Proof Complexity as a Computational Lens

Jakob Nordström

University of Copenhagen and Lund University

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Three Simple Problems...

Colouring

Does the graph $G = (V, E)$ have a colouring with k colours such that all neighbours have distinct colours?

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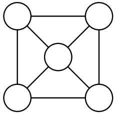

3-colouring?

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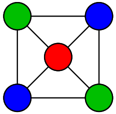

3-colouring? Yes

Three Simple Problems. . .

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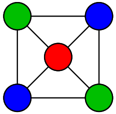

3-colouring? Yes, but no 2-colouring

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Is there a clique in the graph $G = (V, E)$ with k vertices that are all pairwise connected by edges in E?

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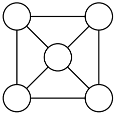

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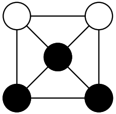

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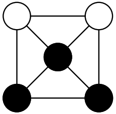

3-clique? Yes, but no 4-clique

Clique

Is there a clique in the graph $G = (V, E)$ with k vertices that are all pairwise connected by edges in E?

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Given propositional logic formula, is there a satisfying assignment?

$$(x \lor z) \land (y \lor \neg z) \land (x \lor \neg y \lor u) \land (\neg y \lor \neg u)$$
$$\land\ (u \lor v) \land (\neg x \lor \neg v) \land (\neg u \lor w) \land (\neg x \lor \neg u \lor \neg w)$$

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- Variables should be set to **true** or **false**
- Constraint $(x \lor \neg y \lor z)$: means x or z should be true or y false
- $\land$ means all constraints should hold simultaneously
- Is there a truth value assignment satisfying all constraints?

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Colouring:	frequency allocation for mobile base stations
Clique:	bioinformatics, computational chemistry
Sat:	easily models these and many other problems

. . . with Huge Practical Implications

- Some more examples of problems that can be encoded as propositional logic formulas:
  - computer hardware verification
  - computer software testing
  - artificial intelligence
  - operations research
  - cryptography
  - bioinformatics
  - et cetera. . .
- Leads to **humongous** formulas (100,000s or even 1,000,000s of variables)
- Can we use computers to solve these problems efficiently?

Solving NP in Theory and Practice

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- SAT problem is NP-complete, so probably very hard [Coo71, Lev73]
- Assuming P $\neq$ NP, even impossible to meaningfully approximate
  - COLOURING [Kho01, Zuc07]
  - CLIQUE [Hås99]
  - SAT [Hås01]

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  - COLOURING [Kho01, Zuc07]
  - CLIQUE [Hås99]
  - SAT [Hås01]
- Except that in practice, there are good algorithms for
  - COLOURING [DLMM08, DLMO09, DLMM11]
  - CLIQUE [Pro12, McC17]

  and amazing conflict-driven clause learning (CDCL) solvers [BS97, MS99, MMZ^{+}01] that solve huge SAT problem instances

How can we understand real-world algorithms for NP-hard problems?

This lecture: Use proof complexity (not only conceivable answer)

Algorithmic View of Proof Complexity

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Study infeasible problems — proofs of feasibility are trivial

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Question 2: Topic for separate lecture(s) — lots of recent exciting progress; mostly negative (worst-case) results that proof search is hard, e.g., [AM20, GKMP20, dRGN$^+$21]

Applications of Proof Complexity

Three applied reasons for proof complexity:

1. Understand real-world applied algorithmic paradigms [**this lecture**]

2. Get ideas for algorithmic improvements
[EN18, EN20, LBD$^+$20, DGD$^+$21, DGN21, KBBN22, MBGN23, MSB$^+$25]
(See, e.g., tutorials youtu.be/VC0CHXoWnS4 and youtu.be/FIJ3k7HWpiQ about RoundingSat)

3. Enhance algorithms to write machine-verifiable certificates of correctness
[EGMN20, GMN20, GMM$^+$20, GN21, GMN22, GMNO22, BBN$^+$23, BGMN23, MM23, BBN$^+$24, DMM$^+$24, GMM$^+$24, HOGN24, IOT$^+$24, MMN24, DHN$^+$25, KLM$^+$25, MM25]
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Or just view this as a convenient excuse to study nice computational complexity problems for their own sake. . . ☺

Outline

1. DPLL, CDCL, and Resolution
   - Davis-Putnam-Logemann-Loveland (DPLL) Method
   - Conflict-Driven Clause Learning (CDCL)
   - Resolution Proof System
2. Algebraic and Semi-algebraic Approaches
   - Nullstellensatz
   - Gröbner Bases and Polynomial Calculus
   - Pseudo-Boolean Solving and Cutting Planes
3. Some More Advanced Proof Systems
   - Sherali-Adams and Sums of Squares
   - Stabbing Planes
   - Extended Resolution

Some Preliminaries

- Variable x: takes value **true** $(= 1)$ or **false** $(= 0)$
- Literal ℓ: variable x or its negation $\overline{x}$ (write $\overline{x}$ instead of $\neg x$)
- Clause $C = \ell_1 \lor \cdots \lor \ell_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- Conjunctive normal form (CNF) formula $F = C_1 \land \cdots \land C_m$: conjunction of clauses

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- Refer to clauses of CNF formula as axioms (as opposed to derived clauses)
- N denotes size of formula (# literals counted with repetitions)
- $\mathcal{O}(f(N))$ grows at most as quickly as $f(N)$ asymptotically
  $\Omega(g(N))$ grows at least as quickly as $g(N)$ asymptotically
  $\Theta(h(N))$ grows equally quickly as $h(N)$ asymptotically

The Sat Problem

The Satisfiability (or just Sat) Problem

Given a formula F in conjunctive normal form (CNF), is it satisfiable?

The SAT Problem

The SATISFIABILITY (or just SAT) Problem

Given a formula F in conjunctive normal form (CNF), is it satisfiable?

For instance, what about our example CNF formula?

$$\begin{gathered}(x \vee z) \wedge (y \vee \overline{z}) \wedge (x \vee \overline{y} \vee u) \wedge (\overline{y} \vee \overline{u}) \\ \wedge\ (u \vee v) \wedge (\overline{x} \vee \overline{v}) \wedge (\overline{u} \vee w) \wedge (\overline{x} \vee \overline{u} \vee \overline{w})\end{gathered}$$

The Same Problem in Three Different Shapes

$$(x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u})$$
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$$\begin{aligned}
(1-x)(1-z) &= 0 \\
(1-y)z &= 0 \\
(1-x)y(1-u) &= 0 \\
yu &= 0 \\
(1-u)(1-v) &= 0 \\
xv &= 0 \\
u(1-w) &= 0 \\
xuw &= 0
\end{aligned}$$

For **true** $= 1$ and **false** $= 0$, is there a $\{0,1\}$-valued solution?

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$$\begin{aligned} 1 - x - z + xz &= 0 \\ z - yz &= 0 \\ y - xy - yu + xyu &= 0 \\ yu &= 0 \\ 1 - u - v + uv &= 0 \\ xv &= 0 \\ u - uw &= 0 \\ xuw &= 0 \end{aligned}$$

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$1 - x - z + xz = 0$	$x + z \geq 1$
$z - yz = 0$	$y + (1 - z) \geq 1$
$y - xy - yu + xyu = 0$	$x + (1 - y) + u \geq 1$
$yu = 0$	$(1 - y) + (1 - u) \geq 1$
$1 - u - v + uv = 0$	$u + v \geq 1$
$xv = 0$	$(1 - x) + (1 - v) \geq 1$
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$$\begin{aligned} x + z &\geq 1 \\ y - z &\geq 0 \\ x - y + u &\geq 0 \\ -y - u &\geq -1 \\ u + v &\geq 1 \\ -x - v &\geq -1 \\ -u + w &\geq 0 \\ -x - u - w &\geq -2 \end{aligned}$$

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Clique and Colouring as CNF Formulas

Clique formula

"The graph $G = (V, E)$ has an m-clique"

$q_{k,1} \lor q_{k,2} \lor \cdots \lor q_{k,n}$	$\lvert V \rvert = n;\ 1 \leq k \leq m$	[some vertex is kth member of clique]
$\overline{q}_{k,u} \lor \overline{q}_{k,v}$	$u \neq v \in V;\ 1 \leq k \leq m$	[clique members are uniquely defined]
$\overline{q}_{k,v} \lor \overline{q}_{k',v}$	$v \in V;\ 1 \leq k < k' \leq m$	[no vertex counted as clique member twice]
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"The graph $G = (V, E)$ is m-colourable"

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(Smarter encodings are possible, but these are good enough for our discussion)

DPLL: Attempting Smart Case Analysis

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4. Set $x = 0$, simplify F and make recursive call
5. Set $x = 1$, simplify F and make recursive call
6. If result in both cases "unsatisfiable", then report "unsatisfiable" and return

A DPLL Toy Example

$$\begin{aligned} F = \quad & (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \\ \land\ & (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w}) \end{aligned}$$

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Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals

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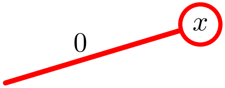

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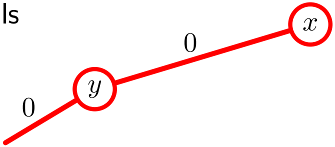


A DPLL Toy Example

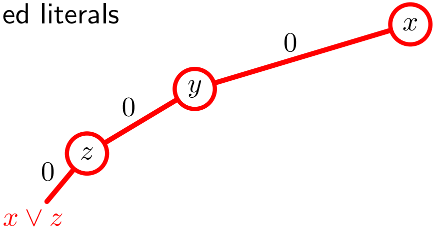
$$F = (x \lor z) \land (z) \land (\overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals



A DPLL Toy Example

$$F = (\quad z) \land (y \lor \overline{z}) \land (\quad \overline{y} \lor u) \land (\overline{y} \lor \overline{u})$$
$$\land\, (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

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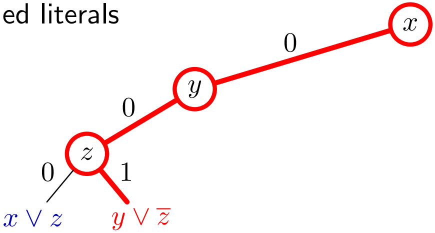

A DPLL Toy Example

$$F = \quad (\quad z) \land (y \lor \overline{z}) \land (\quad u) \land (\quad \overline{u})$$
$$\land\, (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

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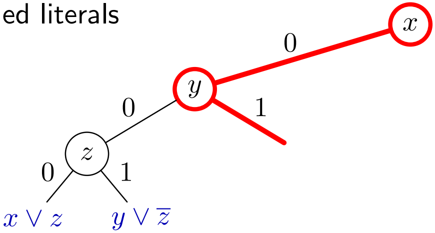

A DPLL Toy Example

$$F = (\quad z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\quad \overline{u}) \land (\quad v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

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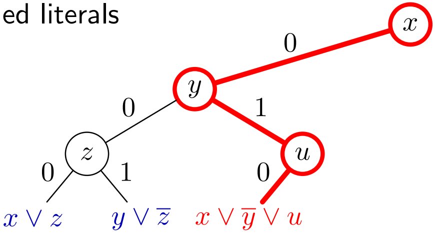

A DPLL Toy Example

$$F = (\quad z) \land (y \lor \overline{z}) \land (\quad u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\quad w) \land (\overline{x} \lor \quad \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

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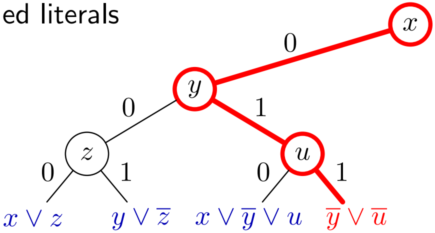

A DPLL Toy Example

$$F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \\ \land (u \lor v) \land (\overline{v}) \land (\overline{u} \lor w) \land (\overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

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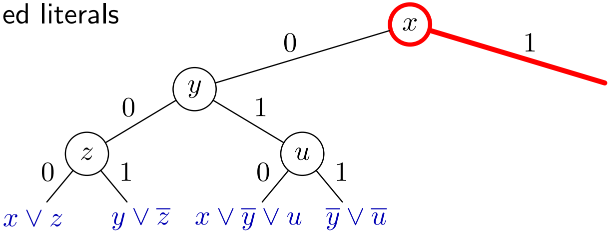

A DPLL Toy Example

$$F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \quad) \land (\overline{y} \lor \overline{u}) \\ \land (\quad v) \land (\quad \overline{v}) \land (\overline{u} \lor w) \land (\quad \overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

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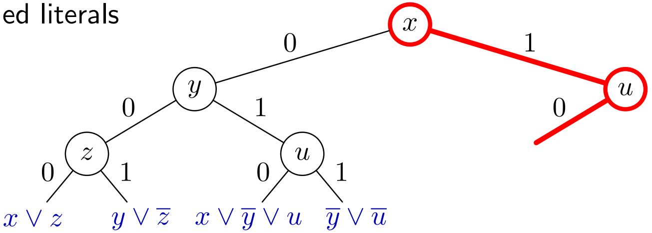

