# Functional Mechanical Design

## Motion Law (2/2)

Simone Cinquemani

# Requirements for motion curves



After defining the principal aspects of the motion curve treatment we have to go into more in detail from the design point of view. Independently of the application, when you are planning a motion curve, there are some design requirements widespread, for example:

- to limit the maximum acceleration
- to limit the maximum velocity
- to limit the mechanical power
- to limit the vibrations

# To limit the maximum velocity



$$\dot{y}_{max} = c_v \frac{h}{t_a} \quad \begin{array}{lll} \Rightarrow & h & \text{small is better} \\ \Rightarrow & t_a & \text{great is better} \end{array}$$

If $h$ and $t_a$ are fixed, the only way to reduce the maximum velocity is to change the shape of the motion curve or, in other words, the value of the velocity coefficient. Note that, from the coefficient definition, it is possible to obtain:

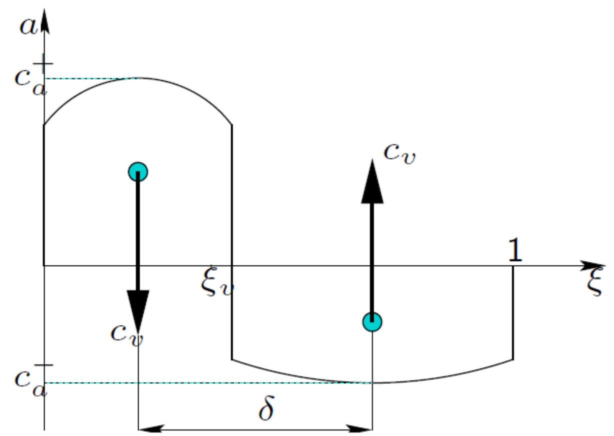
$$c_v = \frac{\dot{y}_{max}}{h/t_a} = \frac{\dot{y}_{max}}{\dot{y}_{med}}$$

from which is immediately obtained for every kind of motion law that $c_v \geq 1$. In Practice values of $c_v$ very close to $1$ cannot be attained because they would require very high values of acceleration.

# To limit the maximum velocity



Furthermore from the distributed loads analogy and looking at the figure it is possible write the following relationship:




$$\left| \dot{y}_{\max} \cdot \frac{\xi_v}{2} - \dot{y}_{\max} \cdot \left( \xi_v + \frac{1-\xi_v}{2} \right) \right| = 1$$

$$\left| \dot{y}_{\max} \cdot \delta \right| = 1$$

$$\left| c_v \frac{h}{t_a} \cdot \delta \right| = 1 \qquad \Rightarrow \qquad c_v \cdot \delta = 1$$

from which the following propriety of velocity coefficient is obtained:

