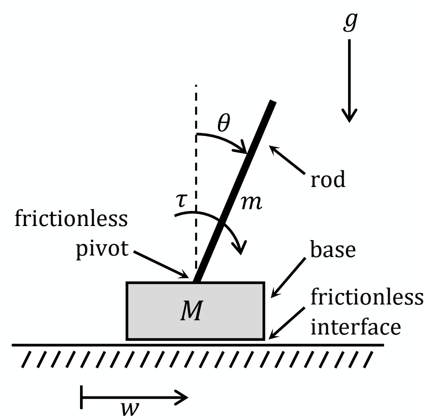
3.9 PROJECT EXERCISES P3 AND P4

Problem P3: Controller design by Youla parameterization for the SISO aiming system

In this chapter we introduced the Youla parameterization, which is a parameterization of all stabilizing LTI controllers for a given plant. There exist several design methods based on the Youla parameterization. In all cases, the idea is to work with $Q(s)$ intead of $C(s)$; this essentially removes the whole issue of worrying about closed-loop stability. Once a "good" $Q(s)$ is found, the corresponding $C(s)$ is easy to compute. In this problem, we briefly explore this approach to controller design. We are still working with the aiming system described back on page 99.



$$P_1(s) = \frac{\theta(s)}{\tau(s)} = \frac{8}{(s-4.427)(s+4.427)}$$

(a) Recall the plant $P_1(s)$ transfer function, shown at the right. Use the Youla parameterization to find a stabilizing controller that gives perfect steady-state tracking for step reference signals.

Important hints:

- We know that, to achieve perfect steady-state tracking of steps, the final $C(s)$ must have an integrator in it. So double check that your final $C(s)$ does indeed have an integrator.
- You will very likely encounter numerical problems. Use "minreal" appropriately, after every single transfer function calculation, possibly with a non-default tolerance. I recommend you explicitly check (using "zpk") that "minreal" worked correctly after every use. Alternatively, do calculations by hand.
- Anecdotal evidence indicates that numerical problems are reduced if, during coprime factorization, the poles of $N(s)$, $M(s)$, $X(s)$, and $Y(s)$ are placed relatively far into the OLHP. (Use the optional "scale" argument in the "coprime" function. Recall that "coprime.m" is available on LEARN.)
  numerical nicer
- If you experience major numerical problems, look for alternative ways of doing transfer function calculations. For example, you can compute the complementary sensitivity function via $T = PC/(1+PC)$ or $T = 1 - 1/(1+PC)$.
- You can likely reduce numerical problems significantly if you simply ignore the stable plant pole. (Why are you justified in ignoring it?)

(b) Plot the closed-loop step response for your controller from part (a).

- Verify that the closed-loop system is stable and that the steady-state tracking error is zero.

a)

$$P_1(s) = \frac{\theta(s)}{\tau(s)} = \frac{8}{(s-4.427)(s+4.427)}$$

Goal:

$$P_1 = \frac{N}{M} \quad , \text{ s.t. } \quad NX + MY = 1$$

Let $N(s) = \frac{(s-4.427)(s+4.427)}{(s+1)(s+2)}$, $M(s) = \frac{8}{(s+1)(s+2)}$

