§ 4 — Spatial - Frequency Domain

§ 4.1 Continuous Fourier Transform

- A signal can be expressed as a weighted sum of sine & cosine

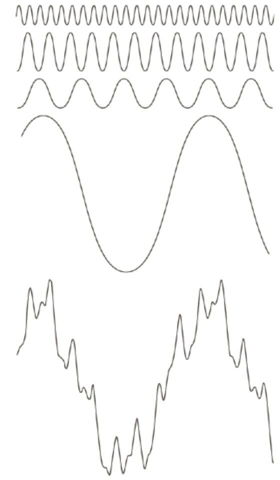


FIGURE 4.1 The function at the bottom is the sum of the four functions above it. Fourier's idea in 1807 that periodic functions could be represented as a weighted sum of sines and cosines was met with skepticism.

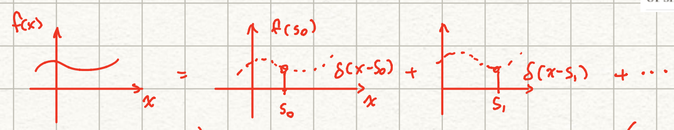
- Euler's Formula: $\boxed{e^{j\theta} = \cos\theta + \sin\theta}$ ← summation of sin & cos can be expresed as a function of a complex exponential.

- Signal Representation:

  Any signal $f(x,y)$ can be expresed as a superposition of unit impulse using the Sifting Property:

$$\boxed{f(x,y) = \iint f(s,t)\,\delta(x-s, y-t)\,ds\,dt}$$

$$f(x) = \int_{-\infty}^{\infty} f(s)\,\delta(x-s)\,ds$$



Σ of weighted impulse.

- Relationship to Convolution.

$$g(x,y) = \overset{\text{Linear Operator (Transformation)}}{T}[f(x,y)]$$

$$= T\left[\iint f(s,t)\,\delta(x-s, y-t)\,ds\,dt\right]$$

$$= \iint f(s,t)\,T[\delta(x-s, y-t)]\,ds\,dt$$

$$= \iint f(s,t)\,h(x-s, y-t)\,ds\,dt$$

$$= f(x,y) * h(x,y). \rightarrow \text{Convolution}$$

Transformation → Convolution

- Transforming a function $f(x,y)$ then involves transforming the impulses, which are a function of (x,y), which leads to ⇒ Convolution.

⇓

- Convolution ⇒ Edge Blur →

⇓ (step response)

- How do we deblur a blurred image?
  - ↳ Deconvolution
    - Inverse System
    - Something Better?

⇓

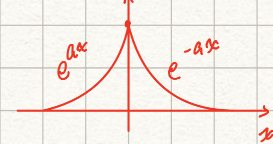
- Inverse System
  - ↳ To undo the effects of an undesirable affect to the image
  - ↳ using: $h(x,y) * h_I(x,y) = \delta(x,y)$

Ex) 1-D Continuous Exponential PSF

- What's step response? ⇒ Edge Blur
- Can we deblur?

$$h(x,y) = \left(\frac{a}{2}\right)^2 e^{-a(|x|+|y|)}$$

$$h(x) = \frac{a}{2} e^{-a|x|}$$

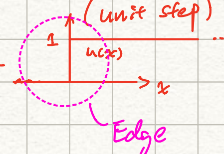


- If $f(x) = \delta(x)$, what's $g(x)$? (unit impulse)

  ↳ $g(x) = h(x)$ (impulse Response)

$$\delta \rightarrow \boxed{h} \rightarrow g = h$$

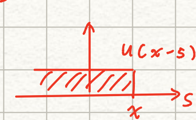
- If $f(x) = u(x)$, what's $g(x)$?



↳ $g(x) = u(x) * h(x)$

$$= \int_{-\infty}^{\infty} h(s)\,u(x-s)\,ds$$

$$= \int_{-\infty}^{x} h(s)\,ds \Rightarrow$$



Blur

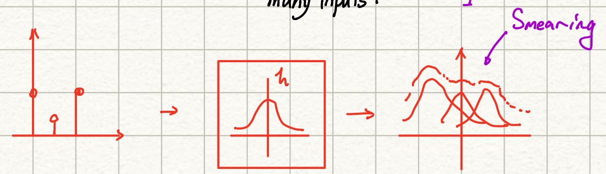
- Two Regions of Interest.

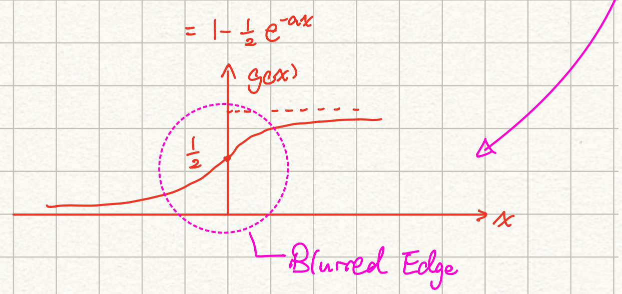
$$\begin{cases} x<0, \; g(x) = \frac{a}{2}\int_{-\infty}^{x} e^{as}\,ds = \frac{1}{2}\left[e^{as}\right]_{-\infty}^{x} = \frac{e^{ax}}{2} \\ x>0, \; g(x) = \frac{1}{2} + \int_{0}^{x} e^{-as}\,ds = \frac{1}{2} - \frac{1}{2}\left[e^{-as}\right]_{0}^{x} \end{cases}$$

↳ Still involves Deconvolution !

↑ It's Difficult!

↳ Each output generally has contribution from many inputs!





Easier Approach ↓

- We prefer to represent $f(x,y)$ with components that are not "smeared" by the LSI (Linear, Shift, Invariant) system

Ex)

$$\{\Phi_{k,l}(x,y)\} \text{ s.t. } \sum f_{kl}\,\Phi_{k,l}(x,y) = f(x,y)$$

$$\& \quad T[\Phi_{k,l}(x,y)] = \lambda_{k,l}\,\Phi_{k,l}(x,y)$$

$$\text{so that: } \Phi_{k,l}(x,y) * h(x,y) = \lambda_{k,l}\,\Phi_{k,l}(x,y)$$

↳ Defined as Eigenfunction of LSI sys

Complex Exponentials

- Complex exponentials, $e^{j2\pi ux}$, will not be altered by a LTI system.
- 'u' is the frequency of the complex exponential
  - 1D: cycles per unit time
  - 2D: cycles per unit distance OR per degree of visual angle

$$e^{j2\pi ux} * h(x) = \int e^{j2\pi u(x-s)} h(s)\, ds$$

$$= \left[\int h(s)\, e^{-j2\pi us}\, ds \right] \cdot e^{j2\pi ux}$$

$$= H(u)\, e^{j2\pi ux}$$

where $H(u)$ rep. the continuous Fourier Transform (FT) of the function $h(x)$

⇓

Continuous Fourier Transform

1D

- Forward: $F(u) = \int f(x)\, e^{-j2\pi ux}\, dx$
- Inverse: $f(x) = \int F(u)\, e^{j2\pi ux}\, du$

- $F(u)$ is complex: $F(u) = \underbrace{|F(u)|}_{\text{magnitude}}\, e^{j\Phi}$ — Phase

2D

- Forward: $F(u,v) = \int f(x,y)\, e^{-j2\pi(ux+vy)}\, dx\,dy$
- Inverse: $f(x,y) = \int F(u,v)\, e^{j2\pi(ux+vy)}\, du\,dv$

⇒

Important Property

- Convolution:

  1D: $g(x) = f(x) * h(x)$

  2D: $g(x,y) = f(x,y) * h(x,y)$

- Transformed Functions can be multiplied in freq (1D) / Spatial Freq (2D):

  1D: $G(u) = F(u)\, H(u)$

  2D: $G(u,v) = F(u,v)\, H(u,v)$

Fourier: Magnitude & Phase (2D)

- Magnitude:

$$|F(u,v)| = \left[R^2(u,v) + I^2(u,v) \right]^{\frac{1}{2}}$$

$$R = Re\{F(u,v)\}, \quad I = Im\{F(u,v)\}$$

- Phase:

$$\Phi(u,v) = \tan^{-1}\left[I(u,v) / R(u,v) \right]$$

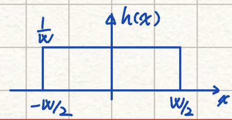
- Power Spectrum:

$$P(u,v) = |F(u,v)|^2$$

§ 4.1.2 Examples

Ex 1: 1D Local Average Over Window W

$$h(x) = \begin{cases} 1/W, & |x| < W/2 \\ 0, & \text{else} \end{cases}$$ [aka, Box Filter]



$$H(u) = \frac{1}{W}\int_{-\infty}^{\infty} e^{-j2\pi ux}\,dx = -\frac{1}{j2\pi u}\left[e^{-j2\pi ux}\right]_{-W/2}^{W/2}$$

Recall: $\sin\theta = \frac{e^{j\theta} - e^{-j\theta}}{2j}$

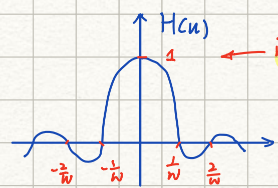
$$= -\frac{1}{j2\pi u}\left[e^{-j\pi uW} - e^{j\pi uW}\right]$$

$$= \boxed{\frac{\sin(\pi uW)}{\pi uW} \triangleq \text{sinc function}}$$

$$DC_{gain} = \partial\left[\frac{\sin(\pi uW)}{\pi uW}\right]/\partial u\Big|_{u=0}$$

(L'Hopital's)

$$\frac{\pi W\cos(\pi uW)}{\pi W}\Big|_{u=0} = 1$$



DC Gain = 1 [H(u) @ x=0, origin]

zeros @ $\pm 1/W, \pm 2/W, \ldots$ (many zeros ⇒ non-invertible)

↳ $H_I(u) = H^{-1}(u)$ → has infinite components.

