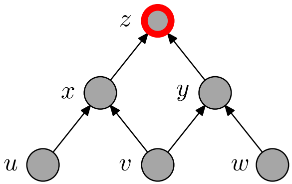

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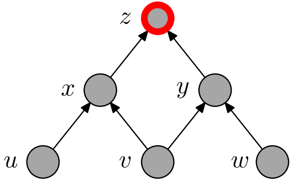

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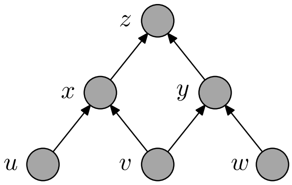

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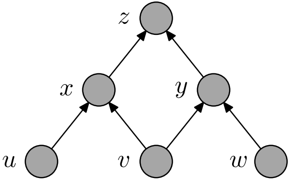

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Easy to refute pebbling formulas in size $\mathcal{O}(n)$ and width $\mathcal{O}(1)$
Pebbling space lower bounds $\Rightarrow$ clause space lower bounds
[Ben-Sasson & Nordström '08, '11]

XOR Substitution with Recycling (1/2)

Suppose

- F CNF formula over variables U
- $\mathcal{G} = (U \mathbin{\dot\cup} V, E)$ bipartite graph

Substituted formula $F[\mathcal{G}]$ over variables V:

- replace every $u \in U$ by $\bigoplus_{v \in N(u)} v$

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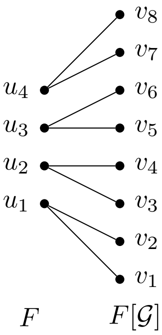

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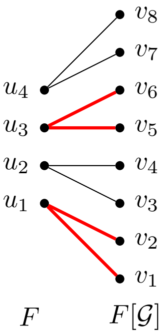

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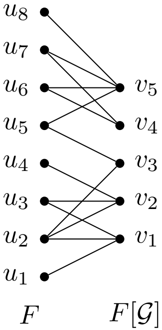

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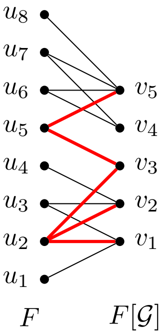

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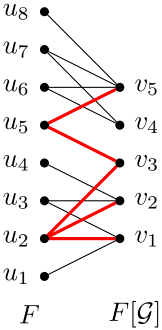

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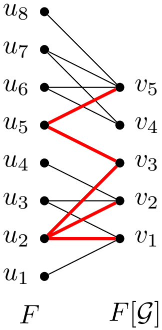

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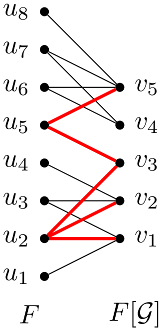

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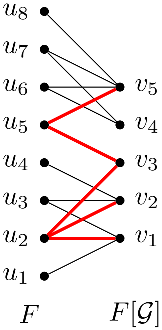

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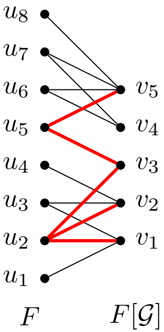

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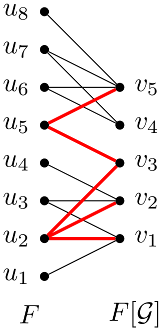

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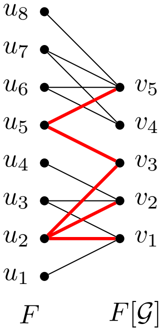

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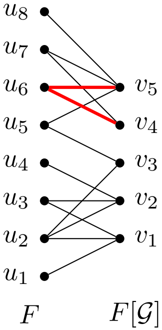

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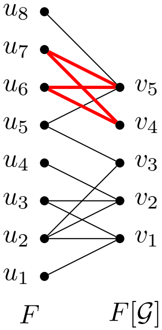

$$\begin{aligned}
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$\lightning$

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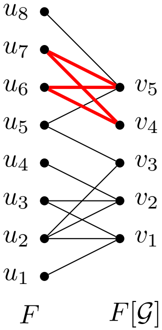

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Solution: Use expander graphs!

- Apply to pebbling formulas F in [Ben-Sasson & Nordström '08]
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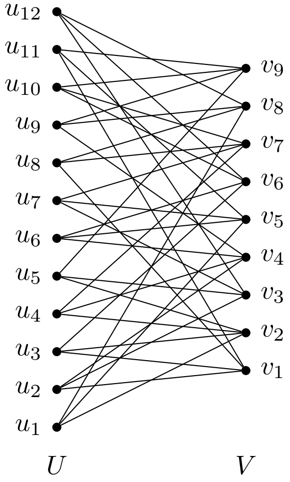
Bipartite Boundary Expander



