

A Generalized 2D and 3D Hilbert Curve

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Abstract

The two and three dimensional Hilbert curves are fractal space filling curves that map the unit interval to the unit square or unit cube. Hilbert curves preserve locality, keeping distance from the unit interval to their respective higher dimensional embeddings. Finite approximations of the Hilbert curve can be constructed in stages by stopping the recursive construction at a specified depth, providing locality from a discrete array of points to points in the embedded dimension. The locality property of the Hilbert curve can help with caching optimizations based on order and finds uses as a companion method in applications ranging from job scheduling to scene rendering. One limitation of the Hilbert curve approximation construction is that the regions need to be exact powers of two, making it difficult to work with in many real world scenarios. In this paper, we present the Gilbert curve, a conceptually straight forward generalization of the Hilbert curve, that works on arbitrary rectangular regions. The construction provides reasonable worst case run-time guarantees for random access lookups for both two and three dimensions. We also provide comparisons to other Hilbert curve generalization methods and investigate overall quality metrics of the Gilbert curve construction.

1. Introduction

1.1. Overview

We present the *Gilbert curve*, a generalized Hilbert curve for 2D and 3D, that works on arbitrary rectangular regions.

An overview of the benefits of the Gilbert curve are that it is:

- Valid on arbitrary rectangular regions
- Equivalent to the Hilbert curve when dimensions are exact powers of 2
- Efficient at random access lookups ($O(\lg N)$)
- A conceptually straight forward algorithm
- Able to realize curves without notches unnecessarily and limits the resulting curve to a single notch when forced
- Similar in measures of locality to the Hilbert curve

Some drawbacks are that:

- Our implementation is recursive, which may undesirable for some applications that require better than $O(\lg N)$ runtime and memory usage
- Might not adequately capture some features of a Hilbert curve

Further, we show:

- Measures of locality are preserved (Section X)
- Trivial extensions to create generalized Moore curves (Section X)
- Comparison metrics to other space filling curves (Section X)

Generalizations of the Hilbert curve to non power of two rectangular cuboid regions has been explored before but are overly complicated algorithmically, create unbalanced curves and often don't generalize well to 3D. Our realization focuses on a conceptually simple algorithm which creates pleasing resulting curves and works in both 2D and 3D.

Space filling curves are a specialization of a more general Hamiltonian path, but have a more stringent requirement of local connectivity. The local connectivity, or locality,

preserves a measure of nearness, where points from the source unit line remain close in the embedded space.

The local connectivity requirements preclude things like zig-zag Hamiltonian paths, as nearby points in the embedded dimension can be far from the origin line.

The Gilbert curve algorithm works by choosing sub rectangular cuboid regions, or blocks, to recursively find a connecting path. If one extent of the cuboid becomes too large or small relative to the other lengths, a special subdivision is performed to try make further subdivisions more cube-like. Subdivisions are performed until a base case is reached and the lowest unit of the curve can be realized.

A Hamiltonian path is not always realizable for certain side lengths and endpoint constraints. In such a case, the Gilbert curve will introduce a single diagonal path move, called a *notch*.

1.2. Definitions

We concern ourselves with a space filling curve, ω , through a rectangular region (N_x, N_y, N_z) , $N = (N_x \cdot N_y \cdot N_z)$:

$$\begin{aligned} k &\in \{0 \dots (N-1)\}, \\ x_k, y_k, z_k &\in \mathbb{N} \\ \omega_k &= (x_k, y_k, z_k) \\ \omega &= (\omega_0, \omega_1, \dots, \omega_{N-1}) \\ k > 1 &\rightarrow |\omega_{k-1} - \omega_k| = 1 \\ \forall i, j \in \{0 \dots (N-1)\}, i \neq j &\rightarrow \omega_i \neq \omega_j \end{aligned}$$

2. Related Work

3. Algorithm

3.1. Overview

For both the 2D and 3D generalized Hilbert (Gilbert) curve, a template is chosen to recursively subdivide the area or volume. When recursively descending the sub-volumes, we will need to translate to a new reference point and rotate by increments of $(\pi/2)$ radians, maintaining cuboid regions that are axis-aligned. We use the convention of labeling $\alpha, \beta, \gamma \in \mathbb{Z}^3$ for the width-like, height-like and depth-like dimension vectors, respectively.

