Complexity Theory for Real-World Computation

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University of Copenhagen and Lund University

Universidade Federal de Minas Gerais
Belo Horizonte, Brazil
September 19, 2025

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Three Simple Problems...

Colouring

Does the graph $G = (V, E)$ have a colouring with k colours such that all neighbours have distinct colours?

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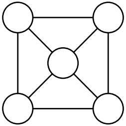

3-colouring?

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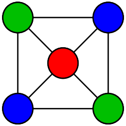

3-colouring? Yes

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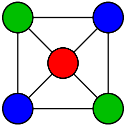


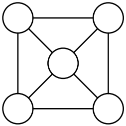
3-colouring? Yes, but no 2-colouring

Three Simple Problems. . .

Clique

Is there a clique in the graph $G = (V, E)$ with k vertices that are all pairwise connected by edges in E?

Three Simple Problems. . .



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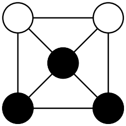

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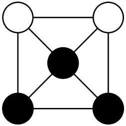

3-clique? Yes, but no 4-clique

Clique

Is there a clique in the graph $G = (V, E)$ with k vertices that are all pairwise connected by edges in E?

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SAT

Given propositional logic formula, is there a satisfying assignment?

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Sat

Given propositional logic formula, is there a satisfying assignment?

$$(x \lor z) \land (y \lor \neg z) \land (x \lor \neg y \lor u) \land (\neg y \lor \neg u)$$
$$\land\ (u \lor v) \land (\neg x \lor \neg v) \land (\neg u \lor w) \land (\neg x \lor \neg u \lor \neg w)$$

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- Variables should be set to **true** or **false**
- Constraint $(x \lor \neg y \lor z)$: means x or z should be true or y false
- $\land$ means all constraints should hold simultaneously
- Is there a truth value assignment satisfying all constraints?

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COLOURING:	frequency allocation for mobile base stations
CLIQUE:	bioinformatics, computational chemistry
SAT:	easily models these and many other problems

. . . with Huge Practical Implications

- Some more examples of problems that can be encoded as propositional logic formulas:
  - computer hardware verification
  - computer software testing
  - artificial intelligence
  - operations research
  - cryptography
  - bioinformatics
  - et cetera. . .
- Leads to **humongous** formulas (100,000s or even 1,000,000s of variables)
- Can we use computers to solve these problems efficiently?

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  - COLOURING [Kho01, Zuc07]
  - CLIQUE [Hås99]
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  and amazing conflict-driven clause learning (CDCL) solvers [BS97, MS99, MMZ^{+}01] that solve huge SAT formulas

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How can we understand real-world algorithms for NP-hard problems?

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How can we understand real-world algorithms for NP-hard problems?

This talk: Use proof complexity (not only conceivable answer)

Algorithmic View of Proof Complexity

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Study infeasible problems — proof of feasibility easy

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Question 2: Separate talk — lots of recent exciting progress; mostly negative (worst-case) results, e.g., [AM20, GKMP20, dRGN$^+$21]

Applications of Proof Complexity

Three applied reasons for proof complexity:

1. Understand real-world applied algorithmic paradigms [**this talk**]
2. Get ideas for algorithmic improvements (e.g., [EN18, EN20, LBD$^+$20, DGD$^+$21, DGN21, KBBN22])
3. Enhance algorithms to write machine-verifiable certificates of correctness (e.g., [EGMN20, GMN20, GMM$^+$20, GN21, GMN22, GMNO22, BBN$^+$23, BGMN23, MM23, BBN$^+$24, DMM$^+$24, GMM$^+$24, HOGN24, IOT$^+$24, MMN24, DHN$^+$25, KLM$^+$25, MM25])

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Or just view this as a convenient excuse to study nice computational complexity problems for their own sake. . . ☺

Outline

1. Conflict-Driven Clause Learning and Resolution
   - The SATISFIABILITY Problem in Different Shapes
   - Conflict-Driven Clause Learning (CDCL)
   - Resolution Proof System
2. Algebraic and Semi-algebraic Methods
   - Nullstellensatz
   - Gröbner Bases and Polynomial Calculus
   - Cutting Planes and Pseudo-Boolean Solving
3. Some Proof Systems We Won't Have Time for
   - Sherali-Adams and Sums of Squares
   - Stabbing Planes
   - Extended Resolution

Formal Description of SAT Problem

- Variable x: takes value **true** $(= 1)$ or **false** $(= 0)$
- Literal ℓ: variable x or its negation $\overline{x}$ (write $\overline{x}$ instead of $\neg x$)
- Clause $C = \ell_1 \vee \cdots \vee \ell_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- Conjunctive normal form (CNF) formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses

The SATISFIABILITY (or just SAT) Problem

Given a CNF formula F, is it satisfiable?

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The SATISFIABILITY (or just SAT) Problem

Given a CNF formula F, is it satisfiable?

Here is our example formula again:

$$(x \vee z) \wedge (y \vee \overline{z}) \wedge (x \vee \overline{y} \vee u) \wedge (\overline{y} \vee \overline{u})$$
$$\wedge\ (u \vee v) \wedge (\overline{x} \vee \overline{v}) \wedge (\overline{u} \vee w) \wedge (\overline{x} \vee \overline{u} \vee \overline{w})$$

The Same Problem in Three Different Shapes

$$(x \lor z) \land (y \lor \neg z) \land (x \lor \neg y \lor u) \land (\neg y \lor \neg u)$$
$$\land\ (u \lor v) \land (\neg x \lor \neg v) \land (\neg u \lor w) \land (\neg x \lor \neg u \lor \neg w)$$

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$$\begin{aligned}
(1-x)(1-z) &= 0 \\
(1-y)z &= 0 \\
(1-x)y(1-u) &= 0 \\
yu &= 0 \\
(1-u)(1-v) &= 0 \\
xv &= 0 \\
u(1-w) &= 0 \\
xuw &= 0
\end{aligned}$$

For **true** $= 1$ and **false** $= 0$, is there a $\{0,1\}$-valued solution?