$X(s) =$, $Y(s) =$

$$C(s) = \frac{X_p + M_p Q(s)}{Y_p - N_p Q(s)}$$

② $\max_\omega |S(j\omega)| \leq 5dB$

$\ln(\max_\omega |S(j\omega)|) \leq \ln(10^{\frac{5}{20}}) = 0.5756$

$10^{\frac{5}{20}} \to 1.778$

$dB = 20 \cdot \log_{10}(\text{ratio})$

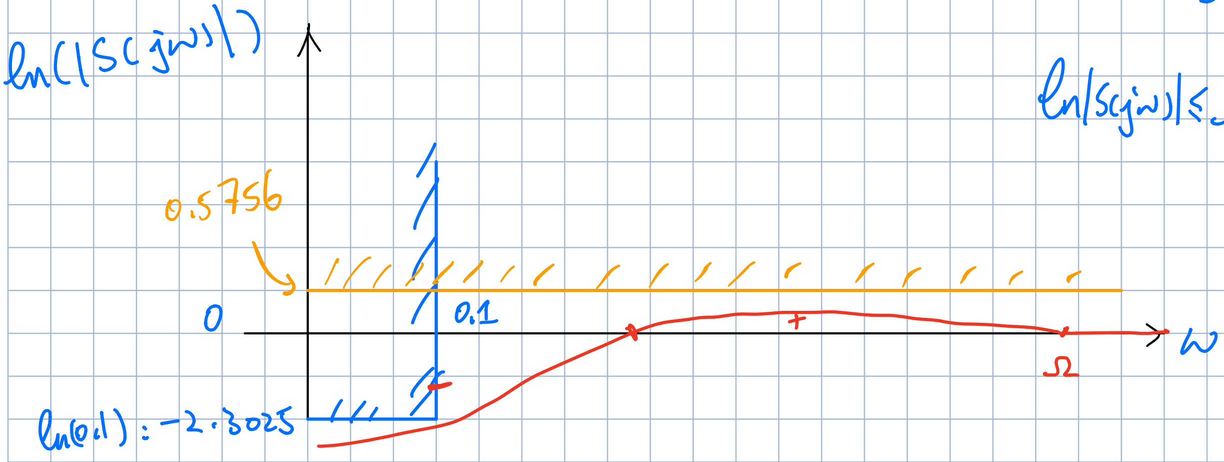
$\Downarrow$

$\text{ratio} = 10^{\left(\frac{dB}{20}\right)}$

① $\to |S(j\omega)| \leq -20dB$

$|S(j\omega)| \leq 10^{\frac{-20}{20}} (= 0.1)$

$\ln|S(j\omega)| \leq \ln(0.1) = -2.3025$



P4)

a)

ii)

Spec: ① $|S(j\omega)| \le -20\text{dB}$ for $0 \le \omega \le 0.1$ [rad/s] $\Rightarrow$ Tracking

② $\max_{\omega} |S(j\omega)| \le 5$ dB $\Rightarrow$ Robust stability

③ $L(j\omega) = 0 \;\; \forall \;\; \omega > \Omega$ — total shut off.

Let

$p = 4.427$. ~~$\Rightarrow BW > 2p \Rightarrow BW = 10$.~~

BSI.

BSI : $$\int_0^{\infty} \ln|S(j\omega)|\, d\omega \ge \pi \sum_{i=1}^{N_p} Re(p_i) \ge 0.$$

$$\ge 4.427\pi \approx 13.9078$$

$$\int_0^{\Omega} \ln|S(j\omega)|\, d\omega \ge \pi p.$$

$$\int_0^{\omega_1 = 0.1} \ln|S(j\omega)|\,d\omega + \int_{\omega_1 = 0.1}^{\Omega} \ln|S(j\omega)|\,d\omega \ge \pi p$$

$$\int_{0.1}^{\Omega} \ln|S(j\omega)|\, d\omega \ge \pi p - \int_0^{0.1} \underbrace{\ln|S(j\omega)|}_{\max = 0.1} d\omega$$

$$\int_{0.1}^{\Omega} \max_{0.1 \le \bar{\omega} \le \Omega} [\ln(|S(j\bar{\omega})|)] \ge \pi p - 0.1 \cdot \ln(0.1)$$

$$(\Omega - 0.1) \max_{0.1 \le \bar{\omega} \le \Omega} [\ln(|S(j\bar{\omega})|)] \ge \pi p - 0.1 \ln 0.1$$

$$\Omega \geq 0.1 + \frac{\pi p - 0.1 \ln(0.1)}{\ln(\underbrace{10^{\frac{5}{20}}}_{1.7783})}$$

$$\Omega \geq 8.05\ [rad/s]$$

$$\Downarrow$$

$$\Omega^{iii}_{(p=0)} = \cancel{0.2295}\ 0.5\ [rad/s]$$

$$\Downarrow$$

Since we don't have ORHP pole,

$$\Downarrow$$

BSI +ve area = −ve area.

area of sensitivity reduction balanced out by the area.

no need ~~to~~ extra sensitivity increase to compensate the sensitivity decrease.

