Zadanie 1

d) $Re\left[(2+i)(-3i+4)-2i-5\right]$

$(2+i)(-3i+4)-2i-5 =$

$-6i+8-3i^2+4i-2i-5 =$

$3-4i-3i^2 = 3-4i+3 = 6-4i$

$Re\left[6-4i\right] = 6$

e) $Im\left[(3-2i)^2-\frac{1+2i}{1-4i}\right]$

$(3-2i)^2-\frac{1+2i}{1-4i} = (3-2i)^2-\frac{1+2i}{1-4i}\cdot\frac{1+4i}{1+4i} =$

$= 9-12i+4i^2-\frac{1+6i+8i^2}{1-16i^2} =$

$= 9-12i-4-\frac{1+6i-8}{1+16} =$

$= 5-12i-\frac{-7+6i}{17} =$

$= 5-12i+\frac{7}{17}\cancel{i}-\frac{6}{17}i =$

$= 5+\frac{7}{17}-\left(12+\frac{6}{17}\right)i =$

$= \frac{85}{17}+\frac{7}{17}-\left(\frac{204}{17}+\frac{6}{17}\right)i =$

$= \frac{92}{17}-\frac{210}{17}i$

$Im\left[\frac{92}{17}-\frac{210}{17}i\right] = -\frac{210}{17}$

Zadanie 2

c) $\quad Re\left(\frac{\bar{z}}{z}\right)$

$$\frac{\bar{z}}{z} = \frac{x-iy}{x+iy} \cdot \frac{x-iy}{x-iy} = \frac{(x-iy)^2}{x^2-(iy)^2} = \frac{(x-iy)^2}{x^2+y^2} =$$

$$= \frac{x^2-2xyi+(yi)^2}{x^2+y^2} = \frac{x^2-2xyi-y^2}{x^2+y^2} = \frac{x^2-y^2-2xyi}{x^2+y^2} =$$

$$= \frac{x^2-y^2}{x^2+y^2} - \frac{2xy}{x^2+y^2}i$$

$$Re\left[\frac{x^2-y^2}{x^2+y^2} - \frac{2xy}{x^2+y^2}i\right] = \frac{x^2-y^2}{x^2+y^2}$$

Zadanie 4

c) Z: $z \neq -1$

T: $\operatorname{Im} z = 0 \Leftrightarrow \operatorname{Im}\left(\frac{z-1}{z+1}\right) = 0$

D: Niech $z = x + iy$, $x, y \in \mathbb{R}$

$\operatorname{Im} z = 0 \Leftrightarrow \operatorname{Im}(x + iy) = 0 \Leftrightarrow y = 0$

$\operatorname{Im}\left(\frac{z-1}{z+1}\right) = 0 \Leftrightarrow \operatorname{Im}\left(\frac{x+iy-1}{x+iy+1}\right) = 0$

$$\frac{x+iy-1}{x+iy+1} = \frac{x-1}{x+iy+1} + \frac{iy}{x+iy+1} = \frac{y}{x+iy+1} \cdot i$$

$$\operatorname{Im}\left(\frac{y}{x+iy+1} \cdot i\right) = y = 0 \quad \square$$

Zadanie 5

a) $w = \frac{2}{2+i}$, $w \in \mathbb{R}$

$(2+i)w = 2 \iff 2w + iw = 2 \iff$

$2w - 2 = -iw \iff 2(w-1) = -iw \iff$

$2 = \frac{-iw}{w-1}$

Zadanie 6

a) $\bar{z}(z+1) - \operatorname{Im}(-z) + 4i^{10} = \frac{2+2i}{1-i}$

Niech $z = x + iy$, $x, y \in \mathbb{R}$

$(x - iy)(x + iy + 1) - \operatorname{Im}(-x - iy) + 4(i^2)^5 = \frac{2+2i}{1-i}$

$x^2 + ixy + x - ixy - i^2y^2 - iy - \operatorname{Im}(-x-iy) - 4 = 2\frac{1+i}{1-i}\cdot\frac{1+i}{1+i}$

