POLITECNICO DI MILANO

Project of:

Mechanical System Dynamics

In plane Finite Element Model of a Truss Bridge

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Introduction

The aim of the following report is to study the dynamic behavior a truss bridge by means of the Finite Element Method (FEM). The bridge is reported in the figure below.

The FEM is a systematic numerical method that allows to study the behavior of complex structures that cannot be solved analytically. It is based on the idea of dividing the structure in small portions, called Finite Elements. In each element some important points are considered, the nodes, and the system is described by using the nodal coordinates as a set of independent coordinates. Then it is possible to obtain the displacement field in any point of the elements by using interpolating functions called shape functions.

The FEM is a discretization technique that allows to pass from solving partial differential equations (characteristic of a continuous system) to solving a system of n_T ordinary differential equations of the second order (characteristic of a discrete system), where n_T is the total number of degrees of freedom considered $n_T = 3N_{nodes}$ since each node has two translational and one rotational degree of freedom.

The procedure is focused on finding the mass matrix [M], the stiffness matrix [K], the damping matrix [C], and the vector of equivalent nodal forces $\underline{F}$, that allow to compute the response of the structure in correspondence of the nodes. The system is partitioned so as to divide the free degrees of freedom $\underline{x_{F_{n_{dof}x1}}}$ and the constrained ones $\underline{x_{C_{n_c}}}$:

$$\begin{bmatrix} [M_{FF}] & [M_{FC}] \\ [M_{CF}] & [M_{CC}] \end{bmatrix} \begin{pmatrix} \underline{\ddot{x}_F} \\ \underline{\ddot{x}_c} \end{pmatrix} + \begin{bmatrix} [C_{FF}] & [C_{FC}] \\ [C_{CF}] & [C_{CC}] \end{bmatrix} \begin{pmatrix} \underline{\dot{x}_F} \\ \underline{\dot{x}_C} \end{pmatrix} + \begin{bmatrix} [K_{FF}] & [K_{FC}] \\ [K_{CF}] & [K_{CC}] \end{bmatrix} \begin{pmatrix} \underline{x_F} \\ \underline{x_C} \end{pmatrix} = \begin{pmatrix} \underline{F_{ext}} \\ \underline{R_C} \end{pmatrix}$$

The unknowns of the system are $\underline{x_F}$ and the constraints reaction forces $\underline{R_C}$.

Bridge modelling

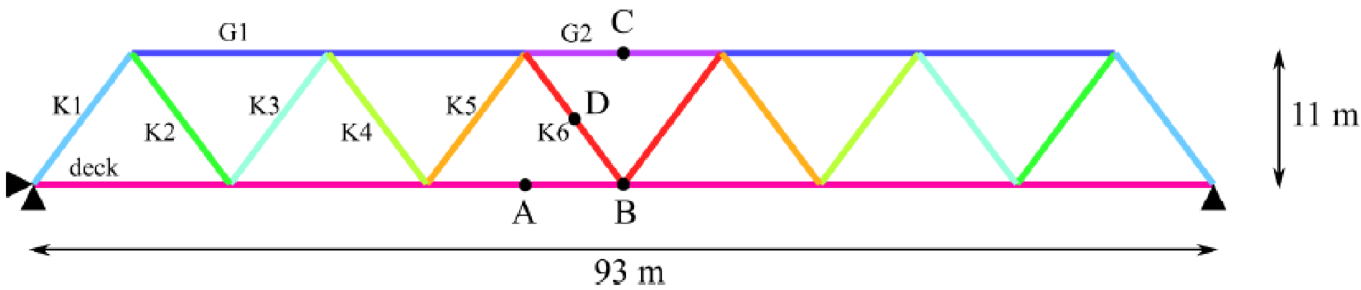
In this case is possible to consider the problem as plane and the structure can be divided in several beam elements (1-D) with two nodes at the ends. The constraints used are a pin on the left end and a roller on the right end (non-symmetric structure).

The hypothesis that should be satisfied to use beam elements are:

1) Each beam can be considered one dimensional (length>>dimensions of the cross section).
2) The axis of each beam must be rectilinear and the cross section must be constant.
3) The material must be homogeneous and linear elastic.

All the hypothesis are valid since the bridge is composed of trusses, and each truss can be assumed as a 1-D beam with constant cross section and made of homogeneous material.

In the image below is reported the schematization of the structure and its properties:



	Colour	m [kg/m]	EA	EJ
Deck Element	Pink	2.36E+03	1.59E+10	1.36E+09
Diag member K1	Light blue	4.42E+02	1.18E+10	2.80E+08
Diag member K2	Green	2.56E+02	6.85E+09	1.36E+08
Diag member K3	Cyan	2.75E+02	7.35E+09	1.51E+08
Diag member K4	Light green	1.75E+02	4.68E+09	4.48E+07
Diag member K5	Orange	1.45E+02	3.89E+09	6.13E+07
Diag member K6	Red	94.2	2.52E+09	1.63E+07
Upper chord G1	Blue	5.34E+02	1.43E+10	1.21E+09
Upper chord G2	Purple	5.93E+02	1.59E+10	1.36E+09

Mesh generation

Generating a mesh means dividing the structure in beam elements. To do this some requirements are needed:

- A node is needed in presence of any kind of discontinuity (change of orientation of the axis, cross section dimension and material characteristics).
- The maximum length of each element is limited: the limit value is the one that allow all the elements to work in quasi static region, since all the equations used to derive the model are valid only in static conditions. It means that the first natural frequency of each beam element ω_1 must be higher than the maximum frequency at which the system can be excited Ω_{max}: $\omega_1 \gg \Omega_{max}$.

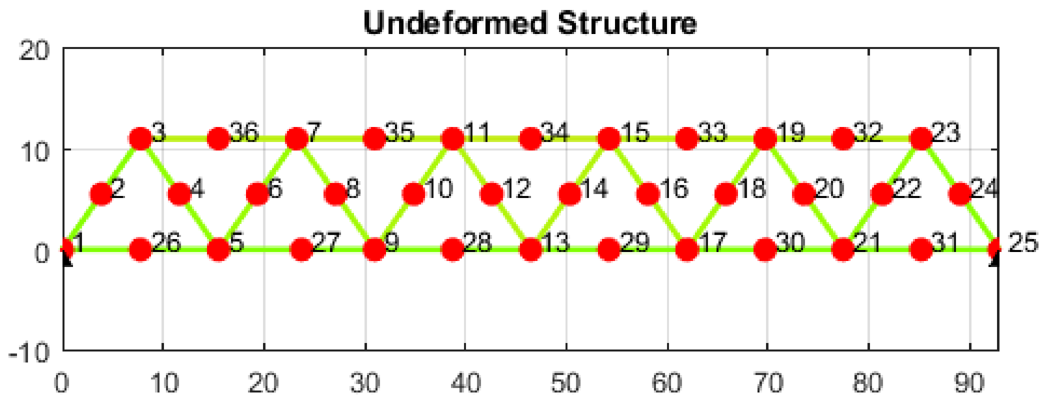
The frequency range considered for the bridge is 0-7 Hz, so $\Omega_{max} = 2\pi f_{max} = 43.98\, rad/s$. The maximum length of each element k can be roughly computed by considering the first natural frequency of a pinned-pinned beam:

$$\omega_{1_k} = \left(\frac{\pi}{L_k}\right)^2 \sqrt{\frac{EJ_k}{m_k}} \qquad \omega_k \gg \Omega_{max} \ \ (\omega_k = \eta\Omega_{max}) \quad \rightarrow \quad L_{k,max} = \pi \sqrt{\frac{1}{\eta\Omega_{max}} \sqrt{\frac{E_k J_k}{m_k}}}$$

Considering the nine k types of elements of which is composed the structure and $\eta = 1.5$:

- The lower maximum length among all the elements is $L_{k,max} = L_{k6} = 7.89\, m$.
- The higher maximum length among all the elements is $L_{k,max} = L_{G1} = 15.05\, m$.

In any case, each truss must be divided into two beam elements.



$N_{nodes} = 36$

$n_{dof} = 105$

$N_{elements} = 46$

Computation of natural frequencies and mode shapes

Once the properties of the system are known, [M] and [K] matrices can be computed. To compute the natural frequencies of the undamped system:

$$[M_{FF}]\underline{\ddot{x}_F} + [K_{FF}]\underline{x_F} = \underline{0} \qquad \underline{x_F} = \underline{x_0} e^{j\omega t}$$

$$(-\omega^2 [M_{FF}] + [K_{FF}])\underline{x_0} e^{j\omega t} = \underline{0}$$

To not have trivial solutions: $\det\left((-\omega^2 [M_{FF}] + [K_{FF}]\right) = 0 \ \rightarrow \ \omega = \omega_i \quad for\ i = 1{:}\,n_{dof}$

The first 6 natural frequencies (the ones that fall in the range 0-7 Hz) are:

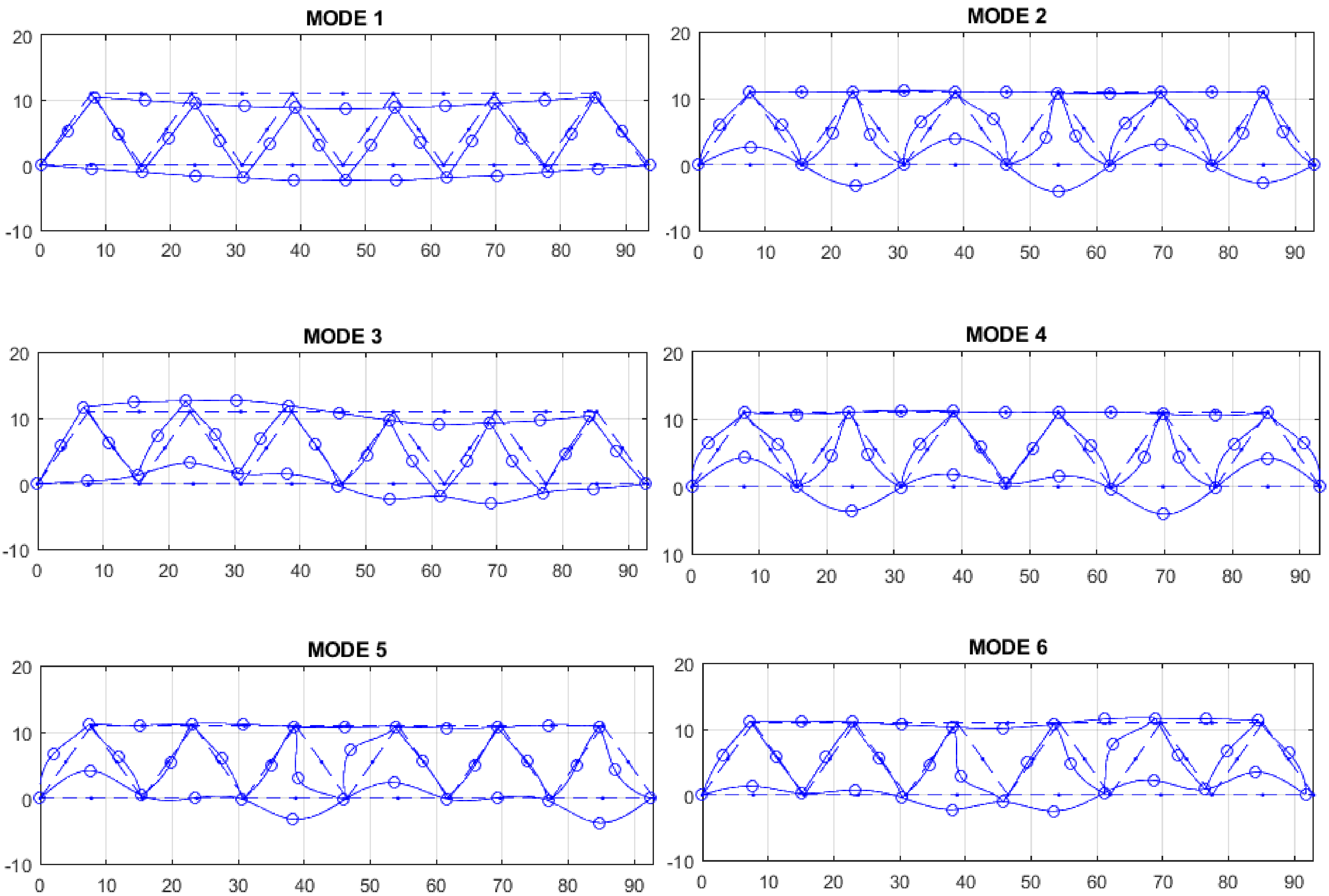
$\omega_1 = 2.52\, Hz$ $\qquad$ $\omega_4 = 5.67\, Hz$

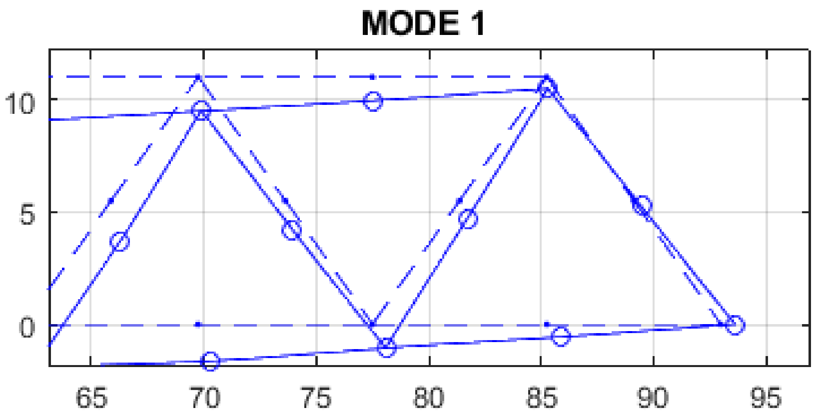
$\omega_2 = 5.24\, Hz$ $\qquad$ $\omega_5 = 6.4\, Hz$

$\omega_3 = 5.56\, Hz$ $\qquad$ $\omega_6 = 6.87\, Hz$

To compute the first 6 mode shapes the following system is solved:

$$(-\omega_i^2 [M_{FF}] + [K_{FF}])\underline{x_0} = \underline{0} \quad \rightarrow \quad \underline{x_0}^i \quad for\ i = 1{:}\,6$$





It is important to notice that the modes are not perfectly symmetric / antisymmetric due to the not symmetric constraints applied. In correspondence to the roller on the right end the structure can translate horizontally. This will have a consequence in the FRFs.

Computation of the damping matrix [C]

It is very common in civil structures that the assumption of structural damping is valid, and so the damping matrix [C] can be written as a linear combination of [M] and [K]: $[C] = \alpha[M] + \beta[K]$. The values of α and β can be computed by knowing the damping ratio ξ_i for at least two modes. ξ_i can be determined experimentally or, as it happens for very big and complex structures, from literature that considers similar cases.

Considering the system in modal coordinates:

$[\widetilde{M}] = [\phi]^T[M][\phi]$; $[\tilde{C}] = [\phi]^T[C][\phi]$; $[\widetilde{K}] = [\phi]^T[K][\phi]$ all diagonal matrices.

The α and β coefficients are still the same: $[\tilde{C}] = \alpha[\tilde{M}] + \beta[\tilde{K}]$, but now the equations are completely decoupled. By considering the first two modes:

$$\begin{cases} c_1 = \alpha m_1 + \beta k_1 \\ c_2 = \alpha m_2 + \beta k_2 \end{cases} ; \qquad \xi_i = \frac{c_i}{2 m_i \omega_i} \; ; \; \omega_i = \sqrt{\frac{k_i}{m_i}}$$

$$\begin{cases} \xi_1 = \dfrac{\alpha}{2\omega_1} + \dfrac{\beta\omega_1}{2} \\ \xi_2 = \dfrac{\alpha}{2\omega_2} + \dfrac{\beta\omega_2}{2} \end{cases} \quad by\ considering \begin{cases} \xi_1 = 0.75\% \\ \xi_2 = 1\% \end{cases} \rightarrow \begin{cases} \alpha = 0.2631 \\ \beta = 0.000213 \end{cases}$$

Finally coming back to the physical coordinates: $[C] = \alpha[M] + \beta[K]$.

Computation of the frequency response functions (FRFs)

1. Vertical displacement and acceleration of nodes A, B, C

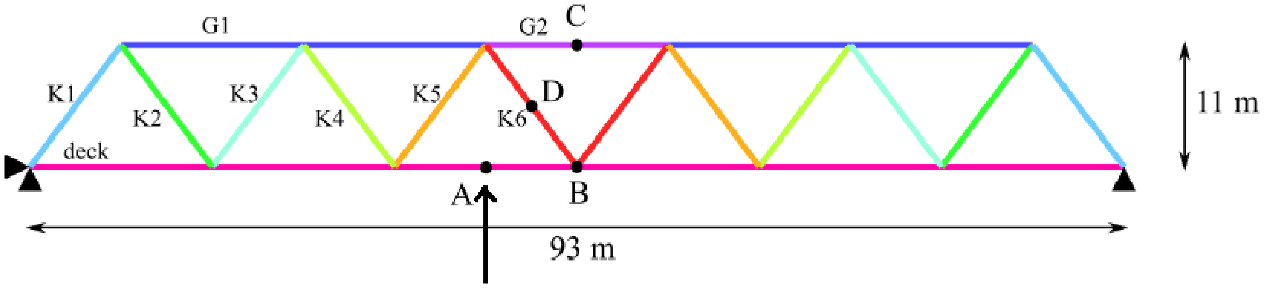
For the computation of the FRFs, a vertical harmonic input force $F = F_0 e^{i\Omega t}$ applied in node A is considered. The response in terms of vertical displacement and acceleration is computed on nodes A, B and C . only free nodes

The steady state response to a harmonic input is computed as:

$$[M_{FF}]\underline{\ddot{x}}_F + [C_{FF}]\underline{\dot{x}}_F[K_{FF}]\underline{x}_F = \underline{b} F_0 e^{j\Omega t} \qquad \underline{x}_F = \underline{x}_0 e^{j\Omega t}$$

$$(-\Omega^2[M_{FF}] + j\Omega[C_{FF}] + [K_{FF}])\underline{x}_0 e^{i\Omega t} = \underline{b} F_0 e^{j\Omega t} \qquad \underline{b} = (0 \ldots \ldots 1 \ldots \ldots 0)^T$$

$$b_i = 1 \qquad i = vertical\ displacement\ of\ node\ A$$



$$F = F_0 e^{i\Omega t}$$

$$\underline{G_{disp}}(j\Omega) = \frac{\underline{x_0}}{F_0} = (-\Omega^2[M_{FF}] + j\Omega[C_{FF}] + [K_{FF}])^{-1}\underline{b}$$

$\underline{G_{disp}}(j\Omega)$ is a $n_{dof}x1$ vector that contains the FRFs of the displacement of all the n_{dof} degrees of freedom of the systems.

