Minimum Spanning Tree Problem

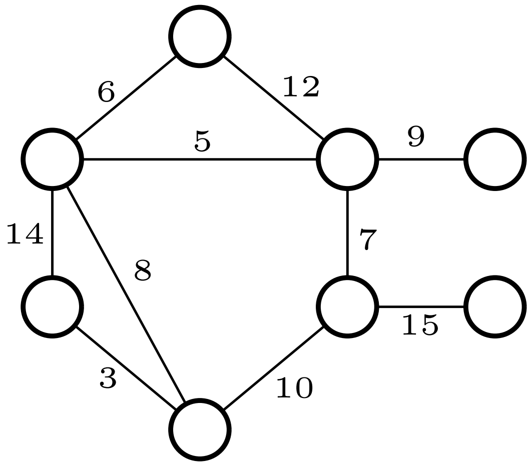
Input: A connected undirected graph $G = (V, E)$ with weight function $w : E \to \mathbb{R}$.

For simplicity, assume that all edge weights are distinct. (the algorithms will still work but the analysis is more difficult)

Output: A spanning tree T (a tree that connects all the vertices) of minimum weight:

$w(T) = \sum_{(u,v)\in T} w(u, v)$

Example: MST



How do we find the minimum spanning tree of a particular graph?

Useful theorem:

Theorem.
Let T be the MST of $G = (V, E)$, and let $A \subseteq V$. Suppose that $(u, v) \in E$ is the least-weight edge connecting A to $V \backslash A$. Then, $(u, v) \in T$.

(Proof skipped: see page 626 for additional details)

Remember, "A locally optimal choice is globally optimal" is the hallmark of a **greedy algorithm**.

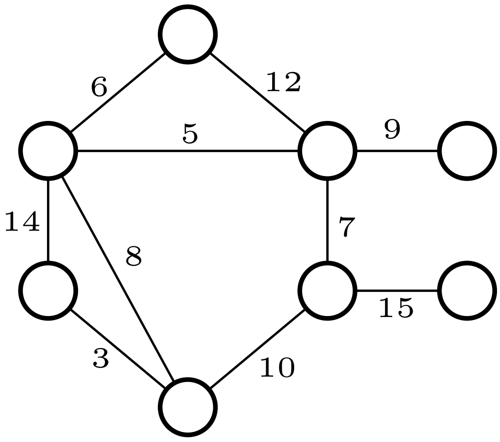
From this theorem, create a greedy algorithm to compute the MST. Roughly the idea is that we can repeatedly choose the least-weight edge that connects a new vertex to our tree.

Prim's Algorithm

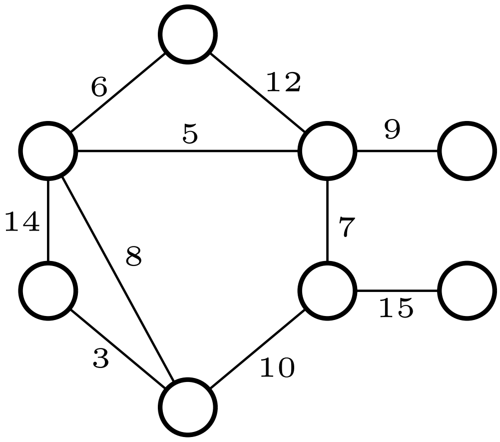
Idea: Maintain $V \backslash A$ as a priority queue Q. Key each vertex in Q with the weight of the least-weight edge connecting it to a vertex in A.

$Q = V$
Set $key[v] = \infty$ for all $v \in V$
Set $key[s] = 0$ for some arbitrary $s \in V$
while Q is not empty **do**
    $u = ExtractMin(Q)$
    for each $v \in Adj[u]$ **do**
        if $v \in Q$ and $w(u, v) < key[v]$ **then**
            $key[v] = w(u, v)$ //Decreases key in priority queue Q
            $\pi[v] = u$
        end if
    end for
end while

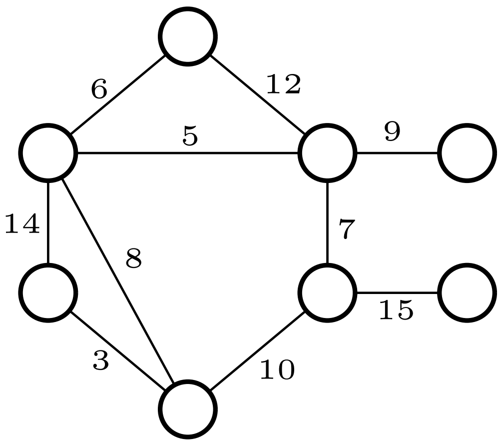
Prim's Example: (1)



Q:

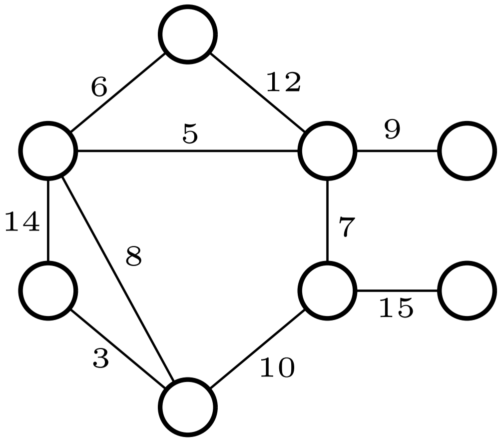


Q:

