

Introduction to Computational Physics

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Course Information

- ▶ Every Thursday at 14.15, online only
- ▶ GitHub link: <https://github.com/jakryd/0800-fizobl/>
- ▶ Slack group: [ncu-students/0800-fizobl-21](#)
- ▶ Zoom meeting ID: [208 317 2371](#)
- ▶ Exams take place in June 2021

Outline

Part 1. Statistical Mechanics

- ▶ Probability
- ▶ Phase space
- ▶ Thermodynamic equilibrium
- ▶ Statistical ensembles

Part 2. Monte Carlo Methods

- ▶ Sampling probability distributions
- ▶ Importance sampling

Part 3. Molecular Dynamics

- ▶ Verlet integrator
- ▶ Force and energy

Outline

Part 4. Enhanced Sampling

- ▶ Rare events
- ▶ Collective variables
- ▶ Free energy

Part 5. Nonequilibrium Statistical Physics

Part 6. Machine Learning

- ▶ Connection between machine learning and statistical physics
- ▶ Unsupervised learning

Literature

- ▶ M. E. Tuckerman, *Statistical Mechanics: Theory and Simulation*, Oxford University Press (2016).
- ▶ D. Chandler, *Introduction to Modern Statistical Mechanics*, Oxford University Press (1987).
- ▶ R. K. Patria, P. D. Beale, *Statistical Mechanics*, Elsevier (2011).
- ▶ M. Toda, R. Kubo, N. Saito, *Statistical Physics I: Equilibrium Statistical Mechanics*, Springer-Verlag (1983).
- ▶ R. Kubo, M. Toda, N. Hashitsume, *Statistical Physics II: Nonequilibrium Statistical Mechanics*, Springer-Verlag (1991).

Statistical Mechanics

Probability

- ▶ Probability is the language of statistical mechanics.
- ▶ Fundamental to the understanding of quantum mechanics.
- ▶ The large number of degrees of freedom of a macroscopic system make it necessary to use statistics.

Probability

Random Variable

A random variable \mathbf{X} is completely defined by the range of values it can take, and its probability distribution $p_{\mathbf{X}}(x_1, \dots, x_k)$. The value $p_{\mathbf{X}}$ is the probability that the random variable \mathbf{X} takes the value $\mathbf{x} = (x_1, \dots, x_k)$.

- $p_{\mathbf{X}}(\mathbf{x}) \equiv \mathbb{P}[\mathbf{X} = \mathbf{x}]$.¹
- $p_{\mathbf{X}}(\mathbf{x})$ is non-negative and satisfies the normalization condition:

$$\int d\mathbf{x} p_{\mathbf{X}}(\mathbf{x}) = 1. \quad (1)$$

- The expectation value of $f(\mathbf{X})$ (or average) is denoted by:

$$\mathbb{E}[f] = \int d\mathbf{x} p_{\mathbf{X}}(\mathbf{x}) f(\mathbf{x}). \quad (2)$$

¹From now on we write $p(\mathbf{x})$ to denote $p_{\mathbf{X}}(\mathbf{x})$.

Probability

Gaussian Random Variable $\mathbf{X} \sim \mathcal{N}(\boldsymbol{\mu}, \boldsymbol{\Sigma})$

A continuous variable $\mathbf{X} \in \mathbb{R}^k$ has a Gaussian distribution of mean $\boldsymbol{\mu}$ and variance $\boldsymbol{\sigma}^2$ if its probability density is:

$$p(\mathbf{x}) = \frac{\exp\left(-\frac{1}{2}(\mathbf{x} - \boldsymbol{\mu})^T \boldsymbol{\Sigma}^{-1} (\mathbf{x} - \boldsymbol{\mu})\right)}{\sqrt{(2\pi)^k \det(\boldsymbol{\Sigma})}}. \quad (3)$$

We have $\mathbb{E}[\mathbf{X}] = \boldsymbol{\mu}$ and $\text{Var}[\mathbf{X}] = \boldsymbol{\Sigma}$.

Probability

- The *entropy* of a random variable \mathbf{X} with probability distribution $p(\mathbf{x})$ is defined as:

$$H_{\mathbf{X}} \equiv - \int d\mathbf{x} p(\mathbf{x}) \log p(\mathbf{x}), \quad (4)$$

where we define $0 \log 0 = 0$.²

- Entropy $H_{\mathbf{X}}$ is a measure of uncertainty of the random variable \mathbf{X} .

²Units for \log_2 : bits and \log_e : nats

Probability

Examples

A fair coin has two values with equal probability. Its entropy is 1 bit.

Consider throwing M fair coins: the number of all possible outcomes is 2^M . The entropy equals M bits.

Bernoulli process. A Bernoulli random variable X can take two values 0, 1 with probabilities $p(0) = q$ and $p(1) = 1 - q$. Its entropy is:

$$H_X = -q \log q - (1 - q) \log(1 - q). \quad (5)$$

The entropy vanishes for $q = 0$ or $q = 1$ because the outcome is certain.

Probability

- The Kullback-Leibler (KL) divergence (or the relative entropy) measures a statistical distance between $p(x)$ and $q(x)$. It is defined as:

$$D_{\text{KL}}(q\|p) \equiv \int dx q(x) \log \frac{q(x)}{p(x)}. \quad (6)$$

- $D_{\text{KL}}(q\|p)$ is convex in $q(x)$,
- $D_{\text{KL}}(q\|p) \geq 0$,
- and $D_{\text{KL}}(q\|p) = 0$ if $q(x) = p(x)$.
- Not symmetric: $D_{\text{KL}}(q\|p) \neq D_{\text{KL}}(p\|q)$.

Dirac δ function

- $\delta(x) = 0$ if $x \neq 0$ and $\delta(x) \rightarrow \infty$ (undefined) if $x = 0$
- $\int_{-\epsilon}^{\epsilon} dx \delta(x) = 1$ for all $\epsilon > 0$
- Gaussian model:

$$\delta_\sigma(x) = \lim_{\sigma \rightarrow 0} \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} \quad (7)$$

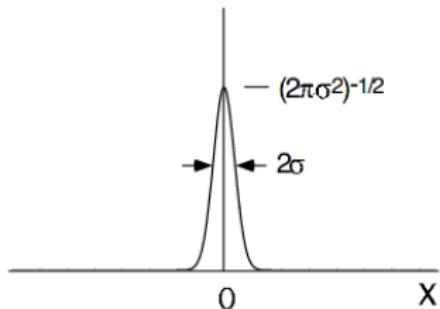


Figure 1: Gaussian model for the Dirac δ function.

Dirac δ function

- δ function times any arbitrary function $f(x)$ is:

$$\int_{-\infty}^{\infty} dx \delta(x) f(x) = f(0). \quad (8)$$

- Other models for δ function include Fourier integral:

$$\delta_\sigma(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} dk e^{ikx - |\sigma|k}, \quad (9)$$

and scaled sinc:

$$\delta_\sigma(x) = \frac{1}{\pi\sigma} \text{sinc}(x/\sigma). \quad (10)$$

- Important for building histograms (notice the change of variables):

$$\int_{-\infty}^{\infty} dx \delta(x - a) f(x) = f(a). \quad (11)$$

Definitions

- ▶ *Microscopic variable*: A variable pertaining to the individual atoms and molecules making up the system.
- ▶ *Macroscopic variable*: A measurable quantity used to describe the state of the system. It depends collectively on the behavior of all the atoms and molecules. These are also referred to as *thermodynamic variables*.
- ▶ *Extensive variables*: The system under consideration is often defined as encompassing some specific N molecules. Then extensive variables are those whose magnitude is proportional to N .
- ▶ *Intensive variables*: Those macroscopic variables whose magnitude is independent of N .

Scales

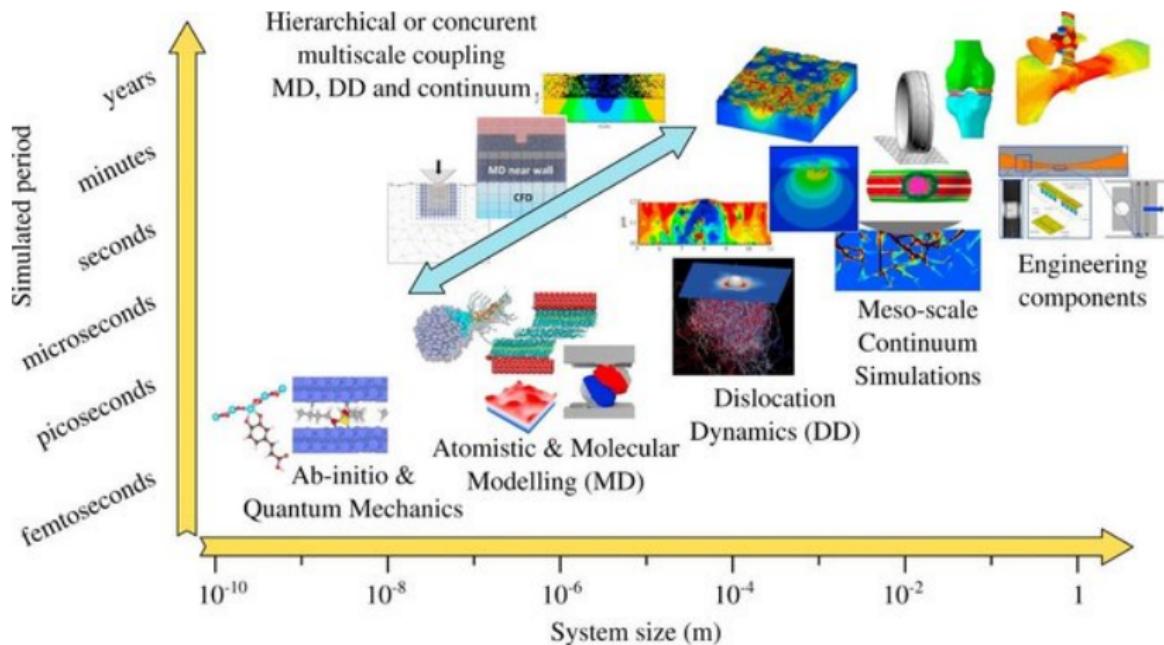


Figure 2: Spatial and temporal scales.

Phase Space



Figure 3: Ludwig Boltzmann (1844–1906)

Phase Space

- ▶ Consider a system with N particles.
- ▶ Microscopic coordinates: $\mathbf{r} \equiv (r_1, \dots, r_{3N})$.
- ▶ Conjugate momenta: $\mathbf{p} \equiv (p_1, \dots, p_{3N})$.
- ▶ We introduce the notion of generalized coordinates by stacking coordinates and momenta:

$$\mathbf{x} \equiv (r_1, \dots, r_{3N}, p_1, \dots, p_{3N}). \quad (12)$$

- ▶ \mathbf{x} evolves in Γ which defines the $6N$ -dimensional *phase space*.

Phase Space

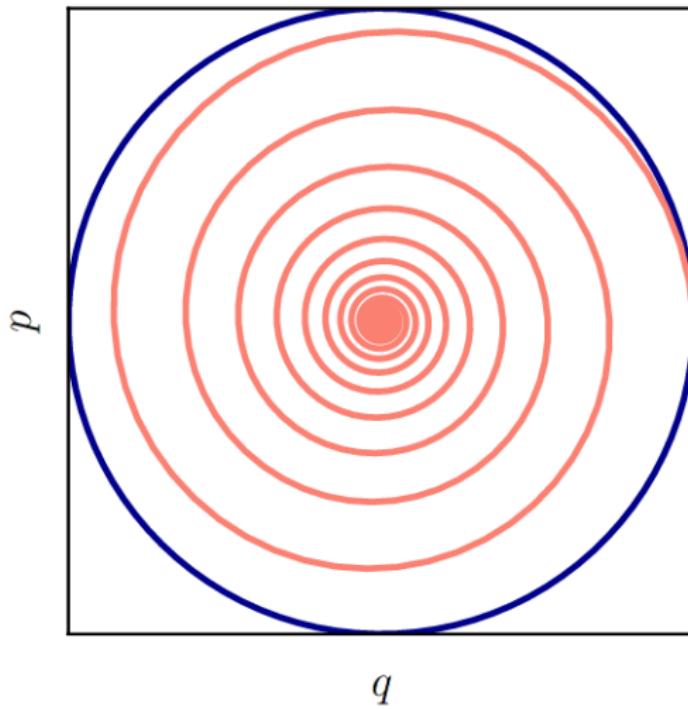


Figure 4: Phase spaces of an undamped (blue) and a damped (orange) harmonic oscillator.

