A Generalized 2D and 3D Hilbert Curve

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Abstract

The two and three dimensional Hilbert curves are fractal space filling curves that map the unit interval to the unit square or unit cube while preserving a notion of closeness, or locality, after the map. We present the Gilbert curve, a conceptually straight forward generalization of the Hilbert curve, that works on arbitrary rectangular regions, overcoming the limitation of the Hilbert curve that requires side lengths be exact powers of two. The construction provides reasonable worst case run-time guarantees for random access lookups for both two and three dimensions. We provide experiments showing comparable quality in locality of the Gilbert curve to the Hilbert curve for a range of cuboid regions and discuss some limitations of the construction algorithm.

Generalized Hilbert Curves

In a website application and scanned note [1], Tautenhahn provided the basis for a generalized 2D space filling curve. Tautenhahn also included policies for when to subdivide regions preferentially in only one dimension that help to create more *harmonious* curve realizations. Tautenhahn's exploration details the parity arguments necessary for when a subdivision scheme can be employed without creating diagonal moves (*notches*).

In this paper, we extend Tautenhahn's ideas to create a 2D and 3D generalized Hilbert curve. We further extend Tautenhahn's core ideas on when to use alternate subdivision schemes when one length is much larger than the rest, what we call *eccentric cases*, and apply them to a 3D generalized Hilbert curve. Tautenhahn's unpublished reasoning behind the constants used in the 2d eccentric split case are briefly discussed later [1].

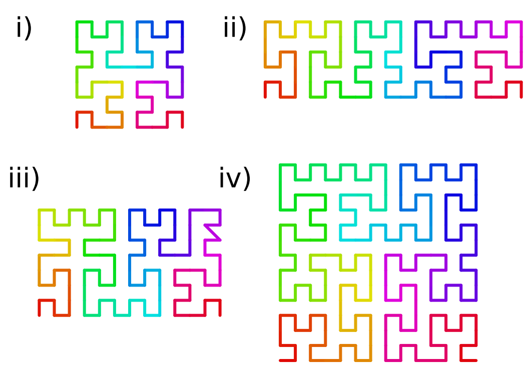

Figure 1: *2D Gilbert curves for i)* 8×8*, ii)* 18×6*, iv)* 13×8 *(with notch), iv)* 14×14

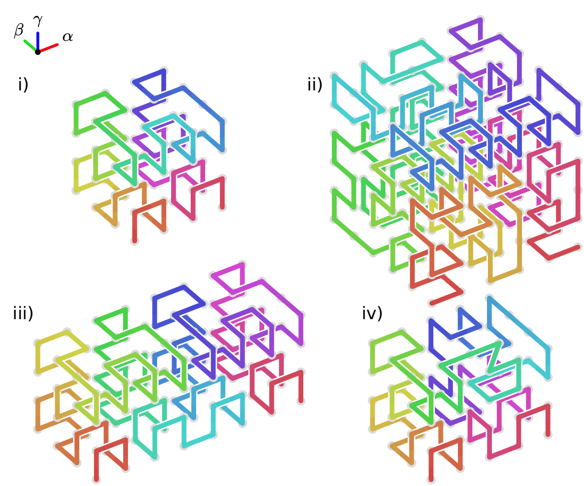

Figure 2: *3D Gilbert curves for i)* $4 \times 4 \times 4$*, ii)* $6 \times 6 \times 6$*, iii)* $8 \times 4 \times 4$*, iv)* $5 \times 4 \times 4$ *(with notch)*

Path Possible		Volume *even*	Volume *odd*
$\lvert\alpha\rvert$	*even*	Yes	Yes
	odd	**No**	Yes

Figure 3: *$|\alpha|$ is the distance of endpoints. A Hamiltonian path is possible only when $|\alpha|$ is even or both $|\alpha|$ and the volume are odd.*