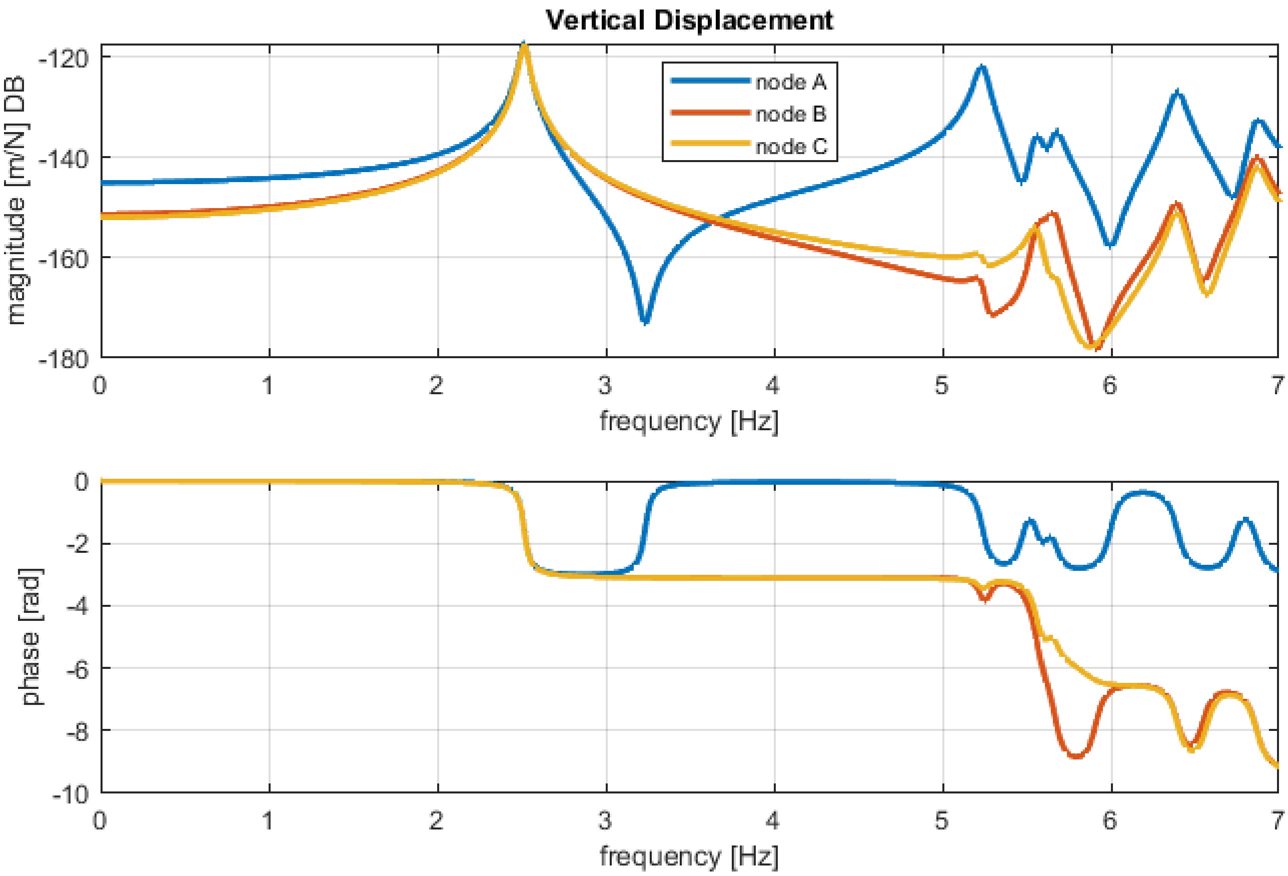
From this vector is possible to extract $\underline{G_{A\,disp,v}}(j\Omega)$, $G_{B\,disp,v}(j\Omega)$, $\underline{G_{C\,disp,v}}(j\Omega)$ of which is computed the correspondent magnitude $|G|$ and phase φ_G.

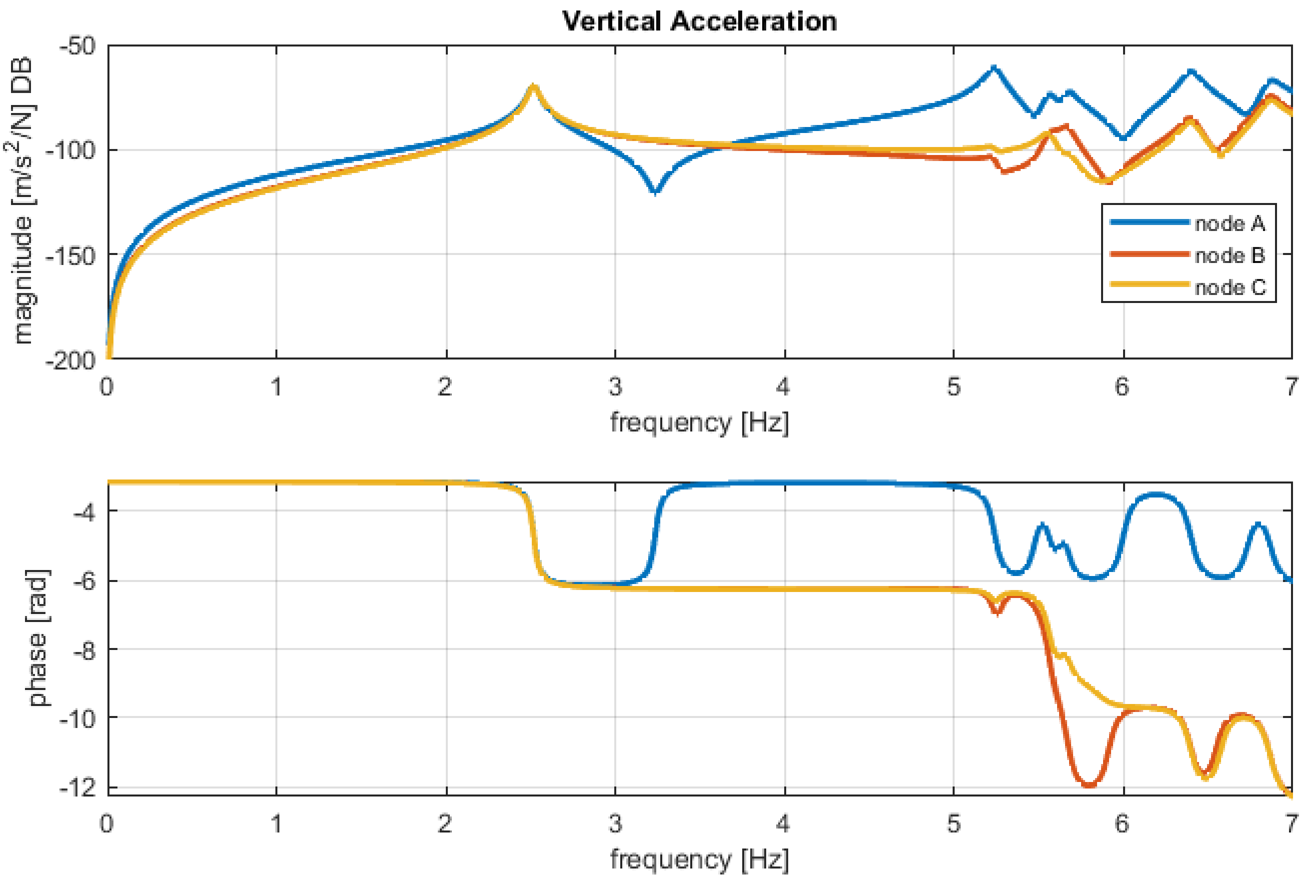
The acceleration is simply given by: $\underline{\ddot{x}_F} = -\Omega^2 \underline{x}_0 e^{j\Omega t}$ and so $\underline{G_{acc}}(j\Omega) = -\Omega^2 \underline{G_{disp}}(j\Omega)$

$(|\underline{G_{acc}}| = \Omega^2 |\underline{G_{disp}}| \; ; \; \varphi_{acc} = \varphi_{disp} - \pi)$.

Below are presented the FRFs obtained in terms of magnitude and phase.

The magnitude is plotted in DB (DeciBell) to better highlight the smaller peaks.





FRF of node A

Both the force and the response are on node A. It is possible to notice that all the six peaks in correspondence of the resonances are clearly visible. This because:

1) In point A all the six mode shapes don't have a node, so the force can provide energy in all the six modes.
2) The response is measured in the same point A, and since all the six modes don't have a node in A the response is not null.

The two previous statement can be proofed considering that near the resonances:

$$G_{jk}(j\Omega) \cong \frac{\mathrm{X}_{\mathrm{j}}^{(i)} X_k^{(i)}/m_i}{-\Omega^2 + j2\Omega\xi_i\omega_i + \omega_i^2} \qquad if \quad \mathrm{X}_{\mathrm{k}}^{(i)} = 0 \quad or \ \mathrm{X}_{\mathrm{j}}^{(i)} = 0 \ \rightarrow \ G_{jk}(j\Omega) = 0$$

Between every resonance there is a double zero, as it is clear from the phase diagram where the phase grows of π. This because since the input and output are located in the same point A:

$$G_{AA}(j\Omega) \cong \frac{{\mathrm{X}_{\mathrm{A}}^{(i)}}^2/m_i}{-\Omega^2 + j2\Omega\xi_i\omega_i + \omega_i^2} \quad \rightarrow \quad G_{AA}(j\omega_\mathrm{i}) \cong -\mathrm{j}\frac{{\mathrm{X}_{\mathrm{A}}^{(i)}}^2/m_i}{2\Omega\xi_i\omega_i}$$

$G_{AA}(\omega_\mathrm{i})$ is always a negative imaginary number, so $\varphi_{AA}(\omega_\mathrm{i}) = -\frac{\pi}{2}$. Since every time a resonance is crossed the phase has a drop of $-\pi$, in order to have always $\varphi_{AA}(\omega_\mathrm{i}) = -\frac{\pi}{2}$ there must be a zero in between each resonance to increase the phase to π and bring it back to 0.

FRF of node B/C

The response is measured in point B and C that are central nodes for the structure.
If the system was symmetric, the modes 2, 3 and 5 would have been antisymmetric and they would have had a node in point B and C: in that case $G_{AB}(j\omega_\mathrm{i}) = G_{AC}(j\omega_\mathrm{i}) = 0 \ for \ i = 2, 3, 5.$
In this case is not like that because the modes are not properly symmetric and antisymmetric due to the constraints applied that make the structure not symmetric.
So the peaks can be seen also in correspondence of the 2nd, 3rd, 5th natural frequencies, even if they are smaller compared to the ones in correspondence to the other frequencies since the response is measured very close to the nodes.