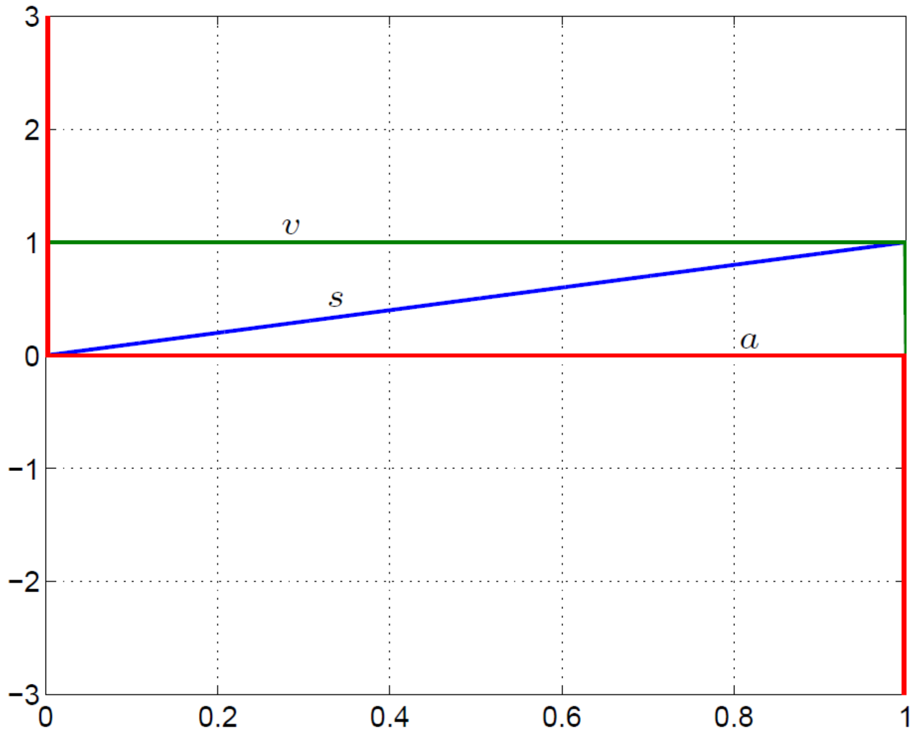
$$c_v = \frac{1}{\delta}$$

That is: to limit $c_v$ we have to increase $\delta$

# To limit the maximum velocity



$c_a^+ = \infty$




$c_v = 1$

$c_a^- = -\infty$

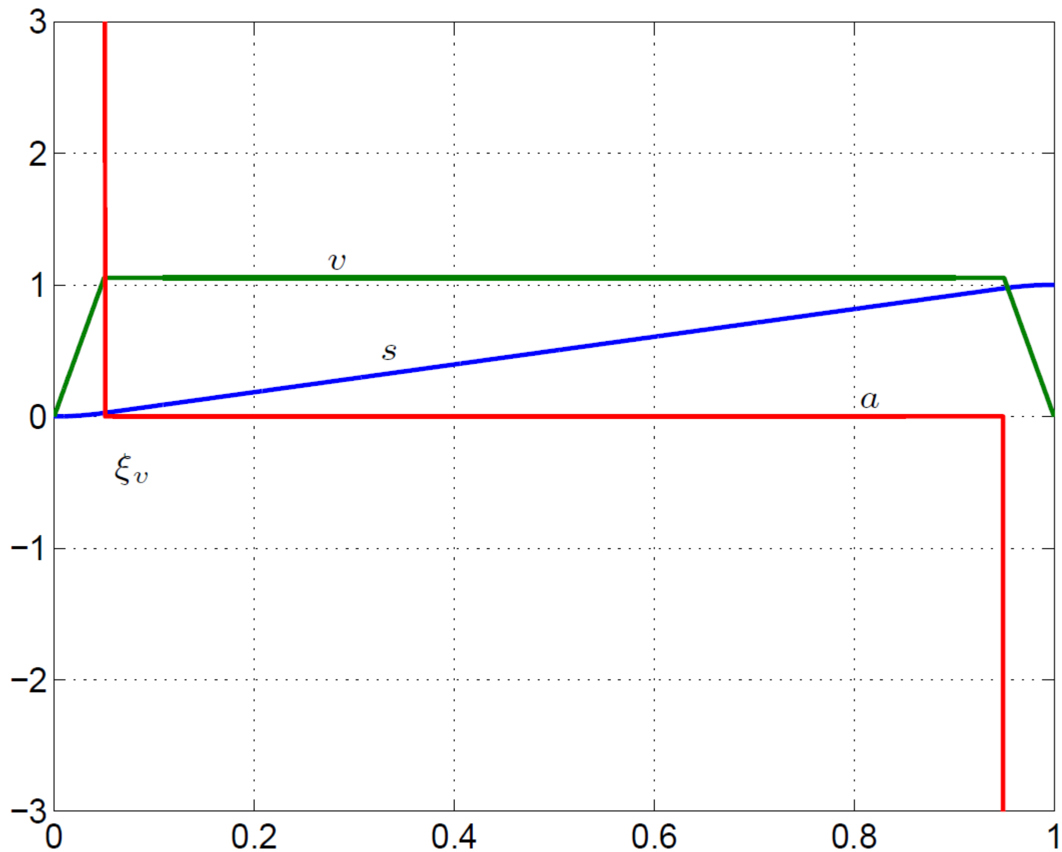
Figure shows the behaviour of a constant velocity curve with $c_v = 1$. The acceleration coefficients are $c_a^+ = c_a^- = \infty$. Note that this kind of motion curve has a polynomial expression as: $y(t) = a_0 + a_1(t - t_0)$.

# To limit the maximum velocity



A suitable motion curve to decrease the cv value is shown in this figure: this is called the **trapezoidal velocity motion curve**.

The case proposed shows a symmetric velocity profile.



# To limit the maximum velocity



If the velocity profile is symmetric, it is possible to use the point at which the acceleration phase ends $\xi_v$, in order to parametrize the mathematical expression of the curve coefficients:

$$\begin{cases} c_v = \dfrac{1}{\delta} = \dfrac{1}{1-\xi_v} \\ a(\xi)_{max}\xi_v = v(\xi_v) = v_{max} = c_v \end{cases} \Rightarrow \quad c_a = \frac{c_v}{\xi_v} = \frac{1}{\xi_v(1-\xi_v)}$$

Being $(\xi_v <= \frac{1}{2})$ it will be $c_v <= 2$. Summarizing:

$$\boxed{c_v = \frac{1}{1-\xi_v}; \quad c_a = \frac{1}{\xi_v(1-\xi_v)}; \quad c_a = \frac{c_v^2}{c_v - 1}}$$

$$c_a - c_v = \frac{1}{\xi_v(1-\xi_v)} - \frac{1}{(1-\xi_v)} = \frac{1-\xi_v}{\xi_v(1-\xi_v)} = \frac{c_a}{c_v} \quad \Rightarrow \quad c_a = \frac{c_v^2}{c_v - 1}$$

# To limit the maximum velocity



| | | |
|---|---|---|
| $\xi_v = 1/2$ | $c_v = 2$ | $c_a = 4$ |
| $\xi_v = 1/3$ | $c_v = 1.5$ | $c_a = 4.5$ |
| $\xi_v = 1/6$ | $c_v = 1.2$ | $c_a = 7.2$ |

Note: a considerable decrease of the velocity coefficient can be obtained without a sharp increase of acceleration coefficient until the value $c_v = 1.2$, after a little reduction is paid with high acceleration values.

# Trapezoidal velocity profile



Displacement, velocity and acceleration of the trapezoidal velocity profile are shown below:

$$0 < \xi < \xi_v \qquad \begin{cases} a = c_a \\ v = c_a \xi \\ s = \dfrac{1}{2} c_a \xi^2 \end{cases}$$

$$\xi_v < \xi < (1 - \xi_v) \qquad \begin{cases} a = 0 \\ v = c_a \xi_v \\ s = c_a \xi_v \xi - \dfrac{1}{2} c_a \xi_v^2 \end{cases}$$

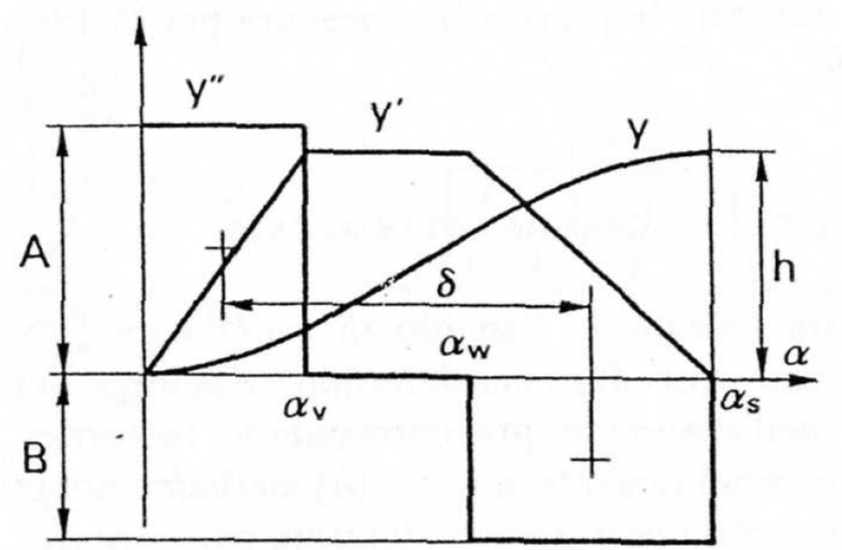
$$(1 - \xi_v) < \xi < 1 \qquad \begin{cases} a = -c_a \\ v = c_a (1 - \xi) \\ s = c_a \left( \xi - \dfrac{\xi^2}{2} + \xi_v - \xi_v^2 - \dfrac{1}{2} \right) \end{cases}$$

Trapezoidal velocity profile is also called modified constant velocity curve (modified = parabolic motion):

# Trapezoidal velocity profile



The modified constant velocity curve can be generalized using asymmetric acceleration segments and two parameters $\xi_v$ and $\xi_w$.



