Combinatorial Solving with Provably Correct Results

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Based on joint work with Jeremias Berg, Bart Bogaerts, Emir Demirović, Jan Elffers, Ambros Gleixner, Stephan Gocht, Alexander Hoen, Hannes Ihalainen, Matti Järvisalo, Ciaran McCreesh, Matthew McIlree, Magnus O. Myreen, Andy Oertel, Tobias Paxian, Konstantin Sidorov, Yong Kiam Tan, and Dieter Vandesande

The Success of Combinatorial Solving (and the Dirty Little Secret)

- Astounding progress last couple of decades on combinatorial solvers for, e.g.:
  - Boolean satisfiability (SAT) solving and optimization [BHvMW21]
  - Constraint programming [RvBW06]
  - Mixed integer linear programming [AW13, BR07]
  - Satisfiability modulo theories (SMT) solving [BHvMW21]
- Solvers very fast, but sometimes wrong (even best commercial ones) [BLB10, CKSW13, AGJ+18, GSD19, GS19, BMN22, BBN+23]
- Even get feasibility of solutions wrong (though this should be straightforward!)
- When can we trust claims of infeasibility?
- Or that a solution is optimal? (Even off-by-one mistakes can snowball into large errors if solver used as subroutine)

What Can Be Done About Solver Bugs?

- **Software testing**
  Hard to get good test coverage for sophisticated solvers
  Progress using fuzzing and delta debugging [BB09, BLB10, KB22, NPB22, PB23]
  But inherently can only detect presence of bugs, not absence

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  Prove that solver implementation adheres to formal specification
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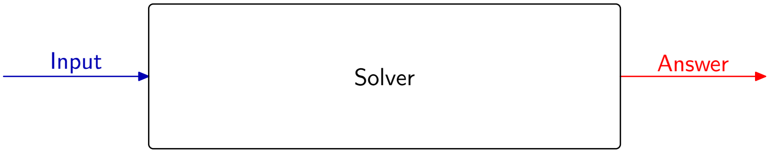
- **Proof logging**
  Make solver certifying [ABM+11, MMNS11] by adding code so that it outputs
  1. not only answer but also
  2. simple, machine-verifiable proof that answer is correct

Proof Logging with Certifying Solvers: Workflow



1. Run combinatorial solving algorithm on problem input

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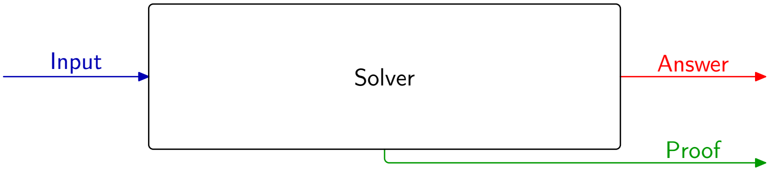

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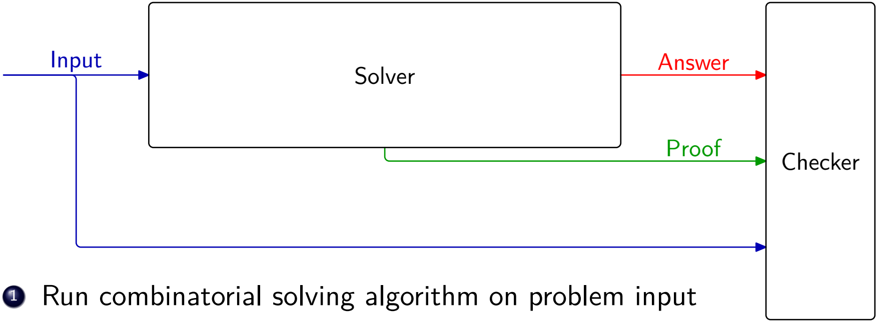

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3. Feed input + answer + proof to proof checker