A DPLL Toy Example

$$F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \quad\quad) \land (\overline{y} \lor \overline{u})$$
$$\land\ (u \lor v) \land (\quad\quad \overline{v}) \land (\overline{u} \lor w) \land (\quad\quad \overline{u} \lor \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

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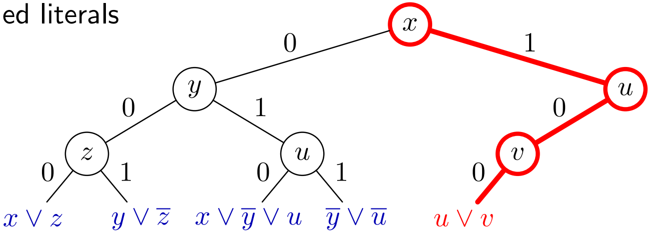


A DPLL Toy Example

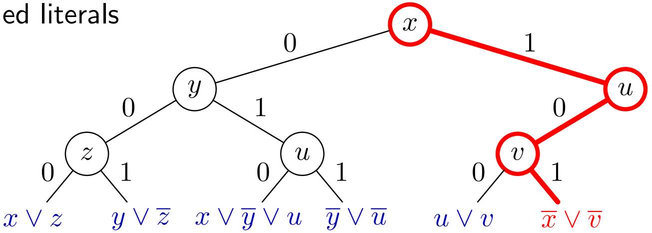
$$F = (x \vee z) \wedge (y \vee \overline{z}) \wedge (x \vee \overline{y} \quad) \wedge (\overline{y} \vee \overline{u}) \wedge (\quad v) \wedge (\overline{x} \vee \overline{v}) \wedge (\overline{u} \vee w) \wedge (\quad \overline{u} \vee \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
- falsified literals



A DPLL Toy Example

$$F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\quad \overline{v}) \land (\quad w) \land (\quad \overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
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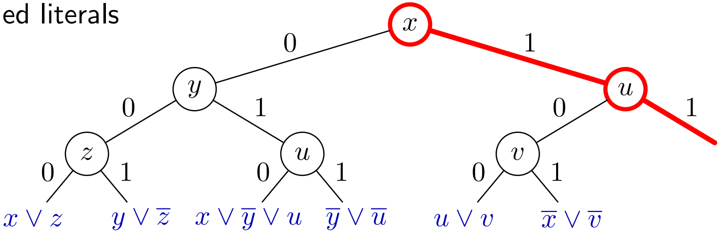

A DPLL Toy Example

$$F = (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \land (u \lor v) \land (\overline{v}) \land (\overline{u} \lor w) \land (\overline{w})$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

- satisfied clauses
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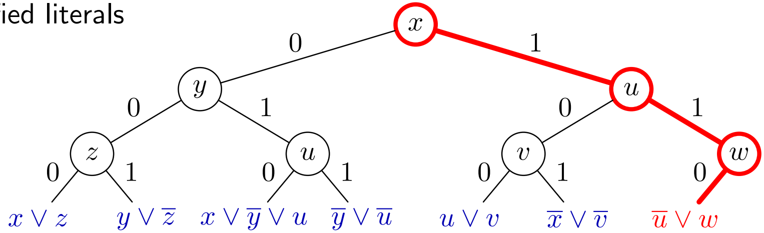

A DPLL Toy Example

$$\begin{aligned} F = \; & (x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u}) \\ & \land (u \lor v) \land (\quad \overline{v}) \land (\quad w) \land (\overline{x} \lor \overline{u} \lor \overline{w}) \end{aligned}$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

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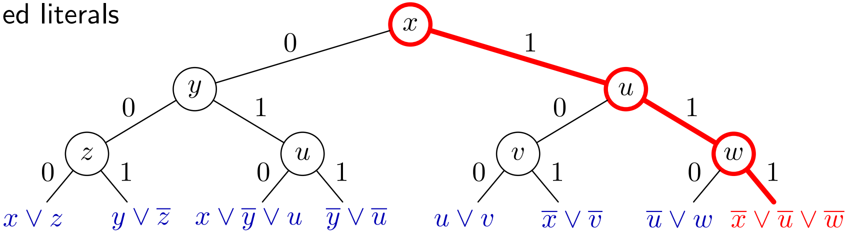

A DPLL Toy Example

$$\begin{aligned} F = \quad & (x \vee z) \wedge (y \vee \overline{z}) \wedge (x \vee \overline{y} \vee u) \wedge (\overline{y} \vee \overline{u}) \\ & \wedge (u \vee v) \wedge (\overline{x} \vee \overline{v}) \wedge (\overline{u} \vee w) \wedge (\overline{x} \vee \overline{u} \vee \overline{w}) \end{aligned}$$

Visualize execution of DPLL algorithm as search tree

Pick variables in internal nodes; terminate in leaves when conflict reached

"Simplify formula" by (mentally) removing

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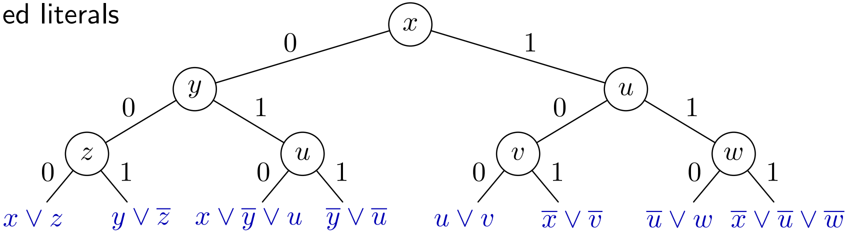

State-of-the-Art SAT Solving in One Slide

High-level description of modern conflict-driven clause learning (CDCL) SAT solving (as pioneered in [BS97, MS99, MMZ$^+$01]):

- Try to build satisfying assignment for formula (branching or decision heuristic crucial)
- When partial assignment violates formula, compute explanation for conflict and add to formula as new clause (clause learning)
- Every once in a while, restart from beginning (but save computed info)

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \overset{\mathsf{d}}{=} 0}$

Decision

Free choice to assign value to variable

Notation $p \overset{\mathsf{d}}{=} 0$

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

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Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

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Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$\boxed{p \overset{\mathsf{d}}{=} 0}$

$\boxed{u \overset{p \vee \overline{u}}{=} 0}$

$\boxed{q \overset{\mathsf{d}}{=} 0}$

$\boxed{r \overset{q \vee r}{=} 1}$

Decision

Free choice to assign value to variable

Notation $p \overset{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \overset{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

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Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

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Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \vee \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \vee r}{=} 1$

$w \stackrel{\overline{r} \vee w}{=} 1$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \vee \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \vee r}{=} 1$

$w \stackrel{\overline{r} \vee w}{=} 1$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$y \stackrel{u \vee x \vee y}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$p \overset{\mathsf{d}}{=} 0$

$u \overset{p \vee \overline{u}}{=} 0$

$q \overset{\mathsf{d}}{=} 0$

$r \overset{q \vee r}{=} 1$

$w \overset{\overline{r} \vee w}{=} 1$

$x \overset{\mathsf{d}}{=} 0$

$y \overset{u \vee x \vee y}{=} 1$

$z \overset{x \vee \overline{y} \vee z}{=} 1$

$\overset{\overline{y} \vee \overline{z}}{\bot}$

Decision

Free choice to assign value to variable

Notation $p \overset{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

Notation $u \overset{p \vee \overline{u}}{=} 0$ ($p \vee \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$
$u \stackrel{p \lor \overline{u}}{=} 0$ — decision level 1

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$
$r \stackrel{q \lor r}{=} 1$
$w \stackrel{\overline{r} \lor w}{=} 1$ — decision level 2

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$
$y \stackrel{u \lor x \lor y}{=} 1$
$z \stackrel{x \lor \overline{y} \lor z}{=} 1$
$\stackrel{\overline{y} \lor \overline{z}}{\bot}$ — decision level 3

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, otherwise decide

Add to assignment trail

Continue until satisfying assignment or conflict

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$$\left.\begin{array}{l} \boxed{p \stackrel{\mathsf{d}}{=} 0} \\ u \stackrel{p \lor \overline{u}}{=} 0 \end{array}\right\} \text{decision level } 1$$

$$\left.\begin{array}{l} \boxed{q \stackrel{\mathsf{d}}{=} 0} \\ r \stackrel{q \lor r}{=} 1 \\ w \stackrel{\overline{r} \lor w}{=} 1 \end{array}\right\} \text{decision level } 2$$

$$\left.\begin{array}{l} \boxed{x \stackrel{\mathsf{d}}{=} 0} \\ y \stackrel{u \lor x \lor y}{=} 1 \\ z \stackrel{x \lor \overline{y} \lor z}{=} 1 \\ \stackrel{\overline{y} \lor \overline{z}}{\bot} \end{array}\right\} \text{decision level } 3$$

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$$\begin{array}{ll}
\boxed{p \stackrel{\mathsf{d}}{=} 0} & \\
u \stackrel{p \lor \overline{u}}{=} 0 & \text{decision level } 1 \\
\boxed{q \stackrel{\mathsf{d}}{=} 0} & \\
r \stackrel{q \lor r}{=} 1 & \text{decision level } 2 \\
w \stackrel{\overline{r} \lor w}{=} 1 & \\
\boxed{x \stackrel{\mathsf{d}}{=} 0} & \\
y \stackrel{u \lor x \lor y}{=} 1 & \\
z \stackrel{x \lor \overline{y} \lor z}{=} 1 & \text{decision level } 3 \\
\stackrel{\overline{y} \lor \overline{z}}{\bot} &
\end{array}$$

Could backtrack by erasing conflict level & flipping last decision

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$
$u \stackrel{p \lor \overline{u}}{=} 0$
decision level 1

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$
$r \stackrel{q \lor r}{=} 1$
$w \stackrel{\overline{r} \lor w}{=} 1$
decision level 2

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$
$y \stackrel{u \lor x \lor y}{=} 1$
$z \stackrel{x \lor \overline{y} \lor z}{=} 1$
$\stackrel{\overline{y} \lor \overline{z}}{\perp}$
decision level 3

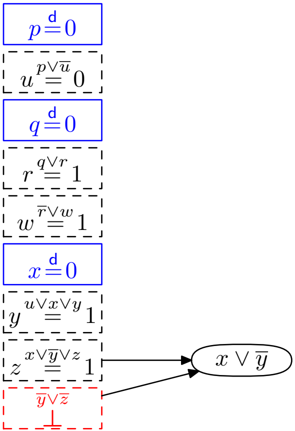
Could backtrack by erasing conflict level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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Case analysis over z for last two clauses:

- $x \lor \overline{y} \lor z$ wants $z = 1$
- $\overline{y} \lor \overline{z}$ wants $z = 0$
- Merge clauses & remove z — must satisfy $x \lor \overline{y}$

Conflict Analysis

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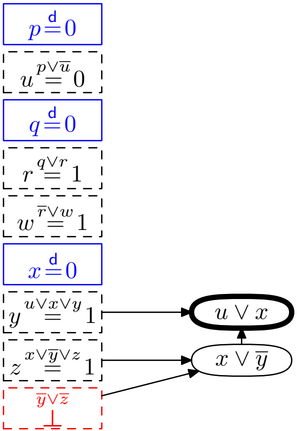

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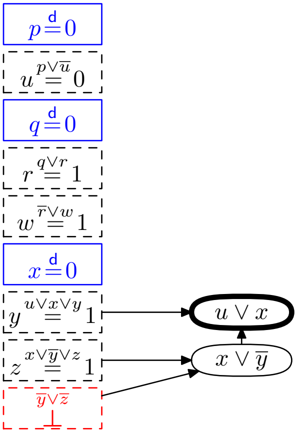
- Merge clauses & remove z — must satisfy $x \lor \overline{y}$

Repeat until UIP clause with only 1 variable at conflict level after last decision — learn and backjump

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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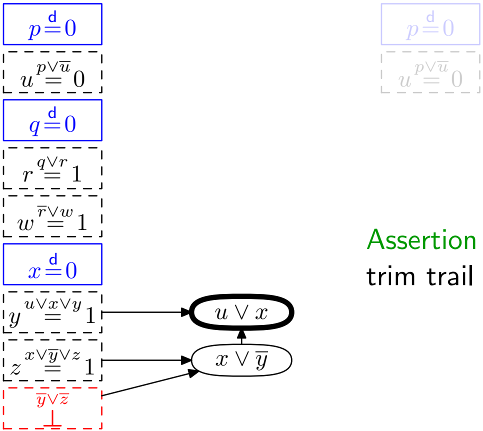
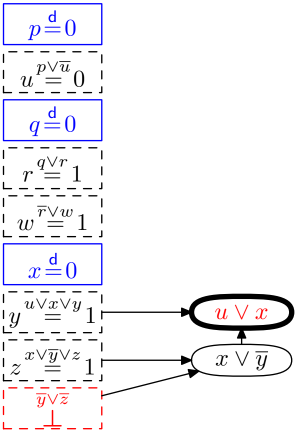


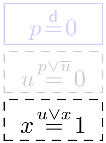
Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$





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Complete Example of CDCL Execution

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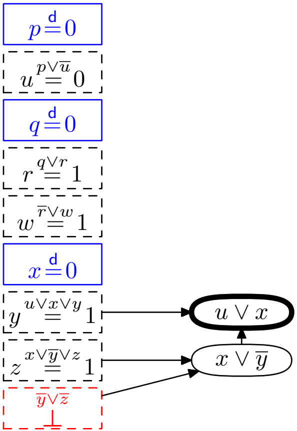

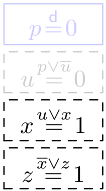

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Now UIP literal guaranteed to flip (assert) — but this is a propagation, not a decision

Then continue as before...

