Understanding Space in Proof Complexity: Separations and Trade-offs via Substitutions

Jakob Nordström

MIT Computer Science and Artificial Intelligence Laboratory

Theory of Computation Colloquium
April 20, 2010

Joint work with Eli Ben-Sasson

A Fundamental Problem in Computer Science

Problem

Given a propositional logic formula F, is it true no matter how we assign values to its variables?

TAUTOLOGY: Fundamental problem in Theoretical Computer Science since Cook's NP-completeness paper (1971)

Last decade or so: also intense applied interest

Enormous progress on algorithms (although still exponential time in worst case)

Proof Complexity

Proof search algorithm: proof system with derivation rules

Proof complexity: study of proofs in such systems

- Lower bounds: no algorithm can do better (even optimal one always guessing the right move)
- Upper bounds: gives hope for good algorithms if we can search for proofs in system efficiently

Resolution

- Prove tautologies $\Leftrightarrow$ refute unsatisfiable formulas in conjunctive normal form (CNF)
- Resolution: proof system for refuting CNF formulas
- Perhaps *the* most studied system in proof complexity
- Basis of current state-of-the-art SAT-solvers (e.g. winners in recent SAT competitions)
- So called DPLL-algorithms (Davis-Putnam-Logemann-Loveland) augmented with clause learning

Trade-offs Between Time and Memory?

- Key bottlenecks for SAT-solvers: time and memory
- What are the connections between these resources? Are they correlated? Are there trade-offs?
- Question ca 1998: Does proof complexity have anything intelligent to say about this? (Corresponding to relation between size and space of proofs)
- This talk: Study these questions for resolution, and also for more general k-DNF resolution proof systems

Outline

1. Resolution-Based Proof Systems
   - Basics
   - Some Previous Work
   - Our Results
2. Outline of Proofs
   - Pebble Games and Pebbling Contradictions
   - Substitution Theorem
   - Putting the Pieces Together
3. Open Problems

Some Notation and Terminology

- Literal a: variable x or its negation $\overline{x}$
- Clause $C = a_1 \vee \cdots \vee a_k$: disjunction of literals
- Term $T = a_1 \wedge \cdots \wedge a_k$: conjunction of literals
- CNF formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses
  k-CNF formula: CNF formula with clauses of size $\leq k$
- DNF formula $D = T_1 \vee \cdots \vee T_m$: disjunction of terms
  k-DNF formula: DNF formula with terms of size $\leq k$

k-DNF Resolution

- Prove that given CNF formula is unsatisfiable
- Proof operates with k-DNF formulas (standard resolution corresponds to 1-DNF formulas, i.e., disjunctive clauses)
- Proof is "presented on blackboard"
- Derivation steps:
  - Write down clauses of CNF formula being refuted (axiom clauses)
  - Infer new k-DNF formulas
  - Erase formulas that are not currently needed (to save space on blackboard)
- Proof ends when contradictory empty clause 0 derived

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \lor y$
3. $\overline{y} \lor z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \vee y$
3. $\overline{y} \vee z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \vee y$
3. $\overline{y} \vee z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \lor y$
3. $\overline{y} \lor z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \vee y$
3. $\overline{y} \vee z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

$$\boxed{\quad x \quad}$$

Write down axiom 1: x

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \vee y$
3. $\overline{y} \vee z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

---

x
$\overline{y} \vee z$

Write down axiom 1: x
Write down axiom 3: $\overline{y} \vee z$

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \vee y$
3. $\overline{y} \vee z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

---

x
$\overline{y} \vee z$

Write down axiom 1: x
Write down axiom 3: $\overline{y} \vee z$
Combine x and $\overline{y} \vee z$
to get $(x \wedge \overline{y}) \vee z$

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \lor y$
3. $\overline{y} \lor z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

---

x
$\overline{y} \lor z$
$(x \land \overline{y}) \lor z$

Write down axiom 1: x
Write down axiom 3: $\overline{y} \lor z$
Combine x and $\overline{y} \lor z$ to get $(x \land \overline{y}) \lor z$

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \lor y$
3. $\overline{y} \lor z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

---

x
$\overline{y} \lor z$
$(x \land \overline{y}) \lor z$

Write down axiom 1: x
Write down axiom 3: $\overline{y} \lor z$
Combine x and $\overline{y} \lor z$
to get $(x \land \overline{y}) \lor z$
Erase the line x

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \lor y$
3. $\overline{y} \lor z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

$\overline{y} \lor z$
$(x \land \overline{y}) \lor z$

Write down axiom 1: x
Write down axiom 3: $\overline{y} \lor z$
Combine x and $\overline{y} \lor z$
to get $(x \land \overline{y}) \lor z$
Erase the line x

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \lor y$
3. $\overline{y} \lor z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

$\overline{y} \lor z$
$(x \land \overline{y}) \lor z$

Write down axiom 3: $\overline{y} \lor z$
Combine x and $\overline{y} \lor z$
to get $(x \land \overline{y}) \lor z$
Erase the line x
Erase the line $\overline{y} \lor z$

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \lor y$
3. $\overline{y} \lor z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

$$\boxed{(x \land \overline{y}) \lor z}$$

Write down axiom 3: $\overline{y} \lor z$
Combine x and $\overline{y} \lor z$
to get $(x \land \overline{y}) \lor z$
Erase the line x
Erase the line $\overline{y} \lor z$

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \lor y$
3. $\overline{y} \lor z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

$(x \land \overline{y}) \lor z$
$\overline{x} \lor y$

Combine x and $\overline{y} \lor z$
to get $(x \land \overline{y}) \lor z$
Erase the line x
Erase the line $\overline{y} \lor z$
Write down axiom 2: $\overline{x} \lor y$

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \lor y$
3. $\overline{y} \lor z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

$(x \land \overline{y}) \lor z$
$\overline{x} \lor y$

Erase the line x
Erase the line $\overline{y} \lor z$
Write down axiom 2: $\overline{x} \lor y$
Infer z from
$\overline{x} \lor y$ and $(x \land \overline{y}) \lor z$

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \lor y$
3. $\overline{y} \lor z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

$(x \land \overline{y}) \lor z$
$\overline{x} \lor y$
z

Erase the line x
Erase the line $\overline{y} \lor z$
Write down axiom 2: $\overline{x} \lor y$
Infer z from
$\overline{x} \lor y$ and $(x \land \overline{y}) \lor z$

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \vee y$
3. $\overline{y} \vee z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

$$(x \wedge \overline{y}) \vee z$$
$$\overline{x} \vee y$$
$$z$$

Erase the line $\overline{y} \vee z$
Write down axiom 2: $\overline{x} \vee y$
Infer z from
$\overline{x} \vee y$ and $(x \wedge \overline{y}) \vee z$
Erase the line $(x \wedge \overline{y}) \vee z$

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \vee y$
3. $\overline{y} \vee z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

$\overline{x} \vee y$

z

Erase the line $\overline{y} \vee z$
Write down axiom 2: $\overline{x} \vee y$
Infer z from
$\overline{x} \vee y$ and $(x \wedge \overline{y}) \vee z$
Erase the line $(x \wedge \overline{y}) \vee z$

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \lor y$
3. $\overline{y} \lor z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

$\overline{x} \lor y$

z

Write down axiom 2: $\overline{x} \lor y$

Infer z from

$\overline{x} \lor y$ and $(x \land \overline{y}) \lor z$

Erase the line $(x \land \overline{y}) \lor z$

Erase the line $\overline{x} \lor y$

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \lor y$
3. $\overline{y} \lor z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

---

z

Write down axiom 2: $\overline{x} \lor y$
Infer z from
$\overline{x} \lor y$ and $(x \land \overline{y}) \lor z$
Erase the line $(x \land \overline{y}) \lor z$
Erase the line $\overline{x} \lor y$

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \lor y$
3. $\overline{y} \lor z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

---

z
$\overline{z}$

Infer z from
$\overline{x} \lor y$ and $(x \land \overline{y}) \lor z$
Erase the line $(x \land \overline{y}) \lor z$
Erase the line $\overline{x} \lor y$
Write down axiom 4: $\overline{z}$

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \lor y$
3. $\overline{y} \lor z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

