Jakub Swistak
Poland
kontakt@jakub0301.net
Stanford SUMAC

Solution to task number 1. $x^2 - y^2 = (x+y)(x-y)$
Firstly let's factorize $2019 = 3 \cdot 673$. We can see that 3 and 673 are prime numbers. We can also notice that $x + y > x - y$ because x and y are positive. Hence,

$$\begin{cases} x + y = 673 \\ x - y = 3 \end{cases}$$

$$\begin{cases} 2x = 673 \\ x - y = 3 \end{cases}$$

$$\begin{cases} x = 338 \\ y = 335 \end{cases}$$

or

$$\begin{cases} x + y = 2019 \\ x - y = 1 \end{cases}$$

$$\begin{cases} 2x = 2020 \\ x - y = 1 \end{cases}$$

$$\begin{cases} x = 1010 \\ y = 1009 \end{cases}$$

$x^3 - y^3 = (x-y)(x^2 + xy + y^2)$

Then simply we can notice that $x - y < x^2 + xy + y^2$

$$\begin{cases} x - y = 3 \\ x^2 + y^2 + xy = 673 \end{cases}$$

$$\begin{cases} x = 3 + y \\ (3+y)^2 + y^2 + (3+y)y = 673 \end{cases}$$

$0 = (3+y)^2 + y^2 + (3+y)y - 673 = 2y^2 + 5y - 667$
$\Delta = 5361$ and $\sqrt[2]{5361} \notin \mathbb{N} \Rightarrow$ there are no solutions. Or

$$\begin{cases} x - y = 1 \\ x^2 + y^2 + xy = 2019 \end{cases}$$

$$\begin{cases} x = 1 + y \\ (1+y)^2 + y^2 + (1+y)y = 2019 \end{cases}$$

$$\begin{cases} x = 1 + y \\ 3y^2 + 3y - 2018 = 0 \end{cases}$$

$\Delta = 24225$ and $\sqrt[2]{24225} \notin \mathbb{N} \Rightarrow$ there are no solutions.

$\square$

Solution to task number 2. Let's suppose $p(\sqrt{7}-\sqrt{3}) = 0$ and $\deg(p) = 4 \Rightarrow p(\sqrt{7}-\sqrt{3}) = a_0 + a_1(\sqrt{7}-\sqrt{3}) + a_2(\sqrt{7}-\sqrt{3})^2 + a_3(\sqrt{7}-\sqrt{3})^3 + a_4(\sqrt{7}-\sqrt{3})^4 = 0$
Then after expending every bracket we obtain:

$$\sqrt{7}(a_1 + 16a_3) + \sqrt{3}(-a_1 - 24a_3) + \sqrt{21}(-2a_2 - 40a_4) + 184a_4 + 10a_2 + a_0 = 0 \tag{1}$$

Observe that equation (1) holds if

$$\begin{cases} a_1 + 16a_3 = 0 \\ a_1 + 24a_3 = 0 \\ a_2 + 20a_4 = 0 \\ a_0 + 10a_2 + 184a_4 = 0 \end{cases}$$

$$\begin{cases} a_3 = 0 \\ a_1 = 0 \\ a_2 = -20a_4 \\ a_0 + 10a_2 + 184a_4 = 0 \end{cases}$$

$$\begin{cases} a_3 = 0 \\ a_1 = 0 \\ a_0 = 16a_4 \\ a_2 = -20a_4 \end{cases}$$

Let $a_4 = 1 \Rightarrow a_0 = 16$ and $a_2 = -20 \Rightarrow p(x) = x^4 - 20x^2 + 4$

$\square$

Solution to task number 3. Firstly let's replace black element with 6.