Complete Example of CDCL Execution

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$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$p \overset{\mathsf{d}}{=} 0$

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$q \overset{\mathsf{d}}{=} 0$

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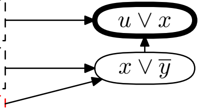

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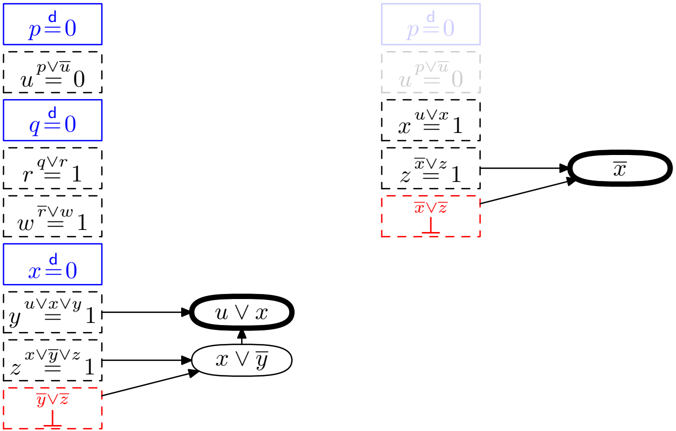
Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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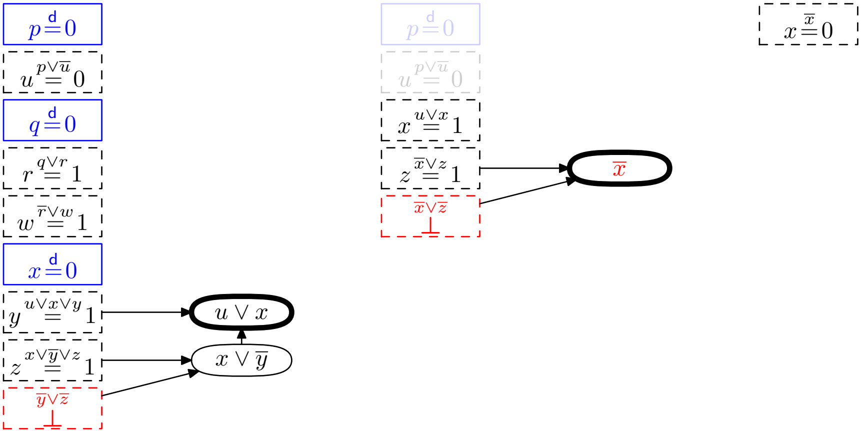

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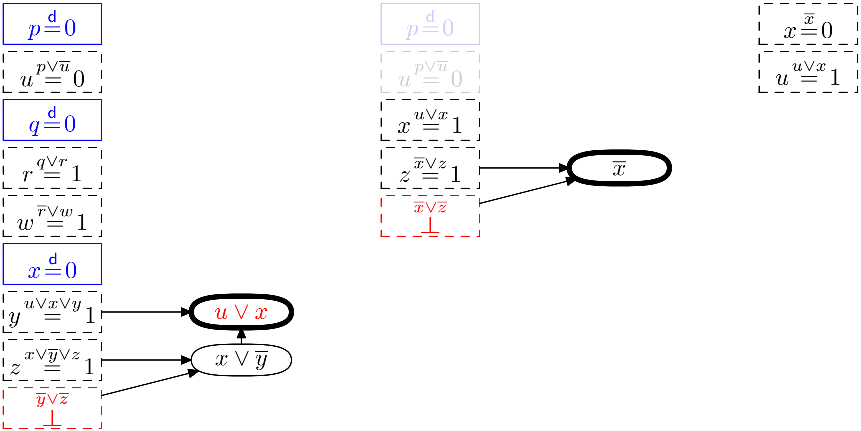

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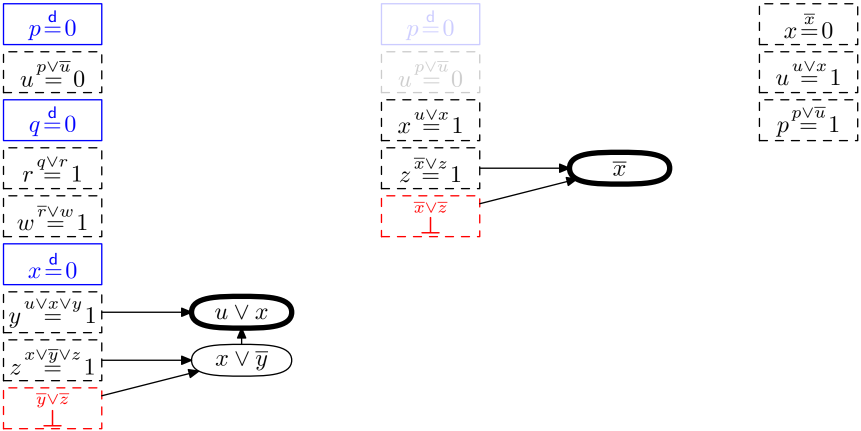

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$\overline{x}$

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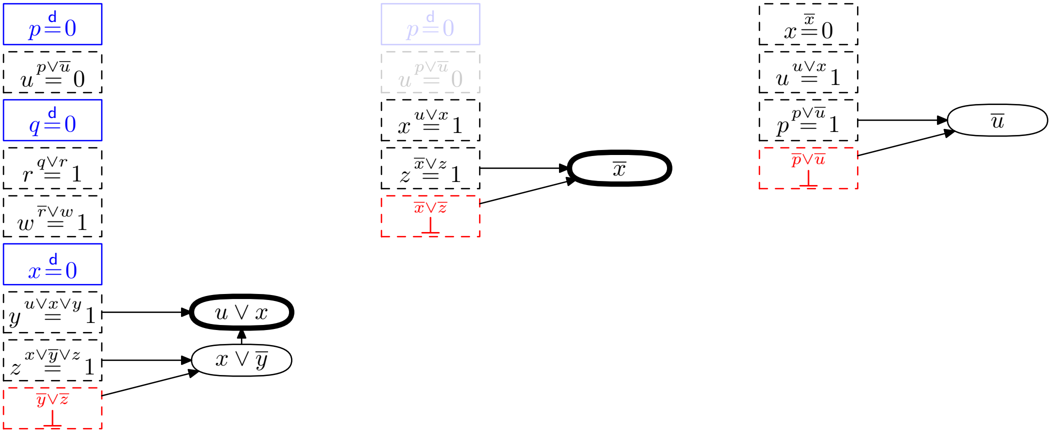
$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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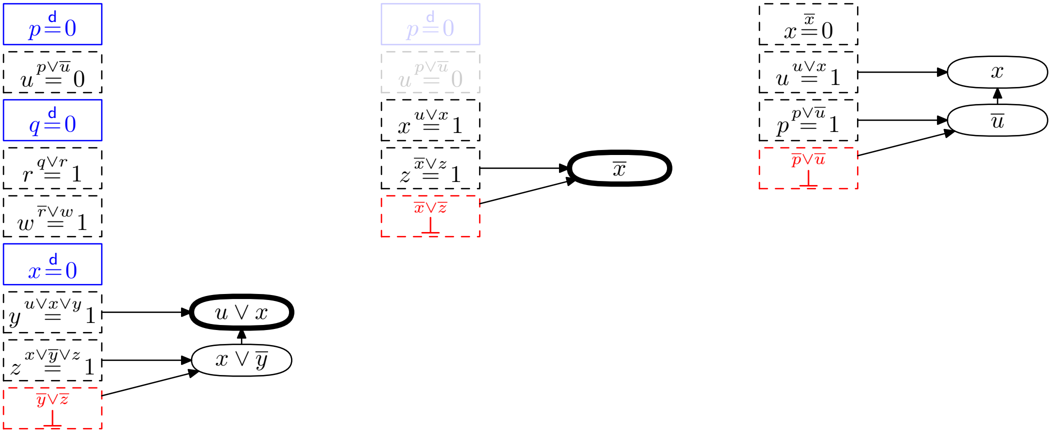

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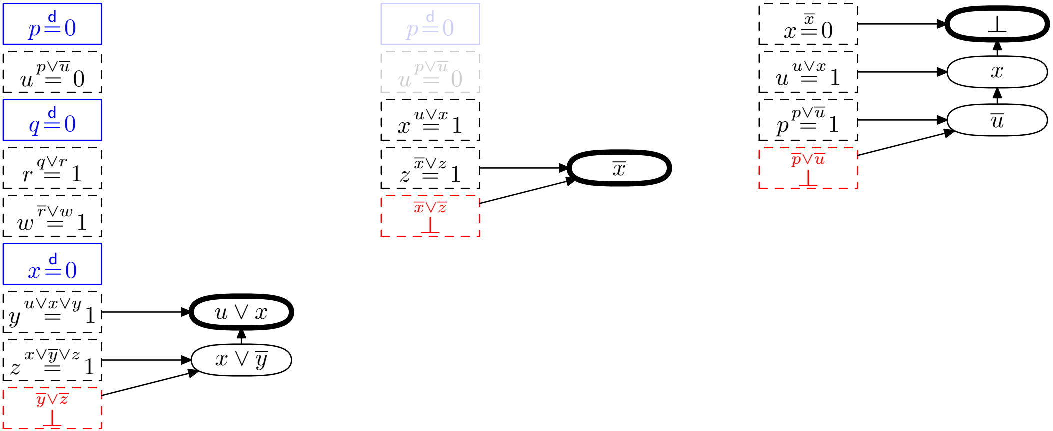

SAT Solver Analysis and the Resolution Proof System

How to make rigorous analysis of SAT solver performance?
Many intricate, hard-to-understand heuristics
So focus instead on underlying method of reasoning

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Resolution proof system [Bla37, Rob65]

- Start with clauses of CNF formula (axioms)
- Derive new clauses by resolution rule

$$\frac{C_1 \lor x \qquad C_2 \lor \overline{x}}{C_1 \lor C_2}$$

Resolution Proofs by Contradiction

Resolution rule:

$$\frac{C_1 \lor x \qquad C_2 \lor \overline{x}}{C_1 \lor C_2}$$

Observation

If F is a satisfiable CNF formula and D is derived from clauses $D_1, D_2 \in F$ by the resolution rule, then $F \land D$ is satisfiable.

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Observation

If F is a satisfiable CNF formula and D is derived from clauses $D_1, D_2 \in F$ by the resolution rule, then $F \land D$ is satisfiable.

So can prove F unsatisfiable by deriving the unsatisfiable empty clause (denoted $\perp$) from F by resolution

Such proof by contradiction also called resolution refutation

DPLL and Resolution Proofs

A DPLL execution is essentially a resolution proof

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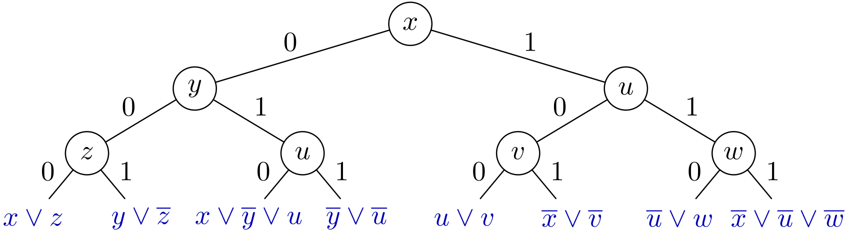
Look at our example again



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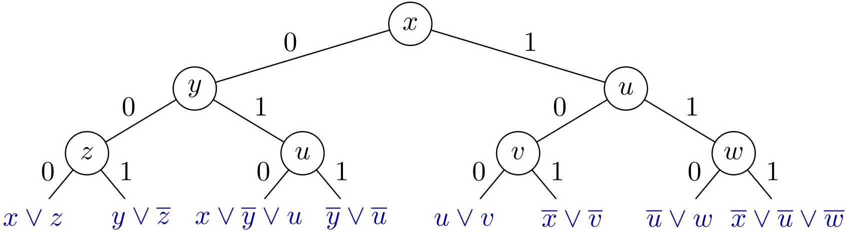

and apply resolution rule $\frac{C_1 \lor x \quad C_2 \lor \overline{x}}{C_1 \lor C_2}$ bottom-up

