Combinatorial Solving with Provably Correct Results

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Belo Horizonte, Brazil
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Based on Joint Work With...

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The Success Story of Combinatorial Solving and Optimization

- Rich field of mathematics and computer science
- Impact in other areas of science and also industry, e.g.:
  - airline scheduling
  - hardware verification
  - donor-recipients matching for kidney transplants [MO12, BvdKM$^+$21]
- Discrete problems — computationally very challenging (NP-complete or worse)
- Lots of effort last couple of decades spent on developing sophisticated so-called combinatorial solvers that often work surprisingly well in practice for, e.g.,
  - Boolean satisfiability (SAT) solving [BHvMW21]
  - Constraint programming [RvBW06]
  - Mixed integer linear programming [AW13, BR07]
  - Satisfiability modulo theories (SMT) solving [BHvMW21]

And the Dirty Little Secret. . .

- Solvers very fast, but sometimes wrong (even best commercial ones) [BLB10, CKSW13, AGJ$^+$18, GSD19, BMN22, GCS23]
- Even worse: No way of knowing for sure when errors happen
- Solvers can propose infeasible "solutions" (but erroneous claims can in principle be checked)
- More challenging: How to achieve reliable claims of infeasibility?
- Or of optimality?
- Even off-by-one mistakes can snowball into large errors if solver used as subroutine

What Can Be Done About Solver Bugs?

- **Software testing**
  Very useful, but bugs slip through even with careful domain-specific testing
  Progress using fuzzing and delta debugging [BB09, BLB10, KB22, NPB22, PB23]
  But testing inherently can only detect presence of bugs, not absence

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- **Formal verification**
  Prove that solver implementation adheres to formal specification
  Current techniques cannot scale to level of complexity in modern solvers
  (Despite valiant efforts in, e.g., [Fle20])

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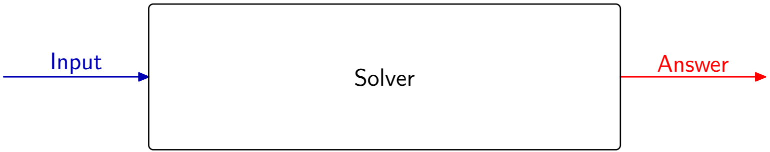
- **Proof logging**
  Make solver certifying [ABM+11, MMNS11] by adding code so that it outputs
  1. not only answer but also
  2. simple, machine-verifiable proof that answer is correct

Proof Logging with Certifying Solvers: Workflow



1. Run combinatorial solving algorithm on problem input

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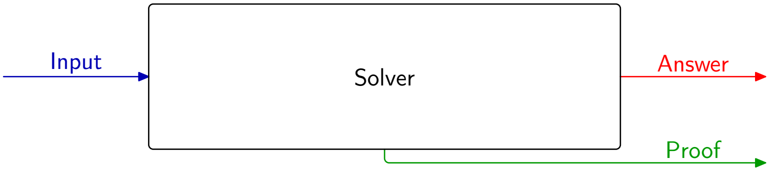

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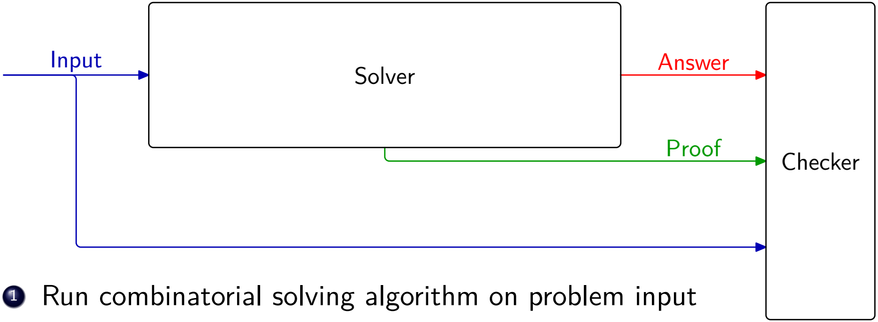

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3. Feed input + answer + proof to proof checker