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$$\begin{aligned}
1 - x - z + xz &= 0 \\
z - yz &= 0 \\
y - xy - yu + xyu &= 0 \\
yu &= 0 \\
1 - u - v + uv &= 0 \\
xv &= 0 \\
u - uw &= 0 \\
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\end{aligned}$$

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$1 - x - z + xz = 0$	$x + z \geq 1$
$z - yz = 0$	$y + (1 - z) \geq 1$
$y - xy - yu + xyu = 0$	$x + (1 - y) + u \geq 1$
$yu = 0$	$(1 - y) + (1 - u) \geq 1$
$1 - u - v + uv = 0$	$u + v \geq 1$
$xv = 0$	$(1 - x) + (1 - v) \geq 1$
$u - uw = 0$	$(1 - u) + w \geq 1$
$xuw = 0$	$(1 - x) + (1 - u) + (1 - w) \geq 1$

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$1 - x - z + xz = 0$	$x + z \geq 1$
$z - yz = 0$	$y - z \geq 0$
$y - xy - yu + xyu = 0$	$x - y + u \geq 0$
$yu = 0$	$-y - u \geq -1$
$1 - u - v + uv = 0$	$u + v \geq 1$
$xv = 0$	$-x - v \geq -1$
$u - uw = 0$	$-u + w \geq 0$
$xuw = 0$	$-x - u - w \geq -2$

For **true** $= 1$ and **false** $= 0$, is there a $\{0,1\}$-valued solution?

State-of-the-Art SAT Solving in One Slide

High-level description of modern conflict-driven clause learning (CDCL) SAT solving (as pioneered in [BS97, MS99, MMZ$^+$01]):

- Try to build satisfying assignment for formula (branching or decision heuristic crucial)
- When partial assignment violates formula, compute explanation for conflict and add to formula as new clause (clause learning)
- Every once in a while, restart from beginning (but save computed info)

Conflict-Driven Clause Learning (CDCL) by Example

Two kinds of assignments — illustrate on example formula:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

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Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

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Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

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Always propagate if possible, else decide

Add to assignment trail

Until satisfying assignment or conflict

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$\boxed{p \overset{\mathsf{d}}{=} 0}$

$u \overset{p \lor \overline{u}}{=} 0$

$\boxed{q \overset{\mathsf{d}}{=} 0}$

$r \overset{q \lor r}{=} 1$

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Two kinds of assignments — illustrate on example formula:

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, else decide

Add to assignment trail

Until satisfying assignment or conflict

Conflict-Driven Clause Learning (CDCL) by Example

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$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$p \overset{\mathsf{d}}{=} 0$

$u \overset{p \vee \overline{u}}{=} 0$

$q \overset{\mathsf{d}}{=} 0$

$r \overset{q \vee r}{=} 1$

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$x \overset{\mathsf{d}}{=} 0$

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$\overset{\overline{y} \vee \overline{z}}{\bot}$

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$\boxed{p \stackrel{\mathsf{d}}{=} 0}$

$u \stackrel{p \lor \overline{u}}{=} 0$

decision level 1

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$

$r \stackrel{q \lor r}{=} 1$

$w \stackrel{\overline{r} \lor w}{=} 1$

decision level 2

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$

$y \stackrel{u \lor x \lor y}{=} 1$

$z \stackrel{x \lor \overline{y} \lor z}{=} 1$

$\stackrel{\overline{y} \lor \overline{z}}{\bot}$

decision level 3

Decision

Free choice to assign value to variable

Notation $p \stackrel{\mathsf{d}}{=} 0$

Unit propagation

Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \lor \overline{u}$ forces $u = 0$

Notation $u \stackrel{p \lor \overline{u}}{=} 0$ ($p \lor \overline{u}$ is reason clause)

Always propagate if possible, else decide

Add to assignment trail

Until satisfying assignment or conflict

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$$\left.\begin{array}{c} \boxed{p \overset{\mathsf{d}}{=} 0} \\ u \overset{p \lor \overline{u}}{=} 0 \end{array}\right\} \text{decision level } 1$$

$$\left.\begin{array}{c} \boxed{q \overset{\mathsf{d}}{=} 0} \\ r \overset{q \lor r}{=} 1 \\ w \overset{\overline{r} \lor w}{=} 1 \end{array}\right\} \text{decision level } 2$$

$$\left.\begin{array}{c} \boxed{x \overset{\mathsf{d}}{=} 0} \\ y \overset{u \lor x \lor y}{=} 1 \\ z \overset{x \lor \overline{y} \lor z}{=} 1 \\ \overset{\overline{y} \lor \overline{z}}{\bot} \end{array}\right\} \text{decision level } 3$$

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$
$u \stackrel{p \vee \overline{u}}{=} 0$
decision level 1

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$
$r \stackrel{q \vee r}{=} 1$
$w \stackrel{\overline{r} \vee w}{=} 1$
decision level 2

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$
$y \stackrel{u \vee x \vee y}{=} 1$
$z \stackrel{x \vee \overline{y} \vee z}{=} 1$
$\stackrel{\overline{y} \vee \overline{z}}{\bot}$
decision level 3

Could backtrack by removing last decision level & flipping last decision

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$

$\boxed{p \stackrel{\mathsf{d}}{=} 0}$
$u \stackrel{p \lor \overline{u}}{=} 0$ } decision level 1

$\boxed{q \stackrel{\mathsf{d}}{=} 0}$
$r \stackrel{q \lor r}{=} 1$
$w \stackrel{\overline{r} \lor w}{=} 1$ } decision level 2

$\boxed{x \stackrel{\mathsf{d}}{=} 0}$
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$\stackrel{\overline{y} \lor \overline{z}}{\bot}$ } decision level 3

Could backtrack by removing last decision level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

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$x \stackrel{\mathsf{d}}{=} 0$

$y \stackrel{u \vee x \vee y}{=} 1$

$z \stackrel{x \vee \overline{y} \vee z}{=} 1$

$\stackrel{\overline{y} \vee \overline{z}}{\bot}$

$x \vee \overline{y}$

Could backtrack by removing last decision level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