Phase Space

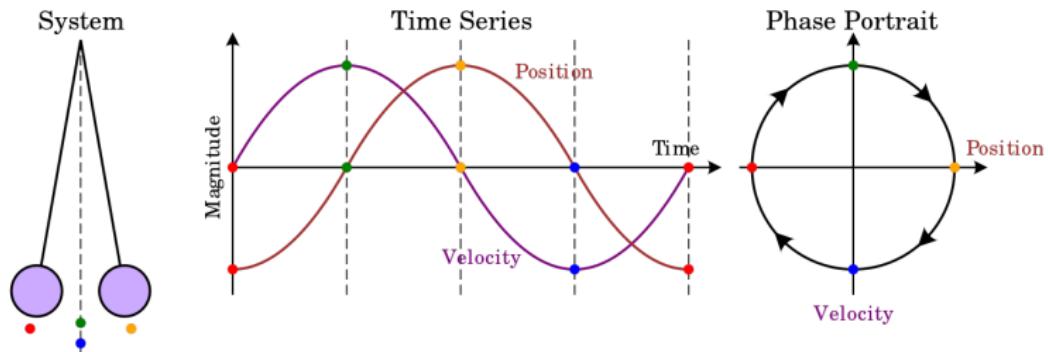


Figure 5: Phase spaces of the system.

Ensemble Average

- ▶ *Ergodicity hypothesis* – the assumption that all states in an ensemble are reached by the time evolution of the corresponding system.
- ▶ We define the ensemble average of a quantity $A(\mathbf{x})$ as:

$$\langle A \rangle = \frac{\int d\mathbf{x} A(\mathbf{x})\rho(\mathbf{x})}{\int d\mathbf{x} \rho(\mathbf{x})}, \quad (13)$$

where $\rho(\mathbf{x})$ is the phase space probability density.

Hamiltonian

- The dynamics of the system under study is described by their *Hamiltonian* $H(\mathbf{x}) = \sum_{i=1}^N \frac{\mathbf{p}_i^2}{2m_i} + U(\mathbf{r})$.
- The equations of motion are:

$$\dot{p}_i = -\frac{\partial H}{\partial r_i} \quad \text{and} \quad \dot{r}_i = \frac{\partial H}{\partial p_i} \quad (i = 1, \dots, 3N). \quad (14)$$

- Can be rewritten as:

$$\dot{p}_i = -\frac{\partial U}{\partial r_i} \quad \text{and} \quad \dot{r}_i = \frac{p_i}{m_i}, \quad (15)$$

where $U(\mathbf{r})$ is the potential energy.

Liouville Theorem

- Temporal evolution of a phase space element of volume V and boundary ∂V is given by:

$$\frac{\partial}{\partial t} \int_V dV \rho + \int_{\partial V} dB \rho \mathbf{v} = 0, \quad (16)$$

where \mathbf{v} is a generalized velocity vector.

- In Eq. 16, ρ satisfies the continuity equation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0, \quad (17)$$

which simplifies to ($\nabla \cdot \mathbf{v} = 0$):

$$\frac{\partial \rho}{\partial t} = \{H, \rho\} \quad (18)$$

Liouville Theorem

- The Liouville Theorem describes the time evolution of the phase space density $\dot{\rho} = \{H, \rho\}$.
- Poisson bracket: $\{u, v\} = \sum_i \left(\frac{\partial u}{\partial r_i} \frac{\partial v}{\partial p_i} - \frac{\partial u}{\partial p_i} \frac{\partial v}{\partial r_i} \right)$.

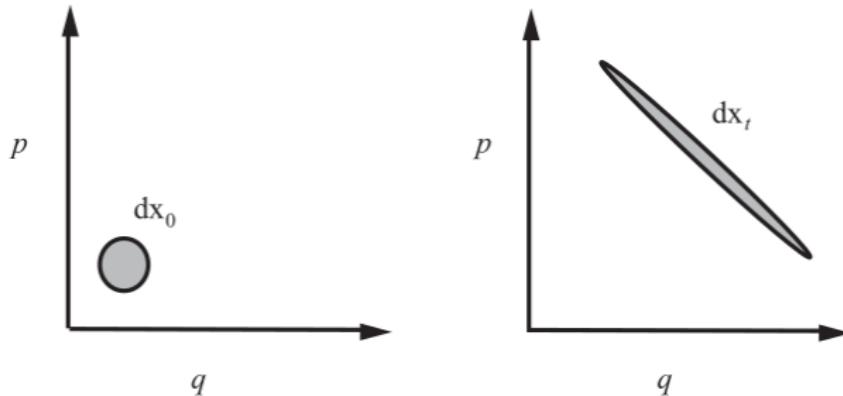


Figure 6: Illustration of phase space volume conservation prescribed by Liouville's theorem.

Thermodynamic Equilibrium

- ▶ In general, if $\rho(\mathbf{x}, t)$ has an explicit time dependence, then so will an observable A .
- ▶ However, a system in thermodynamic equilibrium do not change in time, $\partial\rho/\partial t$ must be equal to 0.
- ▶ In such a case, no external forces act on the system.
- ▶ The Liouville equation reduces to:

$$\{\rho, H\} = 0. \quad (19)$$

Thermodynamic Equilibrium

- The general solution to Eq. 19 is any function of H :

$$\rho(\mathbf{x}) \propto F[H(\mathbf{x})]. \quad (20)$$

- As ρ needs to be properly normalized, we write the solutions as:

$$\rho(\mathbf{x}) = \frac{1}{Z} F[H(\mathbf{x})], \quad (21)$$

where $Z = \int d\mathbf{x} F[H(\mathbf{x})]$ is referred to as the *partition function*.

- The partition function is a measure of the number of microscopic states in the phase space accessible within a given ensemble.
- Each ensemble has a particular partition function that depends on the macroscopic observables used to define the ensemble.

Ensembles

- ▶ N – the number of particles, V – volume, P – pressure, T – temperature, E – internal energy.
- ▶ Microcanonical ensemble: constant NVE .
- ▶ Canonical ensemble: constant NVT .
- ▶ Canonical pressure ensemble: constant NPT .

Microcanonical Ensemble

- NVE are fixed, so also the probability density is constant:

$$\rho(\mathbf{x}) = \frac{1}{Z} \delta[H(\mathbf{x}) - E], \quad (22)$$

where $\delta[\cdot]$ is the Dirac delta function and the microcanonical partition function is given by:

$$Z = \int d\mathbf{x} \delta[H(\mathbf{x}) - E]. \quad (23)$$

- We can see that the function $F[H(\mathbf{x})]$ from Eq. 20 is:

$$F[H(\mathbf{x})] \propto \delta[H(\mathbf{x}) - E]. \quad (24)$$

Microcanonical Ensemble

- ▶ State function is the entropy – a quantity that can be related to the number of the microscopic states of the system.
- ▶ Control variables are:

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E} \right)_{N,V}, \frac{P}{T} = \left(\frac{\partial S}{\partial V} \right)_{N,E}, \frac{\mu}{T} = - \left(\frac{\partial S}{\partial N} \right)_{V,E}. \quad (25)$$

- ▶ Let Ω be the number of microscopic states; the relation of S to Ω is:

$$S(N, V, E) = k_B \log \Omega(N, V, E), \quad (26)$$

where k_B is Boltzmann's constant:

$$k_B = 1.3806505(24) \times 10^{-23} \text{ J/K.} \quad (27)$$

Canonical Ensemble

- NVT are fixed.

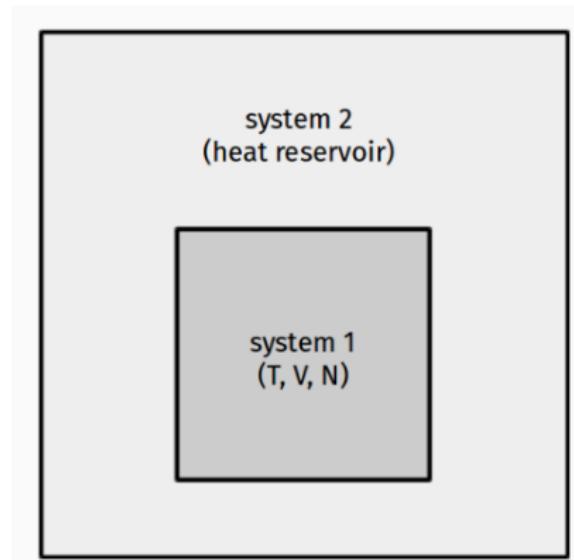


Figure 7: In a canonical ensemble setup, the system we study (system 1) is coupled to a heat reservoir (system 2) that guarantees a constant temperature.

Canonical Ensemble

- At a given temperature T , the probability for a system to be in a certain configuration \mathbf{x} with energy $E(\mathbf{x})$ is:

$$\rho(\mathbf{x}) = \frac{1}{Z} e^{-\beta E(\mathbf{x})}, \quad (28)$$

where $\beta = \frac{1}{k_B T}$ is the *inverse temperature*³, and the canonical partition function is:

$$Z = \int d\mathbf{x} e^{-\beta E(\mathbf{x})}. \quad (29)$$

- The ensemble average of a quantity A is given by:

$$\langle A \rangle = \frac{1}{Z} \int d\mathbf{x} A(\mathbf{x}) e^{-\beta E(\mathbf{x})}. \quad (30)$$

³ $\beta \approx 0.4 \text{ kJ/mol}$ at 300 K.

Canonical Ensemble

- We can derive Eq. 28 by using the maximum entropy principle. We know that:

$$\int d\mathbf{x} \rho(\mathbf{x}) E(\mathbf{x}) = U, \quad (31)$$

and that ρ must be normalized.

- We introduce the augmented function J :

$$J \equiv -k_B \int d\mathbf{x} \rho(\mathbf{x}) \log \rho(\mathbf{x}) \quad (32)$$

$$+ \lambda_1 \left(\int d\mathbf{x} \rho(\mathbf{x}) E(\mathbf{x}) - U \right) \quad (33)$$

$$+ \lambda_0 \left(\int d\mathbf{x} \rho(\mathbf{x}) - 1 \right), \quad (34)$$

where λ_0 and λ_1 are the Lagrange multipliers.

Canonical Ensemble

- Taking the functional derivative w.r.t. to ρ and setting it to zero, we have:

$$\frac{\delta J}{\delta \rho} = -k_B \log \rho(\mathbf{x}) - k_B + \lambda_1 E + \lambda_0 = 0, \quad (35)$$

- which implies that the maximum entropy distribution is:

$$\rho(\mathbf{x}) = \lambda_1 e^{\frac{-k_B + \lambda_0 + \lambda_1 E(\mathbf{x})}{k_B}}, \quad (36)$$

which translates to:

$$\rho(\mathbf{x}) = Z e^{\frac{-k_B + \lambda_0}{k_B}} \quad (37)$$

where Z is the canonical partition function:

$$Z = \int d\mathbf{x} e^{\frac{\lambda_1}{k_B} E}. \quad (38)$$

Canonical Ensemble

- We get that the density must have a form:

$$\rho(\mathbf{x}) = \frac{1}{Z} e^{\frac{\lambda_1}{k_B} E(\mathbf{x})} \quad (39)$$

and the entropy is:

$$S(\mathbf{x}) = -\lambda_1 U + k_B \log Z. \quad (40)$$

- We differentiate S w.r.t. U and apply $dU = TdS - PdV$:

$$\frac{dS}{dU} = -\lambda_1 \equiv \frac{1}{T} \quad (41)$$

from which we get that:

$$\frac{\lambda_1}{k_B} = -\frac{1}{k_B T} = -\beta. \quad (42)$$

Monte Carlo Methods

Monte Carlo Methods

The main steps of the Monte Carlo sampling are:

- ▶ Choose randomly a new configuration in phase space based on a Markov chain.
- ▶ Accept or reject the new configuration, depending on the strategy used.
- ▶ Compute the physical quantity and add it to the averaging procedure.
- ▶ Repeat the previous steps until convergence.

Monte Carlo Methods

$$I = \int_0^1 dx \int_0^{\sqrt{1-x^2}} dy = \frac{\pi}{4} \quad (43)$$

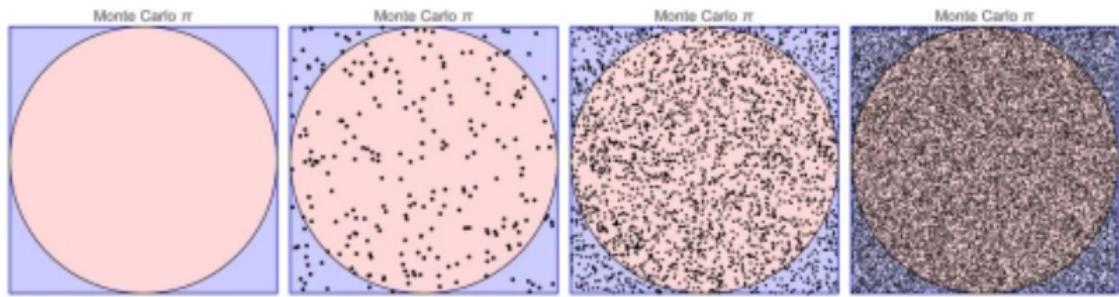


Figure 8: Example of a Monte Carlo method to compute π .