$\mathcal{G} = (U \dot\cup V, E)$ is (d, r, c)-boundary expander if

- left-degree $\leq d$
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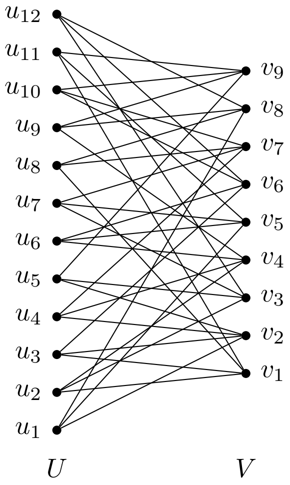

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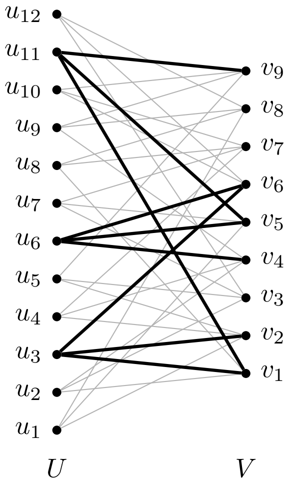

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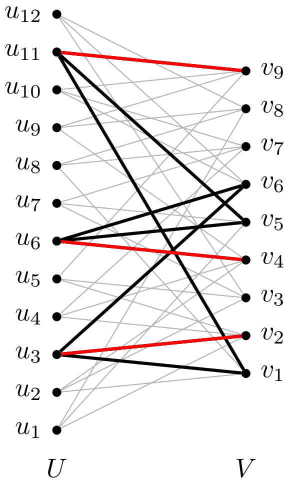

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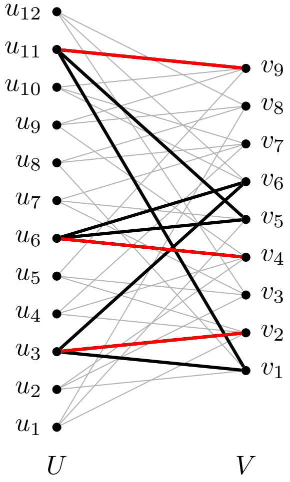

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Lemma ([Razborov '16])

For $\varepsilon > 0$ and n, d with $d \leq |V|^{\frac{1}{2}-\varepsilon}$, $|U| = n$, $|V| = n^{\mathcal{O}(1/d)}$ there are $(d, r, 2)$-boundary expanders $\mathcal{G}$ with $r = d \log n$

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Look at clauses $\mathcal{D}$ in memory in width-w refutation of $F[\mathcal{G}]$

Recover clauses $\mathcal{C}$ in memory in “simulated refutation” of F

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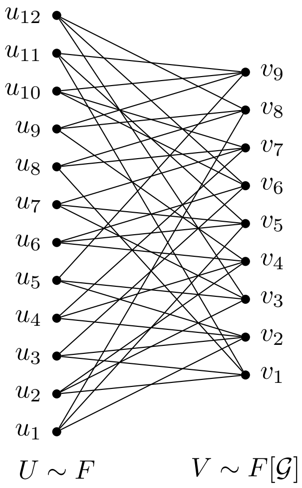

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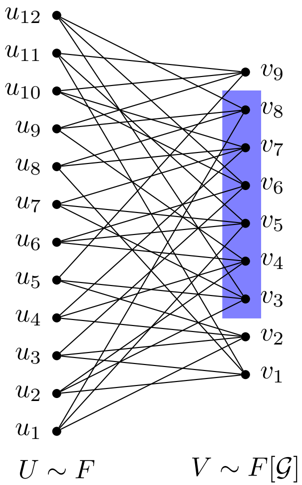

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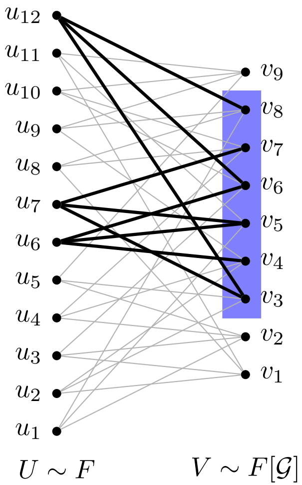

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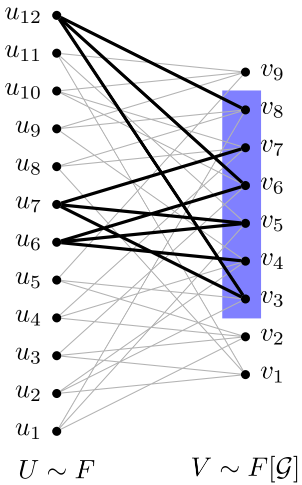

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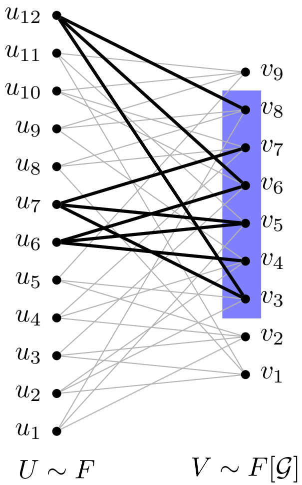

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Where else can this technique be useful?

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