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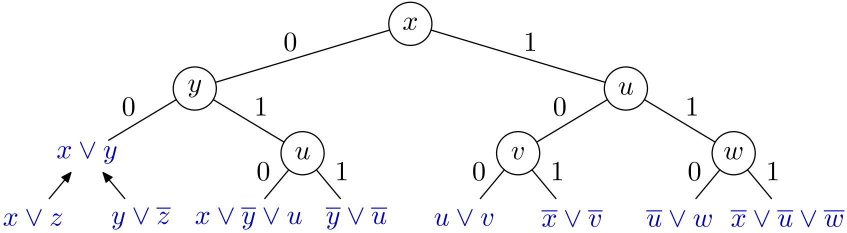

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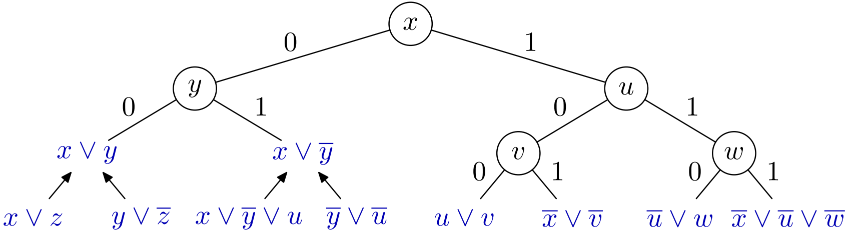

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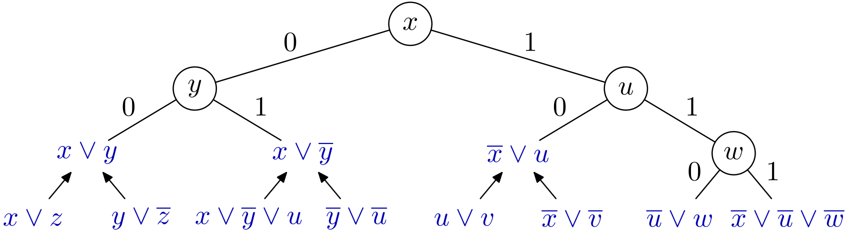

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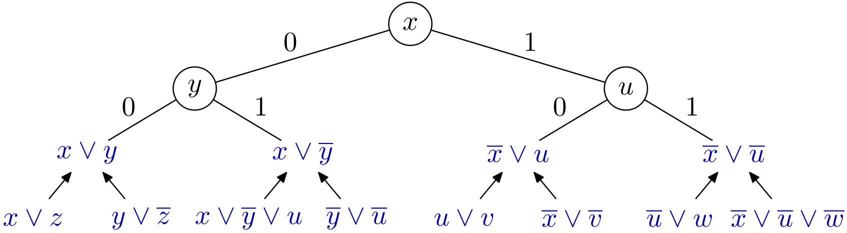

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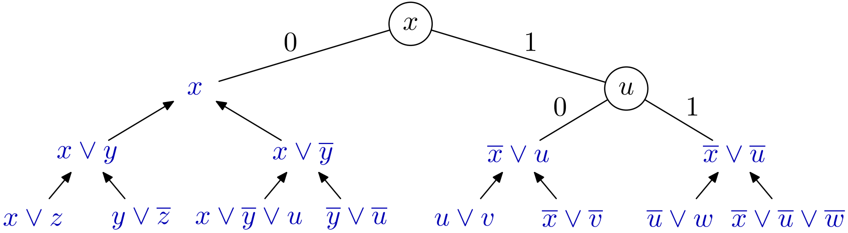

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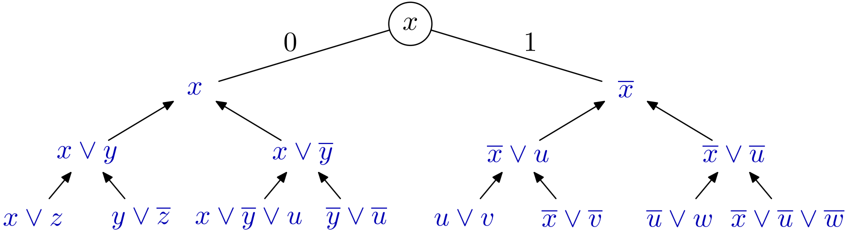

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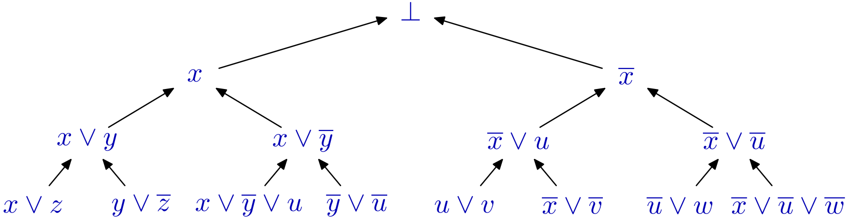

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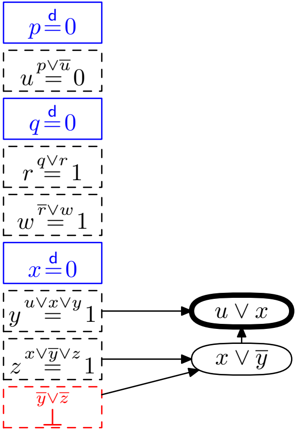
- Hence, lower bounds on tree-like proof size in resolution $\Rightarrow$
  lower bounds on DPLL running time
- Conflict-driven clause learning adds "shortcut edges" in tree, but still yields resolution proof

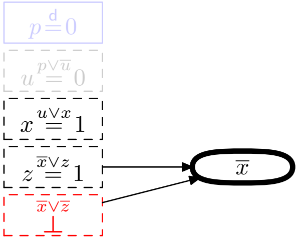
CDCL and Resolution Proofs

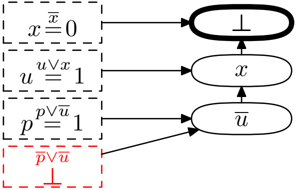
Obtain resolution proof...

CDCL and Resolution Proofs

Obtain resolution proof from our example CDCL execution...

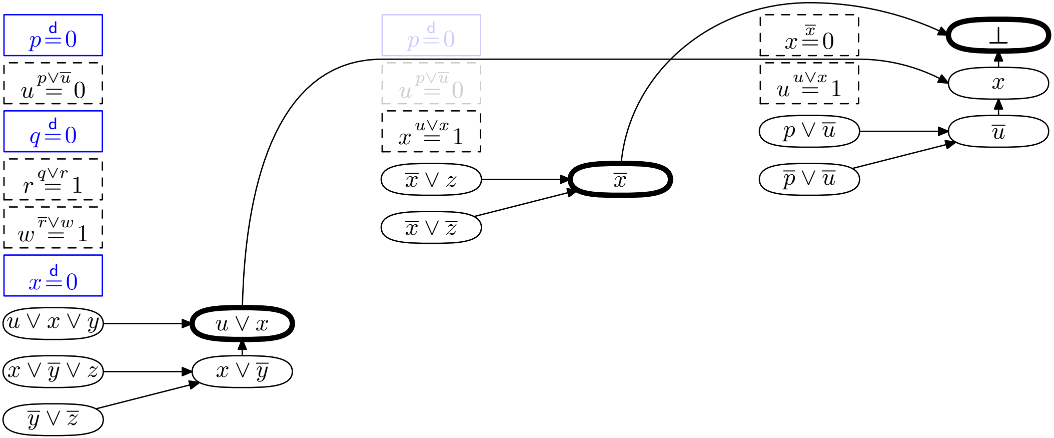






CDCL and Resolution Proofs

Obtain resolution proof from our example CDCL execution by stringing together conflict analyses:



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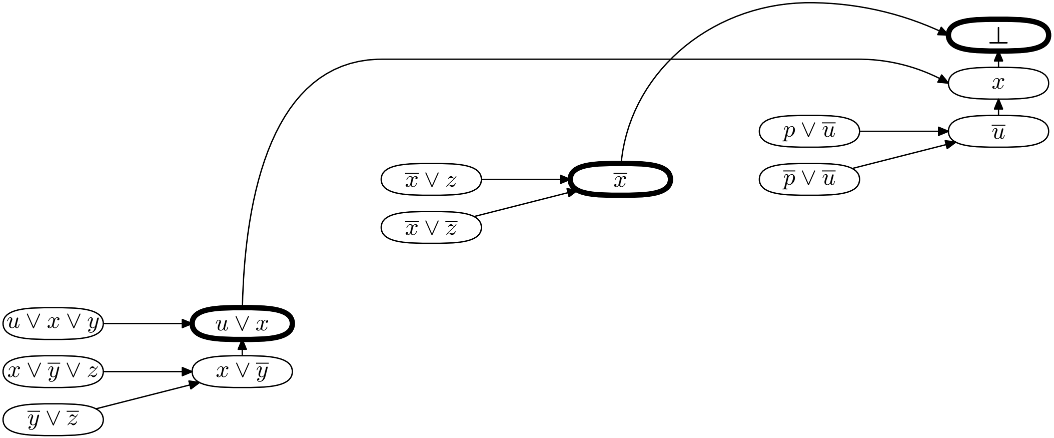

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- Requires an argument, of course, but you have seen enough in this presentation to be able to fill in the required details. . .
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(*) Except for some preprocessing techniques, which is an important omission, but this gets complicated and we don't have time to go into details...

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  - Why do heuristics work?
  - Why are applied instances easy?
- Paradox: resolution quite weak proof system; many strong proof complexity lower bounds for (seemingly) "obvious" formulas

Examples of Hard Formulas for Resolution (1/3)

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Variables $p_{i,j}$ = "pigeon $i \rightarrow$ hole j"; $1 \leq i \leq n+1$; $1 \leq j \leq n$

$p_{i,1} \vee p_{i,2} \vee \cdots \vee p_{i,n}$ [every pigeon i gets a hole]

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Even onto functional PHP hard — **"resolution cannot count"**

Resolution proof requires $\exp(\Omega(n)) = \exp\left(\Omega\left(\sqrt[3]{N}\right)\right)$ clauses
(measured in terms of formula size N, i.e., total number of literals in formula)

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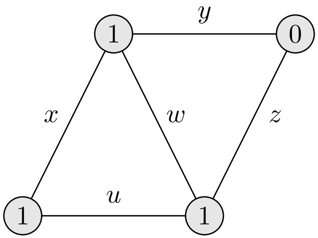

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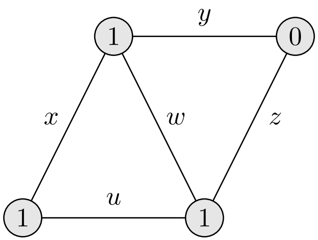

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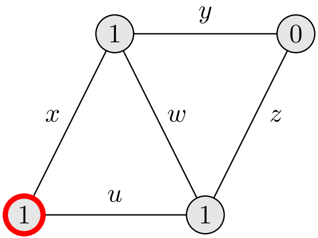

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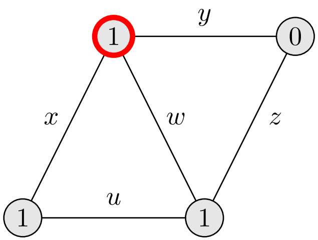

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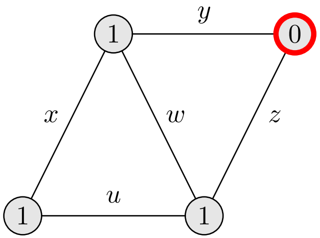

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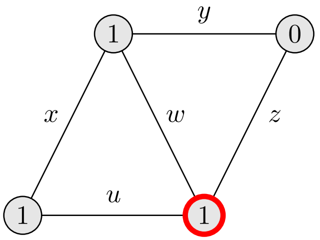

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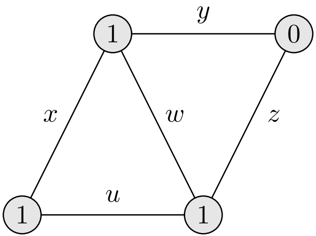

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Requires proof size $\exp(\Omega(N))$ on well-connected so-called expander graphs — **"resolution cannot count** $\mathrm{mod}\ 2$**"**

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Δn randomly sampled k-clauses over n variables

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- COLOURING [BCMM05]
- Zero-one designs [Spe10, VS10, MN14]
- Et cetera... (See, e.g., [BN21] for overview)

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But not CLIQUE!