---

z
$\overline{z}$

Erase the line $(x \land \overline{y}) \lor z$
Erase the line $\overline{x} \lor y$
Write down axiom 4: $\overline{z}$
Infer 0 from
$\overline{z}$ and z

Example 2-DNF Resolution Refutation

Can write down axioms, infer new formulas, and erase used formulas

1. x
2. $\overline{x} \vee y$
3. $\overline{y} \vee z$
4. $\overline{z}$

Rules:

- Infer new formulas only from formulas currently on board
- Only k-DNF formulas can appear on board (for $k = 2$)
- Details about derivation rules won't matter for us

---

z
$\overline{z}$
0

Erase the line $(x \wedge \overline{y}) \vee z$
Erase the line $\overline{x} \vee y$
Write down axiom 4: $\overline{z}$
Infer 0 from
$\overline{z}$ and z

Complexity Measures of Interest: Length and Space

- Length: Lower bound on time for proof search algorithm (length more convenient measure than size for resolution)
- Space: Lower bound on memory for proof search algorithm

Length
\# formulas written on blackboard counted with repetitions

Space
Somewhat less straightforward — several ways of measuring

$$x$$
$$\overline{y} \vee z$$
$$(x \wedge \overline{y}) \vee z$$

Formula space: 3
Total space: 6
Variable space: 3

Complexity Measures of Interest: Length and Space

- Length: Lower bound on time for proof search algorithm (length more convenient measure than size for resolution)
- Space: Lower bound on memory for proof search algorithm

Length

\# formulas written on blackboard counted with repetitions

Space

Somewhat less straightforward — several ways of measuring

$$\boxed{\begin{array}{l} x \\ \overline{y} \lor z \\ (x \land \overline{y}) \lor z \end{array}}$$

Formula space:	3
Total space:	6
Variable space:	3

Complexity Measures of Interest: Length and Space

- Length: Lower bound on time for proof search algorithm (length more convenient measure than size for resolution)
- Space: Lower bound on memory for proof search algorithm

Length

formulas written on blackboard counted with repetitions

Space

Somewhat less straightforward — several ways of measuring

$$\boxed{\begin{array}{l} x \\ \overline{y} \vee z \\ (x \wedge \overline{y}) \vee z \end{array}}$$

Formula space:	3
Total space:	6
Variable space:	3

Complexity Measures of Interest: Length and Space

- Length: Lower bound on time for proof search algorithm (length more convenient measure than size for resolution)
- Space: Lower bound on memory for proof search algorithm

Length

formulas written on blackboard counted with repetitions

Space

Somewhat less straightforward — several ways of measuring

1. x
2. $\overline{y} \vee z$
3. $(x \wedge \overline{y}) \vee z$

Formula space:	3
Total space:	6
Variable space:	3

Complexity Measures of Interest: Length and Space

- Length: Lower bound on time for proof search algorithm (length more convenient measure than size for resolution)
- Space: Lower bound on memory for proof search algorithm

Length
\# formulas written on blackboard counted with repetitions

Space
Somewhat less straightforward — several ways of measuring

$$\boxed{\begin{array}{l} x^{1} \\ \overline{y}^{2} \lor z^{3} \\ (x^{4} \land \overline{y})^{5} \lor z^{6} \end{array}}$$

Formula space:	3
Total space:	6
Variable space:	3

Complexity Measures of Interest: Length and Space

- Length: Lower bound on time for proof search algorithm (length more convenient measure than size for resolution)
- Space: Lower bound on memory for proof search algorithm

Length
formulas written on blackboard counted with repetitions

Space
Somewhat less straightforward — several ways of measuring

$$\boxed{\begin{array}{l} x^{1} \\ \overline{y}^{2} \lor z^{3} \\ (x \land \overline{y}) \lor z \end{array}}$$

Formula space:	3
Total space:	6
Variable space:	3

Complexity Measures of Interest: Length and Space

- Length: Lower bound on time for proof search algorithm (length more convenient measure than size for resolution)
- Space: Lower bound on memory for proof search algorithm

Length

formulas written on blackboard counted with repetitions

Space

Somewhat less straightforward — several ways of measuring

$$\boxed{\begin{array}{l} x \\ \overline{y} \lor z \\ (x \land \overline{y}) \lor z \end{array}}$$

Formula space:	3
Total space:	6
Variable space:	3

Length and Space Bounds for Resolution

Let $n =$ size of formula

Length: at most 2^n
Lower bound $\exp(\Omega(n))$ [Urquhart '87, Chvátal & Szemerédi '88]

Formula space (a.k.a. clause space): at most n
Lower bound $\Omega(n)$ [Torán '99, Alekhnovich et al. '00]

Total space: at most n^2
No better lower bound than $\Omega(n)$!?

Comparing Length and Space

Some "rescaling" is needed to get meaningful comparisons of length and space

- Length exponential in formula size in worst case
- Formula space at most linear
- So natural to compare space to logarithm of length

Length-Space Trade-offs for Resolution?

For restricted system of tree-like resolution: space and (logarithm of) length strongly correlated [Esteban & Torán '99]

So essentially no trade-offs for tree-like resolution

Length-space correlation for general resolution?
Open — even no consensus on likely "right answer"

Nothing known about length-space trade-offs for resolution refutations in the general, unrestricted proof system

(Some trade-off results in restricted settings in [Ben-Sasson '02, Nordström '07])

Length-Space Trade-offs for Resolution?

For restricted system of tree-like resolution: space and (logarithm of) length strongly correlated [Esteban & Torán '99]

So essentially no trade-offs for tree-like resolution

Length-space correlation for general resolution?
Open — even no consensus on likely "right answer"

Nothing known about length-space trade-offs for resolution refutations in the general, unrestricted proof system

(Some trade-off results in restricted settings in [Ben-Sasson '02, Nordström '07])

Length-Space Trade-offs for Resolution?

For restricted system of tree-like resolution: space and (logarithm of) length strongly correlated [Esteban & Torán '99]

So essentially no trade-offs for tree-like resolution

Length-space correlation for general resolution?
Open — even no consensus on likely "right answer"

Nothing known about length-space trade-offs for resolution refutations in the general, unrestricted proof system

(Some trade-off results in restricted settings in [Ben-Sasson '02, Nordström '07])

Previous Work on k-DNF Resolution ($k \geq 2$)

Length: lower bound $\exp(\Omega(n^{1-o(1)}))$ [Segerlind et al. '04, Alekhnovich '05]

Formula space: lower bound $\Omega(n)$ [Esteban et al. '02]

(Suppressing dependencies on k)

($k+1$)-DNF resolution exponentially stronger than k-DNF resolution w.r.t. length [Segerlind et al. '04]

No hierarchy known w.r.t. space
Except for tree-like k-DNF resolution [Esteban et al. '02]
(But tree-like k-DNF weaker than standard resolution)

No trade-off results known

Previous Work on k-DNF Resolution ($k \geq 2$)

Length: lower bound $\exp(\Omega(n^{1-o(1)}))$ [Segerlind et al. '04, Alekhnovich '05]

Formula space: lower bound $\Omega(n)$ [Esteban et al. '02]

(Suppressing dependencies on k)

(k+1)-DNF resolution exponentially stronger than k-DNF resolution w.r.t. length [Segerlind et al. '04]

No hierarchy known w.r.t. space
Except for tree-like k-DNF resolution [Esteban et al. '02]
(But tree-like k-DNF weaker than standard resolution)

No trade-off results known

Previous Work on k-DNF Resolution ($k \geq 2$)

Length: lower bound $\exp(\Omega(n^{1-o(1)}))$ [Segerlind et al. '04, Alekhnovich '05]

Formula space: lower bound $\Omega(n)$ [Esteban et al. '02]

(Suppressing dependencies on k)

($k+1$)-DNF resolution exponentially stronger than k-DNF resolution w.r.t. length [Segerlind et al. '04]

No hierarchy known w.r.t. space
Except for tree-like k-DNF resolution [Esteban et al. '02]
(But tree-like k-DNF weaker than standard resolution)

No trade-off results known

Our results 1: An Optimal Length-Space Separation

Length and space in resolution are “completely uncorrelated”

Theorem (FOCS ’08)

