Numerical Computations - Ex 6, Nov 19, 2020

21. Familiarize yourself with the LU and Cholesky decompositon functions from numpy. Try matrices

    - $A \in \mathbb{R}^{n\times n}$ tridiagonal, where $A_{ii} = 2$, $A_{i,i+1} = A_{i+1,i} = -1$, and all other entries 0.
    - $B \in \mathbb{R}^{n\times n}$, where B_{ij} are random numbers in $[0,1]$.

    Compute the factors L, U, and the permutation P, and L^{-1}, U^{-1} using the library, and then compute A^{-1} from the factors. Same for the factors form the Cholesky decomposition, if applicable. Compute the erros

    $$\|AA^{-1} - I\|_F,$$

    where the Frobenious norm of a matrix M is

    $$\|M\|_F := \sqrt{M : M} = \sqrt{\sum_{i=1}^{n}\sum_{j=1}^{n} M_{i,j}^2}.$$

    Use matrix sizes such that computation time is in the range of seconds.

    voluntary: try LU from eigen and blaze

22. Let $\|\cdot\|$ be some vector norm in $\mathbb{R}^n$ (e.g. $\|\cdot\|_1$, $\|\cdot\|_2$, or $\|\cdot\|_\infty$). Let

    $$\|M\| := \sup_{0\neq v\in\mathbb{R}^n} \frac{\|Mv\|}{\|v\|}$$

    be the associated matrix norm for a matrix $M \in \mathbb{R}^{n\times n}$. Prove that

    - $\|A + B\| \leq \|A\| + \|B\|$ (triangle inequality)
    - $\|AB\| \leq \|A\|\,\|B\|$ (sub-multiplicativity)
    - $\mathrm{cond}(AB) \leq \mathrm{cond}(A)\,\mathrm{cond}(B)$
    - $\mathrm{cond}(A) \geq 1$

    where A and B are matrices, and $\mathrm{cond}(A) = \|A\|\,\|A^{-1}\|$ is the condition number of the (regular) matrix A.

23. Implement a solver for the equation

    $$LX = Y$$

    where $L \in \mathbb{R}^{n\times n}$ is a given lower left triangular matrix, $Y \in \mathbb{R}^{n\times m}$ is the given right hand side, and $X \in \mathbb{R}^{n\times m}$ is the unknown matrix. These can be interpreted as solving

m linear systems for the column vectors of X and Y. Your function may overwrite the right hand side matrix Y by the solution.

Implement the function by recusive blocking, and utilizie fast library functions for matrix-matrix and matrix-vector operations. You may use Python (numpy) or C++ with one of the matrix libraries from Ex 20. What floating-point rate do you obtain? How big systems (with $n = m$) can you solve within 10 sec?

24. Implement a Cholesky factorization for tridiagonal block-matrices

$$A = \begin{pmatrix} A_{1,1} & A_{2,1}^T & 0 & \cdots & 0 \\ A_{2,1} & A_{2,2} & A_{3,2}^T & & \vdots \\ 0 & A_{3,2} & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & A_{n,n-1}^T \\ 0 & \cdots & 0 & A_{n,n-1} & A_{n,n} \end{pmatrix},$$

where $A_{i,i}$ are symmetric, and $A_{i+1,i}$ are upper right triangular $m \times m$ matrices. Find the Cholesky factorization $A = LL^T$ where

$$L = \begin{pmatrix} L_{1,1} & 0 & 0 & \cdots & 0 \\ L_{2,1} & L_{2,2} & 0 & & \vdots \\ 0 & L_{3,2} & \ddots & \ddots & 0 \\ \vdots & & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & L_{n,n-1} & L_{n,n} \end{pmatrix}$$

Store the block-matrix as list of matrices. You should use fast library functions to work with the blocks.

Test your function for solving the Poisson equation (see notes '4 Linear systems', page 12). Choose $f(x, y) = \chi_{[0.2,0.4]^2}(x, y)$ and $u_D = 0$. Plot the solution $u(x, y)$.

What floating point rate do you obtain ? How big systems can you solve in 1 minute?