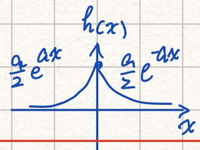
↳ Apply $G(u) = F(u)\cdot H(u)$

↳ Freq. Components of $F(u)$ are zero'd out

↳ Not invertible / recoverable

Ex 2: Exponential Blur

$$h(x) = \frac{a}{2}e^{-a|x|}$$ [aka Low Pass Filter]

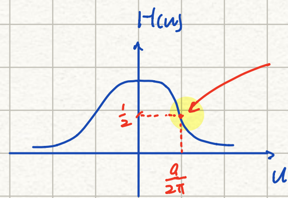


$$H(u) = \frac{a}{2}\left[\int_{-\infty}^{0} e^{(a-j2\pi u)x}\,dx + \int_{0}^{\infty} e^{-(a+j2\pi u)x}\,dx\right]$$

$$= \frac{a}{2}\left\{\frac{e^{(a-j2\pi u)x}}{a-j2\pi u}\Big|_{-\infty}^{0} + \frac{e^{-(a+j2\pi u)x}}{-(a+j2\pi u)}\Big|_{0}^{\infty}\right\}$$

$$= \frac{a}{2}\left[\frac{1}{a-j2\pi u} + \frac{1}{a+j2\pi u}\right]$$

$$= \frac{1}{1+\left(\frac{2\pi u}{a}\right)^2}$$



Half Power ≡ -3dB

To Deblur?

$$\Rightarrow H_I(u) = H^{-1}(u) = 1 + \left(\frac{2\pi u}{a}\right)^2$$

⟶ Let $g(x)$ is processed using $h_I(x)$

$$s(x) = g(x) * h_I(x)$$

$$S(u) = G(u)\cdot H_I(u)$$

$$= G(u)\cdot\left[1+\left(\frac{2\pi u}{a}\right)^2\right]$$

$$= G(u) + \frac{1}{a^2}G(u)(2\pi u)^2$$

↳ $s(x) = g(x) - \frac{1}{a^2}g''(x) \Rightarrow 1 - \nabla^2$

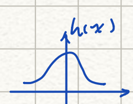
↳ Deblur ≡ Subtracting Scaled 2nd Derivative

• Property: $\frac{df(x)}{dx} \overset{F.T.}{\longleftrightarrow} j2\pi u F(u)$

$$\frac{d^2f(x)}{dx^2} \overset{F.T.}{\longleftrightarrow} -(2\pi u)^2 F(u)$$

Ex 3: Gaussian Blur

$$h(x) = \frac{1}{\sigma\sqrt{2\pi}}e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$



↑σ ⇒ ↑Blur

(⇒ ⟷ Wider h(x)

⇒ more pixels involved)

$$H(u) = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} e^{-j2\pi ux}\,dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\left[\frac{1}{2}\left(\frac{x}{\sigma}\right)^2 + j2\pi ux\right]}\,dx$$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[x^2 + j4\pi ux\sigma^2\right]}\,dx$$

$(x+a)^2 = (x^2 + 2ax + a^2)$

$$= \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{\left\{-\frac{1}{2\sigma^2}\left[x^2 + j4\pi x\sigma^2 - 4\pi^2u^2\sigma^4\right] - 2\pi^2u^2\sigma^2\right\}} = \underbrace{\left\{\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{1}{2\sigma^2}\left[x+j2\pi u\sigma^2\right]^2}dx\right\}}_{\text{Gaussian w/ mean: } \mu = -j2\pi u\sigma^2}\cdot e^{-2\pi^2u^2\sigma^2}$$

$\Downarrow$ $\qquad$ $\Leftarrow$ $\int_{-\infty}^{\infty}$ Gaussian $= 1$

$\therefore H(u) = e^{-2\pi^2 u^2 \sigma^2}$ } ⇒ also a Gaussian (mathematically)

↳ Spatial (-D) $\qquad$ F.T. $\qquad$ Freq (-D)

⇒ σ↑ ⇒ w↑ ⇒ More pixels ⇒ ↑Blur $\qquad$ $\frac{1}{2\pi\sigma}$ → σ↑ ⇒ w↓ ⇒ Attenuate High Freq ⇒ ↑Blur

$g(x) = f(x) * h(x)$ $\qquad$ $G(u) = F(u) \cdot H(u)$

Gaussian $\xrightarrow{\text{become}}$ Gaussian (Mathematically similar shape)

↑σ { in spatial domain, ↑Blur; in freq. domain, ↑ attenuation, ↑Blur } But Technique/Reason is diff.

§4.2 Discrete-Time Fourier Transform (DTFT) & Discrete Fourier Transform (DFT)

§4.2.1 DTFT

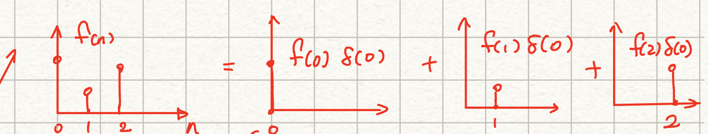
Signal Representation

- As in continuous time, exploit linearity & shift invariance ⇒ to find output as superposition of response to input components.
- Use Sifting Property:

2D: $f(m,n) = \sum\sum f(k,l)\,\delta(m-k, n-l)$

1D: $f(n) = \sum_{l} f(l)\,\delta(n-l)$

Sketch:



- put impulse & get result

⇒ sum all impulse response

Apply Transformation

- $g(m,n) = T[f(m,n)]$

$= T[\sum_k \sum_l f(k,l)\,\delta(m-k,n-l)]$

$= \sum\sum f(k,l) \cdot T[\delta(m-k,n-l)]$

$= \sum\sum f(k,l)\, h(m-k,n-l)$

$= f(m,n) * h(m,n)$ ← impulse response

} we can apply transformation on discrete signals individually ⇒ them sums up.

↳ Once, again, if we want to obtain the original sig $f(m,n)$, deconvolution is a difficult operation.

Another Approach

- Like continuous domain, we can use a discrete version of the freq. domain

↳ use complex exponential: $\boxed{e^{j2\pi un}}$

- A complex exponential remains untouched by an LTI sys, why?

$e^{j2\pi un} * h(n) = \sum_k h(k)\, e^{j2\pi u(n-k)}$

$= \left[\sum_k h(k)\, e^{-j2\pi uk}\right] e^{j2\pi un}$

$= H(e^{j2\pi u})\, e^{j2\pi un}$

, where $H(e^{j2\pi un})$ is known as the discrete-time Fourier transform (DTFT)

DTFT Characteristics

- Forward: $H(e^{j2\pi u}) = \sum_{k=0}^{N-1} h(k) e^{-j2\pi uk}$
- Inverse: $h(n) = \int_0^1 H(e^{j2\pi u}) e^{j2\pi un} du$

1) 'u' represents continuous spatial-frequency
2) $H(e^{j2\pi u})$ is periodic with period 1 since $e^{j2\pi u}$ is periodic with period 1.
3) $H(e^{j2\pi u})$ has no imaginary component, then there is zero phase.
4) If $g(n) = f(n) * h(n)$, then $G(e^{j2\pi u}) = F(e^{j2\pi u}) H(e^{j2\pi u})$
5) Since we only use finite extent signals $h(n)$, where $0 <= n <= N-1$

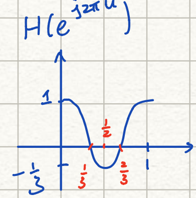
$\Rightarrow$ DTFT is also a finite sum.

$$H(e^{j2\pi u}) = \sum_{k=0}^{N-1} h(k) e^{-j2\pi uk}$$

↳ N samples!

Ex) $h(n) = [\frac{1}{3}\ \frac{1}{3}\ \frac{1}{3}]$

$$H(e^{j2\pi u}) = \frac{1}{3}\sum_{k=-1}^{1} e^{-j2\pi uk}$$
$$= \frac{1}{3}e^{j2\pi u} + \frac{1}{3} + \frac{1}{3}e^{-j2\pi u}$$
$$= \frac{1}{3} + \frac{2}{3}\cos(2\pi u)$$



§4.2.2 DFT

Extending DTFT ⇒ DFT

- We only have N samples, we only need N samples for $F(e^{j2\pi u})$
- Instead of using range $[0,1]$, stretch and sample to range $[0, N-1]$ to create discrete Fourier Transform (DTFT)

- Forward: $F(u) = \sum_{n=0}^{N-1} f(n)\, e^{-j2\pi un/N}$
- Inverse: $f(n) = (1/N) \sum_{u=0}^{N-1} F(u)\, e^{j2\pi un/N}$

- Both $F(u)$ & $f(n)$ are periodic

Verify DFT^{-1}

- Verify $f(n) = DFT^{-1}[DFT[f(n)]]$

$$= \frac{1}{N}\sum_{n=0}^{N-1}\left[\sum_{k=0}^{N-1} f(k) e^{-j2\pi uk/N}\right] e^{j2\pi un/N}$$
$$= \frac{1}{N}\sum_{k=0}^{N-1} f(k)\left[\sum_{u=0}^{N-1} e^{j\frac{2\pi u}{N}(n-k)}\right]$$

Recall

$$\sum_{n=0}^{N-1}\alpha^n = \begin{cases} N, & \alpha = 1 \\ \frac{1-\alpha^N}{1-\alpha}, & \alpha \neq 1\end{cases}$$

But $\sum_{u=0}^{N-1} e^{j\frac{2\pi u}{N}(n-k)} = \begin{cases} N, & \text{for } n=k \\ \boxed{\frac{1-\cancel{e^{j2\pi(n-k)}}}{1-e^{j\frac{2\pi}{N}(n-k)}}} & \text{for } n \neq k \end{cases}$

1 (sth of 2π)