Monte Carlo Methods

- The integrals that must be evaluated in equilibrium statistical mechanics are generally of the form:

$$I = \int d\mathbf{x} \phi(\mathbf{x})p(\mathbf{x}), \quad (44)$$

where \mathbf{x} is an n -dimensional vector, $\phi(\mathbf{x})$ is an arbitrary function, and $p(\mathbf{x})$ is a function satisfying the properties of a probability distribution:

$$p(\mathbf{x}) \geq 0 \quad \text{and} \quad \int d\mathbf{x} p(\mathbf{x}) = 1. \quad (45)$$

- Eq. 44 represents the ensemble average of a physical observable in equilibrium statistical mechanics.

Monte Carlo Methods

- ▶ Let $\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_M$ be a set of M n -dimensional vectors that are sampled from $p(\mathbf{x})$.
- ▶ The problem of sampling from $p(\mathbf{x})$ is a nontrivial one.
- ▶ For now, let us assume that such an algorithm exists. Then, an estimator:

$$\hat{I}_M = \frac{1}{M} \sum_{i=1}^M \phi(\mathbf{x}_i) \quad (46)$$

is such that:

$$\lim_{M \rightarrow \infty} \hat{I}_M = I. \quad (47)$$

- ▶ Eq. 47 is guaranteed by the *central limit theorem*.⁴

⁴ For the derivation, see M. E. Tuckerman, *Statistical Mechanics: Theory and Simulation*, Oxford University Press (2016), p. 281.

Monte Carlo Methods

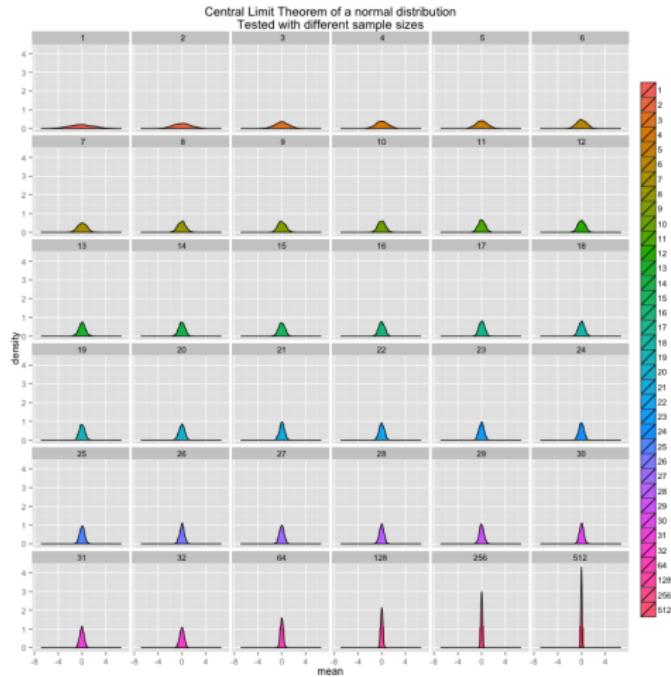


Figure 9: Central limit theorem for a normal distribution.

Source: https://www.reddit.com/r/math/comments/2wrduu/central_limit_theorem/

Monte Carlo Methods

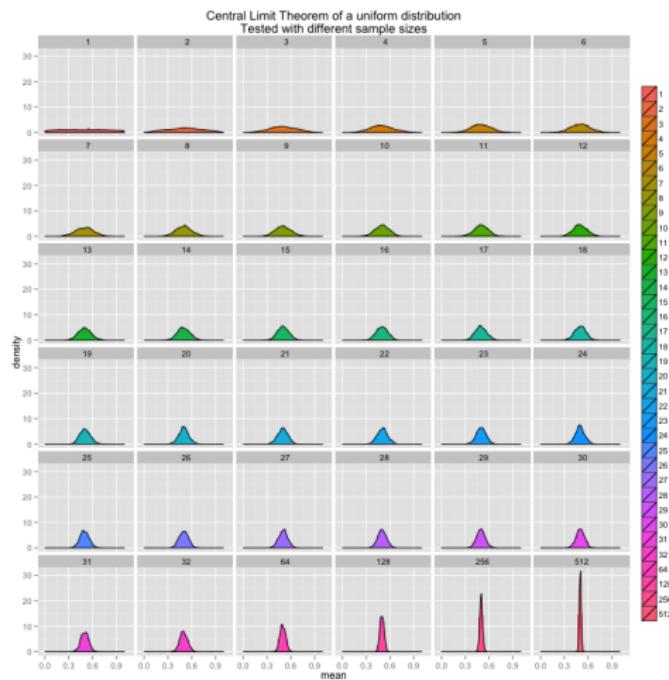


Figure 10: Central limit theorem for a uniform distribution.

Source: https://www.reddit.com/r/math/comments/2wrduu/central_limit_theorem/.

Monte Carlo Methods

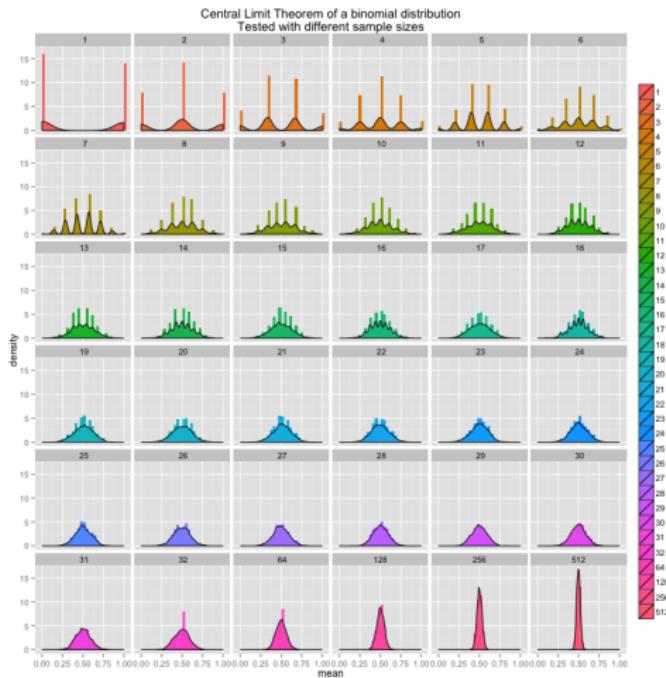


Figure 11: Central limit theorem for a binomial distribution.

Source: https://www.reddit.com/r/math/comments/2wrduu/central_limit_theorem/.

Monte Carlo Methods

Central Limit Theorem

If x_1, x_2, \dots, x_n are n random variables drawn from a population with overall mean μ and finite variance σ^2 , and if \bar{x}_n is the sample mean, the limiting form of the distribution, $\lim_{n \rightarrow \infty} \sqrt{n} \left(\frac{\bar{x}_n - \mu}{\sigma} \right)$, is the standard normal distribution.

Monte Carlo Methods

- Example of sampling a simple distribution given by:

$$p(x) = ce^{-cx}, \quad (48)$$

on the interval $x \in [0, \infty)$.

- To sample from $p(x)$, we need $P(X)$ such that:

$$P(X) = \int_0^X dx ce^{-cx} = 1 - e^{-cX}, \quad (49)$$

and equate Eq. 49 to a random number $\xi \in [0, 1]$, then we can solve for X , which gives:

$$X = -\frac{1}{c} \log(1 - \xi). \quad (50)$$

- In general, we do not have such simple distribution to sample from in statistical mechanics.

Monte Carlo Methods: Examples

Estimator no 1

```
def pi_estimator(n_samples):
    est = list()
    n_inside = 0; n_all = 0
    for i in range(n_samples):
        x = np.random.uniform(-1.0, 1.0)
        y = np.random.uniform(-1.0, 1.0)
        dist = np.sqrt(np.power(x, 2) +
                       np.power(y, 2))
        if dist <= 1.0:
            n_inside += 1
        n_all += 1
        est.append(4.0 * np.float(n_inside) /
                   np.float(n_all))
    return np.array(est)
```

Monte Carlo Methods: Examples

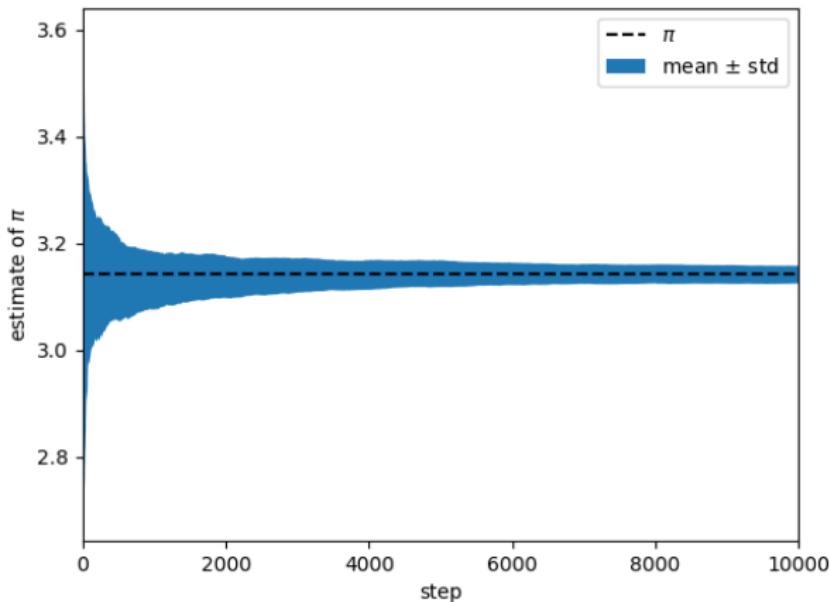


Figure 12: Estimator no 1.

Monte Carlo Methods: Examples

Estimator no 2

```
def pi_estimator_integral(n_samples):
    est = list()
    e = 0.0
    for i in range(n_samples):
        x = np.random.uniform(0.0, 1.0)
        y = 4.0 * np.sqrt(1.0 - np.power(x, 2))
        e += (y - e) / (np.float(i) + 1.0)
        est.append(e)
    return np.array(est)
```

Monte Carlo Methods: Examples

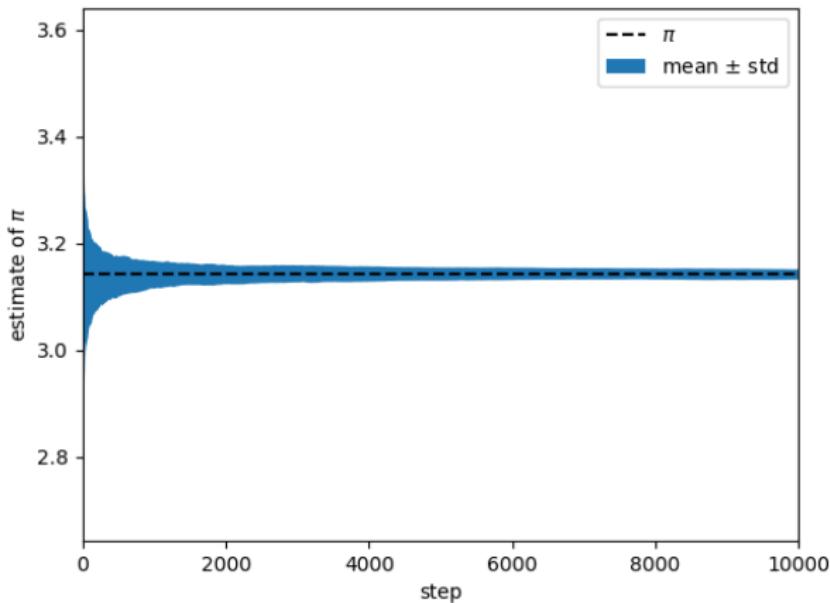


Figure 13: Estimator no 2.