Case analysis over z for last two clauses:

- $x \vee \overline{y} \vee z$ wants $z = 1$
- $\overline{y} \vee \overline{z}$ wants $z = 0$
- Resolve clauses by merging them & removing z — must satisfy $x \vee \overline{y}$

Conflict Analysis

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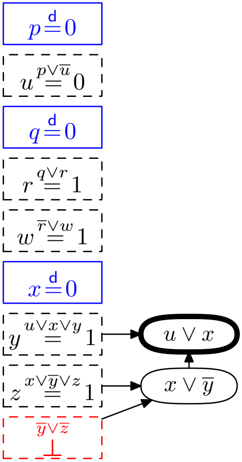
$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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Case analysis over z for last two clauses:

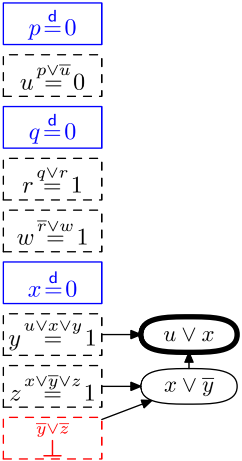
- $x \lor \overline{y} \lor z$ wants $z = 1$
- $\overline{y} \lor \overline{z}$ wants $z = 0$
- Resolve clauses by merging them & removing z — must satisfy $x \lor \overline{y}$

Repeat until UIP clause with only 1 variable after last decision — learn and backjump

Complete Toy Example for CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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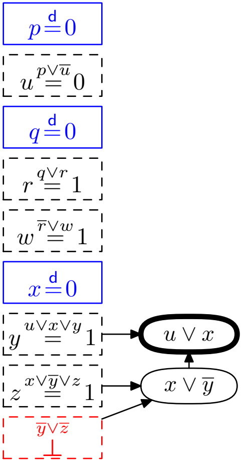
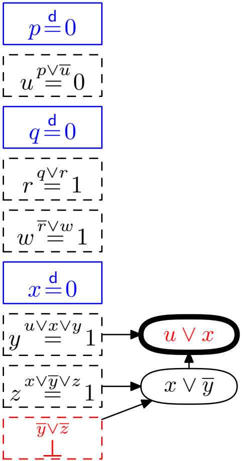


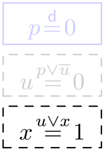
Assertion level 1 (max for non-UIP literal in learned clause) — keep trail to that level

Complete Toy Example for CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$





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Now UIP literal guaranteed to flip (assert) — but this is a propagation, not a decision

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$p \stackrel{\mathsf{d}}{=} 0$

$u \stackrel{p \vee \overline{u}}{=} 0$

$q \stackrel{\mathsf{d}}{=} 0$

$r \stackrel{q \vee r}{=} 1$

$w \stackrel{\overline{r} \vee w}{=} 1$

$x \stackrel{\mathsf{d}}{=} 0$

$y \stackrel{u \vee x \vee y}{=} 1$ → $u \vee x$

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Then continue as before...

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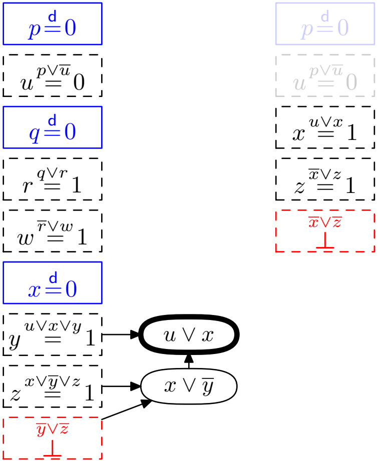
$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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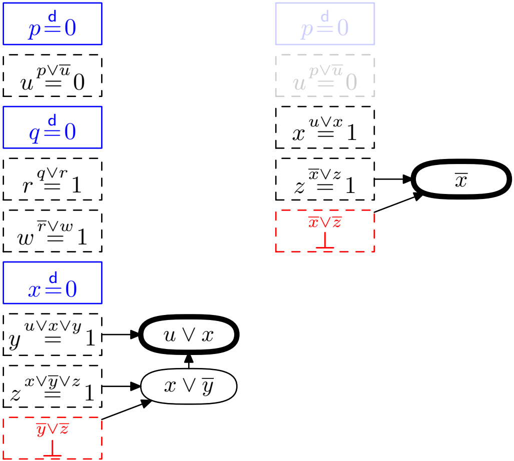

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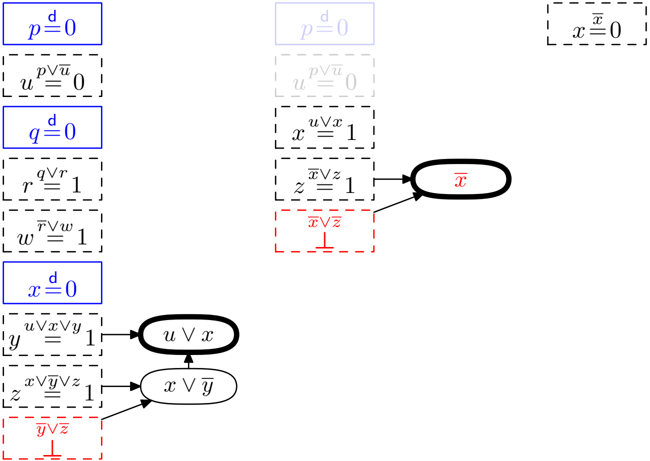

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$\stackrel{\overline{y} \vee \overline{z}}{\perp}$

$p \stackrel{\mathsf{d}}{=} 0$

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$x \stackrel{u \vee x}{=} 1$

$z \stackrel{\overline{x} \vee z}{=} 1$ → $\overline{x}$

$\stackrel{\overline{x} \vee \overline{z}}{\perp}$

$x \stackrel{\overline{x}}{=} 0$

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Complete Toy Example for CDCL Execution