- We would not expect the transient performance to be particularly good since no effort was made to design for good transients. How are the transients of your controller? What is the phase margin? Do you have insight into how to modify $Q(s)$ to improve the transients or phase margin? [You like have zero intuition, demonstrating the point that design in the $Q(s)$ domain, although conceptually appealing and great for some computer-aided design techniques, has a major limitation.]

(c) [Optional] Use "single_pend_fancy_sim.m" to visualize the performance of the controller from part (a). Recall that this Matlab function is available on LEARN and is described on pages 100-102.

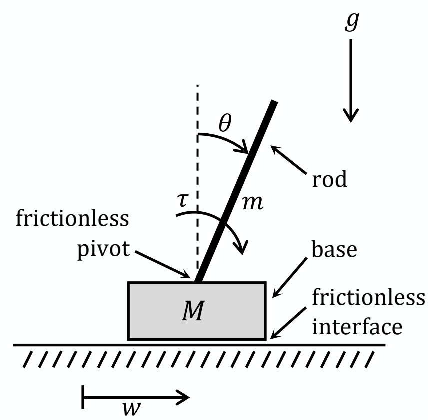
Hard to factor in specifications.

Problem P4: Performance limitations associated with the SISO aiming system

The aiming system is unstable. As we've learned in this chapter, an unstable plant pole leads to various performance limitations. In Problem P4, we explore these limitations quantitatively.

(a) Consider again the SISO aiming system with transfer function

$$P_1(s) = \frac{\theta(s)}{\tau(s)} = \frac{8}{(s-4.427)(s+4.427)}.$$



(i) We established that, because of the unstable plant pole, we need high bandwidth (i.e., a fast response) to get good performance. Assuming that integral control is used to get perfect steady-state step tracking, plot a curve that shows, as a function of rise time (t_r), a minimum bound on the closed-loop step response overshoot (y_{OS}). Assume a 1-DOF topology.

(ii) Suppose that the following three design specifications are imposed:

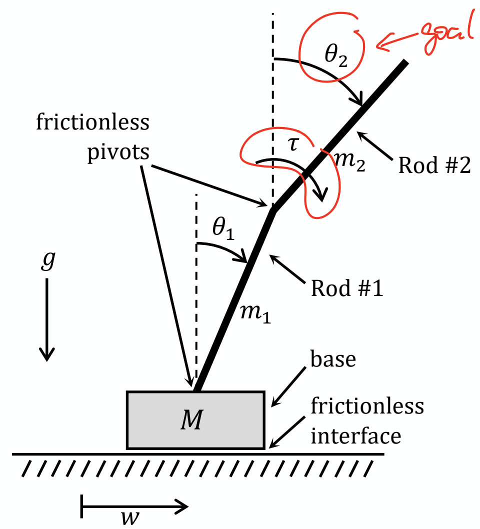
- for tracking purposes, we need $|S(j\omega)| \leq -20$ dB for $0 \leq \omega \leq 0.1$ rad/s,
- for robust stability purposes, we need $\max_{\omega} |S(j\omega)| \leq 5$ dB, and
- due to uncertainty in the plant, the loop gain must be totally "shut off" (i.e., $L(j\omega) = 0$) for all frequencies above Ω rad/s.

[The third specification is mathematically impossible to satisfy exactly. In reality, a finite roll-off rate should be accounted for, but this makes the math a lot more tedious, and it barely affects the final numerical answers.]