2. Constraint forces

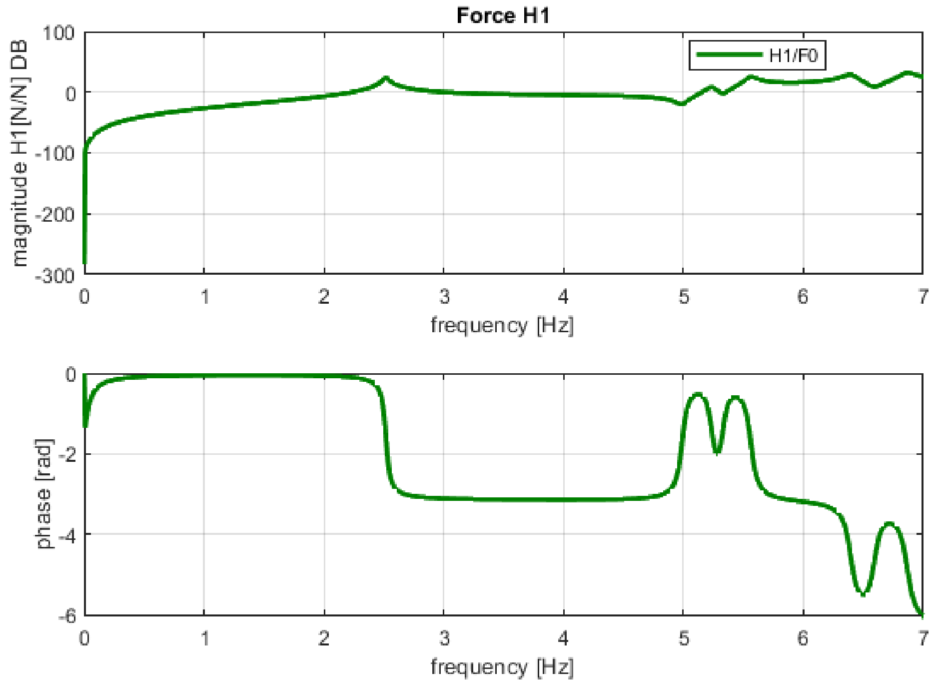
Once the response is known, the constraint forces can be computed as follow:

$$[M_{FC}]\underline{\ddot{x}_F} + [C_{FC}]\underline{\dot{x}_F} + [K_{FC}]\underline{x_F} = \underline{R} \qquad \underline{x_F} = \underline{x_0}e^{j\Omega t} = \underline{G_{disp}}F_0 e^{j\Omega t}\ ; \quad \underline{R} = \underline{R_0}e^{j\Omega t}$$

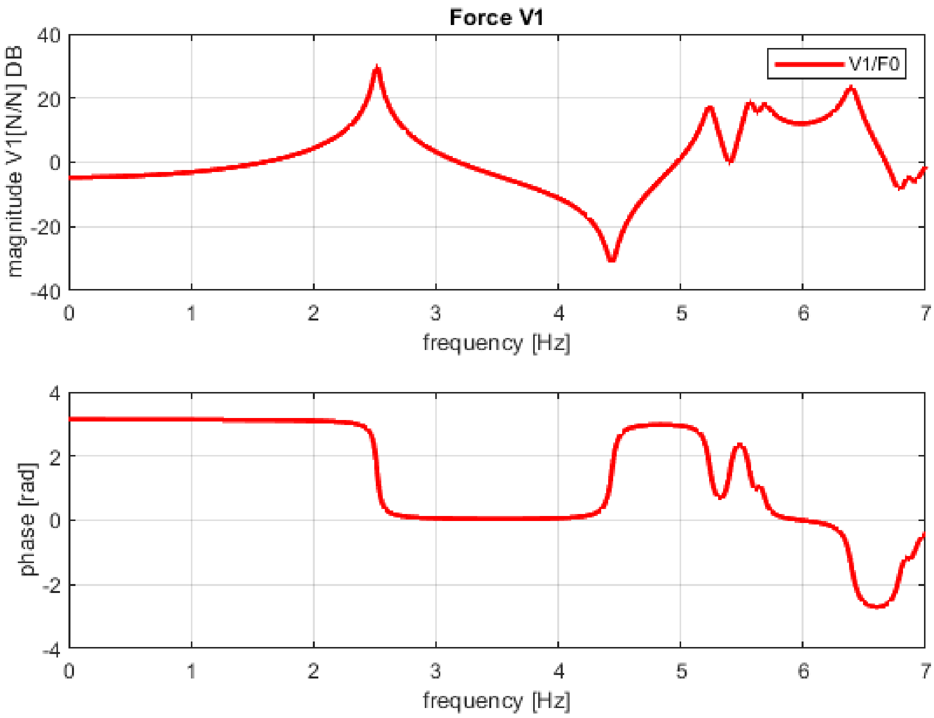
$$(-\Omega^2[M_{FC}] + j\Omega[C_{FC}] + [K_{FC}])\frac{\underline{\underline{x_0}}}{F_0} = \frac{\underline{R}}{F_0} \quad \rightarrow \quad (-\Omega^2[M_{FC}] + j\Omega[C_{FC}] + [K_{FC}])\underline{G_{disp}}(j\Omega) = \frac{\underline{R}}{F_0}$$

$\underline{R}$ directly contains the reaction forces projected in the global reference systems: $\underline{R} = (H_1, V_1, V_2)^T$.

The diagrams of constrain forces in terms of magnitude and phase are reported below.

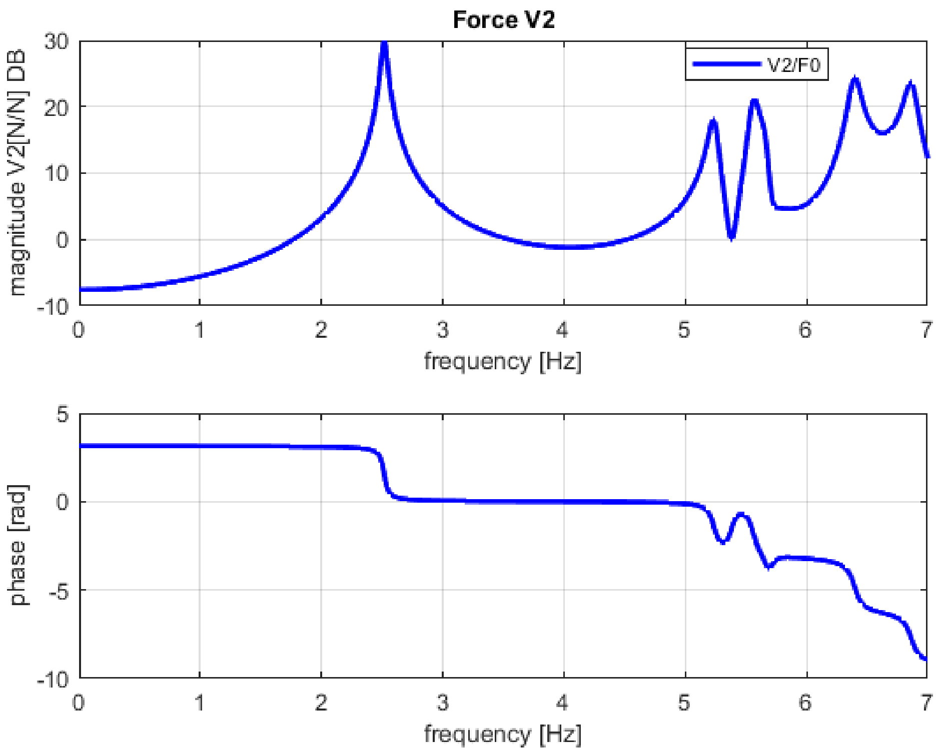


The magnitude of the horizontal constrain force H1 of the left end pin is quite low, also in correspondence of the resonances. It is clear that for $\Omega = 0$ (static condition) H1=0 ($\rightarrow -\infty$ in DB scale) as expected, since F0 is a vertical force.



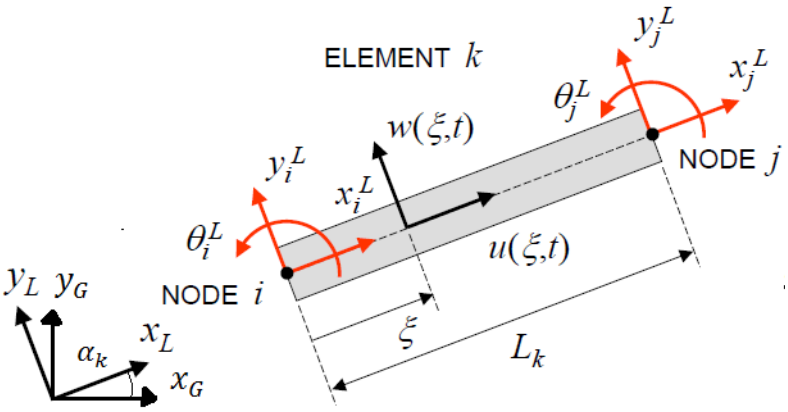
Both the vertical reactions V1 (left end) and V2 (right end) are higher of 10 DB in respect to the horizontal constrain force H1, that means x3.16 higher in linear scale.

Since in static condition the applied force F0 is considered positive upward when $\Omega = 0$ both the forces V1 and V2 are negative (phase=π).



3. Shear force, bending moment and axial force in nodes C and D due to force F

The internal forces are related to the transverse displacement $w(\xi,t)$ and to the axial displacement $u(\xi,t)$ of the element k in which the internal forces want to be computed.



$u(\xi,t)$ and $w(\xi,t)$ are displacements expressed in local coordinates. They are axial and transverse in respect to the beam axis that is rotated of an angle α_k in respect to the global reference system $x_G - y_G$.