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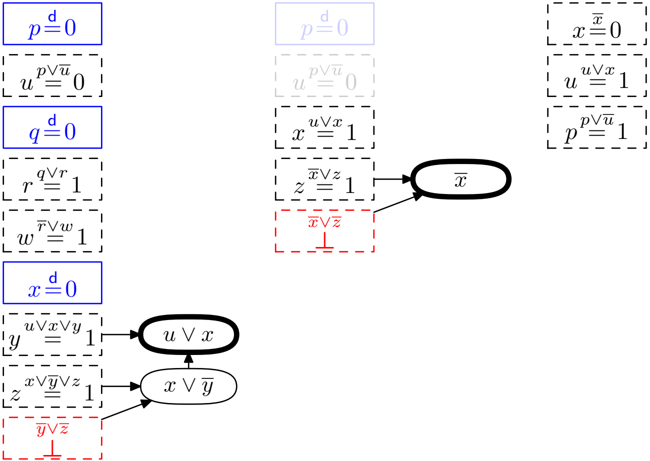
$$(p \lor \overline{u}) \land (q \lor r) \land (\overline{r} \lor w) \land (u \lor x \lor y) \land (x \lor \overline{y} \lor z) \land (\overline{x} \lor z) \land (\overline{y} \lor \overline{z}) \land (\overline{x} \lor \overline{z}) \land (\overline{p} \lor \overline{u})$$



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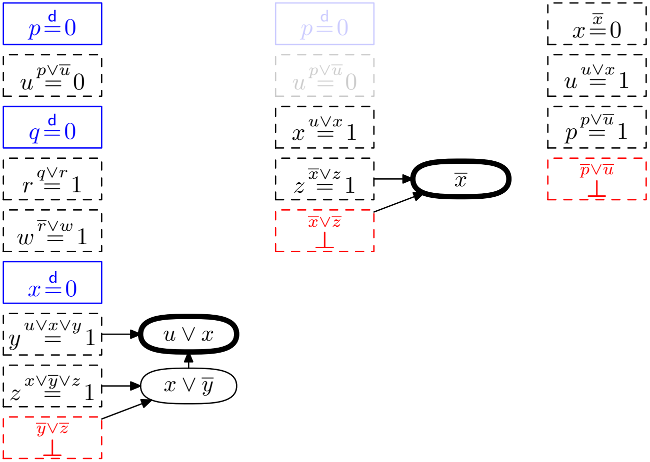

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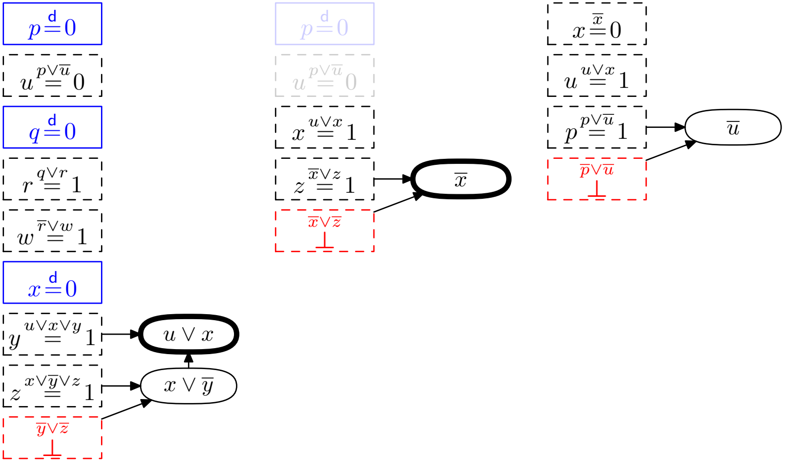

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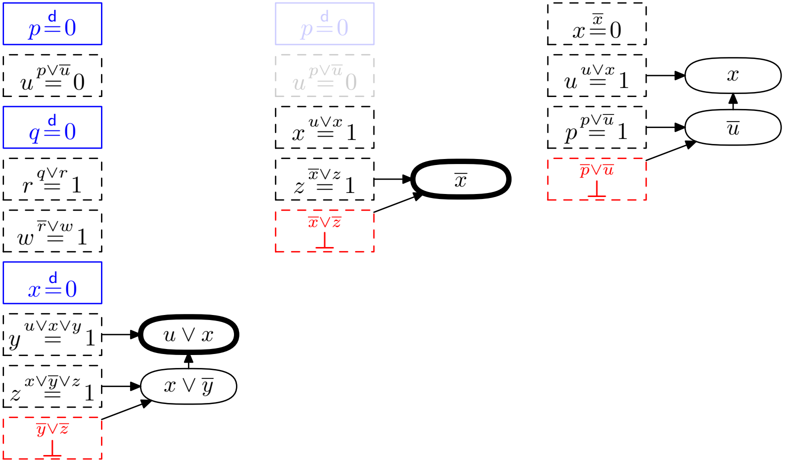

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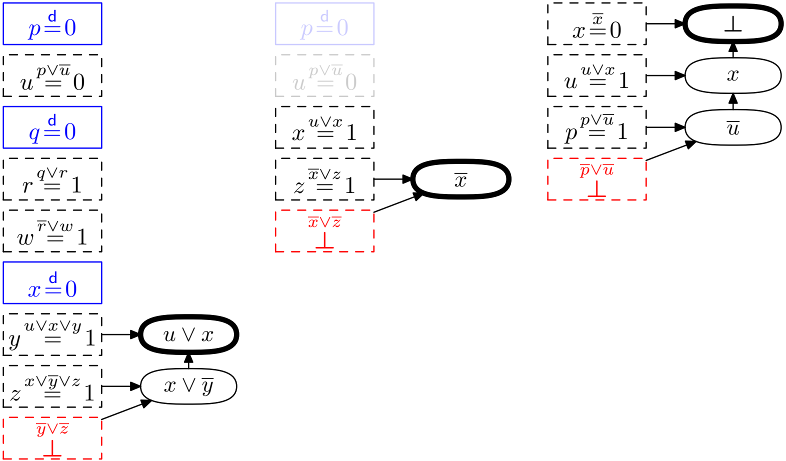

SAT Solver Analysis and the Resolution Proof System

How to make rigorous analysis of SAT solver performance?
Many intricate, hard-to-understand heuristics
So focus instead on underlying method of reasoning

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Resolution proof system [Bla37, Rob65]

- Start with clauses of CNF formula (axioms)
- Derive new clauses by resolution rule

$$\frac{C_1 \lor x \qquad C_2 \lor \overline{x}}{C_1 \lor C_2}$$

Resolution Proofs by Contradction

Resolution rule:

$$\frac{C_1 \lor x \qquad C_2 \lor \overline{x}}{C_1 \lor C_2}$$

Observation

If F is a satisfiable CNF formula and D is derived from clauses $D_1, D_2 \in F$ by the resolution rule, then $F \land D$ is satisfiable.