Each of α, β and γ start off as axis-aligned vectors of integral length and are only rotated by units of $(\pi/2)$ radians, remaining axis aligned throughout. α, β and γ represent

the dimensions of the sub-cuboid region and the local basis when tracing out a curve.

The next sub-section discusses parity arguments for when a valid curve can be traced out in a sub-region. We then discuss the 2D Gilbert curve and end with the 3D Gilbert curve.

When talking about the 2D Gilbert curves, we assume vectors are in \mathbb{Z}^3 so they can be used without alteration for algorithms working in 3D.

3.2. Valid Paths from Grid Parity

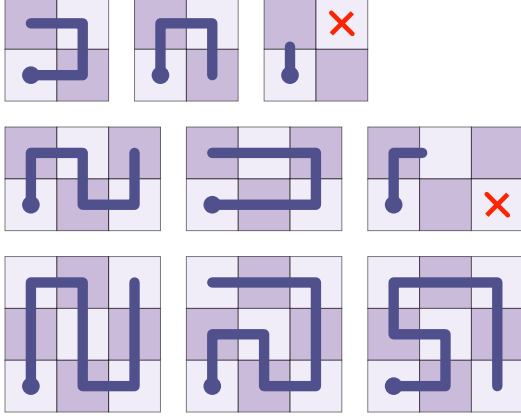


Figure 1: Illustrative examples of Hamiltonian paths height/width that are even/even, even/odd and odd/odd, respectively, when starting from the lower left hand corner

The feasibility of determining whether there exists a Hamiltonian path in a rectangular cuboid grid region can be accomplished through parity arguments. Label grid cell points in a volume as 0 or 1, alternating between labels with every axis-aligned single step move. Any Hamiltonian path that ends at one of the three remaining corners has to have the same parity as the starting point if the volume is odd, or different parity if the volume is even.

Path Possible		Volume	
		even	odd
$ \alpha $	even	Yes	Yes
	odd	No	Yes

Table 1

Table showing when a Hamiltonian cycle is possible. Here, $|\alpha|$ is the absolute difference in start and end position of the path which coincides with one of the axis aligned side length of the cuboid volume. A Hamiltonian path is not possible on only in the case when $|\alpha|$ is odd and the volume is even.

For a path starting at $(0, 0, 0)$ and ending $|\alpha| \in \mathbb{N}$ steps in one of the axis-aligned dimensions, then Table 1 enumerates this condition under which a valid path is possible. Figure 1 illustrates this for starting position $(0, 0)$ with volumes (2×2) , (3×2) and (3×3) , where a red cross indicating a precluded endpoint.

Without loss of generality, we will assume a curve starts from position $p_s = (0, 0, 0)$ and has proposed endpoint at $p_e = ((w-1), 0, 0)$, with a cuboid region as $\alpha = (w, 0, 0)$, $\beta = (0, h, 0)$, $\gamma = (0, 0, d)$. A Hamiltonian path is precluded when $(|\alpha| \cdot (1 - |\beta| \cdot |\gamma|) \equiv 1 \pmod{2})$. That

is, a Hamiltonian from p_s to p_e is precluded when α is odd and at least one of $|\beta|$ or $|\gamma|$ is even.

We state, without proof, that a Hamiltonian path is always possible from p_s to p_e when $|\alpha|$ is even or when $|\alpha|$, $|\beta|$ and $|\gamma|$ are all odd. With this condition, any cuboid subdivision will always have a Hamiltonian path within it and we can recreate a Hamiltonian path in the larger cuboid by connecting endpoints appropriately.

3.3. 2D Generalized Hilbert Function (Gilbert2D)

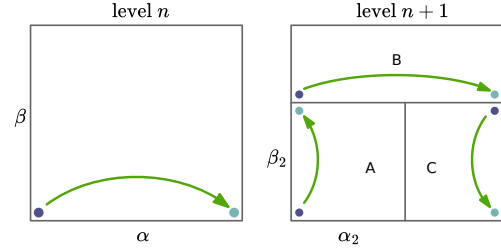


Figure 2: For the 2D Gilbert curve, the main rectangle is subdivided into three sub regions, making sure their paths connect and preferring even side lengths for the first rectangular subdivision. Partitioning the rectangle in this way is what we call a *U-split*.

For the 2D Gilbert curve, a *U-split* template is used, highlighted in figure 2. The *U-split* breaks the region into three sub-blocks, labeled *A*, *B* and *C*. Each of the *A*, *B* and *C* regions have local coordinate systems that are resized, rotated and new endpoints chosen so the process can be recursively re-applied.