Monte Carlo Methods: Examples

Estimator no 3

```
def pi_estimator_markov_chain(n_samples, step=0.1):
    est = list()
    n_inside = 0; n_all = 0;
    x = np.random.uniform(-1.0, 1.0)
    y = np.random.uniform(-1.0, 1.0)
    for i in range(n_samples):
        dx = np.random.uniform(-step, step)
        dy = np.random.uniform(-step, step)
        xn = x + dx; yn = y + dy
        if abs(xn) < 1.0 and abs(yn) < 1.0:
            x = xn; y = yn
        dist = np.sqrt(np.power(x, 2) + np.power(y, 2))
        if dist <= 1.0: n_inside += 1
        n_all += 1
        est.append(4.0 * np.float(n_inside) /
                   np.float(n_all))
    return np.array(est)
```

Monte Carlo Methods: Examples

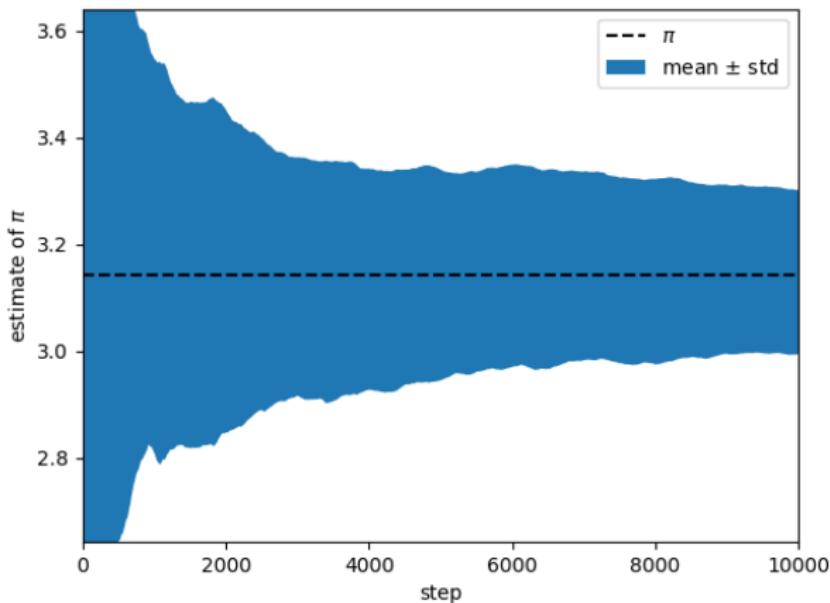


Figure 14: Estimator no 3.

Importance Sampling

- ▶ However, instead of sampling $p(\mathbf{x})$ to estimate $\int d\mathbf{x} \phi(\mathbf{x})p(\mathbf{x})$, we could sample from a different distribution $b(\mathbf{x})$ by rewriting the integral as:

$$I = \int d\mathbf{x} \left[\frac{\phi(\mathbf{x})p(\mathbf{x})}{b(\mathbf{x})} \right] b(\mathbf{x}). \quad (51)$$

- ▶ We set $\psi(\mathbf{x}) = \phi(\mathbf{x})p(\mathbf{x})/b(\mathbf{x})$ which gives us:

$$I = \int d\mathbf{x} \psi(\mathbf{x})b(\mathbf{x}) \quad (52)$$

$$\approx \frac{1}{M} \sum_{i=1}^M \psi(\mathbf{x}_i), \quad (53)$$

where the vectors \mathbf{x}_i are sampled from $b(\mathbf{x})$.

- ▶ This trick is the basis of *importance sampling*.

Importance Sampling

- ▶ Using $b(\mathbf{x})$ as an importance function may lead to easier sampling.
- ▶ But how do we select $b(\mathbf{x})$?
- ▶ Is there an optimal choice of $b(\mathbf{x})$? The best one leads to the smallest possible variance:

$$\text{Var}[b(\mathbf{x})] = \int d\mathbf{x} \psi^2(\mathbf{x})b(\mathbf{x}) - \left(\int d\mathbf{x} \psi(\mathbf{x})b(\mathbf{x}) \right)^2 \quad (54)$$

that gives us:

$$\int d\mathbf{x} \frac{\phi^2(\mathbf{x})p^2(\mathbf{x})}{b^2(\mathbf{x})}b(\mathbf{x}) - \left(\int d\mathbf{x} \phi(\mathbf{x})p(\mathbf{x}) \right)^2 \quad (55)$$

- ▶ Minimize Var w.r.t. $b(\mathbf{x})$ subject to the constraint $\int d\mathbf{x} b(\mathbf{x}) = 1$

Importance Sampling

- From Eq. 55, we have:

$$F[b(\mathbf{x})] = \text{Var}[b(\mathbf{x})] - \lambda \int d\mathbf{x} b(\mathbf{x}), \quad (56)$$

where λ is a Lagrange multiplier.

- Computing the functional derivative $\delta F/\delta b(\mathbf{x})$, we obtain:

$$\frac{\phi^2(\mathbf{x})p^2(\mathbf{x})}{b^2(\mathbf{x})} + \lambda = 0 \quad (57)$$

or

$$b(\mathbf{x}) = \frac{1}{\sqrt{-\lambda}}\phi(\mathbf{x})p(\mathbf{x}). \quad (58)$$

Importance Sampling

- Normalizing $b(\mathbf{x})$, we have:

$$\int d\mathbf{x} b(\mathbf{x}) = \frac{1}{\sqrt{-\lambda}} \int d\mathbf{x} \phi(\mathbf{x}) p(\mathbf{x}) = 1. \quad (59)$$

- Thus, the optimal choice for $b(\mathbf{x})$ is:

$$b(\mathbf{x}) = \frac{\phi(\mathbf{x}) p(\mathbf{x})}{I}.$$

(60)

- But if we knew the integral value I , we would not need to perform the calculation...

Importance Sampling: Examples

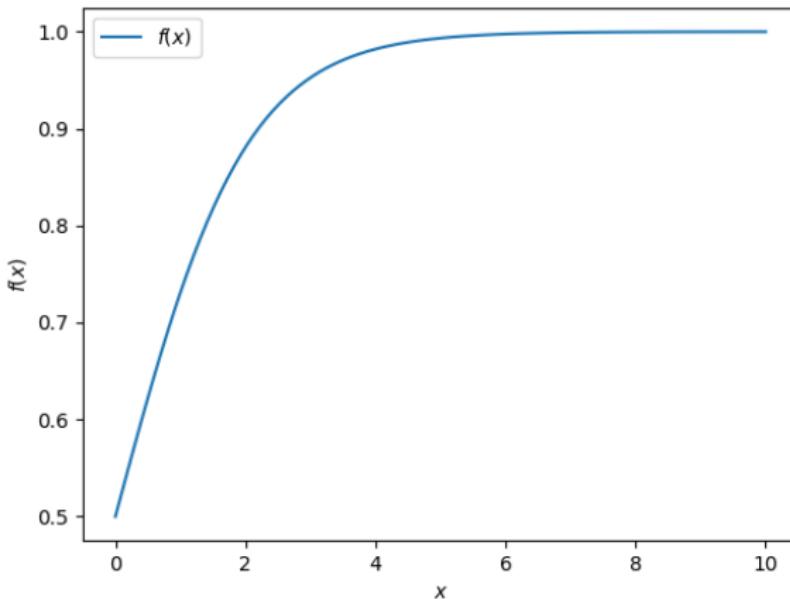


Figure 15: Function $f(x) = 1/(1 + \exp(-x))$.

Importance Sampling: Examples

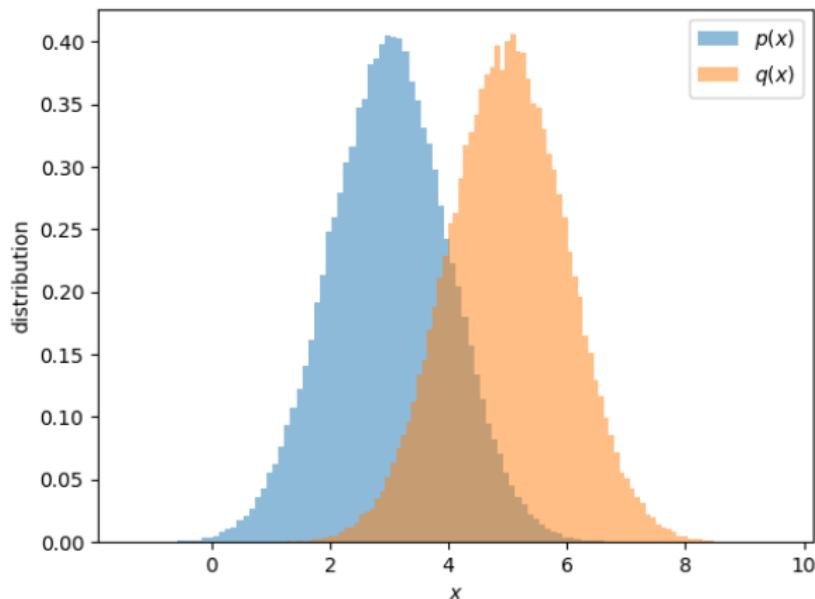


Figure 16: Distributions to sample from.

Importance Sampling: Examples

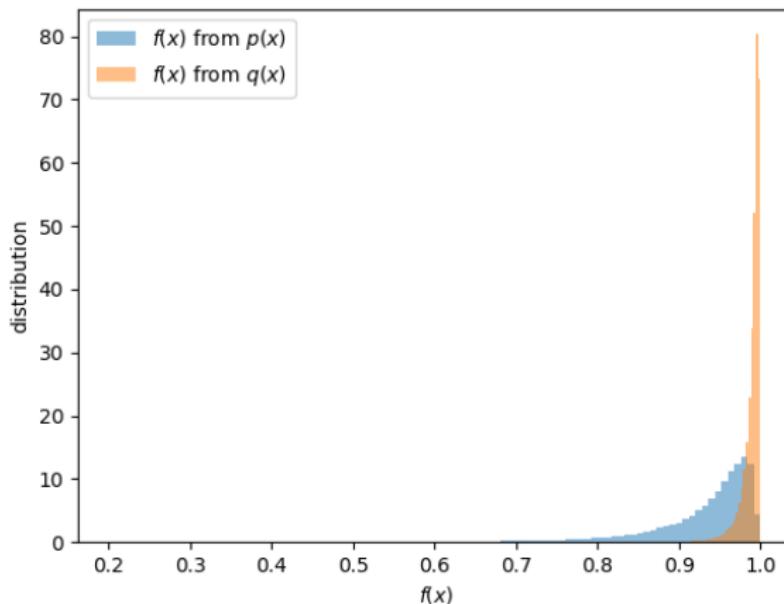


Figure 17: Distributions of evaluated samples.

Importance Sampling: Examples

Results

```
python importance-sampling.py --n_samples 100000  
  
f(x) from p(x): mean=0.9306712389 std=0.0686291035  
f(x) from q(x): mean=0.9891990210 std=0.0132275039  
f(x) from IS:    mean=0.9261305966 std=4.5258393804
```

Markov Chain

If the vectors $\mathbf{x}_1, \dots, \mathbf{x}_M$ are generated sequentially, and \mathbf{x}_{i+1} is generated only based on the knowledge of \mathbf{x}_i . the sequence is called a Makov chain.

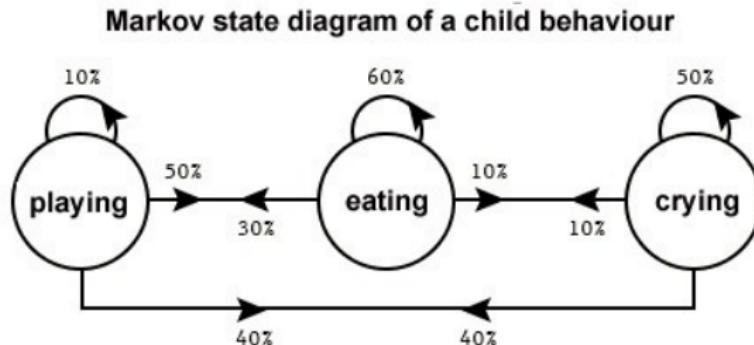


Figure 18: Example of a simple Markov chain.

M(RT)²

- For physical systems, a Markov chain must satisfy the condition of *detailed balance*, which ensures that the Markov process is microscopically reversible.
- M(RT)² is a rejection method.
- With the acceptance probability given by:

$$A(\mathbf{x}_i | \mathbf{x}_j) = \min [1, f(\mathbf{x}_i)/f(\mathbf{x}_j)], \quad (61)$$

where $f(\mathbf{x})$ is the probability of the system being at \mathbf{x} .