The internal forces are computed as follow:

$$M(\xi,t) = EJ\frac{\partial^2 w(\xi,t)}{\partial \xi^2}\ ; \qquad T(\xi,t) = \frac{\partial M(\xi,t)}{\partial \xi} = EJ\frac{\partial^3 w(\xi,t)}{\partial \xi^3}\ ; \qquad N(\xi,t) = EA\frac{\partial u(\xi,t)}{\partial \xi}$$

The transverse and axial displacement fields of the element k are linked to the nodal coordinates $x_k^L(t) = \left(x_i^L,\ y_i^L,\ \theta_i^L,\ x_j^L,\ y_j^L,\ \theta_j^L\right)^T$ by means of the shape functions.

$$u(\xi,t) = \underline{f_u^T}(\xi)\underline{x_k^L}(t)$$

$$w(\xi,t) = \underline{f_w^T}(\xi)\underline{x_k^L}(t)$$

Where:

$$\underline{f_u}(\xi) = \begin{bmatrix} 1-\frac{\xi}{L_k} \\ 0 \\ 0 \\ \frac{\xi}{L_k} \\ 0 \\ 0 \end{bmatrix} \qquad \underline{f_w}(\xi) = \begin{bmatrix} 0 \\ 2\left(\frac{\xi}{L_k}\right)^3 - 3\left(\frac{\xi}{L_k}\right)^2 + 1 \\ L_k\left[\left(\frac{\xi}{L_k}\right)^3 - 2\left(\frac{\xi}{L_k}\right)^2 + \frac{\xi}{L_k}\right] \\ 0 \\ -2\left(\frac{\xi}{L_k}\right)^3 + 3\left(\frac{\xi}{L_k}\right)^2 \\ L_k\left[\left(\frac{\xi}{L_k}\right)^3 - \left(\frac{\xi}{L_k}\right)^2\right] \end{bmatrix}$$

The points C and D coincides with the nodes and they are in position $\xi = 0$ for the elements k considered. In this way for both C and D:

$$\frac{\partial^2 w(0,t)}{\partial \xi^2} = \underline{f^{II}{}^T_w}(0)\underline{x_k^L}(t) = \left(0\ ,\quad -\frac{6}{L_k^2}\ ,\quad -\frac{4}{L_k}\ ,\quad 0\ ,\quad \frac{6}{L_k^2}\ ,\quad -\frac{2}{L_k}\right)^T \underline{x_k^L}(t)$$

$$\frac{\partial^3 w(0,t)}{\partial \xi^3} = \underline{f^{III}{}^T_w}(0)\underline{x_k^L}(t) = \left(0\ ,\quad \frac{12}{L_k^3}\ ,\quad \frac{6}{L_k^2}\ ,\quad 0\ ,\quad -\frac{12}{L_k^3}\ ,\quad \frac{6}{L_k^2}\right)^T \underline{x_k^L}(t)$$

$$\frac{\partial u(0,t)}{\partial \xi} = \underline{f^{I}{}^T_u}(0)\underline{x_k^L}(t) = \left(-\frac{1}{L_k}\ ,\quad 0\ ,\quad 0\ ,\quad \frac{1}{L_k}\ ,\quad 0\ ,\quad 0\right)^T \underline{x_k^L}(t)$$

The nodal displacements $\underline{x_k^L}(t)$ of element k are known, since are related to the applied force F through the FRFs. But the FRFs consider the displacements expressed in global coordinates and so it is necessary to pass from the local to the global coordinates with the rotational matrix $[\Lambda_k]_{LG}$.

$$\underline{x_k^L}(t) = [\Lambda_k]_{LG}\underline{x_k^G}(t) \quad [\Lambda_k]_{LG\,6x6} = \begin{bmatrix} [\lambda_k]_{LG\,3x3} & [0]_{3x3} \\ [0]_{3x3} & [\lambda_k]_{LG\,3x3} \end{bmatrix} \quad [\lambda_k]_{LG} = \begin{bmatrix} \cos(\alpha_k) & \sin(\alpha_k) & 0 \\ -\sin(\alpha_k) & \cos(\alpha_k) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

By relating the nodal displacements with the FRFs: $\frac{\underline{x_k^G}(t)}{F_0} = \underline{G_{disp,k}}\,(\Omega)e^{j\Omega t}$

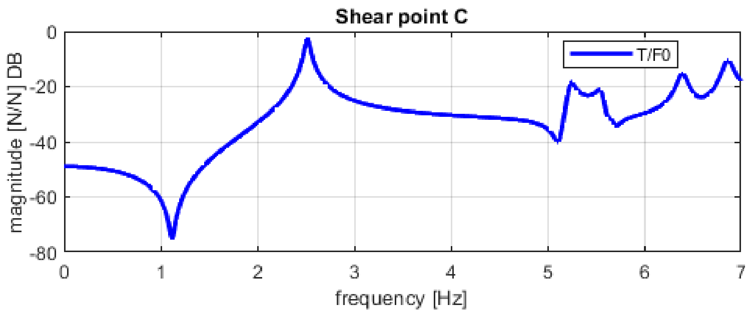
And finally considering: $M_i = M_{i,0}e^{j\Omega t}\,; \quad T_i = T_{i,0}e^{j\Omega t}\,; \quad N_i = N_{i,0}e^{j\Omega t} \quad for\ i = C, D$

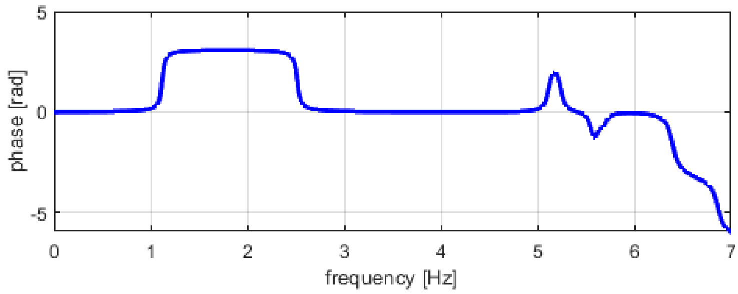
$$\frac{M_{i,0}(\Omega)}{F_0} = EJ\,\underline{{f^{II}}_w^T}(0)\,[\Lambda_k]\,\underline{G_{disp,k}}\,(\Omega) \qquad \frac{T_{i,0}(\Omega)}{F_0} = EJ\,\underline{{f^{III}}_w^T}(0)\,[\Lambda_k]\,\underline{G_{disp,k}}\,(\Omega)$$

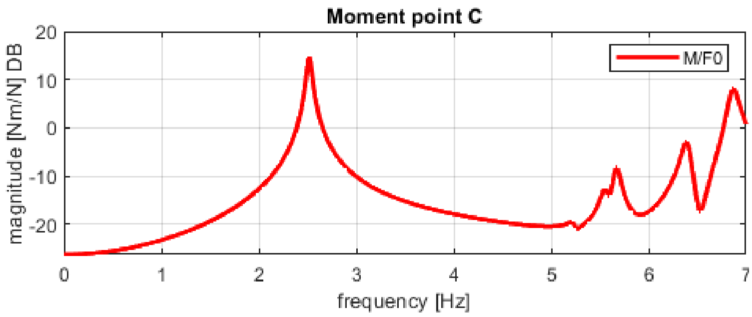
$$\frac{N_{i,0}(\Omega)}{F_0} = EA\,\underline{{f^{I}}_u^T}(0)\,[\Lambda_k]\,\underline{G_{disp,k}}\,(\Omega)$$

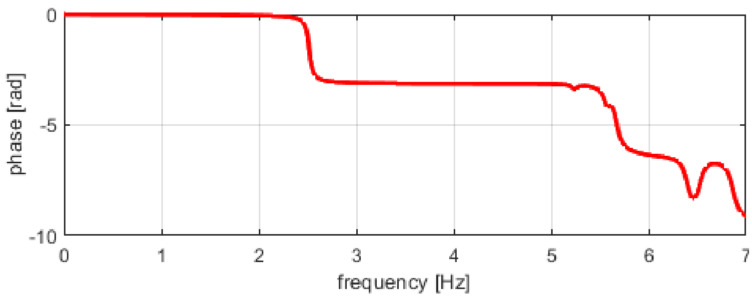
The internal forces obtained in terms of magnitude and phase are reported below:

Point C

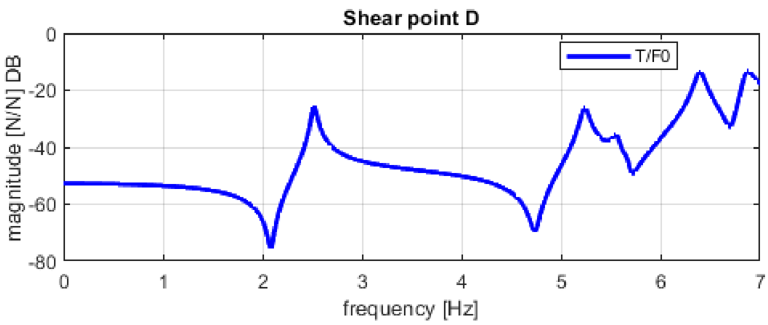


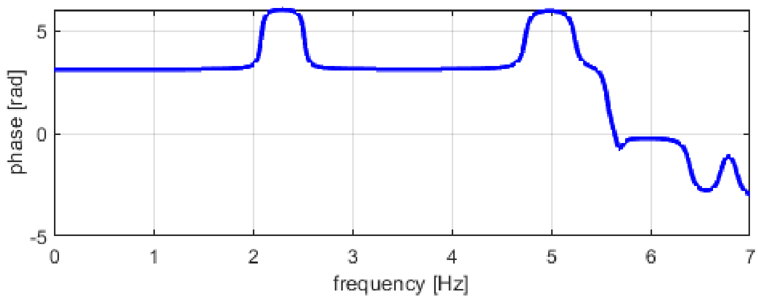


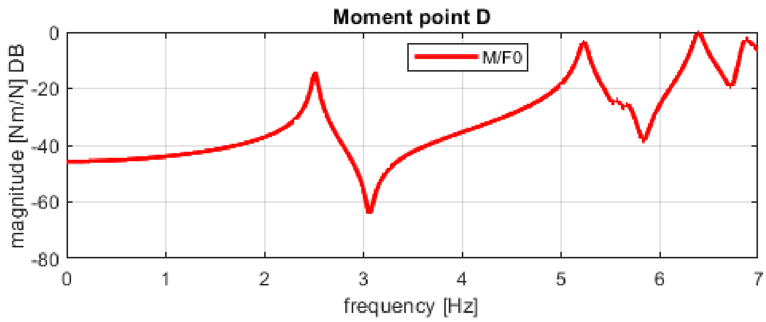


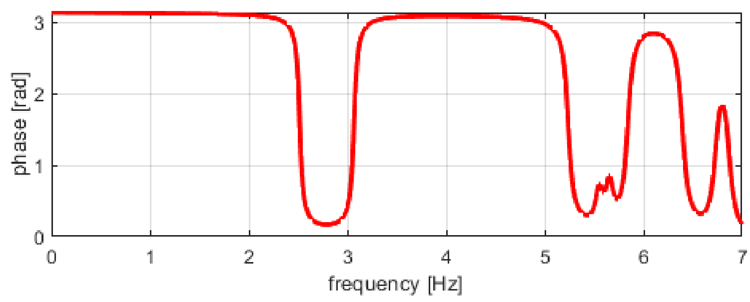


Point D

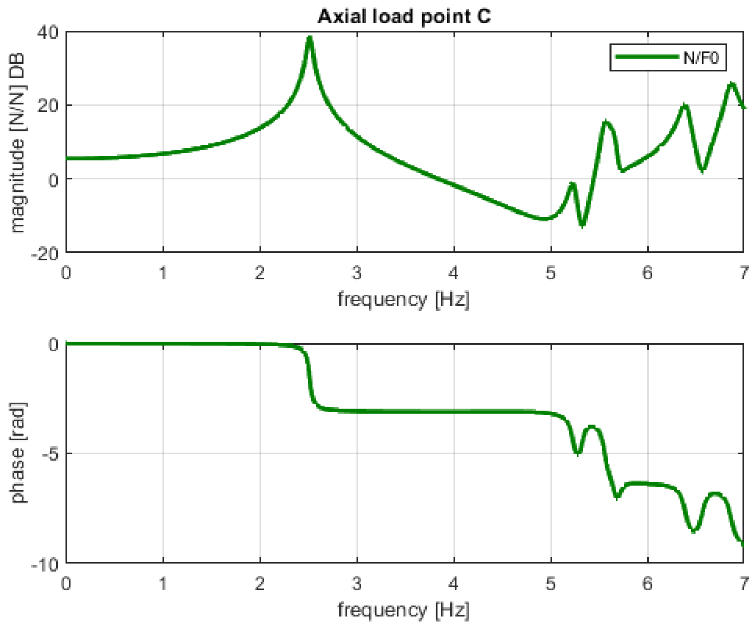




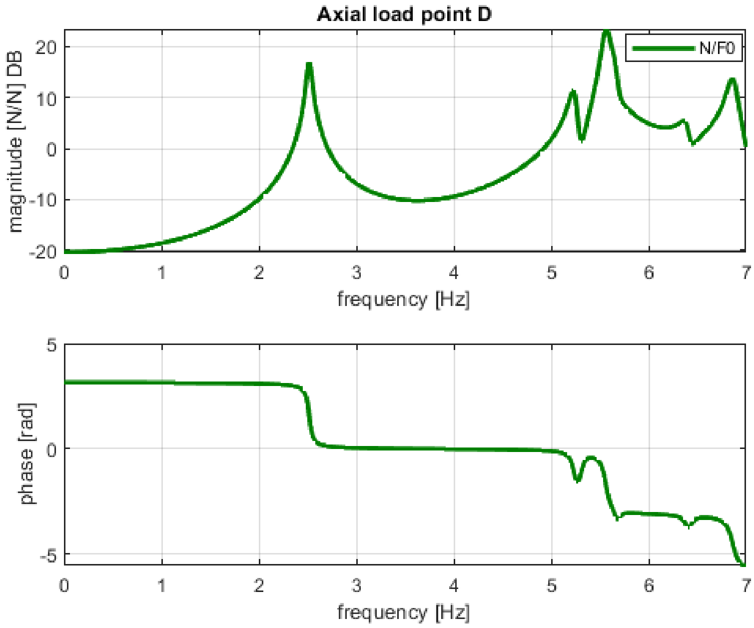




Point C



Point D



FRF with modal superposition approach (MSA)

Computing the FRFs using the MSA can be very useful especially when the structure has a very large number of nodes, and so a large number of degrees of freedom. Since for every d.o.f. a natural frequency is computed, the response will be given by the superposition of a lot of modes, most of them excited at a frequency a lot higher than the frequency range of interest for that structure. This can result in a very slow computation of the response by the FEM software.

In case of the truss bridge considered, the frequency range of interest is 0-7 Hz that cover only the first six modes. It makes sense to compute the response considering only the superposition of that six modes and neglect the other ones, since near the resonances:

$$G_{AA}(j\Omega) \cong \frac{{\mathrm{X}_{\mathrm{A}}^{(i)}}^{2}/m_i}{-\Omega^2 + j2\Omega\xi_i\omega_i + \omega_i^2}$$

By considering only the first three modes: $[\phi]_{nx3} = \left(\underline{\phi}^1, \underline{\phi}^2, \underline{\phi}^3\right)^T$: ?

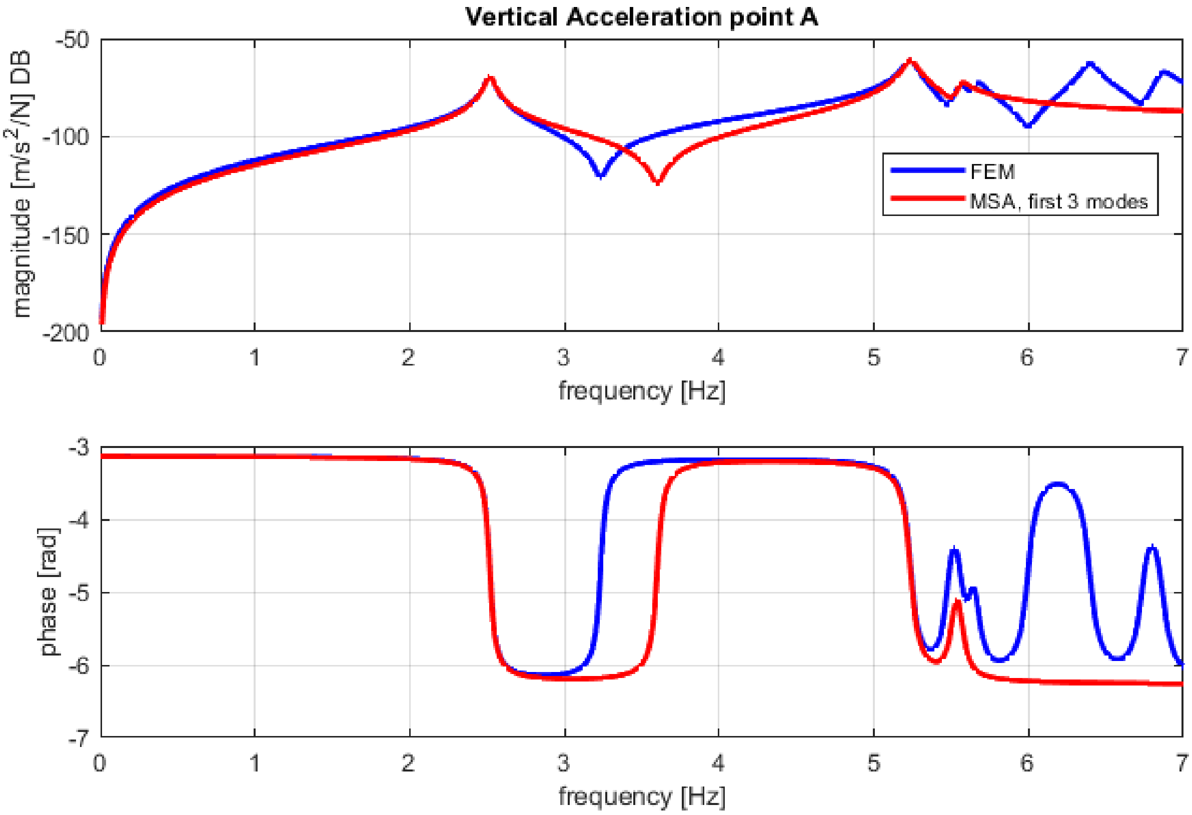
$$\left[\widetilde{M}_{FF}\right]_{3x3} = [\phi]^T[M_{FF}][\phi]\,; \quad \left[\tilde{C}_{FF}\right]_{3x3} = [\phi]^T[C_{FF}][\phi]\,; \quad [\widetilde{K}_{FF}]_{3x3} = [\phi]^T[K_{FF}][\phi]$$

$$\left[\widetilde{M}_{FF}\right]\underline{\ddot{q}} + \left[\tilde{C}_{FF}\right]\underline{\dot{q}} + \left[\widetilde{K}_{FF}\right]\underline{q} = [\phi]^T\underline{b}F_0e^{j\Omega t} \qquad \underline{q_{3x1}} = \underline{q_0}e^{j\Omega t} \text{ firs 3 modal coordinates}$$

