The Joy of Pseudo-Boolean Proof Logging

Jakob Nordström

University of Copenhagen and Lund University

ACP Summer School 2026
CP: Explanation, Verification, and Applications
Changchun, China
August 24–28, 2026

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Or: An Introduction to VeriPB

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Brief Presentation

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University of Copenhagen
and Lund University
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Research interests:

- Computational complexity theory
- Applied combinatorial optimization
- Certifying algorithms / proof logging

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Research interests:

- *Proving exponential lower bounds for* NP*-hard problems*
- *Designing algorithms to solve* NP*-hard problems in (ideally) linear time*
- *Generating machine-verifiable proofs that such algorithms reason correctly*

Tutorial Based on Research Contributions By…

- Berhan Oumer Adame
- Markus Anders
- Jeremias Berg
- Bart Bogaerts
- Benjamin Bogø
- Wolf De Wulf
- Emir Demirović
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Combinatorial Solving and Optimisation

- Revolution last couple of decades in combinatorial solvers for
  - Boolean satisfiability (SAT) solving [BHvMW21][1]
  - Constraint programming (CP) [RvBW06]
  - Mixed integer linear programming (MIP) [AW13, BR07]
- Solve NP-complete problems (or worse) very successfully in practice!
- Except solvers are sometimes wrong… (Even best commercial ones) [BLB10, CKSW13, AGJ+18, GSD19, BMN22, GCS23]
- Solvers can propose infeasible "solutions" (but such errors can be detected)
- More challenging: How to achieve reliable claims of infeasibility?
- Or that a solution is optimal? (Even off-by-one mistakes can snowball into large errors if solver used as subroutine)

[1] See end of slides for all references with bibliographic details

What Can Be Done About Solver Bugs?

- **Software testing**
  Very useful, but bugs slip through even with careful domain-specific testing
  Progress using fuzzing and delta debugging [BB09, BLB10, KB22, NPB22, PB23]
  But testing inherently can only detect presence of bugs, not absence

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  Prove that solver implementation adheres to formal specification
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  (Despite valiant efforts in, e.g., [Fle20])

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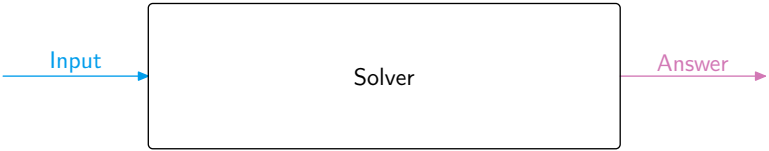
  (Despite valiant efforts in, e.g., [Fle20])
- **Proof logging**
  Make solver certifying [ABM+11, MMNS11] by adding code so that it outputs
  1. not only answer but also
  2. simple, machine-verifiable proof that answer is correct

Proof Logging with Certifying Solvers: Workflow



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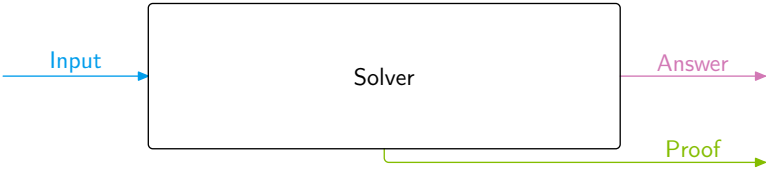

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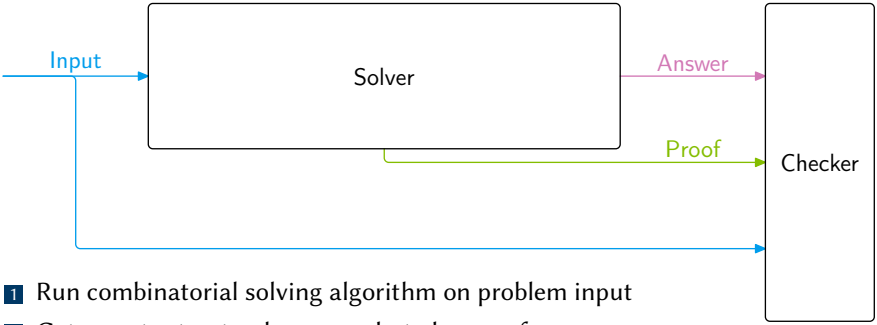

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3. Feed input + answer + proof to proof checker

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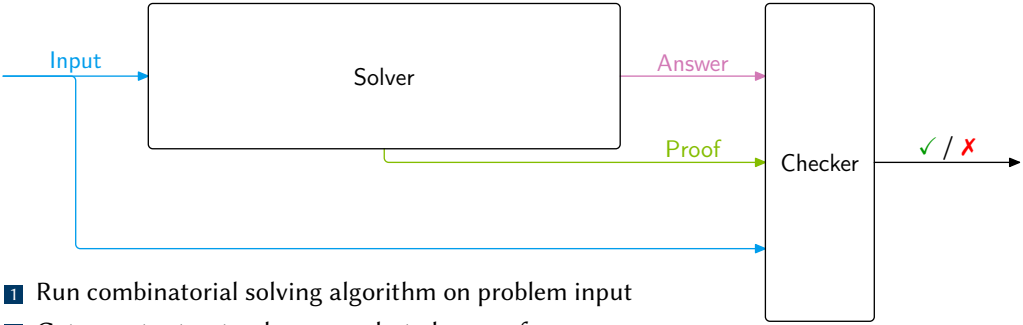

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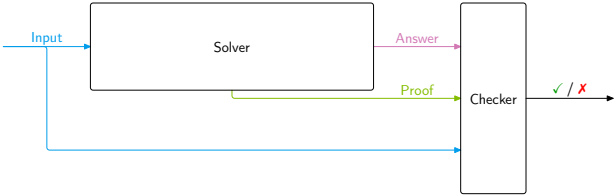
2. Get as output not only answer but also proof
3. Feed input + answer + proof to proof checker
4. Verify that proof checker says answer is correct

Proof Logging Desiderata

Proof format for certifying solver should be



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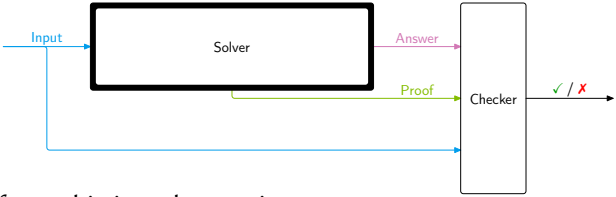

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- **very powerful:** minimal overhead for sophisticated reasoning

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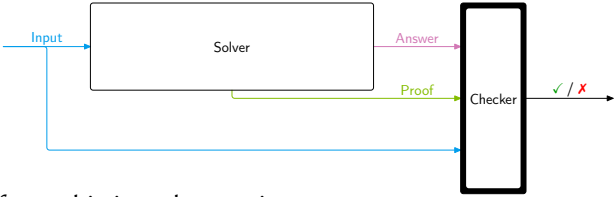

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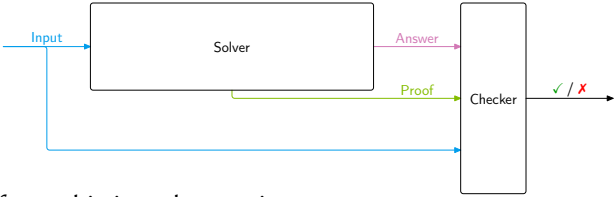

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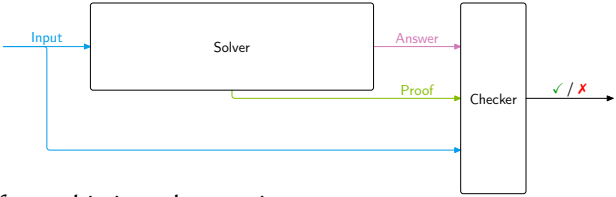

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Clear conflict expressivity vs. simplicity!

Asking for both perhaps a little bit too good to be true?

Some Previous Proof Logging Work

Boolean satisfiability (SAT) solving

- Well established since over decade with several proof formats such as DRAT [HHW13a, HHW13b, WHH14], GRIT [CMS17], LRAT [CHH+17], …
- But no efficient support for most advanced techniques such as
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  - symmetry breaking
- And only for decision problems, not optimization (MaxSAT)

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Constraint programming

- Either have to trust that propagations done correctly [DFS12, OSC09, VS10]
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Mixed integer linear programming

- Work on proof format VIPR [CGS17, EG23]
- But only for exact solving and without support for advanced techniques

Take-Away Message from This Tutorial

Proof logging for sophisticated combinatorial solvers is possible!

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Proof logging for sophisticated combinatorial solvers is possible!

- With single, unified method!
- Producing proofs in simple format calculating with 0–1 integer linear inequalities (a.k.a. pseudo-Boolean constraints)
- Implemented in VeriPB (`https://VeriPB.org`)

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Purpose of this presentation:

1. Marketing pitch ☺

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These slides with references are online at `https://jakobnordstrom.se/presentations/`

Software for Exercises Tomorrow Thursday

Kindling Solver: See `README.md` and files in `slides-proofs/`

https://github.com/ciaranm/kindling

VeriPB (official version): See `README.md` and try to run `veripb justified.opb justified.pbp`

https://gitlab.com/MIAOresearch/software/VeriPB

VeriPB (experimental): Interactive mode (Linux and Mac only?); see `README.md` and then use `cargo run -p veripb-repl -m path-to-veripb-repl/Cargo.toml`

https://github.com/ciaranm/veripb-repl

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7. Serves as stepping stone towards explainability

Design Principles for Proof Logging

Proof logging implementation

- Don't change solver
- Just add proof logging print statements (plus some book-keeping) to solver code

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Performance goals

- Proof logging overhead small constant fraction of running time ($\lessapprox 10\%$)
- Proof checking time within constant factor of solving time (current aim $\lessapprox \times 10$)

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Proof system

- Keep language simple — no XOR constraints, CP propagators, symmetry groups, …
- But reason efficiently about such notions using power of proof system
- Combine proof logging with formally verified proof checker

But Let Us Start from the Beginning…

Review of some basic concepts:

- Satisfiability (SAT) problem
- Unit propagation
- DPLL and CDCL algorithms
- Resolution proof system

The Satisfiability (SAT) Problem

- Variable x: takes value **true** ($=1$) or **false** ($=0$)
- Literal ℓ: variable x or its negation $\overline{x}$
- Clause $C = \ell_1 \vee \cdots \vee \ell_k$: disjunction of literals
  (Consider as sets, so no repetitions and order irrelevant)
- Conjunctive normal form (CNF) formula $F = C_1 \wedge \cdots \wedge C_m$: conjunction of clauses

The SAT Problem

Given a CNF formula F, is it satisfiable?

For instance, what about:

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge$$
$$(x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

Proofs for SAT

For satisfiable instances: just specify satisfying assignment

For unsatisfiability: a sequence of clauses (CNF constraints)

- Each clause follows "obviously" from everything we know so far
- Final clause is empty, meaning contradiction (written $\bot$)
- Means original formula must be inconsistent

What Is Obvious? Unit Propagation

Unit Propagation

Clause C unit propagates ℓ under partial assignment ρ if ρ falsifies all literals in C except ℓ

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- $p \vee \overline{u}$ propagates $u \mapsto 0$

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Proof checker should know how to unit propagate until saturation

Davis-Putman-Logemann-Loveland (DPLL)

DPLL [DP60, DLL62]: Assign variables and propagate; backtrack when clause violated

"Proof trace": when backtracking, write negation of guesses made

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

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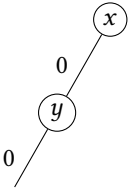

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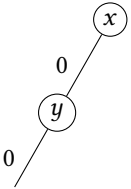

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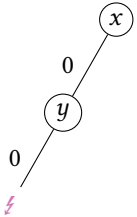
1 $x \vee y$



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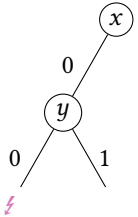

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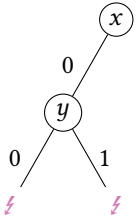

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1. $x \vee y$
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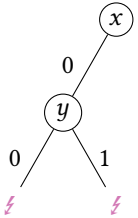

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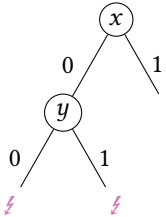

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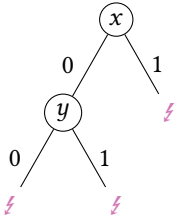

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2. $x \vee \overline{y}$
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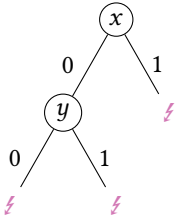
5. $\bot$



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Fact

Backtrack clauses from DPLL solver generate a RUP proof

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Run CDCL [BS97, MS99, MMZ+01] on our favourite CNF formula:

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Forced choice to avoid falsifying clause

Given $p = 0$, clause $p \vee \overline{u}$ forces $u = 0$

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$\overset{\overline{y} \vee \overline{z}}{\bot}$

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decision level 1

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decision level 2

$x \overset{\mathrm{d}}{=} 0$

$y \overset{u \vee x \vee y}{=} 1$

$z \overset{x \vee \overline{y} \vee z}{=} 1$

$\overset{\overline{y} \vee \overline{z}}{\bot}$

decision level 3

Decision

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Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

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$\boxed{p \overset{d}{=} 0}$

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decision level 2

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$\boxed{y \overset{u \vee x \vee y}{=} 1}$

$\boxed{z \overset{x \vee \overline{y} \vee z}{=} 1}$

$\boxed{\overline{y} \vee \overline{z}}$

$\bot$

decision level 3

Could backtrack by erasing conflict level & flipping last decision

Conflict Analysis

Time to analyse this conflict and learn from it!

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$

$\boxed{p \overset{\mathrm{d}}{=} 0}$

$\boxed{u \overset{p \vee \overline{u}}{=} 0}$

decision level 1

$\boxed{q \overset{\mathrm{d}}{=} 0}$

$\boxed{r \overset{q \vee r}{=} 1}$

$\boxed{w \overset{\overline{r} \vee w}{=} 1}$

decision level 2

$\boxed{x \overset{\mathrm{d}}{=} 0}$

$\boxed{y \overset{u \vee x \vee y}{=} 1}$

$\boxed{z \overset{x \vee \overline{y} \vee z}{=} 1}$

$\boxed{\overline{y} \vee \overline{z}}$

$\bot$

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Could backtrack by erasing conflict level & flipping last decision

But want to learn from conflict and cut away as much of search space as possible

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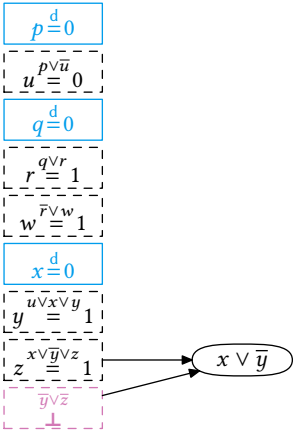

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Case analysis over z for last two clauses:

- $x \vee \overline{y} \vee z$ wants $z = 1$
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- Resolve clauses by merging them & removing z — must satisfy $x \vee \overline{y}$

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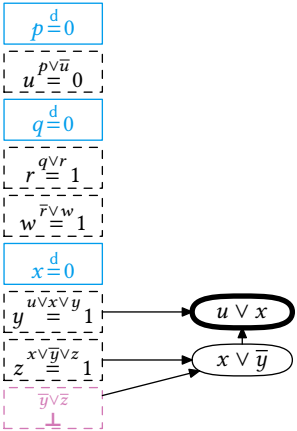

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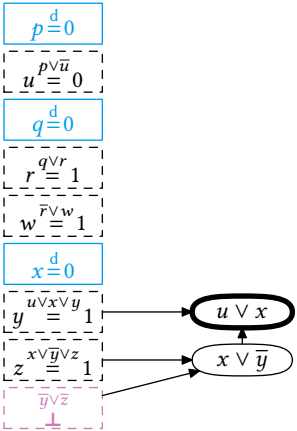
Repeat until UIP clause with only 1 variable at conflict level after last decision — learn and backjump

Complete Example of CDCL Execution

Backjump: undo max #decisions while learned clause propagates

$$(p \vee \overline{u}) \wedge (q \vee r) \wedge (\overline{r} \vee w) \wedge (u \vee x \vee y) \wedge (x \vee \overline{y} \vee z) \wedge (\overline{x} \vee z) \wedge (\overline{y} \vee \overline{z}) \wedge (\overline{x} \vee \overline{z}) \wedge (\overline{p} \vee \overline{u})$$



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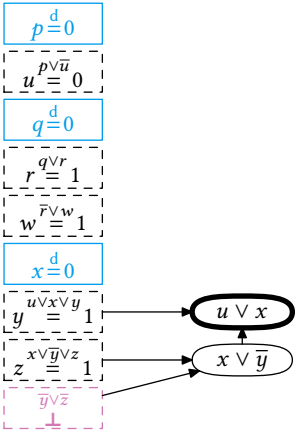

Assertion level 1 (2nd largest level in learned clause) — trim trail to that level