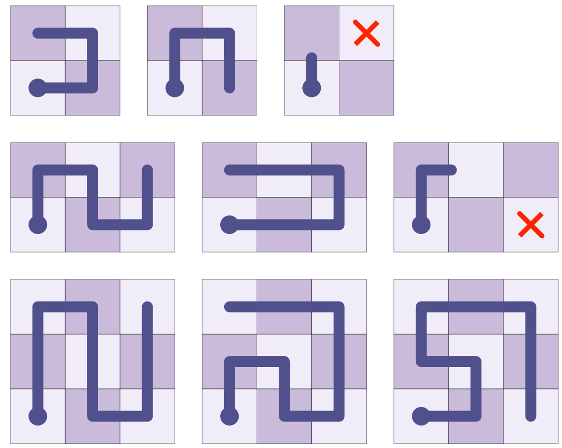

Figure 4: *Examples of Hamiltonian paths for small grid sizes. A red 'x' corresponds to no possible path for the chosen endpoints.*

Valid Paths from Grid Parity

The feasibility of determining whether there exists a non intersecting path, called a *Hamiltonian path*, connecting endpoints on the corners in a rectangular cuboid grid region can be accomplished through parity arguments. Label grid cell points in a volume as 0 or 1, alternating between labels with every axis-aligned single step move. Any Hamiltonian path that ends at one of the three remaining corners has to have the same parity as the starting point if the volume is odd, or different parity if the volume is even.

For a path starting at $(0, 0, 0)$ and ending $|\alpha|$ steps in one of the axis-aligned dimensions, then Table 3 enumerates this condition under which a valid path is possible. Figure 4 illustrates this for starting position $(0, 0)$ with areas (2×2), (3×2) and (3×3), where a red cross indicating a precluded endpoint.

Assuming a curve starts from position $p_s = (0, 0, 0)$ and has proposed endpoint at $p_e = ((w - 1), 0, 0)$, with a cuboid region as $\alpha = (w, 0, 0), \beta = (0, h, 0), \gamma = (0, 0, d)$. We focus on optimizing for providing a notch-free path when all side lengths are even and allow notches when $|\alpha|$ is odd in 2D or when one of the side lengths is odd in 3D.

The 2D Gilbert curve limits the number of notches to one. The 3D Gilbert curve's subdivision strategy creates a notch when the distance between endpoints is odd, potentially creating more than one notch.

In both the 2D and 3D case, when the original side lengths are all even, no notches will be present.

Subdivision Strategies

For both the 2D and 3D Gilbert curve, a subdivision template is chosen to recursively partition the region. Figure 5 shows the 2D template and figure **??** shows the 3D template.

When dividing into sub-regions, side lengths are chosen to be integral and with even side lengths preferred for the first subdivided region. A path is recursively chosen for each of the subregions and, after resolution, endpoints are connected.

When regions have a side length that is much larger than the rest, another subdivision scheme is used.

[1]Through personal communication with the authors, Tautenhahn kindly provided the reasoning for the constants used in the eccentric split

