Tutorial on Conflict-Driven Pseudo-Boolean Optimization

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Outline of Lecture on Pseudo-Boolean Optimization

MaxSAT Problem

Pseudo-Boolean optimization and MaxSAT solving intimately connected, so let's do a detour and define MaxSAT

Weighted partial MaxSAT problem

Input: Soft clauses $C_1, \ldots, C_m$ with weights $w_i \in \mathbb{N}^+$, $i \in [m]$
Hard clauses $C_{m+1}, \ldots, C_M$

Goal: Find assignment ρ such that

- for all hard clauses $C_{m+1}, \ldots, C_M$ it holds that $\rho(C_j) = 1$
- ρ maximizes $\sum_{\rho(C_i)=1, i \in [m]} w_i$

- All hard clauses must be satisfied
- Maximize weight of satisfied soft clauses = Minimize penalty of falsified soft clauses
- Write $(C)_w$ for clause C with weight w ($w = \infty$ for hard clause)

From MaxSAT to Pseudo-Boolean Optimization

MaxSAT instance

$$(\overline{x})_5$$
$$(y \lor \overline{z})_4$$
$$(\overline{y} \lor z)_3$$
$$(x \lor y \lor z)_\infty$$
$$(x \lor \overline{y} \lor \overline{z})_\infty$$

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PBO instance

$$\min \; 5b_1 + 4b_2 + 3b_3$$
$$b_1 + \overline{x} \geq 1$$
$$b_2 + y + \overline{z} \geq 1$$
$$b_3 + \overline{y} + z \geq 1$$
$$x + y + z \geq 1$$
$$x + \overline{y} + \overline{z} \geq 1$$

From MaxSAT to Pseudo-Boolean Optimization

MaxSAT instance	**PBO instance**
	$\min\ 5b_1 + 4b_2 + 3b_3$
$(\overline{x})_5$	$b_1 + \overline{x} \geq 1$
$(y \vee \overline{z})_4$	$b_2 + y + \overline{z} \geq 1$
$(\overline{y} \vee z)_3$	$b_3 + \overline{y} + z \geq 1$
$(x \vee y \vee z)_\infty$	$x + y + z \geq 1$
$(x \vee \overline{y} \vee \overline{z})_\infty$	$x + \overline{y} + \overline{z} \geq 1$

So-called blocking variable transformation

Variables b_i are blocking or relaxation variables

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So-called blocking variable transformation
Variables b_i are blocking or relaxation variables

Optimal solution $\rho = \{x = 0, y = 1, z = 0\}$ with penalty 3

From Pseudo-Boolean Optimization to MaxSAT/WBO

"MaxSAT instance" but with PB constraints:
Weighted Boolean Optimization [MMP09]

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$$\min \sum_{i=1}^{n} w_i \ell_i$$

$$C_1$$

$$C_2$$

$$\vdots$$

$$C_M$$

From Pseudo-Boolean Optimization to MaxSAT/WBO

"MaxSAT instance" but with PB constraints:
Weighted Boolean Optimization [MMP09]

PBO instance	**MaxSAT/WBO instance**
$\min \; \sum_{i=1}^{n} w_i \ell_i$	
C_1	
C_2	$(\bar{\ell}_1)_{w_1}$
$\vdots$	$\vdots$
C_M	$(\bar{\ell}_n)_{w_n}$
	$(C_1)_\infty$
	$\vdots$
	$(C_M)_\infty$

Flavours of MaxSAT

- Partial MaxSAT: Hard and soft clauses
- MaxSAT: Only soft clauses
- Unweighted MaxSAT: Same weight for soft clauses (w.l.o.g. 1)
- Weighted MaxSAT: Different weights for soft clauses

4 different subproblems
But most current solvers deal with the most general problem

Main Approaches for MaxSAT Solving (and PBO)

1. Linear search SAT-UNSAT (LSU) (or model-improving search)
2. Core-guided search
3. Implicit hitting set (IHS) algorithm

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Will describe all of these algorithms as trying to

- minimize $\sum_{i=1}^{n} w_i \ell_i$
- subject to collection of PB constraints $F = C_1 \wedge \cdots \wedge C_m$ (possibly clausal)

Linear Search SAT-UNSAT (LSU) Algorithm

- Minimize $\sum_{i=1}^{n} w_i \ell_i$
- Subject to collection of PB constraints $F = C_1 \wedge \cdots \wedge C_m$

Set $\rho_{\text{best}} = \emptyset$ and repeat the following:

1. Run SAT/PB solver
2. If solver returns `UNSATISFIABLE`, output ρ_{best} and terminate
3. Otherwise, let $\rho_{\text{best}} :=$ returned solution ρ
4. Add solution-improving constraint
   $\sum_{i=1}^{n} w_i \ell_i \leq -1 + \sum_{i=1}^{n} w_i \cdot \rho(\ell_i)$
5. Start over from the top

Linear Search Toy Example

1. Given PB formula F and objective function
$\min x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6$

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2. Solver run on F returns $\rho_1 = \{x_1 = x_2 = x_3 = x_6 = 0; x_4 = x_5 = 1\}$
3. Yields objective value $0 + 2 \cdot 0 + 3 \cdot 0 + 4 \cdot 1 + 5 \cdot 1 + 6 \cdot 0 = 9$, so add
$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \leq 8$$

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$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \leq 8$$
4. Solver run on F plus this new constraint returns $\rho_2 = \{x_1 = x_3 = x_5 = x_6 = 0; x_2 = x_4 = 1\}$

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$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \leq 8$$
4. Solver run on F plus this new constraint returns $\rho_2 = \{x_1 = x_3 = x_5 = x_6 = 0; x_2 = x_4 = 1\}$
5. Yields objective value 6, so add
$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \leq 5$$

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6. Now solver returns `UNSATISFIABLE`

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$$x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \leq 5$$
6. Now solver returns `UNSATISFIABLE`
7. Hence, minimum value of objective function subject to F is 6

CNF Encodings

For SAT solver, need CNF encoding of solution-improving constraint $\sum_{i=1}^{n} w_i \ell_i \leq -1 + \sum_{i=1}^{n} w_i \cdot \rho(\ell_i)$

Lots of work on how to do this in smart ways (with [PRB18] maybe best right now)

For pseudo-Boolean solver, no re-encoding needed

Linear vs. Binary Search?

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Two possible explanations:

1. In theory, objective value could decrease by just 1 every time — in practice, tend to get much larger jumps
2. Potentially very different cost for
   - SAT calls (feasible instances where solver will find solution)
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Properties of linear search SAT-UNSAT:

- Can get **some decent** solution quickly, even if not optimal one
- Important for anytime solving (when time is limited and something is better than nothing)
- But get no estimate of how good the solution is

Core-Guided Search

- Minimize $\sum_{i=1}^{n} w_i \ell_i$
- Subject to collection of PB constraints $F = C_1 \wedge \cdots \wedge C_m$

Think first of this as MaxSAT instance with ℓ_i as blocking variables

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Set $val_{\text{best}} = 0$ and repeat the following:

1. Run SAT solver with assumptions (pre-made decisions) $\ell_i = 0$ for all ℓ_i in objective function

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5. Introduce new variables $z_j \Leftrightarrow \sum_{i=1}^{k} \ell_i \geq j$
6. Update objective function and val_{best} using $\sum_{i=1}^{k} \ell_i = 1 + \sum_{j=2}^{k} z_j$ to cancel at least one literal ℓ_i

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- And rewriting very convenient — just use PB constraints without re-encoding
- Core-guided PB search: assume optimistically that objective can reach best imaginable value; derive contradiction if not possible
- Let us try to explain by concrete example

Core-Guided Search Toy Example (1/5)

1. Given same PB formula F and objective function

$$\min x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6 \tag{1}$$

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3. Run solver on F with assumptions $x_1 = x_2 = \ldots = x_6 = 0$

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2. Set $val_{\text{best}} = 0$
3. Run solver on F with assumptions $x_1 = x_2 = \ldots = x_6 = 0$
4. Suppose solver returns PB core constraint

$$3x_2 + 2x_3 + x_4 + x_5 \geq 4 \tag{2}$$

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3. Run solver on F with assumptions $x_1 = x_2 = \ldots = x_6 = 0$
4. Suppose solver returns PB core constraint
$$3x_2 + 2x_3 + x_4 + x_5 \geq 4 \tag{2}$$
5. Round to nicer-to-work-with cardinality core constraint
$$x_2 + x_3 + x_4 + x_5 \geq 2 \tag{3}$$

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5. Round to nicer-to-work-with cardinality core constraint

$$x_2 + x_3 + x_4 + x_5 \geq 2 \tag{3}$$

6. Introduce new, fresh variables y_3 and y_4 and constraints

$$x_2 + x_3 + x_4 + x_5 = 2 + y_3 + y_4 \tag{4a}$$

$$y_3 \geq y_4 \tag{4b}$$

to enforce that y_j means "$x_2 + x_3 + x_4 + x_5 \geq j$"

Core-Guided Search Toy Example (2/5)