# To limit the mechanical power



At any time $t$, the total virtual power of the external, internal and inertia forces is zero in any admissible virtual state of motion (the principle of virtual power):

$$W_m + W_e + W_i + W_d = 0$$

If no power is lost through friction ($W_d = 0$) and the mechanical power of the external force is marginal ($W_e = 0$), the total virtual power is:

$$W_m = -W_i \quad \text{depending on the sign convention} \quad W_m = W_i$$

In other words the motor power is used only to bear the inertial load power. Thereafter reducing the treatment on a one degree of freedom machine this equation becomes:

$$W_m = C_m \omega_m = m\ddot{y}\dot{y}$$

where $C_m$ and $\omega_m$ are respectively the torque and the angular velocity of the moving element whereas $m$, $\ddot{y}$ and $\dot{y}$ are respectively mass, acceleration and velocity of the follower.

# To limit the mechanical power



Under these hypothesis the power required by the follower depends only on the product of acceleration and velocity. This is why it's worthwhile trying to minimize this product.
The approach is to define a power coefficient capable of describing the power feature of every motion curve. Because the velocity is proportional to $h/t_a$ and the acceleration to $h/t_a^2$, the maximum power is proportional to the product of both:

$$W_{max} = m(\ddot{y}\dot{y})_{max} = (a(\xi)v(\xi))_{max}\frac{mh^2}{t_a^3} = c_k\frac{mh^2}{t_a^3}$$

where $c_k = \max|v(\xi)a(\xi)|$ is a dimensionless coefficient called *power coefficient*.

Considering $m = 1$, the power coefficient definition is:

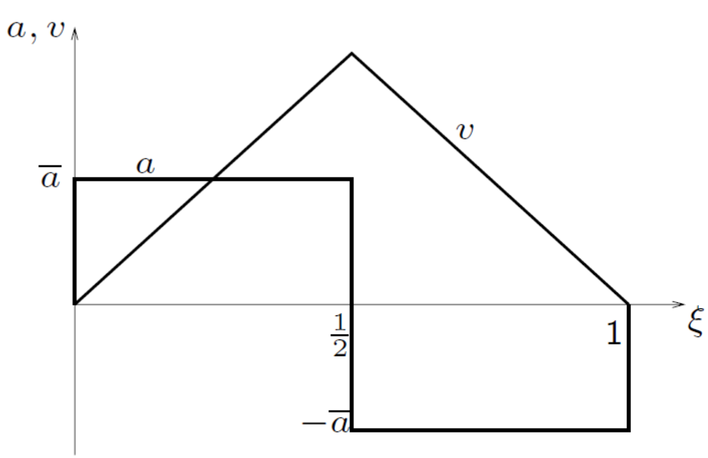
$$\boxed{c_k = \frac{(|\ddot{y}\dot{y}|)_{max}}{h^2/t_a^3}} \quad \Rightarrow$$

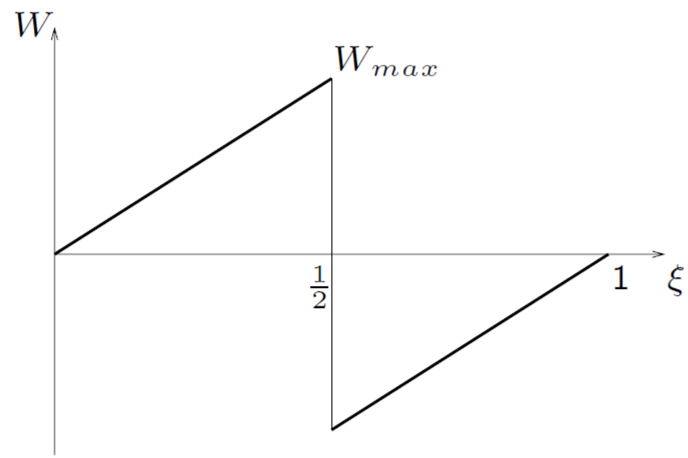
$c_k$ is the maximum absolute value of the following function product of $a(\xi)$ and $v(\xi)$. It is clear that it is less than or equal to the product of the maximum of each function $(c_a \cdot c_v)$, that is $c_k \leq c_a c_v$

# To limit the mechanical power



These figures show the functions $v(\xi)$, $a(\xi)$ and $W(\xi)$ for the symmetric constant acceleration curve.



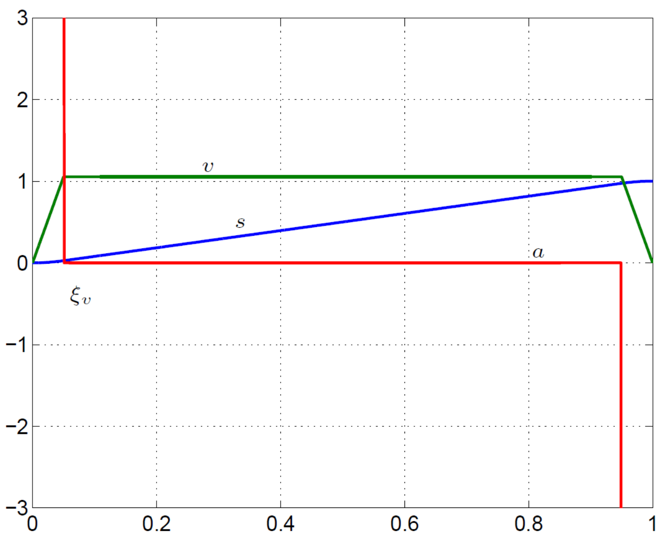




Note that for this curve velocity and acceleration reach their peak at the same time (when $\xi = \xi_v$) then $c_k$ takes its maximum value that is $c_a c_v$.
The velocity coefficient $c_v$ is always equal to $c_v = 2$ for this kind of motion curve.
Then the minimum value of $c_k$ is obtained when the acceleration is at its minimum, that is when is $\xi_v = 1/2$, so $c_a = 4$, and the power coefficient is $c_k = 8$.