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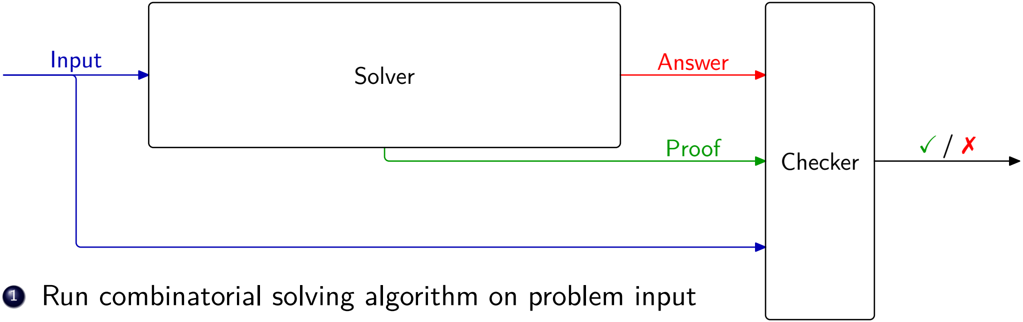

1. Run combinatorial solving algorithm on problem input
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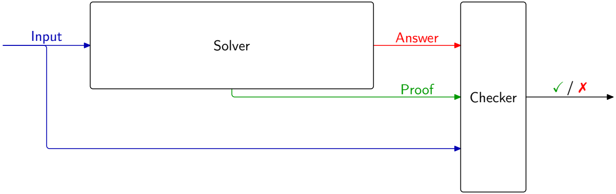
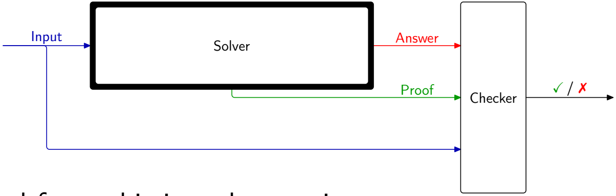
3. Feed input + answer + proof to proof checker
4. Verify that proof checker says answer is correct

Proof Logging Desiderata

Proof format for certifying solver should be



Proof Logging Desiderata



Proof format for certifying solver should be

- **very powerful:** minimal overhead for sophisticated reasoning

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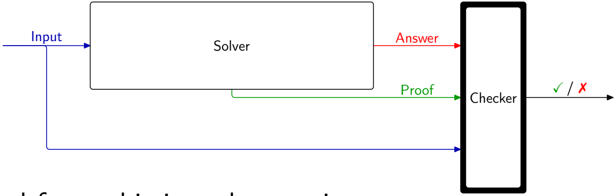

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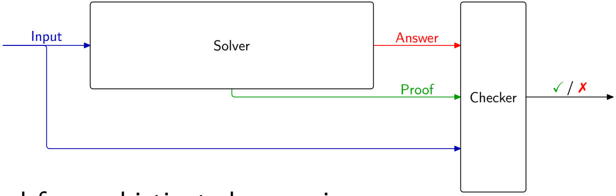

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Clear conflict expressivity vs. simplicity!

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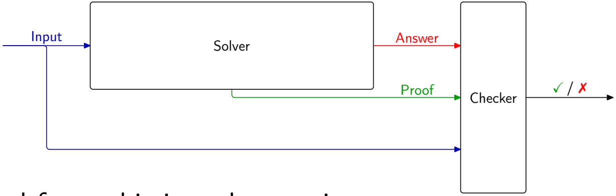

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Asking for both perhaps a little bit too good to be true?

Some Previous Proof Logging Work

Boolean satisfiability (SAT) solving

- Well established since over decade with several proof formats such as
  - DRAT [HHW13a, HHW13b, WHH14]
  - GRIT [CMS17]
  - LRAT [CHH+17]
- But no efficient support for most advanced techniques such as Gaussian elimination and symmetry breaking

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Constraint programming

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Mixed integer linear programming

- Work on proof format VIPR [CGS17, EG23]
- But only for exact solving and without support for advanced techniques

Message of This Talk

Proof logging for combinatorial optimization is possible with single, unified method!

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- Build on successes in proof logging for SAT solving
- But represent constraints as 0–1 integer linear inequalities
- Formalize reasoning using cutting planes [CCT87] proof system
- Add well-chosen strengthening rules [Goc22, GN21, BGMN23]
- Implemented in VeriPB (`https://gitlab.com/MIAOresearch/software/VeriPB`)

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1. Marketing pitch ☺

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2. Overview of what solving paradigms have been covered so far (including SAT solving, constraint programming, and 0-1 ILP presolving)
3. Discuss future challenges and directions