7. Multiply (4a) by 2 to get

$$4 + 2y_3 + 2y_4 - 2x_2 - 2x_3 - 2x_4 - 2x_5 = 0$$

and add to objective function

$$\min x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6$$

in (1) to cancel x_2 and get updated, equivalent objective function

$$\min x_1 + x_3 + 2x_4 + 3x_5 + 6x_6 + 2y_3 + 2y_4 + 4 \qquad (5)$$

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$$\min x_1 + x_3 + 2x_4 + 3x_5 + 6x_6 + 2y_3 + 2y_4 + 4 \qquad (5)$$

8. Update $val_{\text{best}} = 4$
9. Run solver on F assuming all literals in (5) being 0

Core-Guided Search Toy Example (3/5)

10. Suppose solver returns the clausal core constraint

$$x_4 + x_5 + x_6 + y_3 \geq 1 \tag{6}$$

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11. Introduce new variables z_2, z_3, z_4 and the constraints

$$x_4 + x_5 + x_6 + y_3 = 1 + z_2 + z_3 + z_4 \tag{7a}$$
$$z_2 \geq z_3 \tag{7b}$$
$$z_3 \geq z_4 \tag{7c}$$

to enforce that z_j means "$x_4 + x_5 + x_6 + y_3 \geq j$"

Core-Guided Search Toy Example (4/5)

12. Multiply (7a) by 2 to get

$$2 + 2z_2 + 2z_3 + 2z_4 - 2x_4 - 2x_5 - 2x_6 - 2y_3 = 0$$

and add to rewritten objective

$$\min x_1 + x_3 + 2x_4 + 3x_5 + 6x_6 + 2y_3 + 2y_4 + 4$$

in (5) to get 3rd equivalent objective

$$\min x_1 + x_3 + x_5 + 4x_6 + 2y_4 + 2z_2 + 2z_3 + 2z_4 + 6 \tag{8}$$

Core-Guided Search Toy Example (4/5)

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13. Update $val_{\text{best}} = 6$

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in (5) to get 3rd equivalent objective

$$\min x_1 + x_3 + x_5 + 4x_6 + 2y_4 + 2z_2 + 2z_3 + 2z_4 + 6 \tag{8}$$

13. Update $val_{\text{best}} = 6$
14. For 3rd time run solver on F, assuming all literals in (8) being 0

Core-Guided Search Toy Example (5/5)

15. Suppose solver reports it is possible to achieve

$$\rho = \{x_1 = x_3 = x_5 = x_6 = y_4 = z_2 = z_3 = z_4 = 0\} \tag{9}$$

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16. Under assignment (9) the equality (4a) simplifies to

$$x_2 + x_4 = 2 + y_3 \tag{10}$$

which can hold only if $y_3 = 0$ and $x_2 = x_4 = 1$, and this also satisfies (7a).

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17. Hence, have recovered optimal solution yielding objective value 6 (as in LSU example before)

Properties of (Pure) Core-Guided Search

- Can get decent lower bounds on solution quickly
- Helps to cut off parts of search space “too good to be true”
- But find no actual solution until the final, optimal one
- Also, no estimate of how good the lower bound is
- Linear search much better at finding solutions — how to get the best of both worlds?

Improvements of Core-Guided Search (1/2)

Weight stratification [ABGL12]

Set only literals with largest weight in objective to $0 \Rightarrow$

1. More compact core; or
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Core boosting [BDS19]

Start with core-guided search to get good lower bound estimate; then switch to linear search to find optimal solution

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Core boosting [BDS19]
Start with core-guided search to get good lower bound estimate; then switch to linear search to find optimal solution

Hybrid/interleaving search [ADMR15]
Switch back and forth repeatedly between core-guided and linear search — cumbersome in CNF-based solver, but fairly cheap (and efficient) in native pseudo-Boolean solver [DGD+21]

Improvements of Core-Guided Search (2/2)

Core minimization (e.g., [Mar10, MIM15])
In CDCL-based solver, try to get smaller core clauses. For PB solver, not so clear how to do this (constraint minimization also interesting problem in general for PB conflict analysis)

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Lazy variables [MJML14, DGD$^+$21]
For real-world instances, rewriting of objective function can introduce huge numbers of new variables, slowing down the solver — so don't introduce all variables in one go but only lazily as needed

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Inference strength of core-guided search?

- Extension variables very strong in theory, but hard to use in practice
- Core-guided search provides principled way of introducing them
- Can we characterize the power of this method?