Use the Bode Sensitivity Integral "waterbed effect" results in this chapter to derive a lower bound on the value of Ω.

(iii) Repeat part (ii), but this time pretending that the unstable plant pole is not present in the plant. Give a brief explanation about why the resulting Ω bound is larger or smaller than the one you derive in part (ii). when ignoring pole.

(b) Let's modify the plant to make aiming more challenging. Suppose that there are now *two* rods, as shown at the right, and the control objective is to point the upper rod (Rod #2) at some desired angle. At this point we don't care about the angle of Rod #1 or the base position. The plant input is the torque applied to Rod #2 at its pivot point, as indicated in the diagram. You can think of this problem as trying to point a stick accurately when you are on top of a tall ship that can roll and slide freely in frictionless water. Intuitively, adding the second rod makes accurate pointing much more difficult, and the purpose here is to show that the mathematics is consistent with this intuition.



The signals and parameters are summarized in the following table:

Signal	Unit	Interpretation
$\tau(t)$	N·m	Torque applied to Rod #2 (control signal)
$\theta_1(t)$	rad	Rod #1 angle
$\theta_2(t)$	rad	Rod #2 angle (plant output)
$w(t)$	m	Position of the base

Parameter	Value	Interpretation
M	1 kg	Mass of the base
m_1	0.25 kg	Mass of Rod #1
m_2	0.25 kg	Mass of Rod #2
L_1	0.5 m	Length of Rod #1
L_2	0.5 m	Length of Rod #2
g	9.8 m/s^2	Acceleration due to gravity

Assuming both rods are rigid and uniform, mechanical modeling leads to the following set of three nonlinear equations of motion:

$$
\begin{aligned}
&\frac{3}{2}\ddot{w} - \frac{3}{16}\dot{\theta}_1^2 \sin(\theta_1) + \frac{3}{16}\ddot{\theta}_1 \cos(\theta_1) - \frac{1}{16}\dot{\theta}_2^2 \sin(\theta_2) + \frac{1}{16}\ddot{\theta}_2 \cos(\theta_2) = 0 \\
&\frac{1}{12}\ddot{\theta}_1 + \frac{3}{16}\ddot{w}\cos(\theta_1) + \frac{1}{32}\ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \frac{1}{32}(\dot{\theta}_1 - \dot{\theta}_2)\dot{\theta}_2 \sin(\theta_1 - \theta_2) - \frac{3}{16} g \sin(\theta_1) \\
&\qquad\qquad + \frac{1}{32}\dot{\theta}_1\dot{\theta}_2 \sin(\theta_1 - \theta_2) = 0 \\
&\frac{1}{48}\ddot{\theta}_2 + \frac{1}{16}\ddot{w}\cos(\theta_2) + \frac{1}{32}\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \frac{1}{32}(\dot{\theta}_1 - \dot{\theta}_2)\dot{\theta}_1 \sin(\theta_1 - \theta_2) - \frac{1}{16} g \sin(\theta_2) \\
&\qquad\qquad - \frac{1}{32}\dot{\theta}_1\dot{\theta}_2 \sin(\theta_1 - \theta_2) = \tau.
\end{aligned}
$$

Linearization of these equations about the operating point corresponding to where both rods are pointing straight up (i.e., when $\theta_1 = \theta_2 = 0$ and for any value of w) yields the following transfer function for the plant:

$$
\boxed{P_2(s) = \frac{\theta_2(s)}{\tau(s)} = \frac{110.4(s + 5.539)(s - 5.539)}{(s - 4.331)(s + 4.331)(s - 10.52)(s + 10.52)}.} \qquad \text{... (P6)}
$$

Notice that the plant has two ORHP poles and an ORHP zero. Our goal here is to use the results from this chapter to show that there are indeed performance limitations that are more severe than those than we derived for the one-rod aiming system. Note that parameter values have been chosen to ensure any comparision we make is "fair."

