A7-Assignment-CH4.3

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Section 4.3: 3, 5, and 8.b due on 3/31/2025 10:50 AM

3. Derive the approximation formula: $f'(x) \approx \frac{1}{2h}[4f(x+h) - 3f(x) - f(x+2h)]$

- **Goal:** Derive
  $f'(x) \approx \frac{1}{2h}[4f(x+h) - 3f(x) - f(x+2h)]$
- **Expand $f(x+h)$ and $f(x+2h)$ using Taylor series:**
  - For $f(x+h)$:
    - $f(x+h) = f(x) + h, f'(x) + \frac{h^2}{2}, f''(x) + \frac{h^3}{6}, f'''(x) + \mathcal{O}(h^4)$
  - For $f(x+2h)$:
    - $f(x+2h) = f(x) + 2h, f'(x) + 2h^2, f''(x) + \frac{4h^3}{3}, f'''(x) + \mathcal{O}(h^4)$
- **Substitute into the formula:**
  - Write the expression:
    $4f(x+h) - 3f(x) - f(x+2h)$
  - Substitute the series expansions:
    $4\left(f(x) + h, f'(x) + \frac{h^2}{2}, f''(x) + \cdots\right) - 3f(x) - \left(f(x) + 2h, f'(x) + 2h^2, f''(x) + \cdots\right)$
- **Simplify term-by-term:**
  - **$f(x)$ - terms:**
    $4f(x) - 3f(x) - f(x) = 0$
  - **$f'(x)$ -terms:**
    $4h, f'(x) - 2h, f'(x) = 2h, f'(x)$
  - **Higher-order terms:**
    The leading higher-order term is $-\frac{2h^3}{3}, f'''(x) + \mathcal{O}(h^4)$
- **Divide both sides by $2h$:**
  - $\frac{1}{2h}[4f(x+h) - 3f(x) - f(x+2h)] = f'(x) - \frac{h^2}{3}, f'''(x) + \mathcal{O}(h^3)$
- **Result:**
  The approximation formula is proportional to an error of $\mathcal{O}(h^2)$.

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5. Averaging the forward-difference formula: $f'(x) \approx \frac{f(x+h)-f(x)}{h}$ and the backward-difference formula: $f'(x) \approx \frac{f(x)-f(x-h)}{h}$ Each with error term $O(h)$, results in the central-difference formula: $f'(x) \approx \frac{f(x+h)-f(x-h)}{2h}$ with error $O(h^2)$. Show why.

Hint: Determine at least the first term in the error series for each formula.

- Forward Difference (FD):
  - $\frac{f(x+h)-f(x)}{h} = f'(x) + \frac{h}{2}, f''(x) + \mathcal{O}(h^2)$
- Backward Difference (BD):
  - $\frac{f(x)-f(x-h)}{h} = f'(x) - \frac{h}{2}, f''(x) + \mathcal{O}(h^2)$
- Average FD and BD:
  - Compute the average: $\frac{1}{2}\left(\frac{f(x+h)-f(x)}{h} + \frac{f(x)-f(x-h)}{h}\right) = \frac{f(x+h)-f(x-h)}{2h}$
- Error Analysis:
  - FD error term: $+\frac{h}{2}, f''(x)$
  - BD error term: $-\frac{h}{2}, f''(x)$
  - Averaging cancels the $\mathcal{O}(h)$ terms, yielding: $\frac{f(x+h)-f(x-h)}{2h} = f'(x) + \frac{h^2}{6}, f'''(x) + \mathcal{O}(h^3)$
- Result:
  - The central-difference formula $f'(x) \approx \dfrac{f(x+h)-f(x-h)}{2h}$ has an error of $\mathcal{O}(h^2)$
    .
  - The forward and backward formulas both have $O(h)$ error, but their $f'''(x)$ terms are of opposite sign.
  - When averaged, the first-order error cancels, resulting in a central-difference formula with improved accuracy.

8.b. Derive the formula $f''(x) \approx \frac{1}{4h^2}[f(x+2h) - 2f(x) + f(x-2h)]$ and establish formulas for the errors in using them.

- **Goal:** Derive
  $f''(x) \approx \frac{1}{4h^2}[f(x+2h) - 2f(x) + f(x-2h)]$
- **First of, lets expand $f(x+2h)$ and $f(x-2h)$:**
  - $f(x+2h) = f(x) + 2h, f'(x) + 2h^2, f''(x) + \frac{4h^3}{3}, f'''(x) + \frac{2h^4}{3}, f''''(x) + \mathcal{O}(h^5)$
  - $f(x-2h) = f(x) - 2h, f'(x) + 2h^2, f''(x) - \frac{4h^3}{3}, f'''(x) + \frac{2h^4}{3}, f''''(x) + \mathcal{O}(h^5)$
- **Combine the expansions:**
  - Add the two:
    $f(x+2h) + f(x-2h) = 2f(x) + 4h^2, f''(x) + \frac{4h^4}{3}, f''''(x) + \mathcal{O}(h^5)$
- **Subtract $2f(x)$ and divide by $4h^2$:**
  - $\frac{f(x+2h)-2f(x)+f(x-2h)}{4h^2} = f''(x) + \frac{h^2}{3}, f''''(x) + \mathcal{O}(h^3)$
- **Error Term:**
  The leading error is $\frac{h^2}{3}, f''''(x)$, so the formula has an error of $\mathcal{O}(h^2)$.