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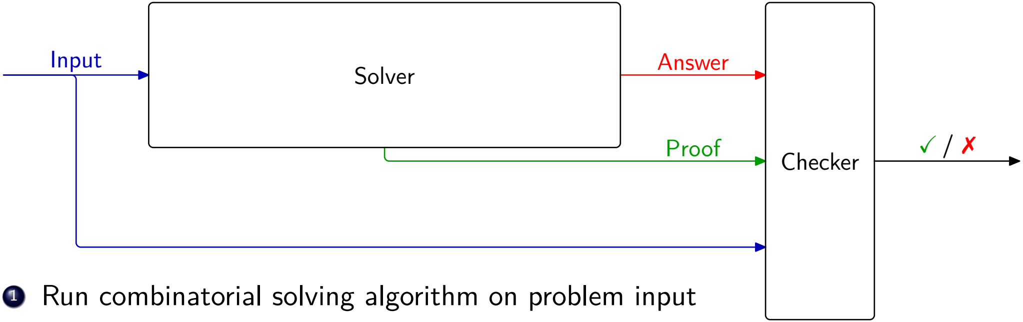
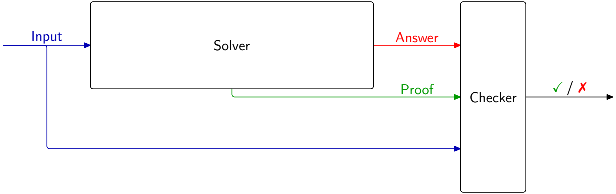


1. Run combinatorial solving algorithm on problem input
2. Get as output not only answer but also proof
3. Feed input + answer + proof to proof checker
4. Verify that proof checker says answer is correct

Proof Logging Desiderata

Proof format for certifying solver should be



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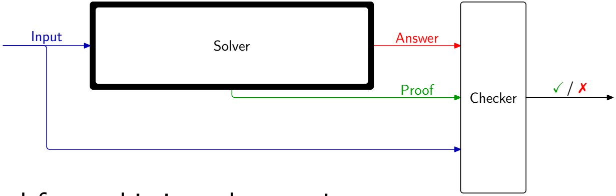

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- **very powerful:** minimal overhead for sophisticated reasoning

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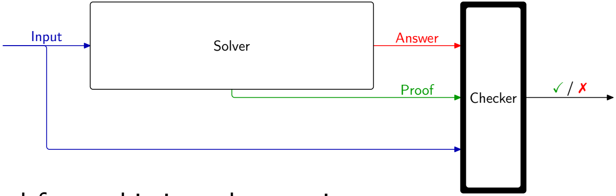

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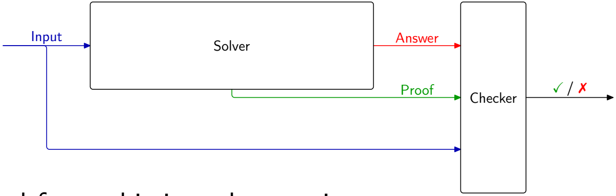

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Clear conflict expressivity vs. simplicity!

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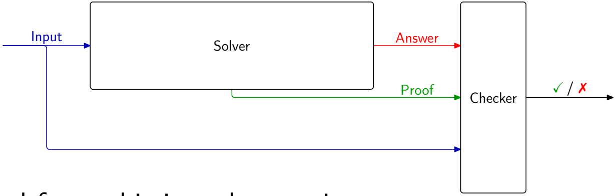

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Clear conflict expressivity vs. simplicity!

Asking for both perhaps a little bit too good to be true?

Some Previous Proof Logging Work

Boolean satisfiability (SAT) solving

- Well established since over decade with several proof formats such as DRAT [HHW13a, HHW13b, WHH14], GRIT [CMS17], LRAT [CHH+17], ...
- But no efficient support for most advanced techniques such as
  - Gaussian elimination
  - symmetry breaking

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Constraint programming

- Either have to trust that propagations done correctly [DFS12, OSC09, VS10]
- Or suffer from exponential slow-down to generate verifiable proofs [GCS23]

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Mixed integer linear programming

- Work on proof format VIPR [CGS17, EG23]
- But only for exact solving and without support for advanced techniques