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Observation

If F is a satisfiable CNF formula and D is derived from clauses $D_1, D_2 \in F$ by the resolution rule, then $F \land D$ is satisfiable.

So can prove F unsatisfiable by deriving the unsatisfiable empty clause (denoted $\perp$) from F by resolution

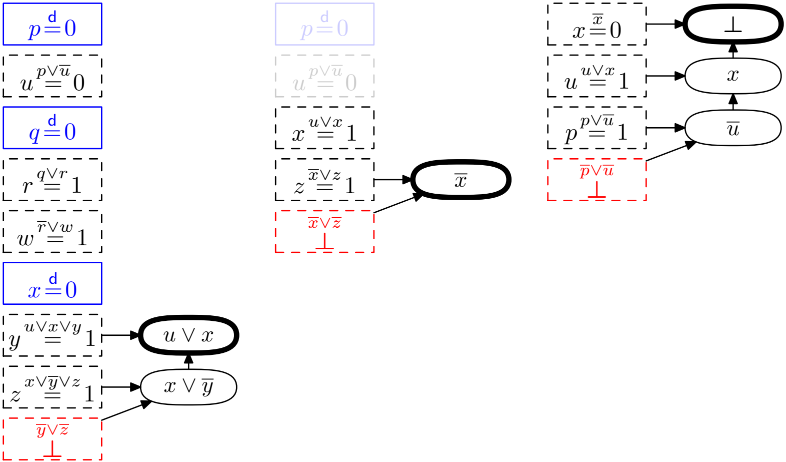
Such proof by contradiction also called resolution refutation

CDCL and Resolution Proofs

Obtain resolution proof...

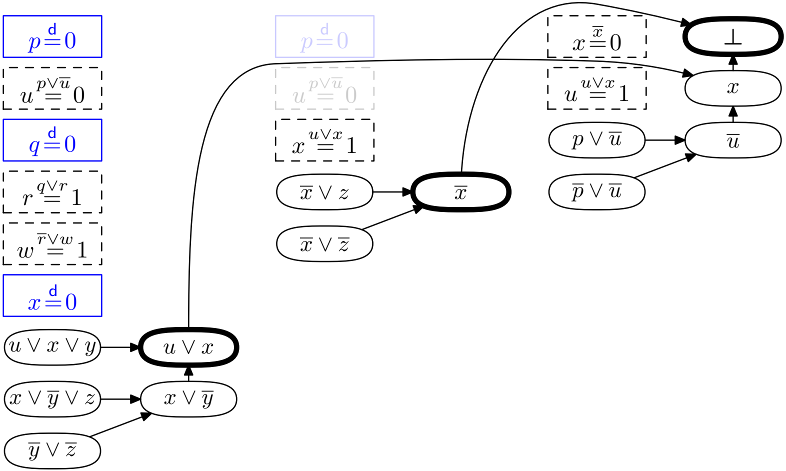
CDCL and Resolution Proofs

Obtain resolution proof from our example CDCL execution. . .



CDCL and Resolution Proofs

Obtain resolution proof from our example CDCL execution by stringing together conflict analyses:



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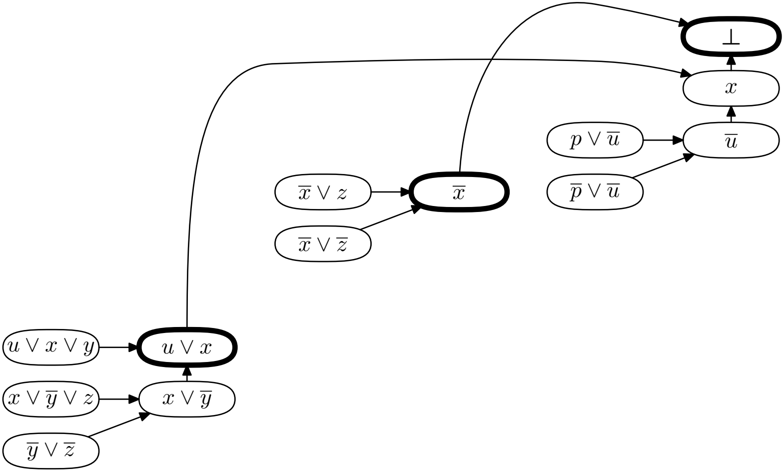

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- Hence, lower bounds on resolution proof size ⇒ lower bounds on CDCL running time
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(*) Except for some preprocessing techniques, which is an important omission, but this gets complicated and we don't have time to go into details...

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- State-of-the-art CDCL solvers often perform amazingly well ("SAT is easy in practice")
- Very poor theoretical understanding:
  - Why do heuristics work?
  - Why are applied instances easy?