There are k-CNF formula families of size $\mathcal{O}(n)$ with

- *refutation length $\mathcal{O}(n)$ requiring*
- *formula space $\Omega(n/\log n)$.*

Optimal separation of length and space — given length n, always possible to achieve space $\mathcal{O}(n/\log n)$

Our Results 2: Length-Space Trade-offs

We prove collection of length-space trade-offs

Results hold for

- resolution (essentially tight analysis)
- k-DNF resolution, $k \geq 2$ (with slightly worse parameters)

Different trade-offs covering (almost) whole range of space from constant to linear

Simple, explicit formulas

One Example: Robust Trade-offs for Small Space

Theorem (ECCC report TR09-034)

For any $\omega(1)$ function and any fixed K there exist explicit CNF formulas of size $\mathcal{O}(n)$

- *refutable in resolution in total space $\omega(1)$*
- *refutable in resolution in length $\mathcal{O}(n)$ and total space $\approx \sqrt[3]{n}$*
- *any resolution refutation in formula space $\lesssim \sqrt[3]{n}$ requires superpolynomial length*
- *any k-DNF resolution refutation, $k \leq K$, in formula space $\lesssim n^{1/3(k+1)}$ requires superpolynomial length*

One Example: Robust Trade-offs for Small Space

Theorem (ECCC report TR09-034)

For any $\omega(1)$ function and any fixed K there exist explicit CNF formulas of size $\mathcal{O}(n)$

- *refutable in resolution in total space $\omega(1)$*
- *refutable in resolution in length $\mathcal{O}(n)$ and total space $\approx \sqrt[3]{n}$*
- *any resolution refutation in formula space $\lesssim \sqrt[3]{n}$ requires superpolynomial length*
- *any k-DNF resolution refutation, $k \leq K$, in formula space $\lesssim n^{1/3(k+1)}$ requires superpolynomial length*

One Example: Robust Trade-offs for Small Space

Theorem (ECCC report TR09-034)

For any $\omega(1)$ function and any fixed K there exist explicit CNF formulas of size $\mathcal{O}(n)$

- *refutable in resolution in total space $\omega(1)$*
- *refutable in resolution in length $\mathcal{O}(n)$ and total space $\approx \sqrt[3]{n}$*
- *any resolution refutation in formula space $\lesssim \sqrt[3]{n}$ requires superpolynomial length*
- *any k-DNF resolution refutation, $k \leq K$, in formula space $\lesssim n^{1/3(k+1)}$ requires superpolynomial length*

One Example: Robust Trade-offs for Small Space

Theorem (ECCC report TR09-034)

For any $\omega(1)$ function and any fixed K there exist explicit CNF formulas of size $\mathcal{O}(n)$

- *refutable in resolution in total space $\omega(1)$*
- *refutable in resolution in length $\mathcal{O}(n)$ and total space $\approx \sqrt[3]{n}$*
- *any resolution refutation in formula space $\lesssim \sqrt[3]{n}$ requires superpolynomial length*
- *any k-DNF resolution refutation, $k \leq K$, in formula space $\lesssim n^{1/3(k+1)}$ requires superpolynomial length*

One Example: Robust Trade-offs for Small Space

Theorem (ECCC report TR09-034)

For any $\omega(1)$ function and any fixed K there exist explicit CNF formulas of size $\mathcal{O}(n)$

- *refutable in resolution in total space $\omega(1)$*
- *refutable in resolution in length $\mathcal{O}(n)$ and total space $\approx \sqrt[3]{n}$*
- *any resolution refutation in formula space $\lesssim \sqrt[3]{n}$ requires superpolynomial length*
- *any k-DNF resolution refutation, $k \leq K$, in formula space $\lesssim n^{1/3(k+1)}$ requires superpolynomial length*

Some Quick Technical Remarks

Upper bounds hold for

- total space (# literals) — larger measure
- standard syntactic rules

Lower bounds hold for

- formula space (# lines) — smaller measure
- semantic rules — exponentially stronger than syntactic

Space definition reminder

$$
\boxed{\begin{array}{l} x \\ \overline{y} \lor z \\ (x \land \overline{y}) \lor z \end{array}}
$$

Formula space:	3
Total space:	6
Variable space:	3

Our Results 3: Space Hierarchy for k-DNF Resolution

We also separate k-DNF resolution from $(k+1)$-DNF resolution w.r.t. formula space

Theorem (ECCC report TR09-047)

For any constant k there are explicit CNF formulas of size $\mathcal{O}(n)$

- *refutable in $(k+1)$-DNF resolution in formula space $\mathcal{O}(1)$ but such that*
- *any k-DNF resolution refutation requires formula space $\Omega\left(\sqrt[k+1]{n/\log n}\right)$*

Rest of This Talk

- Study old combinatorial game from the 70s and 80s
- Prove new theorem about amplification of space hardness via variable substitution
- Combine the two

How to Get a Handle on Time-Space Relations?

Want to find formulas that

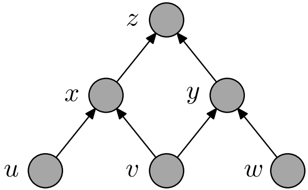
- can be quickly refuted but require large space
- have space-efficient refutations requiring much time

Such time-space trade-off questions well-studied for pebble games modelling calculations described by DAGs ([Cook & Sethi '76] and many others)

- Time needed for calculation: # pebbling moves
- Space needed for calculation: max # pebbles required

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

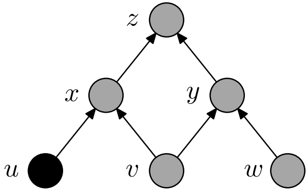


# moves	0
Current # pebbles	0
Max # pebbles so far	0

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

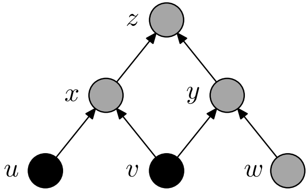


# moves	1
Current # pebbles	1
Max # pebbles so far	1

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

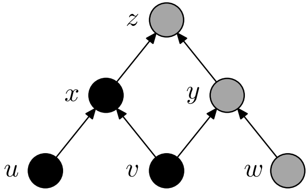


# moves	2
Current # pebbles	2
Max # pebbles so far	2

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

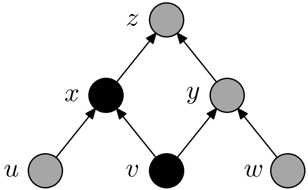


# moves	3
Current # pebbles	3
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

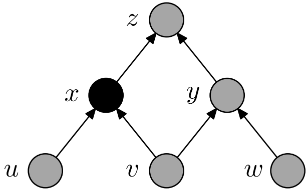


# moves	4
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

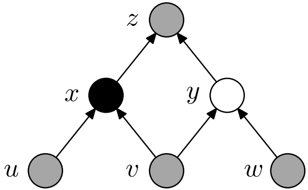


# moves	5
Current # pebbles	1
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

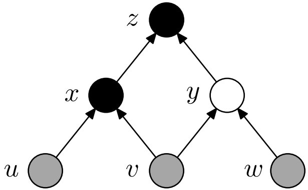


# moves	6
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

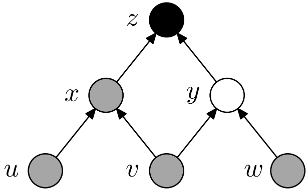


# moves	7
Current # pebbles	3
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of *G*

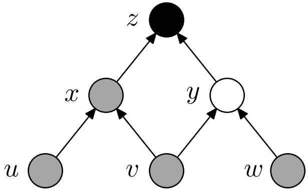


# moves	8
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of *G*

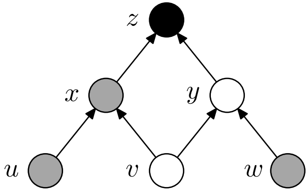


# moves	8
Current # pebbles	2
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of *G*

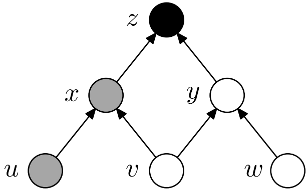


# moves	9
Current # pebbles	3
Max # pebbles so far	3

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of *G*

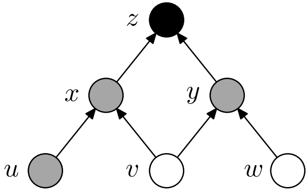


# moves	10
Current # pebbles	4
Max # pebbles so far	4

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

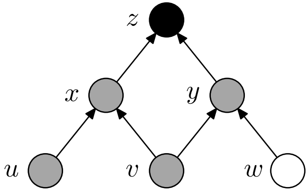


# moves	11
Current # pebbles	3
Max # pebbles so far	4

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G

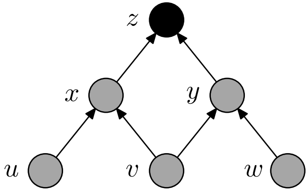


# moves	12
Current # pebbles	2
Max # pebbles so far	4

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

The Black-White Pebble Game

Goal: get single black pebble on sink vertex of G



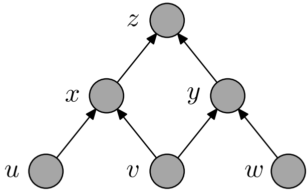
# moves	13
Current # pebbles	1
Max # pebbles so far	4