Message of This Talk

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- Build on successes in proof logging for SAT solving
- But represent constraints as 0–1 integer linear inequalities
- Formalize reasoning using cutting planes [CCT87] proof system
- Add well-chosen strengthening rules [Goc22, GN21, BGMN23]
- Implemented in VeriPB (`https://gitlab.com/MIAOresearch/software/VeriPB`)

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1. Marketing pitch ☺

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2. Describe foundations of proof logging method
3. Discuss future challenges and directions

The Sales Pitch For Proof Logging

1. Certifies correctness of computed results
2. Detects errors even if due to compiler bugs, hardware failures, or cosmic rays
3. Provides debugging support during software development [GMM+20, KM21, BBN+23, EG23, KLM+25]
4. Facilitates performance analysis
5. Helps identify potential for further improvements
6. Enables auditability
7. Serves as stepping stone towards explainability

Design Principles for Proof Logging

Proof logging implementation

- Don't change solver
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Performance goals

- Proof logging overhead small constant fraction of running time ($\lessapprox 10\%$)
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Proof system

- Keep language simple — no XOR constraints, CP propagators, symmetries, ...
- But reason efficiently about such notions using power of proof system
- Combine proof logging with formally verified proof checker

Proof Language: Pseudo-Boolean Constraints

Proof consists of 0–1 integer linear inequalities or pseudo-Boolean constraints:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
- variables x_i take values $0 = \mathit{false}$ or $1 = \mathit{true}$

Sometimes convenient to use normalized form [Bar95] with all a_i, A positive (without loss of generality)

Some Types of Pseudo-Boolean Constraints

1. Disjunctive clauses

$$x \lor \overline{y} \lor z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Pseudo-Boolean Proof Logging Wishlist

Paradigms

- Boolean satisfiability (SAT) solving
- (linear) pseudo-Boolean solving
- subgraph solving
- constraint programming
- automated planning
- mixed integer linear programming
- SMT solving

Problem types

- decision / feasibility
- optimization
- multi-objective optimization
- projected model enumeration
- projected model counting
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Pseudo-Boolean Proof Logging — How and Why?

If problem is (special case of) 0-1 integer linear program

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1. 0-1 ILP expressive formalism for combinatorial problems (including objective)
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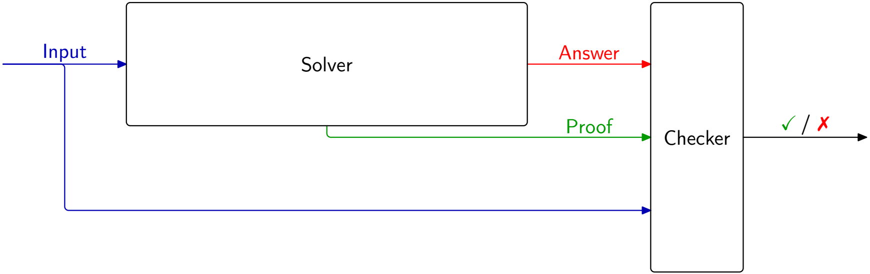
$$r \Rightarrow x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7 \qquad 7\overline{r} + x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

$$r \Leftarrow x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7 \qquad 9r + \overline{x}_1 + 2x_2 + 3\overline{x}_3 + 4x_4 + 5\overline{x}_5 \geq 9$$

Proof Logging with Formally Verified Checking: Full Workflow



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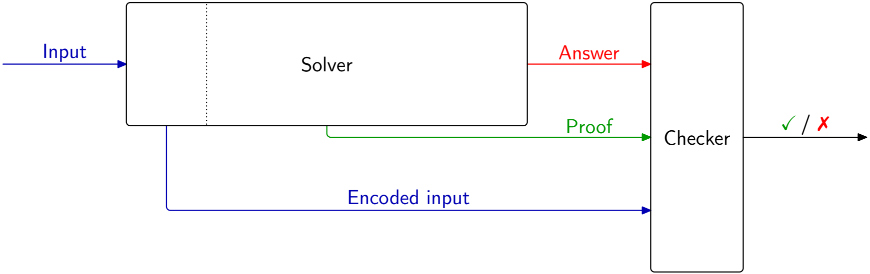

