18.3 Thermonuclear Reaction Rates

away from any resonances, $\xi \rightarrow 1$. In any case, with or without resonances, σ is proportional to $\lambda^2 P_0$, which depends on E as shown by (18.9) and (18.13). Therefore one usually writes

$$\sigma(E) = S E^{-1} \mathrm{e}^{-2\pi\eta}$$

where all remaining effects are contained within the here-defined "astrophysical factor" S. This factor contains all intrinsic nuclear properties of the reaction under consideration and can, in principle, be calculated, although one rather relies on measurements.

The difficulty with laboratory measurements of $S(E)$-if they are possible at allis that, because of the small cross sections, they are usually feasible only at rather high energies, say above 0.1MeV, but this is still roughly a factor 10 larger than those energies which are relevant for astrophysical applications. Therefore one has to extrapolate the measured $S(E)$ downwards over a rather long range of E. This can be done quite reliably for non-resonant reactions, in which case S is nearly constant or a very slowly varying function of E [an advantage of extrapolating $S(E)$ rather than $\sigma(E)$]. The real problems arise from (suspected or unsuspected) resonances in the range over which the extrapolation is to be extended. Then the results can be quite uncertain. Only in underground laboratories, where the experiments are shielded from cosmic rays by hundreds of meters of solid rock, it is sometimes possible to measure the nuclear cross sections of at least a few nuclear reactions at energies as low as $10-30$keV, i.e. at energies relevant for nuclear processes in stellar interiors. The first such measurement was done by Junker et al. (1998) in the Gran Sasso Laboratory and concerned the $^3\mathrm{He}\left(^3\mathrm{He}, 2p\right)^4\mathrm{He}$ reaction (18.62) of the hydrogen burning chains (Sect. 18.5.1). Such experiments sometimes lead to the discovery of resonances, but more importantly reduce the uncertainties of the cross sections considerably, and confirm the near constancy of $S(E)$ at the relevant energies.

Let us denote the types of reacting particles, X and a, by indices j and k respectively. Suppose there is one particle of type j moving with a velocity v relative to all particles of type k. Its cross section σ for reactions with the k sweeps over a volume σv per second. The number of reactions per second will then be $n_k \sigma v$ if there are n_k particles of type k per unit volume. For n_j particles per unit volume the total number of reactions per units of volume and time is

$$\tilde{r}_{jk} = n_j n_k \sigma v$$

This product may also be interpreted by saying that $n_j n_k$ is the number of pairs of possible reaction partners, and σv gives the reaction probability per pair and second. This indicates what we have to do in the case of reactions between identical particles ($j = k$). Then the number of pairs that are possible reaction partners is $n_j \left(n_j - 1\right)/2 \approx n_j^2/2$ for large particle numbers. This has to replace the product $n_j n_k$ in (18.16) so that we can generally write

$$\tilde{r}_{jk} = \frac{1}{1+\delta_{jk}} n_j n_k \sigma v, \quad \delta_{jk} = \left\{ \begin{array}{l} 0, j \neq k \\ 1, j = k \end{array} \right. .$$

Now we have to allow for the fact that particles j and k do not move relatively to each other with uniform velocities, which is important since σ depends strongly on v. Excluding extreme densities (as, e.g. in neutron stars) we can assume that both types have a Maxwell-Boltzmann distribution of their velocities. It is then well known that also their relative velocity v is Maxwellian. If the corresponding energy is

$$E = \frac{1}{2} m v^2$$

with the reduced mass $m = m_j m_k / \left(m_j + m_k\right)$, the fraction of all pairs contained in the interval $[E, E + dE]$ is given by

$$f(E)dE = \frac{2}{\sqrt{\pi}} \frac{E^{1/2}}{(kT)^{3/2}} \mathrm{e}^{-E/kT} dE.$$