$= \emptyset$

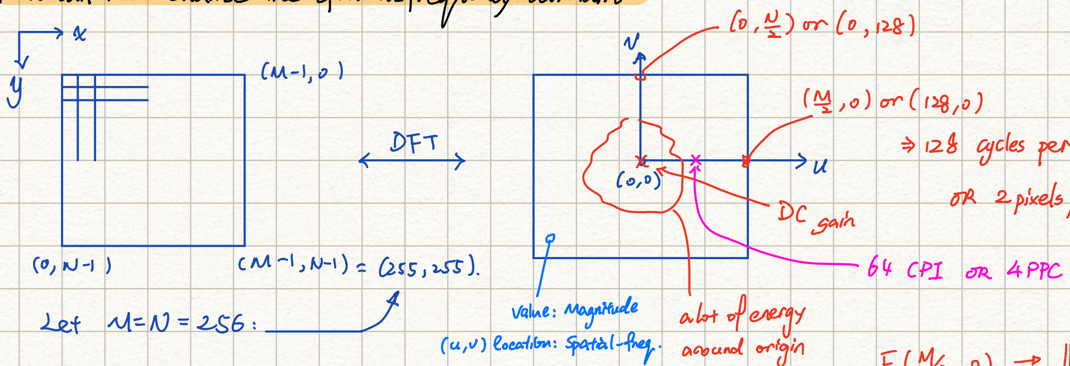
$e^{j2\pi(n-k)}$ INTEGER. $= 1$, $=0$

$$\therefore \frac{1}{N}\sum_{k=0}^{N-1} N \cdot f(k)\,\delta(n-k) = f(n)\ ✓$$

DFT in 2D

- For an $M \times N$ image
  - Forward DFT:
  $$F(u,v) = \sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\, e^{-j2\pi(ux/M + vy/N)}$$
  - Inverse DFT:
  $$f(x,y) = \frac{1}{MN}\sum_{u=0}^{M-1}\sum_{v=0}^{N-1} F(u,v)\, e^{j2\pi(ux/M + vy/N)}$$

How can we visualize the spatial-frequency domain?

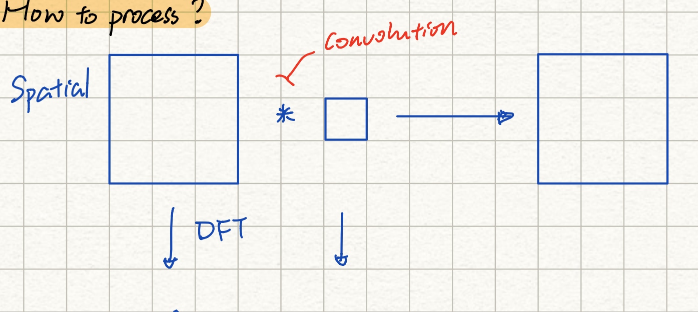


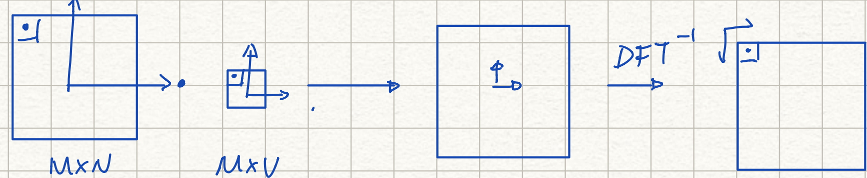
⇒ 128 cycles per image OR 2 pixels per cycle.

$F(M/2, 0) \rightarrow$ ||||||

$F(0, N/2) \rightarrow$ ≡

How to process?





$M \times N$ $\quad$ $M \times V$

freq.D $\quad$ freq.D $\quad$ freq.D $\quad$ $\xrightarrow{DFT^{-1}}$ (Spatial.D)

Implicit Periodicity to $f(n) = DFT^{-1}[F(u)]$

- While we define discrete arrays over a finite interval, the DFT creates an implicit periodicity outside the interval which complicates the use of the DFT to implement LSI systems.

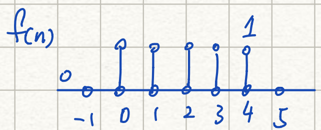
$f(n)$ (finite, Discrete) $\xrightarrow{DTFT}$ $F(e^{j2\pi u})$ (Continuous periodic)

$\downarrow$ Sample DFT

$f(n)$ (periodic discrete) $\xleftarrow{DFT^{-1}}$ $F(u)$ (Discrete Period)

$\Downarrow$

Ex2 $f(n) = h(n) = 1$ for $n = 0, 1, \dots, 4$



$h(n) = f(n)$

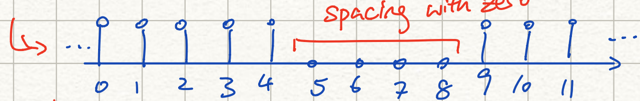
$$g(n) = f(n) * h(n) = [\underbrace{1\ 2\ 3\ 4\ 5\ 4\ 3\ 2\ 1}_{N=9}]$$

- If $N=5$, then $f_{periodic}(n) * h_p(n)$ are 1

then $\Rightarrow$ $f_p(n) \circledast h_p(n) = [\cdots\ 5\ 5\ 5\ 5\ \cdots]$

Cyclic convolution

- Set $N = N_f + N_h - 1 = 5 + 5 - 1 = 9$ $\Rightarrow$ We need to pad '0'

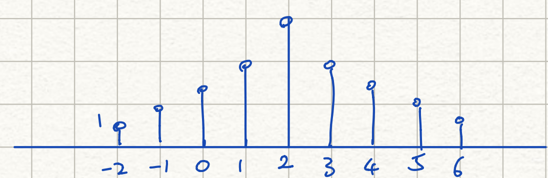


spacing with zero

$*$ same

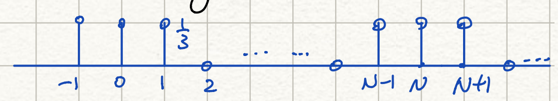
We need to make it periodic to satisfy implicit periodicity assumption.

$=$



$$\Downarrow \quad f_p(n) \circledast h_p(n) = f(n) * h(n) \quad \text{for } 0 \le n \le N-1$$

Ex1 3pt Average (Periodic on N)



$\Rightarrow$

$$h(n) = \begin{cases} \frac{1}{3}, & n = -1, 0, 1 \\ 0, & \text{else} \end{cases} \quad \text{Discrete}$$

$\downarrow$

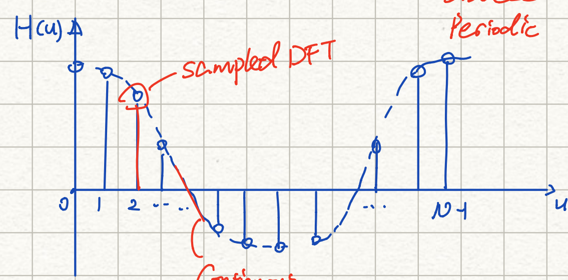
$$H(u) = \sum_{n=0}^{N-1} \frac{1}{3} e^{-j2\pi un/N}$$

$$= \frac{1}{3} \sum_{n=-1}^{1} e^{-j2\pi un/N}$$

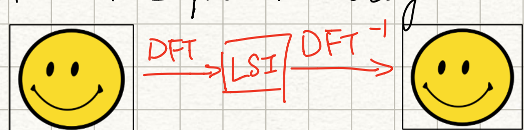
$$= \frac{1}{3}\left[1 + e^{j2\pi un/N} + e^{-j2\pi un/N}\right]$$

$$= \frac{1}{3} + \frac{2}{3}\cos\left(\frac{2\pi u}{N}\right)$$

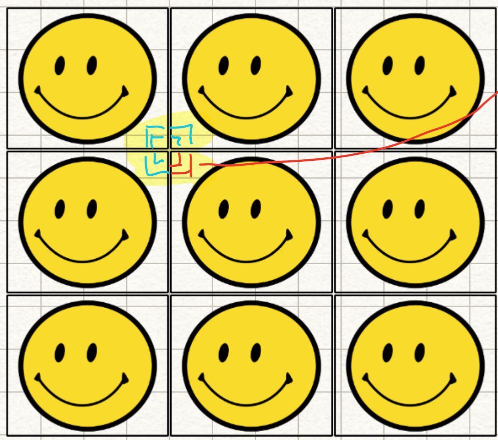
Discrete Periodic



IMPACT OF Implicit Periodicity



X Not like that $\Rightarrow$ gives you same shift



- You need to assume periodicity

needs pixels info from

⇓

Hence, a filter response based on a mask in the top-left uses an image data from all four corners, because of implicit periodicity

§4.2.3 Use of DFT

Fourier Spectra Characteristics of Images

- Most energy resides in low frequency components
  - ↳ Implies that the original image can be reconstructed with good approximation with the low frequency components.