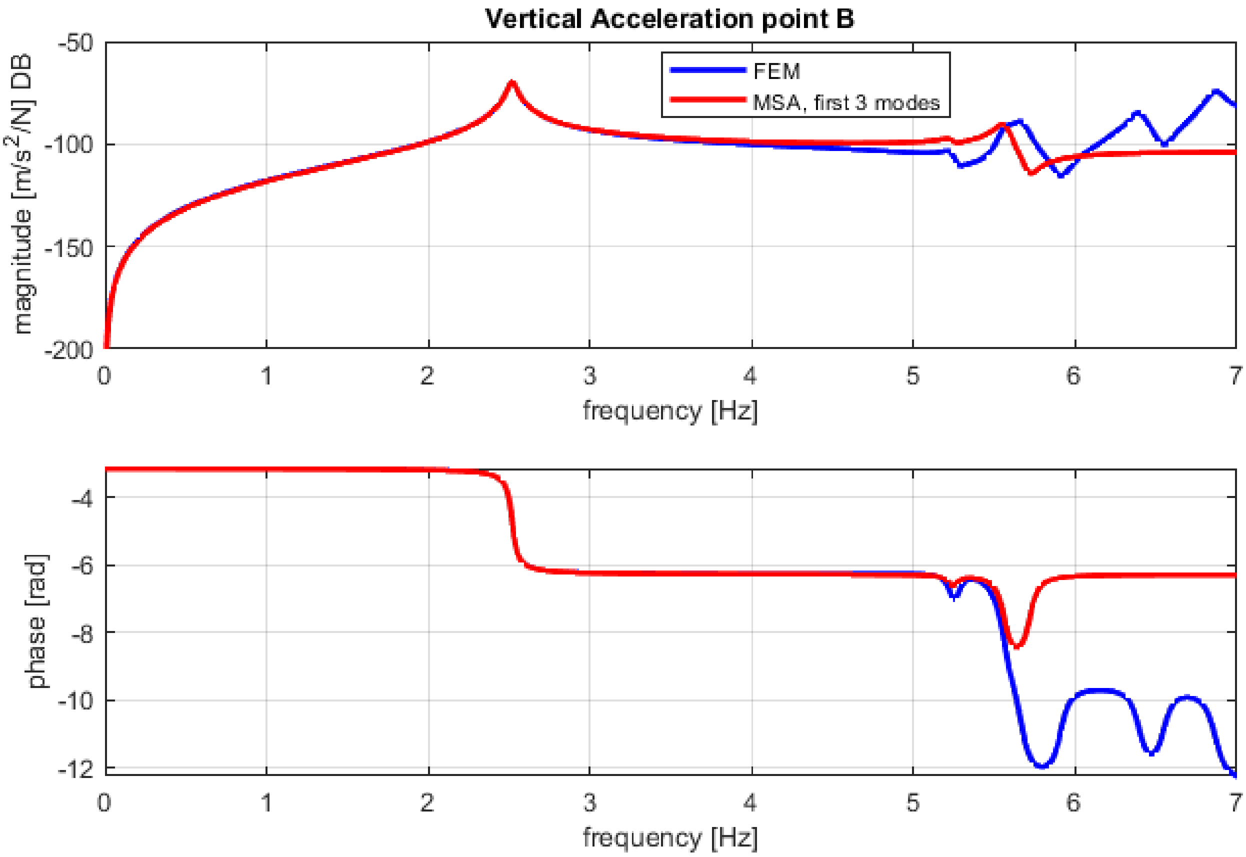
$$\left(-\Omega^2\left[\widetilde{M}_{FF}\right] + i\Omega\left[\tilde{C}_{FF}\right] + \left[\widetilde{K}_{FF}\right]\right)\underline{q_0}e^{j\Omega t} = [\phi]^T\underline{b}F_0e^{i\Omega t}$$

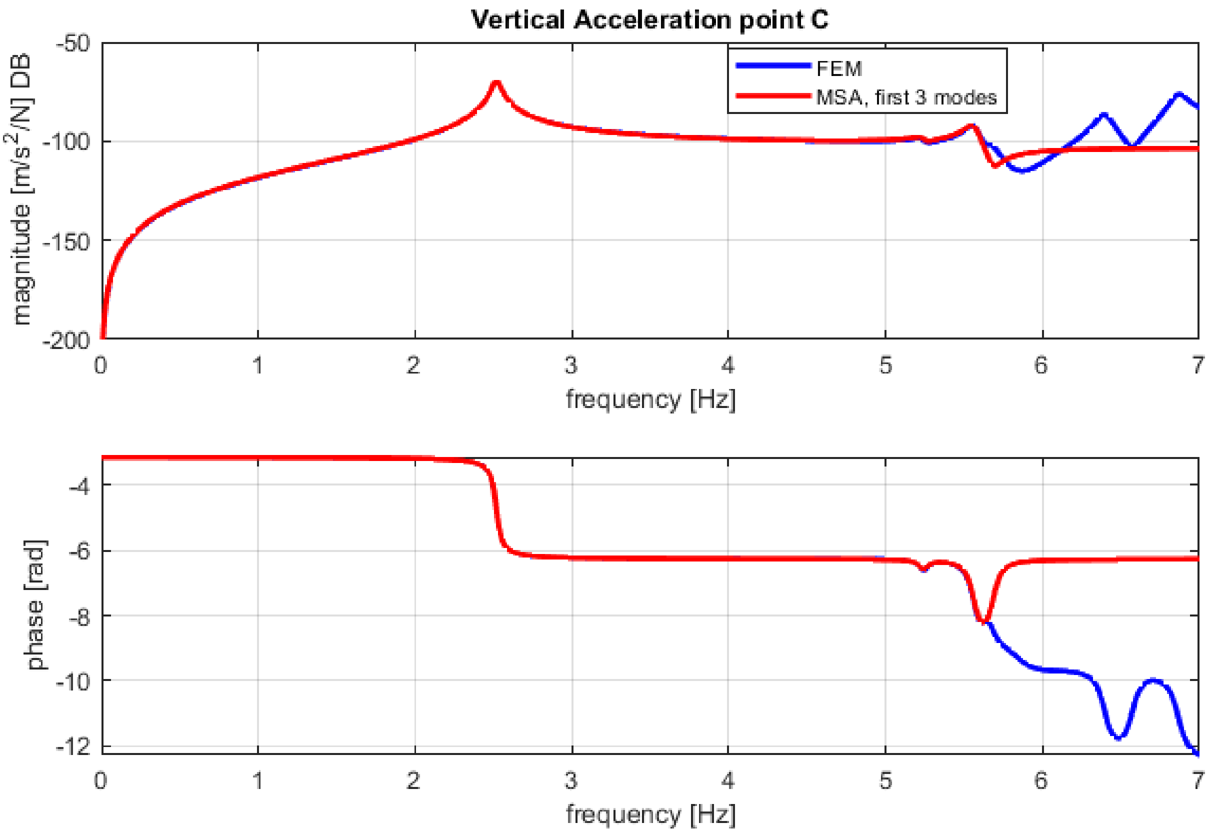
And finally, the response in physical coordinates will be: $\underline{x_{nx1}} = [\phi_{nx3}]\underline{q_{3x1}}$.

Here are reported the results obtained:



The red FRF, obtained using the MSA, approximates well the FRF obtained with the FEM for a frequency lower than ω_3 except in correspondence of the double zero. This is due to the fact that with the MSA the approximation is correct near the resonances, but not away from them because the upper modes (not considered) contribute to the response in the quasi static region, changing the location of the zero.

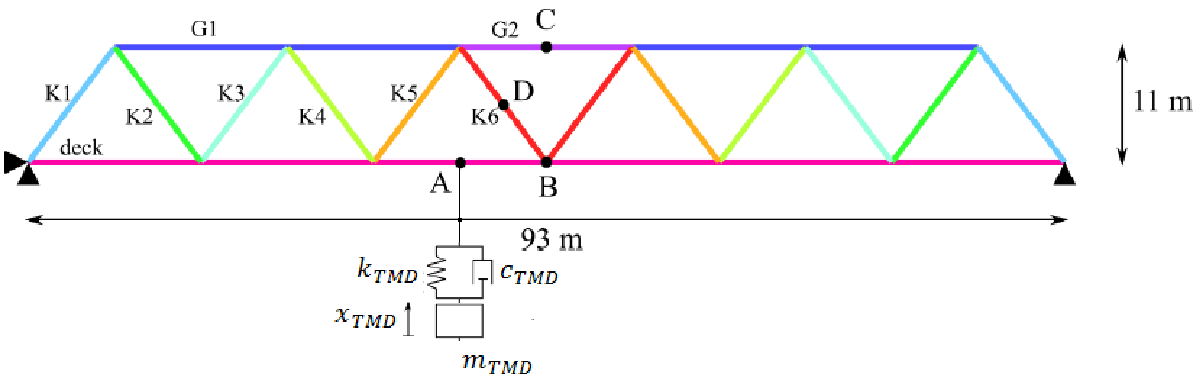




Installation of a Tuned Mass Damper (TMD)

The TMD is a device used to reduce the vibration's amplitude of mechanical systems, especially in very big structures like skyscrapers. It is a harmonic absorber that absorbs the energy from the structure and dissipate it by oscillating itself. The TMD alone is designed to have a natural frequency tuned with the frequency of the most critical resonant mode of the structure that can be excited. In this way when the system is forced by an external harmonic force $F = F_0 e^{j\Omega t}$ with $\Omega = \omega_i$ (natural frequency of the critical resonant mode), the TMD apply a force itself on the system in order to balance instant by instant the external force, reducing the vibrations of the system at that frequency $\Omega = \omega_i$.

In the case of the truss bridge considered, the installation of the TMD has the aim to reduce by at least 15% the amplitude of vertical vibration of point A of the first resonant mode (excited at a frequency $\omega_1 = 2.52\ Hz$), considering a vertical harmonic force $F = F_0 e^{i\Omega t}$. The mass of the TMD is limited to a max of 2% of the total mass of the structure, while the damping ratio of the TMD alone must not exceed the max value $\xi_{TMD_{max}} = 30\%$.
The TMD can be modelled as a one degree of freedom system mounted in correspondence of the most critical point of the structure in which is needed to reduce its vibration amplitudes. In this case, the TMD will be installed on point A and its natural frequency will be tuned to the one of the first mode: $\omega_{TMD} = \sqrt{k_{TMD}/m_{TMD}} = \omega_1 = 15.82\ rad/s$.