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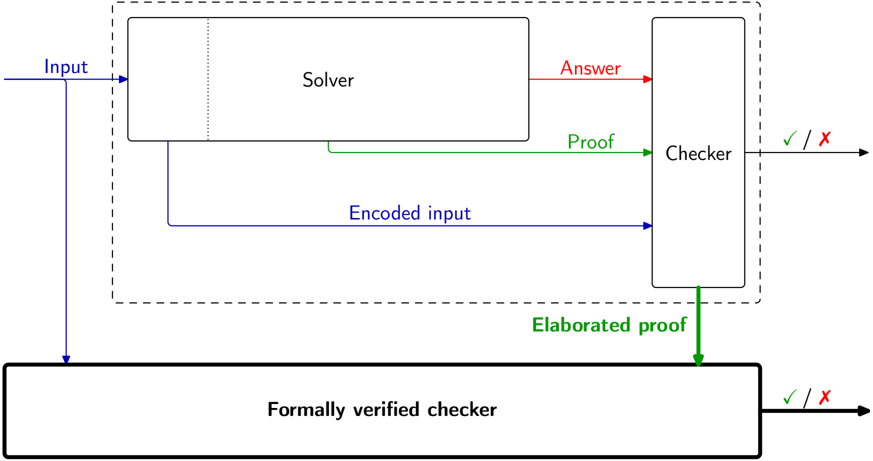

VeriPB Proof Structure

1. **Preamble**
   Load input formula
   Specify settings
2. **Derivation section**
   Derivations of new constraints
   Logging of solutions
3. **Output section**
   Listing of constraints currently in database
   Input to next stage (or for debugging)
4. **Conclusions section**
   Specification of what was established
   - satisfiability / unsatisfiability
   - optimality (or upper and lower bounds)
   - other types of conclusions

VeriPB Proof Configuration (Slightly Simplified)

Core set $\mathcal{C}$

- Contains input formula at the start
- Maintains "equivalence" with input formula

Derived set $\mathcal{D}$

- All constraints derived during search
- Also intermediate constraints used in proof logging [but not used by solver]
- Any satisfying assignment to $\mathcal{C}$ can be extended to $\mathcal{D}$

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Objective $f = \sum_i w_i \ell_i + k$

- 0–1 linear function to minimize
- Or $f = 0$ for decision problem
- Keep track of best known bound; initialize to ∞

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Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input axioms From the input

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Literal axioms
$$\frac{}{\ell_i \geq 0}$$

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Addition

$$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i) \ell_i \geq A + B}$$

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Multiplication for any $c \in \mathbb{N}^+$
$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i c a_i \ell_i \geq cA}$$

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Division for any $c \in \mathbb{N}^+$

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Saturation
(constraint in normalized form) $$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \min(a_i, A) \cdot \ell_i \geq A}$$

Cutting Planes Toy Example

$$w + 2x + y \geq 2$$

Cutting Planes Toy Example

$$\text{Multiply by } 2 \quad \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

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Multiply by 2 $\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$ $\quad w + 2x + 4y + 2z \geq 5$

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$$\text{Add}\;\dfrac{\text{Multiply by } 2\;\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} \qquad \dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\;\text{Multiply by } 2$$

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$$\text{Add}\;\dfrac{\text{Add}\;\dfrac{\text{Multiply by 2}\;\dfrac{w+2x+y\geq 2}{2w+4x+2y\geq 4}\qquad w+2x+4y+2z\geq 5}{3w+6x+6y+2z\geq 9}\qquad \dfrac{\overline{z}\geq 0}{2\overline{z}\geq 0}\;\text{Multiply by 2}}{3w+6x+6y+2z+2\overline{z}\geq 9}$$

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$$

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$$

Cutting Planes Toy Example

$$
\dfrac{\text{Divide by } 3\;\dfrac{\text{Add}\;\dfrac{\text{Add}\;\dfrac{\text{Multiply by } 2\;\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} \qquad \dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\;\text{Multiply by } 2}{3w + 6x + 6y \geq 7}}{}}{w + 2x + 2y \geq 2\frac{1}{3}}
$$

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By referring to constraints by labels and to literal axioms by the literal involved as

$$\begin{aligned}
\texttt{@C1} &\doteq 2x + y + w \geq 2 \\
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\end{aligned}
$$

such a calculation is written in the proof log in reverse Polish notation as

```
pol  @C1  2  *  @C2  +  ~z  2  *  +  3  d  ;
```