- Low frequency components correspond to coarse details
  - ↳ ex: DC component represents average intensity of an image

- High frequency components correspond to fine details
  - ↳ ex: edges, noise ⇒ [remove edges $\xrightarrow{\text{(then)}}$ remove noise]$^{\text{Ideal}}$

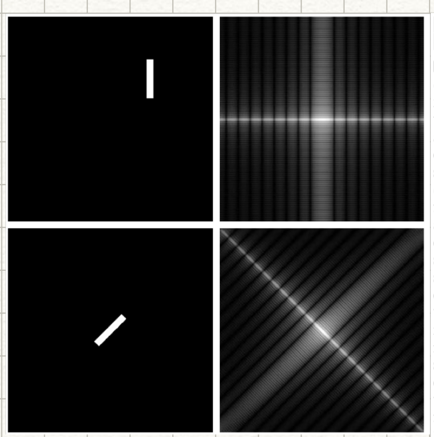
Fourier Analysis: Spectra

- An image coordinates, origin (0, 0) refers to top-left corner
- Results in spectra being centered at corner
- Common to multiply input image by $(-1)^{x+y}$
  - ⇒ Why?

$$\boxed{\mathcal{L}[f(x,y)(-1)^{x+y}] = F(u-M/2,\ v-N/2)}$$

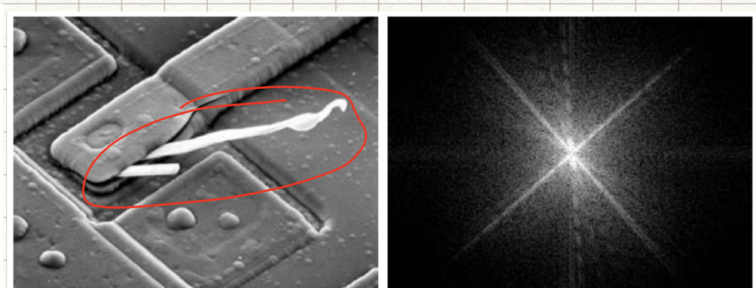
⇓

- Brings origin of spectra ⇒ to center of image



a b
c d

FIGURE 4.25 (a) The rectangle in Fig. 4.24(a) translated, and (b) the corresponding spectrum. (c) Rotated rectangle, and (d) the corresponding spectrum. The spectrum corresponding to the translated rectangle is identical to the spectrum corresponding to the original image in Fig. 4.24(a).



a b

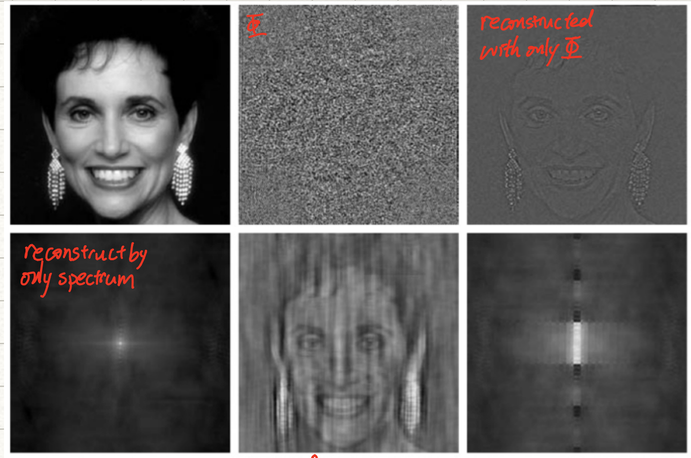
FIGURE 4.29 (a) SEM image of a damaged integrated circuit. (b) Fourier spectrum of (a). (Original image courtesy of Dr. J. M. Hudak, Brockhouse Institute for Materials Research, McMaster University, Hamilton, Ontario, Canada.)

Fourier Analysis: Phase

→ Characterizes structural information within an image

$$\boxed{\Phi(u,v) = \tan^{-1}\left[\frac{I(u,v)}{R(u,v)}\right]}$$

- tell you when things start & stop



a b c
d e f

reconstruction using phase, but with spectrum of rectangle ⇒ still see woman!

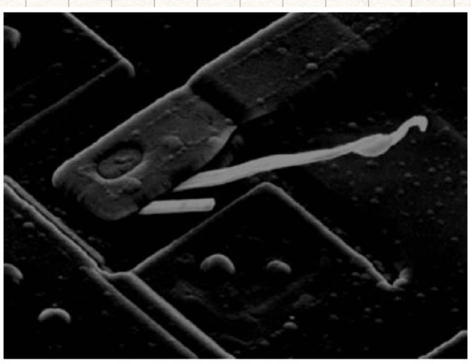
FIGURE 4.27 (a) Woman. (b) Phase angle. (c) Woman reconstructed using only the phase angle. (d) Woman reconstructed using only the spectrum. (e) Reconstruction using the phase angle corresponding to the woman and the spectrum corresponding to the rectangle in Fig. 4.24(a). (f) Reconstruction using the phase of the rectangle and the spectrum of the woman.

Recall: Image Characteristics In Frequency Domain:

- Low freq. responsible for general appearance of image over smooth areas
- High freq. responsible for detail (ex: edges & noise)
- Intuitively, modifying different frequency coefficients affects different characteristics of an image.

⇓

EX) DC component removal



⇒ Why it looks like that?

- DC component characterizes the mean of the image intensities

$$\boxed{\begin{aligned} F(0,0) &= \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y)\,\underbrace{\cancel{e^{-j2\pi(0x/M+0y/N)}}}_{1} \\ &= \frac{1}{MN}\sum_{x=0}^{M-1}\sum_{y=0}^{N-1} f(x,y) = E[f(x,y)] \end{aligned}}$$

Basic Steps of Filtering in Spatial-Frequency Domain

1. Multiply input $f(x,y)$ by $(-1)^{x+y}$ to center transform
2. Compute DFT of image, $F(u,v)$
3. Multiply $F(u,v)$ by filter function $H(u,v)$ to produce $G(u,v)$
4. Compute inverse DFT of $G(u,v)$ to produce $g(x,y)$
5. Multiply $g(x,y)$ by $(-1)^{x+y}$ ⇒ produce filtered image

§4-3 Spatial-Frequency Implementations

§4.3.1 Low Pass Filter

LPF

- **Spatial Filter**: essentially 2-D Discrete Convolution Between an image f & filter function h.

  $g(x,y) = f(x,y) * h(x,y)$ [Convolution]

- **Spatial-Frequency Filter**

  $G(u,v) = F(u,v)\,H(u,v)$ [multiplication]

- **Blurring / Noise Reduction**
  - Noise characterized by sharp transitions in image intensity
  - Such transitions contribute significantly to high freq. components of Fourier transform
  - Intuitively, attenuating certain high freq. components
    $\Rightarrow$ blurring & reduction of image noise.

  - **Visualization**

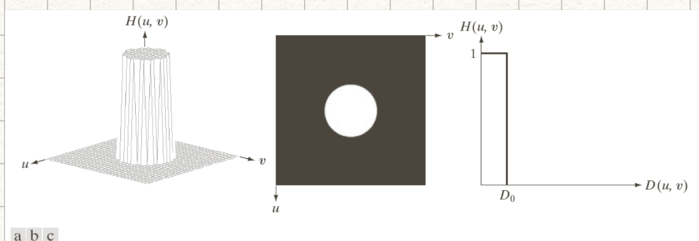


FIGURE 4.40 (a) Perspective plot of an ideal lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross section.

- **Ideal LPF**

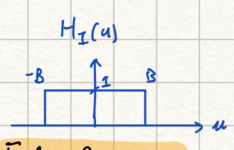
  $$\boxed{H(u,v) = \begin{cases} 1, & \text{if } D(u,v) \le D_0 \\ 0, & \text{if } D(u,v) > D_0 \end{cases}}$$

  Cuts off all high-freq. components at a distance greater than a certain distance from origin. (D_0 : cutoff frequency).

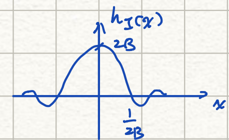
  - **1D Version**

    $H_I(u) = \begin{cases} 1, & |u| < B \\ 0, & \text{else} \end{cases} \Rightarrow h_I(x) = \int_{-B}^{B} e^{j2\pi ux}\,du = \frac{1}{j2\pi x}\left[e^{j2\pi uk}\right]_{-B}^{B}$

    $= \frac{\sin 2\pi Bx}{\pi x} \approxeq$ sinc. func.

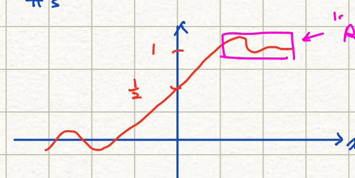


$\Rightarrow$



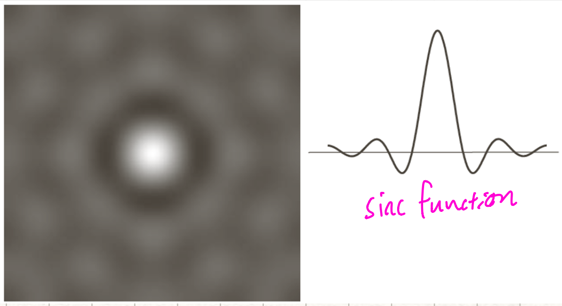
- **Edge Response**

  Edge Response : $g_I(x) = u(x) * h_I(x) = \int_{-\infty}^{x} \frac{\sin 2\pi Bs}{\pi s}\,ds$ $\Rightarrow$ No Closed Form Solution



"Ringing" Artifact / Gibbs Phenomena

- **Ringing Artifacts**
  - Ideal low pass filter function is a rectangular function
  - The inverse F.T. of a rectangular function is a sinc function
  - Convolution of a sinc & a step function $\Rightarrow$ generates ringing on both sides of the edge



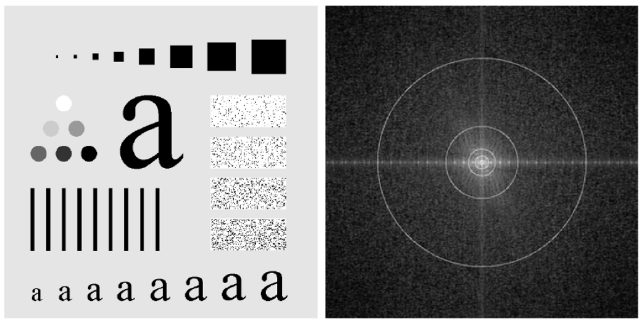
a b

FIGURE 4.43 (a) Representation in the spatial domain of an ILPF of radius 5 and size 1000 × 1000. (b) Intensity profile of a horizontal line passing through the center of the image.