1. Can place black pebble on (empty) vertex if all immediate predecessors have pebbles on them
2. Can always remove black pebble from vertex
3. Can always place white pebble on (empty) vertex
4. Can remove white pebble if all immediate predecessors have pebbles

Pebbling Contradiction

CNF formula encoding pebble game on DAG G

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



- sources are true
- truth propagates upwards
- but sink is false

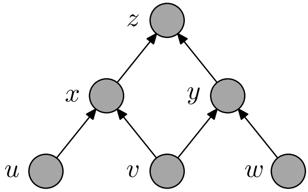
Studied by [Bonet et al. '98, Raz & McKenzie '99, Ben-Sasson & Wigderson '99] and others

Our hope is that pebbling properties of DAG somehow carry over to resolution refutations of pebbling contradictions

Pebbling Contradiction

CNF formula encoding pebble game on DAG G

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



- sources are true
- truth propagates upwards
- but sink is false

Studied by [Bonet et al. '98, Raz & McKenzie '99, Ben-Sasson & Wigderson '99] and others

Our hope is that pebbling properties of DAG somehow carry over to resolution refutations of pebbling contradictions

Interpreting Refutations as Black-White Pebblings

Black-white pebbling models non-deterministic computation

- black pebbles ⇔ computed results
- white pebbles ⇔ guesses needing to be verified

"Know z assuming v, w"

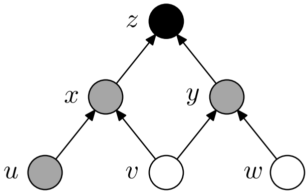
Corresponds to $(v \wedge w) \rightarrow z$, i.e., blackboard clause $\boxed{\overline{v} \vee \overline{w} \vee z}$

So translate clauses to pebbles by:
unnegated variable ⇒ black pebble
negated variable ⇒ white pebble

Interpreting Refutations as Black-White Pebblings

Black-white pebbling models non-deterministic computation

- black pebbles $\Leftrightarrow$ computed results
- white pebbles $\Leftrightarrow$ guesses needing to be verified



"Know z assuming v, w"

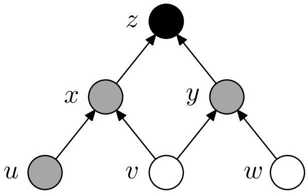
Corresponds to $(v \wedge w) \rightarrow z$, i.e., blackboard clause $\boxed{\overline{v} \vee \overline{w} \vee z}$

So translate clauses to pebbles by:
unnegated variable $\Rightarrow$ black pebble
negated variable $\Rightarrow$ white pebble

Interpreting Refutations as Black-White Pebblings

Black-white pebbling models non-deterministic computation

- black pebbles ⇔ computed results
- white pebbles ⇔ guesses needing to be verified



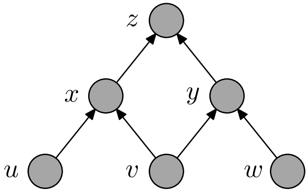
"Know z assuming v, w"

Corresponds to $(v \wedge w) \rightarrow z$, i.e., blackboard clause $\boxed{\overline{v} \vee \overline{w} \vee z}$

So translate clauses to pebbles by:
unnegated variable ⇒ black pebble
negated variable ⇒ white pebble

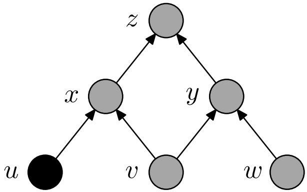
Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

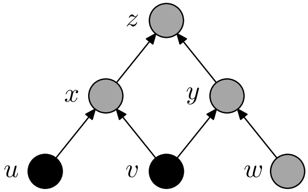


u

Write down axiom 1: u

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

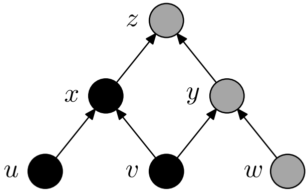


u
v

Write down axiom 1: u
Write down axiom 2: v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

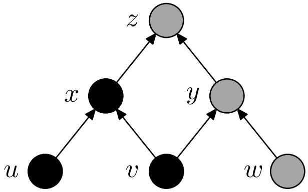


u
v
$\overline{u} \lor \overline{v} \lor x$

Write down axiom 1: u
Write down axiom 2: v
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

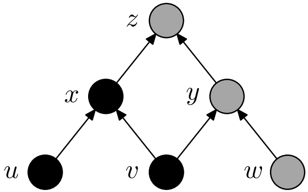


u
v
$\overline{u} \lor \overline{v} \lor x$

Write down axiom 1: u
Write down axiom 2: v
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

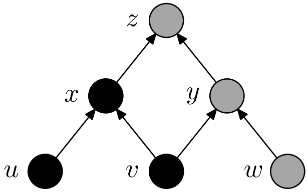


u
v
$\overline{u} \lor \overline{v} \lor x$
$\overline{v} \lor x$

Write down axiom 1: u
Write down axiom 2: v
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



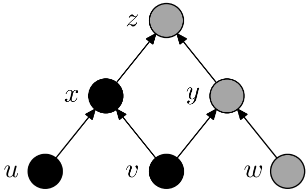
---

u
v
$\overline{u} \lor \overline{v} \lor x$
$\overline{v} \lor x$

Write down axiom 2: v
Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$
Infer $\overline{v} \lor x$ from
u and $\overline{u} \lor \overline{v} \lor x$
Erase the line $\overline{u} \lor \overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

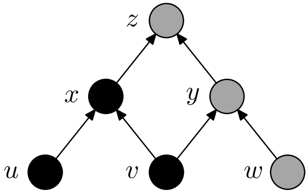


u
v
$\overline{v} \vee x$

Write down axiom 2: v
Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$
Infer $\overline{v} \vee x$ from
u and $\overline{u} \vee \overline{v} \vee x$
Erase the line $\overline{u} \vee \overline{v} \vee x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



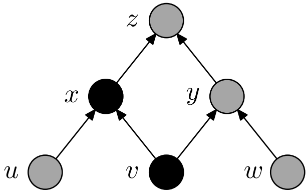
---

u
v
$\overline{v} \vee x$

Write down axiom 4: $\overline{u} \vee \overline{v} \vee x$
Infer $\overline{v} \vee x$ from
u and $\overline{u} \vee \overline{v} \vee x$
Erase the line $\overline{u} \vee \overline{v} \vee x$
Erase the line u

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



---

v

$\overline{v} \lor x$

Write down axiom 4: $\overline{u} \lor \overline{v} \lor x$

Infer $\overline{v} \lor x$ from

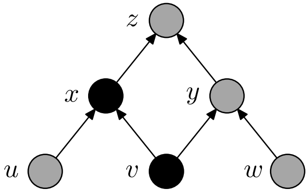
u and $\overline{u} \lor \overline{v} \lor x$

Erase the line $\overline{u} \lor \overline{v} \lor x$

Erase the line u

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

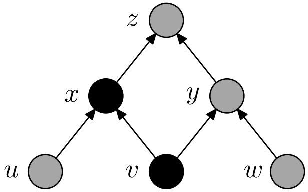


v
$\overline{v} \lor x$

u and $\overline{u} \lor \overline{v} \lor x$
Erase the line $\overline{u} \lor \overline{v} \lor x$
Erase the line u
Infer x from
v and $\overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