This fraction of all pairs has a uniform velocity and contributes the amount $dr_{jk} = \tilde{r}_{jk} f(E) dE$ to the total rate. The total reaction rate per units of volume and time is then given by the integral $\int dr_{jk}$ over all energies, which formally can be written as

$$r_{jk} = \frac{1}{1+\delta_{jk}} n_j n_k \langle \sigma v \rangle,$$

where the averaged probability is

$$\langle \sigma v \rangle = \int_0^\infty \sigma(E) v f(E) dE.$$

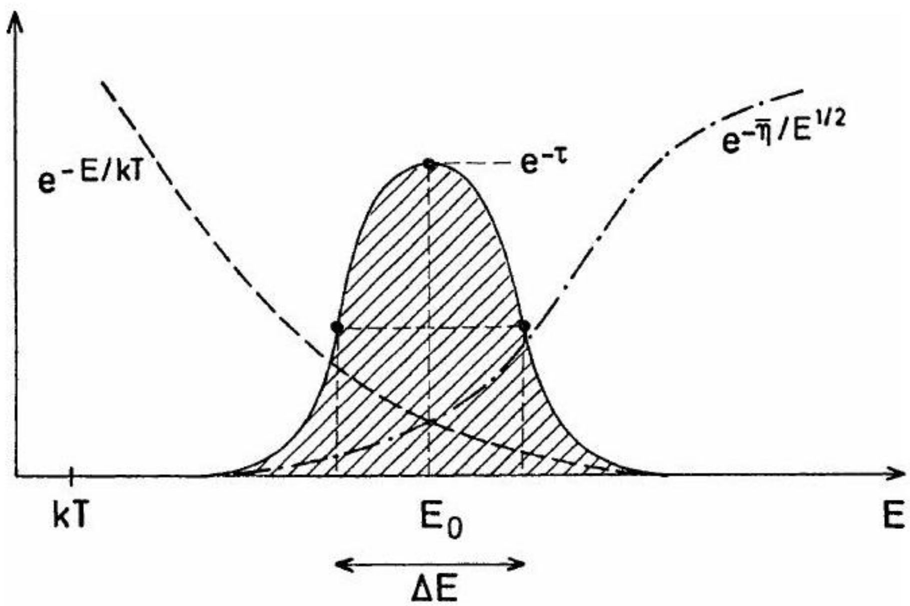
Let us replace the particle numbers per unit volume n_i by the mass fraction X_i with

$$X_i \varrho = n_i m_i$$

cf. (8.2). If the energy Q is released per reaction, then (18.20) gives the energy generation rate per units of mass (instead of unit volume; obtained by dividing by ϱ) and time:

$$\varepsilon_{jk} = \frac{1}{1+\delta_{jk}} \frac{Q}{m_j m_k} \varrho X_j X_k \langle \sigma v \rangle.$$

Fig. 18.5 The Gamow peak (solid curve) as the product of Maxwell distribution (dashed) and penetration factor (dot-dashed). The hatched area under the Gamow peak determines the reaction rate. All three curves are on different scales



Using (18.9), (18.15), (18.18) and (18.19) in (18.21), the average cross section $\langle \sigma v \rangle$ can be written as

$$\langle \sigma v \rangle = \frac{2^{3/2}}{(m\pi)^{1/2}} \frac{1}{(kT)^{3/2}} \int_0^\infty S(E) \mathrm{e}^{-E/kT - \bar{\eta}/E^{1/2}} dE$$

where

$$\bar{\eta} = 2\pi\eta E^{1/2} = \pi(2m)^{1/2} \frac{Z_j Z_k e^2}{\hbar}$$

A further evaluation of $\langle \sigma v \rangle$ requires a specification of $S(E)$. We shall limit ourselves to the simplest but for astrophysical applications very important case of non-resonant reactions. Then we can set $S(E) \approx S_0 =$ constant, and take it out of the integral (18.24), since only a small interval of E will turn out to contribute appreciably. The remaining integral may be written as

$$J = \int_0^\infty \mathrm{e}^{f(E)} dE, \quad \text{with} \quad f(E) = -\frac{E}{kT} - \frac{\bar{\eta}}{E^{1/2}}.$$

The integrand is the product of two exponential functions, one of which drops steeply with increasing E, while the other rises. The integrand will therefore have appreciable values only around a well-defined maximum (see Fig. 18.5), the so-called Gamow peak. This maximum occurs at E_0, where the exponent has a minimum. From the condition $f' = 0$, where f' is the derivative with respect to E, one finds

$$E_0 = \left(\frac{1}{2}\bar{\eta}kT\right)^{2/3} = \left[\left(\frac{m}{2}\right)^{1/2}\pi\frac{Z_i Z_k e^2 kT}{\hbar}\right]^{2/3}.$$