- Refuting existence of k-clique in n-vertex graph should require proof size $n^{\Omega(k)}$
- Only known for restricted so-called regular resolution [ABdR+21]
- For general resolution, best lower bounds are $2^{\Omega(k)}$ for very large k [BIS07]

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There are also other complexity measures of interest such as

- space: memory needed for self-contained presentation of refutation
- depth: longest path in refutation represented as directed acyclic graph (DAG)

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- Add Boolean axioms

$$x_j^2 - x_j = 0$$

  for all variables

Hilbert's Nullstellensatz

Consider any system of polynomial equations

$$
\begin{aligned}
p_1(x_1, \ldots, x_n) &= 0 & \qquad x_1^2 - x_1 &= 0 \\
p_2(x_1, \ldots, x_n) &= 0 & \qquad x_2^2 - x_2 &= 0 \\
&\vdots & &\vdots \\
p_m(x_1, \ldots, x_n) &= 0 & \qquad x_n^2 - x_n &= 0
\end{aligned}
$$

in polynomial ring over field $\mathbb{F}$

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Hilbert's Nullstellensatz

System infeasible $\Leftrightarrow$ exist $q_i, r_j \in \mathbb{F}[x_1, \ldots, x_n]$ such that

$$\sum_{i=1}^{m} q_i(x_1, \ldots, x_n) \cdot p_i(x_1, \ldots, x_n) + \sum_{j=1}^{n} r_j(x_1, \ldots, x_n) \cdot (x_j^2 - x_j) = 1$$

Nullstellensatz Proof System [BIK+94]

Nullstellensatz refutation of

$$p_i(x_1, \ldots, x_n) = 0 \qquad i \in [m]$$
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is (syntactic) equality

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Complexity measures of refutations:

- Size: number of monomials (when all polynomials expanded out)
- Degree: highest total degree of any polynomial

Nullstellensatz Example

$$(x \lor z) \land (y \lor \overline{z}) \land (x \lor \overline{y} \lor u) \land (\overline{y} \lor \overline{u})$$
$$\land\ (u \lor v) \land (\overline{x} \lor \overline{v}) \land (\overline{u} \lor w) \land (\overline{x} \lor \overline{u} \lor \overline{w})$$

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$$(1-x)(1-z)$$
$$(1-y)z$$
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$$yu$$
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$$\begin{aligned}
&(1-y) \cdot (1-x)(1-z) \\
+\ &(1-x) \cdot (1-y)z \\
+\ &1 \cdot (1-x)y(1-u) \\
+\ &(1-x) \cdot yu \\
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$$\begin{aligned}
& 1 - x - y - z + xy + xz + yz - xyz \\
+\ & z - xz - yz + xyz \\
+\ & y - yu - xy + xyu \\
+\ & yu - xyu \\
+\ & x - xu - xv + xuv \\
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=\ & 1
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Size 27
Degree 3
(No use of Boolean axioms)

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$$+\ xuw$$
$$= 1$$

Size 27
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Nullstellensatz Proof Search

- Solve linear system of equations with coefficients of polynomials q_i, r_j as unknowns
- Used successfully to solve, e.g., graph colouring problems [DLMM08, DLMO09, DLMM11]
- Running time grows exponentially with degree, though high-degree refutations can be very small [BCIP02, dRMNR21]

Dual Variables

- Annoying problem: $x_1 \lor x_2 \lor x_3$ translates to polynomial

$$(1 - x_1)(1 - x_2)(1 - x_3) = 1 - x_1 - x_2 - x_3 + x_1x_2 + x_1x_3 + x_2x_3 - x_1x_2x_3$$

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- Translate $C = \bigvee_{i \in \mathcal{P}} x_i \lor \bigvee_{j \in \mathcal{N}} \overline{x}_j$ to monomial equation

$$\prod_{i \in \mathcal{P}} x'_i \cdot \prod_{j \in \mathcal{N}} x_j = 0$$

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- More generally, exponential blow-up in # positive literals
- Fix: introduce dual variables x'_i and axioms $x_i + x'_i - 1 = 0$
- Translate $C = \bigvee_{i \in \mathcal{P}} x_i \vee \bigvee_{j \in \mathcal{N}} \overline{x}_j$ to monomial equation

$$\prod_{i \in \mathcal{P}} x'_i \cdot \prod_{j \in \mathcal{N}} x_j = 0$$

- Doesn't affect degree (obviously), but can decrease size exponentially [dRLNS21] (also for other algebraic proof systems)

Dynamic Construction of Nullstellensatz Certificates

Nullstellensatz again

Infeasibility of

$$\begin{aligned} p_i(x_1,\ldots,x_n) &= 0 & i &\in [m] \\ x_j^2 - x_j &= 0 & j &\in [n] \\ x_j + x_j' - 1 &= 0 & j &\in [n] \end{aligned}$$

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- Ideal $\mathcal{I}$:
  1. $p, q \in \mathcal{I} \Rightarrow p + q \in \mathcal{I}$
  2. $p \in \mathcal{I} \Rightarrow r \cdot p \in \mathcal{I}$ for any r
- Compute polynomials in this ideal $\mathcal{I}$ step by step
- Use "multivariate division" to check whether 1 lies in ideal or not

Gröbner Bases: Admissible Orderings and Leading Terms

Admissible ordering $\preceq$ on monomials m, m', t:

1. $m \preceq m' \Rightarrow t \cdot m \preceq t \cdot m'$
2. $m \preceq t \cdot m$

Examples:

- Lexicographic
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Can write $p = \mathrm{lt}(p) + p'$ for $\mathrm{lt}(p)$ leading term (largest w.r.t. $\preceq$)

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"Multivariate division": Reduce p modulo all q in set of polynomials $\mathcal{G}$ until no further reductions possible

$\mathcal{G}$ is a Gröbner basis if final result uniquely determined

Gröbner Bases: Buchberger's Algorithm

Buchberger's algorithm for computing Gröbner bases (very rough)

1. Let $\mathcal{G} :=$ all axioms
2. Pick unprocessed pair $p, q \in \mathcal{G}$ or terminate if none exists
3. Compute $p' = t_p \cdot p$ and $q' = t_q \cdot q$ to make leading terms cancel
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Facts:

- Buchberger's algorithm computes Gröbner basis
- At termination, $1 \in \mathcal{G} \Leftrightarrow$ polynomial equations infeasible

Polynomial Calculus [CEI96, ABRW02]

- Compute polynomials in ideal $\mathcal{I}$ generated by p_i, $x_j^2 - x_j$, and $x_j + x'_j - 1$ step by step:
  - $p_i \in \mathcal{I}$, $x_j^2 - x_j \in \mathcal{I}$, and $x_j + x'_j - 1 \in \mathcal{I}$ (axioms)
  - If $p, q \in \mathcal{I}$, then $\alpha p + \beta q \in \mathcal{I}$ for any $\alpha, \beta \in \mathbb{F}$ (linear combination)
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  - If $p \in \mathcal{I}$, then $m \cdot p \in \mathcal{I}$ for any monomial $m = \prod_j x_j$ (multiplication)
- A polynomial calculus refutation is a derivation ending with the polynomial 1
- Complexity measures:
  - Size: total number of monomials in all polynomials in derivation expanded out
  - Degree: highest total degree of any polynomial
- Polynomial calculus (much) stronger than Nullstellensatz w.r.t. both size and degree

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$$\frac{x \vee \overline{y} \vee z \qquad \overline{y} \vee \overline{z}}{x \vee \overline{y}}$$

simulated by polynomial calculus derivation

$$\dfrac{x'yz' \qquad \dfrac{\dfrac{yz}{x'yz} \qquad \dfrac{z + z' - 1}{x'yz + x'yz' - x'y}}{-x'yz' + x'y}}{x'y}$$

Polynomial Calculus is Strictly Stronger than Resolution

Polynomial calculus can be exponentially stronger than resolution

For instance:

- Tseitin formulas on expander graphs if $\mathbb{F} = \mathrm{GF}(2)$
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Other hard formulas:

- Tseitin-like formulas for counting $\mathrm{mod}\ p$ if $p \neq$ field characteristic [BGIP01]
- Random k-CNF formulas
  - all characteristics except 2 [BI99]
  - all characteristics [AR03]

COLOURING and CLIQUE for Polynomial Calculus

COLOURING

- Exponential worst-case lower bounds in [LN17]
- Exponential average-case lower bounds in [CdRN+23]

CLIQUE

Almost nothing known! (Except lower bounds for very large cliques)

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If any polynomial calculus refutation of k-CNF formula F over n variables requires degree linear in n, then refuting F in polynomial calculus requires size exponential in n

- Other complexity measures analogous to those for resolution are also studied
- Many results analogous to resolution hold, but are much harder to prove
- Some analogous results are *believed* to hold, but remain open

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- Work in [KFB20, KB20, KBK20a, KBK20b, KB21] on circuit verification quite successful, but struggles with monomial blow-up
- Use dual variables! [KBBN22]

Gröbner bases: Some Problems and Questions

1. Buchberger not a great SAT solving algorithm
   Slow and memory-intensive, and computes too much info (#solutions)
   Possible to use conflict-driven paradigm?!

2. Dual variables increase reasoning power exponentially [dRLNS21]
   But are immediately eliminated by multivariate division
   Possible to design dual-variable-aware Buchberger?!

3. Analysis of polynomial calculus uses degree-lexicographic ordering
   In computational algebra, many other orderings used
   Prove proof complexity separation results for different orderings?

SAT as System of 0–1 Integer Linear Inequalities

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$$C = \bigvee_{i \in \mathcal{P}} x_i \vee \bigvee_{j \in \mathcal{N}} \overline{x}_j$$

to 0-1 integer linear inequalities

$$\sum_{i \in \mathcal{P}} x_i + \sum_{j \in \mathcal{N}} (1 - x_j) \geq 1$$

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$$\sum_{i \in \mathcal{P}} x_i + \sum_{j \in \mathcal{N}} (1 - x_j) \geq 1$$

- Add variable axioms

$$x_j \geq 0$$
$$-x_j \geq -1$$

  for all variables

Cutting Planes Proof System [CCT87]

Cutting planes proof system introduced in [CCT87] to model integer linear programming algorithm in [Gom63, Chv73]

Can be applied to any system of 0-1 integer linear inequalities

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Cutting planes derivation rules

Multiplication $$\frac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA} \quad c \in \mathbb{N}^+$$

Addition $$\frac{\sum a_i x_i \geq A \qquad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$$

Division $$\frac{\sum a_i x_i \geq A}{\sum \lceil a_i / c \rceil x_i \geq \lceil A/c \rceil} \quad c \in \mathbb{N}^+$$

Cutting Planes Derivations and Refutations

- A cutting planes derivation is a sequence of 0-1 integer linear inequalities derived using
  - Axioms (clauses and variable bounds)
  - Multiplication $\sum a_i x_i \geq A \Rightarrow \sum c a_i x_i \geq cA$
  - Addition $\sum a_i x_i \geq A$, $\sum b_i x_i \geq B \Rightarrow \sum (a_i + b_i) x_i \geq A + B$
  - Division $\sum a_i x_i \geq A \Rightarrow \sum \lceil a_i / c \rceil x_i \geq \lceil A/c \rceil$
- A cutting planes refutation ends with the inequality $0 \geq 1$
- Complexity measures:
  - Length: $\#$ inequalities
  - Size: Count also bit size of representing all coefficients

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- Cutting planes can simulate resolution reasoning efficiently and can be exponentially stronger (e.g., for PHP, just count and argue that #pigeons $>$ #holes)
- And 0-1 linear inequalities are similar to but much more concise than CNF

Compare

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 3$$

and

$$\begin{aligned} &(x_1 \lor x_2 \lor x_3 \lor x_4) \land (x_1 \lor x_2 \lor x_3 \lor x_5) \land (x_1 \lor x_2 \lor x_3 \lor x_6) \\ \land &(x_1 \lor x_2 \lor x_4 \lor x_5) \land (x_1 \lor x_2 \lor x_4 \lor x_6) \land (x_1 \lor x_2 \lor x_5 \lor x_6) \\ \land &(x_1 \lor x_3 \lor x_4 \lor x_5) \land (x_1 \lor x_3 \lor x_4 \lor x_6) \land (x_1 \lor x_3 \lor x_5 \lor x_6) \\ \land &(x_1 \lor x_4 \lor x_5 \lor x_6) \land (x_2 \lor x_3 \lor x_4 \lor x_5) \land (x_2 \lor x_3 \lor x_4 \lor x_6) \\ \land &(x_2 \lor x_3 \lor x_5 \lor x_6) \land (x_2 \lor x_4 \lor x_5 \lor x_6) \land (x_3 \lor x_4 \lor x_5 \lor x_6) \end{aligned}$$

Hard Formulas for Cutting Planes

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Variables:

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$q_{k,1} \lor q_{k,2} \lor \cdots \lor q_{k,n}$	$1 \leq k \leq m$	[some vertex is kth member of clique]
$\overline{q}_{k,v} \lor \overline{q}_{k',v}$	$1 \leq v \leq n;\ k \neq k'$	[no vertex counted as clique member twice]
$p_{u,v} \lor \overline{q}_{k,u} \lor \overline{q}_{k',v}$	$1 \leq u < v \leq n;\ k \neq k'$	[clique members are neighbours]
$r_{v,1} \lor r_{v,2} \lor \cdots \lor r_{v,m-1}$	$1 \leq v \leq n;$	[every vertex has a colour]
$\overline{p}_{u,v} \lor \overline{r}_{u,\ell} \lor \overline{r}_{v,\ell}$	$1 \leq u < v \leq n;\ 1 \leq \ell \leq m-1$	[neighbours have distinct colours]

More Hard Formulas for Cutting Planes?

Lower bound for clique-colouring formulas uses interpolation and circuit complexity

- From small cutting planes proof, build small circuit of special type that can decide whether graph has clique
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Some recent developments in [dRMN$^+$20, HP17, FPPR22, GGKS20, Sok24]

- Random $\mathcal{O}(\log n)$-CNF formulas exponentially hard [HP17, FPPR22]
- Lower bound for random k-CNF formulas open
- Surprisingly, Tseitin formulas have refutations of quasi-polynomial size [DT20]!
- Nothing known for COLOURING or CLIQUE

SAT Solvers Based on Cutting Planes?