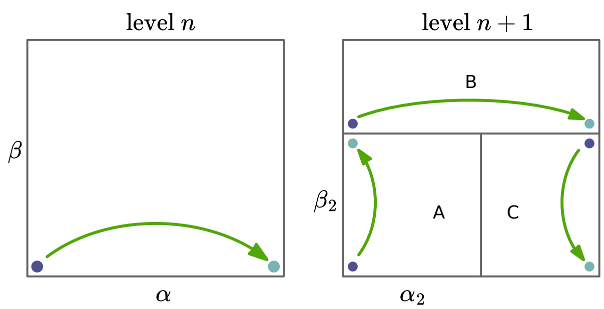

Figure 5: *Subdivision strategy for the 2D Gilbert curve.*

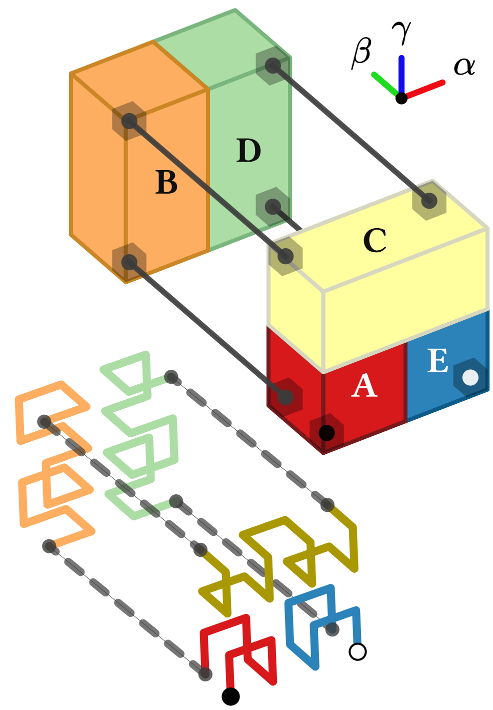

Figure 6: *The main subdivision strategy for the 3D Gilbert curve.*

We call these cases *eccentric subdivisions* as the larger side length makes the region lopsided.

If one side length is much larger than the rest, both the 2D and 3D Gilbert curve split the region in two. For the 3D case, if one side length is much smaller than the rest but with the remaining lengths roughly equal, the region is subdivided as if it were a 2D Gilbert curve, taking the smaller side length as the "flat" dimension.

The eccentric subdivision schemes provide a more visually pleasing solution as regimented subdivisions might paths that are long and skinny or wide and squat. We will briefly go over motivations for choosing how to choose these constants and why the particular value of constants was chosen in the next section.

For the 2D Gilbert curve, if $|\alpha|$ is odd, there will be a diagonal move, or *notch*, for the resulting path. The subdivision scheme will keep the notch in a single region, limiting the notch count to one.

For the 3D Gilbert curve, if any of the subdivided regions has endpoints that lay in a direction with odd length, a notch will appear. This means the number of notches for the 3D Gilbert curve isn't limited to one.

Algorithm **??** shows the pseudo-code for computing the 2D Gilbert curve. Note that α and β are taken to be vectors in 3D, where the third dimension can be ignored if a purely 2D curve is desired. The generalization to 3D allows the Gilbert2D function to be used unaltered when the 3D Gilbert curve needs to trace out in-plane sub-curves.

The $\delta(\cdot)$ function returns one of the six directional vectors indicating which of the major signed axis aligned directions the input vector points to $((\pm 1, 0, 0), (0, \pm 1, 0), (0, 0, \pm 1))$.

Algorithm **??** assumes standard Euclidean two norm ($|v| = \sqrt{v_0^2 + v_1^2 + v_2^2}$) and abuses notation by allowing scalar to vector multiplication ($i \in \mathbb{Z}, v \in \mathbb{Z}^3, i \cdot v \longrightarrow (i \cdot v_0, i \cdot v_1, i \cdot v_2)$). where the context is clear.

