Numerical Computations - Ex 1, Oct 15, 2020

1. Calculate the Taylor series at the expansion point $x = 0$ of the functions $\frac{1}{x-1}$ and $\frac{1}{x^2+1}$, and plot the Taylor polynomials for $n \in \{10, 20, 40\}$ on the interval $[-2, 2]$. Hint: A partial fraction decomposition of the second function explains the bad behaviour of the harmless looking function.

2. Compute and plot interpolation polynomials to $\frac{1}{x^2+1}$ on the interval $[-5, 5]$, for $n \in \{10, 20, 40\}$. Choose uniformly distributed points, and Chebyshev points on $[-5, 5]$: $x_i = 5 \cos \frac{(i+0.5)\pi}{n+1}$ for $i = 0, \ldots, n$.

   Plot the Lagrange interpolation polynomials l_i for both choices of points. Investigate numerically

$$\max_{i \in \{0,\ldots,n\}} \max_{x \in [-5,5]} |l_i(x)| \qquad \text{and} \qquad \max_{x \in [-5,5]} \sum_{i=0}^{n} |l_i(x)|$$

   depending on n.

3. Chebyshev polynomials are recursively defined as

$$\begin{aligned} T_0(x) &= 1 \\ T_1(x) &= x \\ T_{n+1}(x) &= 2xT_n(x) - T_{n-1}(x) \qquad \text{for } n = 1, 2, \ldots \end{aligned}$$

   - Plot the first few polynomials on $[-1, 1]$. What range do you expect?
   - Determine the leading coefficient $lc(T_n)$.
   - Show that there holds

$$T_n(x) = \cos(n \operatorname{acos}(x)) \qquad \text{for } x \in [-1, 1]$$

     Find all roots of T_n.
   - Let q be a polynomial of exact degree n such that $lc(q) = lc(T_n)$. Proof that

$$\max_{x \in [-1,1]} |q(x)| \geq \max_{x \in [-1,1]} |T_n(x)|$$

4. Implement the Aitken-Nevill scheme for polynomial interpolation. Test it for numerical differentiation by extrapolation for $f(x) = \sin(x)$

$$D_f(x, h) := \frac{f(x+h) - f(x)}{h}$$

   and

$$D_{f,sym} := \frac{f(x+h) - f(x-h)}{2h}$$

   Choose $x = \pi$, and interpolation points $h_i = q^i$ for $i \in \{0, \ldots n\}$. Generate convergence plot for the errors $f'(x) - D_f(x, 0)$. Try different refinement ratios $q \in (0, 1)$. How does this effect speed of convergence, and reachable accuracy ? Since $\lim_{h \to 0} D_{f,sym}(x, h) = \lim_{h \to 0} D_{f,sym}(x, \sqrt{h})$, try also extrapolation of the function $h \mapsto D_{f,sym}(x, \sqrt{h})$.