The sides of the subdivided blocks *A*, *B* and *C* are chosen to remain integral. The width-like dimension for subdivided block *A* is chosen to be even (β_2 in Figure 2) by dividing the width-like dimension by two and by adding one if need be. Since the width-like length of the subdivided *A* and *C* block is even, there is still a valid Hamiltonian path in both.

Should the original width-like be even (α in ??), the *C* block will continue to have a valid Hamiltonian path. If the original width-like dimension is odd, there is a forced notch that will appear in the subdivided block *C*.

Figure 3 gives examples of the different parity conditions for the original area, as well as the different conditions when the *A* and *C* subdivided areas are force even parity. Figure 3 also shows when a Hamiltonian path is not possible, indicated by a red cross, and illustrates how the notch is pushed into the *C* block under these conditions.

If a subdivided block becomes too long in its width-like dimension, the block is divided in half and the recursion proceeds as normal. Even sides are only enforced if the length is larger than 2, with a length 2 or smaller representing the base case. The base case enumerates paths in the non trivial direction.

Since *A* and *C* are chosen to have width-like local axis even, no new notches are introduced and notch that is forced is effectively “pushed” into the *C* block.

Algorithm 1 shows the pseudo-code for computing the 2D Gilbert curve. Note that α and β are taken to be vectors in 3D, where the third dimension can be ignored if a purely 2D curve is desired. The generalization to 3D allows the

Algorithm 1 2D Generalized Hilbert Function (Gilbert2D)

```

#  $p, \alpha, \beta \in \mathbb{Z}^3$ 
function GILBERT2D( $p, \alpha, \beta$ )

     $\alpha_2, \beta_2 = (\alpha//2), (\beta//2)$ 

    if ( $|\beta| \equiv 1$ ) then
        yield  $p + i \cdot \delta(\alpha)$  forall  $i \in |\alpha|$ 

    else if ( $|\alpha| \equiv 1$ ) then
        yield  $p + i \cdot \delta(\alpha)$  forall  $i \in |\beta|$ 

    else if ( $2|\alpha| > 3|\beta|$ ) then
        if ( $|\alpha_2| > 2$ ) and ( $|\alpha_2| \bmod 2 \equiv 1$ ) then
             $\alpha_2 \leftarrow \alpha_2 + \delta(\alpha)$ 
        end if

        yield Gilbert2D( $p,$ 
                         $\alpha_2, \beta$ )

        yield Gilbert2D( $p + \alpha_2,$ 
                         $\alpha - \alpha_2, \beta$ )

    else
        if ( $|\beta_2| > 2$ ) and ( $|\beta_2| \bmod 2 \equiv 1$ ) then
             $\beta_2 \leftarrow \beta_2 + \delta(\beta)$ 
        end if

        yield Gilbert2D( $p,$ 
                         $\beta_2, \alpha_2$ )

        yield Gilbert2D( $p + \beta_2,$ 
                         $\alpha, (\beta - \beta_2)$ )

        yield Gilbert2D( $p + \alpha - \delta(\alpha) + \beta_2 - \delta(\beta),$ 
                         $\beta_2, -(\alpha - \alpha_2)$ )

    end if
end function

```

Procedure 2 $\delta(\cdot)$ directional vector function

```

# integral sign function
function SGN( $w \in \mathbb{Z}$ )
    if ( $w < 0$ ) return -1
    if ( $w > 0$ ) return 1
    return 0
end function

# directional vector
function  $\delta(v \in \mathbb{Z}^3)$ 
    return [ $\text{sgn}(v_0), \text{sgn}(v_1), \text{sgn}(v_2)$ ]
end function

```

For each of the six cases, we want to know what relative difference in sizes should be used to determine when an eccentric split is used and how to subdivide the cuboids so as to make progress.

