A3-Assignment-CH3.1

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> Info
>
> All web searches referenced were done with Perplexity app

7.

> Question
>
> Prove Inequality (1).
>
> $$|r - c_n| \leq \frac{b_0 - a_0}{2^{n+1}}$$

> Info
>
> Information from our Text Book
> If the bisection algorithm is now applied and if the computed quantities are denoted by $a_0, b_0, c_0, a_1, b_1, c_1$, and so on, then by the same reasoning,
>
> $$|r - c_n| \leq \frac{b_n - a_n}{2} \quad (n \geq 0)$$
>
> Since the widths of the intervals are divided by 2 in each step, we conclude that
>
> $$|r - c_n| \leq \frac{b_0 - a_0}{2^{n+1}}$$
>
> To summarize, a theorem can be written as follows:
> Bisection Method Theorem: If the bisection algorithm is applied to a continuous function f on an interval $[a, b]$, where $f(a)f(b) < 0$, then, after n steps, an approximate root will have been computed with error at most:
>
> $$\frac{b - a}{2^{n+1}}$$
>
> If an error tolerance has been prescribed in advance, it is possible to determine the number of

steps required in the bisection method. Suppose that we want:

$$|r - c_n| < \epsilon$$

Then it is necessary to solve the following inequality for n:

$$\frac{b-a}{2^{n+1}} < \epsilon$$

By taking logarithms (with any convenient base), we obtain:

$$n > \frac{\log(b-a) - \log(2\epsilon)}{\log 2}$$

Solution

$$|r - c_n| \leq \frac{b_0 - a_0}{2^{n+1}}$$

- The bisection method produces a sequence of intervals $[a_n, b_n]$ that contain the root r, with the midpoint given by $c_n = \frac{a_n+b_n}{2}$. Since r lies in the interval, we have $|r - c_n| \leq \frac{b_n - a_n}{2}$.
- At each step, the interval length is halved, so we get $b_n - a_n = \frac{b_0 - a_0}{2^n}$.
- Substituting $b_n - a_n = \frac{b_0-a_0}{2^n}$ into the error bound gives us $|r - c_n| \leq \frac{1}{2} \cdot \frac{b_0 - a_0}{2^n} = \frac{b_0-a_0}{2^{n+1}}$.
- Sooo, we have proven that $|r - c_n| \leq \frac{b_0-a_0}{2^{n+1}}$.

8.

Question

If $a = 0.1$ and $b = 1.0$, how many steps of the bisection method are needed to determine the root with an error of at most $\frac{1}{2} \times 10^{-8}$?

Solution

- First, what do i have here:
  - Initial interval: $a = 0.1$ and $b = 1.0$, so the width is $b - a = 0.9$.
  - Error tolerance that we want: $\epsilon = \frac{1}{2} \times 10^{-8}$.
- With the bisection method, after n steps the error is bounded by $|r - c_n| \leq \frac{b_0-a_0}{2^{n+1}}$.

- To guarantee an error of at most ϵ, we require $\frac{0.9}{2^{n+1}} \leq \frac{1}{2} \times 10^{-8}$.
- This inequality can be rearranged as follows:
  1. Multiply both sides by 2^{n+1}:
     $0.9 \leq \frac{1}{2} \times 10^{-8} \cdot 2^{n+1}$.
  2. Multiply both sides by 2:
     $1.8 \leq 10^{-8} \cdot 2^{n+1}$.
  3. Rearranging gives:
     $2^{n+1} \geq \frac{1.8}{10^{-8}} = 1.8 \times 10^{8}$.
- Taking logarithms (base 2):
  $n+1 \geq \log_2(1.8 \times 10^8) = \log_2(1.8) + \log_2(10^8)$.
- I remember that:
  - $\log_2(10^8) = 8\log_2(10) \approx 8 \times 3.32193 \approx 26.57544$
  - $\log_2(1.8) \approx 0.848$
- Thus,
  $n+1 \geq 26.57544 + 0.848 \approx 27.42344$.

Since $n+1$ must be an integer, we have
$n+1 \geq 28 \quad \Longrightarrow \quad n \geq 27$.

> ✓ Success
>
> Soo, we need at least 27 steps

14.

> ⓘ Question
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> Denote the successive intervals that arise in the bisection method by $[a_0, b_0]$, $[a_1, b_1]$, $[a_2, b_2]$, and so on. Show that
>
> - a. $a_0 \leq a_1 \leq a_2 \leq \dots$ and $b_0 \geq b_1 \geq b_2 \geq \dots$
> - b. $b_n - a_n = 2^{-n}(b_0 - a_0)$
> - c. $a_n b_n + a_{n-1} b_{n-1} = a_{n-1} b_n + a_n b_{n-1}$ for all n.

(a) $a_0 \leq a_1 \leq a_2 \leq \dots$ and $b_0 \geq b_1 \geq b_2 \geq \dots$

> ⓘ Info

Searching online, i found out that this is called the monotonicity of Endpoints

At each step of the bisection method an interval is chosen that is a subinterval of the previous one. That is, if we start with the interval $[a_n, b_n]$ and compute the midpoint $c_n = \frac{a_n+b_n}{2}$, then either

- the new interval is $[a_n, c_n]$, or
- it is $[c_n, b_n]$.