The Sales Pitch For Proof Logging

1. Certifies correctness of computed results
2. Detects errors even if due to compiler bugs, hardware failures, or cosmic rays
3. Provides debugging support during software development [GMM+20, KM21, BBN+23, EG23]
4. Facilitates performance analysis
5. Helps identify potential for further improvements
6. Enables auditability
7. Serves as stepping stone towards explainability

Design Principles for Proof Logging

Proof logging implementation

- Don't change solver
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Performance goals

- Proof logging overhead small constant fraction of running time ($\lessapprox 10\%$)
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Proof system

- Keep language simple — no XOR constraints, CP propagators, symmetries, . . .
- Reason about such notions using power of proof system
- Combine proof logging with formally verified proof checker

Proof Language: Pseudo-Boolean Constraints

Proof consists of 0-1 integer linear inequalities or pseudo-Boolean constraints:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
- variables x_i take values $0 = \mathit{false}$ or $1 = \mathit{true}$

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Sometimes convenient to use normalized form [Bar95] with all a_i, A positive (without loss of generality)

Some Types of Pseudo-Boolean Constraints

1. Disjunctive clauses

$$x \vee \overline{y} \vee z \quad \Leftrightarrow \quad x + \overline{y} + z \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Pseudo-Boolean Proof Logging Wishlist

Paradigms

- Boolean satisfiability (SAT) solving
- (linear) pseudo-Boolean solving
- graph solving
- constraint programming
- automated planning
- mixed integer linear programming
- SMT solving

Problem types

- decision / feasibility
- optimization
- multi-objective optimization
- projected model enumeration
- projected model counting
- preprocessing / problem reformulation

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Pseudo-Boolean Proof Logging — How and Why?

If problem is (special case of) 0-1 integer linear program

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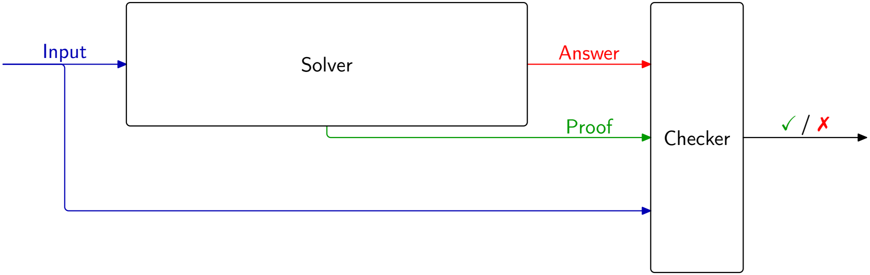
Goldilocks compromise between expressivity and simplicity:

1. 0-1 ILP expressive formalism for combinatorial problems (including objective)
2. Powerful reasoning capturing many combinatorial arguments

Proof Logging with Formally Verified Checking: Full Workflow



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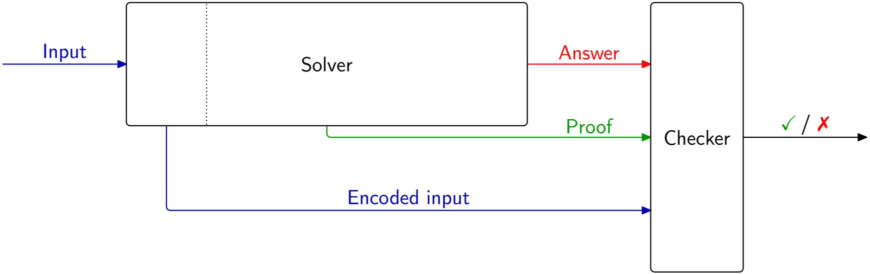

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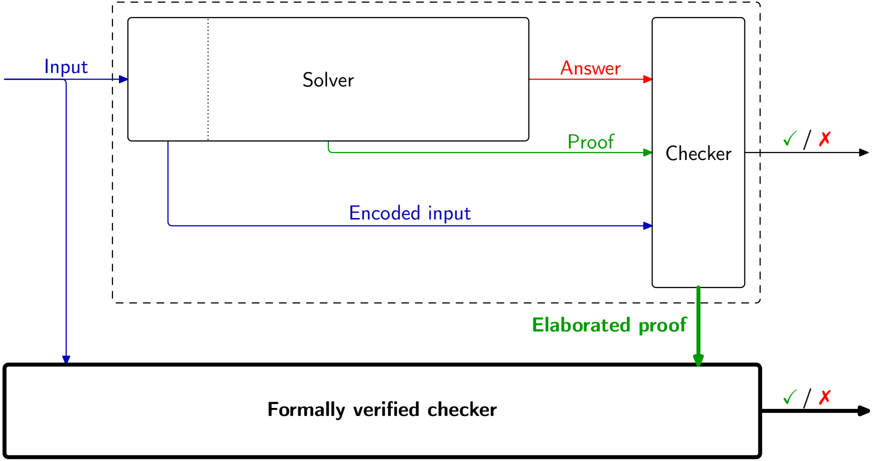