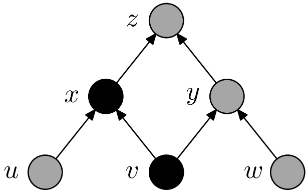
---

v
$\overline{v} \lor x$
x

u and $\overline{u} \lor \overline{v} \lor x$
Erase the line $\overline{u} \lor \overline{v} \lor x$
Erase the line u
Infer x from
v and $\overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



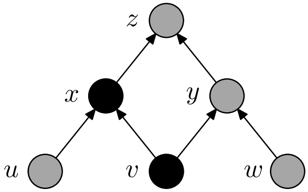
---

v
$\overline{v} \lor x$
x

Erase the line $\overline{u} \lor \overline{v} \lor x$
Erase the line u
Infer x from
v and $\overline{v} \lor x$
Erase the line $\overline{v} \lor x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

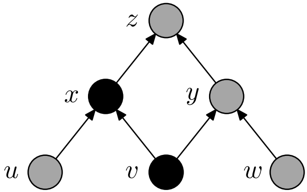


v
x

Erase the line $\overline{u} \vee \overline{v} \vee x$
Erase the line u
Infer x from
v and $\overline{v} \vee x$
Erase the line $\overline{v} \vee x$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



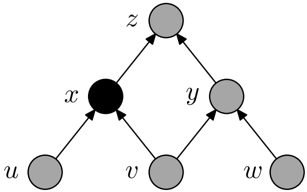
---

v
x

Erase the line u
Infer x from
v and $\overline{v} \lor x$
Erase the line $\overline{v} \lor x$
Erase the line v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

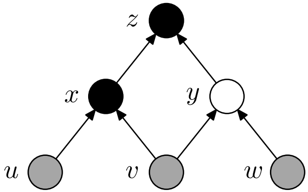


x

Erase the line u
Infer x from
v and $\overline{v} \lor x$
Erase the line $\overline{v} \lor x$
Erase the line v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



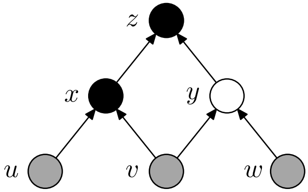
x

$\overline{x} \lor \overline{y} \lor z$

Infer x from
v and $\overline{v} \lor x$
Erase the line $\overline{v} \lor x$
Erase the line v
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



---

x

$\overline{x} \lor \overline{y} \lor z$

Erase the line $\overline{v} \lor x$

Erase the line v

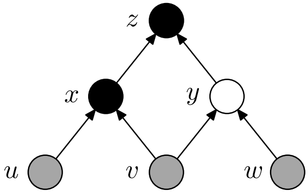
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$

Infer $\overline{y} \lor z$ from

x and $\overline{x} \lor \overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



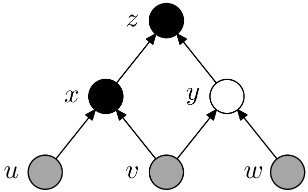
---

x
$\overline{x} \lor \overline{y} \lor z$
$\overline{y} \lor z$

Erase the line $\overline{v} \lor x$
Erase the line v
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



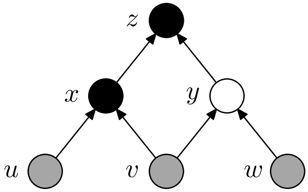
---

x
$\overline{x} \lor \overline{y} \lor z$
$\overline{y} \lor z$

Erase the line v
Write down axiom 6: $\overline{x} \lor \overline{y} \lor z$
Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$
Erase the line $\overline{x} \lor \overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



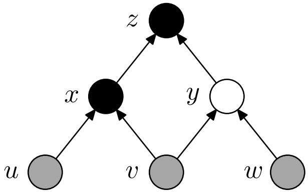
---

x
$\overline{y} \vee z$

Erase the line v
Write down axiom 6: $\overline{x} \vee \overline{y} \vee z$
Infer $\overline{y} \vee z$ from
x and $\overline{x} \vee \overline{y} \vee z$
Erase the line $\overline{x} \vee \overline{y} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



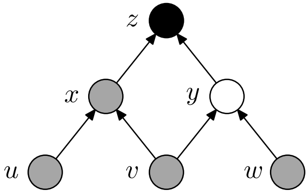
---

x
$\overline{y} \vee z$

Write down axiom 6: $\overline{x} \vee \overline{y} \vee z$
Infer $\overline{y} \vee z$ from
x and $\overline{x} \vee \overline{y} \vee z$
Erase the line $\overline{x} \vee \overline{y} \vee z$
Erase the line x

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

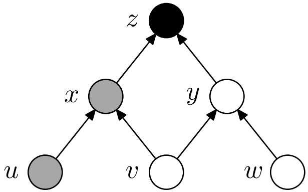


$\overline{y} \vee z$

Write down axiom 6: $\overline{x} \vee \overline{y} \vee z$
Infer $\overline{y} \vee z$ from
x and $\overline{x} \vee \overline{y} \vee z$
Erase the line $\overline{x} \vee \overline{y} \vee z$
Erase the line x

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

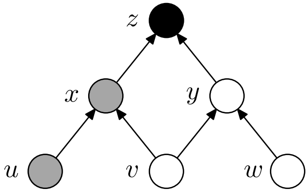


$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$

Infer $\overline{y} \lor z$ from
x and $\overline{x} \lor \overline{y} \lor z$
Erase the line $\overline{x} \lor \overline{y} \lor z$
Erase the line x
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

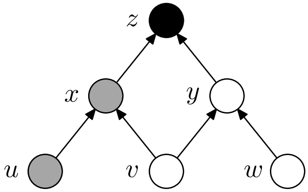


$\overline{y} \vee z$
$\overline{v} \vee \overline{w} \vee y$

Erase the line $\overline{x} \vee \overline{y} \vee z$
Erase the line x
Write down axiom 5: $\overline{v} \vee \overline{w} \vee y$
Infer $\overline{v} \vee \overline{w} \vee z$ from
$\overline{y} \vee z$ and $\overline{v} \vee \overline{w} \vee y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

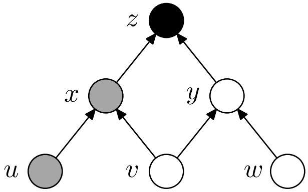


$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$
$\overline{v} \lor \overline{w} \lor z$

Erase the line $\overline{x} \lor \overline{y} \lor z$
Erase the line x
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

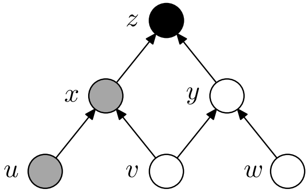


$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor y$
$\overline{v} \lor \overline{w} \lor z$

Erase the line x
Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the line $\overline{v} \lor \overline{w} \lor y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

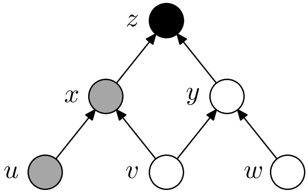


$\overline{y} \vee z$
$\overline{v} \vee \overline{w} \vee z$

Erase the line x
Write down axiom 5: $\overline{v} \vee \overline{w} \vee y$
Infer $\overline{v} \vee \overline{w} \vee z$ from
$\overline{y} \vee z$ and $\overline{v} \vee \overline{w} \vee y$
Erase the line $\overline{v} \vee \overline{w} \vee y$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

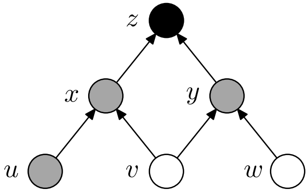


$\overline{y} \lor z$
$\overline{v} \lor \overline{w} \lor z$

Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the line $\overline{v} \lor \overline{w} \lor y$
Erase the line $\overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

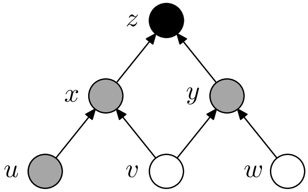


$\overline{v} \lor \overline{w} \lor z$

Write down axiom 5: $\overline{v} \lor \overline{w} \lor y$
Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the line $\overline{v} \lor \overline{w} \lor y$
Erase the line $\overline{y} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