Deriving Non-implied Constraints by Redundance-Based Strengthening

C is said to be "redundant" with respect to F if F and $F \cup \{C\}$ are equisatisfiable

[apologies for the terminology — this is inherited from SAT proof logging]

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Redundance-based strengthening ([BT19, GN21], inspired by [JHB12])

C is redundant with respect to F if and only if there is a substitution ω (mapping variables to truth values or literals), called a witness, for which

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- Proof sketch for interesting direction: If α satisfies F but falsifies C, then α satisfies $(F \cup \{C\})\restriction_\omega$, i.e., $\alpha \circ \omega$ satisfies $F \cup \{C\}$
- In a proof, the implication needs to be efficiently verifiable — every $D \in (F \cup \{C\})\restriction_\omega$ should follow from $F \cup \{\neg C\}$ either
  1. "obviously" or
  2. by explicitly presented derivation

Example: Deriving $r \leftrightarrow (x \land y)$ Using the Redundance Rule

Want to derive

$$2\overline{r} + x + y \geq 2 \qquad\qquad r + \overline{x} + \overline{y} \geq 1$$

using condition $F \cup \{\neg C\} \models (F \cup \{C\})\restriction_\omega$

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   Choose $\omega = \{r \mapsto 1\}$ — F untouched; new constraint satisfied
   Premise $\neg(r + \overline{x} + \overline{y} \geq 1)$ forces $x \mapsto 1$ and $y \mapsto 1$, hence $(2\overline{r} + x + y \geq 2)\restriction_\omega$ is satisfied even though $r \mapsto 1$

Redundance and Dominance Rules in VeriPB (Slightly Simplified)

Redundance-based strengthening, optimization version [BGMN23]

Add constraint C to derived set $\mathcal{D}$ if exists witness substitution ω such that

$$\mathcal{C} \cup \mathcal{D} \cup \{\neg C\} \models (\mathcal{C} \cup \mathcal{D} \cup \{C\})\restriction_\omega \cup \{f\restriction_\omega \leq f\}$$

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- Applying ω should strictly decrease f
- If so, don't need to show that $(\mathcal{D} \cup \{C\})\restriction_\omega$ implied!

Soundness of Dominance Rule

Dominance-based strengthening

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7. ...
8. Can't go on forever, so finally reach α' satisfying $\mathcal{C} \cup \{C\}$