(i) Repeat Problem P4(a)(i), but now for the two-rod aiming system. Find the least conservative bound (i.e., the most severe bound) that you can, and compare the curve to your answer from Problem P4(a)(i). Do your plots demonstrate that the two-rod system is *necessarily* more difficult to control than the one-rod system? Finally, compute a meaningful bound on overshoot for the two-rod system using (1) on page ~~129~~ 161. How does this bound compare to the bounds in your plots?

(ii) As in Problem P4(a)(ii), suppose that the following three design specifications are imposed:

- for tracking purposes, we need $|S(j\omega)| \leq -20$ dB for $0 \leq \omega \leq 0.1$ rad/s,
- for robust stability purposes, we need $\max_{\omega} |S(j\omega)| \leq 5$ dB, and
- due to uncertainty in the plant, the loop gain must be totally "shut off" (i.e., $L(j\omega) = 0$) for all frequencies above Ω rad/s.

Use the BSI to derive a lower bound on Ω. Comparing your answer to that in Problem P4(a)(ii), can we conclude that the two-rod system is definitely more difficult to control than the one-rod system?

(iii) Repeat (ii) above, but this time try using the Poisson Integral instead of the BSI. In particular, use the Poisson Integral to show that, for the two-rod aiming system, it is *impossible* to satisfy the three design specifications given above.

Spec: ① $|S(j\omega)| \le -20\,dB$ for $0 \le \omega \le 0.1$ [rad/s] ⇒ Tracking

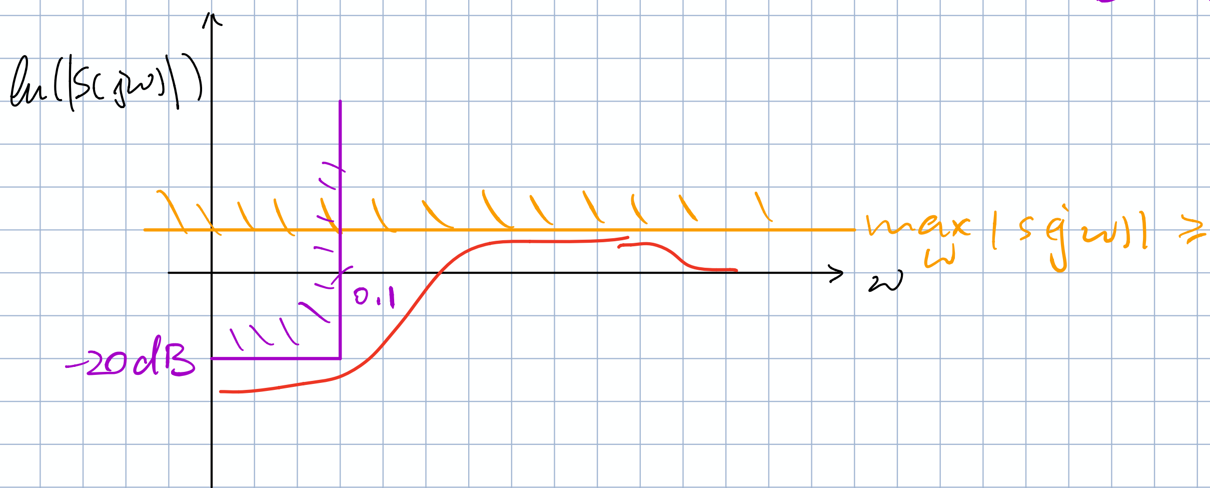
② $\max_{\omega} |S(j\omega)| \le 5\,dB$ ⇒ Robust stability

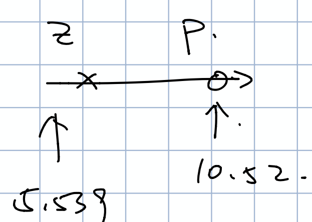
③ $L(j\omega) = 0 \;\; \forall\, \omega > \Omega$ — total shut off.