VeriPB Proof Structure

1. **Preamble**
   Load input formula
   Specify settings
2. **Derivation section**
   Derivations of new constraints
   Logging of solutions
3. **Output section**
   Listing of constraints currently in database
   Input to next stage (or for debugging)
4. **Conclusions section**
   Specification of what was established
   - satisfiability / unsatisfiability
   - optimality (or upper and lower bounds)
   - other types of conclusions

VeriPB Proof Configuration (Slightly Simplified)

Core set $\mathcal{C}$

- Contains input formula at the start
- Maintains "equivalence" with input formula

Derived set $\mathcal{D}$

- All constraints derived during search
- Also intermediate constraints used in proof logging [but not used by solver]

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Objective $f = \sum_i w_i \ell_i + k$

- 0–1 linear function to minimize
- Or $f = 0$ for decision problem
- Keep track of best known bound; initialize to ∞

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Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input axioms From the input

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Addition

$$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i)\ell_i \geq A + B}$$

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Multiplication for any $c \in \mathbb{N}^+$

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i c a_i \ell_i \geq cA}$$

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(constraint in normalized form)

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Saturation
(constraint in normalized form)

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \min(a_i, A) \cdot \ell_i \geq A}$$

Cutting Planes Toy Example

$$w + 2x + y \geq 2$$

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$$\text{Multiply by } 2 \quad \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

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$$w + 2x + 4y + 2z \geq 5$$

Cutting Planes Toy Example

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\text{Multiply by } 2 \; \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5
$$

$$
\text{Add} \; \dfrac{2w + 4x + 2y \geq 4 \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}
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\text{Add}\ \dfrac{\text{Multiply by } 2\ \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5 \qquad \overline{z} \geq 0}{3w + 6x + 6y + 2z \geq 9}
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Multiply by 2
$$\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

$w + 2x + 4y + 2z \geq 5$

Add
$$\dfrac{2w + 4x + 2y \geq 4 \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}$$

$$\dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}$$
Multiply by 2

Add
$$\dfrac{3w + 6x + 6y + 2z \geq 9 \qquad 2\overline{z} \geq 0}{3w + 6x + 6y + 2 \qquad \geq 9}$$

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By naming constraints by integers and literal axioms by the literal involved as

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such a calculation is written in the proof log in reverse Polish notation as

```
pol 1 2 * 2 + ~z 2 * + 3 d
```

Redundance-Based Strengthening

C is said to be "redundant" with respect to F if F and $F \cup \{C\}$ are equisatisfiable

Want to allow adding such "redundant" constraints

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Redundance-based strengthening ([BT19, GN21], inspired by [JHB12])

C is redundant with respect to F if and only if there is a substitution ω (mapping variables to truth values or literals), called a witness, for which

$$F \cup \{\neg C\} \models (F \cup \{C\})\restriction_{\omega}$$

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C is redundant with respect to F if and only if there is a substitution ω (mapping variables to truth values or literals), called a witness, for which

$$F \cup \{\neg C\} \models (F \cup \{C\})\restriction_\omega$$

- Proof sketch for interesting direction: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \cup \{C\}$

Redundance-Based Strengthening

C is said to be "redundant" with respect to F if F and $F \cup \{C\}$ are equisatisfiable
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- Proof sketch for interesting direction: If α satisfies F but falsifies C, then $\alpha \circ \omega$ satisfies $F \cup \{C\}$
- In a proof, the implication needs to be efficiently verifiable — every $D \in (F \cup \{C\})\restriction_{\omega}$ should follow from $F \cup \{\neg C\}$ either
  1. "obviously" or
  2. by explicitly presented derivation

Redundance and Dominance Rules in VeriPB (Slightly Simplified)

Redundance-based strengthening, optimization version [BGMN23]

Add constraint C to derived set $\mathcal{D}$ if exists witness substitution ω such that

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- Applying ω should strictly decrease f
- If so, don't need to show that $(\mathcal{D} \cup \{C\})\restriction_\omega$ implied!