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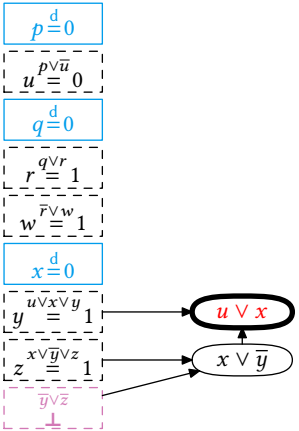

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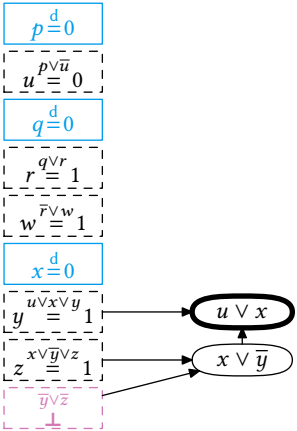

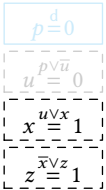

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Then continue as before…

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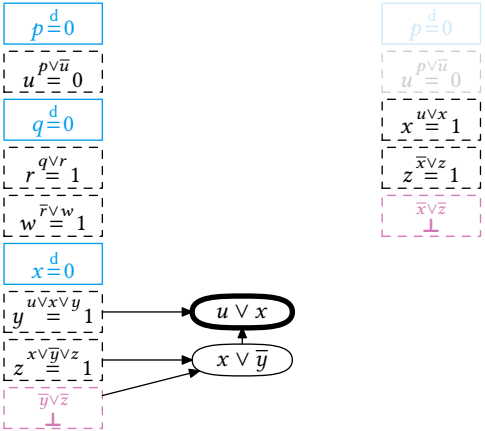

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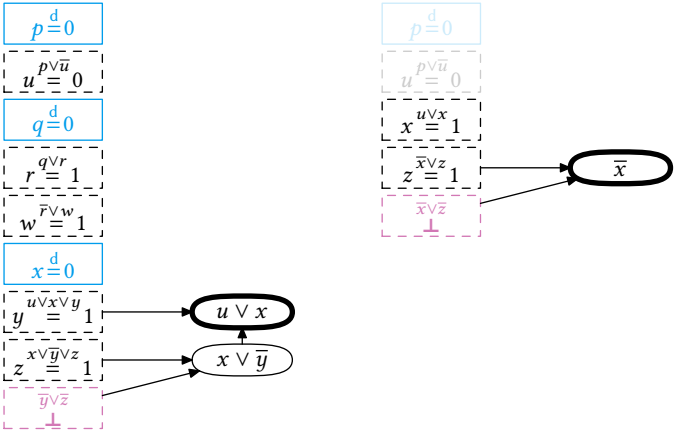

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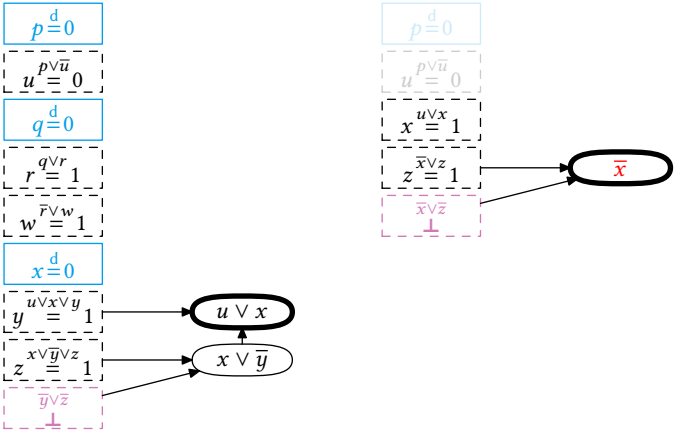

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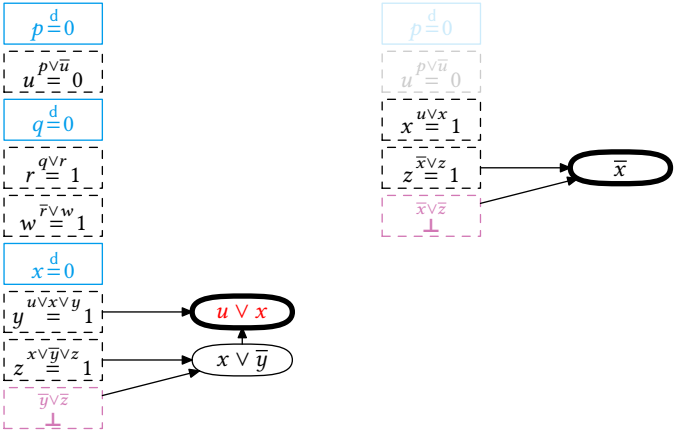

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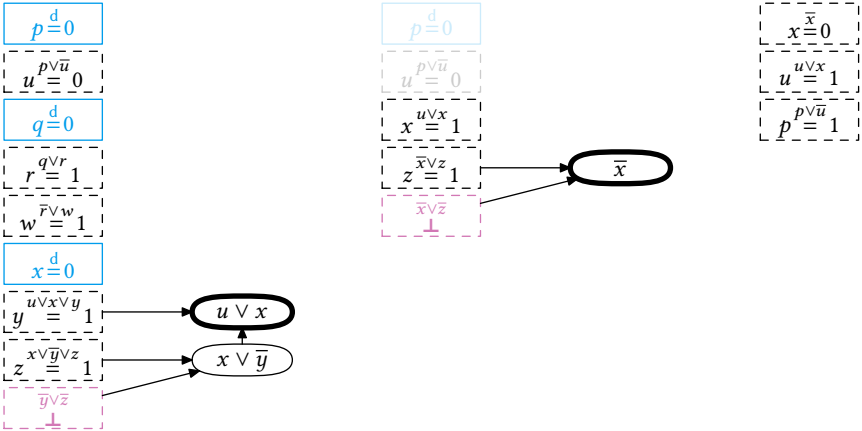

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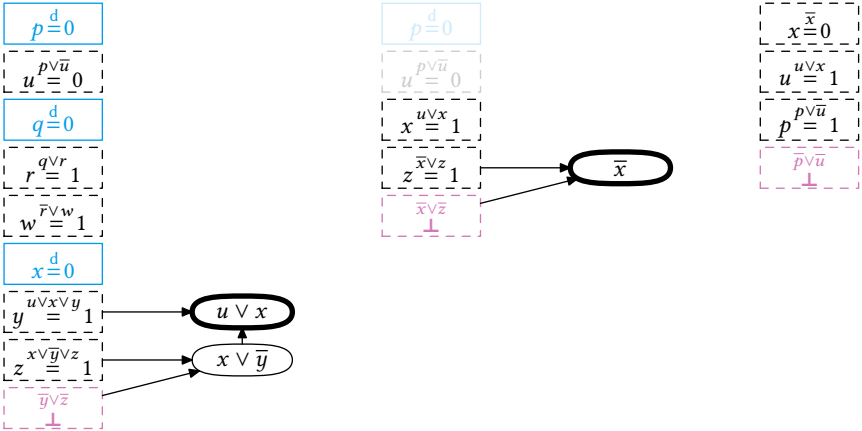

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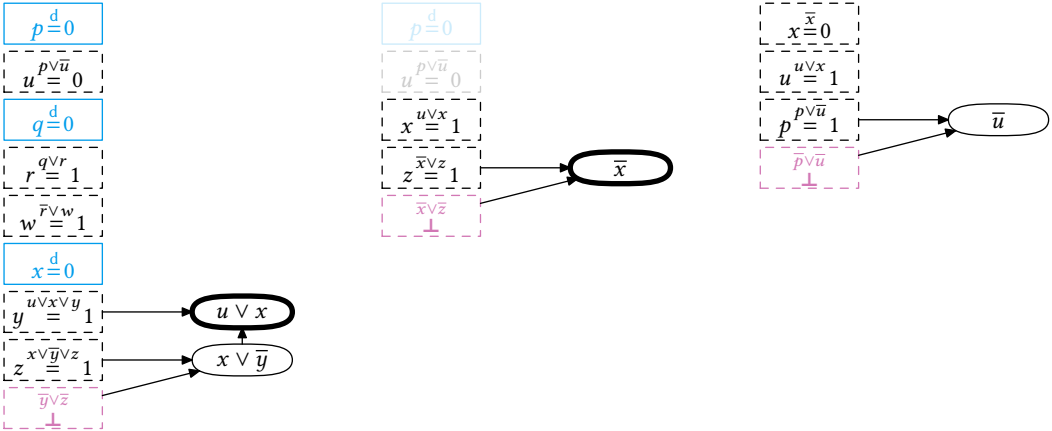

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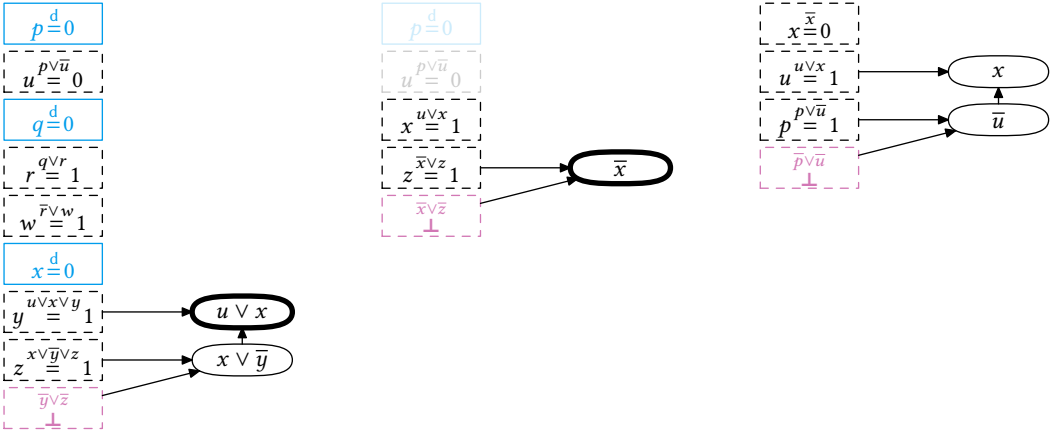

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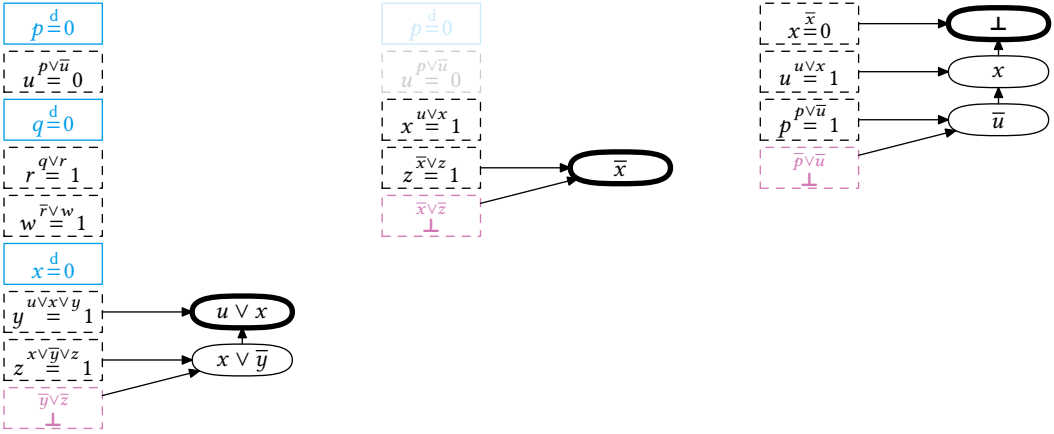

CDCL Reasoning and the Resolution Proof System

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Resolution proof system [Bla37, Rob65]

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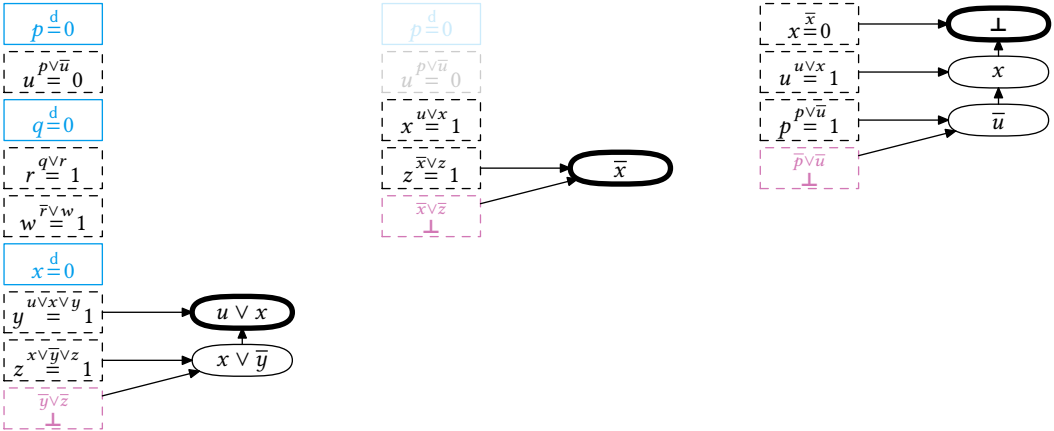

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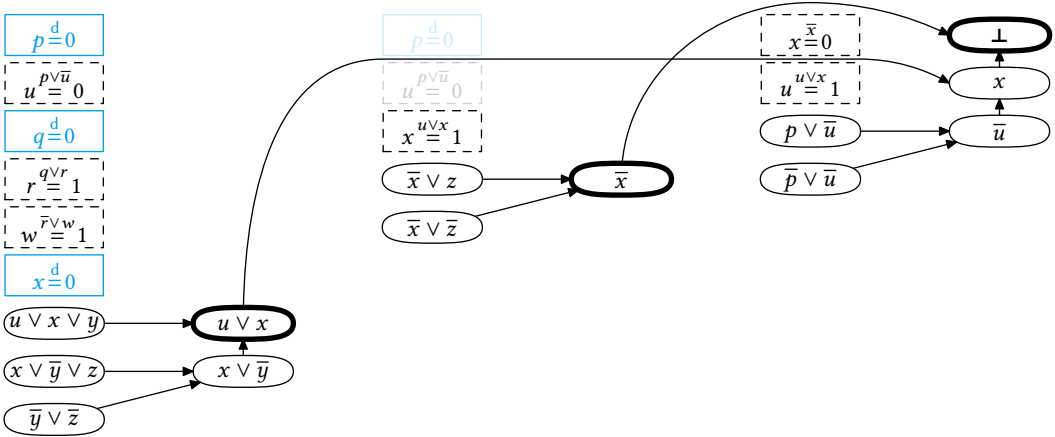
Obtain resolution proof from our example CDCL execution by stringing together conflict analyses:



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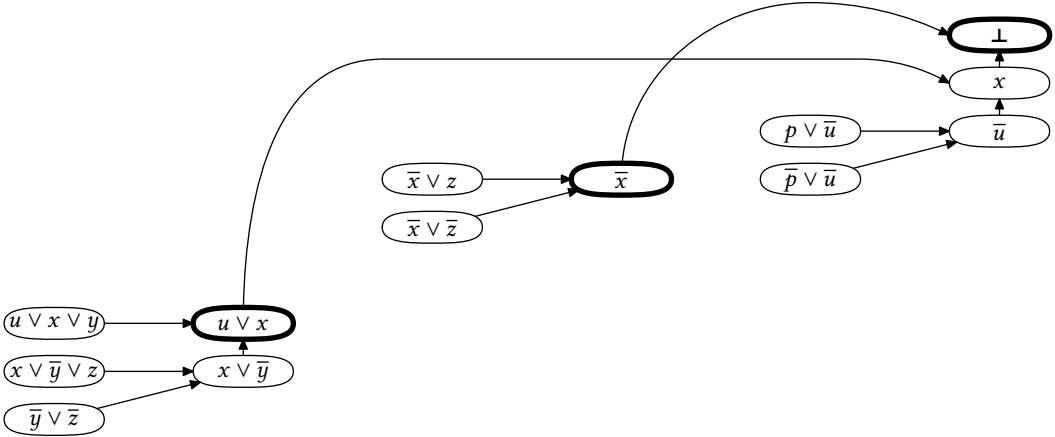

RUP Proofs and CDCL

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More Ingredients in Proof Logging for SAT

Fact

RUP proofs can be viewed as shorthand for resolution proofs

See proof complexity and SAT solving survey [BN21] for more on this

But RUP and resolution are not enough for preprocessing, inprocessing, and some other kinds of reasoning

Extension Variables, Part 1

Suppose we want a reified variable r encoding

$$r \Leftrightarrow (x \wedge y)$$

Extended resolution [Tse68]

Resolution rule plus extension rule introducing clauses

$$r \vee \overline{x} \vee \overline{y} \qquad \overline{r} \vee x \qquad \overline{r} \vee y$$

for fresh variable r (this is fine since r doesn't appear anywhere previously)

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Suppose we want a reified variable r encoding

$$r \Leftrightarrow (x \wedge y)$$

Extended resolution [Tse68]

Resolution rule plus extension rule introducing clauses

$$r \vee \overline{x} \vee \overline{y} \qquad \overline{r} \vee x \qquad \overline{r} \vee y$$

for fresh variable r (this is fine since r doesn't appear anywhere previously)