# To limit the mechanical power



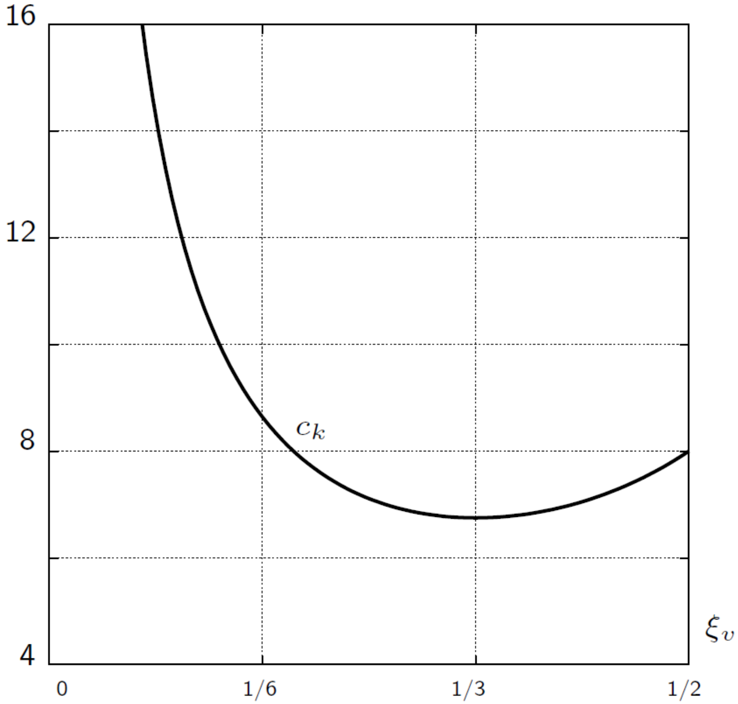
Better values of $c_k$ can be obtained by a trapezoidal velocity curve: in fact, for these kind of curves, diminishing $\xi_v$ we have firstly a reduction of $c_v$ with little increase of $c_a$.




$$c_v = \frac{1}{1-\xi_v}; \quad c_a = \frac{1}{\xi_v(1-\xi_v)}$$

For these motion curves it is possible to define the power coefficient as follows:

$$c_k = c_a \cdot c_v = \frac{1}{\xi_v(1-\xi_v)^2},$$



# To limit the mechanical power



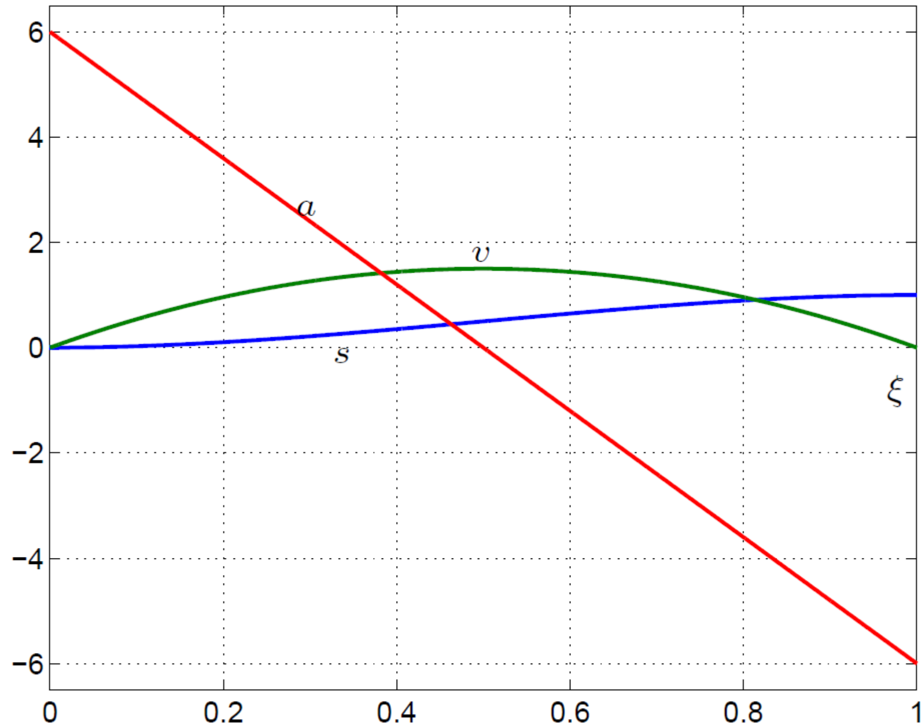
The mimimum of $c_k$ is obtained solving the equation:

$$\frac{\mathrm{d}}{\mathrm{d}\xi_v}\left(\frac{1}{\xi_v(1-\xi_v)^2}\right) = 0 \quad \Rightarrow \quad -\frac{(1-\xi_v)^2 - 2\xi_v(1-\xi_v)}{\xi_v(1-\xi_v)^2} = 0 \quad \Rightarrow \quad 3\xi_v^2 - 4\xi_v - 1 = 0$$

it is easy to calculate that the mimimum is when $\xi_v = 1/3$ and the coefficient value is $c_k = 27/4 = 6.75$.

Motion curves with a constant acceleration are not satisfactory in term of the reduction of power coefficient because they have at the same time velocity and acceleration maximums.

It is evident that, to reduce the value of $c_k$, the motion curves must have the maximum velocity and acceleration not at the same time. A profile with this property is the cubic curve:



# Cubic profile

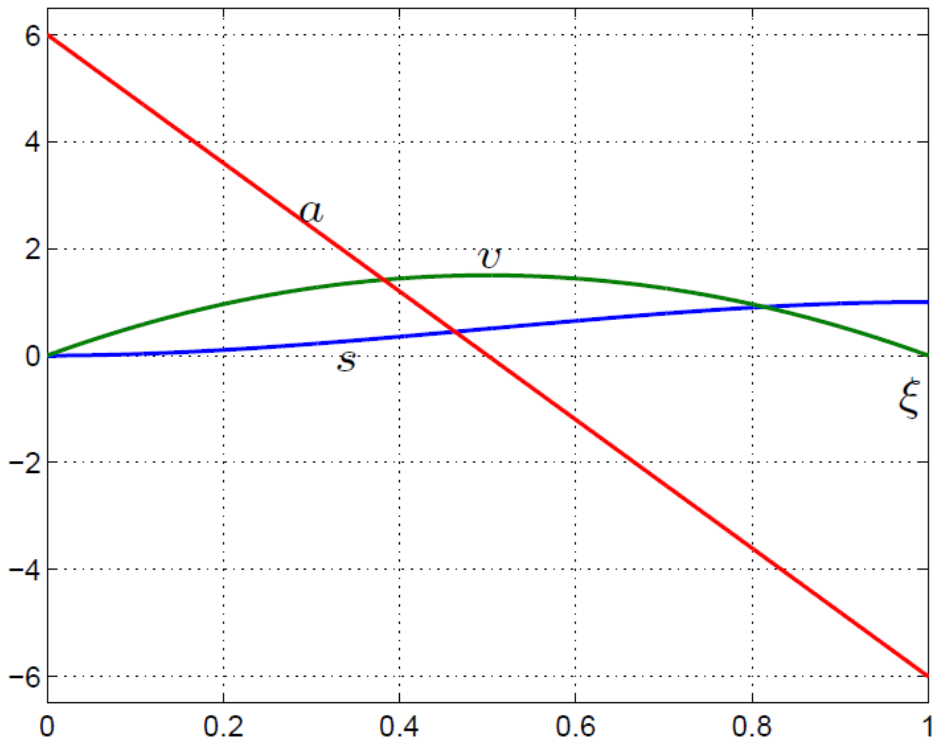


It is easy to calculate for the cubic curve the velocity coefficient:

$$\delta = \frac{2}{3} \quad \Rightarrow \quad c_v = \frac{1}{\delta} = 1.5$$

While the acceleration coefficient is $c_a$ ($c_a^+ = c_a^- = c_a$):

$$\frac{c_a \frac{1}{2}}{2} = c_v \quad \Rightarrow \quad c_a = 6$$



# Cubic profile



The acceleration expression is:

$$\ddot{y} = at + b \quad \begin{cases} t = 0; \quad \ddot{y} = \ddot{y}_{max} = b; \quad b = \ddot{y}_{max} \\ t = t_a; \quad \ddot{y} = -\ddot{y}_{max} = at_a + \ddot{y}_{max}; \;\; a = \dfrac{-2\ddot{y}_{max}}{t_a} \end{cases}$$

Then:

$$\ddot{y} = c_a \frac{h}{t_a^2}\left(1 - \frac{2}{t_a}t\right) \quad \Rightarrow \quad a(\xi) = 6(1 - 2\xi)$$

Integrating twice, we obtain velocity and displacement:

$$\dot{y} = c_a \frac{h}{t_a^2} t \left(1 - \frac{t}{t_a}\right) \qquad \Rightarrow \quad v(\xi) = 6\xi(1 - \xi)$$

$$y = c_a \frac{h}{t_a^2} t^2 \left(\frac{1}{2} - \frac{2}{3}\frac{t}{t_a}\right) \quad \Rightarrow \quad s(\xi) = \xi(3\xi - 2\xi^2)$$

# Cubic profile



From the definition of the power coefficient:

$$c_k = \frac{(\ddot{y}\dot{y})_{max}}{\frac{h^2}{t_a^3}} = \frac{\left[c_a \frac{h}{t_a^2}\left(1 - \frac{2}{t_a}t\right) c_a \frac{h}{t_a^2} t \left(1 - \frac{t}{t_a}\right)\right]_{max}}{\frac{h^2}{t_a^3}}$$

or using a dimensionless approach, we have to minimize the following function:

$$f(\xi) = a(\xi)v(\xi) = 6(1-2\xi)6\xi(1-\xi) = 36\xi(1-2\xi)(1-\xi) = 36\xi - 108\xi + 72\xi$$

It is possible to calculate the value of $\xi$ in which the product of $v(\xi)$ and $a(\xi)$ is maximum, nullifying the derivative of $f(\xi)$ with respect to $\xi$:

$$\dot{f}(\xi) = 0 \quad \Rightarrow \quad 1 - 6\xi + 6\xi^2 = 0 \quad \Rightarrow \quad \xi = \frac{1}{2} - \frac{\sqrt{3}}{6}$$

Substituting the obtained $\xi$ value into the expression of $c_k$ it is possible to evaluate the power coefficient for the cubic curve: $c_k = 2\sqrt{3} \simeq 3.4$

# To limit the vibrations



The vibration phenomenon is directly linked to the discontinuity on the acceleration: abrupt variation on the acceleration creates an abrupt variation on the inertial forces and these, if the system is flexible, can generate oscillations.

To reduce the vibration problem it is useful to use a motion curve without discontinuity on $\ddot{y}$. Furthermore the acceleration must be nil at the beginnig and at the end of the movement $\ddot{y}(0) = \ddot{y}(t_a)$, where the vibrations are most dangerous. A motion curve with these requirements is the cycloidal curve, provided by a sinusoidal acceleration trend.

$$\ddot{y} = c_a \frac{h}{t_a^2} \sin\left(\frac{2\pi t}{t_a}\right)$$

It's clear that the value of its coefficient $c_v$ is equal to 2.

# Cycloidal profile

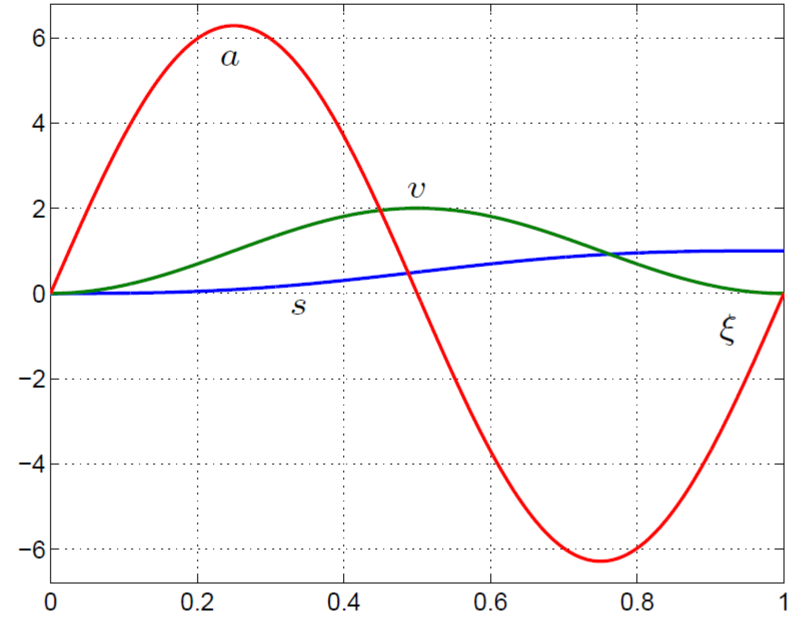


Using the accleretaion diagram proprieties it is possible to evaluate the coefficient $c_a$:

$$\int_0^{\frac{t_a}{2}} c_a \frac{h}{t_a^2} \sin\left(\frac{2\pi t}{t_a}\right) = c_v \frac{h}{t_a} = 2\frac{h}{t_a} \quad \Rightarrow \quad c_a \frac{h}{t_a^2}\frac{t_a}{2\pi}[-\cos(\pi) + \cos(0)] = 2\frac{h}{t_a}$$

from which we obtain the value $c_a = 2\pi$.