Effect of Different Cutoff Frequencies

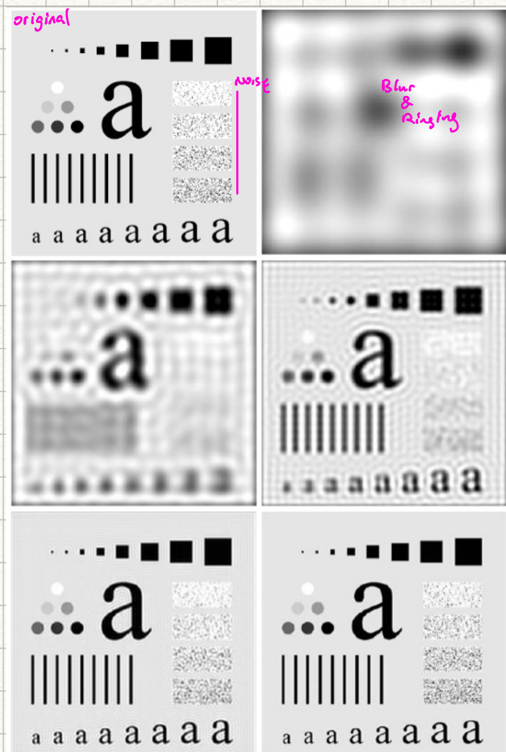
- As cutoff freq ↓ decreases :
  ⇒ Image : more blurred & more ringing artifacts
  Noise : more reduced
  Analogous to larger spatial filter sizes

- Noticeable Ringing Artifacts that increase ↑ as the amount of high-freq. components removed is increasing ↑



a b

FIGURE 4.41 (a) Test pattern of size 688 × 688 pixels, and (b) its Fourier spectrum. The spectrum is double the image size due to padding but is shown in half size so that it fits in the page. The superimposed circles have radii equal to 10, 30, 60, 160, and 460 with respect to the full-size spectrum image. These radii enclose 87.0, 93.1, 95.7, 97.8, and 99.2% of the padded image power, respectively.



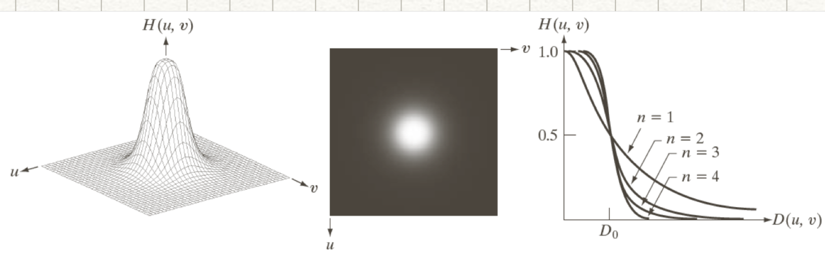
a b
c d
e f

FIGURE 4.42 (a) Original image. (b)–(f) Results of filtering using ILPFs with cutoff frequencies set at radii values 10, 30, 60, 160, and 460, as shown in Fig. 4.41(b). The power removed by these filters was 13, 6.9, 4.3, 2.2, and 0.8% of the total, respectively.

Butterworth LPF

$$\boxed{H(u,v) = \frac{1}{1+[D(u,v)/D_0]^{2n}}}$$

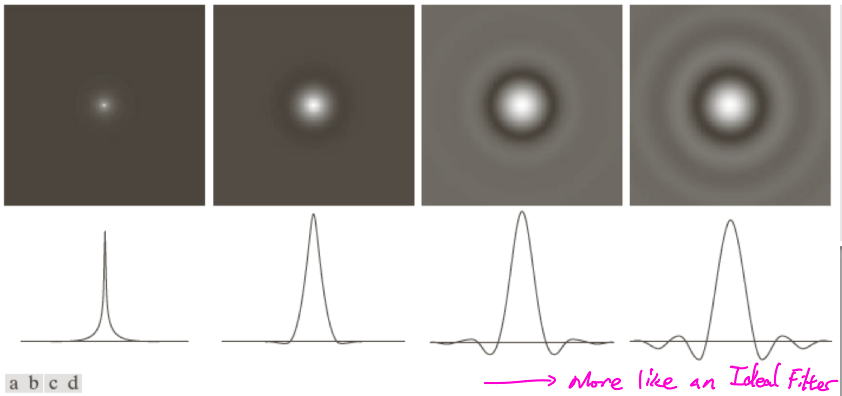
- Transfer Function : does not have sharp discontinuity, establishing cutoff between passed & filtered frequencies.
- D_0 = Cutoff freq. : defines point at which $H(u,v) = 0.5$
- ≈ similar to Exponential LPF



a b c

FIGURE 4.44 (a) Perspective plot of a Butterworth lowpass-filter transfer function. (b) Filter displayed as an image. (c) Filter radial cross sections of orders 1 through 4.

Spatial Rep : Tradeoff btwn amount of smoothing & ringing



→ More like an Ideal Filter

FIGURE 4.46 (a)–(d) Spatial representation of BLPFs of order 1, 2, 5, and 20, and corresponding intensity profiles through the center of the filters (the size in all cases is 1000 × 1000 and the cutoff frequency is 5). Observe how ringing increases as a function of filter order.

FIGURE 4.45 (a) Original image. (b)–(f) Results of filtering using BLPFs of order 2, with cutoff frequencies at the radii shown in Fig. 4.41. Compare with Fig. 4.42.

Gaussian LPF

$$\boxed{H(u,v) = e^{-D^2(u,v)/2D_0^2}}$$

- Another form of Gaussian filter
- T.F. is smooth, like Butterworth filter
- Gaussian in Freq. Domain remains a Gaussian is Spatial Domain
- Pro: No Ringing Artifacts

§ 4.3.2 High Pass Filter

HPF

Impact:

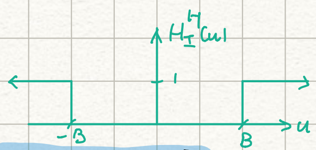
- Edges & Fine Detail characterized by sharp transitions in Image Intensity
- Such transitions contribute significantly to high freq. components of F.T.
- Intuitively, Attenuating Low-Freq. components, & Preserving High-Freq. components ⇒ will Retain Image Intensity Edges

Spatial-Freq. HPFs

- HPFs are effectively the opposite of LPFs.
- $H_{HP}(u,v) = 1 - H_{LP}(u,v)$
  $h_{HP}(x,y) = \delta(x,y) - h_{LP}(x,y)$ — HPF is related to LPF
- $DC^{HPF}_{gain} = 0$.

Ideal HPF

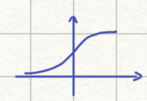
$$H_I^H(u) = 1 - H_I(u) \longrightarrow h_I^H(x) = \delta(x) - h_I(x)$$



Edge Response: $g_E(x) = \int_{-\infty}^{x} h_E^H(s)\,ds$

$= \int_{-\infty}^{x} [\delta(s) - \frac{a}{2}e^{-a|s|}]\,ds$

$= u(x) -$

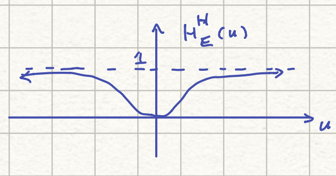


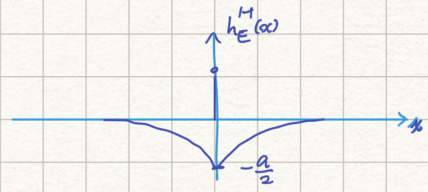
$=$

∴ Edge Detector

Exponential HPF

$$H_E^H(u) = 1 - \overbrace{\frac{1}{1+(\frac{2\pi u}{a})^2}}^{\text{Exp. LPF}} = \frac{(\frac{2\pi u}{a})^2}{1+(\frac{2\pi u}{a})^2} \rightarrow h_E^H(x) = \delta(x) - \frac{a}{2}e^{-a|x|}$$





HPFs:

Ideal:

$$H(u,v) = \begin{cases} 0 & \text{if } D(u,v) \le D_0 \\ 1 & \text{if } D(u,v) > D_0 \end{cases}$$

Butterworth:

$$H(u,v) = \frac{1}{1 + [D_0/D(u,v)]^{2n}}$$

Gaussian:

$$H(u,v) = 1 - e^{-D^2(u,v)/2D_0^2}$$

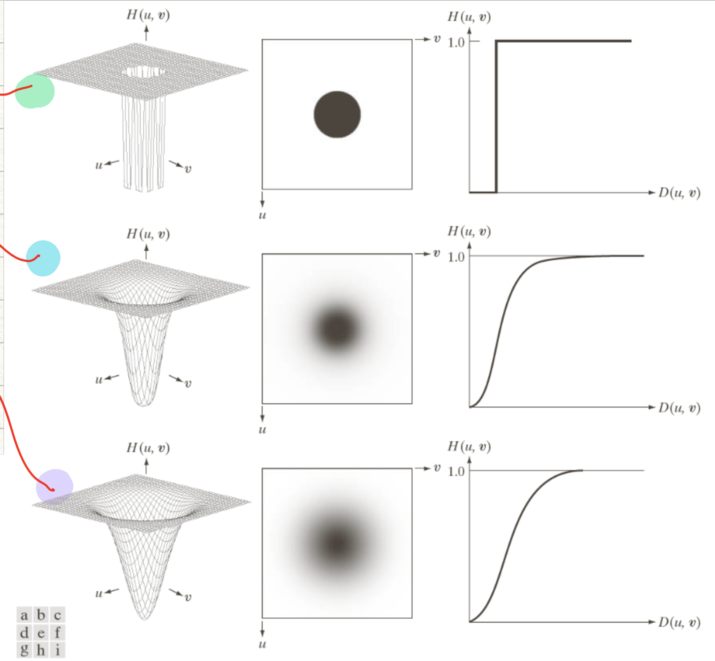


FIGURE 4.52 Top row: Perspective plot, image representation, and cross section of a typical ideal highpass filter. Middle and bottom rows: The same sequence for typical Butterworth and Gaussian highpass filters.