Fact

Extended resolution (RUP + definition of new variables) is essentially equivalent to the DRAT proof logging system most commonly used for SAT solving [KRH18]

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Practical limitations of current SAT proof logging technology:

- Difficulties dealing with stronger reasoning efficiently (even for SAT solving)
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- And more…

Pseudo-Boolean Constraints

0–1 integer linear inequalities or (linear) pseudo-Boolean constraints:

$$\sum_i a_i \ell_i \geq A$$

- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)

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Sometimes convenient to use normalised form [Bar95] with all a_i, A positive (without loss of generality)

Some Types of Pseudo-Boolean Constraints

1. Clauses

$$x_1 \vee \overline{x}_2 \vee x_3 \quad \Leftrightarrow \quad x_1 + \overline{x}_2 + x_3 \geq 1$$

2. Cardinality constraints

$$x_1 + x_2 + x_3 + x_4 \geq 2$$

3. General pseudo-Boolean constraints

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

Pseudo-Boolean Reasoning: Cutting Planes [CCT87]

Input/model axioms From the input

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Input/model axioms From the input

Literal axioms $\overline{\ell_i \geq 0}$

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Input/model axioms From the input

Literal axioms

$$\frac{}{\ell_i \geq 0}$$

Addition

$$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i)\ell_i \geq A + B}$$

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Multiplication for any $c \in \mathbb{N}^+$

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Saturation for any $c \in \mathbb{N}^+$
(assumes normalised form)

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \min(a_i, A) \cdot \ell_i \geq A}$$

Cutting Planes Toy Example

$$w + 2x + y \geq 2$$

Cutting Planes Toy Example

$$\text{Multiply by 2}\quad \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}$$

Cutting Planes Toy Example

$$\text{Multiply by 2}\ \frac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5$$

Cutting Planes Toy Example

$$\text{Add}\ \frac{\text{Multiply by 2}\ \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}$$

Cutting Planes Toy Example

$$\dfrac{\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}\ \text{Multiply by 2} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}\ \text{Add} \qquad \overline{z} \geq 0$$

Cutting Planes Toy Example

$$\text{Add}\ \frac{\text{Multiply by 2}\ \dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} \qquad \frac{\overline{z} \geq 0}{2\overline{z} \geq 0}\ \text{Multiply by 2}$$

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$$

Cutting Planes Toy Example

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$$

Cutting Planes Toy Example

$$
\dfrac{\dfrac{\dfrac{\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}\ \text{Multiply by 2} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}\ \text{Add} \qquad \dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\ \text{Multiply by 2}}{3w + 6x + 6y \qquad \geq 7}\ \text{Add}}{w + 2x + 2y \geq 3}\ \text{Divide by 3}
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Cutting Planes Toy Example

$$\text{Divide by 3}\;\dfrac{\text{Add}\;\dfrac{\text{Add}\;\dfrac{\text{Multiply by 2}\;\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9} \qquad \dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\;\text{Multiply by 2}}{3w + 6x + 6y \qquad \geq 7}}{w + 2x + 2y \geq 3}$$

Naming constraints by integers and literal axioms by the literal involved (with ~ for negation) as

$$\begin{aligned} \text{Constraint 1} &\doteq 2x + y + w \geq 2 \\ \text{Constraint 2} &\doteq 2x + 4y + 2z + w \geq 5 \\ \sim\text{z} &\doteq \overline{z} \geq 0 \end{aligned}$$

Cutting Planes Toy Example

$$\dfrac{\dfrac{\dfrac{\dfrac{w + 2x + y \geq 2}{2w + 4x + 2y \geq 4}\ \text{Multiply by 2} \qquad w + 2x + 4y + 2z \geq 5}{3w + 6x + 6y + 2z \geq 9}\ \text{Add} \qquad \dfrac{\overline{z} \geq 0}{2\overline{z} \geq 0}\ \text{Multiply by 2}}{3w + 6x + 6y \qquad \geq 7}\ \text{Add}}{w + 2x + 2y \geq 3}\ \text{Divide by 3}$$

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such a calculation is written in the proof log in reverse Polish notation as*

```
pol 1 2 * 2 + ~z 2 * + 3 d ;
```

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(*) Except VeriPB variables have to be at least two characters long, but we abuse syntax slightly here for simplicity

Resolution and Cutting Planes

To simulate resolution step such as

$$\frac{\overline{y} \vee \overline{z} \qquad x \vee \overline{y} \vee z}{x \vee \overline{y}}$$

we can perform the cutting planes steps

$$\text{Divide by 2}\ \frac{\text{Add}\ \dfrac{\overline{y} + \overline{z} \geq 1 \qquad x + \overline{y} + z \geq 1}{x + 2\overline{y} \geq 1}}{x + \overline{y} \geq 1}$$

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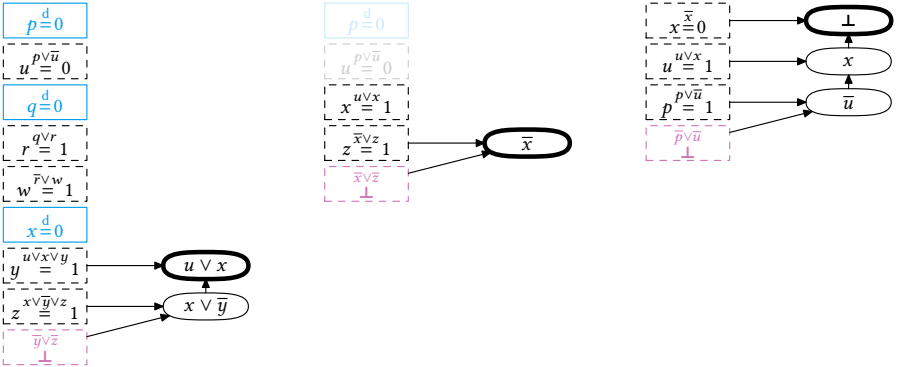
Given that the premises are clauses 7 and 5 in our example CNF formula, using references

$$\text{Constraint 7} \doteq \overline{y} + \overline{z} \geq 1$$
$$\text{Constraint 5} \doteq x + \overline{y} + z \geq 1$$

we can write this in the proof log as

```
pol 7 5 + 2 d ;
```

Pseudo-Boolean Proof Logging for Example CDCL Conflict Analyses



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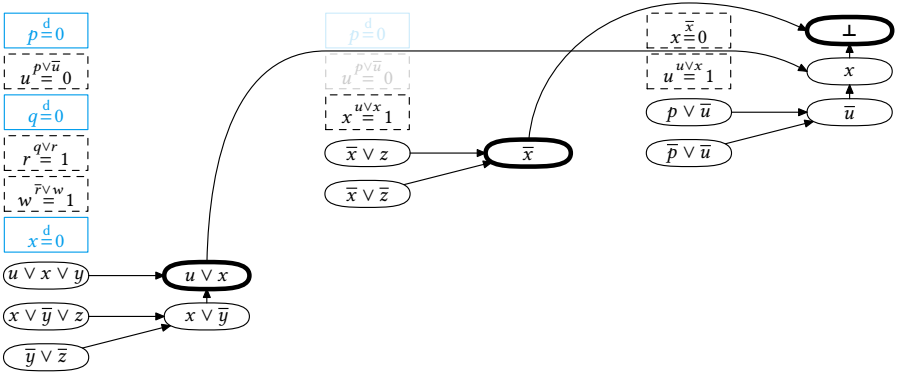

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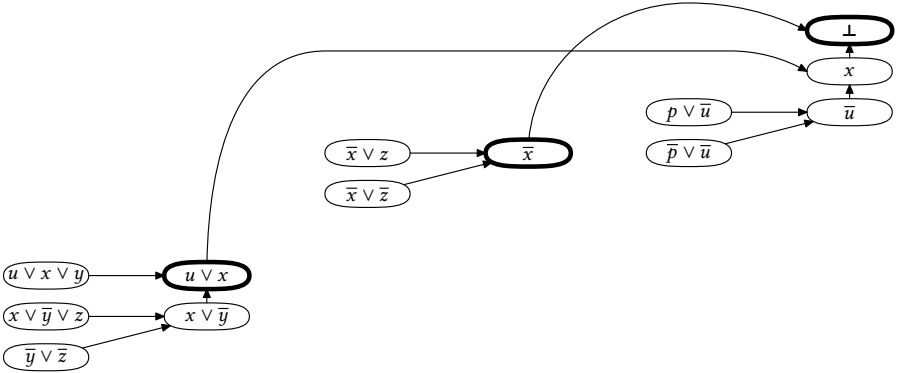

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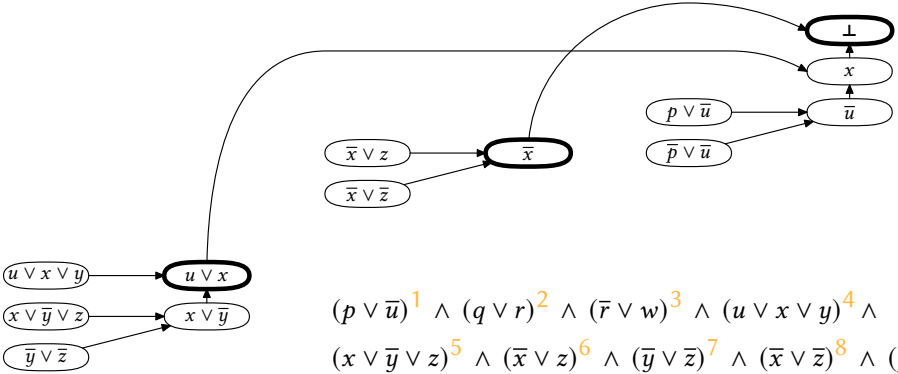

$$(p \vee \overline{u})^{1} \wedge (q \vee r)^{2} \wedge (\overline{r} \vee w)^{3} \wedge (u \vee x \vee y)^{4} \wedge$$
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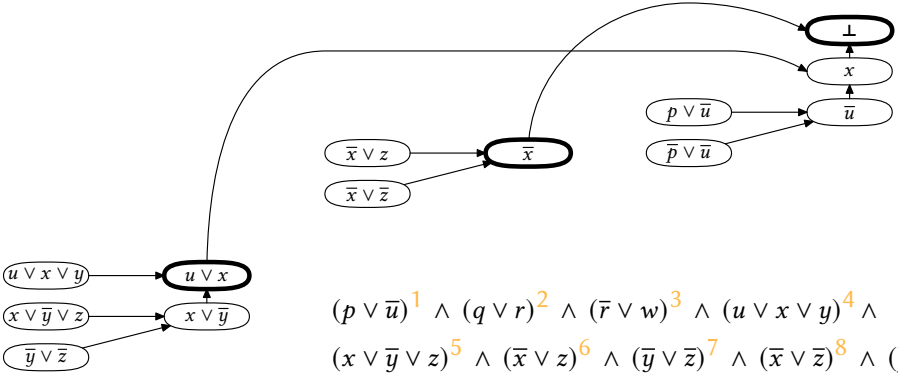

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```
pol 7 5 + 2 d 4 + 2 d ;
pol 8 6 + 2 d ;
pol 9 1 + 2 d 10 + 2 d 11 + 2 d ;
```

$\rightsquigarrow$ Constraint 10 $\doteq u + x \geq 1$

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RUP Revisited

Can define (reverse) unit propagation in a pseudo-Boolean setting [EGMN20]

Constraint C propagates variable x if setting x to "wrong value" would make C unsatisfiable

E.g., if x_5 is false,

$$x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

propagates $\overline{x}_4$ (since other coefficients do not add up to 7)

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Risk for confusion!

- Constraint programming people might call this (reverse) integer bounds consistency
  - Does the same thing if we're working with clauses
  - More interesting for general pseudo-Boolean constraints
- SAT people beware: constraints can propagate multiple times and multiple variables

Propagation, Conflict, and Slack

Slack measures how far assignment ρ is from falsifying $\sum_i a_i \ell_i \geq A$
Assuming normalised form:

$$slack\left(\sum_i a_i \ell_i \geq A; \rho\right) = \sum_{i\,:\,\rho(\ell_i) \neq 0} a_i \; - A$$

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ρ	$slack(C; \rho)$	comment

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$\{\}$	8	

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$\{\overline{x}_5\}$	3	propagates $\overline{x}_4$ (coefficient > slack)

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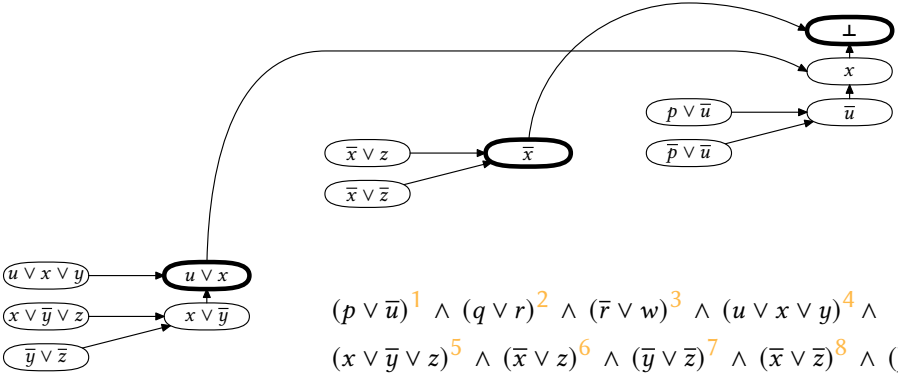
$\{\overline{x}_5, \overline{x}_4, \overline{x}_3, x_2\}$	-2	conflict (slack < 0)

Note: constraint can be conflicting though not all variables assigned

Pseudo-Boolean Proof Logging for Example CDCL Execution with RUP



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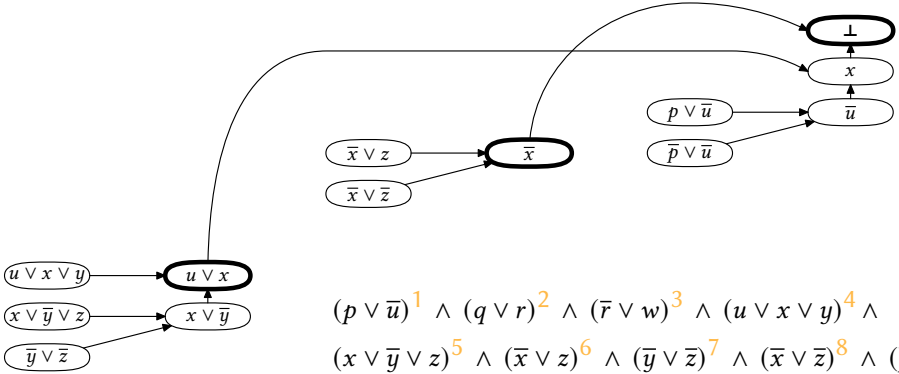
Pseudo-Boolean Proof Logging for Example CDCL Execution with RUP



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```
rup 1 u 1 x >= 1 ;
rup 1 ~x >= 1 ;
rup >= 1 ;
```

$\rightsquigarrow$ Constraint 10 $\doteq u + x \geq 1$

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Extension Variables, Part 2

Suppose we want new, fresh reified variable r encoding

$$r \Leftrightarrow (3x + 2y + z + w \geq 3)$$

This time, introduce constraints

$$3\overline{r} + 3x + 2y + z + w \geq 3 \qquad 5r + 3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \geq 5$$

Again, needs support from the proof system in the form of strengthening rules

Proof Logs for "Cutting Planes with Strengthening"

For satisfiable instances: just specify a satisfying assignment.

For unsatisfiability: a sequence of pseudo-Boolean constraints in (slight extension of) OPB format [RM16]

- Each constraint follows "obviously" from what is known so far
- Either implicitly, by RUP…
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(*) Not actually implemented this way — more details in a few slides …

Optimisation and Enumeration Problems

Optimisation

- Define an objective $f = \sum_i w_i \ell_i$, $w_i \in \mathbb{Z}$, to minimise subject to the contraints in the formula
- To maximise, negate objective
- Log solution α with `soli` rule $\Rightarrow$ objective function value $UB = \sum_i w_i \alpha(\ell_i)$
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Enumeration

- When a solution is found, can log it with `solx` rule
- Introduces a new constraint saying "not this solution"
- So the proof semantics is "infeasible, except for all the solutions I told you about"

Deleting Constraints

In practice, important to erase constraints to save memory and time during verification

Unsatisfiability proofs: fairly straightforward to deal with from point of view of proof logging

Optimisation and enumeration proofs: significantly more delicate

We will mostly ignore deletions during this tutorial for simplicity and clarity

Pseudo-Boolean Proof Logging — How and Why?