Strengthening Rules: Proof Format

```
red ⟨Constraint C⟩ : ⟨var1⟩ -> ⟨val1⟩ ... ⟨varN⟩ -> ⟨valN⟩ : subproof
    subproofs for proof goals
qed;

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- Witness ω should be explicitly specified in proof log
- Subproofs of proof goals should also be explicit
- But can be skipped for proof goals “obvious” to proof checker

Successful Applications of VeriPB Proof Logging

Surprisingly, pseudo-Boolean reasoning with strengthening rules sufficient to efficiently certify wide range of combinatorial solving techniques:

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1. Boolean satisfiability (SAT) solving including advanced techniques such as
   - Gaussian elimination [GN21]
   - symmetry breaking [BGMN23]
2. SAT-based optimization (MaxSAT) [VDB22, BBN^{+}23, BBN^{+}24, IOT^{+}24]
3. (Linear) Pseudo-Boolean solving [GMNO22, KLM^{+}25]
4. Subgraph solving (max clique, subgraph isomorphism, max common connected subgraph) [GMN20, GMM^{+}20, GMM^{+}24]
5. Dynamic programming and decision diagrams [DMM^{+}24]
6. Presolving in 0–1 integer linear programming [HOGN24]
7. Constraint programming [EGMN20, GMN22, MM23, MMN24, MM25]
8. Automated planning [DHN^{+}25]

Three Pseudo-Boolean Proof Logging Vignettes

1. Symmetry breaking [BGMN23]
2. Graph solving (subgraph isomorphism) [GMN20, GMM$^+$20, GMM$^+$24]
3. Constraint programming [EGMN20, GMN22, MM23, MMN24, MM25]

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3. Derive symmetry breaking clauses from this PB constraint:

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$$\overline{y}_j \vee y_{j-1} \qquad y_j \vee \overline{y}_{j-1} \vee \sigma(x_j)$$

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VeriPB can certify fully general SAT symmetry breaking [BGMN23]

The Subgraph Isomorphism Problem

Input

- Pattern graph $\mathcal{P}$ with vertices $V(\mathcal{P}) = \{a, b, c, \ldots\}$
- Target graph $\mathcal{T}$ with vertices $V(\mathcal{T}) = \{u, v, w, \ldots\}$

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Task

- Find all subgraph isomorphisms $\varphi : V(\mathcal{P}) \to V(\mathcal{T})$
- I.e., one-to-one mappings φ such that if
  1. $\varphi(a) = u$
  2. $\varphi(b) = v$
  3. $(a, b) \in E(\mathcal{P})$

  then must have $(u, v) \in E(\mathcal{T})$

Pseudo-Boolean Proof Logging for Subgraph Isomorphism Solving

All reasoning steps in Glasgow Subgraph Solver [ADH+19, GSS] can be formalized efficiently in the cutting planes proof system [GMN20]

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Means that

1. Solver can justify each step by writing local formal derivation
2. Local derivations can be chained into global correctness proof
3. Proof checkable by stand-alone verifier that knows nothing about graphs
4. With end-to-end fully formally verified result [GMM+24]

Subgraph Isomorphism as a Pseudo-Boolean Formula

- Pattern graph $\mathcal{P}$ with $V(\mathcal{P}) = \{a, b, c, \ldots\}$
- Target graph $\mathcal{T}$ with $V(\mathcal{T}) = \{u, v, w, \ldots\}$
- No loops (for simplicity)

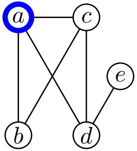
Pseudo-Boolean encoding

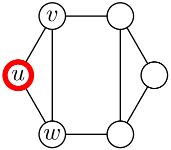
$$\sum_{v \in V(\mathcal{T})} x_{a,v} = 1 \qquad \text{[every } a \text{ maps somewhere]}$$

$$\sum_{b \in V(\mathcal{P})} \overline{x}_{b,u} \geq |V(\mathcal{P})| - 1 \qquad \text{[mapping is one-to-one]}$$

$$\overline{x}_{a,u} + \sum_{v \in N(u)} x_{b,v} \geq 1 \qquad \text{[edge } (a,b) \text{ maps to edge } (u,v)\text{]}$$

Pseudo-Boolean Proof Logging Example: Degree Preprocessing





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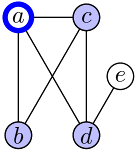

$$\overline{x}_{a,u} + x_{b,v} + x_{b,w} \geq 1$$
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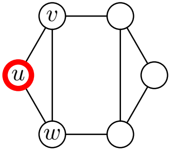

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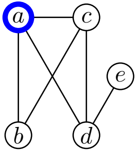

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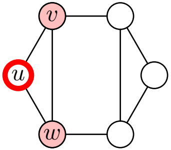

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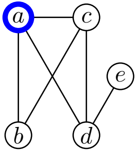

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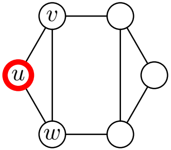

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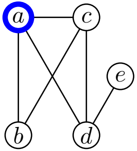

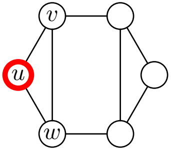

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Sum up all constraints & divide by 3 to obtain