Spatial Rep

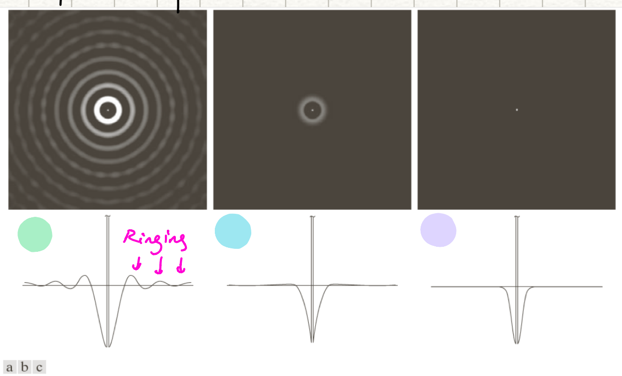


FIGURE 4.53 Spatial representation of typical (a) ideal, (b) Butterworth, and (c) Gaussian frequency domain highpass filters, and corresponding intensity profiles through their centers.

HPFs

Ideal

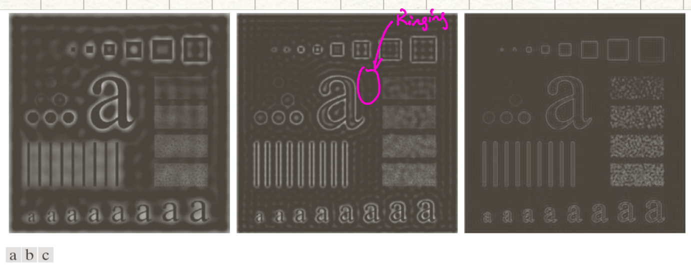


FIGURE 4.54 Results of highpass filtering the image in Fig. 4.41(a) using an IHPF with D_0 = 30, 60, and 160.

Butterworth

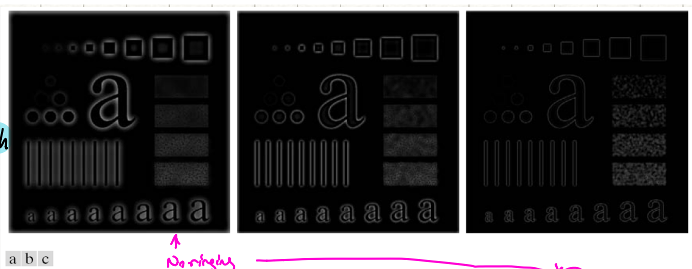


FIGURE 4.55 Results of highpass filtering the image in Fig. 4.41(a) using a BHPF of order 2 with D_0 = 30, 60, and 160, corresponding to the circles in Fig. 4.41(b). These results are much smoother than those obtained with an IHPF.

Gaussian

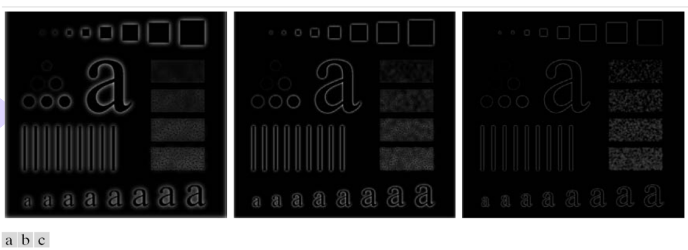


FIGURE 4.56 Results of highpass filtering the image in Fig. 4.41(a) using a GHPF with D_0 = 30, 60, and 160, corresponding to the circles in Fig. 4.41(b). Compare with Figs. 4.54 and 4.55.

Observations

1) As with Ideal LPF, Ideal HPFs shows significant Ringing Artifacts!!
2) 2^{nd} order Butterworth HPF shows sharp edges with minor ringing artifacts.
3) Gaussian HPF show good sharpness in edges with NO ringing artifacts.

§4.3.3 Edge Enhancement

- can be performed directly in the spatial-freq domain
- Ex: High boost filtering (unsharp masking)

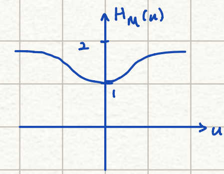
Unsharp Masking: $g(x) = 2f(x) - f_s(x)$

$h(x) = 2\delta(x) - h_s(x)$

Use Exponential LPF : $h_M(x) = 2\delta(x) - \frac{a}{2}e^{-a|x|}$ ⟶ Edge Response:

$$H_M(u) = 2 - \frac{1}{1+(\frac{2\pi u}{a})^2}$$

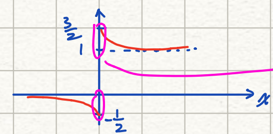
$$= \frac{1+2(\frac{2\pi u}{a})^2}{1+(\frac{2\pi u}{a})^2}$$



$$g_M(x) = u(x) * h_M(x)$$

$$= \int_{-\infty}^{x} [2\delta(s) - \frac{a}{2}e^{-a|s|}]\,ds$$

$$= \underbrace{2u(x)}_{2\times \text{ unit step}} - \underbrace{g_E(x)}_{\text{Exponential response}}$$



Edge Enhancement (Mach Band).

High Freq. Emphasis

- Advantageous to accentuate enhancements by high-freq. components of image in certain situations (ex. image visualization)
- Solution: Multiply HPF with a constant & offset so zero freq. term not eliminated.

$$g(x,y) = f(x,y) + k\,g_{HPF}(x,y)$$

↳ refer to High-Boost Filter   High Pass Filter

High Boost Filter

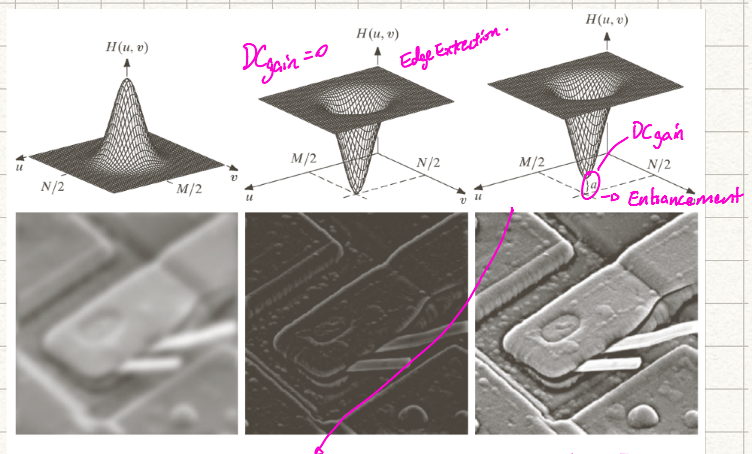
- Spatial Domain: $g(x,y) = f(x,y) + K g_{HPF}(x,y)$
  - Impulse Response: $h(x,y) = \delta(x,y) + k\,h_{HPF}(x,y)$
- Transfer function in spatial-freq. domain:

$$H(u,v) = 1 + kH_{HPF}(u,v) \quad \text{HPF-version}$$

OR:

$$H(u,v) = 1 + k(1 - H_{LPF}(u,v))$$
$$= (1+k) - kH_{LPF}(u,v) \quad \text{LPF-version}$$

↳ if k=1 ⇒ unsharp masking.

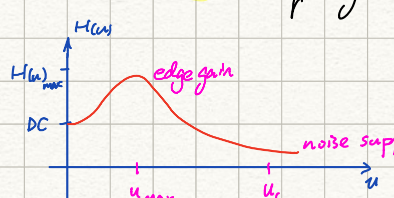


Note: Minor / Suttle change in FqDomain → Significant Impact on SpDomain

FIGURE 4.31 Top row: frequency domain filters. Bottom row: corresponding filtered images obtained using Eq. (4.7-1). We used $a = 0.85$ in (c) to obtain (f) (the height of the filter itself is 1). Compare (f) with Fig. 4.29(a).

§4.3.4 HVS modelling

- For a generic Spatial-freq. Img Enhancement Filter, what should the T.F. look like?
  1) DC gain is typically reduced ⇒ so $0 < H(u) < 1$
  2) $H(u) \to 0$ as $u \nearrow$ increases
  3) $H(u) > 1$ for freq. range, where signal dominates



$u_{max} = 8$ cpd

$u_c = 30$ cpd.

$H(\phi) = \frac{1}{2}$

$H(8) = 5$

$H(30) \approx \phi$

- Model of HVS
  - Light entering the eye is processed by two steps:

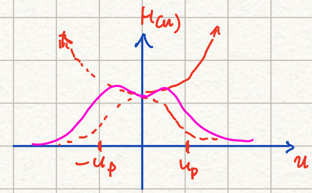
1) Cornea / Lens $H_1(u)$: modelled as LPF (ex. Gaussian)

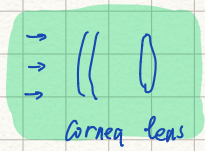
2) Retina $H_2(u)$: modelled as edge enhancement (ex. 1-Laplacian)

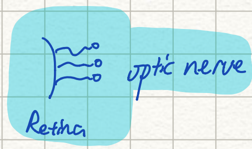
⇓

Combined:

$$H(u) = H_1(u)\,H_2(u) = \left(1 + (2\pi u/a)^2\right) e^{-2\pi^2 u^2 \sigma^2}$$







physiological optics

$H_1(u) = e^{-2\pi^2 u^2 \sigma^2}$ $\qquad$ $H_2(u) = 1 + \left(\frac{2\pi u}{a}\right)^2$

↳ @ $u = 30\,\text{cpd} = \frac{1}{2\sigma} \Rightarrow \sigma = \frac{1}{60}$ deg

⇒

$$H'(u) = \left[1 + \left(\frac{2\pi u}{a}\right)^2\right] e^{-2\pi^2 u^2 \sigma^2}$$

$$= \left(1 + \frac{4\pi^2 u^2}{a^2}\right)\left(-4\pi^2 \sigma^2 u\right) e^{-2\pi^2 u^2 \sigma^2}$$

$$= -4\pi^2\sigma^2 - \frac{16\pi^2 \sigma^2 u^2}{a^2} + \frac{8\pi^2}{a^2}$$

Let $H'(u) = 0$

$$u^2 = \frac{2 - \sigma^2 a^2}{4\pi^2 \sigma^2} \Rightarrow u = \pm \frac{1}{2\pi\sigma}\sqrt{2 - \sigma^2 a^2}$$

⇓

For real roots $2 - \sigma^2 a^2 > 0$, $a^2\sigma^2 < 2$

⇓

$a\sigma < \sqrt{2}$

⇓

$\therefore$ If $\sigma = \frac{1}{60}°$, $a < 60\sqrt{2}$

HVS Model in Spatial Domain.