Soundness of Dominance Rule

Dominance-based strengthening

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Why is this sound? Let $\mathcal{D} = \emptyset$ for simplicity

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7. ...
8. Can't go on forever, so finally reach α' satisfying $\mathcal{C} \cup \{C\}$

Soundness of Dominance Rule (Continued)

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- Same inductive proof as before, but also nested forward induction over derivation
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Further extensions:

- Define dominance rule with respect to order $\mathcal{O}$ independent of objective function
- Switch between different orders in same proof
- See [BGMN23] for details

Three Pseudo-Boolean Proof Logging Vignettes

1. Symmetry breaking [BGMN23]
2. Graph solving (subgraph isomorphism) [GMN20, GMM^{+}20, GMM^{+}24]
3. Constraint programming [EGMN20, GMN22, MM23, MMN24]

Symmetry Breaking in SAT Solving

1. Pretend to solve optimisation problem minimizing $f \doteq \sum_{i=1}^{n} 2^{n-i} \cdot x_i$ (search for lexicographically smallest assignment satisfying formula)

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$$f \leq f\restriction_\sigma \quad \doteq \quad \sum_{i=1}^{n} 2^{n-i} \cdot (\sigma(x_i) - x_i) \geq 0$$

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3. Derive symmetry breaking clauses from this PB constraint:

$$y_0 \qquad\qquad \overline{y}_j \vee \overline{\sigma(x_j)} \vee x_j$$

$$\overline{y}_{j-1} \vee \overline{x}_j \vee \sigma(x_j) \qquad\qquad y_j \vee \overline{y}_{j-1} \vee \overline{x}_j$$

$$\overline{y}_j \vee y_{j-1} \qquad\qquad y_j \vee \overline{y}_{j-1} \vee \sigma(x_j)$$

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$$y_0 \geq 1 \qquad \overline{y}_j + \overline{\sigma(x_j)} + x_j \geq 1$$

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VeriPB can certify fully general SAT symmetry breaking [BGMN23]

The Subgraph Isomorphism Problem

Input

- Pattern graph $\mathcal{P}$ with vertices $V(\mathcal{P}) = \{a, b, c, \ldots\}$
- Target graph $\mathcal{T}$ with vertices $V(\mathcal{T}) = \{u, v, w, \ldots\}$

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Task

- Find all subgraph isomorphisms $\varphi : V(\mathcal{P}) \to V(\mathcal{T})$
- I.e., if
  1. $\varphi(a) = u$
  2. $\varphi(b) = v$
  3. $(a, b) \in E(\mathcal{P})$

  then must have $(u, v) \in E(\mathcal{T})$

Pseudo-Boolean Proof Logging for Subgraph Isomorphism Solving

All reasoning steps in Glasgow Subgraph Solver [ADH$^+$19, GSS] can be formalized efficiently in the cutting planes proof system [GMN20]

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Means that

1. Solver can justify each step by writing local formal derivation
2. Local derivations can be chained into global correctness proof
3. Proof checkable by stand-alone verifier that knows nothing about graphs
4. With end-to-end fully formally verified result [GMM+24]

Subgraph Isomorphism as a Pseudo-Boolean Formula

- Pattern graph $\mathcal{P}$ with $V(\mathcal{P}) = \{a, b, c, \ldots\}$
- Target graph $\mathcal{T}$ with $V(\mathcal{T}) = \{u, v, w, \ldots\}$
- No loops (for simplicity)

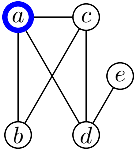
Pseudo-Boolean encoding

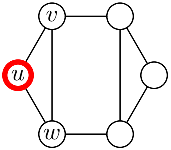
$$\sum_{v \in V(\mathcal{T})} x_{a,v} = 1 \qquad \text{[every } a \text{ maps somewhere]}$$

$$\sum_{b \in V(\mathcal{P})} \overline{x}_{b,u} \geq |V(\mathcal{P})| - 1 \qquad \text{[mapping is one-to-one]}$$

$$\overline{x}_{a,u} + \sum_{v \in N(u)} x_{b,v} \geq 1 \qquad \text{[edge } (a,b) \text{ maps to edge } (u,v)\text{]}$$

