# Probability Extra Practice

// flip coin $K$ times; heads on $i^{th}$ flip $= \text{win } 5 \cdot i$ \$

// $X_i :=$ amount money won from $i^{th}$ flip

otherwise $= \text{win } -i^2$ \$

// $X :=$ random variable for amount of money won across all $K$ flips

## 1) $K = 1$

### 1.a) Sample Space

$$S = \{H, T\}$$

### 1.b) Compute expected value $X_1$

// $S = \{H, T\}$

$\Rightarrow i = 1$

$5 \cdot 1 = \$5$

$-1^2 = -\$1$

① $\Rightarrow X_1(s) = \begin{cases} \$5 & \text{if } s == H \\ -\$1 & \text{otherwise} \end{cases}$

outcome

$p(H) = 1/2$
$p(T) = 1/2$

// probability of H or T of our outcomes (s) is 1/2

② $\Rightarrow E(X_1) = \sum_{s \in S} p(s) \cdot X(s)$

$= p(H) \cdot X(H) + p(T) \cdot X(T)$

$= \frac{1}{2} \cdot 5 + \frac{1}{2} \cdot (-1)$

$= \$2$

$\Rightarrow p(X_1 = \$2) = 0\%$

// $X_1$ can be \$5 or \$-1

2) $k = 2$

## 2.a) Sample size

$2^2 = 4$ possible outcomes

$\Rightarrow S_2 = \{HH, HT, TH, TT\}$ // $S_2 = S_1 \times S_1$

$= \{HT\} \times \{HT\}$

## 2.b) random var. X as sum of two random variables

$\Rightarrow X_2 :=$ all money won from flip #2

$\Rightarrow$ 1 $X_2(s) = \begin{cases} \$10 & \text{if } s == H \\ -\$4 & \text{otherwise} \end{cases}$ // $5 \cdot i$ i $=$ heads

$(-i^2) =$ otherwise

$-2^2 = \$4$

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$\Rightarrow E(X_2) = P(H)\,X_2(H) + P(T)\,X_2(T)$

$= (1/2 \cdot \$10) + (1/2 \cdot (-\$4))$

$= \$3$

// can now find X a random variable as the sum of two random variables

$X :=$ total winnings across $k$-flips

// $s \in \{HH, HT, TH, TT\}$

$$\Rightarrow X(s) = X_1(s) + X_2(s)$$ // add amount from 1st flip to 2nd flip

ex: $X(HT) = X_1(HT) + X_2(HT)$

$= \$5 + (-\$4)$ // its $-4$ because we got $H$ on $X_1$ then $T$ on $X_2$

$= \$1$ ; $X = X_1 + X_2$

## 2.c) Compute EV of this RV using linearity of expectation

$$\Rightarrow E(X) = E(X_1 + X_2)$$

$= E(X_1) + E(X_2)$

$= \$2 + \$3$

$= \$5$ (EV on average for both flips)

# 3.) Consider a general formula for K-flips

## 3.a) give formula for EV of the RV $X_i$

// $(1 \le i \le k)$

$\Rightarrow E(X_i) = \frac{1}{2}5i - \frac{1}{2}i^2$ // got here by seeing pattern in previous problems $(E(X_1)$ & $E(X_2))$

### - Sample Space

$\Rightarrow \underbrace{HHHH\ldots H}_{k \text{ flips}}$ $S_1 = \{H, T\}$ // define $S_1$ ($i=1$ sample space)

$\Rightarrow S = \underbrace{S_1 \times S_1 \times S_1 \times \ldots \times S_1}_{k} = S_1^k$ // sample space of k-dimensions

## 3.b) describe RV (X) as sum of K RV's

$$\Rightarrow X = \sum_{i=1}^{K} X_i = X_1 + X_2 + X_3 + \dots + X_K$$

## 3.c) Give formula for EV of this RV

$$\Rightarrow E(X) = \sum_{i=1}^{K} E(X_i)$$

// just plug & chug now

$\Rightarrow$ from 3.a) $E(X_i) = \frac{1}{2}5i - \frac{1}{2}i^2$

$$= \sum_{i=1}^{K} \frac{1}{2}5i - \frac{1}{2}i^2 = \frac{5}{2}\sum_{i=1}^{K} i - \frac{1}{2}\sum_{i=1}^{K} i^2$$

// can use sum rules to simplify

$$= \frac{5}{2} \cdot \frac{K(K+1)}{2} - \frac{1}{2} \cdot \frac{K(K+1)(2K+1)}{6}$$

$$= \frac{5K(K+1)}{4} - \frac{K(K+1)(2K+1)}{12}$$

// $(K^2+K)(2K+1)$

$= 2K^3 + K^2 + 2K^2 + K$

$= 2K^3 + 3K^2 + K$

$$= \frac{5K^2 + 5K}{4} \qquad = \frac{2K^3 + 3K^2 + K}{12}$$

$$\Rightarrow \in \Theta(K^2) \qquad \Rightarrow \in \Theta(K^3)$$

$\Rightarrow$ This tells us as K grows, the negative summation grows faster

## 3. d) How many times would we want to flip coin?

⇒ We test $K$ values until summation returns 0

$$X_s(s) = \begin{cases} \$25 & \text{if } s == H \\ -\$25 & \text{otherwise} \end{cases}$$

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⇒ let $K = 5$ ; $E(X_s) = \sum_{s \in S} p(s) X(s)$

$$\Rightarrow E(X_s) = \frac{1}{2} \cdot 25 - \frac{1}{2} \cdot 25$$

$$= \frac{25}{2} - \frac{25}{2}$$

$$= 0$$ // This indicates that any flip after $K=5$ will be negative

⇒ Now plug this value into $E\left(\sum_{i=1}^{K} X_i\right)$ $\therefore K=5$

↓

$$\Rightarrow E(X) = \sum_{i=1}^{5} E(X_i)$$

// $E(X_i) = \frac{1}{2}5i - \frac{1}{2}i^2$

// we know $X_1 = \$2$

$X_2 = \$3$

$$\Rightarrow \underset{i=1}{2} + \underset{i=2}{3} + \left(\frac{1}{2}5(3) - \frac{1}{2}(3)^2\right) + \left(\frac{1}{2}5(4) - \frac{1}{2}(4)^2\right) + \left(\frac{1}{2}5(5) - \frac{1}{2}(5)^2\right)$$

$$= 0$$

// Should $= 10$, then finished. D

A Can also plug 5 in for K

$$\Rightarrow \frac{K(K+1)}{2} - \frac{K(K+1)(2K+1)}{6} = \$10$$