- Sampling in the canonical ensemble:⁵

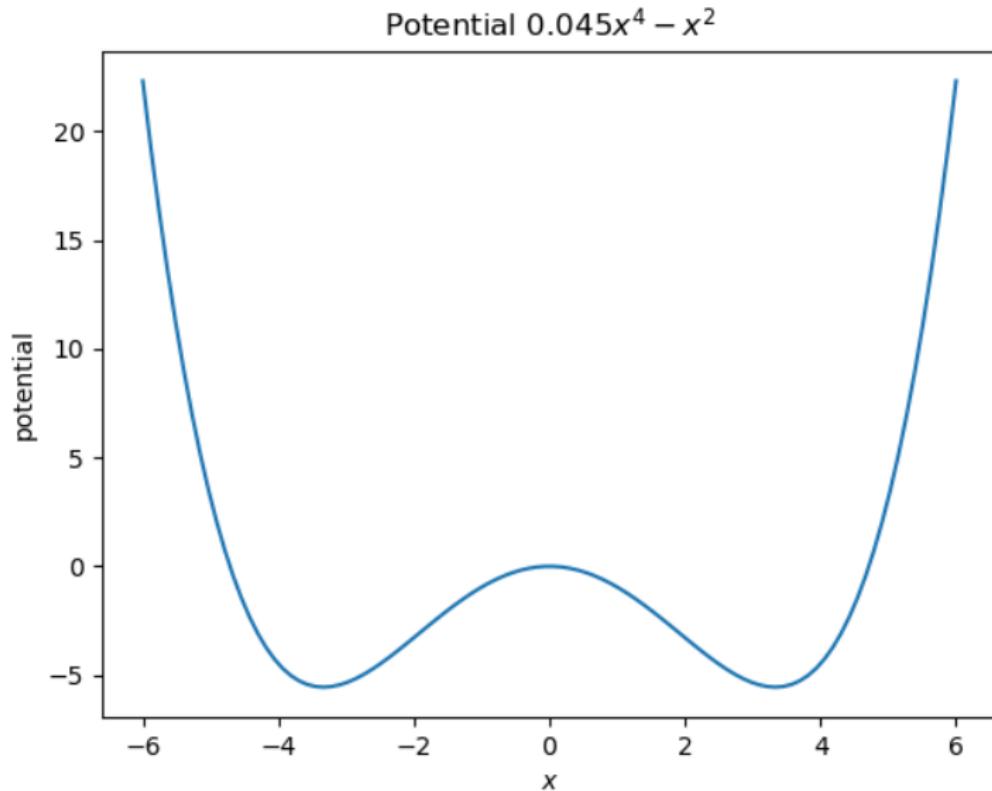
$$A(\mathbf{r}_i | \mathbf{r}_j) = \min \left[1, e^{-\beta[U(\mathbf{r}_i) - U(\mathbf{r}_j)]} \right], \quad (62)$$

as $f(\mathbf{r}) \propto e^{-\beta U(\mathbf{r})}$.

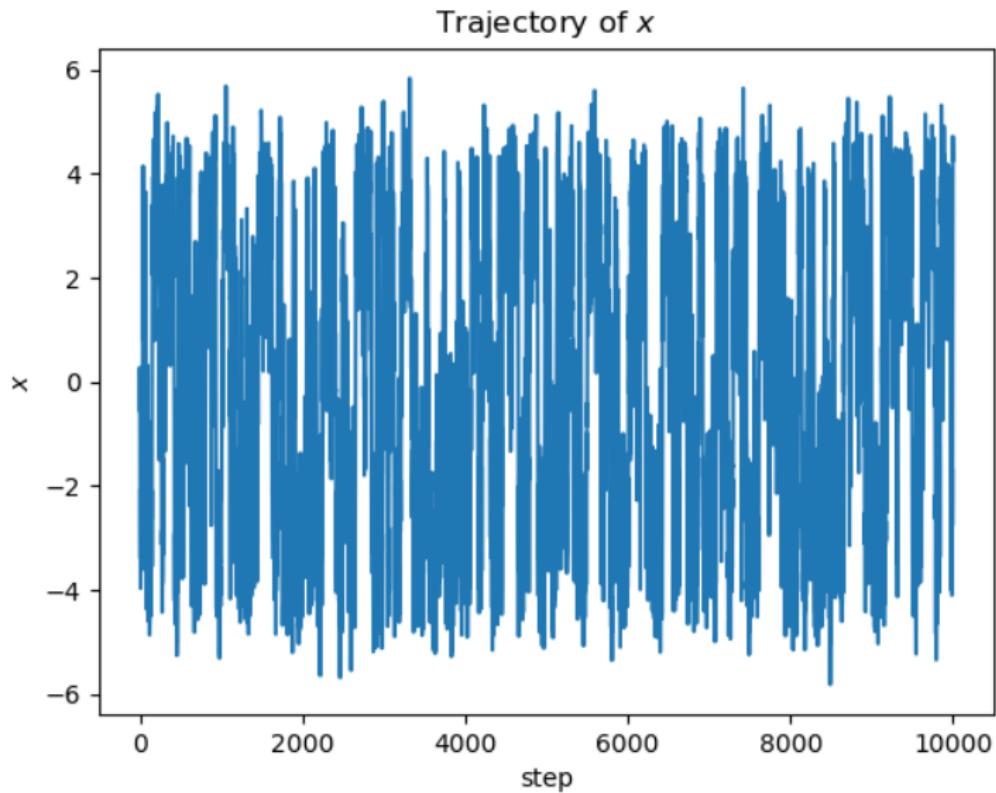
⁵Reminder: \mathbf{r} are microscopic coordinates.

Particle on Double-Well Potential

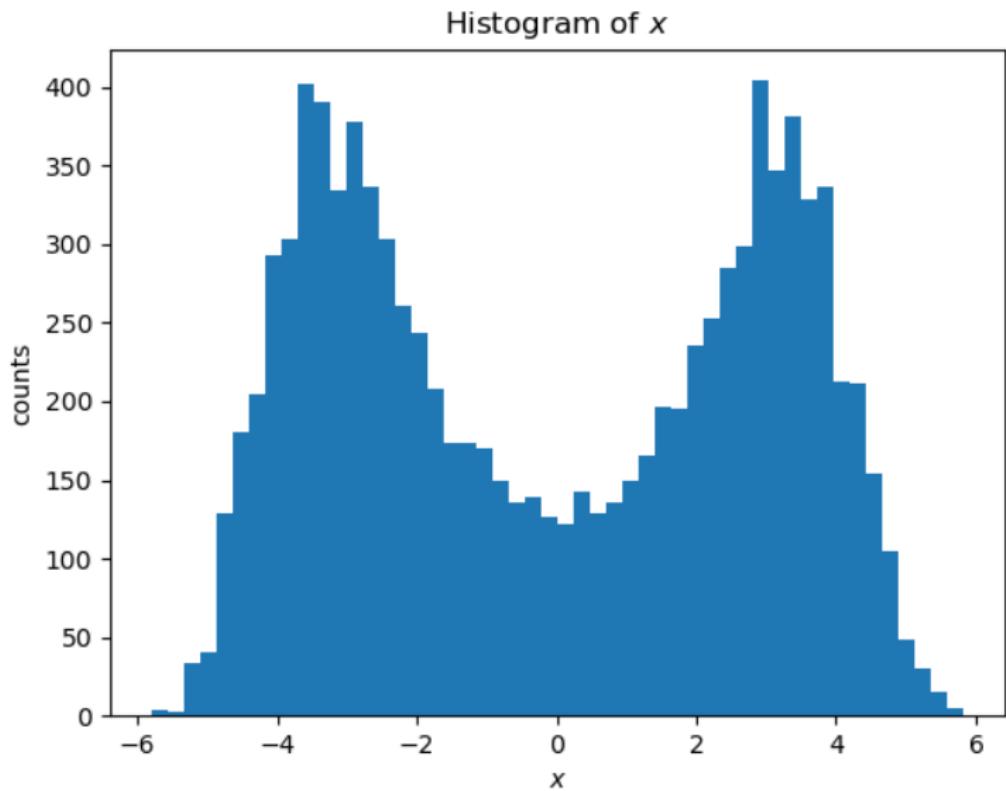
```
python mc-double-well.py --n_samples 10000 --temp 5
```



Particle on Double-Well Potential

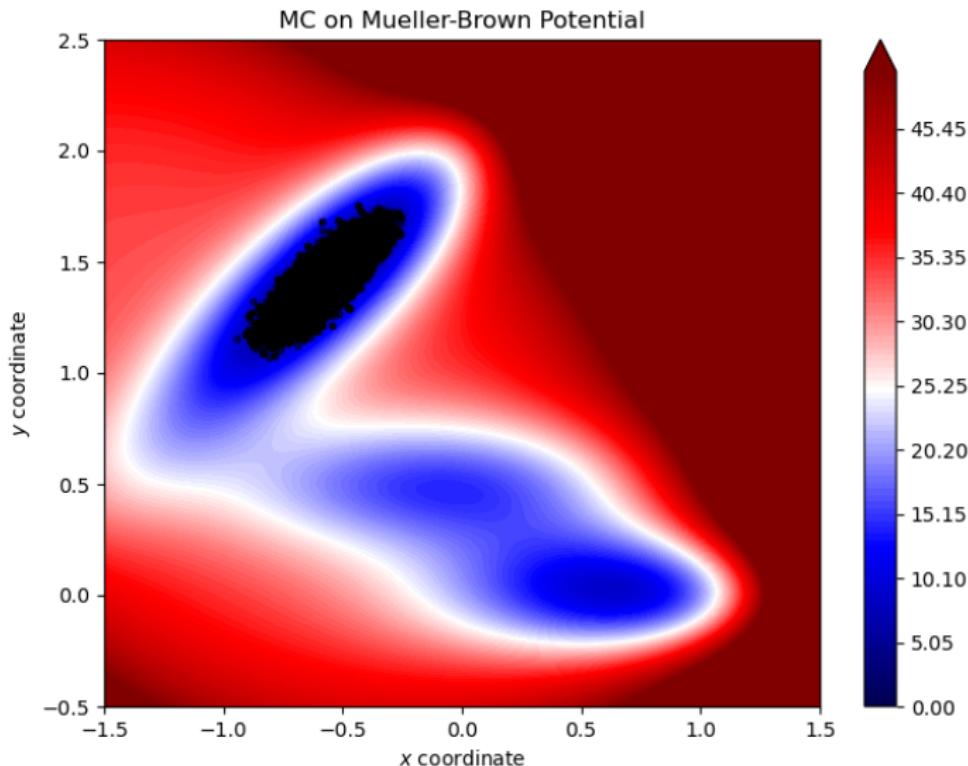


Particle on Double-Well Potential



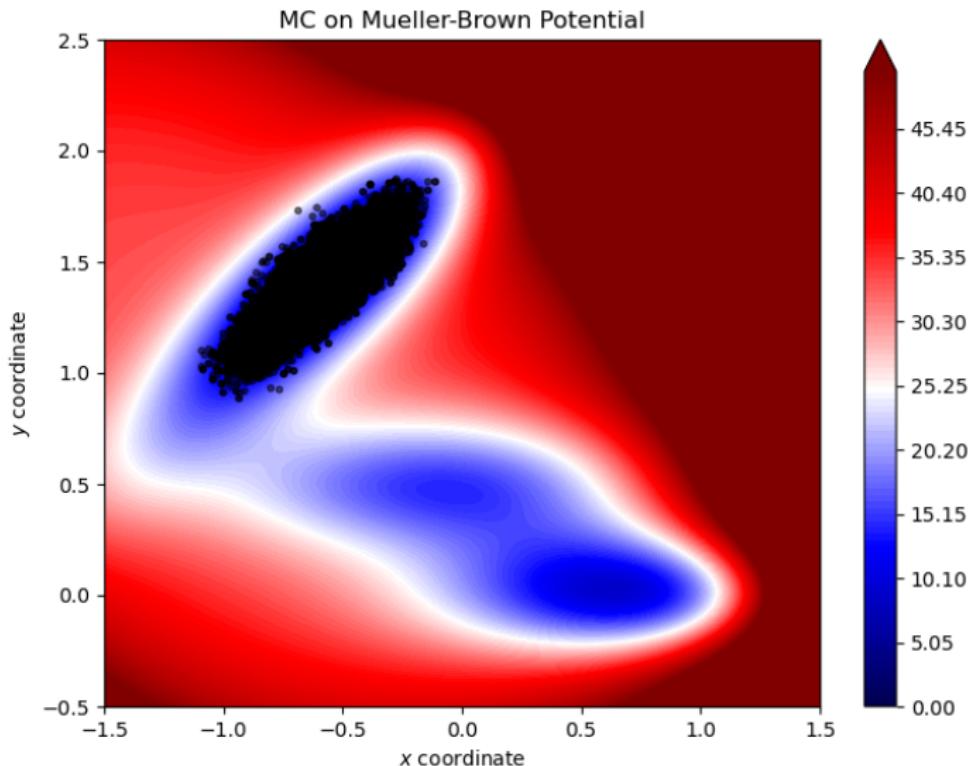
Particle on the Müller-Brown Potential

```
python mc-potential.py --n_samples 10000 --temp 1.0
```