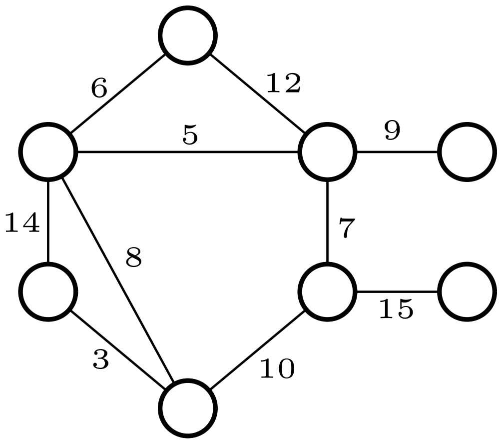


Q:

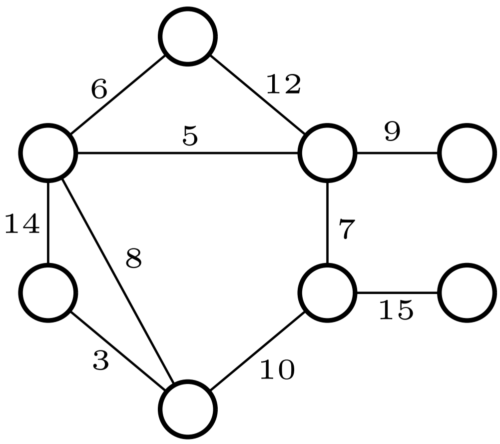
Prim's Example: (2)



Q:

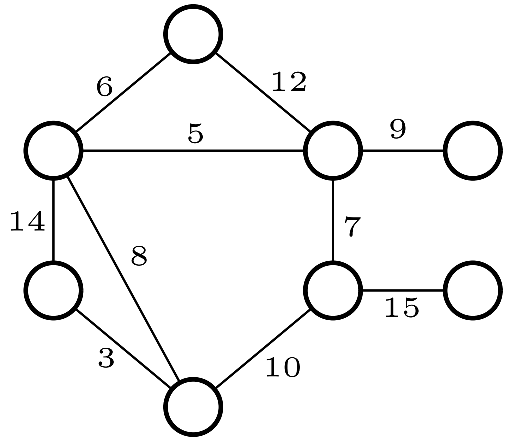


Q:

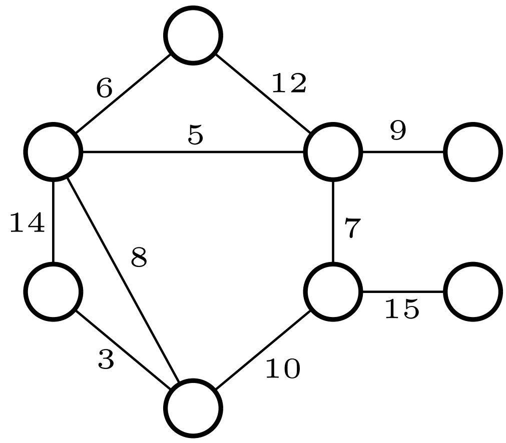


Q:

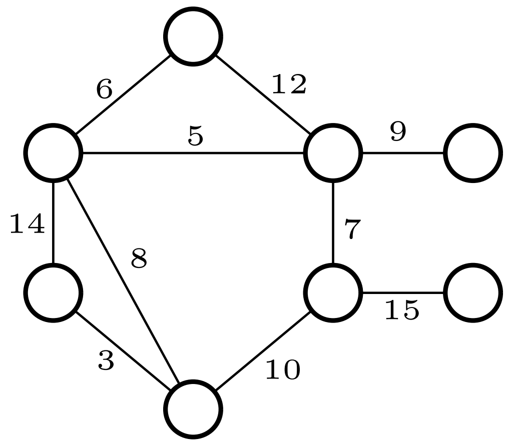
Prim's Example: (3)



Q:



Q:



Q:

Prim's Algorithm Runtime

```
Q = V
Set key[v] = ∞ for all v ∈ V
Set key[s] = 0 for some arbitrary s ∈ V
while Q is not empty do
    u = ExtractMin(Q)
    for each v ∈ Adj[u] do
        if v ∈ Q and w(u, v) < key[v] then
            key[v] = w(u, v)      //Decreases key in priority queue Q
            π[v] = u
        end if
    end for
end while
```

Q	$T_{EXTRACT-MIN}$	$T_{DECREASE-KEY}$	Total Runtime
array			
binary heap			
Fibonacci heap (not covered)			

Kruskal's Algorithm

Another greedy algorithm for computing a MST is Kruskal's.

Idea: Repeatedly pick edge with smallest weight as long as it does not form a cycle.

- The algorithm creates a set of trees (a forest)
- During the algorithm the added edges merge the trees together, such that in the end only one tree remains