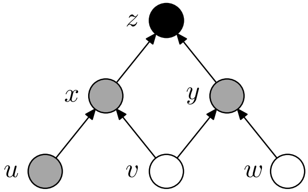
$\overline{v} \lor \overline{w} \lor z$

v

Infer $\overline{v} \lor \overline{w} \lor z$ from
$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$
Erase the line $\overline{v} \lor \overline{w} \lor y$
Erase the line $\overline{y} \lor z$
Write down axiom 2: v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



$\overline{v} \lor \overline{w} \lor z$

v

w

$\overline{y} \lor z$ and $\overline{v} \lor \overline{w} \lor y$

Erase the line $\overline{v} \lor \overline{w} \lor y$

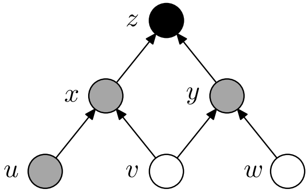
Erase the line $\overline{y} \lor z$

Write down axiom 2: v

Write down axiom 3: w

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

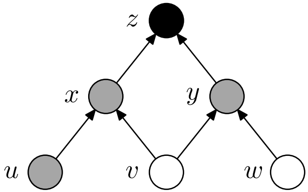


$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$

Erase the line $\overline{v} \lor \overline{w} \lor y$
Erase the line $\overline{y} \lor z$
Write down axiom 2: v
Write down axiom 3: w
Write down axiom 7: $\overline{z}$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

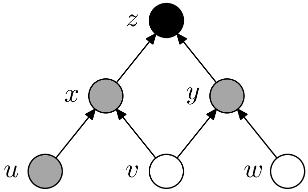


$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$

Write down axiom 2: v
Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$

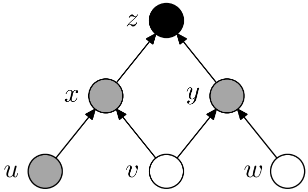


$\overline{v} \vee \overline{w} \vee z$
v
w
$\overline{z}$
$\overline{w} \vee z$

Write down axiom 2: v
Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \vee z$ from
v and $\overline{v} \vee \overline{w} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

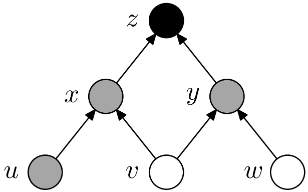


$\overline{v} \lor \overline{w} \lor z$
v
w
$\overline{z}$
$\overline{w} \lor z$

Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$
Erase the line v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

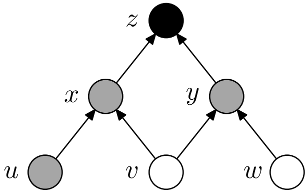


$\overline{v} \lor \overline{w} \lor z$
w
$\overline{z}$
$\overline{w} \lor z$

Write down axiom 3: w
Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$
Erase the line v

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

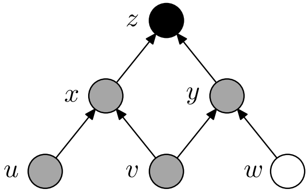


$\overline{v} \lor \overline{w} \lor z$
w
$\overline{z}$
$\overline{w} \lor z$

Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$
Erase the line v
Erase the line $\overline{v} \lor \overline{w} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



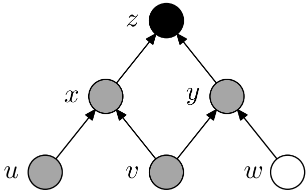
---

w
$\overline{z}$
$\overline{w} \lor z$

Write down axiom 7: $\overline{z}$
Infer $\overline{w} \lor z$ from
v and $\overline{v} \lor \overline{w} \lor z$
Erase the line v
Erase the line $\overline{v} \lor \overline{w} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



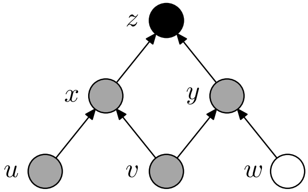
---

w
$\overline{z}$
$\overline{w} \lor z$

v and $\overline{v} \lor \overline{w} \lor z$
Erase the line v
Erase the line $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



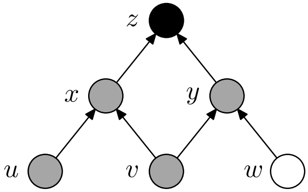
---

w
$\overline{z}$
$\overline{w} \lor z$
z

v and $\overline{v} \lor \overline{w} \lor z$
Erase the line v
Erase the line $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



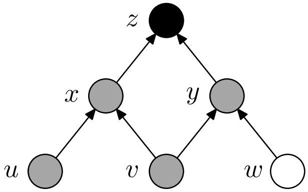
---

w
$\overline{z}$
$\overline{w} \vee z$
z

Erase the line v
Erase the line $\overline{v} \vee \overline{w} \vee z$
Infer z from
w and $\overline{w} \vee z$
Erase the line w

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

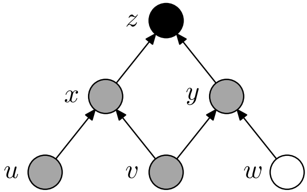


$\overline{z}$
$\overline{w} \lor z$
z

Erase the line v
Erase the line $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$
Erase the line w

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$

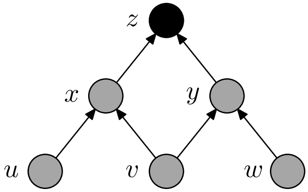


$\overline{z}$
$\overline{w} \lor z$
z

Erase the line $\overline{v} \lor \overline{w} \lor z$
Infer z from
w and $\overline{w} \lor z$
Erase the line w
Erase the line $\overline{w} \lor z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \vee \overline{v} \vee x$
5. $\overline{v} \vee \overline{w} \vee y$
6. $\overline{x} \vee \overline{y} \vee z$
7. $\overline{z}$



$\overline{z}$

z

Erase the line $\overline{v} \vee \overline{w} \vee z$

Infer z from

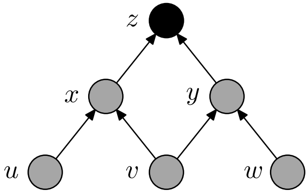
w and $\overline{w} \vee z$

Erase the line w

Erase the line $\overline{w} \vee z$

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



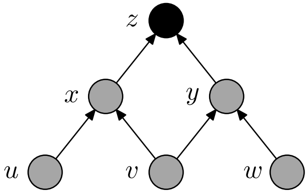
$\overline{z}$

z

w and $\overline{w} \lor z$
Erase the line w
Erase the line $\overline{w} \lor z$
Infer 0 from
$\overline{z}$ and z

Example of Refutation-Pebbling Correspondence

1. u
2. v
3. w
4. $\overline{u} \lor \overline{v} \lor x$
5. $\overline{v} \lor \overline{w} \lor y$
6. $\overline{x} \lor \overline{y} \lor z$
7. $\overline{z}$



$\overline{z}$
z
0

w and $\overline{w} \lor z$
Erase the line w
Erase the line $\overline{w} \lor z$
Infer 0 from
$\overline{z}$ and z

Formal Refutation-Pebbling Correspondence

Theorem (Ben-Sasson '02)

Any refutation translates into black-white pebbling with

- *# moves $\leq$ refutation length*
- *# pebbles $\leq$ variable space*

Observation (Ben-Sasson et al. '00)

Any black-pebbles-only pebbling translates into refutation with

- *refutation length $\leq$ # moves*
- *total space $\leq$ # pebbles*

Unfortunately pebbling contradictions are extremely easy w.r.t. formula space!

Formal Refutation-Pebbling Correspondence

Theorem (Ben-Sasson '02)

Any refutation translates into black-white pebbling with

- *# moves $\leq$ refutation length*
- *# pebbles $\leq$ variable space*

Observation (Ben-Sasson et al. '00)

Any black-pebbles-only pebbling translates into refutation with

- *refutation length $\leq$ # moves*
- *total space $\leq$ # pebbles*

Unfortunately pebbling contradictions are extremely easy w.r.t. formula space!