1	2	3
4	5	6

Let's define $\pi(x)$ gives us position of the element, then for example $\pi(4) = 4$. Let's notice that first matrix is identity. Let's also notice that we have to "go"with 6 to element and then "go back", then we will make $2n$ moves.
$sgn(\pi_1 \circ\ pi_2) = sgn(\pi_1) * sgn(\pi_2)$
Then for first matrix $sgn(t_1 \circ ... \circ t_{2n}) = 1$

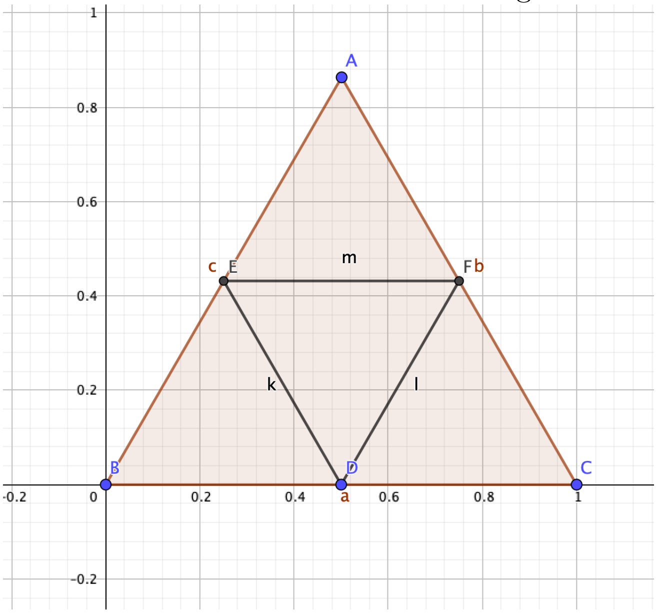
2	1	3
4	5	6

For second matrix $sgn(t_1 \circ ... \circ t_{2n+1}) = -1$
which gives us a contradiction

$\square$

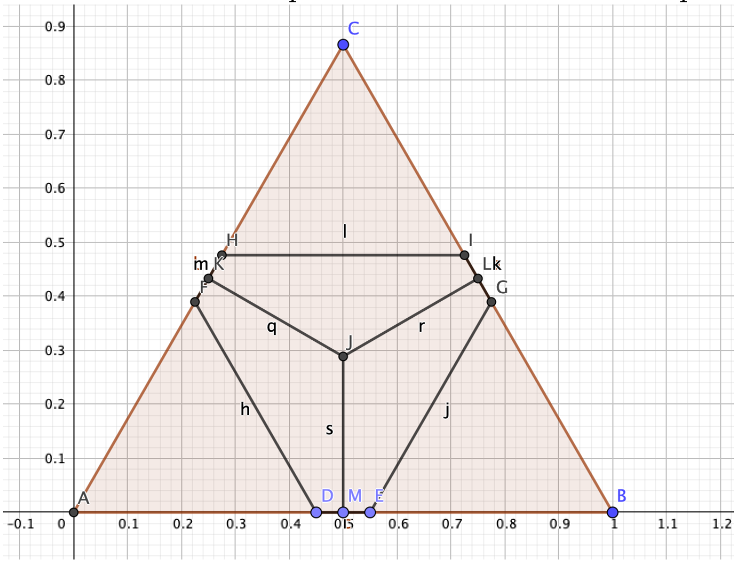
Solution to task number 4.

a) Let's divide the triangle to 4 congruent equilateral triangles with side-length 0.5 cm each (see the picture below). If we select 5 points then since there are only 4 triangles, thus one of the triangles will contain two points. The distance between these two points is not greater then 0.5 cm which is the diameter of the triangle.



b) Let's take points A,B,C,D,E and F from the picture. They are the vertices of the 0.5 cm triangles and the picture shows that the distance between each pair will be exactly 0.5 cm.

c) Let's divide triangle as on the picture below to 3 congruent equilateral triangles with side-length 0.45 cm and to 3 congruent quadrangles. With the longest diameter equal 0.48 cm $\Rightarrow$ It will work for 7 points. $\Rightarrow$ It will work for 8 points.



$\square$

Solution to task number 5.

a) The number of islands equals to the number of $0-1$ sequences of length n which is 2^n. For any island we have n islands which differ on a one bit, each island is connected with remaining islands with n routes (cause it can be diffrent in n positions), then $\frac{2^n * n}{2}$

[Here I was looking for some more ideas (to solve for $n=5$ and $n=6$) and found some articles about Hypercube and Hamming distance] Let K be the gratest subset of vertices of Hypercube Q=(V,E), such that for any vertices v_1, v_2 in K occurs $dist(v_1, v_2) \geq 3$ (Hamming distance).