If problem is (special case of) 0-1 integer linear program

- just do proof logging [basically: add print statements to solver code]

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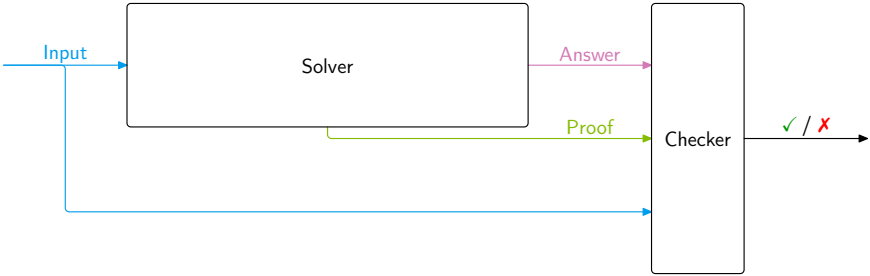
$$r \Rightarrow x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7 \qquad 7\overline{r} + x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7$$

$$r \Leftarrow x_1 + 2\overline{x}_2 + 3x_3 + 4\overline{x}_4 + 5x_5 \geq 7 \qquad 9r + \overline{x}_1 + 2x_2 + 3\overline{x}_3 + 4x_4 + 5\overline{x}_5 \geq 9$$

Proof Logging with Formally Verified Checking: Full Workflow



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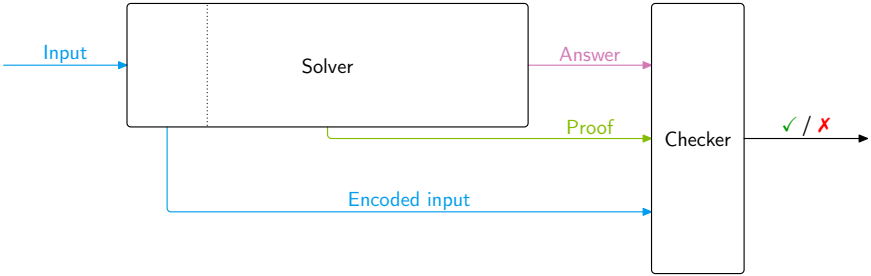

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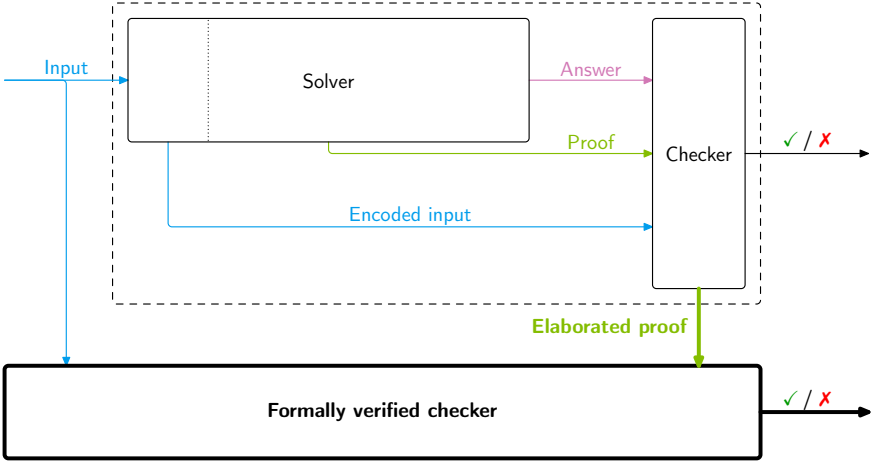

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we introduced pseudo-Boolean constraints

$$3\overline{r} + 3x + 2y + z + w \geq 3 \qquad 5r + 3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \geq 5$$

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Cutting planes method inherently cannot certify such constraints — they are not implied!

Wish to allow without-loss-of-generality arguments that can derive non-implied constraints

Redundance-Based Strengthening

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C is redundant with respect to F if and only if there is a substitution ω (mapping variables to truth values or literals), called a witness, for which

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2. by explicitly presented derivation

Deriving $r \Leftrightarrow (3x + 2y + z + w \geq 3)$ Using the Redundance Rule

Want to derive

$$3\overline{r} + 3x + 2y + z + w \geq 3 \qquad 5a + 3\overline{x} + 2\overline{y} + \overline{z} + \overline{w} \geq 5$$

using condition $F \cup \{\neg C\} \models (F \cup \{C\})\restriction_{\omega}$

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VeriPB code:

```
red 3 ~r 3 x 2 y 1 z 1 w >= 3 : r -> 0 ;
red 5 r 3 ~x 2 ~y 1 ~z 1 ~w >= 5 : r -> 1 ;
```

Redundance and Dominance Rules for Optimisation

Redundance-based strengthening, optimisation version

Add constraint C to formula F if exists witness substitution ω s.t.

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- Applying ω should strictly decrease f
- If so, don't need to show that $C\restriction_\omega$ holds!

Soundness of Dominance Rule

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7. ...

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5. If $(\alpha \circ \omega) \circ \omega$ satisfies C, we're done
6. Otherwise $\big((\alpha \circ \omega) \circ \omega\big) \circ \omega$ satisfies F and $f\big(((\alpha \circ \omega) \circ \omega) \circ \omega\big) < f\big((\alpha \circ \omega) \circ \omega\big)$
7. …
8. Can't go on forever, so finally reach α' satisfying both F and C

Strength of Dominance Rule

Dominance-based strengthening (stronger, still simplified)

If $C_1, C_2, \ldots, C_{m-1}$ have been derived from F (maybe using dominance), then can derive C_m if exists witness substitution ω s.t.

$$F \cup \{C_1, \ldots, C_{m-1}\} \cup \{\neg C_m\} \models F\restriction_\omega \cup \{f\restriction_\omega < f\}$$

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Further extensions:

- Define dominance rule w.r.t. order independent of objective
- Switch between different orders in same proof
- See [BGMN23, ABB+26b] fore more details

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- Witness ω should be explicitly specified in proof log
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- Except can be skipped for proof goals that "obviously" follow, e.g.,
  - by reverse unit propagation (RUP)
  - by simple syntactic implication from other constraint (as in our reification example)
  - if the proof goal is not affected by the witness substitution ω

Successful Applications of Pseudo-Boolean Proof Logging with VeriPB

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1. Boolean satisfiability (SAT) solving including advanced techniques such as
   - Gaussian elimination [GN21]
   - symmetry breaking [BGMN23, ABB+26b]
2. SAT-based optimization (MaxSAT) [VDB22, BBN+23, BBN+24, IOT+24, VCB26]
3. Pseudo-Boolean solving and optimization [GMNO22, KLM+25]
4. Graph solving [GMN20, GMM+20, GMM+24, DKK+26]
5. Dynamic programming and decision diagrams [DMM+24]
6. Presolving in 0–1 integer linear programming [HOGN24]
7. Constraint programming [EGMN20, GMN22, MM23, MMN24, MM25]
8. Automated planning [DHN+25]
9. Enumeration of solutions/models [MNOT26]

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Lots of concrete example files at https://gitlab.com/MIAOresearch/software/VeriPB

Summing up the First Lecture

- Combinatorial solving and optimisation is a true success story
- But ensuring correctness is a crucial, and not yet satisfactorily addressed, concern
- Certifying solvers producing machine-verifiable proofs of correctness seems like most promising approach
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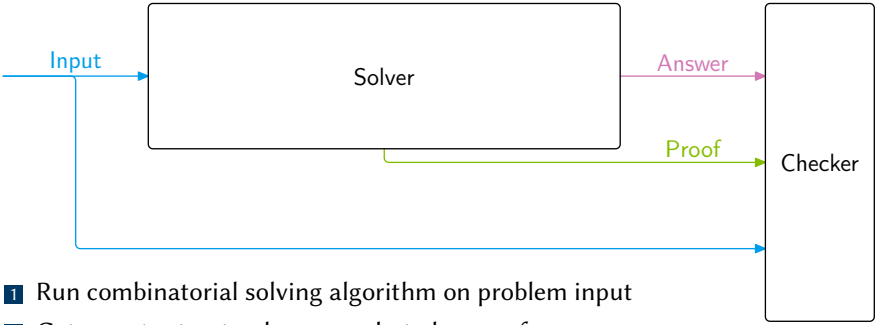
Questions so far?

Recap of Proof Logging Workflow (1/2)



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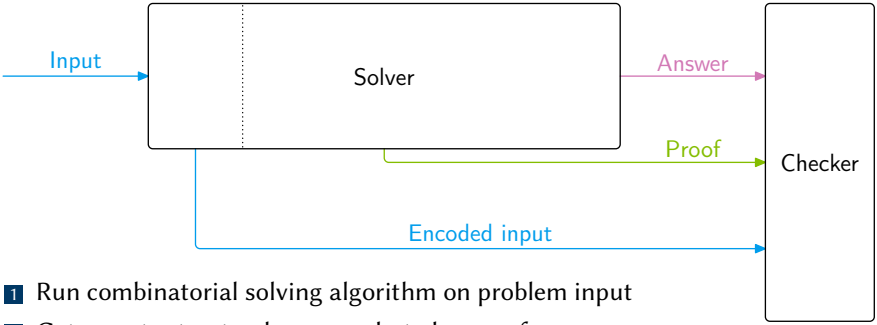

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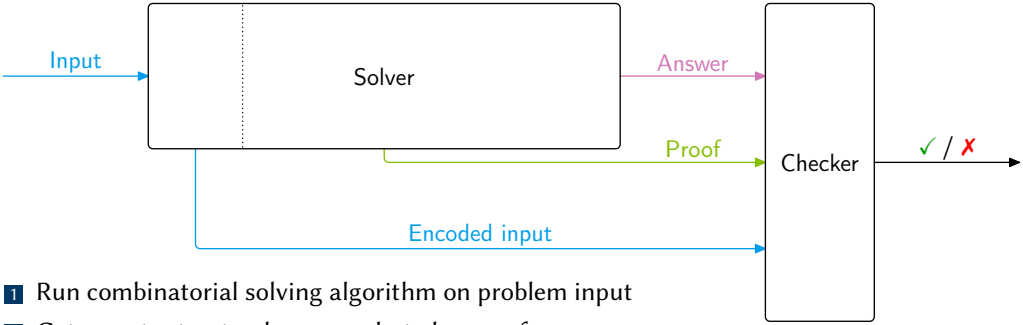

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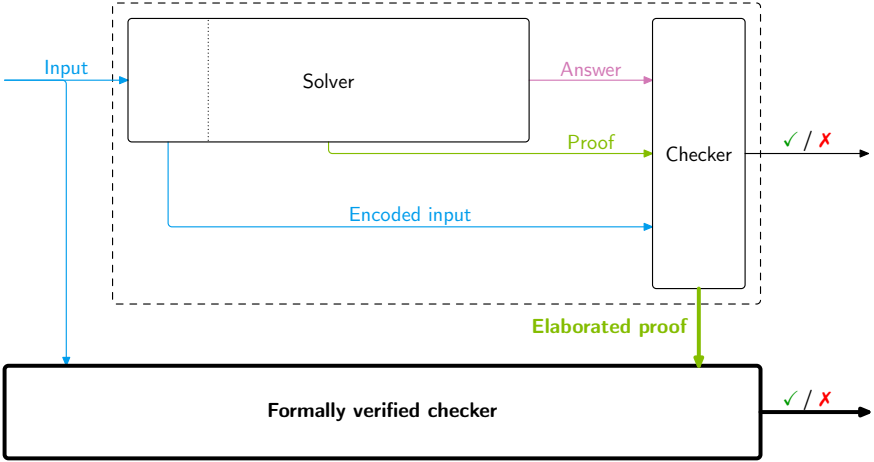
4. Verify that proof checker says answer is correct

Recap of Proof Logging Workflow (2/2)



Implementation and Performance

Proof logging implementation

- Don't change solver
- Just add proof logging statements (plus some book-keeping)
- Proof generation should be very efficient

Performance goals for proof checking

Want linear(ish) scaling in terms of original solver running time for

- proof size
- proof checking time

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   Specify
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4. **Conclusions section**
   Specification of what was established
   - satisfiability / unsatisfiability
   - optimality (or upper and lower bounds)
   - enumeration
   - other types of conclusions (future)

VeriPB Proof Structure: Syntax

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pseudo-Boolean proof version 3.0
f ⟨M⟩ ;
⟨derivation part⟩
output ⟨output part⟩
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Core set $\mathscr{C}$

- Contains input formula at the start
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Order $\mathscr{O}$

- Pseudo-Boolean formula encoding pre-order (reflexive and transitive)
- Syntactic proof of properties required
- Applied to specified variable set $\vec{z}$

Cutting Planes Rules [CCT87]

Input/model axioms — From the input

Literal axioms

$$\frac{}{\ell_i \geq 0}$$

Addition

$$\frac{\sum_i a_i \ell_i \geq A \qquad \sum_i b_i \ell_i \geq B}{\sum_i (a_i + b_i)\ell_i \geq A + B}$$

Multiplication for any $c \in \mathbb{N}^+$

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i c a_i \ell_i \geq cA}$$

Division for any $c \in \mathbb{N}^+$

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \left\lceil \frac{a_i}{c} \right\rceil \ell_i \geq \left\lceil \frac{A}{c} \right\rceil}$$

Saturation for any $c \in \mathbb{N}^+$
(assumes normalised form)

$$\frac{\sum_i a_i \ell_i \geq A}{\sum_i \min(a_i, A) \cdot \ell_i \geq A}$$

Strengthening Rules in Their (Almost) Full Formal Glory

Witness ω: substitution mapping variables to truth values or literals

Redundance-based strengthening (witness ω show how to "patch assignment")

Derive constraint C from $\mathscr{C} \cup \mathscr{D}$ if exists witness ω such that

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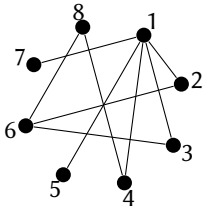

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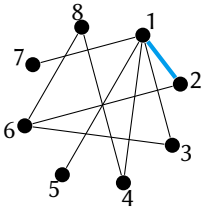

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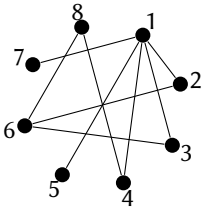

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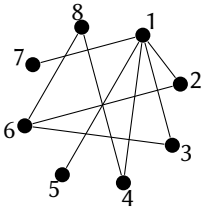

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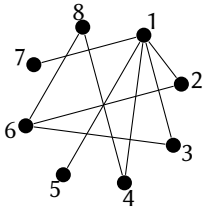

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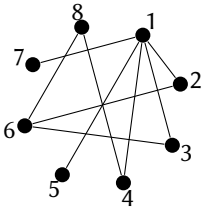

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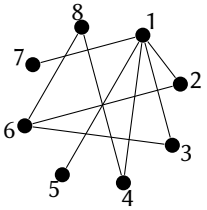

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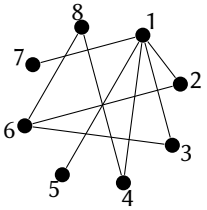

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Without loss of generality, node 1 is blue (otherwise we can swap the two colours)

Derive $b_1 \geq 1$ with redundance; witness $\omega : b_i \mapsto \overline{b}_i$ for all i

$$F \doteq F\restriction_\omega \quad \text{and} \quad \neg(b_1 \geq 1) = (b_1 \geq 1)\restriction_\omega$$

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- Start the proof
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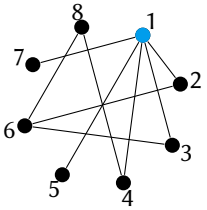

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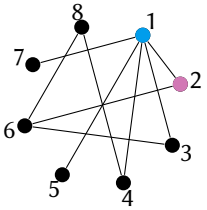

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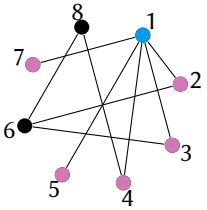

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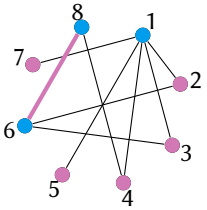

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Enumeration: deletion can reintroduce already enumerated solutions

Solution: distinguish between deletion from core set $\mathscr{C}$ and derived set $\mathscr{D}$

Deletion, Core Transfer, and Order Change

Deletion

1. Deletion of constraint C always OK from derived set $\mathscr{D}$
2. OK from core set $\mathscr{C}$ only if C can be rederived from $\mathscr{C} \setminus \{C\}$ with redundance rule (otherwise unchecked deletion — special conditions apply)

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Change of order

Possible to change order only if $\mathscr{D} = \emptyset$

Conclusions for Decision Problems

`NONE`
Status is undetermined

`SAT [ : ⟨assignment⟩]`
Check provided assignment against original formula
If no assignment given, then

- solution should have been logged
- no unchecked deletion must have occurred

`UNSAT [ : ⟨constraint ID⟩]`
Only valid if no solution has been logged
Check that specified constraint is contradictory (technically: negative slack)
If no constraint specified, check that database contains contradictory constraint

Optimization Problems

Any solution α found is logged with `soli` "log solution and improve" command

- provided solution α checked against current core set $\mathscr{C}$
- Objective-improving constraint $\sum_i w_i \ell_i \leq -1 + \sum_i w_i \cdot \alpha(\ell_i)$ added to core set (forces search for better solutions)

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Proof of optimality: Contradiction derived from objective-improving constraint

Proof format supports not just optimality, but also non-tight upper and lower bounds