Current State of Affairs in SAT Solving

- State-of-the-art CDCL solvers often perform amazingly well ("SAT is easy in practice")
- Very poor theoretical understanding:
  - Why do heuristics work?
  - Why are applied instances easy?
- Paradox: resolution quite weak proof system; many strong proof complexity lower bounds for (seemingly) "obvious" formulas

Examples of Hard Formulas For Resolution (1/3)

Pigeonhole principle (PHP) formulas [Hak85]

"$n + 1$ pigeons don't fit into n holes"

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Variables $p_{i,j}$ = "pigeon $i \rightarrow$ hole j"; $1 \leq i \leq n+1$; $1 \leq j \leq n$

$p_{i,1} \lor p_{i,2} \lor \cdots \lor p_{i,n}$	every pigeon i gets a hole
$\overline{p}_{i,j} \lor \overline{p}_{i',j}$	no hole j gets two pigeons $i \neq i'$

Can also add "functionality" and "onto" axioms

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Even onto functional PHP hard — **"resolution cannot count"**

Resolution proof requires $\exp(\Omega(n)) = \exp\left(\Omega\left(\sqrt[3]{N}\right)\right)$ clauses
(measured in terms of formula size N)

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Tseitin formulas [Urq87]

"Sum of degrees of vertices in graph is even"

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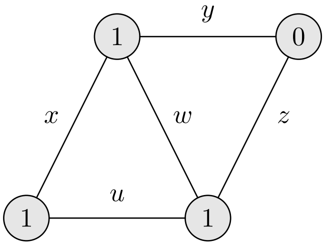

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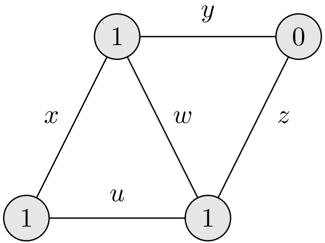

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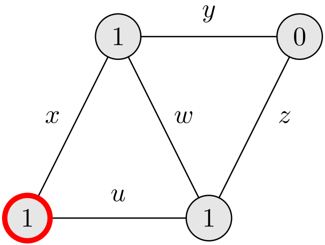

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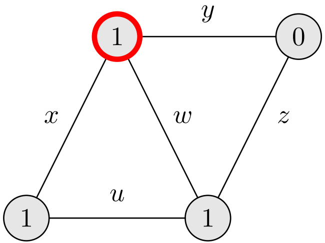

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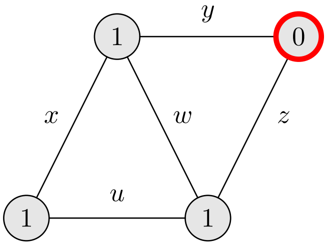

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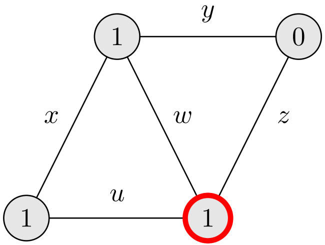

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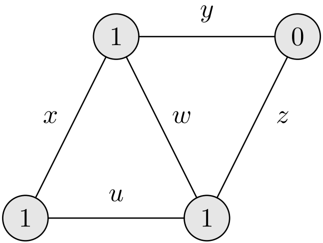

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Requires proof size $\exp(\Omega(N))$ on well-connected so-called expander graphs — **"resolution cannot count** $\mathrm{mod}\ 2$**"**

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Random k-CNF formulas [CS88]

Δn randomly sampled k-clauses over n variables

($\Delta \gtrsim 4.5$ sufficient to get unsatisfiable 3-CNF almost surely)

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And more...

- COLOURING [BCMM05]
- Zero-one designs [Spe10, VS10, MN14]
- Et cetera... (See, e.g., [BN21] for overview)

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But not CLIQUE!

- Refuting existence of k-clique should require proof size $n^{\Omega(k)}$
- Only known for restricted so-called regular resolution [ABdR+21]

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- Given CNF formula $F = \bigwedge_{i=1}^{m} C_i$

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to polynomial equations

$$\prod_{i \in \mathcal{P}} (1 - x_i) \cdot \prod_{j \in \mathcal{N}} x_j = 0$$

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- Add Boolean axioms

$$x_j^2 - x_j = 0$$

for all variables

Hilbert's Nullstellensatz

Consider any system of polynomial equations

$$
\begin{aligned}
p_1(x_1, \ldots, x_n) &= 0 \qquad & x_1^2 - x_1 &= 0 \\
p_2(x_1, \ldots, x_n) &= 0 \qquad & x_2^2 - x_2 &= 0 \\
&\vdots & &\vdots \\
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in polynomial ring over field $\mathbb{F}$

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Hilbert's Nullstellensatz

System infeasible $\Leftrightarrow$ exist $q_i, r_j \in \mathbb{F}[x_1,\ldots,x_n]$ such that

$$\sum_{i=1}^{m} q_i(x_1,\ldots,x_n) \cdot p_i(x_1,\ldots,x_n) + \sum_{j=1}^{n} r_j(x_1,\ldots,x_n) \cdot (x_j^2 - x_j) = 1$$

Nullstellensatz Proof System [BIK+94]

Nullstellensatz refutation of

$$p_i(x_1, \ldots, x_n) = 0 \qquad i \in [m]$$
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Complexity measures of refutations:

- Size: number of monomials (when all polynomials expanded out)
- Degree: highest total degree of any polynomial

Nullstellensatz Example (Not Expanded out)

$$(x \lor z) \land (y \lor \neg z) \land (x \lor \neg y \lor u) \land (\neg y \lor \neg u)$$
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$$(1-x)(1-z)$$
$$(1-y)z$$
$$(1-x)y(1-u)$$
$$yu$$
$$(1-u)(1-v)$$
$$xv$$
$$u(1-w)$$
$$xuw$$

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$$\begin{aligned}
& (1-y) \cdot (1-x)(1-z) \\
+ \ & (1-x) \cdot (1-y)z \\
+ \ & 1 \cdot (1-x)y(1-u) \\
+ \ & (1-x) \cdot yu \\
+ \ & x \cdot (1-u)(1-v) \\
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Size 27
Degree 3

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Size 27
Degree 3
(No use of Boolean axioms)

Nullstellensatz Proof Search

- Solve linear system of equations with coefficients of polynomials q_i, r_j as unknowns
- Used successfully to solve, e.g., graph colouring problems [DLMM08, DLMO09, DLMM11]
- Running time grows exponentially with degree, though high-degree refutations can be very small [BCIP02, dRMNR21]

