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Business Analytics & Machine Learning
Homework sheet 11: Convex Optimization – Solution

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Exercise H11.1 *Convex function*

You are given the following function:

$$f(x, y) = \exp(ax + by^2)$$

Determine all parameters $a, b \in \mathbb{R}$ such that f is convex.

Solution

The gradient of f is

$$\nabla f(x, y) = \begin{pmatrix} a\exp(ax + by^2) \\ 2by\exp(ax + by^2) \end{pmatrix}$$

The Hessian of f is

$$\nabla^2 f(x, y) = \begin{pmatrix} a^2\exp(ax + by^2) & 2aby\exp(ax + by^2) \\ 2aby\exp(ax + by^2) & (4b^2y^2 + 2b)\exp(ax + by^2) \end{pmatrix}$$

The first principal minor is $H_1(x, y) = a^2\exp(ax + by^2) \geq 0$ for all $x, y \in \mathbb{R}$. The second principal minor is $H_2(x, y) = (4b^2y^2 + 2b)\exp(ax + by^2) \geq 0$ for all $x, y \in \mathbb{R}$ if $b \geq 0$. The third principal minor is $H_2(x, y) = 2a^2b\exp(2ax + 2by^2)$. $H_3(x, y) \geq 0$ for all $x, y \in \mathbb{R}$ iff $b \geq 0$. In this case, $\nabla^2 f(x, y)$ is positive semidefinite.
Thus $f(x, y)$ is convex for all $a \in \mathbb{R}, b \geq 0$.

Exercise H11.2 *Convex functions*

Determine if the following functions are convex.

a) $f(x, y) = \exp(3x + 2y^2)$

b) $f(x, y) = \frac{1}{2}x^2 + \exp(-y) + 3xy$

c) $f(x) = |x| + \cos(x)$

d) $f(x) = 3x^{5n}$ for even n

Solution

a) $f(x, y) = \exp(3x + 2y^2)$

The gradient of f is

$$\nabla f(x, y) = \begin{pmatrix} 3\exp(3x + 2y^2) \\ 4y\exp(3x + 2y^2) \end{pmatrix}$$

The Hessian of f is

$$\nabla^2 f(x, y) = \begin{pmatrix} 9\exp(3x + 2y^2) & 12y\exp(3x + 2y^2) \\ 12y\exp(3x + 2y^2) & (16y^2 + 4)\exp(3x + 2y^2) \end{pmatrix}$$

The first principal minor is $H_1(x, y) = 9\exp(3x + 2y^2) \geq 0$ for all $x, y \in \mathbb{R}$. The second principal minor is $H_2(x, y) = (16y^2 + 4)\exp(3x + 2y^2) \geq 0$ for all $x, y \in \mathbb{R}$. The third principal minor is $H_3(x, y) = 36\exp(3x + 2y^2)^2 \geq 0$ for all $x, y \in \mathbb{R}$. $\nabla^2 f(x, y)$ is positive semidefinite and $f(x, y)$ is thus convex.

b) $f(x, y) = \frac{1}{2}x^2 + \exp(-y) + 3xy$ The gradient of f is

$$\nabla f(x, y) = \begin{pmatrix} x + 3y \\ -\exp(-y) + 3x \end{pmatrix}$$

The Hessian of f is

$$\nabla^2 f(x, y) = \begin{pmatrix} 1 & 3 \\ 3 & \exp(-y) \end{pmatrix}$$

The first principal minor is $H_1(x, y) = 1 \geq 0$ for all $x, y \in \mathbb{R}$. The second principal minor is $H_2(x, y) = \exp(-y) \geq 0$ for all $x, y \in \mathbb{R}$. The third principal minor is $H_3(x, y) = \exp(-y) - 9$. $H_3(x, y) \geq 0$ is not satisfied for all $x, y \in \mathbb{R}$ and therefore $f(x, y)$ is not convex.

c) $f(x) = |x| + \cos(x)$

f is not convex. Consider $x_1 = \pi$ and $x_2 = 3\pi$. Then

$$f(\frac{1}{1}x_1 + \frac{1}{2}x_2) = f(2\pi) = 1 + 2\pi > 2\pi - 1 = \frac{1}{2}f(x_1) + \frac{1}{2}f(x_2)$$

This violates the standard condition for convexity.

d) $f(x) = 3x^{5n}$ for even n

We take the second derivative $f''(x) = (75n^2 - 15n)x^{5n-2}$. Since n is even, $x^{5n-2} \geq 0$ for all $x \in \mathbb{R}$. Thus $f''(x) \geq 0$ for all $x \in \mathbb{R}$ and f is convex.