Conclusions for Optimization Problems

`NONE`

No solution or lower bound found

`BOUNDS` ⟨`LB`⟩ `[ :` ⟨`constraint ID`⟩ `]` ⟨`UB`⟩ `[ :` ⟨`assignment`⟩ `]`

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Constraint ⟨*constraint ID*⟩, if specified, should imply lower bound

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Upper bound

Check provided assignment against original formula

If no assignment specified, then

- solution with value $\langle UB \rangle$ should have been logged
- no unchecked deletion must have occurred

More About VeriPB Proof Syntax

Variables

- start with a letter in `A-Z` or `a-z`
- continue with characters in `A-Z`, `a-z`, `0-9`, or `[]{}-_^`
  (square and curly brackets, hyphen, underscore, and caret)
- contain at least two characters

Constraints

Are referred to by positive integers (constraint IDs)

Derivation rules and requirements

Come in two flavours

1. kernel format for formally verified proof checker
2. augmented format with convenience rules such as reverse unit propagation (RUP)

See `https://VeriPB.org` for more details

Syntactic Sugar: Labels for Constraints

Constraint identifiers are consecutive numbers 1, 2, 3, …

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Labels simplify things — just aliases @⟨label⟩ for constraint IDs (in input file or proof file)

```
* input file (.opb)
@foo 1 x1 1 x2 >= 1 ;
@bar 1 x1 1 x3 >= 1 ;
@baz 1 x1 1 x3 >= 1 ;
```

```
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@sum pol @foo @bar + @baz + ;
pol @sum 2 d ;
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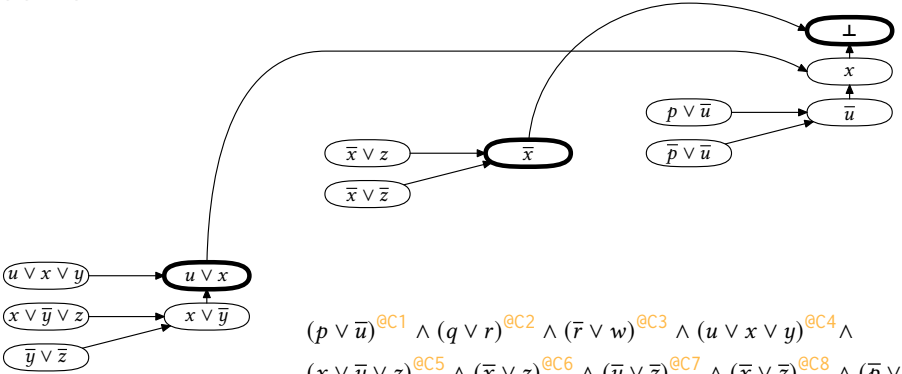
- Can be used anywhere a constraint ID is expected
- Prefixing of proof rule assigns label to derived constraint
- Defining a label again just overwrites it — can be convenient when referring to simplified form of "the same constraint"

Proof Logging with Labels for CDCL Example



$$(p \vee \overline{u})^{@C1} \wedge (q \vee r)^{@C2} \wedge (\overline{r} \vee w)^{@C3} \wedge (u \vee x \vee y)^{@C4} \wedge$$
$$(x \vee \overline{y} \vee z)^{@C5} \wedge (\overline{x} \vee z)^{@C6} \wedge (\overline{y} \vee \overline{z})^{@C7} \wedge (\overline{x} \vee \overline{z})^{@C8} \wedge (\overline{p} \vee \overline{u})^{@C9}$$

```
@C10 pol @C7 @C5 + 2 d @C4 + 2 d ;
@C11 pol @C8 @C6 + 2 d ;
@C12 pol @C9 @C1 + 2 d @C10 + 2 d @C11 + 2 d ;
```

$\rightsquigarrow u + x \geq 1$

$\rightsquigarrow \overline{x} \geq 1$

$\rightsquigarrow 0 \geq 1$ ↯

Linear Pseudo-Boolean Constraints and Normalised Form

Usually, pseudo-Boolean constraints are any 0–1 integer linear constraints on the form

$$\sum_i a_i \ell_i \bowtie A$$

- $\bowtie \in \{\geq, \leq, =, >, <\}$
- $a_i, A \in \mathbb{Z}$
- literals ℓ_i: x_i or $\overline{x}_i$ (where $x_i + \overline{x}_i = 1$)
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For us, convenient to use normalised form [Bar95] (without loss of generality)

$$\sum_i a_i \ell_i \geq A$$

- constraint always greater-than-or-equal
- $a_i, A \in \mathbb{N}$ non-negative
- $A = \mathit{deg}(\sum_i a_i \ell_i \geq A)$ referred to as degree (of falsity)

Conversion to Normalised Form: Example

Normalised form is used for convenience and without loss of generality

$$-x_1 + 2x_2 - 3x_3 + 4x_4 - 5x_5 < 0$$

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3. Replace $-\ell$ by $-(1 - \overline{\ell})$ [where we define $\overline{\overline{x}} \doteq x$]

$$x_1 - 2(1 - \overline{x}_2) + 3x_3 - 4(1 - \overline{x}_4) + 5x_5 \geq 1$$
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4. Replace "=" by two inequalities "≥" and "≤"

Conversion to Normalised Form: Formal Details

Given (normalised) linear form $\sum_i a_i \ell_i$ with coefficient sum $\sum_i a_i = W$

Syntactic sugar	**Meaning**
$\sum_i a_i \ell_i > A$	$\sum_i a_i \ell_i \geq A + 1$
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$\sum_i a_i \ell_i = A$	$\sum_i a_i \ell_i \geq A$ and $\sum_i a_i \overline{\ell}_i \geq W - A$

In what follows:

- Use syntactic sugar when convenient
- Assume (implicit) normalisation whenever it matters
- Write $\doteq$ for syntactic equality

Some Convenient Notation for Constraints

Given constraint $C \doteq \sum_i a_i \ell_i \geq A$ with $\sum_i a_i = W$

Negation

$$\neg C \doteq \sum_i a_i \overline{\ell}_i \geq W - A + 1$$

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Reification

$$r \Rightarrow C \doteq A \cdot \overline{r} + \sum_i a_i \ell_i \geq A$$

$$r \Leftarrow C \doteq (W - A + 1) \cdot r + \sum_i a_i \overline{\ell}_i \geq W - A + 1$$

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Some calculations

$$r \Leftarrow C \doteq \overline{r} \Rightarrow \neg C$$

$$C + \neg C \doteq 0 \geq 1$$

$$deg(C) \cdot (r \geq 1) + (r \Rightarrow C) \doteq C$$

$$C + (r \Leftarrow C) \doteq deg(\neg C) \cdot r \geq 1$$

More Syntactic Sugar: Reification in VeriPB Syntax

For big-M reification constraints, can be annoying to compute the right constant
So VeriPB allows to write reified constraints with arrows

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For $C \doteq \sum_i a_i \ell_i \geq A$ and literals $r_1, r_2, \ldots, r_n$, we have

$$r_1 \wedge r_2 \wedge \cdots \wedge r_n \Rightarrow C \doteq A\overline{r}_1 + A\overline{r}_2 + \cdots + A\overline{r}_n + \sum_i a_i \ell_i \geq A$$

Reification in VeriPB Syntax: Concrete Examples

For $C \doteq x_1 + 2x_2 + 3x_3 + 4x_4 \geq 5$ we have

- $deg(C) = 5$
- $W = \sum_i a_i = 10$
- $\neg C \doteq \overline{x}_1 + 2\overline{x}_2 + 3\overline{x}_3 + 4\overline{x}_4 \geq 10 - 5 + 1$

Reification in VeriPB Syntax: Concrete Examples

For $C \doteq x_1 + 2x_2 + 3x_3 + 4x_4 \geq 5$ we have

- $deg(C) = 5$
- $W = \sum_i a_i = 10$
- $\neg C \doteq \overline{x}_1 + 2\overline{x}_2 + 3\overline{x}_3 + 4\overline{x}_4 \geq 6$

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Syntactic sugar	**Meaning**
`r ==> 1 x1 2 x2 3 x3 4 x4 >= 5 ;`	$5\overline{r} + x_1 + 2x_2 + 3x_3 + 4x_4 \geq 5$
`r <== 1 x1 2 x2 3 x3 4 x4 >= 5 ;`	$6r + \overline{x}_1 + 2\overline{x}_2 + 3\overline{x}_3 + 4\overline{x}_4 \geq 6$
`r <==> 1 x1 2 x2 3 x3 4 x4 >= 5 ;`	1st and 2nd constraint, in that order
`r1 ~r2 r3 ==> 1 x1 2 x2 3 x3 4 x4 >= 5 ;`	$5\overline{r}_1 + 5r_2 + 5\overline{r}_3 + x_1 + 2x_2 + 3x_3 + 4x_4 \geq 5$

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- Right reification: negate reification literal(s) and add with coefficient $deg(C) = 5$
- Left reification: negate constraint and add reification literal with coefficient $deg(\neg C) = 6$

Calculations with Reified Constraints: Concrete Examples

Still with $C \doteq x_1 + 2x_2 + 3x_3 + 4x_4 \geq 5$, so that $\neg C \doteq \overline{x}_1 + 2\overline{x}_2 + 3\overline{x}_3 + 4\overline{x}_4 \geq 6$
(and recall that $\ell + \overline{\ell} = 1$, so that opposing literals cancel and leave 1)

$$\begin{aligned} C + \neg C &\doteq (x_1 + \overline{x}_1) + 2(x_2 + \overline{x}_2) + 3(x_3 + \overline{x}_3) + 4(x_4 + \overline{x}_4) \geq 11 \\ &\doteq 0 \geq 11 - 10 \\ &\doteq 0 \geq 1 \end{aligned}$$

$$\begin{aligned} deg(C) \cdot (r \geq 1) + (r \Rightarrow C) &\doteq 5(r + \overline{r}) + x_1 + 2x_2 + 3x_3 + 4x_4 \geq 10 \\ &\doteq x_1 + 2x_2 + 3x_3 + 4x_4 \geq 10 - 5 \\ &\doteq C \end{aligned}$$

$$\begin{aligned} C + (r \Leftarrow C) &\doteq 6r + (x_1 + \overline{x}_1) + 2(x_2 + \overline{x}_2) + 3(x_3 + \overline{x}_3) + 4(x_4 + \overline{x}_4) \geq 11 \\ &\doteq 6r \geq 11 - 10 \\ &\doteq deg(\neg C) \cdot r \geq 1 \end{aligned}$$

Cutting Planes Derivations Under Assumptions

Suppose we have the constraints

$$r_1 + x_1 + x_2 \geq 1$$
$$r_1 + x_1 + x_3 \geq 1$$
$$x_2 + x_3 \geq 1$$
$$\overline{r}_2 + \overline{x}_1 + \overline{x}_2 + \overline{x}_3 \geq 2$$

Cutting Planes Derivations Under Assumptions

Suppose we have the constraints

$$\cancel{r_1 +} x_1 + x_2 \geq 1 \quad \rightsquigarrow \quad x_1 + x_2 \geq 1$$

$$\cancel{r_1 +} x_1 + x_3 \geq 1 \quad \rightsquigarrow \quad x_1 + x_3 \geq 1$$

$$x_2 + x_3 \geq 1 \quad \rightsquigarrow \quad x_2 + x_3 \geq 1$$

$$\cancel{\overline{r}_2 +} \overline{x}_1 + \overline{x}_2 + \overline{x}_3 \geq 2 \quad \rightsquigarrow \quad \overline{x}_1 + \overline{x}_2 + \overline{x}_3 \geq 2$$

Assume $r_1 = 0$ and $r_2 = 1$ and simplify

Cutting Planes Derivations Under Assumptions

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$$\begin{aligned}
\cancel{r_1 +}\ x_1 + x_2 &\geq 1 &\rightsquigarrow\quad & x_1 + x_2 \geq 1\\
\cancel{r_1 +}\ x_1 + x_3 &\geq 1 &\rightsquigarrow\quad & x_1 + x_3 \geq 1\\
x_2 + x_3 &\geq 1 &\rightsquigarrow\quad & x_2 + x_3 \geq 1\\
\cancel{\overline{r}_2 +}\ \overline{x}_1 + \overline{x}_2 + \overline{x}_3 &\geq 2 &\rightsquigarrow\quad & \overline{x}_1 + \overline{x}_2 + \overline{x}_3 \geq 2
\end{aligned}$$

Assume $r_1 = 0$ and $r_2 = 1$ and simplify — then cutting planes can derive contradiction

$$\dfrac{\dfrac{\dfrac{\dfrac{x_1 + x_2 \geq 1 \qquad x_1 + x_3 \geq 1}{2x_1 + x_2 + x_3 \geq 2}\ \text{Add} \qquad x_2 + x_3 \geq 1}{2x_1 + 2x_2 + 2x_3 \geq 3}\ \text{Add}}{x_1 + x_2 + x_3 \geq 2}\ \text{Divide by 2} \qquad \overline{x}_1 + \overline{x}_2 + \overline{x}_3 \geq 2}{0 \geq 1}\ \text{Add}$$

Cutting Planes Derivations Under Assumptions

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This means that the original constraints imply $r_1 + \overline{r}_2 \geq 1$

The Same Derivation Without Assumptions

Now forget about the assumptions and run exactly the same derivation on the original constraints:

$$\text{Add}\ \dfrac{\text{Add}\ \dfrac{\text{Divide by 2}\ \dfrac{\text{Add}\ \dfrac{r_1 + x_1 + x_2 \geq 1 \qquad r_1 + x_1 + x_3 \geq 1}{2r_1 + 2x_1 + x_2 + x_3 \geq 2} \qquad x_2 + x_3 \geq 1}{2r_1 + 2x_1 + 2x_2 + 2x_3 \geq 3}}{r_1 + x_1 + x_2 + x_3 \geq 2} \qquad \overline{r}_2 + \overline{x}_1 + \overline{x}_2 + \overline{x}_3 \geq 2}{r_1 + \overline{r}_2 \geq 1}$$

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This yields

$$r_1 + \overline{r}_2 \geq 1$$

[VeriPB syntax: `1 r1 1 ~r2 >= 1 ;`]

or equivalently

$$\overline{r}_1 \wedge r_2 \Rightarrow (0 \geq 1)$$

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or equivalently

$$\overline{r}_1 \wedge r_2 \Rightarrow (0 \geq 1)$$

[VeriPB syntax: `~r1 r2 ==> >= 1 ;`]

That is, our assumptions imply contradiction

Derivations Under Assumptions: The General Claim

This works in general!

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- Suppose that we have
  1. constraints $C_1, C_2, \ldots, C_m$
  2. assumptions $\rho = \{\ell_1 = 1, \ell_2 = 1, \ldots, \ell_n = 1\}$
  3. cutting planes derivation of D from the restricted constraints $C_1\restriction_\rho, C_2\restriction_\rho, \ldots, C_m\restriction_\rho$ (i.e., without strengthening)

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$$\ell_1 \wedge \ell_2 \wedge \cdots \wedge \ell_n \Rightarrow D \;\doteq\; deg(D) \cdot \textstyle\sum_{i=1}^{n} \overline{\ell}_i \;+\; D$$

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Give a convenient way to construct derivations:

- Apply assumptions in ρ to constraints C_j and simplify to get $C_j\restriction_\rho$

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- Remove assumptions to get reified version $\rho \Rightarrow D$ of constraints

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- Remove assumptions to get reified version $\rho \Rightarrow D$ of constraints
- Important special case: Backtracking clause from decisions/guesses leading to contradiction

Progress So Far

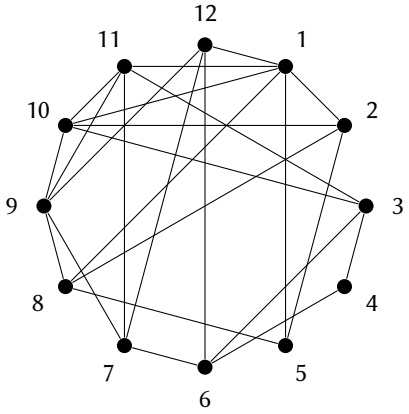
We've seen proof logging, and how it works for SAT

We've learned about

- pseudo-Boolean constraints (0–1 linear inequalities)
- cutting planes reasoning
- strengthening rules

Coming next, some worked examples from dedicated graph solvers

The Maximum Clique Problem



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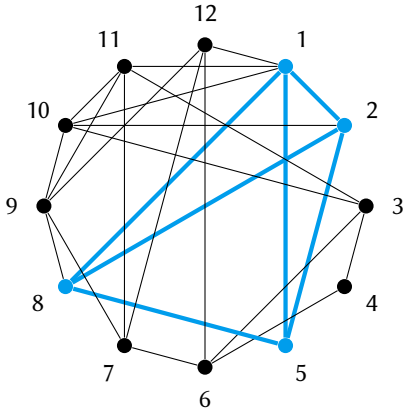


Maximum Clique Solvers

There are a lot of dedicated solvers for clique problems

But there are issues:

- "State-of-the-art" solvers have been buggy.
- Often undetected: error rate of around 0.1% [MPP19]

Often used inside other solvers

- An off-by-one result can cause much larger errors

A Brief and Incomplete Guide to Clique Solving (1/4)

Recursive maximum clique algorithm:

- Pick a vertex v
- Either v is in the clique...
  - Throw away every vertex not adjacent to v
  - If vertices remain, recurse
- ...or v is not in the clique
  - Throw away v (and all incident edges) and pick another vertex

A Brief and Incomplete Guide to Clique Solving (2/4)

Key data structures:

- Growing clique C
- Shrinking set of potential vertices P
  - All the vertices we haven't thrown away yet
  - Every $v \in P$ is adjacent to every $w \in C$

A Brief and Incomplete Guide to Clique Solving (2/4)

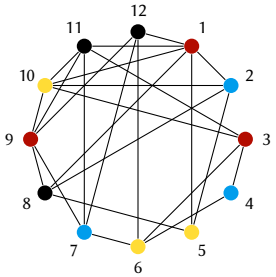
Key data structures:

- Growing clique C
- Shrinking set of potential vertices P
  - All the vertices we haven't thrown away yet
  - Every $v \in P$ is adjacent to every $w \in C$

Branch and bound:

- Remember the biggest clique $C^\star$ found so far
- If $|C| + |P| \leq |C^\star|$, no need to keep going

A Brief and Incomplete Guide to Clique Solving (3/4)



Given a k-colouring of a subgraph, that subgraph cannot have a clique of more than k vertices

We can use $|C| + \#colours(P)$ as a bound, for any colouring

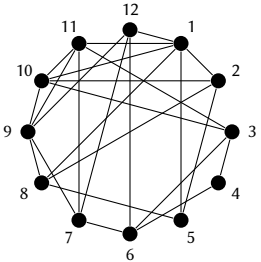
A Brief and Incomplete Guide to Clique Solving (4/4)

- This brings us to 1997
- Many improvements since then
  - better bound functions
  - clever vertex selection heuristics
  - efficient data structures
  - local search
  - ...
- But key ideas for proof logging can be explained without worrying about such things

Making a Certifying Clique Solver

1. Output a pseudo-Boolean encoding of the problem
   - Clique problems have several standard file formats
   - Give the constraints labels, so the proof can refer to them by name
2. Make solver generate proof log for search tree
   - Output a small header
   - Output something on every backtrack
   - Output something every time a solution is found
   - Output a small footer
3. Figure out how to certify the bound function

A Pseudo-Boolean Encoding for Clique (in OPB Format)