It is usual to introduce now a quantity τ defined by

$$\tau = 3\frac{E_0}{kT} = 3\left[\pi\left(\frac{m}{2kT}\right)^{1/2}\frac{Z_j Z_k e^2}{\hbar}\right]^{2/3}$$

and to represent $f(E)$ near the maximum by the series expansion

$$\begin{aligned} f(E) &= f_0 + f_0' \cdot (E - E_0) + \frac{1}{2}f_0'' \cdot (E - E_0)^2 + \cdots \\ &= -\tau - \frac{1}{4}\tau\left(\frac{E}{E_0} - 1\right)^2 + \cdots \end{aligned}$$

from which we retain only these two terms (the linear term vanishes since $f_0' = 0$ at the maximum). Their substitution in (18.26) means to approximate the Gamow peak of the integrand by a Gaussian, as will become particularly clear when we transform J to the new variable of integration $\xi = (E/E_0 - 1)\sqrt{\tau}/2$:

$$J = \int_0^\infty \exp\left[-\tau - \frac{\tau}{4}\left(\frac{E}{E_0} - 1\right)^2\right] dE = \frac{2}{3}kT\tau^{1/2}\mathrm{e}^{-\tau}\int_{-\sqrt{\tau/2}}^{\infty}\mathrm{e}^{-\xi^2}d\xi$$

The main contribution to J comes from a range close to $E = E_0$, i.e. $\xi = 0$, so that no large errors are introduced when extending the range of integration to $-\infty$, the integral over the Gaussian becoming $\sqrt{\pi}$.

We then have

$$J \approx kT\frac{2}{3}\pi^{1/2}\tau^{1/2}\mathrm{e}^{-\tau}$$

and for non-resonant reactions (18.24) becomes

$$\langle\sigma v\rangle = \frac{4}{3}\left(\frac{2}{m}\right)^{1/2}\frac{1}{(kT)^{1/2}}S_0\tau^{1/2}\mathrm{e}^{-\tau}$$

From (18.28) one has $(kT)^{-1/2} \sim \tau^{3/2}$; hence the kT can be substituted in (18.32), which then gives $\langle\sigma v\rangle \sim \tau^2\mathrm{e}^{-\tau}$.

The properties of the Gamow peak are so important that we should inspect some of them a bit further. In order to have convenient numerical values, we count the temperature in units of 10^7 K (which is typical for many stellar centres) and denote this dimensionless temperature by $T_7 = T/10^7$ K or generally

$$T_n := \frac{T}{10^n \text{ K}}$$

We then have the following relations (some of which will be derived below):

$$\begin{aligned}
W &= Z_j^2 Z_k^2 A = Z_j^2 Z_k^2 \frac{A_j A_k}{A_j + A_k} \\
\tau &= 19.721 W^{1/3} T_7^{-1/3} \\
E_0 &= 5.665\text{keV} \cdot W^{1/3} T_7^{2/3} \\
\frac{E_0}{kT} &= \frac{\tau}{3} = 6.574 W^{1/3} T_7^{-1/3} \\
\Delta E &= 4.249\text{keV} \cdot W^{1/6} T_7^{5/6}, \\
\frac{\Delta E}{E_0} &= 4(\ln 2)^{1/2} \tau^{-1/2} = 0.750 W^{-1/6} T_7^{1/6}, \\
v &= \partial \ln\langle\sigma v\rangle / \partial \ln T = (\tau - 2)/3 = 6.574 W^{1/3} T_7^{-1/3} - 2/3
\end{aligned}$$

The value of W is determined by the reaction partners and is at least of order unity. Large W discriminates against the reactions of heavy nuclei so much that only the lighter nuclei can react with appreciable rate. The Gamow peak occurs as a compromise in the counteraction between Maxwell distribution and penetration probability with a maximum at $E = E_0$, which is roughly 5-100 times the average thermal energy kT. This "effective stellar energy range" is, on the other hand, far below the $\gtrsim$ 100keV available to most laboratory experiments. With increasing T, E_0 increases moderately, while the maximum height of the peak $H_0 = \mathrm{e}^{-\tau}$ increases very steeply owing to the decreasing τ.