The figure shows position, velocity and acceleration of the cycloidal motion curve.



# Cycloidal profile



The following there are the expressions (dimension and dimensionless) of the cycloidal motion curve:

$$\begin{cases} \ddot{y} = c_a \dfrac{h}{t_a^2} \sin\left(\dfrac{2\pi t}{t_a}\right) \\[2ex] \dot{y} = \dfrac{h}{t_a}\left[1 - \cos\left(\dfrac{2\pi t}{t_a}\right)\right] \\[2ex] y = h\left[\dfrac{t}{t_a} - \dfrac{1}{2\pi}\sin\left(\dfrac{2\pi t}{t_a}\right)\right] \end{cases}$$

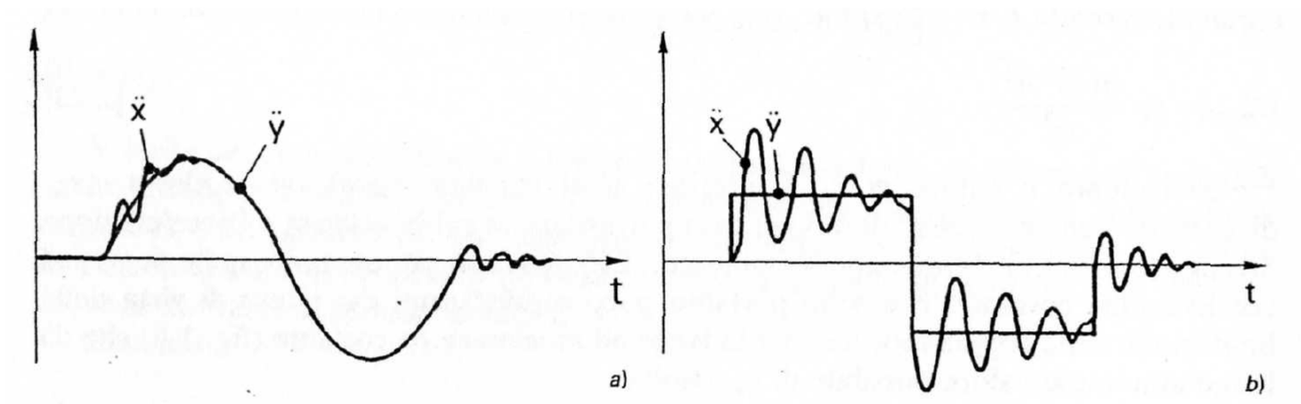
$$\begin{cases} a(\xi) = 2\pi \sin(2\pi\xi) \\[2ex] v(\xi) = 1 - \cos(2\pi\xi) \\[2ex] s(\xi) = \xi - \dfrac{1}{2\pi}\sin(2\pi\xi) \end{cases}$$

Velocity and accelerations coefficients are : $c_v = 2,\ c_a = c_a^+ = c_a^- = 2\pi$

# Cycloidal profile



The figure shows the comparison between constant acceleration and cycloidal motion curves:




Note that the constant acceleration curve shows an actual acceleration ($\ddot{x}$) greater than that of the cycloidal even if the theoretical coefficient is lower ($4 < 2\pi$)

# Resuming…



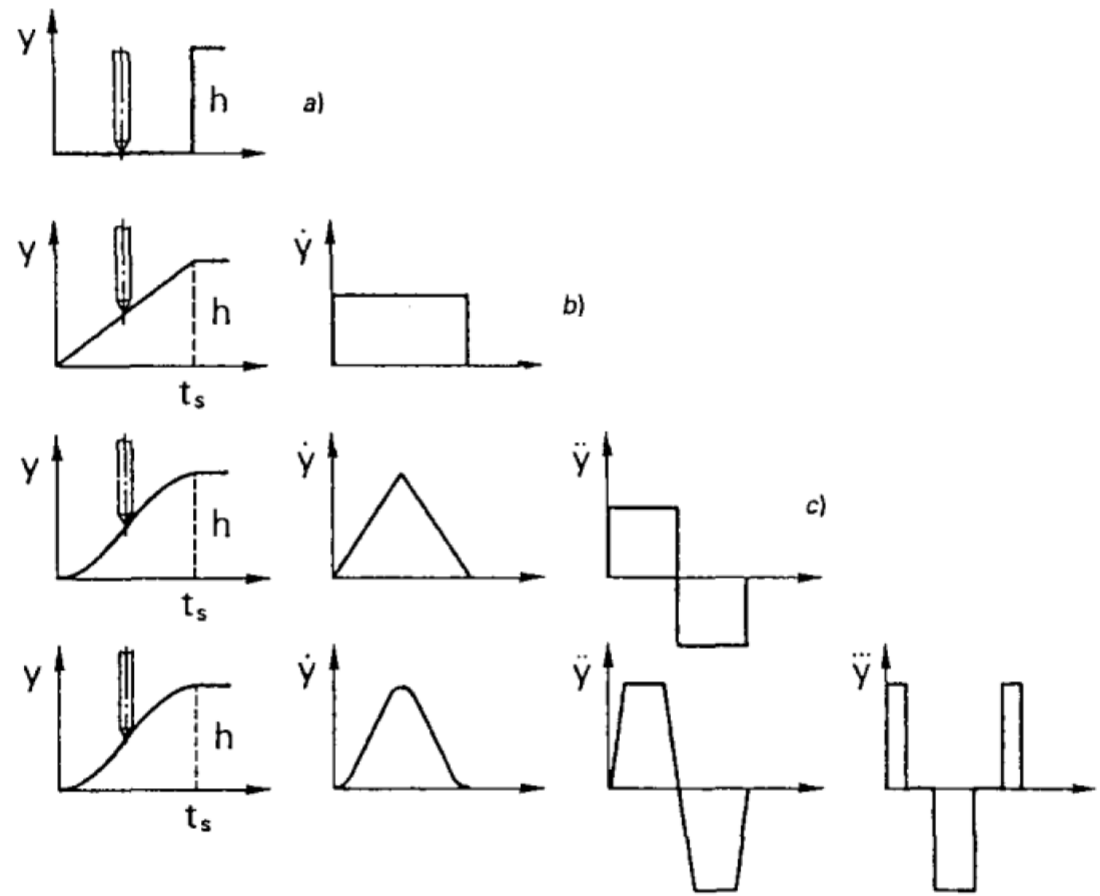
Generally the objective of trajectory planning is to find motion curve that minimize all the coefficients described before. However it is necessary to reach a certain level of compromise because usually reducing one coefficient means increasing the others.