```
* #variable= 12 #constraint= 41
min: 1 ~x1 1 ~x2 1 ~x3 1 ~x4 . . . and so on. . .  1 ~x11 1 ~x12 ;
@noedge1_3 -1 x3 -1 x1 >= -1 ;
@noedge2_3 -1 x3 -1 x2 >= -1 ;
@noedge1_4 -1 x4 -1 x1 >= -1 ;
* . . . and a further 38 similar lines for the remaining non-edges
```

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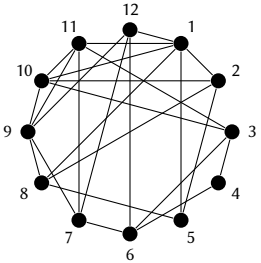


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* . . . and a further 38 similar lines for the remaining non-edges
```
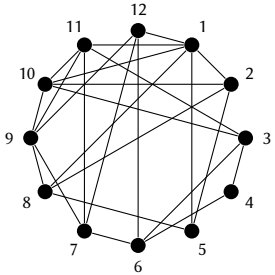

$x_i = 1$ means "vertex i is in the clique"

VeriPB proof minimises number of vertices not selected, so 4-clique yields objective value $12 - 4 = 8$

First Attempt at a Proof

```
pseudo-Boolean proof version 3.0
@obj soli x7 x9 x12 ;
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
@obj soli x1 x2 x5 x8 ;
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE ;
conclusion BOUNDS 8 8 ;
end pseudo-Boolean proof ;
```



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@obj soli x7 x9 x12 ;
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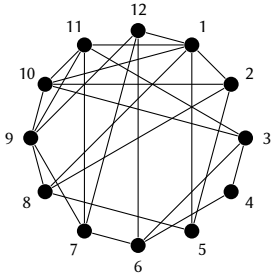

Start with a header
The problem constraints are loaded for us

First Attempt at a Proof

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pseudo-Boolean proof version 3.0
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rup 1 ~x12 1 ~x7 >= 1 ;
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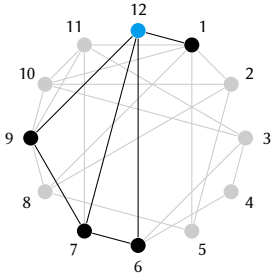

Branch accepting 12
Throw away non-adjacent vertices

First Attempt at a Proof

```
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@obj soli x7 x9 x12 ;
rup 1 ~x12 1 ~x7 >= 1 ;
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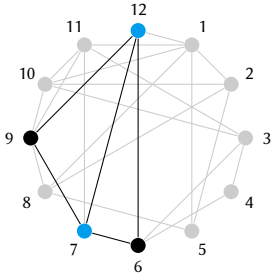

Branch also accepting 7
Throw away non-adjacent vertices

First Attempt at a Proof

```
pseudo-Boolean proof version 3.0
@obj soli x7 x9 x12 ;
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
@obj soli x1 x2 x5 x8 ;
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
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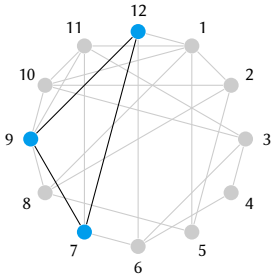
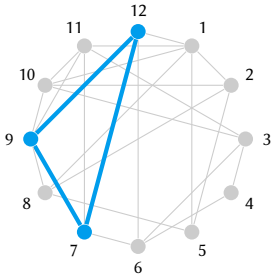
conclusion BOUNDS 8 8 ;
end pseudo-Boolean proof ;
```



Branch also accepting 9
Throw away non-adjacent vertices

First Attempt at a Proof

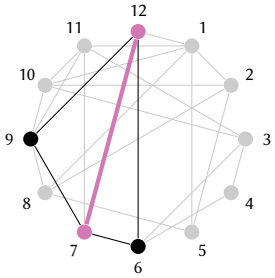
```
pseudo-Boolean proof version 3.0
@obj soli x7 x9 x12 ;
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
@obj soli x1 x2 x5 x8 ;
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE ;
conclusion BOUNDS 8 8 ;
end pseudo-Boolean proof ;
```



We branched on 12, 7, 9
Found a new incumbent
Now looking for a ≥ 4 vertex clique
"@obj" names the constraint this creates

First Attempt at a Proof

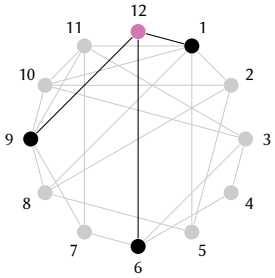
```
pseudo-Boolean proof version 3.0
@obj soli x7 x9 x12 ;
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
@obj soli x1 x2 x5 x8 ;
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE ;
conclusion BOUNDS 8 8 ;
end pseudo-Boolean proof ;
```



Backtrack from 12, 7
9 explored already, only 6 feasible
No $\geq$ 4 vertex clique possible
Effectively this deletes the 7–12 edge

First Attempt at a Proof

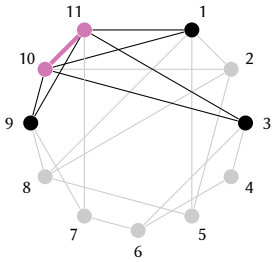
```
pseudo-Boolean proof version 3.0
@obj soli x7 x9 x12 ;
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
@obj soli x1 x2 x5 x8 ;
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE ;
conclusion BOUNDS 8 8 ;
end pseudo-Boolean proof ;
```



Backtrack from 12
Only 1, 6 and 9 feasible (1-colourable)
No $\geq$ 4 vertex clique possible
Effectively this deletes vertex 12

First Attempt at a Proof

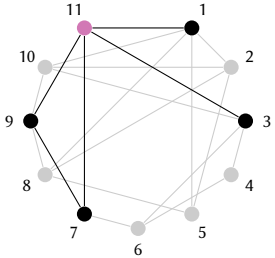
```
pseudo-Boolean proof version 3.0
@obj soli x7 x9 x12 ;
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
@obj soli x1 x2 x5 x8 ;
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE ;
conclusion BOUNDS 8 8 ;
end pseudo-Boolean proof ;
```



Branch on 11 then 10
Only 1, 3 and 9 feasible (1-colourable)
No $\geq$ 4 vertex clique possible
Backtrack, deleting the edge

First Attempt at a Proof

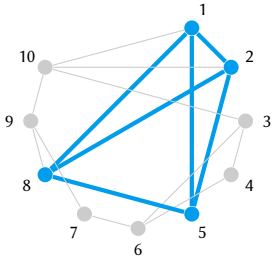
```
pseudo-Boolean proof version 3.0
@obj soli x7 x9 x12 ;
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
@obj soli x1 x2 x5 x8 ;
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE ;
conclusion BOUNDS 8 8 ;
end pseudo-Boolean proof ;
```



Backtrack from 11
2-colourable, so no ≥ 4 clique
Delete the vertex

First Attempt at a Proof

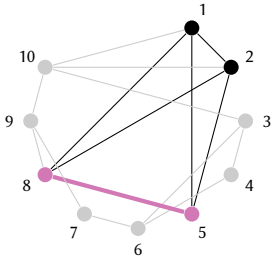
```
pseudo-Boolean proof version 3.0
@obj soli x7 x9 x12 ;
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
@obj soli x1 x2 x5 x8 ;
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE ;
conclusion BOUNDS 8 8 ;
end pseudo-Boolean proof ;
```



Branch on 8, 5, 1, 2
Find a new incumbent
Now looking for a ≥ 5 vertex clique

First Attempt at a Proof

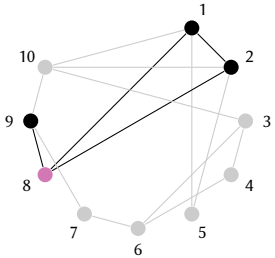
```
pseudo-Boolean proof version 3.0
@obj soli x7 x9 x12 ;
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
@obj soli x1 x2 x5 x8 ;
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE ;
conclusion BOUNDS 8 8 ;
end pseudo-Boolean proof ;
```



Backtrack from 8, 5
Only 4 vertices; can't have a ≥ 5 clique
Delete the edge

First Attempt at a Proof

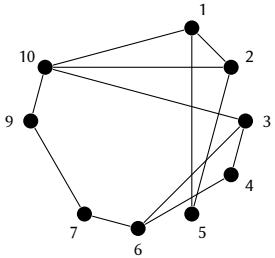
```
pseudo-Boolean proof version 3.0
@obj soli x7 x9 x12 ;
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
@obj soli x1 x2 x5 x8 ;
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE ;
conclusion BOUNDS 8 8 ;
end pseudo-Boolean proof ;
```



Backtrack from 8
Still not enough vertices
Delete the vertex

First Attempt at a Proof

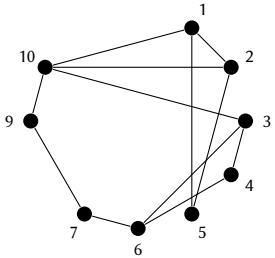
```
pseudo-Boolean proof version 3.0
@obj soli x7 x9 x12 ;
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
@obj soli x1 x2 x5 x8 ;
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE ;
conclusion BOUNDS 8 8 ;
end pseudo-Boolean proof ;
```



Remaining graph is 3-colourable
Backtrack from root node

First Attempt at a Proof

```
pseudo-Boolean proof version 3.0
@obj soli x7 x9 x12 ;
rup 1 ~x12 1 ~x7 >= 1 ;
rup 1 ~x12 >= 1 ;
rup 1 ~x11 1 ~x10 >= 1 ;
rup 1 ~x11 >= 1 ;
@obj soli x1 x2 x5 x8 ;
rup 1 ~x8 1 ~x5 >= 1 ;
rup 1 ~x8 >= 1 ;
rup >= 1 ;
output NONE ;
conclusion BOUNDS 8 8 ;
end pseudo-Boolean proof ;
```



Finish with what we've concluded
We specify a lower and an upper bound
Remember we're minimising the number of vertices we *don't* select, so a 4-clique has an objective value of $12 - 4 = 8$
("output" is for features we're not using)

Verifying This Proof (Or Not...)

```
$ veripb clique.opb clique-attempt-one.pbp
Running VeriPB version 3.0.2
Error: Checking error at clique-attempt-one.pbp:5

Caused by:
        The constraint is not implied by reverse unit propagation
        (RUP) from core and derived database. Rerun with the option
        '--trace-failed' to see the propagations. . . .
```

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```
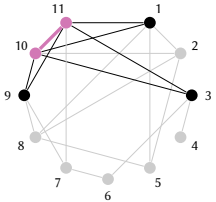



Verifying This Proof (Or Not...)

```
$ veripb --trace clique.opb clique-attempt-one.pbp
Running VeriPB version 3.0.2
  Objective: min 1 ~x1 1 ~x2 . . .  1 ~x11 1 ~x12 0
  ConstraintId 1: 1 ~x1 1 ~x3 >= 1
  ConstraintId 2: 1 ~x2 1 ~x3 >= 1
  . . .
  ConstraintId 41: 1 ~x11 1 ~x12 >= 1
line    1: pseudo-Boolean proof version 3.0
line    2: @obj soli x7 x9 x12 ;
  ConstraintId 42: 1 x1 1 x2 . . .  1 x11 1 x12 >= 4
line    3: rup 1 ~x12 1 ~x7 >= 1 ;
  ConstraintId 43: 1 ~x7 1 ~x12 >= 1
line    4: rup 1 ~x12 >= 1 ;
  ConstraintId 44: 1 ~x12 >= 1
line    5: rup 1 ~x11 1 ~x10 >= 1 ;
Error: Checking error at clique-attempt-one.pbp:5
```

Verifying This Proof (Or Not...)

```
$ veripb --trace-failed clique.opb clique-attempt-one.pbp
Running VeriPB version 3.0.2
Propagation check failed! The propagation had the following trail:
  propagations in format: <assignment> (<reason constraint>)
    ~x12 (1 ~x12 >= 1)
    x10 (1 x10 1 x11 >= 2)
    x11 (1 x10 1 x11 >= 2)
    ~x4 (1 ~x4 1 ~x10 >= 1)
    ~x5 (1 ~x5 1 ~x10 >= 1)
    ~x6 (1 ~x6 1 ~x10 >= 1)
    ~x7 (1 ~x7 1 ~x10 >= 1)
    ~x8 (1 ~x8 1 ~x10 >= 1)
    ~x2 (1 ~x2 1 ~x11 >= 1)
Error: Checking error at clique-attempt-one.pbp:5
```

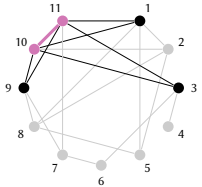
The Problem of Counting

The proof checker isn't smart enough to work out that

- vertices 1, 3 and 9 can all be given the same colour…
- and each colour class contributes at most one vertex to a clique…
- so "find me a clique with more than three vertices" can't be satisfied once we've accepted both 10 and 11

Unit propagation never sees this:

- The bound needs counting
- But the encoding only ever talks about pairs of vertices



Dealing With Colourings

So the colour bound doesn't follow by reverse unit propagation...

But we can lazily recover at-most-one constraints for each colour class!

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$$
\begin{aligned}
&(\overline{x}_1 + \overline{x}_6 \geq 1) && \\
+\,&(\overline{x}_1 + \overline{x}_9 \geq 1) && = 2\overline{x}_1 + \overline{x}_6 + \overline{x}_9 \geq 2 \\
+\,&(\overline{x}_6 + \overline{x}_9 \geq 1) && = 2\overline{x}_1 + 2\overline{x}_6 + 2\overline{x}_9 \geq 3 \\
&/\,2 && = \overline{x}_1 + \overline{x}_6 + \overline{x}_9 \geq 2 \\
& && \text{i.e. } x_1 + x_6 + x_9 \leq 1
\end{aligned}
$$

Dealing With Colourings

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&/\,2 && = \overline{x}_1 + \overline{x}_6 + \overline{x}_9 \geq 2 \\
& && \text{i.e. } x_1 + x_6 + x_9 \leq 1
\end{aligned}
$$

This generalises to colour classes of any size v

- Each non-edge is used exactly once, $v(v-1)$ additions
- $v-3$ multiplications and $v-2$ divisions

Solvers don't need to "understand" cutting planes to write out this derivation

Details on At-Most-One Constraint Recovery (3 Variables)

- Derive constraint $x_1 + \ldots + x_n \leq 1$ from $\binom{n}{2}$ constraints $x_i + x_j \leq 1$
- I.e., $\overline{x}_1 + \ldots + \overline{x}_n \geq n - 1$ from constraints $\overline{x}_i + \overline{x}_j \geq 1$ in normalised form
- Need this to prove that at most one vertex per colour class can be a member of the clique

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$$\text{Add}\ \frac{\overline{x}_1 + \overline{x}_2 \geq 1 \qquad \overline{x}_1 + \overline{x}_3 \geq 1}{2\overline{x}_1 + \overline{x}_2 + \overline{x}_3 \geq 2}$$

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$$\text{Add}\ \dfrac{\text{Add}\ \dfrac{\overline{x}_1 + \overline{x}_2 \geq 1 \qquad \overline{x}_1 + \overline{x}_3 \geq 1}{2\overline{x}_1 + \overline{x}_2 + \overline{x}_3 \geq 2} \qquad \overline{x}_2 + \overline{x}_3 \geq 1}{2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3}$$

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$$\text{Divide by 2}\ \dfrac{\text{Add}\ \dfrac{\text{Add}\ \dfrac{\overline{x}_1 + \overline{x}_2 \geq 1 \qquad \overline{x}_1 + \overline{x}_3 \geq 1}{2\overline{x}_1 + \overline{x}_2 + \overline{x}_3 \geq 2} \qquad \overline{x}_2 + \overline{x}_3 \geq 1}{2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3}}{\overline{x}_1 + \overline{x}_2 + \overline{x}_3 \geq \frac{3}{2}}$$

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- Derive constraint $x_1 + \ldots + x_n \leq 1$ from $\binom{n}{2}$ constraints $x_i + x_j \leq 1$
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- Derive constraint $x_1 + \ldots + x_n \leq 1$ from $\binom{n}{2}$ constraints $x_i + x_j \leq 1$
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$$\text{Divide by 2}\ \dfrac{\text{Add}\ \dfrac{\text{Add}\ \dfrac{\overline{x}_1 + \overline{x}_2 \geq 1 \qquad \overline{x}_1 + \overline{x}_3 \geq 1}{2\overline{x}_1 + \overline{x}_2 + \overline{x}_3 \geq 2} \qquad \overline{x}_2 + \overline{x}_3 \geq 1}{2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 3}}{\overline{x}_1 + \overline{x}_2 + \overline{x}_3 \geq 2}$$

Assuming constraint $\overline{x}_i + \overline{x}_j \geq 1$ has label `@Cij`, get VeriPB code