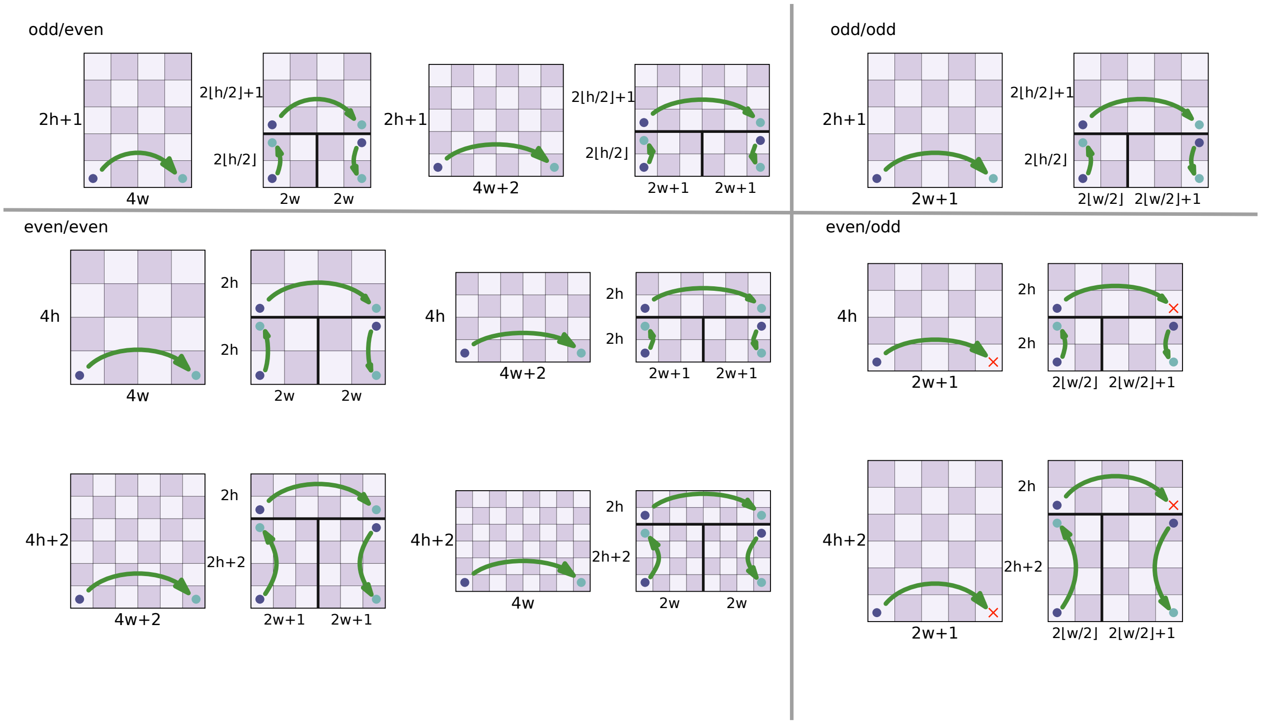

Figure 7: *Enumeration of the subdivision template depending on different parities of α and β dimensions.*

 # $p, \alpha, \beta \in \mathbb{Z}^3$
function GILBERT2D(p, α, β)
  $\alpha_2, \beta_2 = \mathrm{div}(\alpha, 2), \mathrm{div}(\beta, 2)$
  if $(|\beta| \equiv 1)$ **then**
    yield $p + i \cdot \delta(\alpha)$ **forall** $i \in |\alpha|$
  else if $(|\alpha| \equiv 1)$ **then**
    yield $p + i \cdot \delta(\alpha)$ **forall** $i \in |\beta|$
  else if $(2|\alpha| > 3|\beta|)$ **then**
    if $(|\alpha_2| > 2)$ and $(|\alpha_2| \bmod 2 \equiv 1)$ **then**
      $\alpha_2 \leftarrow \alpha_2 + \delta(\alpha)$
    end if
    yield Gilbert2D(p, α_2, β)
    yield Gilbert2D($p + \alpha_2$, $\alpha - \alpha_2$, β)
  else
    if $(|\beta_2| > 2)$ and $(|\beta_2| \bmod 2 \equiv 1)$ **then**
      $\beta_2 \leftarrow \beta_2 + \delta(\beta)$
    end if
    yield Gilbert2D(p,
          β_2, α_2)
    yield Gilbert2D($p + \beta_2$,
          α, $(\beta - \beta_2)$)
    yield Gilbert2D($p + \alpha - \delta(\alpha) + \beta_2 - \delta(\beta)$,
          β_2, $-(\alpha - \alpha_2)$)
  end if
end function

 # $p, \alpha, \beta, \gamma \in \mathbb{Z}^3$
function GILBERT3D(p, α, β, γ)

  # Parity of dimensions
  $\alpha_0 \leftarrow (|\alpha| \bmod 2)$
  $\beta_0 \leftarrow (|\beta| \bmod 2)$
  $\gamma_0 \leftarrow (|\gamma| \bmod 2)$

  # Base cases
  if $((|\alpha| \equiv 2)$ and $(|\beta| \equiv 2)$ and $(|\gamma| \equiv 2))$
    return Hilbert3D(p,α,β,γ)
  return Gilbert2D(p,β,γ) **if** $(|\alpha| \equiv 1)$
  return Gilbert2D(p,α,γ) **if** $(|\beta| \equiv 1)$
  return Gilbert2D(p,α,β) **if** $(|\gamma| \equiv 1)$