So-called pseudo-Boolean (PB) solvers using (subset of) cutting planes reasoning developed in, e.g., [CK05, SS06, LP10, EN18]

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Is it truly harder to build good pseudo-Boolean solvers?
Or has just so much more work has been put into CDCL solvers?

Division Versus Saturation

Use negated literals as needed to get all a_i, A positive (normalized form)

Boolean derivation rules for 0–1 integer linear inequalities

Division $$\frac{\sum a_i \ell_i \geq A}{\sum \lceil a_i / c \rceil \ell_i \geq \lceil A/c \rceil} \quad c \in \mathbb{N}^+$$

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- . . . And most often also in practice [EN18], though not always [LBD$^+$20]

Separating Division from Saturation?

Even colouring formulas [Mar06]

"$\exists$ $0/1$-colouring of edges so that every vertex has equal number of 0-edges and 1-edges"

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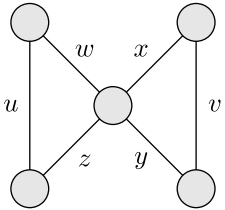

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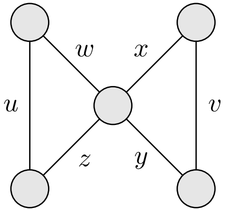

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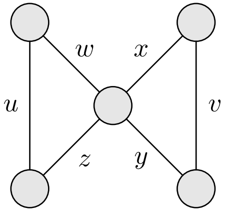

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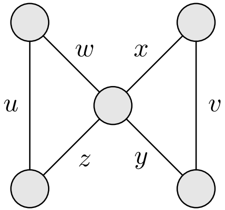

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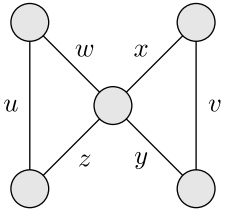
- Very easy for (tree-like) cutting planes
- CNF encoding exponentially hard for resolution (and PB solvers) for expander graphs

Separating Division from Saturation?

Even colouring formulas [Mar06]

"$\exists$ $0/1$-colouring of edges so that every vertex has equal number of 0-edges and 1-edges"



$$\begin{aligned} u + w &\geq 1 & -u - w &\geq -1 \\ u + z &\geq 1 & -u - z &\geq -1 \\ w + x + y + z &\geq 2 & -w - x - y - z &\geq -2 \\ v + x &\geq 1 & -v - x &\geq -1 \\ v + y &\geq 1 & -v - y &\geq -1 \end{aligned}$$

Unsatisfiable $\Leftrightarrow$ total $\#$ edges odd

- Very easy for (tree-like) cutting planes
- CNF encoding exponentially hard for resolution (and PB solvers) for expander graphs
- Pseudo-Boolean encoding hard in practice for pseudo-Boolean solvers [EGNV18]

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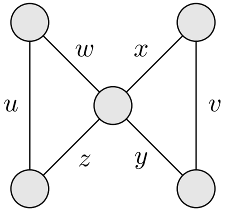

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- Very easy for (tree-like) cutting planes
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- Possible to prove lower bounds for cutting planes with saturation instead of division?

The Subgraph Isomorphism Problem

Input

- Pattern graph $\mathcal{P}$ with vertices $V(\mathcal{P}) = \{a, b, c, \ldots\}$
- Target graph $\mathcal{T}$ with vertices $V(\mathcal{T}) = \{u, v, w, \ldots\}$
- No loops (for simplicity)

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Task

- Find all subgraph isomorphisms $\varphi : V(\mathcal{P}) \to V(\mathcal{T})$
- I.e., one-to-one mappings φ such that if
  1. $\varphi(a) = u$
  2. $\varphi(b) = v$
  3. $(a, b) \in E(\mathcal{P})$

  then must have $(u, v) \in E(\mathcal{T})$

Cutting Planes Lower Bounds for Subgraph Isomorphism?

Subgraph isomorphism formula

$\sum_{v \in V(\mathcal{T})} x_{a,v} \geq 1$	[every pattern vertex $a \in V(\mathcal{P})$ maps somewhere]
$\sum_{v \in V(\mathcal{T})} -x_{a,v} \geq -1$	[... but only to one target vertex $u \in V(\mathcal{T})$]
$\sum_{b \in V(\mathcal{P})} -x_{b,u} \geq -1$	[mapping is one-to-one]
$-x_{a,u} + \sum_{v \in N(u)} x_{b,v} \geq 0$	[edge $(a,b) \in E(\mathcal{P})$ maps to edge $(u,v) \in E(\mathcal{T})$]

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- All reasoning steps in Glasgow Subgraph Solver [ADH$^+$19, GSS] can be formalized efficiently in cutting planes [GMN20, GMM$^+$24]
- So lower bounds for any graph pairs $(\mathcal{P}, \mathcal{T})$ would establish theoretical limitations on state-of-the-art algorithms

Sherali–Adams (SA) and Sum of Squares (SoS)

Refutation of $p_i \in \mathbb{R}[x_1, \ldots, x_n]$, $i \in [m]$, and $x_j^2 - x_j$, $j \in [n]$

Nullstellensatz

$$\sum_{i=1}^{m} q_i \cdot p_i + \sum_{j=1}^{n} r_j \cdot (x_j^2 - x_j) = 1$$

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Sum of squares (SoS) $(s_k \in \mathbb{R}[x_1, \ldots, x_n])$

$$\sum_{i=1}^{m} q_i \cdot p_i + \sum_{j=1}^{n} r_j \cdot (x_j^2 - x_j) + \sum_{k=1}^{s} s_k^2 = -1$$

Sherali–Adams, Sum of Squares, and Relations to Other Proof Systems

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Survey [FKP19] recommended for more reading

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Complexity measures:

- Length: $\#$ branching nodes / sets $\mathcal{S}$
- Size: Count also bit size for representing all coefficients

Stabbing Planes and Cutting Planes

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Recent news: Interpolation and circuit complexity can be used to get similar lower bounds for stabbing planes as for cutting planes! [GP24]

Still possible that stabbing planes is exponentially more powerful than cutting planes, but hard to know what to believe

Extended Resolution [Tse68]

Resolution rule

$$\frac{C_1 \lor x \qquad C_2 \lor \overline{x}}{C_1 \lor C_2}$$

Extension rule introducing clauses

$$a \lor \overline{x} \lor \overline{y} \qquad \overline{a} \lor x \qquad \overline{a} \lor y$$

for fresh variable a (encoding that $a \leftrightarrow (x \land y)$ must hold)

Extended Resolution and SAT Solving

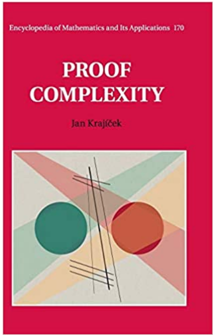
- Closely related (and equivalent) to *DRAT* proof system used to justify correctness of some SAT preprocessing techniques [JHB12]
- *DRAT* also used for SAT solver proof logging [HHW13a, HHW13b, WHH14]
- Attempts to combine extended resolution with CDCL in, e.g., [AKS10, Hua10]
- Without restrictions, as powerful as extremely strong extended Frege system [CR79] — pretty much no lower bounds known
- To analyse solvers using extended resolution, would need to:
  - Describe heuristics/rules actually used
  - See if possible to reason about such restricted proof system

Some More References for Further Reading

Handbook of Satisfiability
(Especially chapter 7 ☺)

[BHvMW21]

Proof Complexity
by Jan Krajíček



[Kra19]

Summing up This Lecture

Overview of some proof systems used in combinatorial solving:

- Resolution $\longleftrightarrow$ conflict-driven clause learning (CDCL)
- Nullstellensatz and polynomial calculus $\longleftrightarrow$ Gröbner bases
- Cutting planes $\longleftrightarrow$ pseudo-Boolean solving

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Thank you for your attention!

References I

[ABdR+21] Albert Atserias, Ilario Bonacina, Susanna F. de Rezende, Massimo Lauria, Jakob Nordström, and Alexander Razborov. Clique is hard on average for regular resolution. *Journal of the ACM*, 68(4):23:1–23:26, August 2021. Preliminary version in *STOC '18*.

[ABRW02] Michael Alekhnovich, Eli Ben-Sasson, Alexander A. Razborov, and Avi Wigderson. Space complexity in propositional calculus. *SIAM Journal on Computing*, 31(4):1184–1211, April 2002. Preliminary version in *STOC '00*.

[ADH+19] Blair Archibald, Fraser Dunlop, Ruth Hoffmann, Ciaran McCreesh, Patrick Prosser, and James Trimble. Sequential and parallel solution-biased search for subgraph algorithms. In *Proceedings of the 16th International Conference on the Integration of Constraint Programming, Artificial Intelligence, and Operations Research (CPAIOR '19)*, volume 11494 of *Lecture Notes in Computer Science*, pages 20–38. Springer, June 2019.

[AFT11] Albert Atserias, Johannes Klaus Fichte, and Marc Thurley. Clause-learning algorithms with many restarts and bounded-width resolution. *Journal of Artificial Intelligence Research*, 40:353–373, January 2011. Preliminary version in *SAT '09*.

References II

[AKS10] Gilles Audemard, George Katsirelos, and Laurent Simon. A restriction of extended resolution for clause learning SAT solvers. In *Proceedings of the 24th AAAI Conference on Artificial Intelligence (AAAI '10)*, pages 15–20, July 2010.

[AM20] Albert Atserias and Moritz Müller. Automating resolution is NP-hard. *Journal of the ACM*, 67(5):31:1–31:17, October 2020. Preliminary version in *FOCS '19*.

[AR03] Michael Alekhnovich and Alexander A. Razborov. Lower bounds for polynomial calculus: Non-binomial case. *Proceedings of the Steklov Institute of Mathematics*, 242:18–35, 2003. Available at `http://people.cs.uchicago.edu/~razborov/files/misha.pdf`. Preliminary version in *FOCS '01*.

[BBN+23] Jeremias Berg, Bart Bogaerts, Jakob Nordström, Andy Oertel, and Dieter Vandesande. Certified core-guided MaxSAT solving. In *Proceedings of the 29th International Conference on Automated Deduction (CADE-29)*, volume 14132 of *Lecture Notes in Computer Science*, pages 1–22. Springer, July 2023.

References III

[BBN$^+$24] Jeremias Berg, Bart Bogaerts, Jakob Nordström, Andy Oertel, Tobias Paxian, and Dieter Vandesande. Certifying without loss of generality reasoning in solution-improving maximum satisfiability. In *Proceedings of the 30th International Conference on Principles and Practice of Constraint Programming (CP '24)*, volume 307 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 4:1–4:28, September 2024.

[BCIP02] Joshua Buresh-Oppenheim, Matthew Clegg, Russell Impagliazzo, and Toniann Pitassi. Homogenization and the polynomial calculus. *Computational Complexity*, 11(3-4):91–108, 2002. Preliminary version in *ICALP '00*.

[BCMM05] Paul Beame, Joseph C. Culberson, David G. Mitchell, and Cristopher Moore. The resolution complexity of random graph k-colorability. *Discrete Applied Mathematics*, 153(1-3):25–47, December 2005.

[Ber18] Christoph Berkholz. The relation between polynomial calculus, Sherali-Adams, and sum-of-squares proofs. In *Proceedings of the 35th Symposium on Theoretical Aspects of Computer Science (STACS '18)*, volume 96 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 11:1–11:14, February 2018.

References IV

[BFI+18] Paul Beame, Noah Fleming, Russell Impagliazzo, Antonina Kolokolova, Denis Pankratov, Toniann Pitassi, and Robert Robere. Stabbing planes. In *Proceedings of the 9th Innovations in Theoretical Computer Science Conference (ITCS '18)*, volume 94 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 10:1–10:20, January 2018.

[BGIP01] Samuel R. Buss, Dima Grigoriev, Russell Impagliazzo, and Toniann Pitassi. Linear gaps between degrees for the polynomial calculus modulo distinct primes. *Journal of Computer and System Sciences*, 62(2):267–289, March 2001. Preliminary version in *CCC '99*.

[BGMN23] Bart Bogaerts, Stephan Gocht, Ciaran McCreesh, and Jakob Nordström. Certified dominance and symmetry breaking for combinatorial optimisation. *Journal of Artificial Intelligence Research*, 77:1539–1589, August 2023. Preliminary version in *AAAI '22*.

[BHvMW21] Armin Biere, Marijn J. H. Heule, Hans van Maaren, and Toby Walsh, editors. *Handbook of Satisfiability*, volume 336 of *Frontiers in Artificial Intelligence and Applications*. IOS Press, 2nd edition, February 2021.

[BI99] Eli Ben-Sasson and Russell Impagliazzo. Random CNF's are hard for the polynomial calculus. In *Proceedings of the 40th Annual IEEE Symposium on Foundations of Computer Science (FOCS '99)*, pages 415–421, October 1999. Journal version in [BI10].