```
pol @C12 @C13 + @C23 + 2 d ;
```

Details on At-Most-One Constraint Recovery (4 and More Variables)

- Derive constraint $x_1 + \ldots + x_n \leq 1$ from $\binom{n}{2}$ constraints $x_i + x_j \leq 1$
- I.e., $\overline{x}_1 + \ldots + \overline{x}_n \geq n - 1$ from constraints $\overline{x}_i + \overline{x}_j \geq 1$ in normalised form
- Need this to prove that at most one vertex per colour class can be a member of the clique

$$\overline{x}_1 + \overline{x}_2 + \overline{x}_3 \geq 2$$

Details on At-Most-One Constraint Recovery (4 and More Variables)

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- Need this to prove that at most one vertex per colour class can be a member of the clique

$$\text{Multiply by 2} \quad \frac{\overline{x}_1 + \overline{x}_2 + \overline{x}_3 \geq 2}{2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 4}$$

Details on At-Most-One Constraint Recovery (4 and More Variables)

- Derive constraint $x_1 + \ldots + x_n \leq 1$ from $\binom{n}{2}$ constraints $x_i + x_j \leq 1$
- I.e., $\overline{x}_1 + \ldots + \overline{x}_n \geq n - 1$ from constraints $\overline{x}_i + \overline{x}_j \geq 1$ in normalised form
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$$\dfrac{\dfrac{\dfrac{\dfrac{\overline{x}_1 + \overline{x}_2 + \overline{x}_3 \geq 2}{2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 4}\ \text{Multiply by 2} \qquad \overline{x}_1 + \overline{x}_4 \geq 1}{3\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 + \overline{x}_4 \geq 5}\ \text{Add} \qquad \overline{x}_2 + \overline{x}_4 \geq 1}{3\overline{x}_1 + 3\overline{x}_2 + 2\overline{x}_3 + 2\overline{x}_4 \geq 6}\ \text{Add} \qquad \overline{x}_3 + \overline{x}_4 \geq 1}{3\overline{x}_1 + 3\overline{x}_2 + 3\overline{x}_3 + 3\overline{x}_4 \geq 7}\ \text{Add}$$

Details on At-Most-One Constraint Recovery (4 and More Variables)

- Derive constraint $x_1 + \ldots + x_n \leq 1$ from $\binom{n}{2}$ constraints $x_i + x_j \leq 1$
- I.e., $\overline{x}_1 + \ldots + \overline{x}_n \geq n - 1$ from constraints $\overline{x}_i + \overline{x}_j \geq 1$ in normalised form
- Need this to prove that at most one vertex per colour class can be a member of the clique

$$\text{Multiply by 2}\ \overline{x}_1 + \overline{x}_2 + \overline{x}_3 \geq 2$$

$$\text{Add}\ 2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 4 \qquad \overline{x}_1 + \overline{x}_4 \geq 1$$

$$\text{Add}\ 3\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 + \overline{x}_4 \geq 5 \qquad \overline{x}_2 + \overline{x}_4 \geq 1$$

$$\text{Add}\ 3\overline{x}_1 + 3\overline{x}_2 + 2\overline{x}_3 + 2\overline{x}_4 \geq 6 \qquad \overline{x}_3 + \overline{x}_4 \geq 1$$

$$\text{Divide by 3}\ 3\overline{x}_1 + 3\overline{x}_2 + 3\overline{x}_3 + 3\overline{x}_4 \geq 7$$

$$\overline{x}_1 + \overline{x}_2 + \overline{x}_3 + \overline{x}_4 \geq \left\lceil \frac{7}{3} \right\rceil$$

Details on At-Most-One Constraint Recovery (4 and More Variables)

- Derive constraint $x_1 + \ldots + x_n \leq 1$ from $\binom{n}{2}$ constraints $x_i + x_j \leq 1$
- I.e., $\overline{x}_1 + \ldots + \overline{x}_n \geq n - 1$ from constraints $\overline{x}_i + \overline{x}_j \geq 1$ in normalised form
- Need this to prove that at most one vertex per colour class can be a member of the clique

$$\text{Multiply by 2}\ \frac{\overline{x}_1 + \overline{x}_2 + \overline{x}_3 \geq 2}{2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 4}$$

$$\text{Add}\ \frac{2\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 \geq 4 \qquad \overline{x}_1 + \overline{x}_4 \geq 1}{3\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 + \overline{x}_4 \geq 5}$$

$$\text{Add}\ \frac{3\overline{x}_1 + 2\overline{x}_2 + 2\overline{x}_3 + \overline{x}_4 \geq 5 \qquad \overline{x}_2 + \overline{x}_4 \geq 1}{3\overline{x}_1 + 3\overline{x}_2 + 2\overline{x}_3 + 2\overline{x}_4 \geq 6}$$

$$\text{Add}\ \frac{3\overline{x}_1 + 3\overline{x}_2 + 2\overline{x}_3 + 2\overline{x}_4 \geq 6 \qquad \overline{x}_3 + \overline{x}_4 \geq 1}{3\overline{x}_1 + 3\overline{x}_2 + 3\overline{x}_3 + 3\overline{x}_4 \geq 7}$$

$$\text{Divide by 3}\ \frac{3\overline{x}_1 + 3\overline{x}_2 + 3\overline{x}_3 + 3\overline{x}_4 \geq 7}{\overline{x}_1 + \overline{x}_2 + \overline{x}_3 + \overline{x}_4 \geq 3}$$

Details on At-Most-One Constraint Recovery (4 and More Variables)

- Derive constraint $x_1 + \ldots + x_n \leq 1$ from $\binom{n}{2}$ constraints $x_i + x_j \leq 1$
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pol @C12 @C13 + @C23 + 2 d 2 * @C14 + @C24 + @C34 + 3 d ;
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```

(and this pattern generalizes to any number of variables)

What The Revised Proof with Explicit Counting Looks Like

```
pseudo-Boolean proof version 3.0
@obj soli x7 x9 x12 ;
rup 1 ~x12 1 ~x7 >= 1 ;
pol @noedge1_6 @noedge1_9 + @noedge6_9 + 2 d @obj + ;
rup 1 ~x12 >= 1 ;
pol @noedge1_3 @noedge1_9 + @noedge3_9 + 2 d @obj + ;
rup 1 ~x11 1 ~x10 >= 1 ;
pol @noedge1_3 @noedge1_7 + @noedge3_7 + 2 d @obj + ;
rup 1 ~x11 >= 1 ;
@obj soli x1 x2 x5 x8 ;
rup 1 ~x8 1 ~x5 >= 1 ;
pol @obj @noedge1_9 + ;
rup 1 ~x8 >= 1 ;
pol @noedge1_3 @noedge1_7 + @noedge3_7 + 2 d @noedge2_4 @noedge2_9 + @noedge4_9 + 2 d +
    @noedge5_6 @noedge5_10 + @noedge6_10 + 2 d + @obj + ;
rup >= 1 ;
output NONE ;
conclusion BOUNDS 8 8 ;
end pseudo-Boolean proof ;
```

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Failing RUP steps have become pol derivations: add three non-adjacency constraints together, divide by two, then add the result to the solution-improving constraint

The labels come straight out of the OPB file

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    @noedge5_6 @noedge5_10 + @noedge6_10 + 2 d + @obj + ;
rup >= 1 ;
output NONE ;
conclusion BOUNDS 8 8 ;
end pseudo-Boolean proof ;
```

Three colour classes here, so three at-most-one constraints, all added to the solution-improving constraint

Verifying This Proof (For Real, This Time)

```
$ veripb clique.opb clique-new-attempt.pbp
Running VeriPB version 3.0.2
s VERIFIED BOUNDS 8 <= obj <= 8
```

No errors this time — every step is justified

Note that the checker knows nothing about

- cliques,
- colourings, or
- bound functions

Verifying This Proof (For Real, This Time)

```
$ veripb --trace clique.opb clique-new-attempt.pbp
Running VeriPB version 3.0.2
Objective: min 1 ~x1 1 ~x2 . . .  1 ~x11 1 ~x12 0
ConstraintId 1: 1 ~x1 1 ~x3 >= 1
. . .
ConstraintId 41: 1 ~x11 1 ~x12 >= 1
line    2: @obj soli x7 x9 x12 ;
ConstraintId 42: 1 x1 1 x2 . . .  1 x11 1 x12 >= 4
line    3: rup 1 ~x12 1 ~x7 >= 1 ;
ConstraintId 43: 1 ~x7 1 ~x12 >= 1
line    4: pol @noedge1_6 @noedge1_9 + @noedge6_9 + 2 d @obj + ;
ConstraintId 44: 1 x2 1 x3 1 x4 1 x5 1 x7 1 x8 1 x10 1 x11 1 x12 >= 3
  . . .
line   16: rup >= 1 ;
ConstraintId 55: >= 1
s VERIFIED BOUNDS 8 <= obj <= 8
```

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Cutting planes to the rescue!

- Turns out to be strong enough for all of these cases
- Proof length scales linearly with solver reasoning time (asymptotically)

Different Clique Algorithms

Different search orders?

- ✓ Irrelevant for proof logging

Using local search to initialise?

- ✓ Just log the incumbent

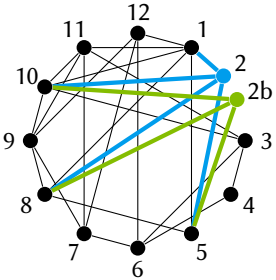
Different bound functions?

- Is cutting planes strong enough to justify every useful bound function ever invented?
- So far, seems like it…

Weighted cliques?

- ✓ Multiply a colour class by its largest weight
- ✓ Also works for vertices "split between colour classes"

Lazy Global Domination for Maximum Clique [MP16]



Can ignore vertex 2b

- Every neighbour of 2b is also a neighbour of 2
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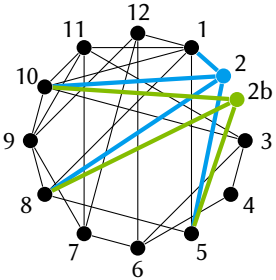

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Dominance rule can justify this

- Even when detected dynamically during search

Reducing the Trust Base

We are still trusting rather a lot

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  - reads the original graph file
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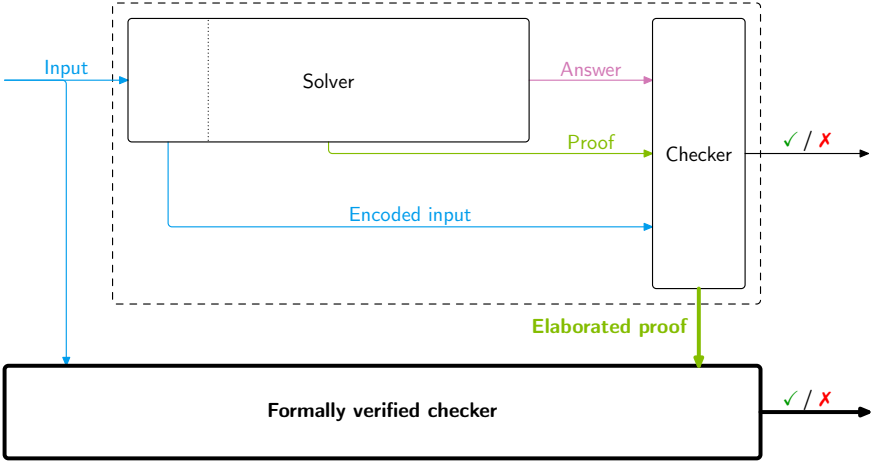
  - checks the kernel proof against this translation
- So the answer is verified in terms of the original input

End-to-End Verification for Maximum Clique

```
$ glasgow_clique_solver brock200_4.clq
recursions = 56613
clique = 17 (12 19 28 29 38 54 65 71 79 93 117 127 139 161 165 186 192)
runtime = 0.02153s

$ glasgow_clique_solver brock200_4.clq --prove brock200_4
runtime = 0.266475s

$ cake_pb_clique brock200_4.clq > brock200_4.verifiedopb

$ veripb --elaborate brock200_4.corepb brock200_4.verifiedopb brock200_4.pbp
s VERIFIED BOUNDS 183 <= obj <= 183

$ cake_pb_clique brock200_4.clq brock200_4.corepb
s VERIFIED MAX CLIQUE SIZE |CLIQUE| = 17
```

Workflow for Solving with End-to-End Certified Results



What's Left to Trust?

Still have to trust

- The HOL4 theorem prover
- That the formal model of the CakeML environment matches the hardware it runs on
- The HOL4 definition of what it means to be a maximum clique
- Input parsing and output formatting

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- The HOL4 theorem prover
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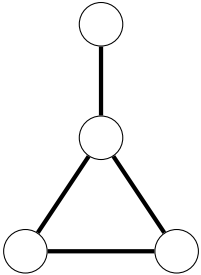
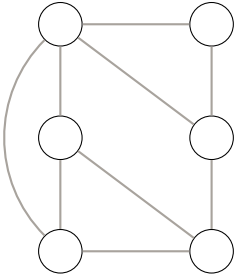
No need to trust, or even to know about

- How the solver works
- That the VeriPB proof system is sound
- Or even what "pseudo-Boolean" means

Subgraph Isomorphism





- Find the pattern inside the target
- Applications in compilers, biochemistry, model checking, pattern recognition, …
- Often want to find all matches

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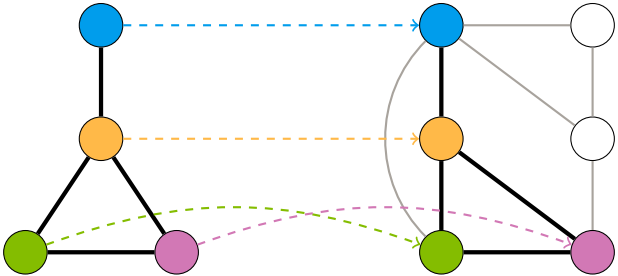

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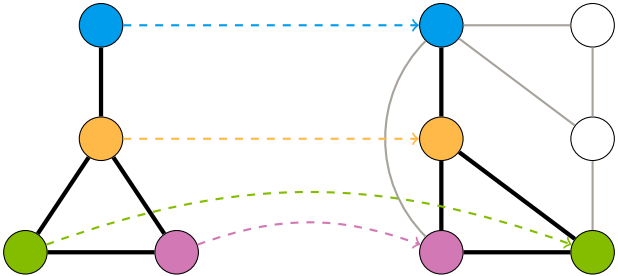

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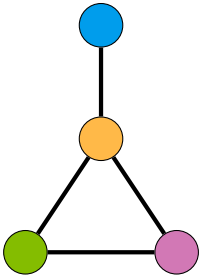

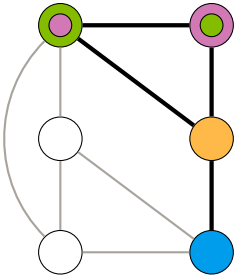

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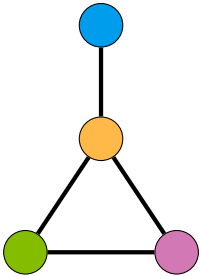

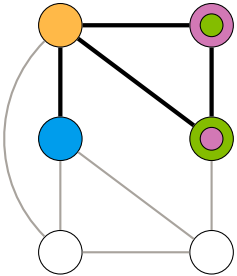

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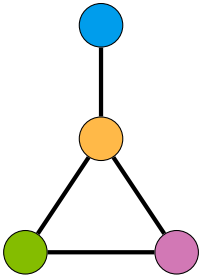