Pseudo-Boolean Proof Logging Example: Degree Preprocessing





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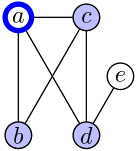

$$\overline{x}_{a,u} + x_{b,v} + x_{b,w} \geq 1$$
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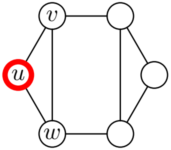

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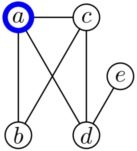

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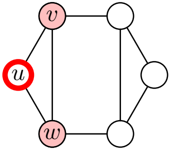

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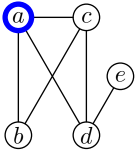

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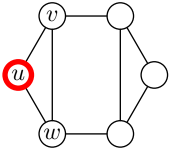

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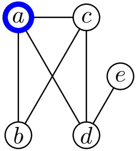

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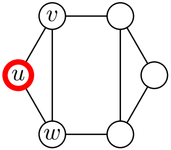

Sum up all constraints & divide by 3 to obtain

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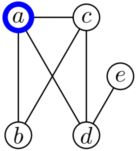

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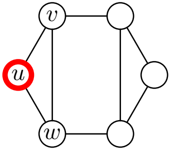
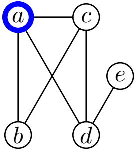


Sum up all constraints & divide by 3 to obtain

$$3\overline{x}_{a,u} + 10 \geq 11$$

Pseudo-Boolean Proof Logging Example: Degree Preprocessing



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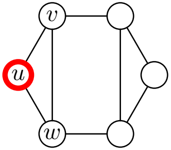

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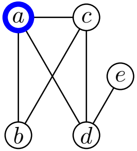
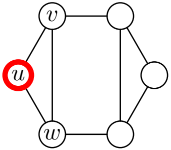
$$3\overline{x}_{a,u} \geq 1$$

Pseudo-Boolean Proof Logging Example: Degree Preprocessing



$$
\begin{aligned}
\overline{x}_{a,u} + x_{b,v} + x_{b,w} &\geq 1 \\
\overline{x}_{a,u} + x_{c,v} + x_{c,w} &\geq 1 \\
\overline{x}_{a,u} + x_{d,v} + x_{d,w} &\geq 1 \\
\overline{x}_{a,v} + \overline{x}_{b,v} + \overline{x}_{c,v} + \overline{x}_{d,v} + \overline{x}_{e,v} &\geq 4 \\
\overline{x}_{a,w} + \overline{x}_{b,w} + \overline{x}_{c,w} + \overline{x}_{d,w} + \overline{x}_{e,w} &\geq 4 \\
x_{a,v} &\geq 0 \\
x_{a,w} &\geq 0 \\
x_{e,v} &\geq 0 \\
x_{e,w} &\geq 0
\end{aligned}
$$



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$$
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$$

Integer Variables in Constraint Programming (1/2)

How to deal with integer variables?

Given $A \in \{-3 \ldots 9\}$, the direct encoding is:

$$\begin{aligned} a_{=-3} + a_{=-2} + a_{=-1} + a_{=0} + a_{=1} + a_{=2} + a_{=3} \\ + a_{=4} + a_{=5} + a_{=6} + a_{=7} + a_{=8} + a_{=9} = 1 \end{aligned}$$

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This doesn't work for large domains. . .

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This doesn't work for large domains...

We can instead use a binary encoding:

$$\begin{aligned} -16a_{\mathrm{neg}} + 1a_{\mathrm{b0}} + 2a_{\mathrm{b1}} + 4a_{\mathrm{b2}} + 8a_{\mathrm{b3}} &\geq -3 \qquad \text{and} \\ 16a_{\mathrm{neg}} + -1a_{\mathrm{b0}} + -2a_{\mathrm{b1}} + -4a_{\mathrm{b2}} + -8a_{\mathrm{b3}} &\geq -9 \end{aligned}$$

Doesn't propagate much, but that isn't a problem for proof logging

Integer Variables in Constraint Programming (2/2)

