A4-Assignment-CH3.2

John Akujobi - Math 374

1. Verify that when Newton's method is used to compute $\sqrt{R}$ (by solving the equation $x^2 = R$), the sequence of iterates is defined by $x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$

I want to solve $X^2 - R = 0$

- First, for Newton's method we set
  - $f(x) = x^2 - R, \quad f'(x) = 2x$
- Iterating the formula
  - $X_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
  - $= x_n - \frac{x_n^2 - R}{2x_n}$
  - $= \frac{2x_n^2 - (x_n^2 - R)}{2x_n}$
  - $= \frac{x_n^2 + R}{2x_n}$
  - $= \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$

> ✓ Success
> Our formula is gonna be
>
> $$x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$$

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2. (Continuation) Show that if the sequence x_n is defined as in the preceding exercise, then $x_{n+1}^2 - R = \frac{x_n^2 - R}{2x_n}$

- Using the iteration formula:
    - $X_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$
- Square both sides:-
    - $X_{n+1}^2 = \frac{1}{4}\left(x_n^2 + 2R + \frac{R^2}{x_n^2}\right)$
- Subtract (R)
    - $X_{n+1}^2 - R = \frac{1}{4}\left(x_n^2 - 2R + \frac{R^2}{x_n^2}\right)$
    - $= \frac{(x_n^2 - R)^2}{4x_n^2}$

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13. Each of the following functions has $\sqrt[3]{R}$ as a zero for any positive real number R. Determine the formulas for Newton's method for each and any necessary restrictions on the choice for x_0.

b. $b(x) = 1/x^3 - 1/R$

- Find the derivative:
    - $b'(x) = -\frac{3}{x^4}$
- Apply Newton's method:
    - $x_{n+1} = x_n - \frac{b(x_n)}{b'(x_n)}$
    - $= x_n - \frac{\frac{1}{x_n^3} - \frac{1}{R}}{-\frac{3}{x_n^4}}$
    - $= x_n + \frac{x_n^4}{3}\left(\frac{1}{x_n^3} - \frac{1}{R}\right).$
    - $= x_n + \frac{x_n}{3}\left(1 - \frac{x_n^3}{R}\right).$
    - $= x_n + \frac{x_n}{3} - \frac{x_n^4}{3R}).$
- Simplified to
  - $= x_n \cdot \frac{4R - x_n^3}{3R}$

✓ Success

So our answer is

$$x_{n+1} = x_n \cdot \frac{4R - x_n^3}{3R}$$

- Restriction: Since the derivative $b'(x) = -3/x^4$ is undefined at x=0, we must choose an initial guess $x_0 \neq 0$. To converge to the positive cube root, choose $x_0 > 0$.

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d. $d(x) = x - R/x^2$

- First, for Newton's method, we need the derivative
  - $d'(x) = 1 + 2R/x^3$
- Using the formula
  - $x_{n+1} = x_n - \frac{d(x_n)}{d'(x_n)}$
  - $= x_n - \frac{x_n - R/x_n^2}{1+2R/x_n^3}$
  - $= x_n - \frac{x_n^3 - R/x_n^2 \cdot x_n^3}{x_n^3+2R}$
  - $= x_n - \frac{x_n^4 - R}{x_n^3+2R} = x_n - \frac{x_n(x_n^3-R)}{x_n^3+2R}$
  - $= \frac{x_n(x_n^3+2R) - x_n(x_n^3-R)}{x_n^3+2R} = \frac{x_n(x_n^3+2R+R-x_n^3)}{x_n^3+2R}$
  - = $\frac{x_n(3R)}{x_n^3+2R}$
  - = $\frac{3Rx_n}{x_n^3+2R}$

✓ Success

Our formula is $X_{n+1} = \frac{3Rx_n}{x_n^3+2R}$

Restrictions:$(x_0 > 0)$.

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14. Determine the formulas for Newton's method for finding a root of the function $f(x) = x - e^{x/l}$. What is the behavior of the iterates?

Getting the formula

- First, get the derivative
  - $f'(x) = 1 - \frac{1}{l}e^{x/l}$
- The Newton's iteration:
  - $x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}$
  - $= x_n - \frac{x_n - e^{x_n/l}}{1-\frac{1}{l}e^{x_n/l}}$

> ✓ Success
> So, our formula is $x_{n+1} = x_n - \frac{x_n - e^{x_n/l}}{1-\frac{1}{l}e^{x_n/l}}$.

- Behavior of the Iterates:
  - For (l > e), two roots exist.
  - Convergence depends on (x_0). For ($l \leq e$), iterates might diverge

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18. Determine Newton's iteration formula for computing the cube root of N/M for nonzero integers N and M.

To compute the cube root of NM\frac{N}{M} (with NN and MM nonzero integers), we solve
$f(x) = x^3 - \frac{N}{M} = 0$,

So that
$f'(x) = 3x^2$

Newton's iteration is:

- $X_{n+1} = x_n - \frac{x_n^3 - \frac{N}{M}}{3x_n^2}$
- $= \frac{3x_n^3 - (x_n^3 - \frac{N}{M})}{3x_n^2}$
- $= \frac{2x_n^3 + \frac{N}{M}}{3x_n^2}$
- Rearranged this to be $\frac{2}{3}\,x_n + \frac{N}{3M\,x_n^2}$.

> ✓ Success

$$x_{n+1} = \frac{2}{3}\,x_n + \frac{N}{3M\,x_n^2}$$

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25. Newton's method for finding $\sqrt{R}$ is $x_{n+1} = \frac{1}{2}\left(x_n + \frac{R}{x_n}\right)$. Perform three iterations of this scheme for computing $\sqrt{2}$, starting with $x_0 = 1$, and of the bisection method for $\sqrt{2}$, starting with interval $[1, 2]$. How many iterations are needed for each method in order to obtain 10^{-6} accuracy?

Newton's Method

Starting with x_0 = 1.

- First iteration;
  - $X_1 = \frac{1}{2}\left(1 + \frac{2}{1}\right)$
  - $= \frac{1}{2}(3)$
  - = 1.5
- Iteration 2:
  - $X_2 = \frac{1}{2}\left(1.5 + \frac{2}{1.5}\right)$
  - $= \frac{1}{2}\left(1.5 + 1.33333\right)$
  - $\approx \frac{2.83333}{2}$
  - = 1.41667
- Iteration 3:
  - $X_3 = \frac{1}{2}\left(1.41667 + \frac{2}{1.41667}\right)$.
  - Since $\frac{2}{1.41667} \approx 1.41176$,
  - then $X_3 \approx \frac{1.41667+1.41176}{2}$
  - = 1.414215$.
- Iteration 4:
  This one will produce an answer accurate to better than 10^{-6}. The error will be on the

order of 10^{-9}. A lot better that what i need.

> ✓ Success
>
> So, we need about of Newton's method to guarantee an accuracy of 10^{-6}.

Bisection Method

Interval ([1, 2]), each iteration halves the error.

The bisection method for $\sqrt{2}$ starts with the interval [1,2][1, 2]. At each step the error (half the interval length) is reduced by a factor of 2.
So, ill use logarithms,

- $N > \log_2(10^6)$
- $\approx 6\log_2(10)$
- $\approx 6(3.32193)$
- ≈ 19.93

> ✓ Success
> So, we need at least 20 iterations to get an accuracy of 10^{-6}.