$x^2 + x + y^2 - iy - \operatorname{Im}(-x-iy) - 4 = 2\frac{1+2i+i^2}{1-i^2}$

$x^2 + x + y^2 - iy - \operatorname{Im}(-x-iy) - 4 = 2\frac{1+2i-1}{1+1}$

$x^2 + x + y^2 - iy - \operatorname{Im}(-x-iy) = 4 + 2i$

Zadanie 7

a) $|z - 2i| \leqslant \sqrt{2}\,\mathrm{Im}\,(z - 2i)$

Niech $z = x + iy$, $x, y \in \mathbb{R}$

$|x + iy - 2i| \leqslant \sqrt{2}\,\mathrm{Im}\,(x + iy - 2i)$

$|x + iy - 2i| \leqslant \sqrt{2}\,\mathrm{Im}\,(x + (y-2)i)$

$|x + iy - 2i| \leqslant \sqrt{2} \cdot (y-2)$

$|x + (y-2)i| \leqslant \sqrt{2} \cdot (y-2)$

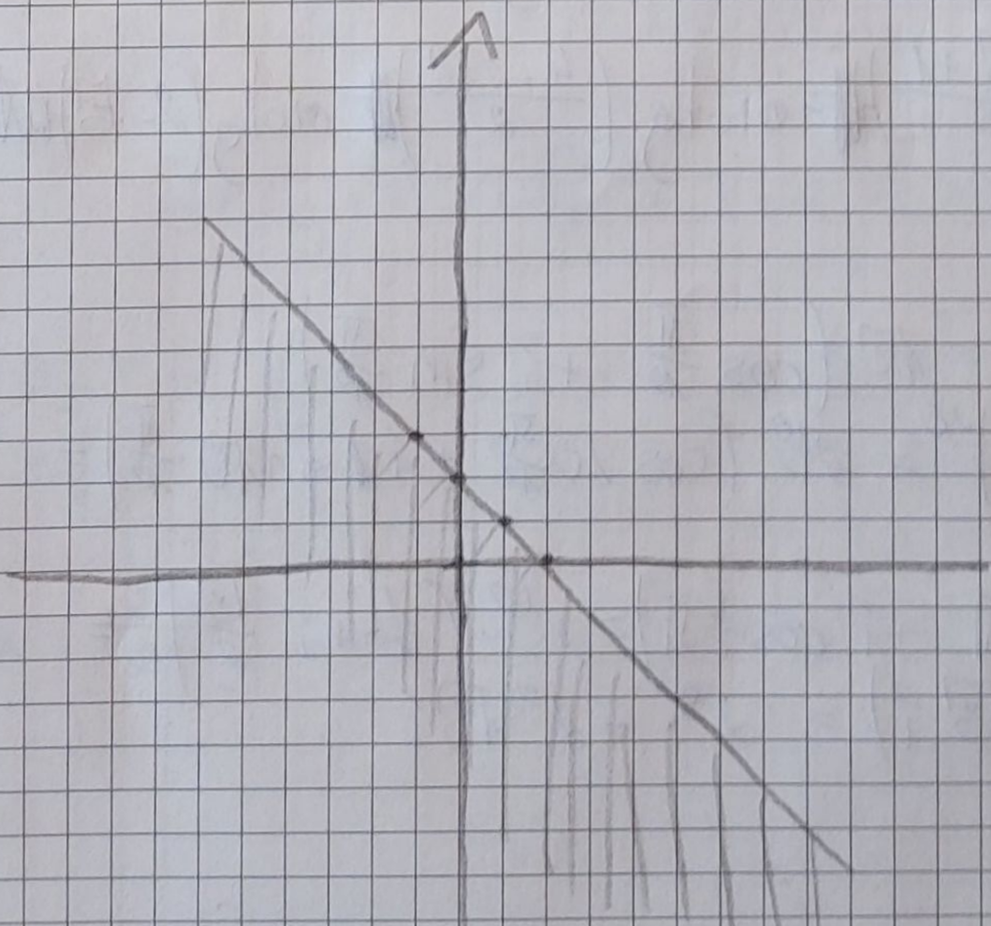
$\sqrt{x^2 + ((y-2)\cancel{i})^2} \leqslant \sqrt{2} \cdot (y-2)$