For $1 + \frac{4\pi^2 u^2}{\sigma^2}$ has an equivalent operation:

$$h(x) = f(x) - \frac{1}{a^2} f''(x)$$

⇓

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$

$$f'(x) = \frac{-x}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$

$$f''(x) = \frac{x^2}{\sigma^5\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2} - \frac{1}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$

⇓

$$\frac{1}{a^2} f''(x) = \frac{1}{\sigma\sqrt{2\pi}}\left[\frac{x^2}{\sigma^4 a^2} - \frac{1}{a^2\sigma^2}\right] e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$

$$h(x) = \frac{1}{\sigma\sqrt{2\pi}}\left[1 + \frac{1}{a^2\sigma^2} - \frac{x^2}{\sigma^4 a^2}\right] e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$

⇓

↳ Lateral Inhibition

Difference of Gaussians

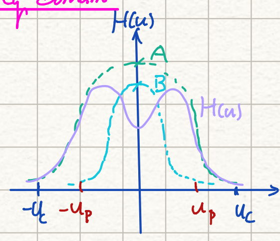
- a number of Gaussian-based function that mimic lateral inhibition
- takes the diff of 2 Gaussians with different σ.

  $H(u) = A e^{-2\pi^2 u^2 \sigma_1^2} - B e^{-2\pi^2 u^2 \sigma_2^2}$

  with $A > B$, & $\sigma_1 < \sigma_2$

- Can vary σ_1 & σ_2 ⇒ create filter bank with varying peak frequencies

Freq. Domain



$u_p = 8\,\text{cpd} = \frac{1}{2\sigma_2} \Rightarrow \sigma_2 = \frac{1}{16}$ [deg]

$u_c = 30\,\text{cpd} = \frac{1}{2\sigma_1} \Rightarrow \sigma_1 = \frac{1}{60}$ [deg]

$H(u_p) \approx A = 5$

$H(\phi) = \frac{1}{2}$, $B = 4.5$

$H(u_c) \approx \phi$

⇓

Spatial Domain

Gabor Filters

- A Gaussian Band-Pass Filter

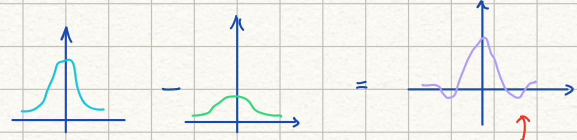
Freq Domain: $H(u) = (A/2)\, e^{-2\pi^2 u^2 \sigma_1^2} \underbrace{*}_{\text{Convolution}} [\,\underbrace{\delta(u-u_p)}_{\text{shifted}} + \underbrace{\delta(u+u_p)}_{\text{symmetrical shift.}}\,]$ 2 spike

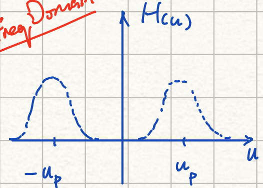
- Time-Domain, a Gaussian-modulated sinusoid (real part of Gabor filter)

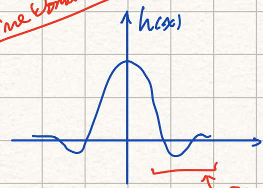
Time Domain: $h(x) = A/(\sigma(2\pi)^{0.5}) \cdot e^{-0.5/(x/\sigma)^2} \cdot \boxed{\cos(2\pi u_p x)}$ cosine

- Sketch in freq & time
  - similar shape as Diff of Gaussians, But with Ringing
  - Note: Complex form of filter used for texture feature extractions

$$h(x) = \frac{A_1}{\sigma_1\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma_1}\right)^2} - \frac{B}{\sigma_2\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma_2}\right)^2}$$





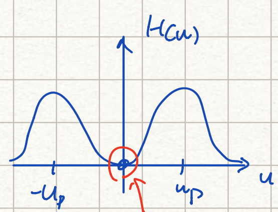


Laplacian of a Gaussian (Edge Detector)

- Consider the Marr-Hildreth Operator

$$H(u) = (-j2\pi u)^2 e^{-2\pi^2 u^2 \sigma^2} = 4\pi^2 u^2 e^{-2\pi^2 u^2 \sigma^2}$$

- Impact:



DC gain = 0 ⇒ Edge Detector.

Impulse Response

$$h(x) = -\frac{\partial^2}{\partial x^2}\left[\frac{1}{\sigma\sqrt{2}\pi} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}\right]$$

$$= \frac{1}{\sigma^3\sqrt{2\pi}}\left[1 - \frac{x^2}{\sigma^2}\right] e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$

Peaks?

$$H'(u) = 4\pi^2[2u - 4\pi u^3\sigma^2]\, e^{-2\pi^2 u^2\sigma^2}$$

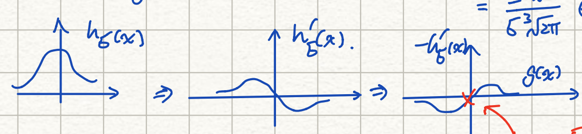
$$\Downarrow$$

$$2u = 4\pi u^3\sigma^2 \Rightarrow u_p = \pm\frac{1}{\pi\sigma\sqrt{2}}$$

Step Response

$$g(x) = -h_\sigma''(x) * u(x), \quad h_\sigma'(x) = \frac{1}{\sigma\sqrt{2\pi}}\left(\frac{-x}{\sigma^2}\right) e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$

$$= -h_\sigma'(x) * \delta(x) \qquad = \frac{-x}{\sigma^3\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x}{\sigma}\right)^2}$$



Edge Detector, zero crossing / DC gain = 0

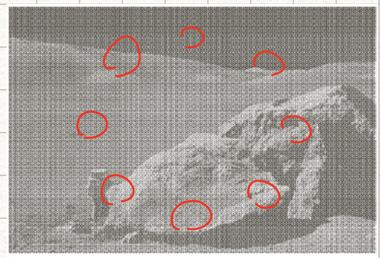
David Marr

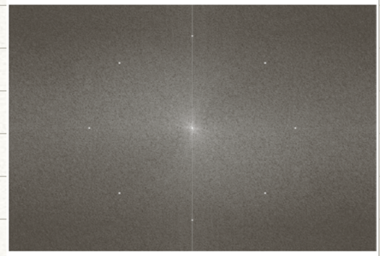
- Laplacian of a Gaussian motivated & applied by David Marr. (1982)
- 1970s, Marr was the first to transcend biological vision to a formal math model
- Combined math & neurobiology.

§4.3.5 Periodic Noise

- Typically occurs from electrical / electromechanical Interference during Image Acquisition
- Spatially dependent noise
  - Ex: Spatial Sinusoidal Noise

- Observations :
  - Symmetric pairs of bright spots appear in Fourier Spectra
  - Why?
    - ↳ F.T. of cosine function is the sum of a pair of impulse function
    - $\cos(2\pi u_0 x) \leftrightarrow 0.5\,[\delta(u+u_0) + \delta(u-u_0)]$
  - Intuitively, sinusoidal noise can be reduced by attenuating these bright spots.





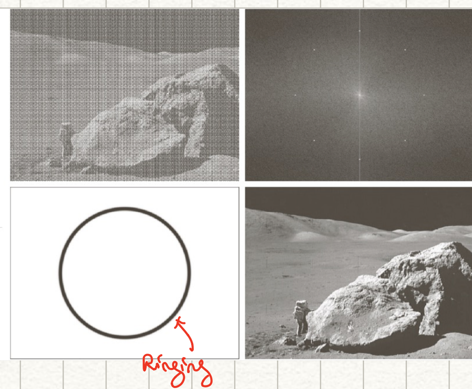
- Bandreject Filter
  - Removes / Attenuates a band of frequencies about the origin of the F.T.
  - Sinusoidal Noise may be reduced by filtering the band of frequencies upon which the bright spots associated with period noise appear

- Ideal Bandreject Filter

$$H(u,v) = \begin{cases} 1, & \text{if } D(u,v) < D_0 - \frac{W}{2} \\ 0, & \text{if } D_0 - \frac{W}{2} \le D(u,v) \le D_0 + \frac{W}{2} \\ 1, & \text{if } D(u,v) > D_0 + \frac{W}{2} \end{cases}$$

a b
c d

FIGURE 5.16 (a) Image corrupted by sinusoidal noise. (b) Spectrum of (a). (c) Butterworth bandreject filter (white represents 1). (d) Result of filtering. (Original image courtesy of NASA.)

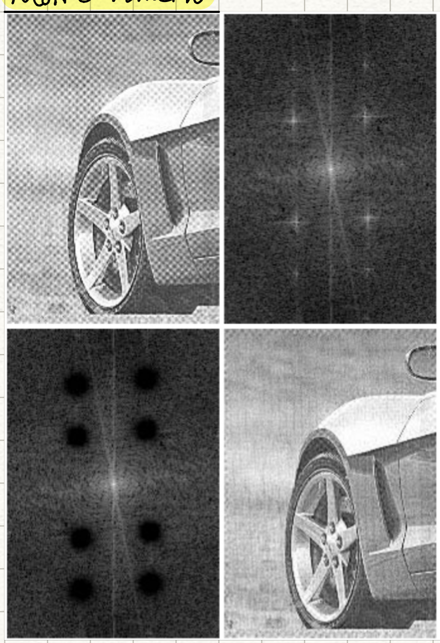


Ringing

- Notch Reject Filter
  - Idea :
    - Sinusoidal Noise appears as bright spots in Fourier Spectra
    - Reject freq. in predefined neighborhoods about a center freq.
    - In this case, center notch reject filters around frequencies coinciding with the bright spots.