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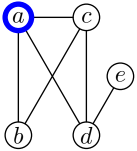

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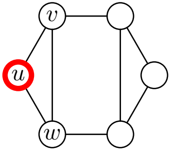

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$$3\overline{x}_{a,u} + 10 \geq 11$$

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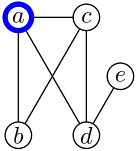

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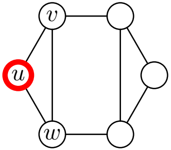

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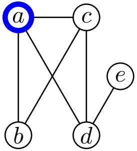

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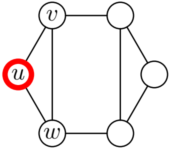

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Constraint Programming: Integer Variables (1/2)

How to deal with integer variables in constraint programming?

Given $A \in \{-3 \ldots 9\}$, the direct encoding is:

$$\begin{aligned} a_{=-3} + a_{=-2} + a_{=-1} + a_{=0} + a_{=1} + a_{=2} + a_{=3} \\ + a_{=4} + a_{=5} + a_{=6} + a_{=7} + a_{=8} + a_{=9} = 1 \end{aligned}$$

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This doesn't work for large domains...

We can instead use a binary encoding:

$$\begin{aligned} -16a_{\text{neg}} + 1a_{\text{b0}} + 2a_{\text{b1}} + 4a_{\text{b2}} + 8a_{\text{b3}} &\geq -3 \qquad \text{and} \\ 16a_{\text{neg}} + -1a_{\text{b0}} + -2a_{\text{b1}} + -4a_{\text{b2}} + -8a_{\text{b3}} &\geq -9 \end{aligned}$$

Bad properties for solver propagation, but that isn't a problem for proof logging

Constraint Programming: Integer Variables (2/2)

We can mix binary and order encodings! Define big-M linear inequalities encoding

$$a_{\geq 4} \Leftrightarrow -16a_{\mathrm{neg}} + 1a_{\mathrm{b0}} + 2a_{\mathrm{b1}} + 4a_{\mathrm{b2}} + 8a_{\mathrm{b3}} \geq 4$$

$$a_{\geq 5} \Leftrightarrow -16a_{\mathrm{neg}} + 1a_{\mathrm{b0}} + 2a_{\mathrm{b1}} + 4a_{\mathrm{b2}} + 8a_{\mathrm{b3}} \geq 5$$

$$a_{=4} \Leftrightarrow a_{\geq 4} \land \overline{a}_{\geq 5}$$

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When creating $a_{\geq i}$, also introduce pseudo-Boolean constraints encoding

$$a_{\geq i} \Rightarrow a_{\geq j} \quad \text{and} \quad a_{\geq h} \Rightarrow a_{\geq i}$$

for the closest values $j < i < h$ that already exist

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We can do this:

- Inside the pseudo-Boolean model where needed
- Otherwise lazily during proof logging

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Given table constraint

$$(A, B, C) \in [(1,2,3), (1,3,4), (2,2,5)]$$

define

$$\begin{aligned} 3\bar{t}_1 + a_{=1} + b_{=2} + c_{=3} &\geq 3 && \text{i.e.,} \quad t_1 \Rightarrow (a_{=1} \land b_{=2} \land c_{=3}) \\ 3\bar{t}_2 + a_{=1} + b_{=4} + c_{=4} &\geq 3 && \text{i.e.,} \quad t_2 \Rightarrow (a_{=1} \land b_{=4} \land c_{=4}) \\ 3\bar{t}_3 + a_{=2} + b_{=2} + c_{=5} &\geq 3 && \text{i.e.,} \quad t_3 \Rightarrow (a_{=2} \land b_{=2} \land c_{=5}) \end{aligned}$$

using tuple selector variables

$$t_1 + t_2 + t_3 = 1$$

A Constraint Programming Solver with Pseudo-Boolean Proof Logging

Proof-of-concept CP solver at `github.com/ciaranm/glasgow-constraint-solver` supports proof logging for global constraints:

- All-different
- Integer linear inequality (including for very large domains)
- Smart table and regular
- Minimum / maximum of an array
- Element (kind of array indexing)
- Absolute value
- (Hamiltonian) Circuit
- and more...

Details in [EGMN20, GMN22, MM23, MMN24, MM25]

Future Research Directions

Performance and reliability of pseudo-Boolean proof logging and checking

- Trim proof while verifying (as in DRAT-TRIM [HHW13a])
- Compress proof file using binary format
- Formally verifed end-to-end checking (as in [GMM+24, IOT+24, KLM+25])
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- We're hiring! Talk to me to join the pseudo-Boolean proof logging revolution! ☺

VeriPB Resources

VeriPB tutorials

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- Video tutorial at `https://youtu.be/s_5BIi4I22w`
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Lots of concrete example files at `gitlab.com/MIAOresearch/software/VeriPB`

Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity
- **Action point:** What problems can VeriPB solve for you? ☺

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Thank you for your attention!

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