Dual Variables

- Annoying problem: $x_1 \lor x_2 \lor x_3$ translates to polynomial

$$(1-x_1)(1-x_2)(1-x_3) = 1-x_1-x_2-x_3+x_1x_2+x_1x_3+x_2x_3-x_1x_2x_3$$

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- Fix: introduce dual variables x'_i and axioms $x_i + x'_i - 1 = 0$
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- Doesn't affect degree (obviously), but can decrease size exponentially [dRLNS21] (also for other algebraic proof systems)

Dynamic Construction of Nullstellensatz Certificates

Nullstellensatz again

Infeasibility of

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$$\Updownarrow$$

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- Ideal $\mathcal{I}$:
  1. $p, q \in \mathcal{I} \Rightarrow p + q \in \mathcal{I}$
  2. $p \in \mathcal{I} \Rightarrow r \cdot q \in \mathcal{I}$ for any r
- Compute polynomials in this ideal $\mathcal{I}$ step by step
- Use "multivariate division" to check whether 1 lies in ideal or not

Gröbner Bases: Admissible Orderings and Leading Terms

Admissible ordering $\preceq$ on monomials m, m', t:

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2. $m \preceq t \cdot m$

Examples:

- Lexicographic
- Degree-lexicographic

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"Multivariate division": Reduce p modulo all q in set of polynomials $\mathcal{G}$ until no further reductions possible

$\mathcal{G}$ is a Gröbner basis if final result uniquely determined

Gröbner Bases: Buchberger's Algorithm

Buchberger's algorithm for computing Gröbner bases (very rough)

1. Let $\mathcal{G} :=$ all axioms
2. Pick unprocessed pair $p, q \in \mathcal{G}$ or terminate if none exists
3. Compute $p' = t_p \cdot p$ and $q' = t_q \cdot q$ to make leading terms cancel
4. Set $S := p' - q'$; reduce $S \bmod \mathcal{G}$ with multivariate division; add result to $\mathcal{G}$ if non-zero
5. Go to 2

Gröbner Bases: Buchberger's Algorithm

Buchberger's algorithm for computing Gröbner bases (**very** rough)

1. Let $\mathcal{G} :=$ all axioms
2. Pick unprocessed pair $p, q \in \mathcal{G}$ or terminate if none exists
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Facts:

- Buchberger's algorithm computes Gröbner basis
- At termination, $1 \in \mathcal{G} \Leftrightarrow$ polynomial equations infeasible

Polynomial Calculus [CEI96, ABRW02]

- Compute polynomials in ideal $\mathcal{I}$ generated by p_i, $x_j^2 - x_j$, and $x_j + x'_j - 1$ step by step:
  - $p_i \in \mathcal{I}$, $x_j^2 - x_j \in \mathcal{I}$, and $x_j + x'_j - 1 \in \mathcal{I}$ (axioms)
  - If $p, q \in \mathcal{I}$, then $\alpha p + \beta q \in \mathcal{I}$ for any $\alpha, \beta \in \mathbb{F}$ (linear combination)
  - If $p \in \mathcal{I}$, then $m \cdot p \in \mathcal{I}$ for any monomial $m = \prod_j x_j$ (multiplication)

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  - If $p \in \mathcal{I}$, then $m \cdot p \in \mathcal{I}$ for any monomial $m = \prod_j x_j$ (multiplication)
- A refutation is a derivation ending with the polynomial 1
- Complexity measures:
  - Size: total number of monomials in all polynomials in derivation expanded out
  - Degree: highest total degree of any polynomial
- Polynomial calculus (much) stronger than Nullstellensatz w.r.t. both size and degree

Polynomial Calculus Can Simulate Resolution

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simulated by polynomial calculus derivation

$$\dfrac{x'yz' \qquad \dfrac{\dfrac{yz}{x'yz} \qquad \dfrac{z + z' - 1}{x'yz + x'yz' - x'y}}{-x'yz' + x'y}}{x'y}$$

Polynomial Calculus is Strictly Stronger than Resolution

Polynomial calculus can be exponentially stronger than resolution

For instance:

- Tseitin formulas on expander graphs if $\mathbb{F} = \mathrm{GF}(2)$
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Other hard formulas:

- Tseitin-like formulas for counting $\mathrm{mod}\ p$ if $p \neq$ field characteristic [BGIP01]
- Random k-CNF formulas
  - all characteristics except 2 [BI99]
  - all characteristics [AR03]

COLOURING and CLIQUE for Polynomial Calculus

COLOURING

- Exponential worst-case lower bounds in [LN17]
- Exponential average-case lower bounds in [CdRN+23]

CLIQUE

Essentially nothing known!

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- Use dual variables! [KBBN22]

Gröbner bases: Some Problems and Questions

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   But are immediately eliminated by multivariate division
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3. Analysis of polynomial calculus uses degree-lexicographic ordering
   In computational algebra, many other orderings used
   Prove proof complexity separation results for different orderings?

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to 0-1 integer linear inequalities

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- Add variable axioms

$$\begin{aligned} x_j &\geq 0 \\ -x_j &\geq -1 \end{aligned}$$

for all variables

Cutting Planes Proof System [CCT87]

Cutting planes introduced in [CCT87] to model integer linear programming algorithm in [Gom63, Chv73]

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Cutting planes derivation rules

Multiplication $$\frac{\sum a_i x_i \geq A}{\sum c a_i x_i \geq cA} \quad c \in \mathbb{N}^+$$

Addition $$\frac{\sum a_i x_i \geq A \qquad \sum b_i x_i \geq B}{\sum (a_i + b_i) x_i \geq A + B}$$

Division $$\frac{\sum a_i x_i \geq A}{\sum \lceil a_i / c \rceil x_i \geq \lceil A/c \rceil} \quad c \in \mathbb{N}^+$$

Cutting Planes Derivations and Refutations

- A cutting planes derivation is a sequence of 0-1 integer linear inequalities derived using
  - Axioms (input inequalities and variable bounds)
  - Derivation rules applied to previous inequalities
- A refutation ends with the inequality $0 \geq 1$
- Complexity measures:
  - Length: $\#$ inequalities
  - Size: Count also bit size of representing all coefficients

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  (e.g., for PHP, just count and argue that #pigeons > #holes)
- And 0-1 linear inequalities are similar to but much more concise than CNF