Specifically, we want to find the ratio, σ or η , of when one dimension is proportionally larger or smaller, respectively, than the others and the ratio, ρ , of where to choose the split point of subdivision. For simplicity, we might want to subdivide at the half way point ($\rho = \frac{1}{2}$) but as we will see, this might give lopsided sub-cuboid regions and using

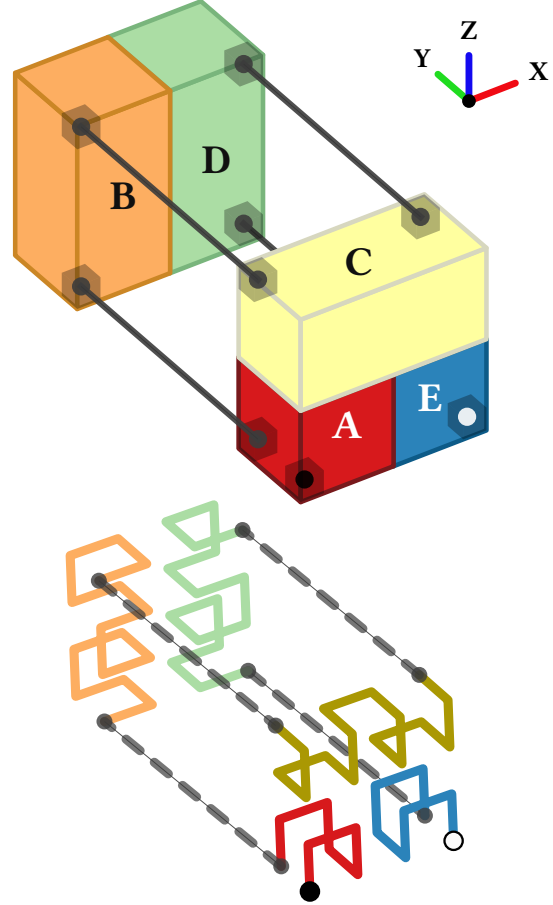


Figure 4: The J_0 -split subdivision, representing the main subdivision of the bulk recursion for the 3D Gilbert curve case.

Procedure 3 S_0 -Split function (eccentric split)

```

# split halfway on  $\alpha$ 
function  $S_0(p, \alpha, \beta, \gamma)$ 
     $\alpha_2 \leftarrow (\alpha//2)$ 
    if ( $|\alpha| > 2$ ) and ( $|\alpha_2| \bmod 2 \equiv 1$ ) then
         $\alpha_2 \leftarrow \alpha_2 + \delta(\alpha)$ 
    end if

    yield Gilbert3D( $p,$ 
                     $\alpha_2, \beta, \gamma$ )

    yield Gilbert3D( $p + \alpha_2,$ 
                     $(\alpha - \alpha_2), \beta, \gamma$ )

end function

```

a better split point is desirable.

In what follows, our goal is to choose a subdivision that will reduce the average defect. We assume that the start and end of the path lie in the w dimension with the local start point at $(0, 0, 0)$ and endpoint at $(w - 1, 0, 0)$.

Because we want to avoid adding unnecessary notches, we are forced split in the w axis. We commit to always splitting the w axis in two. If we want to subdivide into three cuboid regions, we also need to pick the split point in the other dimension another dimension

