* Least Square Circle *

Suppose, we have a set of points in R^2 e.x. $\{(x_i, y_i) \mid 0 \leq i \leq N)$

→ The objective is to find the circle that "best" fits the points.

→ $\bar{X} = 1/N(\Sigma x_i)$ and
$\bar{Y} = 1/N(\Sigma Y_i)$

→ let, $u_i = x_i - \bar{x}$ and,
$v_i = y_i - \bar{y}$
} for $0 \leq i < N$

* The circle having center (u_c, v_c) and Radius R, the objective is to minimize $S = \Sigma_i (g(u_i, v_i))^2$, where,
$g(u,v) = (u - u_c)^2 + (v - v_c)^2 - \alpha$.
where $\alpha = R^2$.

Here, we need to find,

$\frac{\partial S}{\partial \alpha}$, $\frac{\partial S}{\partial u_c}$ & $\frac{\partial S}{\partial v_c}$ for $S(\alpha, u_c, v_c)$

$$\frac{\partial s}{\partial z} = 2 \sum_i g(u_i, v_i) \frac{\partial g}{\partial z}(u_i, v_i)$$

$$= -2 \sum_i g(u_i, v_i)$$

Thus, $\frac{\partial s}{\partial z} = 0$ iff

$$\sum_i g(u_i, v_i) = 0 \qquad - \text{eq. 1}$$

Same way,

$$\frac{\partial s}{\partial u_c} = 2 \sum_i g(u_i, v_i) \frac{\partial g}{\partial u_c}(u_i, v_i)$$

$$= 2 \sum_i g(u_i, v_i)\, 2(u_i - u_c)(-1)$$

$$= -4 \sum_i (u_i - u_c)\, g(u_i, v_i)$$

$$= -4 \sum_i u_i\, g(u_i, v_i) + 4u_c \sum_i g(u_i, v_i)$$

Here, based on e.q. 1

$$\sum g(u_i, v_i) = 0$$

So, based on eq. 1, $\frac{\partial s}{\partial u_c} = 0$

iff

$$\sum_i u_i g(u_i, v_i) = 0 \quad \rightarrow \text{e.q. 2}$$

Similarly, $\frac{\partial s}{\partial v_c} = 0$ gives

$$\sum_i v_i g(u_i, v_i) = 0 \quad \rightarrow \text{e.q. 3}$$

Here, Expanding e.q. 2 gives,

$$\sum_i u_i \left[u_i^2 - 2u_i u_c + u_c^2 + v_i^2 - 2v_i v_c + v_c^2 - \alpha \right] = 0$$

Suppose, $S_u = \sum_i u_i$, $S_{uu} = \sum_i u_i^2$

etc the above e.q. becomes

$$S_{uuu} - 2u_c S_{uu} + u_c^2 S_u + S_{uvv} - 2v_c S_{uv} + v_c^2 S_u - \alpha S_u = 0$$

Putting $S_u = 0$ gives,

$$u_c S_{uu} + v_c S_{uv} = \frac{1}{2}(S_{uuu} + S_{uvv})$$

$\hookrightarrow$ e.q. 4

using $S_v = 0$ gives,

$$u_c S_{uv} + v_c S_{vv} = \frac{1}{2}(S_{vvv} + S_{vuu})$$

$\hookrightarrow$ e.q. 5

by Solving e.q. 4 and 5, one can get $(u_c \& v_c)$

$\longrightarrow$ Find Radius R by expanding e.q. 1

$$\sum_i [u_i^2 - 2u_i u_c + u_c^2 + v_i^2 - 2v_i v_c + v_c^2 - \alpha] = 0$$

using $S_u = S_v = 0$; we get

$$N(u_c^2 + v_c^2 - \alpha) + S_{uu} + S_{vv} = 0$$

$$\alpha = u_c^2 + v_c^2 + \frac{S_{uu} + S_{vv}}{N}$$

Here, $R = \sqrt{\alpha}$.

Finally,

$$\boxed{\begin{aligned} &R = \sqrt{\alpha} \\ &x\text{-center} = u_c + \overline{x} \\ &y\text{-center} = v_c + \overline{y} \end{aligned}}$$