In the first case, the left endpoint remains the same ($a_{n+1} = a_n$) and the right endpoint becomes c_n with $a_n < c_n \leq b_n$. In the second case, the right endpoint remains the same ($b_{n+1} = b_n$) and the left endpoint becomes c_n with $a_n \leq c_n < b_n$. Thus, in every step we have

$a_n \leq a_{n+1}$ and $b_{n+1} \leq b_n$.

✓ Success

This shows that the sequence $a_0, a_1, a_2, \ldots$ is non-decreasing and $b_0, b_1, b_2, \ldots$ is non-increasing.

(b) $b_n - a_n = 2^{-n}(b_0 - a_0)$

ⓘ Info

Searching online, i found out that this is called the Length of the Intervals

- The initial interval has length `$b_0-a_0$`. At each step the interval is halved, so that after one step $b_1 - a_1 = \frac{b_0-a_0}{2}$
- By induction, after n steps the length of the interval is $b_n - a_n = \frac{b_0-a_0}{2^n}$.

(c) $a_n b_n + a_{n-1} b_{n-1} = a_{n-1} b_n + a_n b_{n-1}$ for all n.

ⓘ Info

Searching online, i found out that this is called the Endpoint Product Identity

- We need to show that for all n
  $A_n b_n + a_{n-1}b_{n-1} = a_{n-1}b_n + a_n b_{n-1}$
- Subtract the right-hand side from the left-hand side:
  $A_n b_n + a_{n-1}b_{n-1} - a_{n-1}b_n - a_n b_{n-1}$
- Group the terms as follows:
  $\left(a_n b_n - a_n b_{n-1}\right) - \left(a_{n-1}b_n - a_{n-1}b_{n-1}\right) = a_n(b_n - b_{n-1}) - a_{n-1}(b_n - b_{n-1}).$
- Factor out $b_n - b_{n-1}$:
  $(a_n - a_{n-1})(b_n - b_{n-1}).$
- In the bisection method, in each iteration only one endpoint changes (either $a_n = a_{n-1}$ or $b_n = b_{n-1}$), so one of the factors is zero. Hence,
  $(a_n - a_{n-1})(b_n - b_{n-1}) = 0,$
- Which means
  $A_n b_n + a_{n-1}b_{n-1} = a_{n-1}b_n + a_n b_{n-1}.$

15.

> Question
>
> (Continuation) Can $a_0 = a_1 = a_2 = \ldots$ happen?

I searched online that constant left endpoints are possible.

It happens if at every step the interval selected is of the form $[a_n, c_n]$ instead of $[c_n, b_n]$.
In that case the left endpoint stays the same throughout the iterations (i.e. $a_{n+1} = a_n$ for all n). Here the, function values meet $f(a_n)f(c_n) < 0$ at every step, which makes the algorithm to choose the left half of the interval every time.

Soo, constant Left endpoints like a0=a1=a3=.... are possible

16.

> Question
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> (Continuation) Let $c_n = (a_n + b_n)/2$. Show that >
>
> $$\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$$

Define the midpoint $c_n = \frac{a_n+b_n}{2}$. We have already shown that the sequence a_n is non-decreasing and the sequence b_n is non-increasing. Since each a_n is bounded above (by any b_n) and each b_n is bounded below (by any a_n), both sequences converge. Denote
$\lim_{n\to\infty} a_n = \ell_a \quad \text{and} \quad \lim_{n\to\infty} b_n = \ell_b.$

- From part (b) we know
  $B_n - a_n = \frac{b_0-a_0}{2^n}$
- Taking the limit as $n \to \infty$ yields
  $\lim_{n\to\infty}(b_n - a_n) = 0,$
- So that
  $\ell_b - \ell_a = 0 \quad \text{or} \quad \ell_a = \ell_b$
- Since $c_n = \frac{a_n+b_n}{2}$, its limit is
  $lim_{n\to\infty} c_n = \frac{\ell_a+\ell_b}{2} = \ell_a.$
- So, we conclude that
  $\lim_{n\to\infty} c_n = \lim_{n\to\infty} a_n = \lim_{n\to\infty} b_n.$

> **Info**
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> Searching online, i found out that this is called the Convergence of the Endpoints and Midpoints

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22.

> **Question**
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> If the bisection method is applied with starting interval $[2^m, 2^{m+1}]$, where m is a positive or negative integer, how many steps should be taken to compute the root to full machine precision on a 32-bit word-length computer?

Solution

- We are given a starting interval of the form $[2^m, 2^{m+1}]$, where m is any integer. The width of this interval is
  $b_0 - a_0 = 2^{m+1} - 2^m = 2^m.$
- After n steps, the error bound is
  $|r - c_n| \leq \frac{2^m}{2^{n+1}} = 2^{m-n-1}.$
- On a 32-bit word-length computer (using IEEE single precision), the number is represented with a 24-bit significand (including the implicit bit). This means that for numbers of magnitude about 2^m, the spacing between adjacent representable numbers is approximately
  $2^{m-23}.$

- To achieve full machine precision, the error must be no larger than this spacing. Hence, we require $2^{m-n-1} \leq 2^{m-23}$.
- Canceling 2^m from both sides yields:
  $2^{-n-1} \leq 2^{-23}$.
- Taking logarithms (base 2):
  $-n-1 \leq -23 \quad \Longrightarrow \quad n+1 \geq 23 \quad \Longrightarrow \quad n \geq 22.$

> ✓ **Success**
>
> Sooo, we need 22 steps to compute the root to full machine precision.