Compare

$$x_1 + x_2 + x_3 + x_4 + x_5 + x_6 \geq 3$$

and

$$\begin{aligned}
&(x_1 \lor x_2 \lor x_3 \lor x_4) \land (x_1 \lor x_2 \lor x_3 \lor x_5) \land (x_1 \lor x_2 \lor x_3 \lor x_6) \\
\land &(x_1 \lor x_2 \lor x_4 \lor x_5) \land (x_1 \lor x_2 \lor x_4 \lor x_6) \land (x_1 \lor x_2 \lor x_5 \lor x_6) \\
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\end{aligned}$$

Hard Formulas for Cutting Planes

Clique-colouring formulas [Pud97]
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Variables

- $p_{i,j}$ indicators of the edges in graph; $1 \leq i < j \leq n$
- $q_{k,i}$ identify members of m-clique; $1 \leq k \leq m$, $1 \leq i \leq n$
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$q_{k,1} \vee q_{k,2} \vee \cdots \vee q_{k,n}$	some vertex is the kth member of clique
$\overline{q}_{k,i} \vee \overline{q}_{k',i}$	clique members are uniquely defined $(k \neq k')$
$p_{i,j} \vee \overline{q}_{k,i} \vee \overline{q}_{k',j}$	clique members are connected by edges
$r_{i,1} \vee r_{i,2} \vee \cdots \vee r_{i,m-1}$	every vertex i has a colour
$\overline{p}_{i,j} \vee \overline{r}_{i,\ell} \vee \overline{r}_{j,\ell}$	neighbours have distinct colours

More Hard Formulas for Cutting Planes?

Lower bound for clique-colouring formulas uses interpolation and circuit complexity

- From small cutting planes proof, build small circuit of special type that can decide whether graph has clique
- Prove separately that no such small circuits can exist
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Some more recent developments in
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Nothing known for COLOURING or CLIQUE
Surprisingly, Tseitin formulas are at most quasi-polynomially hard for cutting planes [DT20]!

SAT Solvers Based on Cutting Planes?

So-called pseudo-Boolean (PB) solvers using (subset of) cutting planes reasoning developed in, e.g., [CK05, SS06, LP10, EN18]

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Challenge 1: Conjunctive normal form

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- Solvers can rewrite CNF to more helpful 0-1 linear inequalities [BLLM14, EN20], but this doesn't work so well in practice
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Is it truly harder to build good pseudo-Boolean solvers?
Or has just so much more work has been put into CDCL solvers?

Division Versus Saturation

Normalized form: Use negated literals as needed to make positive linear combination with all a_i, $A > 0$

Boolean derivation rules for 0–1 integer linear inequalities

$$\text{Division} \quad \frac{\sum a_i \ell_i \geq A}{\sum \lceil a_i / c \rceil \ell_i \geq \lceil A/c \rceil} \quad c \in \mathbb{N}^+$$

$$\text{Saturation} \quad \frac{\sum a_i \ell_i \geq A}{\sum \min\{a_i, A\} \cdot \ell_i \geq A}$$

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- ... And most often also in practice [EN18]

Sherali-Adams (SA) and Sums of Squares (SoS)

Refutation of $p_i \in \mathbb{R}[x_1, \ldots, x_n]$, $i \in [m]$, and $x_j^2 - x_j$, $j \in [n]$

Nullstellensatz

$$\sum_{i=1}^{m} q_i \cdot p_i + \sum_{j=1}^{n} r_j \cdot (x_j^2 - x_j) = 1$$

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Sums of squares (SoS) $(s_k \in \mathbb{R}[x_1, \ldots, x_n])$

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SA, SoS, and Other Proof Systems

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Sums of squares is strictly stronger than polynomial calculus (over $\mathbb{R}$) while Sherali-Adams and polynomial calculus are incomparable [Ber18]

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Sums of squares very strong proof system, except it cannot do parity reasoning efficiently [GV01, Gri01]

Survey [FKP19] is recommended for more reading

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Intended to model modern 0-1 integer linear programming

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Stabbing planes with polynomial-size coefficient can be simulated by cutting planes with quasi-polynomial overhead [DT20, FGI+21]

Extended Resolution [Tse68]

Resolution rule

$$\frac{C_1 \lor x \qquad C_2 \lor \overline{x}}{C_1 \lor C_2}$$

Extension rule introducing clauses

$$a \lor \overline{x} \lor \overline{y} \qquad \overline{a} \lor x \qquad \overline{a} \lor y$$

for fresh variable a (encoding that $a \leftrightarrow (x \land y)$ must hold)

Extended Resolution and SAT Solving

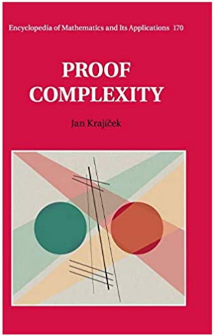
- Closely related (and equivalent) to *DRAT* system used to justify correctness of some SAT preprocessing techniques [JHB12]
- *DRAT* also used for SAT solver proof logging
- Attempts to combine extended resolution with CDCL in, e.g., [AKS10, Hua10]
- Without restrictions, corresponds to extremely strong extended Frege system [CR79] — pretty much no lower bounds known
- To analyse solvers using extended resolution, would need to:
    - Describe heuristics/rules actually used
    - See if possible to reason about such restricted proof system

Some More References for Further Reading

Handbook of Satisfiability
(Especially chapter 7 ☺)

[BHvMW21]

Proof Complexity
by Jan Krajíček



[Kra19]

Summing up This Presentation

Overview of some proof systems used in combinatorial solving:

- Resolution $\longleftrightarrow$ Conflict-driven clause learning
- Nullstellensatz and polynomial calculus $\longleftrightarrow$ Gröbner bases
- Cutting planes $\longleftrightarrow$ pseudo-Boolean solving

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Very brief (or non-existent) discussion of some other proof systems:

- Sherali-Adams
- Sums of squares
- Stabbing planes
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- Give ideas for new approaches
- Be a fun playground for theory-practice interaction!

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Thank you for your attention!

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