  # Eccentric cases
  if $(3|\alpha| > 5|\beta|)$ and $(3|\alpha| > 5|\gamma|)$
    return $S_0(p,\alpha,\beta,\gamma)$
  if $(2|\beta| > 3|\gamma|)$ or $(2|\beta| > 3|\alpha|)$
    return $S_2(p,\alpha,\beta,\gamma)$
  if $(2|\gamma| > 3|\beta|)$
    return $S_1(p,\alpha,\beta,\gamma)$

  # Bulk recursion
  return $J_0(p,\alpha,\beta,\gamma)$ **if** $(\gamma_0 \equiv 0)$
  return $J_1(p,\alpha,\beta,\gamma)$ **if** $(\alpha_0 \equiv 0)$or$(\beta_0 \equiv 0)$
  return $J_2(p,\alpha,\beta,\gamma)$
end function

Eccentric Subdivision

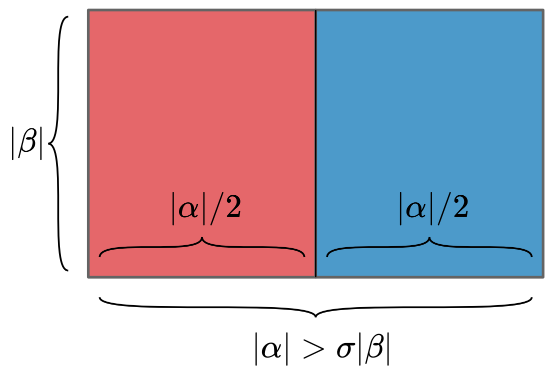

Figure 8: *When the length of the width-like dimension ($|\alpha|$) gets larger than a threshold of the height-like dimension ($\sigma|\beta|$), then the width like dimension is split into two.*

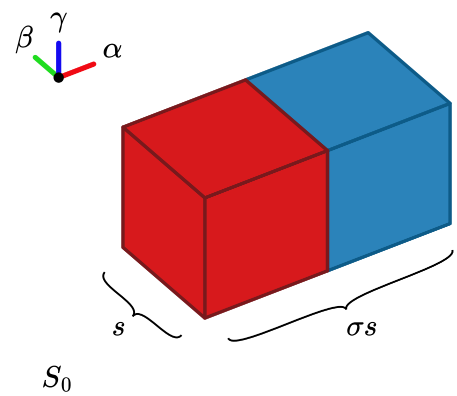

Figure 9: *An S_0 eccentric subdivision showing the threshold, σ, used to determine when we should use this split.*

When side lengths become too lopsided, an alternate subdivision scheme is chosen to split along the larger side length or lengths.

For each eccentric subdivision, a proportion threshold needs to be chosen to determine when a side lengths become too lopsided to warrant an alternate subdivision. Here we motivate the eccentric subdivision and the constants chosen for it.

Consider a *defect* function, $\lambda_d : \mathbb{N}^d \mapsto \mathbb{N}$, that measures the area or volume relative to what the area or volume would be if it just the minimum side length were taken:

$$\lambda_2(|\alpha|, |\beta|) = \frac{|\alpha| \cdot |\beta|}{\min(|\alpha|, |\beta|)^2}$$
$$\lambda_3(|\alpha|, |\beta|, |\gamma|) = \frac{|\alpha| \cdot |\beta| \cdot |\gamma|}{\min(|\alpha|, |\beta|, |\gamma|)^3}$$

If there is a disjoint subdivision of a volume V_0 to $V_1 = (V_{0,0}, V_{0,1}, \ldots, V_{0,m-1})$, $V_0 = \cup_k V_{0,k}$, define the *average defect* of the subdivided volume to be the sum of defects weighted by their proportional volumee:

$$\lambda_s(V_1) = \sum_k \frac{\mathrm{Vol}(V_{0,k})}{\mathrm{Vol}(V_0)} \cdot \lambda(V_{0,k})$$

The defect gives a coarse idea of how lopsided or *eccentric* a cuboid region is. If the defect is too high, we might want to split the larger sides while keeping the smaller sides the same size.

Reducing the average defect attempts to make each subdivided cuboid more cube-like. When subdivided cuboids are more cube-like, we say that the subdivided regions are more *harmonious*. The same idea is applied for 2D rectangles and rectangular regions that are more square like are said to be more harmonious.