⇓

Moiré Pattern

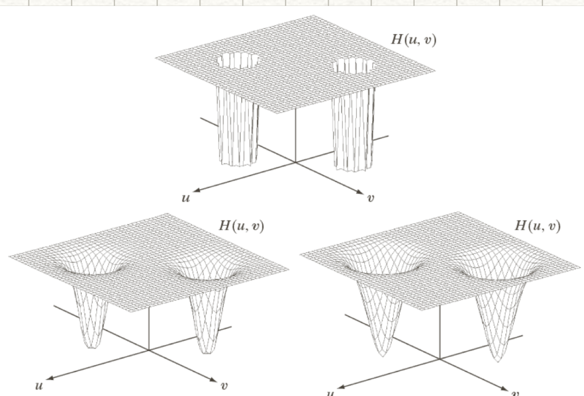


a b
c d

FIGURE 4.64 (a) Sampled newspaper image showing a moiré pattern. (b) Spectrum. (c) Butterworth notch reject filter multiplied by the Fourier transform. (d) Filtered image.

a
b c

FIGURE 5.18 Perspective plots of (a) ideal, (b) Butterworth (of order 2), and (c) Gaussian notch (reject) filters.



§4.4. Homomorphic Filtering

- Image can be modeled as a product of illumination (i) & reflectance (r)

  $$f(x,y) = i(x,y)\, r(x,y)$$

  Multiplicative Noise

- Unlike Additive Noise, cannot operate on freq. components of illumination & reflectance separately

  $$\mathcal{F}\{f(x,y)\} \neq \mathcal{F}\{i(x,y)\}\, \mathcal{F}\{r(x,y)\}$$

- Idea: ① Take log. of the image / signal : $\ln f(x,y) = \ln i(x,y) + \ln r(x,y)$ Take log

  ② Now freq. components of i & r can be operate separately : $\mathcal{F}\{\ln f(x,y)\} = \mathcal{F}\{\ln i(x,y)\} + \mathcal{F}\{\ln r(x,y)\}$ Transform to Additive Noise.

  ⇓ Now, apply LPF.

  ⇓

- System Diagram

  $$s_{[0,1]} \to \otimes \to f \to \boxed{\ln} \xrightarrow{\hat{f} = \hat{s} + \hat{n}} \boxed{h} \xrightarrow{\hat{g}} \boxed{\exp} \to g = s' \cdot n'$$

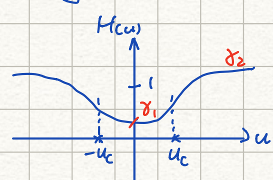
  with $n_{[0,\infty]}$ into $\otimes$.

  $$\Rightarrow g = T[f]$$
  $$= \exp[h * \ln f] = \exp[h * (\ln s + \ln n)]$$
  $$= \exp[h * \ln s] \cdot \exp[h * \ln n]$$
  $$= T[s] \cdot T[n]$$

- Freq. Domain Model ⟶ Assume Img has regions of low freq. illumination.

  Homomorphic ≙ "same form"
  ↳ algebraic form of input + output are the same.

  Density Domain : Log density = optical density



  $$H(u) = \begin{cases} \gamma_1 < 1, & u < u_c \\ \gamma_2 > 1, & u > u_c \end{cases}$$
  ↑ cutoff freq.

  ⇓

  - If $\hat{s}(x)$ is mainly frequencies $> u_c$.
    $\Rightarrow h(x) * \hat{s}(x) \approx \gamma_2 \hat{s}(x)$
  - If $\hat{n}(x)$ is mainly frequencies $< u_c$
    $\Rightarrow h(x) * \hat{n}(x) \approx \gamma_1 \hat{n}(x)$

  $$\hat{G}(u) = \gamma_2 \hat{S}(u) + \gamma_1 \hat{N}(u)$$
  $$\hat{g}(x) = \gamma_2 \hat{s}(x) + \gamma_1 \hat{n}(x)$$
  $$= \gamma_2 \ln(s) + \gamma_1 \ln(n)$$
  $$= \ln s^{\gamma_2} + \ln n^{\gamma_1}$$

  ⇓

  $$g(x) = \exp[\hat{g}(x)]$$
  $$= \exp[\ln s^{\gamma_2} + \ln n^{\gamma_1}]$$
  $$= e^{\ln s^{\gamma_2}} \cdot e^{\ln s^{\gamma_1}}$$
  $$= [s(x)]^{\gamma_2} [n(x)]^{\gamma_1}$$

  ↳ $\gamma_1 < 1 \Rightarrow$ compact illumination
  $\gamma_2 > 1 \Rightarrow$ ↑ signal content, ↑ edges, ↑ local contrast

- Discrete Implementation

  $$f \to \boxed{\ln} \xrightarrow{\hat{f}} \boxed{h} \xrightarrow{\hat{g}} \boxed{\exp} \to g$$

  $$\hat{g}(n) = \sum_{k=-\infty}^{\infty} \hat{f}(n-k)\, h(k)$$
  $$= \sum_{k=-\infty}^{\infty} h(k) \ln[f(n-k)] \Rightarrow g(n) = \exp\left\{\sum_{k=-\infty}^{\infty} h(k) \ln[f(n-k)]\right\}$$
  $$= \exp\left\{\sum_{k=-\infty}^{\infty} \ln[f(n-k)]^{h(k)}\right\}$$
  $$= \prod_{k=-\infty}^{\infty} [f(n-k)]^{h(k)}$$

  ⇓

  Smoothing:
  $$h(n) = [\tfrac{1}{3}\ \tfrac{1}{3}\ \tfrac{1}{3}] \qquad g(n) = [f(n+1) f(n) f(n-1)]^{\frac{1}{3}}$$

  Sharpening:
  $$h(n) = [-1\ 3\ -1] \quad g(n) = \frac{f^3(n)}{f(n-1)\, f(n+1)}$$

  Unsharp Mask:
  $$h(n) = [-\tfrac{1}{3}\ \tfrac{5}{3}\ -\tfrac{1}{3}] \quad g(n) = \frac{f(n)}{f(n-1)^{1/3}\ f(n+1)^{1/3}}$$

- Filtering Network (Alternative, Do in freq. Domain)

FIGURE 4.60 Summary of steps in homomorphic filtering.

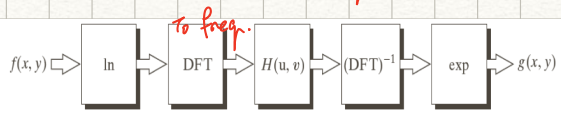


Image Enhancement

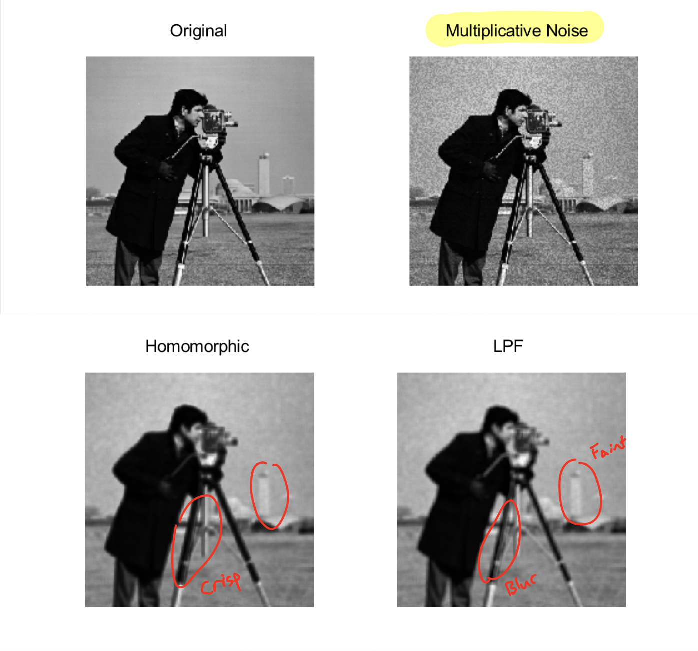
- Simultaneous Dynamic Range compression ⇒ Reduce Illumination Variation
- Contrast Enhancement ⇒ Increase Reflectance Variance

- Components
  - Illumination : characterized by slow spatial variations (Low spatial freq)
  - Reflectance : characterized by abrupt spatial variations (High Spatial freq)

- Can be accomplished using a high freq emphasis filter in log space.
  - DC gain of 0.5 (reduce illumination variations ↓)
  - High-freq gain of 2 (increase reflectance variations ↑)

- Output of Homomorphic Filter

$$g(x,y) \Rightarrow \sqrt{i(x,y)}\,(r(x,y))^2$$



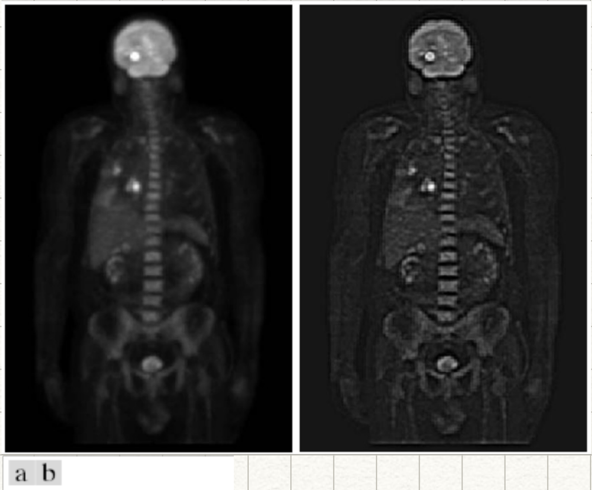


FIGURE 4.62 (a) Full body PET scan. (b) Image enhanced using homomorphic filtering. (Original image courtesy of Dr. Michael E. Casey, CTI PET Systems.)