Formal Refutation-Pebbling Correspondence

Theorem (Ben-Sasson '02)

Any refutation translates into black-white pebbling with

- *# moves $\leq$ refutation length*
- *# pebbles $\leq$ variable space*

Observation (Ben-Sasson et al. '00)

Any black-pebbles-only pebbling translates into refutation with

- *refutation length $\leq$ # moves*
- *total space $\leq$ # pebbles*

Unfortunately pebbling contradictions are extremely easy w.r.t. formula space!

Key Idea: Variable Substitution

Make formula harder by substituting $x_1 \oplus x_2$ for every variable x (also works for other Boolean functions with "right" properties):

$$\overline{x} \vee y$$

$$\Downarrow$$

$$\neg(x_1 \oplus x_2) \vee (y_1 \oplus y_2)$$

$$\Downarrow$$

$$\begin{aligned} & (x_1 \vee \overline{x}_2 \vee y_1 \vee y_2) \\ \wedge\ & (x_1 \vee \overline{x}_2 \vee \overline{y}_1 \vee \overline{y}_2) \\ \wedge\ & (\overline{x}_1 \vee x_2 \vee y_1 \vee y_2) \\ \wedge\ & (\overline{x}_1 \vee x_2 \vee \overline{y}_1 \vee \overline{y}_2) \end{aligned}$$

Key Technical Result: Substitution Theorem

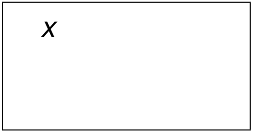
Let $F[\oplus]$ denote formula with XOR $x_1 \oplus x_2$ substituted for x

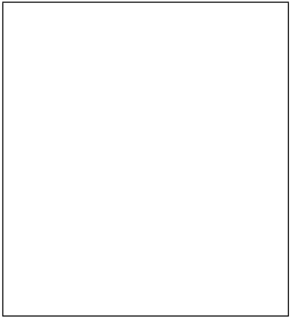
Obvious approach for $F[\oplus]$: mimic refutation of F

Key Technical Result: Substitution Theorem

Let $F[\oplus]$ denote formula with XOR $x_1 \oplus x_2$ substituted for x

Obvious approach for $F[\oplus]$: mimic refutation of F





Key Technical Result: Substitution Theorem

Let $F[\oplus]$ denote formula with XOR $x_1 \oplus x_2$ substituted for x

Obvious approach for $F[\oplus]$: mimic refutation of F

x
$\overline{x} \vee y$

Key Technical Result: Substitution Theorem

Let $F[\oplus]$ denote formula with XOR $x_1 \oplus x_2$ substituted for x

Obvious approach for $F[\oplus]$: mimic refutation of F

x
$\overline{x} \lor y$
y

Key Technical Result: Substitution Theorem

Let $F[\oplus]$ denote formula with XOR $x_1 \oplus x_2$ substituted for x

Obvious approach for $F[\oplus]$: mimic refutation of F

x
$\overline{x} \vee y$
y

$x_1 \vee x_2$
$\overline{x}_1 \vee \overline{x}_2$

Key Technical Result: Substitution Theorem

Let $F[\oplus]$ denote formula with XOR $x_1 \oplus x_2$ substituted for x

Obvious approach for $F[\oplus]$: mimic refutation of F

x
$\overline{x} \vee y$
y

$x_1 \vee x_2$
$\overline{x}_1 \vee \overline{x}_2$
$x_1 \vee \overline{x}_2 \vee y_1 \vee y_2$
$x_1 \vee \overline{x}_2 \vee \overline{y}_1 \vee \overline{y}_2$
$\overline{x}_1 \vee x_2 \vee y_1 \vee y_2$
$\overline{x}_1 \vee x_2 \vee \overline{y}_1 \vee \overline{y}_2$

Key Technical Result: Substitution Theorem

Let $F[\oplus]$ denote formula with XOR $x_1 \oplus x_2$ substituted for x

Obvious approach for $F[\oplus]$: mimic refutation of F

x
$\overline{x} \lor y$
y

$x_1 \lor x_2$
$\overline{x}_1 \lor \overline{x}_2$
$x_1 \lor \overline{x}_2 \lor y_1 \lor y_2$
$x_1 \lor \overline{x}_2 \lor \overline{y}_1 \lor \overline{y}_2$
$\overline{x}_1 \lor x_2 \lor y_1 \lor y_2$
$\overline{x}_1 \lor x_2 \lor \overline{y}_1 \lor \overline{y}_2$
$y_1 \lor y_2$
$\overline{y}_1 \lor \overline{y}_2$

Key Technical Result: Substitution Theorem

Let $F[\oplus]$ denote formula with XOR $x_1 \oplus x_2$ substituted for x

Obvious approach for $F[\oplus]$: mimic refutation of F

$$\boxed{\begin{array}{l} x \\ \overline{x} \vee y \\ y \end{array}}$$

$$\boxed{\begin{array}{l} x_1 \vee x_2 \\ \overline{x}_1 \vee \overline{x}_2 \\ x_1 \vee \overline{x}_2 \vee y_1 \vee y_2 \\ x_1 \vee \overline{x}_2 \vee \overline{y}_1 \vee \overline{y}_2 \\ \overline{x}_1 \vee x_2 \vee y_1 \vee y_2 \\ \overline{x}_1 \vee x_2 \vee \overline{y}_1 \vee \overline{y}_2 \\ y_1 \vee y_2 \\ \overline{y}_1 \vee \overline{y}_2 \end{array}}$$

For such refutation of $F[\oplus]$:

- length $\geq$ length for F
- formula space $\geq$ variable space for F

Key Technical Result: Substitution Theorem

Let $F[\oplus]$ denote formula with XOR $x_1 \oplus x_2$ substituted for x

Obvious approach for $F[\oplus]$: mimic refutation of F

$$x \qquad \overline{x} \lor y \qquad y$$

$$\begin{array}{l} x_1 \lor x_2 \\ \overline{x}_1 \lor \overline{x}_2 \\ x_1 \lor \overline{x}_2 \lor y_1 \lor y_2 \\ x_1 \lor \overline{x}_2 \lor \overline{y}_1 \lor \overline{y}_2 \\ \overline{x}_1 \lor x_2 \lor y_1 \lor y_2 \\ \overline{x}_1 \lor x_2 \lor \overline{y}_1 \lor \overline{y}_2 \\ y_1 \lor y_2 \\ \overline{y}_1 \lor \overline{y}_2 \end{array}$$

For such refutation of $F[\oplus]$:

- length $\geq$ length for F
- formula space $\geq$ variable space for F

Prove that this is (sort of) best one can do for $F[\oplus]$!

Sketch of Proof of Substitution Theorem

Given refutation of $F[\oplus]$, extract "shadow refutation" of F

XOR formula $F[\oplus]$	**Original formula** F
If XOR blackboard implies e.g. $\neg(x_1 \oplus x_2) \vee (y_1 \oplus y_2)$...	write $\overline{x} \vee y$ on shadow blackboard
For consecutive XOR blackboard configurations...	can get between corresponding shadow blackboards by legal derivation steps
... (sort of) upper-bounded by XOR derivation length	Length of shadow blackboard derivation ...
... is at most # clauses on XOR blackboard	# variables mentioned on shadow blackboard...

Sketch of Proof of Substitution Theorem

Given refutation of $F[\oplus]$, extract "shadow refutation" of F

XOR formula $F[\oplus]$	**Original formula** F
If XOR blackboard implies e.g. $\neg(x_1 \oplus x_2) \vee (y_1 \oplus y_2)$...	write $\overline{x} \vee y$ on shadow blackboard
For consecutive XOR blackboard configurations...	can get between corresponding shadow blackboards by legal derivation steps
... (sort of) upper-bounded by XOR derivation length	Length of shadow blackboard derivation ...
... is at most # clauses on XOR blackboard	# variables mentioned on shadow blackboard...

Sketch of Proof of Substitution Theorem

Given refutation of $F[\oplus]$, extract "shadow refutation" of F

XOR formula $F[\oplus]$	**Original formula** F
If XOR blackboard implies e.g. $\neg(x_1 \oplus x_2) \vee (y_1 \oplus y_2) \ldots$	write $\overline{x} \vee y$ on shadow blackboard
For consecutive XOR blackboard configurations...	can get between corresponding shadow blackboards by legal derivation steps
... (sort of) upper-bounded by XOR derivation length	Length of shadow blackboard derivation ...
... is at most # clauses on XOR blackboard	# variables mentioned on shadow blackboard...