Let

.427. ⇒ $BW > 2p$ ⇒ $BW = 10$.

BSI.





BSI: $\displaystyle\int_0^{\infty} \ln|S(j\omega)|\, d\omega \ge \pi \sum_{i=1}^{N_p} Re(p_i) \ge 0.$

$$\int_{0.1}^{\Omega} \ln|S(j\omega)|\,d\omega \geq \pi\Sigma p - \int_0^{0.1} \underbrace{\ln|S(j\omega)|}_{max=0.1} d\omega$$

$$\int_{0.1}^{\Omega} \max_{0.1\leq\bar{\omega}\leq\Omega} [\ln(|S(j\bar{\omega})|)] \geq \pi\Sigma p - 0.1 \cdot \ln(0.1)$$

$$(\Omega - 0.1) \max_{0.1\leq\bar{\omega}\leq\Omega} [\ln(|S(j\bar{\omega})|)] \geq \pi p - 0.1 \ln 0.1$$

BSI: $\Omega \geq \dfrac{\pi\Sigma p - 0.1\ln(0.1)}{\max_{0.1\leq\bar{\omega}\leq\Omega} [\ln(|S(j\bar{\omega})|)] \to 10^{\frac{5}{20}}} + 0.1 = 26.466.$

$K = \ln(5dB) = 0.5756$

ORHP poles.

Poison I: $\int_0^{\infty} \ln(|S(j\omega)|) W(\omega)\,d\omega \geq \pi \cdot \sum_{i=1}^{N_p} \ln\left|\dfrac{P_i + Z}{P_i^* - Z}\right| \geq 0$

$$W(\omega) = \frac{2Z}{Z^2 + \omega^2}$$

$$\int_0^{0.1} K\, W(\omega)\,d\omega + \int_{0.1}^{\Omega} K\, W(\omega)\,d\omega \geq 0$$

$$\int_{0.1}^{\Omega} K\, W(\omega)\,d\omega \geq -\int_0^{0.1} K\, W(\omega)\,d\omega$$

$$\int_0^{0.1} (\ln \varepsilon)\, W(\omega)\, d\omega + \int_{0.1}^{\Omega} (\ln M)\, W(\omega)\, d\omega \geq 0$$

$$(\ln \varepsilon) \int_0^{0.1} \frac{2z}{z^2+\omega^2}\, d\omega + (\ln M) \int_{0.1}^{\Omega} \frac{2z}{z^2+\omega^2}\, d\omega \geq 0$$

$$2z(\ln \varepsilon) \int_0^{0.1} \frac{1}{z^2+\omega^2}\, d\omega$$

$$= 2z \left\{ (\ln \varepsilon) \left[\frac{1}{z} \tan^{-1}\left(\frac{\omega}{z}\right) \right]_0^{0.1} + (\ln M) \left[\frac{1}{z} \tan^{-1}\left(\frac{\omega}{z}\right) \right]_{0.1}^{\Omega} \right\} \geq 0$$

$$(\ln M) \geq \frac{-\ln(\varepsilon) \left[\frac{1}{z} \tan^{-1}\left(\frac{\omega}{z}\right) \right]_{\omega=0}^{0.1}}{\left[\frac{1}{z} \tan^{-1}\left(\frac{\omega}{z}\right) \right]_{0.1}^{\Omega}}$$

$$M \geq \exp \left\{ \frac{-\ln(\varepsilon) \left[\cancel{\frac{1}{z}} \tan^{-1}\left(\frac{0.1}{z}\right) \right]}{\left[\cancel{\frac{1}{z}} \tan^{-1}\left(\frac{\Omega}{z}\right) \right] - \left[\cancel{\frac{1}{z}} \tan^{-1}\left(\frac{0.1}{z}\right) \right]} \right\}$$