$|K| \leq \frac{2^n}{n+1}$ (because every vertex from K will remove himself and n connected with direct edge)

$|K| * (deg(v) + 1) \leq |V|$

n=3

$K = \{000, 111\}, |K| = 2$ (cube)

n=4

without loss of generality K contain $\{0000\}$, so $\{v : dist(v, 0000) \geq 3\} = \{1111, 1110, 1101, 1011, 0111\}$ Let's notice that in the set above distance between every pair of points is ≤ 2, so we can pick only two points $\Rightarrow K = \{0000, 1111\}, |K| = 2$

n=5

We are using very similar idea like in $n = 4$ without loss of generality K contain $\{00000\}$, so $\{v : dist(v, 00000) \geq 3\} = \{binary\ strings\ with\ 3\ or\ 4\ ones\}$ If we add $\{11111\}$, we will not be able to add because $\{v : dist(v, 11111) \geq 3\} = \{strings\ with\ 3,\ 4\ or\ 5\ zeros\}$ therefore $v : dist(v, 11111) \geq 3 \cap v : dist(v, 00000) \geq 3 = \emptyset$

WLG-without loss of generality

Now let's consider when we will add string with 4 ones WLG it will be 11110 $\Rightarrow \{v : dist(v, 00000) \geq 3$ and $dist(v, 11110) \geq 3\} =$ $\{00111, 10011, 11001, 01011, 10101, 01101\}$ Distance between every two vertices listed above is 2 or 4 and it's impossible to pick 3, so as to distance between will be 4.$\Rightarrow$ we can add only 2 more vertices $\Rightarrow K = \{00000, 11110, 00111, 11001\}$

Case when K doesn't contain 4 ones will be similar, but $|K|$ will be smaller

n=3

M=000,111 |M|=2

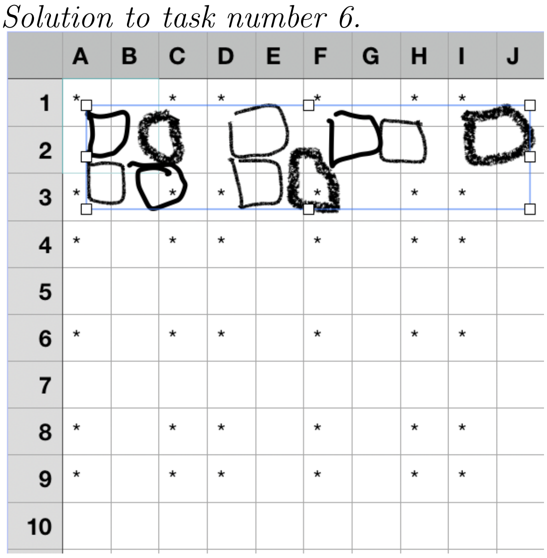
n=4

$|M| \geq \frac{16}{5}$, therefore $|M| \geq 4$ Easily we can check, that $\{0000, 1001, 1110, 0111\}$ $|M| = 4$

n=5

$|M| \geq \frac{32}{6}$, therefore $|M| \geq 6$ The greatest set which I have found is $\{00000, 11110, 00111, 11001, 10011, 01011, 01100, 10100\}$. $\square$

Solution to task number 6.



a) Let's consider the following construction(image above) by star denote area covered by only one 2x2 card and by (1A,2B) denote 2x2 card covering filed from A1 to B2.
There are 36 stars and every block are covering exactly one star(blocks on (A1,B2),(B1,C2),(D1,E2),...,(2A,3B),...(9I,10J). (there are some marked on the picture)

	A	B	C	D	E	F	G	H	I	J
1	*		*	*		*		*	*	
2										

Let's also notice that this construction will generate the greatest score for non-redundant to prove let's consider 2x10 table and construct greedily, then the maximum score which will avoid redundant cards. We can also use the same way to table 10x2 and "connect both ideas", which will give construction mentioned earlier. It also proves that every covering with 55 cards is redundant. □