| **Motion curve** | $c_v$ | $c_a$ | $c_w$ |
|---|---|---|---|
| Const. symm. acc. | $2$ | $4$ | $8$ |
| Trap. vel. $(\frac{1}{3}\frac{1}{3}\frac{1}{3})$ | $1.5$ | $4.5$ | $6.75$ |
| Cubic | $1.5$ | $6$ | $3.46$ |
| Cycloidal | $2$ | $2\pi$ | $8.13$ |

# Improvements of motion curves

The motion curve improvement is based on the concept of "softness of movement": that is everything linked to the reduction of over acceleration, shock and the like.

The idea is to shift the discontinuity of the motion curve at the higher derivatives. Usually this is linked to the velocity of the motion element and/or the elastic proprieties of the system.



# Improvements of motion curves



- It is impossible to remove every motion curve discontinuity because the movement profile for industrial applications is always characterized by stop and go situations.
- It is sufficient to shift the discontinuities to higher derivatives without an in depth elastodynamics analysis of the system.

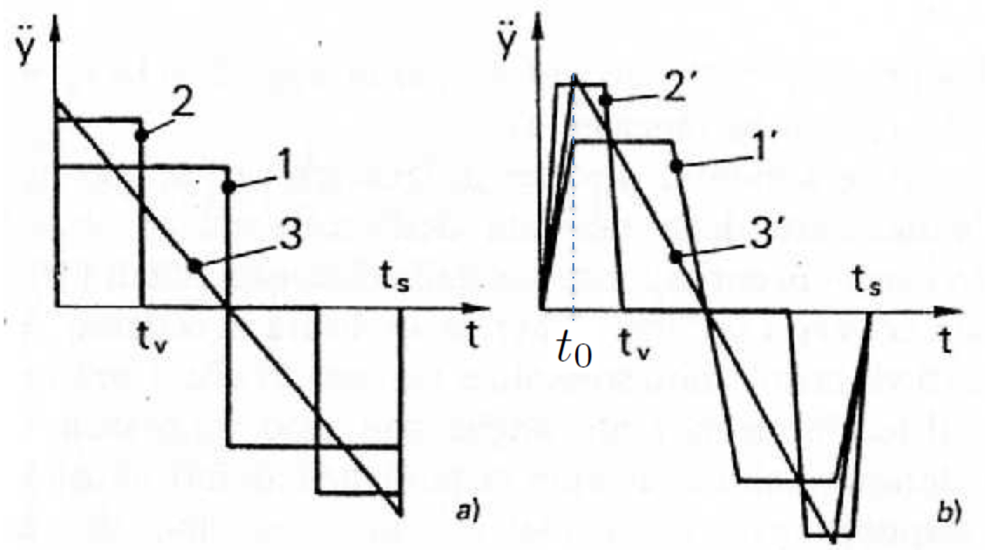
The principal improvement motion curves are called:

- Trapezoidal curves
- Polynomial curves

# Trapezoidal Motion curves

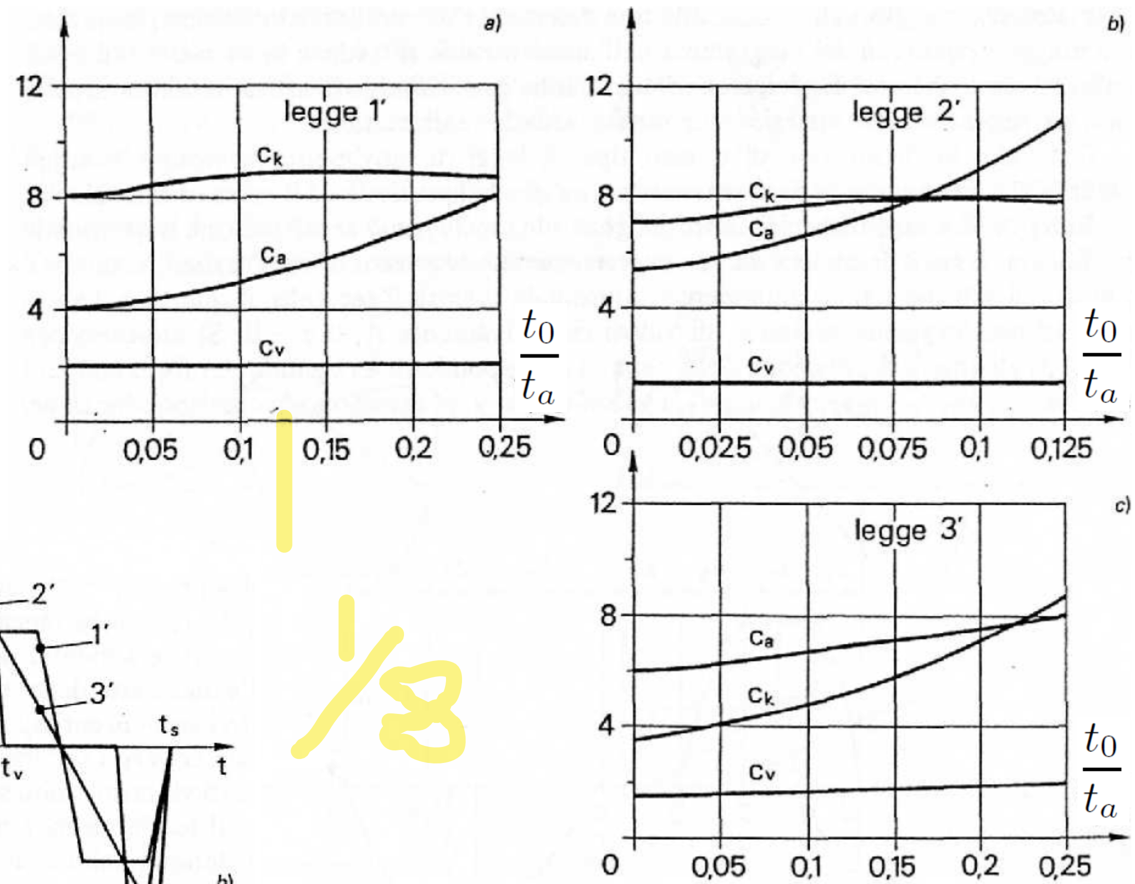


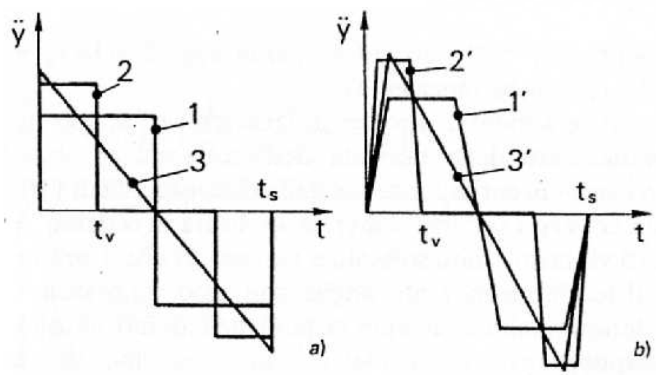
Trapezoidal motion curves derive from the elementary motion profiles. The difference is that they use linear *chamfer* (with the length equal to $t_0$) to remove discontinuity on the acceleration diagram (in other terms eliminating the sharp corners in the acceleration profile).



Note that an increase of the time $t_0$ means increasing the maximum acceleration value, but a $t_0$ too short does not have any effect in terms of the "softness of movement"

The figure shows some diagrams with the trend of some coefficients against the ratio $\frac{t_0}{t_a}$.

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# Trapezoidal Motion curves

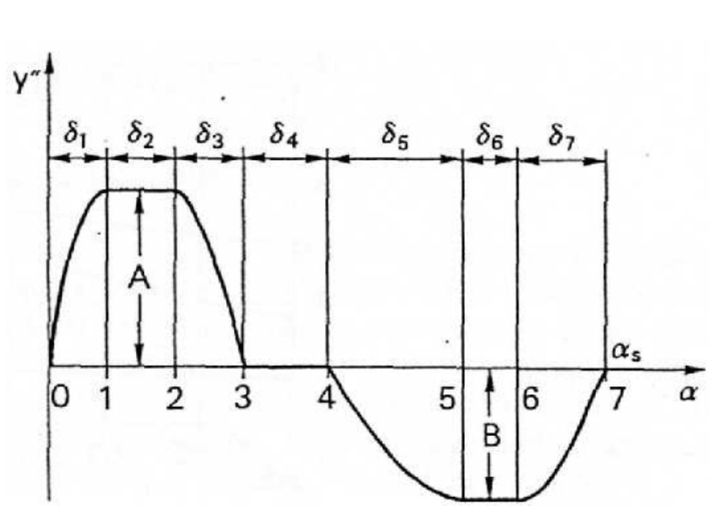


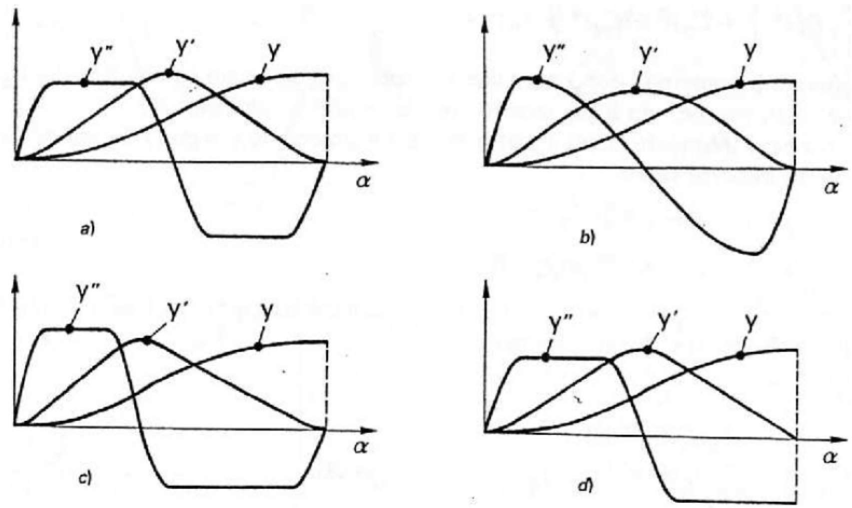
We can state that a good value of $t_0$ is $\Rightarrow$ $\dfrac{t_0}{t_a} = \dfrac{1}{8}$

| **Motion profile** | $c_a$ |
|---|---|
| Constant acceleration | 4 |
| Modified constant acceleration | 5.33 |
| Trapezoidal velocity ($\xi_v = \frac{1}{4}$) | 5.33 |
| Modified trapezoidal velocity | 9.67 |
| Cubic | 6 |
| Modified cubic | 6.86 |

# Improvements of motion curves






The shape of the diagram of the modified trapezoidal motion curves is defined by means of seven parameters: $\delta_1 \ldots \delta_7$.
In the figures there are some modified trapezoidal motion curves:

a) Motion: $\delta = 0.125 - 0.25 - 0.125 - 0 - 0.125 - 0.25 - 0.125$
b) Motion: $\delta = 0.125 - 0 - 0.375 - 0 - 0.375 - 0 - 0.125$
c) Motion: $\delta = 0.1 - 0.2 - 0.1 - 0 - 0.1 - 0.4 - 0.1$
d) Motion: $\delta_7 = 0$

**Check equations on pages 109-111**

# Improvements of motion curves



In practice, the use of an improved motion curve might not give the desired results, because we are not taking into consideration:

- system elasticity, backlash and construction tolerances