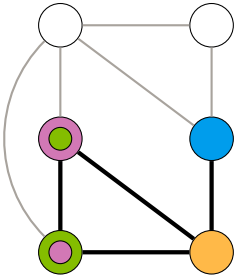

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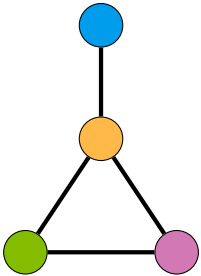

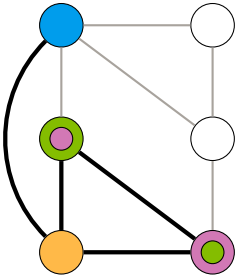

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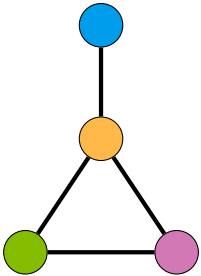

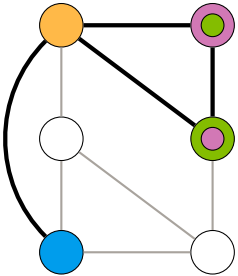

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Subgraph Isomorphism in Pseudo-Boolean Form

Each pattern vertex gets a target vertex:

$$\sum_{t \in \mathrm{V}(T)} x_{p,t} = 1 \qquad p \in \mathrm{V}(P)$$

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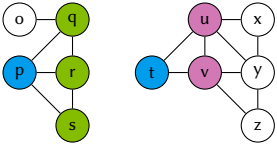
Adjacency constraints: if p maps to t, then neighbours of p map to neighbours of t:

$$\overline{x}_{p,t} + \sum_{u \in \mathrm{N}(t)} x_{q,u} \geq 1 \qquad p \in \mathrm{V}(P),\ q \in \mathrm{N}(p),\ t \in \mathrm{V}(T)$$

Degree Reasoning in Cutting Planes

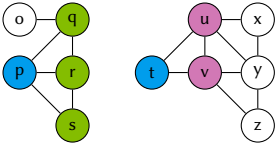


Pattern vertex p of degree $\deg(p)$ can never be mapped to target vertex t of degree $< \deg(p)$

Observe $\mathrm{N}(p) = \{q, r, s\}$ and $\mathrm{N}(t) = \{u, v\}$

We wish to derive $\overline{x}_{p,t} \geq 1$

Degree Reasoning in Cutting Planes



Adjacency:

$$\overline{x}_{p,t} + x_{q,u} + x_{q,v} \geq 1$$
$$\overline{x}_{p,t} + x_{r,u} + x_{r,v} \geq 1$$
$$\overline{x}_{p,t} + x_{s,u} + x_{s,v} \geq 1$$

Injectivity:

$$-x_{o,u} + -x_{p,u} + -x_{q,u} + -x_{r,u} + -x_{s,u} \geq -1$$
$$-x_{o,v} + -x_{p,v} + -x_{q,v} + -x_{r,v} + -x_{s,v} \geq -1$$

Literal axioms:

$$x_{o,u} \geq 0$$
$$x_{o,v} \geq 0$$
$$x_{p,u} \geq 0$$
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Add these together …

$$3 \cdot \overline{x}_{p,t} \geq 1$$

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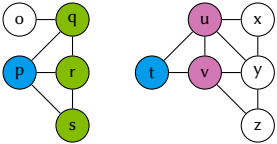

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Literal axioms:

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Add these together and divide by 3 to get

$$\overline{x}_{p,t} \geq 1$$

Degree Reasoning in VeriPB

```
pol @adj_p_t_q @adj_p_t_r + @adj_p_t_s +   % sum adjacency constraints
    @inj_u + @inj_v +                      % sum injectivity constraints
    xo_u + xo_v +                          % cancel stray xo_* >= 0
    xp_u + xp_v +                          % cancel stray xp_* >= 0
    3 d ;                                  % divide, and we're done
```

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Or we can let VeriPB do the last bit of tidying up for us:

```
pol @adj_p_t_q @adj_p_t_r + @adj_p_t_s +   % sum adjacency constraints
    @inj_u + @inj_v + ;                    % sum injectivity constraints
rup 1 ~xp_t >= 1 ;                         % implied by the above
```

Proof Logging for Other Forms of Reasoning

Possible to certify (almost) all of the other things state-of-the-art subgraph solvers do:

- Conditional injectivity reasoning and filtering
- Distance filtering
- Neighbourhood degree sequences
- Path filtering
- Supplemental graphs

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Code for the certifying *Glasgow Subgraph Solver* at
https://github.com/ciaranm/glasgow-subgraph-solver

Limitations

Why trust the encoding?

- ✓ For clique and subgraph isomorphism this is now taken care of [GMM+24]
- Has to be done again for each new problem, but not too bad

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Works up to moderately-sized hard instances

- Even an $O(n^3)$ encoding is painful
- Particularly bad when the pseudo-Boolean encoding talks about "non-edges" but large sparse graphs are "easy"

What About Constraint Programming?

Non-Boolean variables?

Constraints?

- Encoding constraints in pseudo-Boolean form?
- Justifying inferences?

Reformulations?

Integer Variables in Constraint Programming: (1/2)

How to deal with integer variables in constraint programming?

Given $A \in \{-3, \ldots, 9\}$, the direct encoding is:

$$a_{=-3} + a_{=-2} + a_{=-1} + a_{=0} + a_{=1} + a_{=2} + a_{=3} + a_{=4} + a_{=5} + a_{=6} + a_{=7} + a_{=8} + a_{=9} = 1$$

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We could use a binary encoding:

$$-16a_{\text{neg}} + 1a_{\text{b0}} + 2a_{\text{b1}} + 4a_{\text{b2}} + 8a_{\text{b3}} \geq -3 \qquad \text{and}$$

$$16a_{\text{neg}} + -1a_{\text{b0}} + -2a_{\text{b1}} + -4a_{\text{b2}} + -8a_{\text{b3}} \geq -9$$

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Convention in what follows:

- Upper-case A, B, C are CP variables used by solver
- Lower-case a, b, c are corresponding Boolean variables used for PB encoding in proof log

Integer Variables in Constraint Programming: (2/2)

We can mix binary and order encodings! Define big-M linear inequalities encoding

$$a_{\geq 4} \Leftrightarrow -16a_{\text{neg}} + 1a_{\text{b0}} + 2a_{\text{b1}} + 4a_{\text{b2}} + 8a_{\text{b3}} \geq 4$$

$$a_{\geq 5} \Leftrightarrow -16a_{\text{neg}} + 1a_{\text{b0}} + 2a_{\text{b1}} + 4a_{\text{b2}} + 8a_{\text{b3}} \geq 5$$

$$a_{=4} \Leftrightarrow a_{\geq 4} \wedge \overline{a}_{\geq 5}$$

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When creating $a_{\geq i}$, also introduce pseudo-Boolean constraints encoding

$$a_{\geq i} \Rightarrow a_{\geq j} \quad \text{and} \quad a_{\geq h} \Rightarrow a_{\geq i}$$

for the closest values $j < i < h$ that already exist

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We can do this:

- Inside the pseudo-Boolean model where needed
- Otherwise lazily during proof logging

Compiling Constraints

- Also need to compile every constraint to pseudo-Boolean form
- Doesn't need to be a propagating encoding
- Can use additional variables

Compiling Linear Inequalities

Given inequality

$$2A + 3B + 4C \geq 42$$

where $A, B, C \in \{-3, \ldots, 9\}$

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Encode in pseudo-Boolean form as

$$\begin{aligned} &-32a_{\text{neg}} + 2a_{\text{b0}} + 4a_{\text{b1}} + 8a_{\text{b2}} + 16a_{\text{b3}} \\ +&-48b_{\text{neg}} + 3b_{\text{b0}} + 6b_{\text{b1}} + 12b_{\text{b2}} + 24b_{\text{b3}} \\ +&-64c_{\text{neg}} + 4c_{\text{b0}} + 8c_{\text{b1}} + 16c_{\text{b2}} + 32c_{\text{b3}} \geq 42 \end{aligned}$$

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Constraints can be specified extensionally as list of feasible tuples, called a table

Variable assignments must match some row in table

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Variable assignments must match some row in table

Given table constraint

$$(A, B, C) \in [(1, 2, 3), (1, 3, 4), (2, 2, 5)]$$

define

$$3\bar{t}_1 + a_{=1} + b_{=2} + c_{=3} \geq 3 \qquad \text{i.e., } t_1 \Rightarrow (a_{=1} \wedge b_{=2} \wedge c_{=3})$$
$$3\bar{t}_2 + a_{=1} + b_{=3} + c_{=4} \geq 3 \qquad \text{i.e., } t_2 \Rightarrow (a_{=1} \wedge b_{=3} \wedge c_{=4})$$
$$3\bar{t}_3 + a_{=2} + b_{=2} + c_{=5} \geq 3 \qquad \text{i.e., } t_3 \Rightarrow (a_{=2} \wedge b_{=2} \wedge c_{=5})$$

using newly introduced tuple selector variables

$$t_1 + t_2 + t_3 = 1$$

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Tuple variables used for proof logging — need not be known by solver (except for printing proofs)

Encoding Constraint Definitions

Already know how to do it for any constraint with a sane encoding using some combination of

- CNF
- Integer linear inequalities
- Table constraints
- Auxiliary variables

Simplicity is important, propagation strength isn't

Justifying Search

Mostly this works out of the box

Restarts are easy

No need to justify guesses or decisions — only justify backtracking

Justifying Inference

Key idea

Anything the constraint programming solver knows must follow from unit propagation of guessed assignments on constraints in proof log

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- A lot of propagators are effectively "doing a little bit of lookahead" but in an efficient way

A few need explicit cutting planes justifications written to the proof log

- Linear inequalities just need to multiply and add
- All-different needs a bit more

Justifying All-Different Failures

$$
\begin{array}{rcl}
V & \in & \{\ 1 \quad\quad\ \ 4 \quad 5\ \} \\
W & \in & \{\ 1 \quad 2 \quad 3 \quad\quad\ \}\\
X & \in & \{\ \quad\ \ 2 \quad 3 \quad\quad\ \}\\
Y & \in & \{\ 1 \quad\quad 3 \quad\quad\ \}\\
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\end{array}
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W \in & \{\ 1\ \ 2\ \ 3\ \quad\ \ \}\quad w_{=1} + \ w_{=2} + \ w_{=3} \quad \geq \ 1 \quad [\ W \text{ takes some value}\ ] \\
X \in & \{\quad\ \ 2\ \ 3\ \quad\ \ \} \\
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\end{array}
\qquad
\begin{array}{rcrcrcl}
w_{=1} & + & w_{=2} & + & w_{=3} & \geq & 1 \\
 & & x_{=2} & + & x_{=3} & \geq & 1 \\
y_{=1} & & & + & y_{=3} & \geq & 1 \\
z_{=1} & & & + & z_{=3} & \geq & 1
\end{array}
\qquad
\begin{array}{l}
[\ W \text{ takes some value}\ ] \\
[\ X \text{ takes some value}\ ] \\
[\ Y \text{ takes some value}\ ] \\
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\end{array}
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\begin{array}{rlrl}
V \in \{\ 1 & \quad\ \ 4\ \ 5\ \} & & \\
W \in \{\ 1\ \ 2\ \ 3 & \quad\ \ \ \ \ \ \ \ \ \ \} & w_{=1} + w_{=2} + w_{=3} \geq 1 & [\ W \text{ takes some value}\ ] \\
X \in \{\ \ \ \ 2\ \ 3 & \quad\ \ \ \ \ \ \ \ \ \ \} & x_{=2} + x_{=3} \geq 1 & [\ X \text{ takes some value}\ ] \\
Y \in \{\ 1\ \ \ \ \ 3 & \quad\ \ \ \ \ \ \ \ \ \ \} & y_{=1} + y_{=3} \geq 1 & [\ Y \text{ takes some value}\ ] \\
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$$
\begin{array}{rlrl}
\rightarrow & & -v_{=1} + -w_{=1} + -y_{=1} + -z_{=1} \geq -1 & [\ \text{At most one variable} = 1\ ] \\
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\\
 & -v_{=1} & \geq 1 & [\ \text{Sum all constraints so far}\ ]
\end{array}
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V & \in \{\ 1 \quad\quad\ \ 4\ \ 5\ \} & & \\
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\begin{array}{rlrrrrrrrl}
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$$

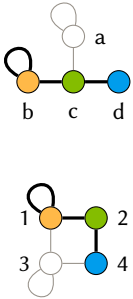
$$
\begin{array}{rlrl}
 & -v_{=1} & \geq 1 & [\ \text{Sum all constraints so far}\ ] \\
 & v_{=1} & \geq 0 & [\ \text{Variable } v_{=1} \text{ non-negative}\ ] \\
 & 0 & \geq 1 & [\ \text{Sum above two constraints}\ ]
\end{array}
$$

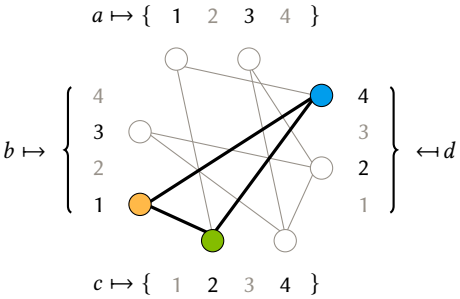
Reformulation

Auto-tabulation is possible

- Heavy use of extension variables

Can re-encode max common subgraph as clique problem without changing PB encoding





High Level Modelling Languages?

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How do we know we're giving a proof for the problem the user actually specified?

We don't

Need modelling language with formally specified semantics…

The *Glasgow Constraint Solver*

Support for 37 (families of) constraints (and counting), with all inferences certified:

Abs, AllDifferent, AllEqual, Among, AtMostOne, BinPacking, Circuit, Comparison, Count, Cumulative, Diffn, Disjunctive, DivMod, Element, Equals, GCC, In, Increasing, Inverse, Knapsack, Lex, Linear, Logical, MDD, MinDistance, MinMax, Multiply, NValue, Parity, PlusMinus, Power, Regular, SeqPrecedeChain, SmartTable, Sort, Table, ValuePrecede

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Support for 37 (families of) constraints (and counting), with all inferences certified:

Abs, AllDifferent, AllEqual, Among, AtMostOne, BinPacking, Circuit, Comparison, Count, Cumulative, Diffn, Disjunctive, DivMod, Element, Equals, GCC, In, Increasing, Inverse, Knapsack, Lex, Linear, Logical, MDD, MinDistance, MinMax, Multiply, NValue, Parity, PlusMinus, Power, Regular, SeqPrecedeChain, SmartTable, Sort, Table, ValuePrecede

Also certifies reformulations not written in but inferred from input:

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Details in [EGMN20, GMN22, MM23, MMN24, MM25] *(or, better still, talk to Ciaran)*
Code at `https://github.com/ciaranm/glasgow-constraint-solver`

Future Research Directions

Performance of pseudo-Boolean proof logging and checking

- Faster proof logging and checking!
- Trim proofs while checking as in SAT solving [HHW13a] *(upcoming work [ABB+26a])*
- Compress proof files using binary format

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- Lots of other challenging problems and interesting ideas
- Talk to us if you want to join the pseudo-Boolean proof logging revolution! ☺
  We're happy to collaborate, and we're hiring

Reminder about VeriPB Resources

VeriPB tutorials, technical documentation, and more information at
https://VeriPB.org

Tutorial series with videos and slides from *WHOOPS '25* workshop at
https://jakobnordstrom.se/WHOOPS25/

Technical details on different proof logging techniques covered in research papers [EGMN20, GMN20, GMM+20, GN21, GMN22, GMNO22, VDB22, BBN+23, BGMN23, MM23, BBN+24, DMM+24, GMM+24, HOGN24, IOT+24, MMN24, DHN+25, JBBJ25, KLM+25, MM25, ABB+26a, ABB+26b, DKK+26, MNOT26, VCB26] and PhD theses [Goc22, McI26, Oer26]

Lots of concrete example files at https://gitlab.com/MIAOresearch/software/VeriPB

Reminder about software for Exercises Tomorrow Thursday

Kindling Solver: See `README.md` and files in `slides-proofs/`

https://github.com/ciaranm/kindling

VeriPB (official version): See `README.md` and try to run `veripb justified.opb justified.pbp`

https://gitlab.com/MIAOresearch/software/VeriPB

VeriPB (experimental): Interactive mode (Linux and Mac only?); see `README.md` and then use `cargo run -p veripb-repl -m path-to-veripb-repl/Cargo.toml`

https://github.com/ciaranm/veripb-repl

Summing up

Second Half of Lecture

- More technical details on VeriPB (mostly for reference)
- Some convenient syntactic sugar
- Example applications
  1. Maximum clique
  2. Subgraph isomorphism
  3. Constraint programming *(much more about this tomorrow, including hands-on exercises)*

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- Certifying solvers producing machine-verifiable proofs is the most promising approach
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Thank you for your attention!

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