Exercise H11.3 *Extreme points*

You are given the following function:

$$f(x, y) = 2xy^3 - 3x^2 - 6xy - 1$$

a) Determine all local minima and maxima of f in $\mathbb{R}^2$.

b) Determine all local minima and maxima of f in the square $[0, 1] \times [0, 1]$. Consider the edges as well.

Solution

We first determine the gradient ∇f and the Hessian matrix $\nabla^2 f$.

$$\nabla f(x,y) = \begin{pmatrix} 2y^3 - 6x - 6y \\ 6xy^2 - 6x \end{pmatrix}$$

$$\nabla^2 f(x,y) = \begin{pmatrix} -6 & 6y^2 - 6 \\ 6y^2 - 6 & 12xy \end{pmatrix}$$

a) The gradient $\nabla f(x,y) = 0$ at each critical point is zero. It thus follows

$$2y^3 - 6x - 6y = 0 \Leftrightarrow x = \frac{1}{3}y^3 - y \quad (1)$$
$$6xy^2 - 6x = 0 \Leftrightarrow x(y^2-1) = 0 \quad (2)$$

Inserting 1 in 2 yields

$$(\frac{1}{3}y^3 - y)(y^2-1) = 0 \Leftrightarrow y(\frac{1}{3}y^2 - 1)(y^2-1) = 0$$

And we obtain $y^* = \{0, \pm\sqrt{3}, \pm 1\}$. Inserting into 1 yields the critical points:

$$P_1 = (0,0); P_2 = (0,-\sqrt{3}); P_3 = (0,\sqrt{3}); P_4 = (\frac{2}{3},-1); P_5 = (-\frac{2}{3},1)$$

We insert these points into the Hessian matrix to determine if each point is a local minimum (positive definite Hessian), a local maximum (negative definite Hessian), or a saddle point (indefinite Hessian). If the matrix is positive or negative *semi*-definite, the test is inconclusive.

$$P_1 : \nabla^2 f(0,0) = \begin{pmatrix} -6 & -6 \\ -6 & 0 \end{pmatrix} \qquad \text{Indefinite matrix} \Rightarrow \text{Saddle point}$$

$$P_2 : \nabla^2 f(0,-\sqrt{3}) = \begin{pmatrix} -6 & 12 \\ 12 & 0 \end{pmatrix} \qquad \text{Indefinite matrix} \Rightarrow \text{Saddle point}$$

$$P_3 : \nabla^2 f(0,\sqrt{3}) = \begin{pmatrix} -6 & 12 \\ 12 & 0 \end{pmatrix} \qquad \text{Indefinite matrix} \Rightarrow \text{Saddle point}$$

$$P_4 : \nabla^2 f(\frac{2}{3},-1) = \begin{pmatrix} -6 & 0 \\ 0 & -8 \end{pmatrix} \qquad \text{Negative definite} \Rightarrow P_4 \text{ is a local maximum}$$

$$P_5 : \nabla^2 f(-\frac{2}{3},1) = \begin{pmatrix} -6 & 0 \\ 0 & -8 \end{pmatrix} \qquad \text{Negative definite} \Rightarrow P_5 \text{ is a local maximum}$$

b) Among the solutions obtained in a.), only $P_1 = (0,0)$ is within the square and is part of the edges. We consider the edges:

- $f(0,y) = -1$.
- $f(x,0) = -3x^2 - 1$. This function is maximal for $x = 0$ with $f = -1$ and minimal for $x = 1$ with $f = -4$.

- $f(1,y) = 2y^3 - 6y - 4$. This function is maximal for $y = 0$ with $f = -4$ and minimal for $y = 1$ with $f = -8$.
- $f(x,1) = -3x^2 - 4x - 1$. This function is maximal for $x = 0$ with $f = -1$ and minimal for $x = 1$ with $f = -8$.

Thus, within the square, f attains its maximum at $(0, y), \forall y \in [0, 1]$ and its minimum at $(1, 1)$