$\sqrt{x^2 + (y-2)^2} \leqslant \sqrt{2} \cdot (y-2) \quad |()^2$

$x^2 + (y-2)^2 \leqslant 2 \cdot (y-2)^2$

$x^2 \leqslant (y-2)^2$

$\cancel{x^2 \leqslant y^2 - 4y}$



Zadanie 9

c) $\left(\frac{1+i\sqrt{3}}{1+i}\right)^{20} = \left(\frac{1+i\sqrt{3}}{1+i}\cdot\frac{1-i}{1-i}\right)^{20} =$

$= \left(\frac{1-i+i\sqrt{3}-i^2\sqrt{3}}{1-i^2}\right)^{20} = \left(\frac{1}{2}\left(1-i+i\sqrt{3}+\sqrt{3}\right)\right)^{20} =$

$= \left(\frac{1+\sqrt{3}}{2} + \frac{\sqrt{3}-1}{2}i\right)^{20}$

Niech $z = \frac{1+\sqrt{3}}{2} + \frac{\sqrt{3}-1}{2}i = \frac{1+\sqrt{3}}{2} - \frac{1-\sqrt{3}}{2}i$

$\begin{cases} x = \frac{1+\sqrt{3}}{2} \\ y = -\frac{1-\sqrt{3}}{2} \end{cases}$ $|z| = \sqrt{\left(\frac{1+\sqrt{3}}{2}\right)^2 + \left(-\frac{1-\sqrt{3}}{2}\right)^2} =$

$= \sqrt{\frac{(1+\sqrt{3})^2+(\sqrt{3}-1)^2}{4}} = \frac{1}{2}\sqrt{1+2\sqrt{3}+3+3-2\sqrt{3}+1} =$

$= \frac{1}{2}\sqrt{8} = \sqrt{2}$

$\varphi_0 = \operatorname{arctg}\left(\frac{\sqrt{3}-1}{2}\cdot\frac{2}{\sqrt{3}+1}\right) = \operatorname{arctg}\left(\frac{\sqrt{3}-1}{\sqrt{3}+1}\right) =$

$= \operatorname{arctg}\left(\frac{3-2\sqrt{3}+1}{3-1}\right) = \operatorname{arctg}\left(\frac{4-2\sqrt{3}}{2}\right) = \operatorname{arctg}(2-\sqrt{3})$

$= \frac{\pi}{12}$

$\left(\frac{1+\sqrt{3}}{2} + \frac{\sqrt{3}-1}{2}\right) = \sqrt{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right)$

$\left(\frac{1+\sqrt{3}}{2} + \frac{\sqrt{3}-1}{2}\right)^{20} = 2^{10}\left(\cos 10\frac{\pi}{12} + i\sin 10\cdot\frac{\pi}{12}\right) =$

$= 2^{10}\left(\cos\frac{5}{6}\pi + i\cos\frac{5}{6}\pi\right) = 2^{10}\cdot\left(\frac{1}{2} - i\frac{\sqrt{3}}{2}\right) =$

$= 2^{9}\left(1-\sqrt{3}i\right) = 2^{9} - 2^{9}\sqrt{3}i$

d) $\operatorname{Im}\left[\frac{(1+i)^{22}}{(-1-\sqrt{3}\,i)^{6}}\right]$

$z_1 = (1+i)^{22}$

$x_1 = 1 > 0$

$y_1 = 1 > 0 \qquad |z_1| = \sqrt{2}$

$\varphi_0 = \operatorname{arctg} 1 = \frac{\pi}{4}$

$z_1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)$

$z_1^{22} = 2^{11}\left(\cos\frac{11\pi}{2} + i\sin\frac{11\pi}{2}\right) =$

$2^{11}\left(0 - i\right) = -2^{11} i$

$z_2 = (-1-\sqrt{3}\,i)^6$

$x_1 = -1 < 0$

$y_1 = -\sqrt{3} < 0 \quad |z_2| = \sqrt{1+3} = 2$

$\varphi_2 = \operatorname{arctg}\sqrt{3} + \pi = \frac{\pi}{3} + \pi = \frac{4}{3}\pi$

$z_2 = 2\left(\cos\frac{4}{3}\pi + i\sin\frac{4}{3}\pi\right)$

$z_2^6 = 2^6\left(\cos 8\pi + i\sin 8\pi\right) = 2^6(1 + i\cdot 0) =$

$= 2^6$

$\operatorname{Im}\left[\frac{-2^{11} i}{2^6}\right] = \operatorname{Im}\left[-2^5 i\right] = -2^5 = -32$