The width of the effective energy range is described by ΔE, which is the full width of the Gamow peak at half maximum (see Fig. 18.5), i.e. between the points with height $0.5\mathrm{e}^{-\tau}$. Equating this to the integrand in the first form of (18.30), we obtain

$$\frac{\Delta E}{E_0} = 4 \frac{(\ln 2)^{1/2}}{\tau^{1/2}}$$

According to (18.34), this is always below unity, and therefore one has a welldefined energy range in which the reactions occur effectively. With ΔE increasing with T only slightly more than E_0, the relative form of the peak remains nearly constant.

The most striking feature of thermonuclear reactions is their strong sensitivity to the temperature. In order to demonstrate this, one represents the T dependence of $\langle\sigma v\rangle$ (and thus of r_{jk} and ε_{jk}) around some value $T = T_0$ by a power law such as

$$\langle\sigma v\rangle = \langle\sigma v\rangle_0 \left(\frac{T}{T_0}\right)^v, \quad v = \frac{\partial \ln\langle\sigma v\rangle}{\partial \ln T}.$$

From (18.28) we have $\tau \sim T^{-1/3}$, and then from (18.32) $\langle\sigma v\rangle \sim T^{-2/3}\mathrm{e}^{-\tau}$. Therefore

$$\ln\langle\sigma v\rangle = \text{constant} - \frac{2}{3}\ln T - \tau$$

and

$$\frac{\partial \ln\langle\sigma v\rangle}{\partial \ln T} = -\frac{2}{3} - \frac{\partial\tau}{\partial \ln T} = -\frac{2}{3} - \tau\frac{\partial \ln\tau}{\partial \ln T}$$

Since $\tau \sim T^{-1/3}$, we have $\partial \ln\tau/\partial \ln T = -1/3$, so that finally

$$v \equiv \frac{\partial \ln\langle\sigma v\rangle}{\partial \ln T} = \frac{\tau}{3} - \frac{2}{3}$$

where for most reactions $\tau/3$ is much larger than $2/3$ and $v \approx \tau/3$. Then v decreases with T as $v \sim T^{-1/3}$. From (18.34) we see that even for reactions between the lightest nuclei, $v \approx 5$, and it can easily attain values around (and even above) $v \approx 20$. With such values for the exponent (!) of T, the thermonuclear reaction rate is about the most strongly varying function treated in physics, and this temperature sensitivity has a clear influence on stellar models. Also, since small fluctuations of T (which will certainly be present) must result in drastic changes in the energy production, we have to assume that there exists an effective stabilizing mechanism (a thermostat) in stars (Sect. 25.3.5).

We may easily see how the large v values are related to the change of the Gamow peak with T : the value $\langle\sigma v\rangle$ is proportional to the integral J in (18.30), and this is given by the area under the Gamow peak, which is roughly $J \approx \Delta E \cdot H_0$ ($H_0 = \mathrm{e}^{-\tau}$ is the height of the peak). According to (18.34), $\Delta E \sim T^{5/6}$, while H_0 increases strongly with T. In fact it is this height H_0 which provides the exponential $\mathrm{e}^{-\tau}$ in the expressions for $\langle\sigma v\rangle$ and is therefore responsible for the large values of v.

We should briefly mention a few corrections to the derived formulae for the reaction rates. The first concerns inaccuracies made by evaluating the integral in (18.24) with constant S and with an integrand approximated by a Gaussian. This is usually corrected for by multiplying $\langle\sigma v\rangle$ with a factor

$$g_{jk} = 1 + \frac{5}{12\tau} + \frac{S'}{S}E_0\left(1 + \frac{105}{36\tau}\right) + \frac{1}{2}\frac{S''}{S}E_0^2\left(1 + \frac{267}{36\tau}\right),$$

where S and its derivatives with respect to E have to be taken at $E = 0$ (Eq. (17.206b) in Weiss et al. 2004, p. 601).

Another correction factor, f_{jk}, allows for a partial shielding of the Coulomb potential of the nuclei, owing to the negative field of neighbouring electrons. This plays a role only at very high densities; it will be treated separately in Sect. 18.4.

Concerning resonant reactions we shall only remark that the situation depends very much on the location of the resonance. For example, the integral in (18.24) can be dominated by a strong peak at the resonance energy. However, once $S(E)$ is given, (18.24) can in principle always be evaluated.