References V

[BI10] Eli Ben-Sasson and Russell Impagliazzo. Random CNF's are hard for the polynomial calculus. *Computational Complexity*, 19(4):501–519, 2010. Preliminary version in *FOCS '99*.

[BIK+94] Paul Beame, Russell Impagliazzo, Jan Krajíček, Toniann Pitassi, and Pavel Pudlák. Lower bounds on Hilbert's Nullstellensatz and propositional proofs. In *Proceedings of the 35th Annual IEEE Symposium on Foundations of Computer Science (FOCS '94)*, pages 794–806, November 1994.

[BIS07] Paul Beame, Russell Impagliazzo, and Ashish Sabharwal. The resolution complexity of independent sets and vertex covers in random graphs. *Computational Complexity*, 16(3):245–297, October 2007. Preliminary version in *CCC '01*.

[Bla37] Archie Blake. *Canonical Expressions in Boolean Algebra*. PhD thesis, University of Chicago, 1937.

[BLLM14] Armin Biere, Daniel Le Berre, Emmanuel Lonca, and Norbert Manthey. Detecting cardinality constraints in CNF. In *Proceedings of the 17th International Conference on Theory and Applications of Satisfiability Testing (SAT '14)*, volume 8561 of *Lecture Notes in Computer Science*, pages 285–301. Springer, July 2014.

[BN21] Samuel R. Buss and Jakob Nordström. Proof complexity and SAT solving. In Biere et al. [BHvMW21], chapter 7, pages 233–350.

References VI

[BS97] Roberto J. Bayardo Jr. and Robert Schrag. Using CSP look-back techniques to solve real-world SAT instances. In *Proceedings of the 14th National Conference on Artificial Intelligence (AAAI '97)*, pages 203–208, July 1997.

[BW01] Eli Ben-Sasson and Avi Wigderson. Short proofs are narrow—resolution made simple. *Journal of the ACM*, 48(2):149–169, March 2001. Preliminary version in *STOC '99*.

[CCT87] William Cook, Collette Rene Coullard, and György Turán. On the complexity of cutting-plane proofs. *Discrete Applied Mathematics*, 18(1):25–38, November 1987.

[CdRN+23] Jonas Conneryd, Susanna F. de Rezende, Jakob Nordström, Shuo Pang, and Kilian Risse. Graph colouring is hard on average for polynomial calculus and Nullstellensatz. In *Proceedings of the 64th Annual IEEE Symposium on Foundations of Computer Science (FOCS '23)*, pages 1–11, November 2023.

[CEI96] Matthew Clegg, Jeffery Edmonds, and Russell Impagliazzo. Using the Groebner basis algorithm to find proofs of unsatisfiability. In *Proceedings of the 28th Annual ACM Symposium on Theory of Computing (STOC '96)*, pages 174–183, May 1996.

References VII

[Chv73] Vašek Chvátal. Edmonds polytopes and a hierarchy of combinatorial problems. *Discrete Mathematics*, 4(1):305–337, 1973.

[CK05] Donald Chai and Andreas Kuehlmann. A fast pseudo-Boolean constraint solver. *IEEE Transactions on Computer-Aided Design of Integrated Circuits and Systems*, 24(3):305–317, March 2005. Preliminary version in *DAC '03*.

[Coo71] Stephen A. Cook. The complexity of theorem-proving procedures. In *Proceedings of the 3rd Annual ACM Symposium on Theory of Computing (STOC '71)*, pages 151–158, May 1971.

[CR79] Stephen A. Cook and Robert A. Reckhow. The relative efficiency of propositional proof systems. *Journal of Symbolic Logic*, 44(1):36–50, March 1979. Preliminary version in *STOC '74*.

[CS88] Vašek Chvátal and Endre Szemerédi. Many hard examples for resolution. *Journal of the ACM*, 35(4):759–768, October 1988.

[DGD+21] Jo Devriendt, Stephan Gocht, Emir Demirović, Jakob Nordström, and Peter Stuckey. Cutting to the core of pseudo-Boolean optimization: Combining core-guided search with cutting planes reasoning. In *Proceedings of the 35th AAAI Conference on Artificial Intelligence (AAAI '21)*, pages 3750–3758, February 2021.

References VIII

[DGN21] Jo Devriendt, Ambros Gleixner, and Jakob Nordström. Learn to relax: Integrating 0-1 integer linear programming with pseudo-Boolean conflict-driven search. *Constraints*, 26(1–4):26–55, October 2021. Preliminary version in *CPAIOR '20*.

[DHN+25] Simon Dold, Malte Helmert, Jakob Nordström, Gabriele Röger, and Tanja Schindler. Pseudo-Boolean proof logging for optimal classical planning. In *Proceedings of the 35th International Conference on Automated Planning and Scheduling (ICAPS '25)*, November 2025. To appear.

[DLL62] Martin Davis, George Logemann, and Donald Loveland. A machine program for theorem proving. *Communications of the ACM*, 5(7):394–397, July 1962.

[DLMM08] Jesús A. De Loera, Jon Lee, Peter N. Malkin, and Susan Margulies. Hilbert's Nullstellensatz and an algorithm for proving combinatorial infeasibility. In *Proceedings of the 21st International Symposium on Symbolic and Algebraic Computation (ISSAC '08)*, pages 197–206, July 2008.

[DLMM11] Jesús A. De Loera, Jon Lee, Peter N. Malkin, and Susan Margulies. Computing infeasibility certificates for combinatorial problems through Hilbert's Nullstellensatz. *Journal of Symbolic Computation*, 46(11):1260–1283, November 2011.

References IX

[DLMO09] Jesús A. De Loera, Jon Lee, Susan Margulies, and Shmuel Onn. Expressing combinatorial problems by systems of polynomial equations and Hilbert's Nullstellensatz. *Combinatorics, Probability and Computing*, 18(4):551–582, July 2009.

[DMM+24] Emir Demirović, Ciaran McCreesh, Matthew McIlree, Jakob Nordström, Andy Oertel, and Konstantin Sidorov. Pseudo-Boolean reasoning about states and transitions to certify dynamic programming and decision diagram algorithms. In *Proceedings of the 30th International Conference on Principles and Practice of Constraint Programming (CP '24)*, volume 307 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 9:1–9:21, September 2024.

[DP60] Martin Davis and Hilary Putnam. A computing procedure for quantification theory. *Journal of the ACM*, 7(3):201–215, 1960.

[dRGN+21] Susanna F. de Rezende, Mika Göös, Jakob Nordström, Toniann Pitassi, Robert Robere, and Dmitry Sokolov. Automating algebraic proof systems is NP-hard. In *Proceedings of the 53rd Annual ACM Symposium on Theory of Computing (STOC '21)*, pages 209–222, June 2021.

References X

[dRLNS21] Susanna F. de Rezende, Massimo Lauria, Jakob Nordström, and Dmitry Sokolov. The power of negative reasoning. In *Proceedings of the 36th Annual Computational Complexity Conference (CCC '21)*, volume 200 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 40:1–40:24, July 2021.

[dRMN+20] Susanna F. de Rezende, Or Meir, Jakob Nordström, Toniann Pitassi, Robert Robere, and Marc Vinyals. Lifting with simple gadgets and applications to circuit and proof complexity. In *Proceedings of the 61st Annual IEEE Symposium on Foundations of Computer Science (FOCS '20)*, pages 24–30, November 2020.

[dRMNR21] Susanna F. de Rezende, Or Meir, Jakob Nordström, and Robert Robere. Nullstellensatz size-degree trade-offs from reversible pebbling. *Computational Complexity*, 30:4:1–4:45, February 2021. Preliminary version in *CCC '19*.

[DT20] Daniel Dadush and Samarth Tiwari. On the complexity of branching proofs. In *Proceedings of the 35th Annual Computational Complexity Conference (CCC '20)*, volume 169 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 34:1–34:35, July 2020.

References XI

[EGMN20] Jan Elffers, Stephan Gocht, Ciaran McCreesh, and Jakob Nordström. Justifying all differences using pseudo-Boolean reasoning. In *Proceedings of the 34th AAAI Conference on Artificial Intelligence (AAAI '20)*, pages 1486–1494, February 2020.

[EGNV18] Jan Elffers, Jesús Giráldez-Cru, Jakob Nordström, and Marc Vinyals. Using combinatorial benchmarks to probe the reasoning power of pseudo-Boolean solvers. In *Proceedings of the 21st International Conference on Theory and Applications of Satisfiability Testing (SAT '18)*, volume 10929 of *Lecture Notes in Computer Science*, pages 75–93. Springer, July 2018.

[EN18] Jan Elffers and Jakob Nordström. Divide and conquer: Towards faster pseudo-Boolean solving. In *Proceedings of the 27th International Joint Conference on Artificial Intelligence (IJCAI '18)*, pages 1291–1299, July 2018.

[EN20] Jan Elffers and Jakob Nordström. A cardinal improvement to pseudo-Boolean solving. In *Proceedings of the 34th AAAI Conference on Artificial Intelligence (AAAI '20)*, pages 1495–1503, February 2020.

[FGI+21] Noah Fleming, Mika Göös, Russell Impagliazzo, Toniann Pitassi, Robert Robere, Li-Yang Tan, and Avi Wigderson. On the power and limitations of branch and cut. In *Proceedings of the 36th Annual Computational Complexity Conference (CCC '21)*, volume 200 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 6:1–6:30, July 2021.

References XII

[FKP19] Noah Fleming, Pravesh Kothari, and Toniann Pitassi. Semialgebraic proofs and efficient algorithm design. *Foundations and Trends in Theoretical Computer Science*, 14(1–2):1–221, December 2019.

[FPPR22] Noah Fleming, Denis Pankratov, Toniann Pitassi, and Robert Robere. Random $\theta(\log n)$-CNFs are hard for cutting planes. *Journal of the ACM*, 69(3):19:1–19:32, June 2022. Preliminary version in *FOCS '17*.

[GGKS20] Ankit Garg, Mika Göös, Pritish Kamath, and Dmitry Sokolov. Monotone circuit lower bounds from resolution. *Theory of Computing*, 16(13):1–30, 2020. Preliminary version in *STOC '18*.

[GKMP20] Mika Göös, Sajin Koroth, Ian Mertz, and Toniann Pitassi. Automating cutting planes is NP-hard. In *Proceedings of the 52nd Annual ACM Symposium on Theory of Computing (STOC '20)*, pages 68–77, June 2020.

[GMM^{+}20] Stephan Gocht, Ross McBride, Ciaran McCreesh, Jakob Nordström, Patrick Prosser, and James Trimble. Certifying solvers for clique and maximum common (connected) subgraph problems. In *Proceedings of the 26th International Conference on Principles and Practice of Constraint Programming (CP '20)*, volume 12333 of *Lecture Notes in Computer Science*, pages 338–357. Springer, September 2020.

References XIII

[GMM+24] Stephan Gocht, Ciaran McCreesh, Magnus O. Myreen, Jakob Nordström, Andy Oertel, and Yong Kiam Tan. End-to-end verification for subgraph solving. In *Proceedings of the 38th AAAI Conference on Artificial Intelligence (AAAI '24)*, pages 8038–8047, February 2024.

[GMN20] Stephan Gocht, Ciaran McCreesh, and Jakob Nordström. Subgraph isomorphism meets cutting planes: Solving with certified solutions. In *Proceedings of the 29th International Joint Conference on Artificial Intelligence (IJCAI '20)*, pages 1134–1140, July 2020.

[GMN22] Stephan Gocht, Ciaran McCreesh, and Jakob Nordström. An auditable constraint programming solver. In *Proceedings of the 28th International Conference on Principles and Practice of Constraint Programming (CP '22)*, volume 235 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 25:1–25:18, August 2022.

[GMNO22] Stephan Gocht, Ruben Martins, Jakob Nordström, and Andy Oertel. Certified CNF translations for pseudo-Boolean solving. In *Proceedings of the 25th International Conference on Theory and Applications of Satisfiability Testing (SAT '22)*, volume 236 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 16:1–16:25, August 2022.

References XIV

[GN21] Stephan Gocht and Jakob Nordström. Certifying parity reasoning efficiently using pseudo-Boolean proofs. In *Proceedings of the 35th AAAI Conference on Artificial Intelligence (AAAI '21)*, pages 3768–3777, February 2021.

[GNY19] Stephan Gocht, Jakob Nordström, and Amir Yehudayoff. On division versus saturation in pseudo-Boolean solving. In *Proceedings of the 28th International Joint Conference on Artificial Intelligence (IJCAI '19)*, pages 1711–1718, August 2019.

[Gom63] Ralph E. Gomory. An algorithm for integer solutions of linear programs. In R.L. Graves and P. Wolfe, editors, *Recent Advances in Mathematical Programming*, pages 269–302. McGraw-Hill, New York, 1963.

[GP24] Max Gläser and Marc E. Pfetsch. Sub-exponential lower bounds for branch-and-bound with general disjunctions via interpolation. In *Proceedings of the 35th Annual ACM-SIAM Symposium on Discrete Algorithms (SODA '24)*, pages 3747–3764, January 2024.