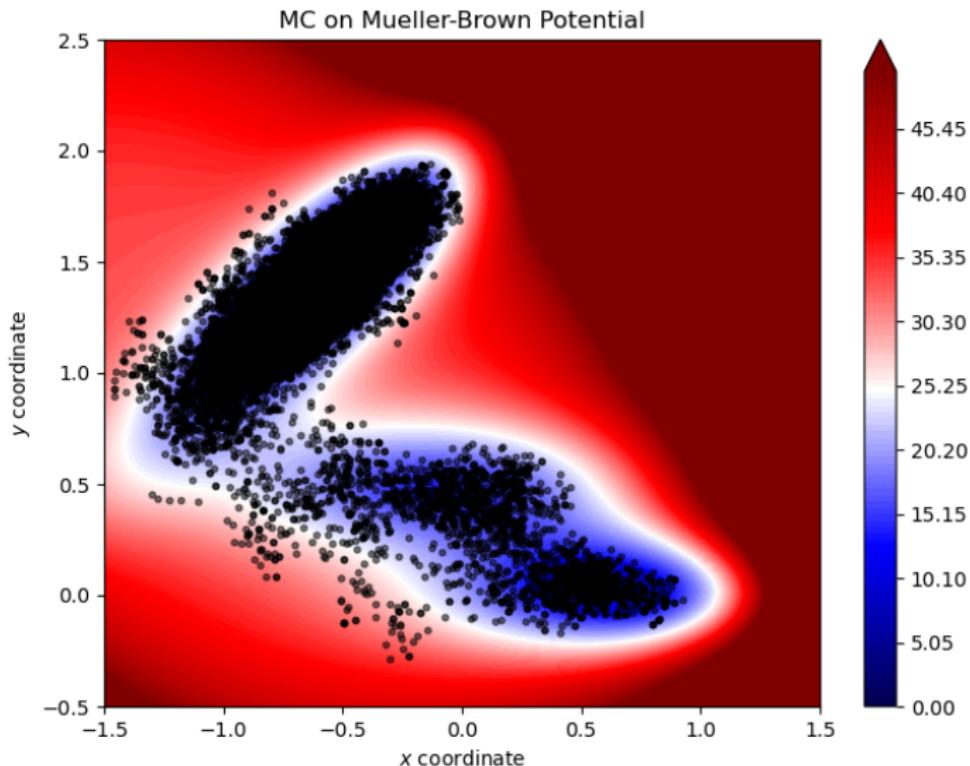
Particle on the Müller-Brown Potential

```
python mc-potential.py --n_samples 10000 --temp 2.0
```



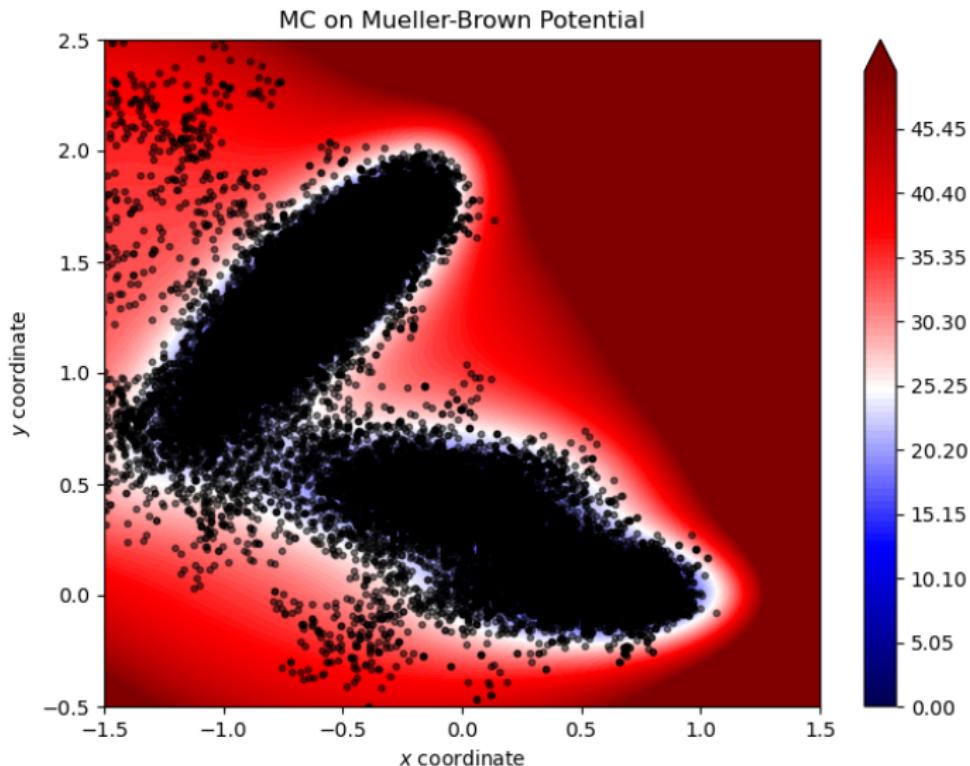
Particle on the Müller-Brown Potential

```
python mc-potential.py --n_samples 10000 --temp 3.0
```



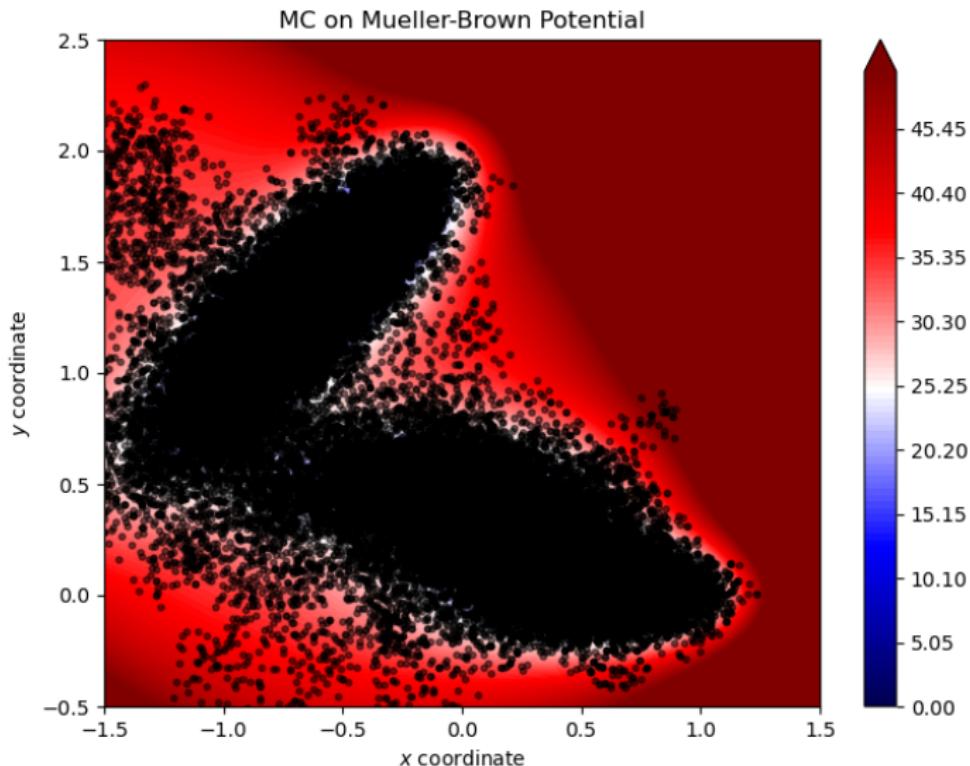
Particle on the Müller-Brown Potential

```
python mc-potential.py --n_samples 10000 --temp 4.0
```



Particle on the Müller-Brown Potential

```
python mc-potential.py --n_samples 10000 --temp 5.0
```



Replica Exchange Monte Carlo

- ▶ Barrier crossing are frequently rarely encountered during a simulation.
- ▶ For a barrier height of 15 kJ/mol at 300 K, the Boltzmann factor is approximately 3×10^{-3} ; for 30 kJ/mol is 6×10^{-6} .

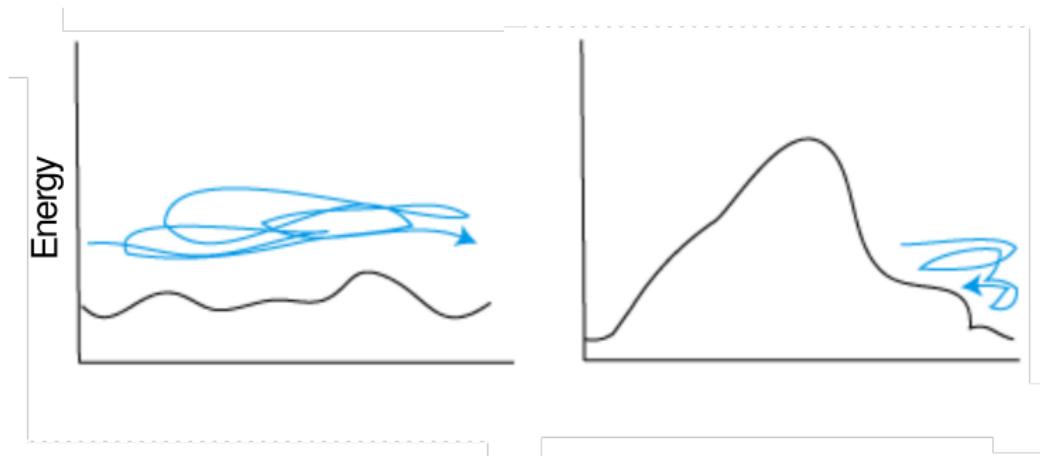


Figure 19: Barrier crossing showing a trajectory in the case of a low and a high energy barrier, respectively.

Parallel Tempering⁶

- ▶ Temperature is used as the control variable, and different temperatures are assigned to the *replicas*.
- ▶ In the parallel tempering scheme, a set of temperatures T_1, \dots, T_M such that:

$$T_1 < T_2 < \dots < T_M \quad (63)$$

are assigned to the M replicas, where T_1 is the temperature T of the canonical distribution.

- ▶ The high-temperature replicas easily cross potential energy barriers.
- ▶ We attempt exchanges between the neighboring replicas.

⁶Marinari and Parisi. *Simulated tempering: A new Monte Carlo scheme* Europhys. Lett. 19 (1992)

Parallel Tempering

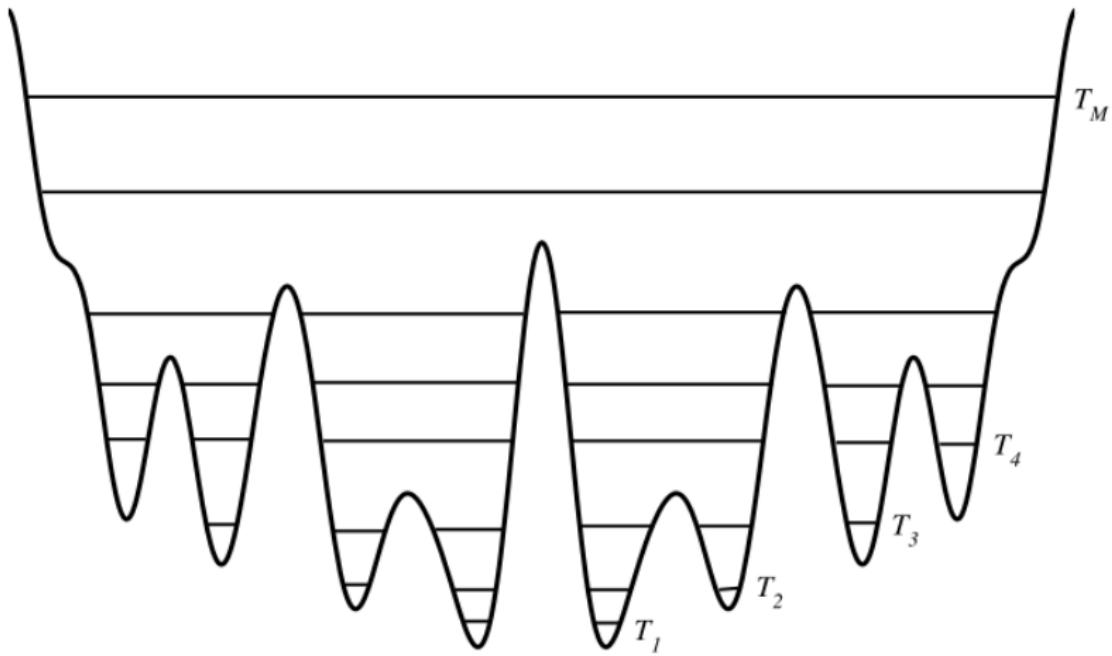


Figure 20: Schematic of the parallel-tempering replica exchange Monte Carlo.⁷

⁷Source: M. E. Tuckerman, *Statistical Mechanics: Theory and Simulation*, Oxford University Press (2016)