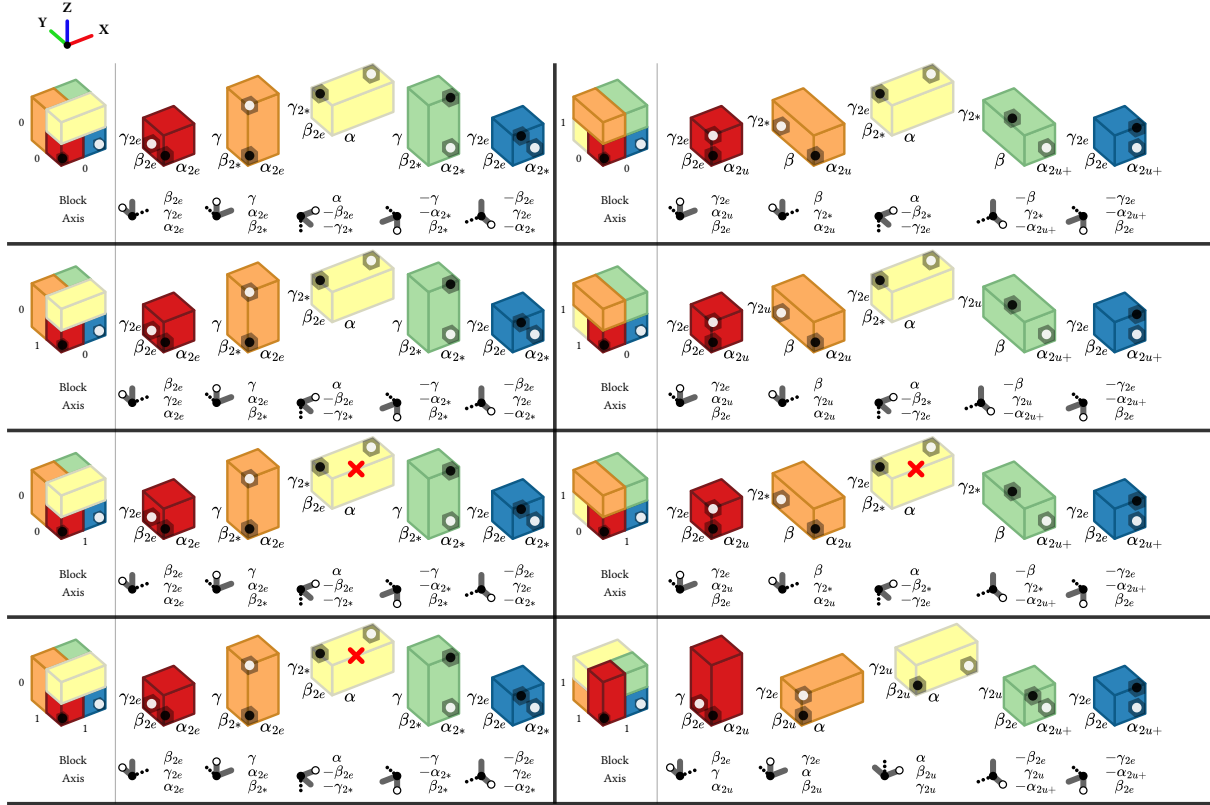


Figure 5: Bulk recursion J-split atlas for the 3D Gilbert algorithm

Procedure 4 S_1 -Split functions (eccentric split)

```
# split  $\frac{1}{3}$  on  $\gamma$  and halfway on  $\alpha$ 
function  $S_1(p, \alpha, \beta, \gamma)$ 
   $\alpha_2, \gamma_3 \leftarrow (\alpha/2), (\gamma/3)$ 
  if  $(|\alpha| > 2)$  and  $((|\alpha_2| \bmod 2) \equiv 1)$  then
     $\alpha_2 \leftarrow \alpha_2 + \delta(\alpha)$ 
  end if
  if  $(|\gamma| > 2)$  and  $((|\gamma_3| \bmod 2) \equiv 1)$  then
     $\gamma_3 \leftarrow \gamma_3 + \delta(\gamma)$ 
  end if

  yield Gilbert3D( $p$ ,
     $\gamma_3, \alpha_2, \beta$ )

  yield Gilbert3D( $p + \gamma_3$ ,
     $\alpha, \beta, (\gamma - \gamma_3)$ )

  yield Gilbert3D( $p + \alpha - \delta(\alpha) + \gamma_3 - \delta(\gamma)$ ,
     $\gamma_3, (\alpha - \alpha_2), \beta$ )

end function
```

Procedure 5 S_2 -Split function (eccentric split)

```
# split  $\frac{1}{3}$  on  $\beta$  and halfway on  $\alpha$ 
function  $S_2(p, \alpha, \beta, \gamma)$ 
   $\alpha_2, \beta_3 \leftarrow (\alpha/2), (\beta/3)$ 
  if  $(|\alpha| > 2)$  and  $((|\alpha_2| \bmod 2) \equiv 1)$  then
     $\alpha_2 \leftarrow \alpha_2 + \delta(\alpha)$ 
  end if
  if  $(|\beta| > 2)$  and  $((|\beta_3| \bmod 2) \equiv 1)$  then
     $\beta_3 \leftarrow \beta_3 + \delta(\beta)$ 
  end if

  yield Gilbert3D( $p$ ,
     $\beta_3, \gamma, \alpha_2$ )

  yield Gilbert3D( $p + \beta_3$ ,
     $\alpha, (\beta - \beta_3), \gamma$ )

  yield Gilbert3D( $p + \alpha - \delta(\alpha) + \beta_3 - \delta(\beta)$ ,
     $-\beta_3, \gamma, -\alpha$ )

end function
```

A.3. $w \gg h \sim d$

Take $w = \sigma s$ and $h = d = s$. We commit to splitting the width dimension in half, subdividing the original volume into two equal volumes.

The defect of the original volume is $\lambda(w, h, d) = \sigma$.