We work out the threshold value for the eccentric split of the 2D case. Threshold values for the 3D case are not provided in this paper.

For the 2D Gilbert curve, if the α side length is significantly longer than the β side length, we want to subdivide the rectangle into two nearly equal regions and recursively find a Gilbert curve in each region. We

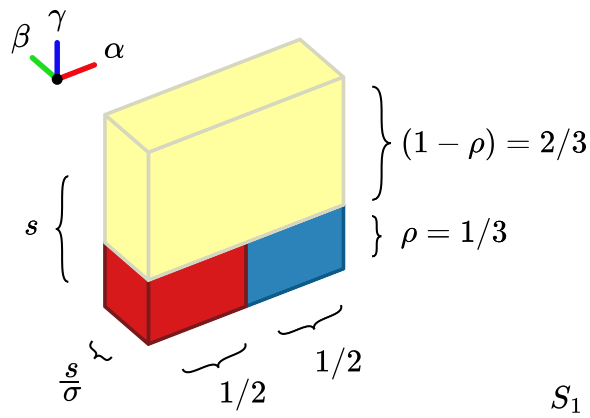

Figure 10: *An S_1 eccentric subdivision showing the threshold, σ, used to determine when we should use this split and the split point ratio, ρ, to reduce the defect for a more harmonious subdivision.*

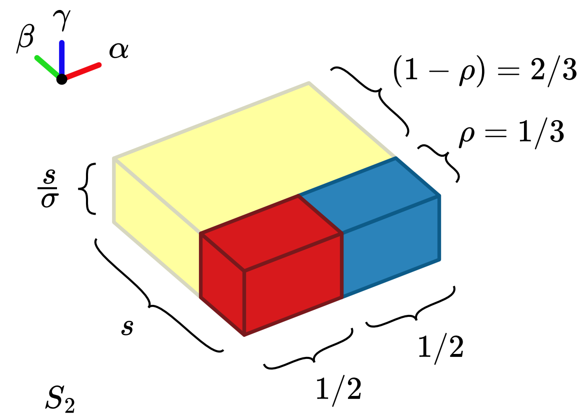

Figure 11: *An S_1 eccentric subdivision showing the threshold, σ, used to determine when we should use this split and the split point ratio, ρ, to reduce the defect for a more harmonious subdivision.*

will justify the constant $(3/2)$ as the ratio threshold to split on using an argument originally developed by L. Tautenhahn [2].

The defect of a rectangle of side length $|\alpha|$ and $|\beta|$ is, with $|\alpha| > |\beta|$:

$$\begin{aligned} \lambda_2(|\alpha|, |\beta|) &= |\alpha| \cdot |\beta| / |\beta|^2 \\ &= |\alpha| / |\beta| \end{aligned}$$

After a subdivision, if we assume $|\alpha| < 2|\beta|$, the defect is:

$$\begin{aligned} \lambda_2(|\alpha|/2, |\beta|) &= \frac{|\alpha|}{2} \cdot |\beta| / (\frac{|\alpha|}{2})^2 \\ &= 2|\beta| / |\alpha| \end{aligned}$$

We're looking for the condition when there's a defect reduction, so

$$\begin{aligned} & \lambda_2(|\alpha|/2, |\beta|) < \lambda_2(|\alpha|, |\beta|) \\ \rightarrow \quad & 2|\beta|/|\alpha| < |\alpha|/|\beta| \\ \rightarrow \quad & \sqrt{2} < |\alpha|/|\beta| \end{aligned}$$

Since $\sqrt{2} \approx 1.4142 < (3/2)$, if we choose $|\alpha|/|\beta| > (3/2)$ we can be assured a defect reduction.

In the case when $|\alpha| > 2|\beta|$, it is easy to verify that the defect is always reduced [3].

References

[1] L. Tautenhahn. "Draw a space-filling curve of arbitrary size." 2003. https://lutanho.net/pic2html/draw_sfc.html.

[2] Given to us through personal communication with L. Tautenhahn

[3] The ratio of the defects before and after the split is $(1/2)$