Parallel Tempering

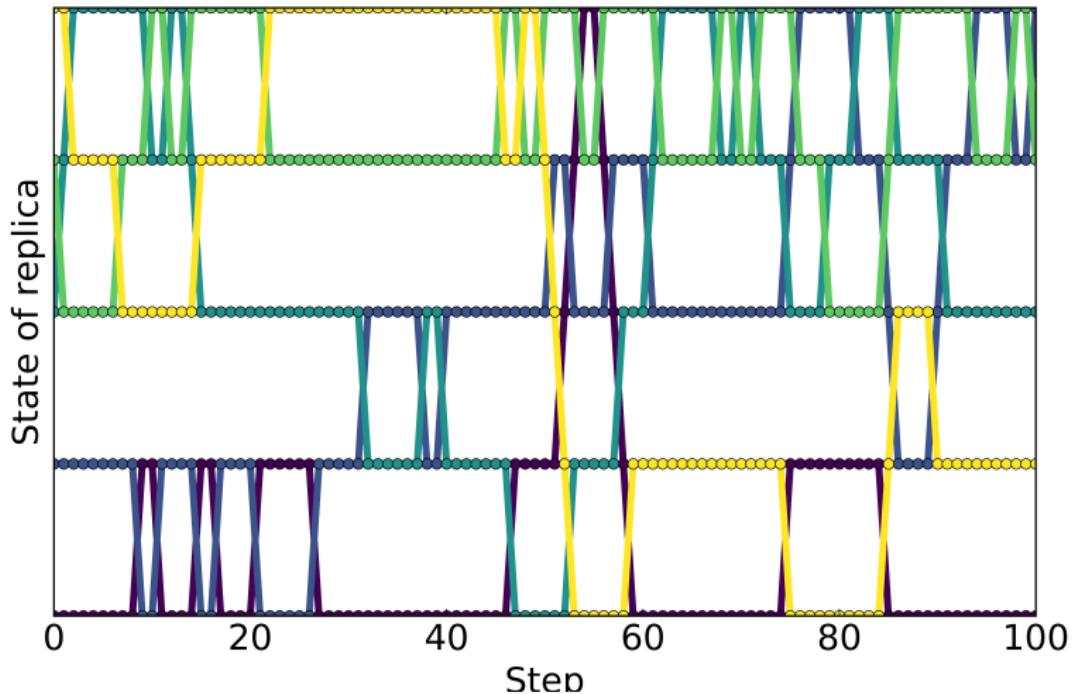


Figure 21: An example of a parallel tempering trajectory.⁸

⁸Source: <https://coulomb.umontpellier.fr/perso/daniele.coslovich/pt/>

Paralell Tempering

- ▶ Let $\mathbf{r}^{(1)}, \dots, \mathbf{r}^{(M)}$ be the complete configurations of the M replicas, i.e., $\mathbf{r}^{(K)} \equiv (\mathbf{r}_1^{(K)}, \dots, \mathbf{r}_N^{(K)})$.
- ▶ The replicas are independent, so the total probability distribution $F(\mathbf{r}^{(1)}, \dots, \mathbf{r}^{(M)})$ is:

$$F(\mathbf{r}^{(1)}, \dots, \mathbf{r}^{(M)}) = \prod_{K=1}^M f_K(\mathbf{r}^{(K)}), \quad (64)$$

where:

$$f_K(\mathbf{r}^{(K)}) = \frac{1}{Z} e^{-\beta_K U(\mathbf{r}^{(K)})}, \quad (65)$$

in which $\beta_K = (k_B T_K)^{-1}$.

Parallel Tempering

- ▶ Periodically, a neighboring pair of replicas K and $K + 1$ is selected, and an attempted switch is made with probability:

$$A_s = A \left[(\mathbf{r}^{(K+1)}, \mathbf{r}^{(K)}) | (\mathbf{r}^{(K)}, \mathbf{r}^{(K+1)}) \right] \quad (66)$$

$$= \min \left[1, \frac{f_K(\mathbf{r}^{(K+1)})}{f_K(\mathbf{r}^{(K)})} \cdot \frac{f_{K+1}(\mathbf{r}^{(K)})}{f_{K+1}(\mathbf{r}^{(K+1)})} \right] \quad (67)$$

$$= \min \left[1, e^{-\Delta_{K,K+1}} \right], \quad (68)$$

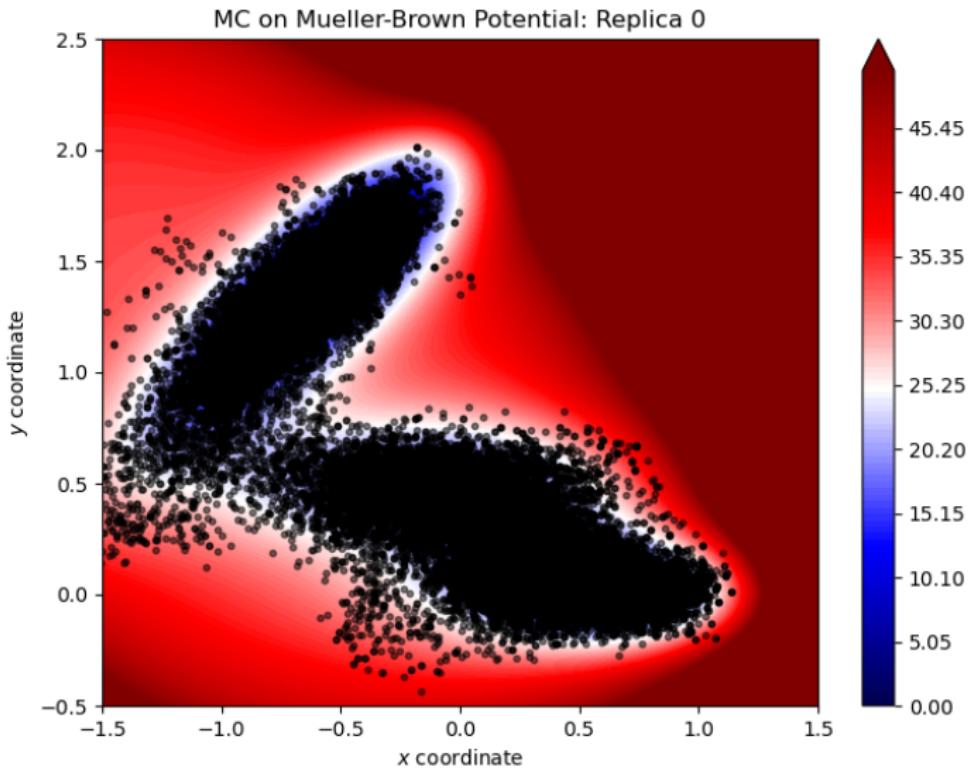
where:

$$\Delta_{K,K+1} = (\beta_K - \beta_{K+1}) \left[U(\mathbf{r}^{(K)}) - U(\mathbf{r}^{(K+1)}) \right] \quad (69)$$

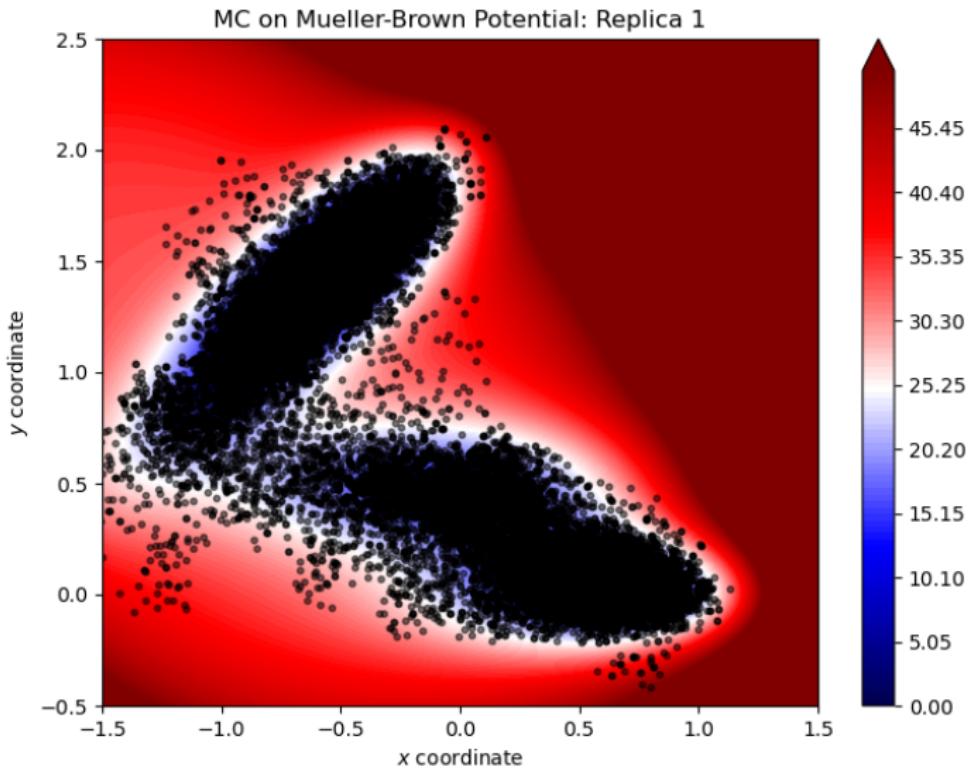
Particle on the Müller-Brown Potential

```
python mc-pt-potential.py --n_samples 10000  
                      --temp 1,2,3,4,5  
                      --n_replicas 5  
                      --exchange 1000
```