If $w > 2h = 2s$, then $\min(\frac{w}{2}, h, d) = s$ and we have an average defect $\lambda_s(V_1) = \frac{\sigma}{2}$. Intuitively, this is validation that if the width is more than twice the length of the depth

and height, we make progress if we split the width in half and recursively process each sub-cuboid.

If $w < 2h$ but $\min(\frac{w}{2}, h, d) = \frac{w}{2}$, the average defect $\lambda_s(V_1) = 2(\frac{1}{2})(\frac{\frac{\sigma}{2} \cdot s^3}{(\frac{\sigma}{2}s)^3}) = \frac{4}{\sigma^2}$. We want to reduce the average defect relative to the original defect, so:

Procedure 6 J_0 -Split function

$|\gamma|$ even**function** $J_0(p, \alpha, \beta, \gamma)$

$$\alpha_2, \beta_2, \gamma_2 \leftarrow (\alpha/2), (\beta/2), (\gamma/2)$$

prefer initial block even

$$\alpha_2 = \alpha_2 + \delta(\alpha) \text{ if } (|\alpha_2| > 2) \text{ and } (|\alpha_2| \bmod 2 \equiv 1)$$

$$\beta_2 = \beta_2 + \delta(\beta) \text{ if } (|\beta_2| > 2) \text{ and } (|\beta_2| \bmod 2 \equiv 1)$$

$$\gamma_2 = \gamma_2 + \delta(\gamma) \text{ if } (|\gamma_2| > 2) \text{ and } (|\gamma_2| \bmod 2 \equiv 1)$$

yield Gilbert3D(p ,

$$\beta_2, \gamma_2, \alpha_2)$$

yield Gilbert3D($p + \beta_2$,

$$\gamma, \alpha_2, \beta - \beta_2)$$

yield Gilbert3D($p + \beta_2 - \delta(\beta_2) + \gamma - \delta(\gamma)$,

$$\alpha, -\beta_2, -(\gamma - \gamma_2))$$

yield Gilbert3D($p + \alpha - \delta(\alpha) + \beta_2 + \gamma - \delta(\gamma)$,

$$-\gamma, -(\alpha - \alpha_2), (\beta - \beta_2))$$

yield Gilbert3D($p + \alpha - \delta(\alpha) + \beta_2 - \delta(\beta)$,

$$-\beta_2, \gamma_2, -(\alpha - \alpha_2))$$

end function

Procedure 7 J_1 -Split function

$|\gamma|$ odd, one of $|\alpha|$ or $|\beta|$ even**function** $J_1(p, \alpha, \beta, \gamma)$

$$\alpha_2, \beta_2, \gamma_2 \leftarrow (\alpha/2), (\beta/2), (\gamma/2)$$

prefer β_2, γ_2 even but force α_2 odd

$$\alpha_2 = \alpha_2 + \delta(\alpha) \text{ if } (|\alpha_2| > 2) \text{ and } (|\alpha_2| \bmod 2 \equiv 0)$$

$$\beta_2 = \beta_2 + \delta(\beta) \text{ if } (|\beta_2| > 2) \text{ and } (|\beta_2| \bmod 2 \equiv 1)$$

$$\gamma_2 = \gamma_2 + \delta(\gamma) \text{ if } (|\gamma_2| > 2) \text{ and } (|\gamma_2| \bmod 2 \equiv 1)$$

yield Gilbert3D(p ,

$$\gamma_2, \alpha_2, \beta_2)$$

yield Gilbert3D($p + \gamma_2$,

$$\beta, \gamma - \gamma_2, \alpha_2)$$

yield Gilbert3D($p + \gamma_2 - \delta(\gamma) + \beta - \delta(\beta)$,

$$\alpha, -(\beta - \beta_2), -\gamma_2)$$

yield Gilbert3D($p + \alpha - \delta(\alpha) + \beta - \delta(\beta) + \gamma_2 - \delta(\gamma)$,

$$\beta, \gamma - \gamma_2, -(\alpha - \alpha_2))$$

yield Gilbert3D($p + \alpha - \delta(\alpha) + \gamma_2 - \delta(\gamma)$,

$$-\gamma_2, -(\alpha - \alpha_2), \beta_2)$$

end function

Procedure 8 J_2 -Split function

$|\alpha|, |\beta|, |\gamma|$ odd**function** $J_2(p, \alpha, \beta, \gamma)$

$$\alpha_2, \beta_2, \gamma_2 \leftarrow (\alpha/2), (\beta/2), (\gamma/2)$$

prefer β_2, γ_2 even but force α_2 odd

$$\alpha_2 = \alpha_2 + \delta(\alpha) \text{ if } (|\alpha_2| > 2) \text{ and } (|\alpha_2| \bmod 2 \equiv 0)$$

