Numerical Computations - Ex 4, Nov 5, 2020

13. Polynomial multiplication using FFT.
Given polynomials $a(x) = \sum_{i=0}^{n} a_i x^i$ and $b(x) = \sum_{i=0}^{m} b_i x^i$, the product $c(x) = a(x)b(x)$ is a polynomial $\sum_{i=0}^{n+m} c_i x^i$ with coefficients $c_i = \sum_{j=0}^{i} a_j b_{i-j}$. Implement a function for polynomial multiplication with $O(N \log N)$ run-time complexity using FFT (Notes Ex 1.53, Lecture fom Oct 20). Test it for $a(x) = \sum_{i=0}^{100} x^i$ and $b(x) = \sum_{i=0}^{100} ix^i$.

14. Solve the heat equation using FFT. See lecture from Oct 27. Find $u : [0,1] \times [0,T] \to \mathbb{R}$ such that

$$\begin{aligned} \frac{\partial u}{\partial t}(x,t) - \frac{\partial^2 u}{\partial x^2}(x,t) &= 0 \qquad \forall\, x \in [0,1] \forall\, t \in [0,T] \\ u(x,0) &= u_0(x) \qquad \forall\, x \in [0,1] \end{aligned}$$

with u periodic in x. Try $u_0 = \chi_{[0.3,0.7]}$ and $u_0 = \max\{0, 1 - 10|x - 0.5|\}$.

Plot $u(x, t = 0.01)$, $u(x, t = 0.1)$, and $u(x, t = 1)$.

Be careful to choose Fourier indices in $[-n/2, n/2)$.

Voluntary: Solve the wave equation, same u_0, and $v_0 = 0$.

15. Implement a function for numerical integration by a composite integration rule. Input is the function to be integrated, the interval $[a, b]$, the amount of subintervals m, and a quadrature rule $[(c_0, \omega_0), \ldots, (c_n, \omega_n)]$ with reference points $c_i \in [0,1]$ and weights ω_i. Study convergence for the functions $f(x) = \int_0^1 e^x \, dx$ and $f(x) = \sqrt{x}$, and integration rules

    (a) trapezoidal: $[(0, 0.5), (1, 0.5)]$
    (b) Simpson: $[(0, 1/6), (0.5, 2/3), (1, 1/6)]$
    (c) mid-point: $[(0.5, 1)]$
    (d) 2-point Gaussian rule: $[(0.5 - 0.5/\sqrt{3}, 0.5), (0.5 + 0.5/\sqrt{3}, 0.5)]$

16. Determine the 3-point Gaussian quadrature rule.

    (a) Calculate the Legendre polynomial L_3, either by orthogonalizaton by hand, or by Rodrigues' formula, or by using the recurrence relation.
    (b) Find the roots of L_3 to define the quadrature points (in $[-1, 1]$).
    (c) Find the weights by integrating monomials x^k, and solving a 2×2 system.
    (d) Use the 3-point Gauss rule in Ex 15. Be careful with the reference interval, either $[0, 1]$ or $[-1, 1]$.