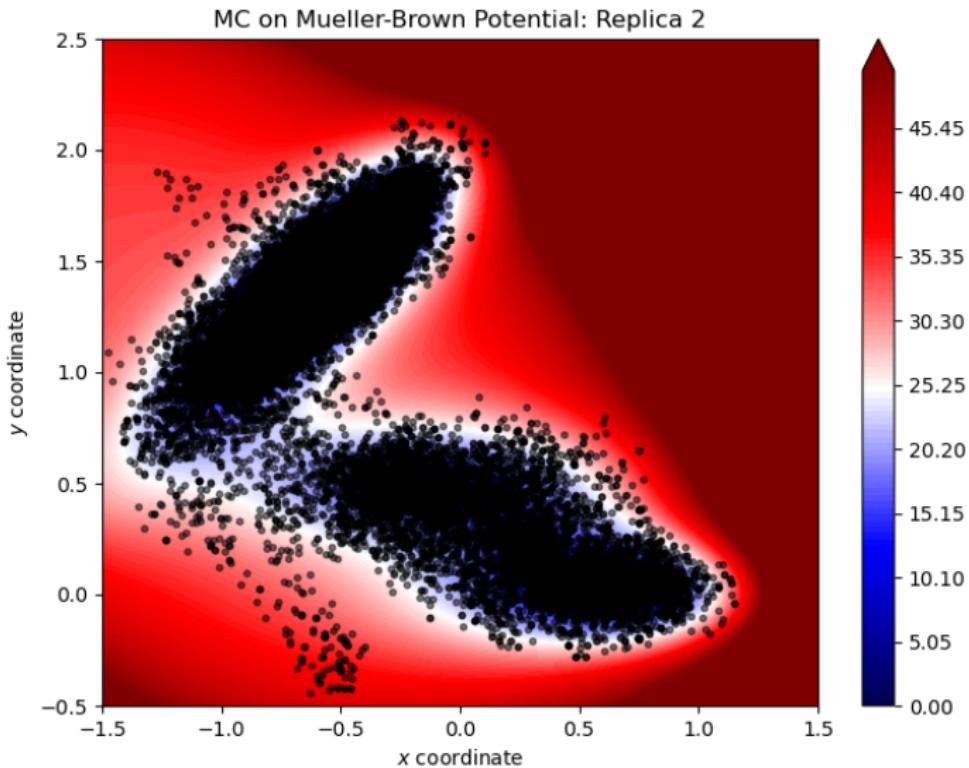
Particle on the Müller-Brown Potential



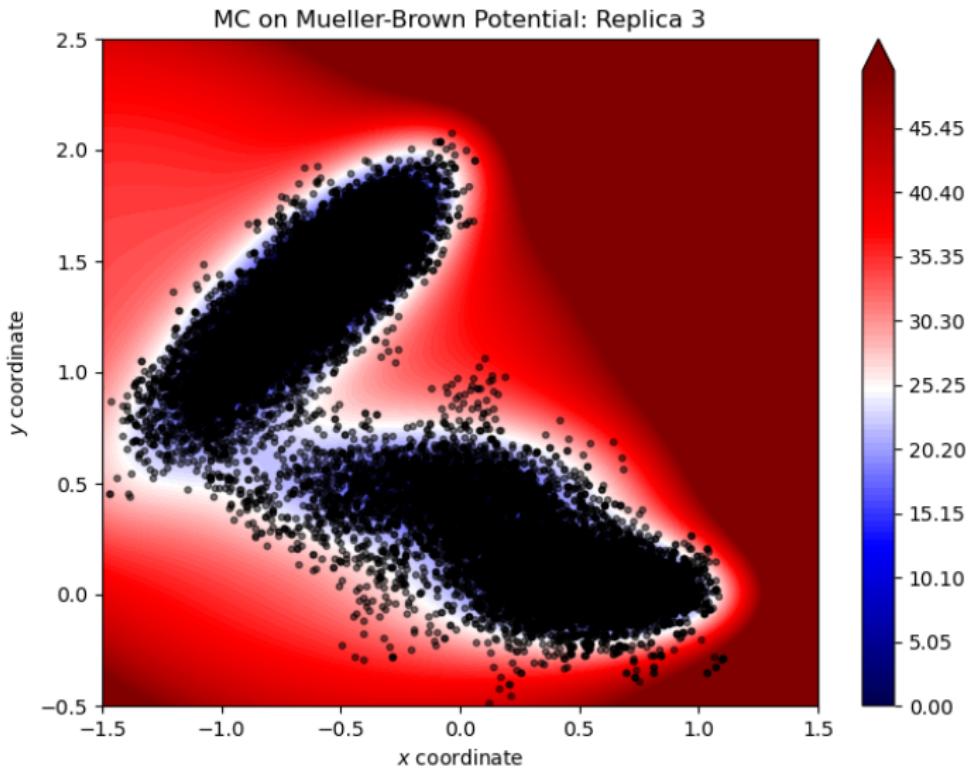
Particle on the Müller-Brown Potential



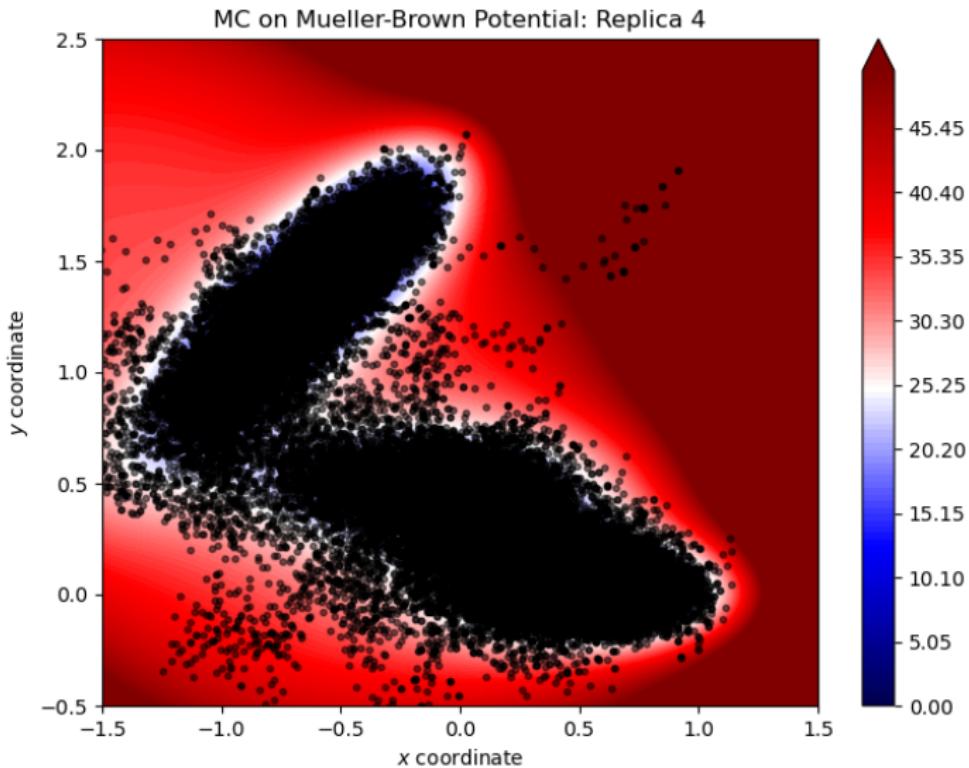
Particle on the Müller-Brown Potential



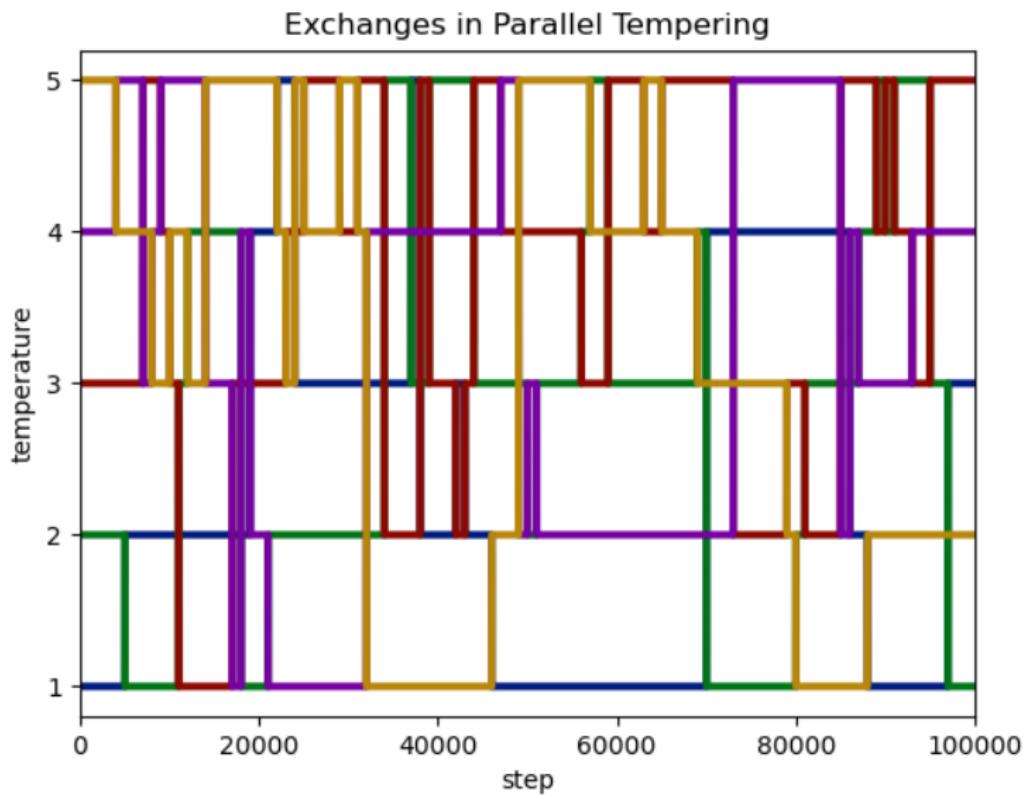
Particle on the Müller-Brown Potential



Particle on the Müller-Brown Potential



Particle on the Müller-Brown Potential



Wang-Landau Sampling

- Recall that the canonical partition function $Z(N, V, T)$ can be expressed in terms of the microcanonical partition function $\Omega(N, V, E)$:

$$Z(N, V, T) = \frac{1}{E_0} \int_0^{\infty} dE e^{-\beta E} \Omega(N, V, E), \quad (70)$$

where E_0 is an arbitrary reference energy.

- Setting $E_0 = 1$ and dropping fixed V and N , we have:

$$Z(\beta) = \int_0^{\infty} dE e^{-\beta E} \Omega(E). \quad (71)$$

- Eq. 71 suggests a procedure to compute the canonical partition function, and thus, all thermodynamic quantities.

Wang-Landau Sampling

Wang-Landau Algorithm⁹

1. Set $\Omega(E)$ for all values of E .
2. Attempt a trial move from E to E' with the acceptance probability given by:

$$A(E'|E) = \min \left[1, \frac{\Omega(E)}{\Omega(E')} \right]. \quad (72)$$

3. Modify the bin such that $\Omega(E) \rightarrow \Omega(E)f$, where $f > 1$.
4. Accumulate the histogram of energy, $h(E) \rightarrow h(E) + 1$.
5. If $h(E)$ is flat enough, then $f \rightarrow \sqrt{f}$.
6. If not converged, move to 2; else return $\Omega(E)$.

⁹ Numerical implementations require that $\Omega(E)$ be discretized into a number of energy bins.

Wang-Landau Sampling

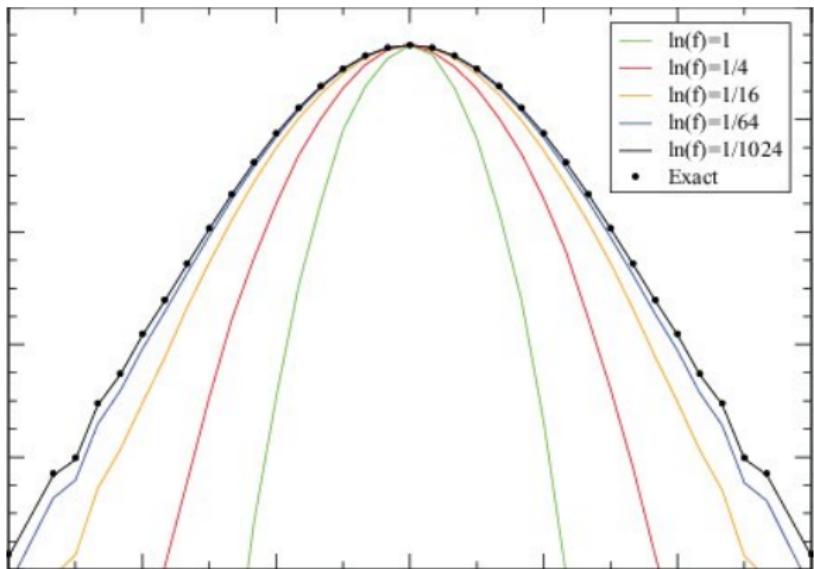


Figure 22: Convergence of the Wang-Landau algorithm.¹⁰

¹⁰ Source: Brown, Odbadrakh, Nicholson, and Eisenbach, *Convergence for the Wang-Landau density of states*, Phys. Rev. E 84, 065702(R) (2011)

Molecular Dynamics

Molecular Dynamics

There are three main ingredients in molecular dynamics (MD):

- ▶ The algorithm to integrate the equations of motion,
- ▶ The model describing the interparticle interactions,
- ▶ The calculation of forces and energies from the model.

Verlet Algorithm

- The simplest approach to obtain a numerical integration scheme is to use a Taylor series:

$$\mathbf{r}_i(t + \Delta t) \approx \mathbf{r}_i(t) + \Delta t \mathbf{v}_i(t) + \frac{\Delta t^2}{2m_i} \mathbf{F}_i(t), \quad (73)$$

and

$$\mathbf{r}_i(t - \Delta t) \approx \mathbf{r}_i(t) - \Delta t \mathbf{v}_i(t) + \frac{\Delta t^2}{2m_i} \mathbf{F}_i(t). \quad (74)$$

- Adding Eq. 73 to Eq. 74, we get a velocity-independent Verlet algorithm:

$$\mathbf{r}_i(t + \Delta t) = 2\mathbf{r}_i(t) - \mathbf{r}_i(t - \Delta t) + \frac{\Delta t^2}{m_i} \mathbf{F}_i(t). \quad (75)$$

Velocity Verlet Algorithm

- Consider a shift in time in comparison to Eq. 73:

$$\mathbf{r}_i(t) \approx \mathbf{r}_i(t + \Delta t) - \Delta t \mathbf{v}_i(t + \Delta t) + \frac{\Delta t^2}{2m_i} \mathbf{F}_i(t + \Delta t), \quad (76)$$

which after some rearrangements takes the following form:

$$\mathbf{v}_i(t + \Delta t) = \mathbf{v}_i(t) + \frac{\Delta t}{2m_i} [\mathbf{F}_i(t) + \mathbf{F}_i(t + \Delta t)]. \quad (77)$$

- This scheme allows to use both evolution of the positions and velocities.

Initial Conditions

- ▶ Initial coordinates.
- ▶ Initial velocities are sampled from a Maxwell-Boltzmann distribution, taking care to ensure that the sampled velocities are consistent with any constraints imposed on the system:

$$f(v) = \frac{m}{2\pi k_B T}^{1/2} e^{-mv^2/2k_B T}, \quad (78)$$

where v is the velocity of a particle of mass m at temperature T .

- ▶ An example of a Gaussian probability distribution.

Potential Energy Model

- ▶ Commonly used model of potential energy in MD for biological macromolecules is the following:

$$U(\mathbf{r}) = \sum_{\text{bonds}} \frac{1}{2} K_{\text{bond}} (r - r_0)^2 \quad (79)$$

$$+ \sum_{\text{bends}} \frac{1}{2} K_{\text{bends}} (\theta - \theta_0)^2 \quad (80)$$

$$+ \sum_{\text{tors}} \sum_{n=0}^6 A_n [1 + \cos(C_n \phi + \delta_n)] \quad (81)$$

$$+ \sum_{i,j \in \text{non-bonds}} \left(4\epsilon_{ij} \left[\frac{\sigma_{ij}^{12}}{r_{ij}} - \frac{\sigma_{ij}^6}{r_{ij}} \right] + \frac{q_i q_j}{r_{ij}} \right). \quad (82)$$

- ▶ Force is given by $\mathbf{F} = -\partial U / \partial \mathbf{r}$.
- ▶ Called *force field* in MD.