$$\beta_2 = \beta_2 + \delta(\beta) \text{ if } (|\beta_2| > 2) \text{ and } (|\beta_2| \bmod 2 \equiv 1)$$

$$\gamma_2 = \gamma_2 + \delta(\gamma) \text{ if } (|\gamma_2| > 2) \text{ and } (|\gamma_2| \bmod 2 \equiv 1)$$

yield Gilbert3D(p ,

$$\beta_2, \gamma, \alpha_2)$$

yield Gilbert3D($p + \beta_2$,

$$\gamma_2, \alpha, (\beta - \beta_2))$$

yield Gilbert3D($p + \beta_2 + \gamma_2$,

$$\alpha, (\beta - \beta_2), (\gamma - \gamma_2))$$

yield Gilbert3D($p + \alpha - \delta(\alpha) + \beta_2 - \delta(\beta) + \gamma_2$,

$$-\beta_2, (\gamma - \gamma_2), -(\alpha - \delta(\alpha)))$$

yield Gilbert3D($p + \alpha - \delta(\alpha) + \gamma_2 - \delta(\gamma)$,

$$-\gamma_2, -(\alpha - \alpha_2), \beta_2)$$

end function

For this case, $\sigma = \frac{5}{3}$ and $\rho = \frac{1}{2}$. That is, we split the w axis by half when the width axis exceed $\frac{5}{3}$ of both the depth, d , and height, h , dimension.

A.4. $h \gg w \sim d$

$$\begin{aligned} & \lambda_s(V_1) < \lambda(V_0) \\ \rightarrow & \frac{4}{\sigma^2} < \sigma \\ \rightarrow & \sigma > 4^{\frac{1}{3}} \\ \rightarrow & \frac{5}{3} > \sigma > 4^{\frac{1}{3}} = 1.58740 \dots \end{aligned}$$

We've chosen the constant $\frac{5}{3}$ as a simple fraction above $4^{\frac{1}{3}}$ to know when to split w .

Algorithm 9 3D Generalized Hilbert Function (Gilbert3D)

```
#  $p, \alpha, \beta, \gamma \in \mathbb{Z}^3$ 
function GILBERT3D( $p, \alpha, \beta, \gamma$ )

    # Parity of dimensions
     $\alpha_0 \leftarrow (|\alpha| \bmod 2)$ 
     $\beta_0 \leftarrow (|\beta| \bmod 2)$ 
     $\gamma_0 \leftarrow (|\gamma| \bmod 2)$ 

    # Base cases
    if  $((|\alpha| \equiv 2) \text{ and } (|\beta| \equiv 2) \text{ and } (|\gamma| \equiv 2))$ 
        return Hilbert3D( $p, \alpha, \beta, \gamma$ )
    return Gilbert2D( $p, \beta, \gamma$ ) if  $(|\alpha| \equiv 1)$ 
    return Gilbert2D( $p, \alpha, \gamma$ ) if  $(|\beta| \equiv 1)$ 
    return Gilbert2D( $p, \alpha, \beta$ ) if  $(|\gamma| \equiv 1)$ 

    # Eccentric cases
    if  $(3|\alpha| > 5|\beta|) \text{ and } (3|\alpha| > 5|\gamma|)$ 
        return  $S_0(p, \alpha, \beta, \gamma)$ 
    if  $(2|\beta| > 3|\gamma|) \text{ or } (2|\beta| > 3|\alpha|)$ 
        return  $S_2(p, \alpha, \beta, \gamma)$ 
    if  $(2|\gamma| > 3|\beta|)$ 
        return  $S_1(p, \alpha, \beta, \gamma)$ 

    # Bulk recursion
    return  $J_0(p, \alpha, \beta, \gamma)$  if  $(\gamma_0 \equiv 0)$ 
    return  $J_1(p, \alpha, \beta, \gamma)$  if  $(\alpha_0 \equiv 0) \text{ or } (\beta_0 \equiv 0)$ 
    return  $J_2(p, \alpha, \beta, \gamma)$ 
end function
```