We can mix binary and order encodings! Define big-M linear inequalities encoding

$$a_{\geq 4} \Leftrightarrow -16a_{\text{neg}} + 1a_{\text{b0}} + 2a_{\text{b1}} + 4a_{\text{b2}} + 8a_{\text{b3}} \geq 4$$

$$a_{\geq 5} \Leftrightarrow -16a_{\text{neg}} + 1a_{\text{b0}} + 2a_{\text{b1}} + 4a_{\text{b2}} + 8a_{\text{b3}} \geq 5$$

$$a_{=4} \Leftrightarrow a_{\geq 4} \land \overline{a}_{\geq 5}$$

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$$a_{=4} \Leftrightarrow a_{\geq 4} \wedge \overline{a}_{\geq 5}$$

When creating $a_{\geq i}$, also introduce pseudo-Boolean constraints encoding

$$a_{\geq i} \Rightarrow a_{\geq j} \quad \text{and} \quad a_{\geq h} \Rightarrow a_{\geq i}$$

for the closest values $j < i < h$ that already exist

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We can do this:

- Inside the pseudo-Boolean model where needed
- Otherwise lazily during proof logging

Table Constraints

Constraints can be specified extensionally as list of feasible tuples, called a table

Variable assignments must match some row in table

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Given table constraint

$$(A, B, C) \in [(1,2,3),(1,3,4),(2,2,5)]$$

define

$$3\bar{t}_1 + a_{=1} + b_{=2} + c_{=3} \geq 3 \qquad \text{i.e.,} \quad t_1 \Rightarrow (a_{=1} \wedge b_{=2} \wedge c_{=3})$$
$$3\bar{t}_2 + a_{=1} + b_{=4} + c_{=4} \geq 3 \qquad \text{i.e.,} \quad t_2 \Rightarrow (a_{=1} \wedge b_{=4} \wedge c_{=4})$$
$$3\bar{t}_3 + a_{=2} + b_{=2} + c_{=5} \geq 3 \qquad \text{i.e.,} \quad t_3 \Rightarrow (a_{=2} \wedge b_{=2} \wedge c_{=5})$$

using tuple selector variables

$$t_1 + t_2 + t_3 = 1$$

A Constraint Programming Solver with Pseudo-Boolean Proof Logging

Proof-of-concept constraint programming solver at

`https://github.com/ciaranm/glasgow-constraint-solver`

Supports proof logging for global constraints including:

- All-different
- Integer linear inequality (including for very large domains)
- Smart table and regular
- Minimum / maximum of an array
- Element (kind of array indexing)
- Absolute value
- (Hamiltonian) Circuit

Details in [EGMN20, GMN22, MM23, MMN24]

Future Research Directions

Performance of pseudo-Boolean proof logging and checking with VeriPB

- Trim proof while verifying (as in DRAT-TRIM [HHW13a])
- Compress proof file using binary format
- More careful software engineering in proof checker (such as faster propagation)

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Proof logging for other combinatorial problems and techniques

- Model enumeration and counting
- SMT solving *(work on solvers* CVC5, SMTINTERPOL, Z3, . . . *[BBC+23, HS22])*
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And more. . .

- Use proof logs for algorithm analysis or explainability purposes
- Lots of other challenging problems and interesting ideas
- We're hiring! Talk to me to join the pseudo-Boolean proof logging revolution! ☺

VeriPB Documentation

VeriPB tutorial at *CP '22* [BMN22]

- video at `youtu.be/s_5BIi4I22w`
- updated slides for *IJCAI '23* tutorial [BMN23]

Description of VeriPB and CakePB [BMM^{+}23] for SAT 2023 competition

- Available at `satcompetition.github.io/2023/checkers.html`

Specific details on different proof logging techniques covered in research papers [EGMN20, GMN20, GMM^{+}20, GN21, GMN22, GMNO22, VDB22, BBN^{+}23, BGMN23, MM23, BBN^{+}24, DMM^{+}24, GMM^{+}24, HOGN24, IOT^{+}24, MMN24]

Lots of concrete example files at `gitlab.com/MIAOresearch/software/VeriPB`

Summing up

- Combinatorial solving and optimization is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
- Cutting planes reasoning with pseudo-Boolean constraints seems to hit a sweet spot between simplicity and expressivity
- **Action point:** What problems can VeriPB solve for you? ☺

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Thank you for your attention!

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