[Gri01] Dima Grigoriev. Linear lower bound on degrees of Positivstellensatz calculus proofs for the parity. *Theoretical Computer Science*, 259(1–2):613–622, May 2001.

[GSS] The Glasgow subgraph solver. `https://github.com/ciaranm/glasgow-subgraph-solver`.

References XV

[GV01] Dima Grigoriev and Nicolai Vorobjov. Complexity of Null- and Positivstellensatz proofs. *Annals of Pure and Applied Logic*, 113(1–3):153–160, December 2001.

[Hak85] Armin Haken. The intractability of resolution. *Theoretical Computer Science*, 39(2-3):297–308, August 1985.

[Hås99] Johan Håstad. Clique is hard to approximate within $n^{1-\epsilon}$. *Acta Mathematica*, 182:105–142, 1999. Preliminary version in *FOCS '96*.

[Hås01] Johan Håstad. Some optimal inapproximability results. *Journal of the ACM*, 48(4):798–859, July 2001. Preliminary version in *STOC '97*.

[HHW13a] Marijn J. H. Heule, Warren A. Hunt Jr., and Nathan Wetzler. Trimming while checking clausal proofs. In *Proceedings of the 13th International Conference on Formal Methods in Computer-Aided Design (FMCAD '13)*, pages 181–188, October 2013.

[HHW13b] Marijn J. H. Heule, Warren A. Hunt Jr., and Nathan Wetzler. Verifying refutations with extended resolution. In *Proceedings of the 24th International Conference on Automated Deduction (CADE-24)*, volume 7898 of *Lecture Notes in Computer Science*, pages 345–359. Springer, June 2013.

References XVI

[HOGN24] Alexander Hoen, Andy Oertel, Ambros Gleixner, and Jakob Nordström. Certifying MIP-based presolve reductions for 0–1 integer linear programs. In *Proceedings of the 21st International Conference on the Integration of Constraint Programming, Artificial Intelligence, and Operations Research (CPAIOR '24)*, volume 14742 of *Lecture Notes in Computer Science*, pages 310–328. Springer, May 2024.

[HP17] Pavel Hrubeš and Pavel Pudlák. Random formulas, monotone circuits, and interpolation. In *Proceedings of the 58th Annual IEEE Symposium on Foundations of Computer Science (FOCS '17)*, pages 121–131, October 2017.

[Hua10] Jinbo Huang. Extended clause learning. *Artificial Intelligence*, 174(15):1277–1284, October 2010.

[IOT+24] Hannes Ihalainen, Andy Oertel, Yong Kiam Tan, Jeremias Berg, Matti Järvisalo, Magnus O. Myreen, and Jakob Nordström. Certified MaxSAT preprocessing. In *Proceedings of the 12th International Joint Conference on Automated Reasoning (IJCAR '24)*, volume 14739 of *Lecture Notes in Computer Science*, pages 396–418. Springer, July 2024.

[IPS99] Russell Impagliazzo, Pavel Pudlák, and Jiří Sgall. Lower bounds for the polynomial calculus and the Gröbner basis algorithm. *Computational Complexity*, 8(2):127–144, 1999.

References XVII

[JHB12] Matti Järvisalo, Marijn J. H. Heule, and Armin Biere. Inprocessing rules. In *Proceedings of the 6th International Joint Conference on Automated Reasoning (IJCAR '12)*, volume 7364 of *Lecture Notes in Computer Science*, pages 355–370. Springer, June 2012.

[KB20] Daniela Kaufmann and Armin Biere. Nullstellensatz-proofs for multiplier verification. In *Proceedings of the 22nd International Workshop on Computer Algebra in Scientific Computing (CASC' 20)*, volume 12291 of *Lecture Notes in Computer Science*, pages 368–389. Springer, September 2020.

[KB21] Daniela Kaufmann and Armin Biere. AMulet 2.0 for verifying multiplier circuits. In *Proceedings of the 27th International Conference on Tools and Algorithms for the Construction and Analysis of Systems (TACAS '21)*, volume 12652 of *Lecture Notes in Computer Science*, pages 357–364. Springer, March-April 2021.

[KBBN22] Daniela Kaufmann, Paul Beame, Armin Biere, and Jakob Nordström. Adding dual variables to algebraic reasoning for circuit verification. In *Proceedings of the 25th Design, Automation and Test in Europe Conference (DATE '22)*, pages 1435–1440, March 2022.

[KBK20a] Daniela Kaufmann, Armin Biere, and Manuel Kauers. From DRUP to PAC and back. In *Proceedings of the Design, Automation & Test in Europe Conference & Exhibition (DATE '20)*, pages 654–657, March 2020.

References XVIII

[KBK20b] Daniela Kaufmann, Armin Biere, and Manuel Kauers. Incremental column-wise verifiation of arithmetic circuits using computer algebra. *Formal Methods in Systems Design*, 56(1–3):22–54, 2020. Preliminary version in *FMCAD '17*.

[KFB20] Daniela Kaufmann, Mathias Fleury, and Armin Biere. The proof checkers Pacheck and Pastèque for the practical algebraic calculus. In *Proceedings of the 20th Conference on Formal Methods in Computer-Aided Design (FMCAD '20)*, pages 264–269, September 2020.

[Kho01] Subhash Khot. Improved inapproximability results for MaxClique, chromatic number and approximate graph coloring. In *Proceedings of the 42nd Annual IEEE Symposium on Foundations of Computer Science (FOCS '01)*, pages 600–609, October 2001.

[KLM$^+$25] Wietze Koops, Daniel Le Berre, Magnus O. Myreen, Jakob Nordström, Andy Oertel, Yong Kiam Tan, and Marc Vinyals. Practically feasible proof logging for pseudo-Boolean optimization. In *Proceedings of the 31st International Conference on Principles and Practice of Constraint Programming (CP '25)*, volume 340 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 21:1–21:27, August 2025.

[Kra19] Jan Krajíček. *Proof Complexity*, volume 170 of *Encyclopedia of Mathematics and Its Applications*. Cambridge University Press, March 2019.

References XIX

[LBD+20] Vincent Liew, Paul Beame, Jo Devriendt, Jan Elffers, and Jakob Nordström. Verifying properties of bit-vector multiplication using cutting planes reasoning. In *Proceedings of the 20th Conference on Formal Methods in Computer-Aided Design (FMCAD '20)*, pages 194–204, September 2020.

[Lev73] Leonid A. Levin. Universal sequential search problems. *Problemy peredachi informatsii*, 9(3):115–116, 1973. In Russian. Available at `http://mi.mathnet.ru/ppi914`.

[LN17] Massimo Lauria and Jakob Nordström. Graph colouring is hard for algorithms based on Hilbert's Nullstellensatz and Gröbner bases. In *Proceedings of the 32nd Annual Computational Complexity Conference (CCC '17)*, volume 79 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 2:1–2:20, July 2017.

[LP10] Daniel Le Berre and Anne Parrain. The Sat4j library, release 2.2. *Journal on Satisfiability, Boolean Modeling and Computation*, 7:59–64, July 2010.

[Mar06] Klas Markström. Locality and hard SAT-instances. *Journal on Satisfiability, Boolean Modeling and Computation*, 2(1-4):221–227, 2006.

References XX

[MBGN23] Gioni Mexi, Timo Berthold, Ambros Gleixner, and Jakob Nordström. Improving conflict analysis in MIP solvers by pseudo-Boolean reasoning. In *Proceedings of the 29th International Conference on Principles and Practice of Constraint Programming (CP '23)*, volume 280 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 27:1–26:19, August 2023.

[McC17] Ciaran McCreesh. *Solving Hard Subgraph Problems in Parallel*. PhD thesis, University of Glasgow, 2017.

[MM23] Matthew McIlree and Ciaran McCreesh. Proof logging for smart extensional constraints. In *Proceedings of the 29th International Conference on Principles and Practice of Constraint Programming (CP '23)*, volume 280 of *Leibniz International Proceedings in Informatics (LIPIcs)*, pages 26:1–26:17, August 2023.

[MM25] Matthew McIlree and Ciaran McCreesh. Certifying bounds propagation for integer multiplication constraints. In *Proceedings of the 39th AAAI Conference on Artificial Intelligence (AAAI '25)*, pages 11309–11317, February-March 2025.

References XXI

[MMN24] Matthew McIlree, Ciaran McCreesh, and Jakob Nordström. Proof logging for the circuit constraint. In *Proceedings of the 21st International Conference on the Integration of Constraint Programming, Artificial Intelligence, and Operations Research (CPAIOR '24)*, volume 14743 of *Lecture Notes in Computer Science*, pages 38–55. Springer, May 2024.

[MMZ+01] Matthew W. Moskewicz, Conor F. Madigan, Ying Zhao, Lintao Zhang, and Sharad Malik. Chaff: Engineering an efficient SAT solver. In *Proceedings of the 38th Design Automation Conference (DAC '01)*, pages 530–535, June 2001.

[MN14] Mladen Mikša and Jakob Nordström. Long proofs of (seemingly) simple formulas. In *Proceedings of the 17th International Conference on Theory and Applications of Satisfiability Testing (SAT '14)*, volume 8561 of *Lecture Notes in Computer Science*, pages 121–137. Springer, July 2014.

[MN24] Mladen Mikša and Jakob Nordström. A generalized method for proving polynomial calculus degree lower bounds. *Journal of the ACM*, 71(6):37:1–37:43, November 2024. Preliminary version in *CCC '15*.

[MS99] João P. Marques-Silva and Karem A. Sakallah. GRASP: A search algorithm for propositional satisfiability. *IEEE Transactions on Computers*, 48(5):506–521, May 1999. Preliminary version in *ICCAD '96*.

References XXII

[MSB+25] Gioni Mexi, Felipe Serrano, Timo Berthold, Ambros Gleixner, and Jakob Nordström. Cut-based conflict analysis in mixed integer programming. *INFORMS Journal on Computing*, 2025. To appear.

[PD11] Knot Pipatsrisawat and Adnan Darwiche. On the power of clause-learning SAT solvers as resolution engines. *Artificial Intelligence*, 175(2):512–525, February 2011. Preliminary version in *CP '09*.

[Pro12] Patrick Prosser. Exact algorithms for maximum clique: A computational study. *Algorithms*, 5(4):545–587, November 2012.

[Pud97] Pavel Pudlák. Lower bounds for resolution and cutting plane proofs and monotone computations. *Journal of Symbolic Logic*, 62(3):981–998, September 1997.

[Raz98] Alexander A. Razborov. Lower bounds for the polynomial calculus. *Computational Complexity*, 7(4):291–324, December 1998.

[Rii93] Søren Riis. *Independence in Bounded Arithmetic*. PhD thesis, University of Oxford, 1993.

[Rob65] John Alan Robinson. A machine-oriented logic based on the resolution principle. *Journal of the ACM*, 12(1):23–41, January 1965.

References XXIII

[Sch08] Grant Schoenebeck. Linear level Lasserre lower bounds for certain k-CSPs. In *Proceedings of the 49th Annual IEEE Symposium on Foundations of Computer Science (FOCS '08)*, pages 593–602, October 2008.

[Sok24] Dmitry Sokolov. Random $(\log n)$-CNF are hard for cutting planes (again). In *Proceedings of the 56th Annual ACM Symposium on Theory of Computing (STOC '24)*, pages 2008–2015, June 2024.

[Spe10] Ivor Spence. sgen1: A generator of small but difficult satisfiability benchmarks. *Journal of Experimental Algorithmics*, 15:1.2:1–1.2:15, March 2010.

[SS06] Hossein M. Sheini and Karem A. Sakallah. Pueblo: A hybrid pseudo-Boolean SAT solver. *Journal on Satisfiability, Boolean Modeling and Computation*, 2(1-4):165–189, March 2006. Preliminary version in *DATE '05*.

[Tse68] Grigori Tseitin. On the complexity of derivation in propositional calculus. In A. O. Silenko, editor, *Structures in Constructive Mathematics and Mathematical Logic, Part II*, pages 115–125. Consultants Bureau, New York-London, 1968.

[Urq87] Alasdair Urquhart. Hard examples for resolution. *Journal of the ACM*, 34(1):209–219, January 1987.

References XXIV

[VS10] Allen Van Gelder and Ivor Spence. Zero-one designs produce small hard SAT instances. In *Proceedings of the 13th International Conference on Theory and Applications of Satisfiability Testing (SAT '10)*, volume 6175 of *Lecture Notes in Computer Science*, pages 388–397. Springer, July 2010.

[WHH14] Nathan Wetzler, Marijn J. H. Heule, and Warren A. Hunt Jr. DRAT-trim: Efficient checking and trimming using expressive clausal proofs. In *Proceedings of the 17th International Conference on Theory and Applications of Satisfiability Testing (SAT '14)*, volume 8561 of *Lecture Notes in Computer Science*, pages 422–429. Springer, July 2014.

[Zuc07] David Zuckerman. Linear degree extractors and the inapproximability of max clique and chromatic number. *Theory of Computing*, 3(6):103–128, August 2007. Preliminary version in *STOC '06*.