Sketch of Proof of Substitution Theorem

Given refutation of $F[\oplus]$, extract "shadow refutation" of F

XOR formula $F[\oplus]$	**Original formula** F
If XOR blackboard implies e.g. $\neg(x_1 \oplus x_2) \vee (y_1 \oplus y_2)$...	write $\overline{x} \vee y$ on shadow blackboard
For consecutive XOR blackboard configurations...	can get between corresponding shadow blackboards by legal derivation steps
... (sort of) upper-bounded by XOR derivation length	Length of shadow blackboard derivation ...
... is at most # clauses on XOR blackboard	# variables mentioned on shadow blackboard...

Sketch of Proof of Substitution Theorem

Given refutation of $F[\oplus]$, extract "shadow refutation" of F

XOR formula $F[\oplus]$	**Original formula** F
If XOR blackboard implies e.g. $\neg(x_1 \oplus x_2) \vee (y_1 \oplus y_2)\ldots$	write $\overline{x} \vee y$ on shadow blackboard
For consecutive XOR blackboard configurations...	can get between corresponding shadow blackboards by legal derivation steps
... (sort of) upper-bounded by XOR derivation length	Length of shadow blackboard derivation ...
... is at most # clauses on XOR blackboard	# variables mentioned on shadow blackboard...

Sketch of Proof of Substitution Theorem

Given refutation of $F[\oplus]$, extract "shadow refutation" of F

XOR formula $F[\oplus]$	**Original formula** F
If XOR blackboard implies e.g. $\neg(x_1 \oplus x_2) \lor (y_1 \oplus y_2)$...	write $\overline{x} \lor y$ on shadow blackboard
For consecutive XOR blackboard configurations...	can get between corresponding shadow blackboards by legal derivation steps
... (sort of) upper-bounded by XOR derivation length	Length of shadow blackboard derivation ...
... is at most # clauses on XOR blackboard	# variables mentioned on shadow blackboard...

Sketch of Proof of Substitution Theorem

Given refutation of $F[\oplus]$, extract "shadow refutation" of F

XOR formula $F[\oplus]$	**Original formula** F
If XOR blackboard implies e.g. $\neg(x_1 \oplus x_2) \vee (y_1 \oplus y_2)\ldots$	write $\overline{x} \vee y$ on shadow blackboard
For consecutive XOR blackboard configurations…	can get between corresponding shadow blackboards by legal derivation steps
… (sort of) upper-bounded by XOR derivation length	Length of shadow blackboard derivation …
… is at most # clauses on XOR blackboard	# variables mentioned on shadow blackboard…

Sketch of Proof of Substitution Theorem

Given refutation of $F[\oplus]$, extract "shadow refutation" of F

XOR formula $F[\oplus]$	**Original formula** F
If XOR blackboard implies e.g. $\neg(x_1 \oplus x_2) \lor (y_1 \oplus y_2)$...	write $\overline{x} \lor y$ on shadow blackboard
For consecutive XOR blackboard configurations...	can get between corresponding shadow blackboards by legal derivation steps
... (sort of) upper-bounded by XOR derivation length	Length of shadow blackboard derivation ...
... is at most # clauses on XOR blackboard	# variables mentioned on shadow blackboard...

Sketch of Proof of Substitution Theorem

Given refutation of $F[\oplus]$, extract "shadow refutation" of F

XOR formula $F[\oplus]$	**Original formula** F
If XOR blackboard implies e.g. $\neg(x_1 \oplus x_2) \lor (y_1 \oplus y_2)\ldots$	write $\overline{x} \lor y$ on shadow blackboard
For consecutive XOR blackboard configurations...	can get between corresponding shadow blackboards by legal derivation steps
... (sort of) upper-bounded by XOR derivation length	Length of shadow blackboard derivation ...
... is at most # clauses on XOR blackboard	# variables mentioned on shadow blackboard...

Pieces Together: Substitution + Pebbling Formulas

Making variable substitutions in pebbling formulas

- lifts lower bound from variable space to formula space
- maintains upper bound in terms of total space and length

Substitution with XOR over $k + 1$ variables works against k-DNF resolution

Get our results by

- using known pebbling results from literature of 70s and 80s
- proving a couple of new pebbling results
- to get tight trade-offs, showing that resolution proofs can sometimes do better than black-only pebblings

(Work in last two bullets to appear in Complexity '10)

Pieces Together: Substitution + Pebbling Formulas

Making variable substitutions in pebbling formulas

- lifts lower bound from variable space to formula space
- maintains upper bound in terms of total space and length

Substitution with XOR over $k+1$ variables works against k-DNF resolution

Get our results by

- using known pebbling results from literature of 70s and 80s
- proving a couple of new pebbling results
- to get tight trade-offs, showing that resolution proofs can sometimes do better than black-only pebblings

(Work in last two bullets to appear in Complexity '10)

Pieces Together: Substitution + Pebbling Formulas

Making variable substitutions in pebbling formulas

- lifts lower bound from variable space to formula space
- maintains upper bound in terms of total space and length

Substitution with XOR over $k + 1$ variables works against k-DNF resolution

Get our results by

- using known pebbling results from literature of 70s and 80s
- proving a couple of new pebbling results
- to get tight trade-offs, showing that resolution proofs can sometimes do better than black-only pebblings

(Work in last two bullets to appear in Complexity '10)

Stronger Results for k-DNF resolution?

Gap of $(k+1)$st root between upper and lower bounds for k-DNF resolution

Open Question

Can the loss of a $(k+1)$st root in the k-DNF resolution lower bounds be diminished? Or even eliminated completely?

Conceivable that same bounds as for resolution could hold

However, any improvement beyond kth root requires fundamentally different approach [Nordström & Razborov '09]

Stronger Results for *k*-DNF resolution?

Gap of $(k+1)$st root between upper and lower bounds for k-DNF resolution

Open Question

Can the loss of a $(k+1)$st root in the k-DNF resolution lower bounds be diminished? Or even eliminated completely?

Conceivable that same bounds as for resolution could hold

However, any improvement beyond kth root requires fundamentally different approach [Nordström & Razborov '09]

Trade-offs for Stronger Proof Systems?

Recall key technical theorem: amplify space lower bounds through variable substitution

Almost completely oblivious to which proof system is being studied—maybe can be made to work for stronger systems?

Open Question

Can the Substitution Theorem be proven for, say, Cutting Planes or Polynomial Calculus (with/without Resolution), thus yielding time-space trade-offs for these proof systems as well?

Approach in previous works provably will not work, but there are other (related but different) ideas one could try

Trade-offs for Stronger Proof Systems?

Recall key technical theorem: amplify space lower bounds through variable substitution

Almost completely oblivious to which proof system is being studied—maybe can be made to work for stronger systems?

Open Question

Can the Substitution Theorem be proven for, say, Cutting Planes or Polynomial Calculus (with/without Resolution), thus yielding time-space trade-offs for these proof systems as well?

Approach in previous works provably will not work, but there are other (related but different) ideas one could try

Trade-offs for Stronger Proof Systems?

Recall key technical theorem: amplify space lower bounds through variable substitution

Almost completely oblivious to which proof system is being studied—maybe can be made to work for stronger systems?

Open Question

Can the Substitution Theorem be proven for, say, Cutting Planes or Polynomial Calculus (with/without Resolution), thus yielding time-space trade-offs for these proof systems as well?

Approach in previous works provably will not work, but there are other (related but different) ideas one could try

Empirical Results?

Open Question

Do our trade-off phenomena show up in real life for state-of-the-art SAT-solvers run on pebbling contradictions?

Number of different possibilities to try out:

- Base formulas on different graph families
- Do substitution with $\vee$, $\oplus$, or other Boolean functions
- Possibly add some redundant "noise clauses" to make structural analysis a bit harder

Summing up

- Optimal time-space separation in resolution
- Strong time-space trade-offs for resolution and k-DNF resolution for wide range of parameters
- Strict space hierarchy for k-DNF resolution
- Many remaining open questions about space in proof complexity (see survey *Pebble Games, Proof Complexity, and Time-Space Trade-offs* at my webpage for details)

Thank you for your attention!