Evaluation of Core-Guided PB Solver in [DGD$^+$21]

RoundingSat with core-guided (CG) and linear search (LSU)

#instances solved to optimality; highlighting **1st**, **2nd**, and **3rd** best

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	PB16opt (1600)	MIPopt (291)	KNAP (783)	CRAFT (985)
Hybrid (interleave CG & LSU)	**968**	**78**	306	**639**
HybridCl (w/ clausal cores)	937	75	298	**618**
HybridNL (w/ non-lazy variables)	936	70	186	607
HybridClNL (w/ both)	917	67	203	612
RoundingSat (only LSU)	853	75	341	309
CoreGuided (only CG)	911	61	43	595
CoreBoosted (10% CG, then LSU)	**959**	**80**	**344**	580
Sat4j	773	61	**373**	105
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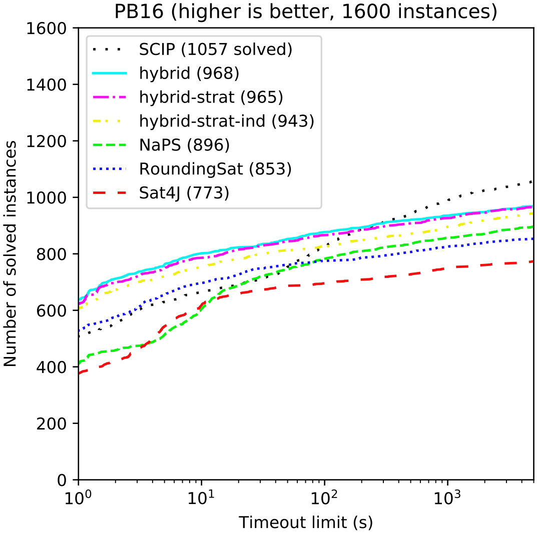
Significant improvement over PB state of the art, but MIP still better

Core-Guided PB Solving for PB16 benchmarks [DGD+21]

Cumulative plot for solver performance on PB16 optimization benchmarks

Also including

- weight stratification (strat)
- disjoint/independent cores (ind)



Implicit Hitting Set (IHS) Algorithm (1/2)

- Minimize $\sum_{i=1}^{n} w_i \ell_i$
- Subject to collection of PB constraints $F = C_1 \land \cdots \land C_m$ (consider clausal constraints)

As in core-guided search, use solving with assumptions, but maintain collection $\mathcal{K}$ of learned core clauses

$$
\begin{aligned}
C_1 &\doteq \ell_{1,1} \lor \ell_{1,2} \lor \cdots \lor \ell_{1,k_s} \\
C_2 &\doteq \ell_{2,1} \lor \ell_{2,2} \lor \cdots \lor \ell_{2,k_s} \\
&\vdots \\
C_s &\doteq \ell_{s,1} \lor \ell_{s,2} \lor \cdots \lor \ell_{s,k_s}
\end{aligned}
$$

Implicit Hitting Set (IHS) Algorithm (2/2)

Set $\mathcal{K} = \emptyset$ and repeat the following:

1. Compute minimum hitting set for $\mathcal{K}$, i.e., $H = \{\ell_i\}$ s.t.
   - $H \cap C \neq \emptyset$ for all $C \in \mathcal{K}$ (H is hitting set)
   - $\sum_{\ell_i \in H} w_i$ minimal among H with this property.
2. Run the solver with assumptions $\{\ell_j = 0 \mid \ell_j \notin H\}$
3. If solver found solution, it must be optimal (since hitting set is optimal), so return solution with value $\sum_{\ell_i \in H} w_i$
4. Otherwise, solver returns new core C_{s+1} — add it to $\mathcal{K}$ and start over from top

More About the Hitting Sets

- Minimality is actually not needed except in the very final step
- Save time by computing "decent" hitting sets earlier on in the search
- How to find hitting set?
- This is itself a pseudo-Boolean optimization problem
  - Run IP solver [standard approach]
  - Or PB solver?
  - Or local search?!

Combine IHS with Pseudo-Boolean Optimization?

IHS and PB Optimization

- In PB setting, cores will not be subsets of clauses but PB constraints $C_1, \ldots, C_s$ over objective function literals
- "Hitting set" H is partial assignment guaranteed to satisfy all constraints $C_1, \ldots, C_s$
- Want to find minimum-cost set H of literals (w.r.t. objective function) with this property

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- Explored by CoReO group in Helsinki in [SBJ21, SBJ22]
- Using RoundingSat version in [DGN21] as pseudo-Boolean "oracle solver"

IHS Algorithm for PB Optimization (Simplified)

- Minimize $\sum_{i=1}^{n} w_i \ell_i$
- Subject to collection of PB constraints $F = C_1 \wedge \cdots \wedge C_m$

Set $\mathcal{K} = \emptyset$ and repeat the following:

1. Run optimization solver to minimize $\sum_{i=1}^{n} w_i \ell_i$ under $\mathcal{K}$, yielding solution ρ to objective variables
2. Run decision solver with assumptions ρ on decision problem F
3. If decision solver returns `SATISFIABLE`, we have found optimal solution extending ρ with value $\sum_{i=1}^{n} w_i \cdot \rho(\ell_i)$
4. Otherwise, solver returns new core C — add it to $\mathcal{K}$ and start over from top

IHS Toy Example (1/2)

1. Given same PB formula F and objective function

$$\min x_1 + 2x_2 + 3x_3 + 4x_4 + 5x_5 + 6x_6$$

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2. For

$$\mathcal{K}_1 = \emptyset$$

optimization solver returns minimal solution
$\rho_1 = \{x_1 = x_2 = \ldots = x_6 = 0\}$

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$$3x_2 + 2x_3 + x_4 + x_5 \geq 4$$

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4. For

$$\mathcal{K}_2 = \{3x_2 + 2x_3 + x_4 + x_5 \geq 4\}$$

optimization solver returns minimal solution
$\rho_2 = \{x_2 = x_3 = 1;\ x_1 = x_4 = \ldots = x_6 = 0\}$

IHS Toy Example (2/2)

5. Decision solver with assumptions ρ_2 returns PB core constraint

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6. For

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7. Decision solver with assumptions ρ_2 returns `SATISFIABLE`

IHS Toy Example (2/2)

5. Decision solver with assumptions ρ_2 returns PB core constraint

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$$\mathcal{K}_3 = \{3x_2 + 2x_3 + x_4 + x_5 \geq 4,\ x_2 + x_4 + x_5 + x_6 \geq 2\}$$

   optimization solver returns minimal solution
   $\rho_3 = \{x_2 = x_4 = 1;\ x_1 = x_3 = x_5 = x_6 = 0\}$
7. Decision solver with assumptions ρ_2 returns SATISFIABLE
8. Hence, we have found an optimal solution with objective value 6 (as for LSU and core-guided search)

Comparison of Core-Guided Search and IHS

Suppose solver with assumptions returns core

$$C \doteq x_1 + x_2 + x_3 + x_4 \geq 2$$

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Core-guided search

- Introduce new variables by $x_1 + x_2 + x_3 + x_4 = 2 + y_3 + y_4$
- Ignore all x_i with smallest weight in objective in next call (get cancelled when objective rewritten)
- Instead assume that "somehow $x_1 + x_2 + x_3 + x_4 \leq 2$ holds" (i.e., $y_3 = 0$)

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IHS

- Add C to collection of cores $\mathcal{K}$
- Find concrete assignment satisfying all of $\mathcal{K}$ as cheaply as possible
- Try that assignment as starting point for next call to decision solver

Competitive Advantages of Core-Guided vs. IHS

- IHS and core-guided approaches for MaxSAT orthogonal [Bac21]
- For MaxSAT problems with many interchangeable soft clauses core-guided seems better (i.e., when it is not important exactly which of these clauses end up in the core)
- For MaxSAT problems with many distinct weights, IHS seems better

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Theoretical relations between IHS and core-guided search?

Provide a more precise theoretical comparison of IHS and core-guided search with simulations and/or separations

(Some theoretical work on related problems in, e.g., [FMSV20, MIB+19])

More Questions Aboute Core-Guided Search and IHS

1. Use assumptions $\{\ell_j = 0 \mid \ell_j \notin H\}$ or add also $\{\ell_i = 1 \mid \ell_i \in H\}$ for pseudo-Boolean IHS? (The latter done in [SBJ21, SBJ22])
2. Use cores in pseudo-Boolean core-guided search for objective reformulation without converting to cardinality constraints first?
3. How to do core minimization/strengthening in a PB setting?
4. Use something other than IP solver for pseudo-Boolean "hitting set problem"?
5. Abstract cores [BBP20] used to get IHS plus core counting variables — is it possible to do full integration of core-guided search and IHS in same solver in meaningful way?
6. Certify correctness using proof logging? [work in progress]

Summing up

- MaxSAT problems can be attacked with combination of powerful tools
  - Core-guided search
  - Implicit hitting set (IHS) search
  - Integer linear programming
- Approaches with complementary strengths — room for synergies?
- Lifting core-guided and IHS search to a pseudo-Boolean setting presents opportunities and challenges
  - No need for CNF re-encoding
  - More powerful pseudo-Boolean reasoning
  - But also slower than clausal reasoning
  - And more degrees of freedom in algorithm design
- Should provide rich pickings of low-hanging fruit!

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Thank you for your attention!

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