The system with the TMD is a different system in respect to the original one: the matrices [M], [C] and [K] must be modified because of the TMD, and 1 degree of freedom must be added to the structure. The matrices will be modified as follow:

$$[M^*]_{106x106} = \begin{bmatrix} [M_{FF}] & \underline{0}_{105x1} \\ \underline{0}^T_{1x105} & m_{TMD} \end{bmatrix};$$?

$$[K^*]_{106x106} = \begin{bmatrix} [K_{FF}] + [K_{TMD}] & \underline{k}_T \\ \underline{k}^T{}_T & k_{TMD} \end{bmatrix}; \quad [C^*] = \begin{bmatrix} [C_{FF}] + [C_{TMD}] & \underline{c}^T{}_T \\ \underline{c}^T{}_T & c_{TMD} \end{bmatrix}; \quad \text{where:}$$

$$[K_{TMD}] = \begin{array}{c} \begin{array}{ccccccccc} & & & y_A & & & & & \end{array} \\ \left[\begin{array}{ccc|c|ccc} 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & k_{TMD} & 0 & \cdots & 0 \\ \hline 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \end{array}\right] \end{array} \begin{array}{c} \\ \\ \\ \\ y_A \\ \\ \\ \\ \end{array}$$

$$\begin{array}{c} \qquad\qquad\qquad y_A \\ \underline{k}_T = (0 \quad \cdots \quad \boxed{-k_{TMD}} \quad \cdots \quad 0)^T \end{array}$$

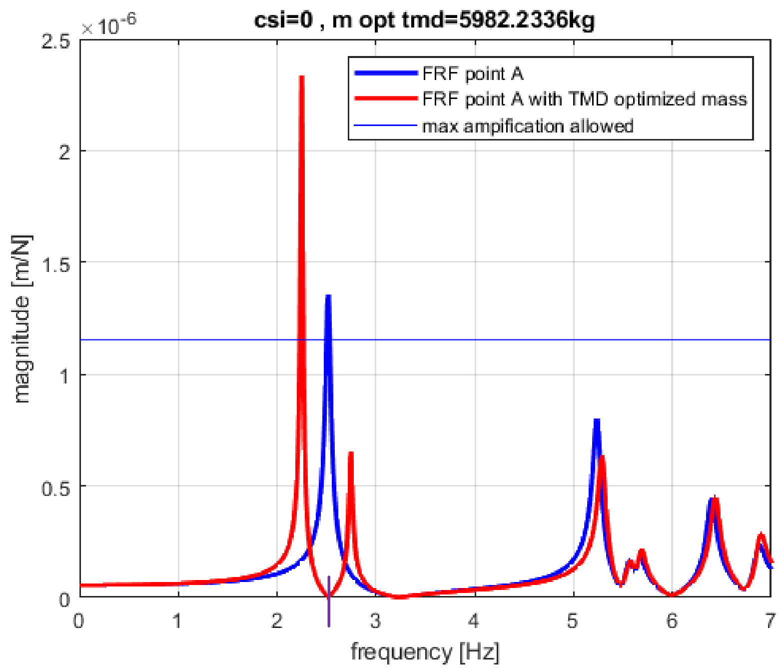
(same for [C])

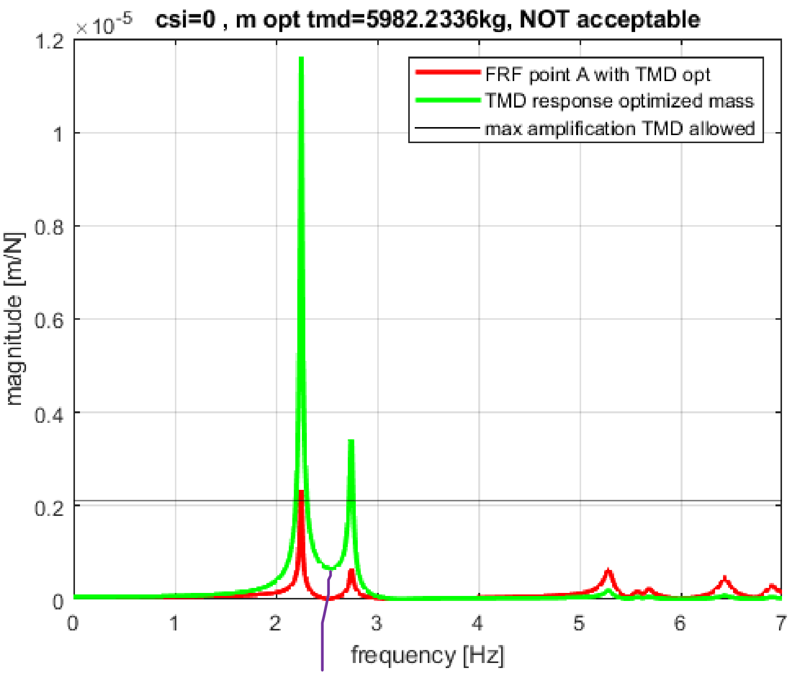
In order to choose the best TMD, an optimization analysis has been done: for different values of $0 < \xi_{TMD} < 0.3$ it is selected the minimum mass of the TMD that can satisfy the requirements previously reported (it is used a safety coefficient of $\eta = 2$ to reduce the max amplification of point A from the 15% required to $2x15\% = 30\%$). An additional constraint is imposed on the max vibration amplitude of the TMD. The constraints are:

$$\begin{cases} \xi_{TMD} < 0.3 & max\ TMD\ damping \\ m_{TMD} < 2\% m_{tot} = 5982kg & max\ TMD\ mass \\ \left|G_{A_{TMD}}\right|_{\omega_{1max}} = 0.85|G_A|_{\omega_1} = 1.2x10^{-6}\ mm/N & max\ FRF\ amplitude \rightarrow\ m_{TMD_{opt}}\ (minimum) \\ |G_{TMD}|_{max} < 1.5|G_A|_{\omega_1} = 2.1x10^{-6}\ mm & max\ TMD\ amplification \end{cases}$$?

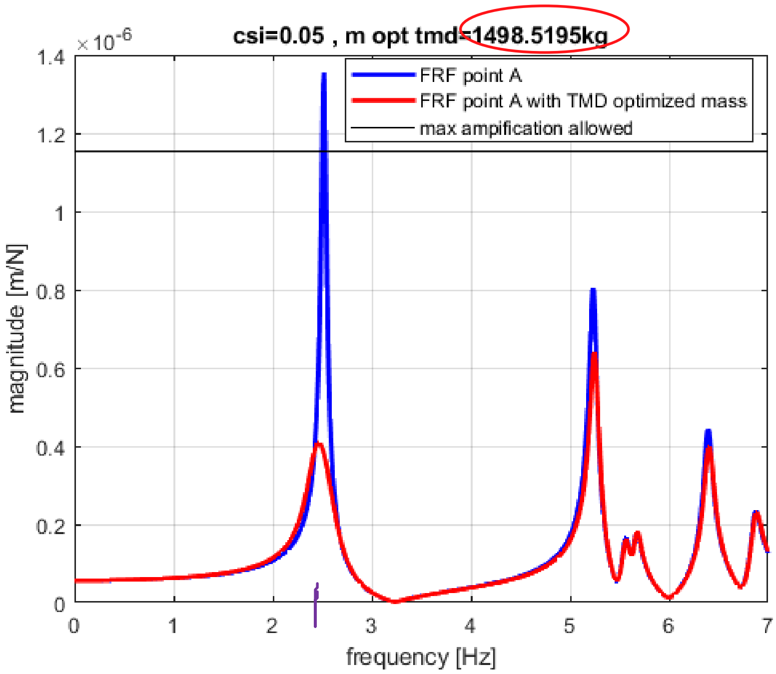
Once the value of the mass is known: $k_{TMD} = \omega_1^2 m_{opt_{TMD}}$.

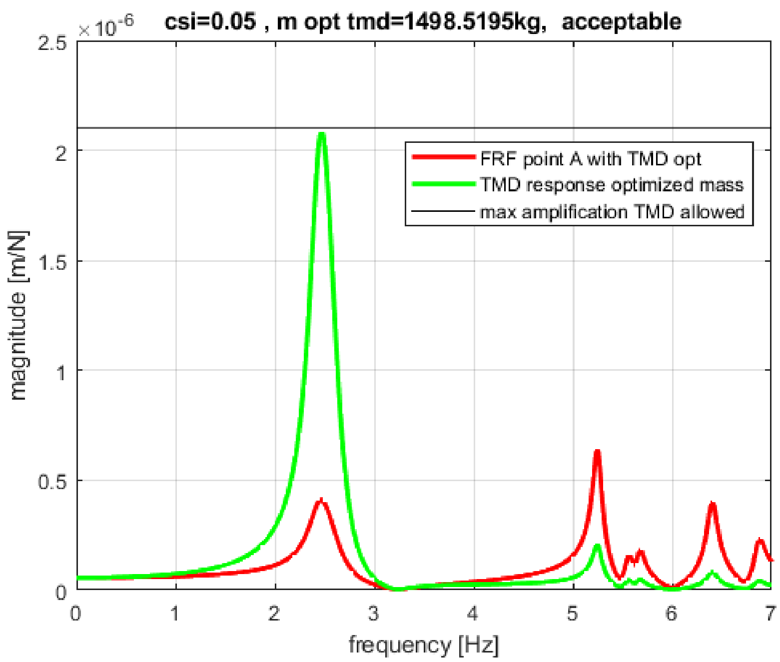
Here are reported the results obtained. The plots are in linear scale.



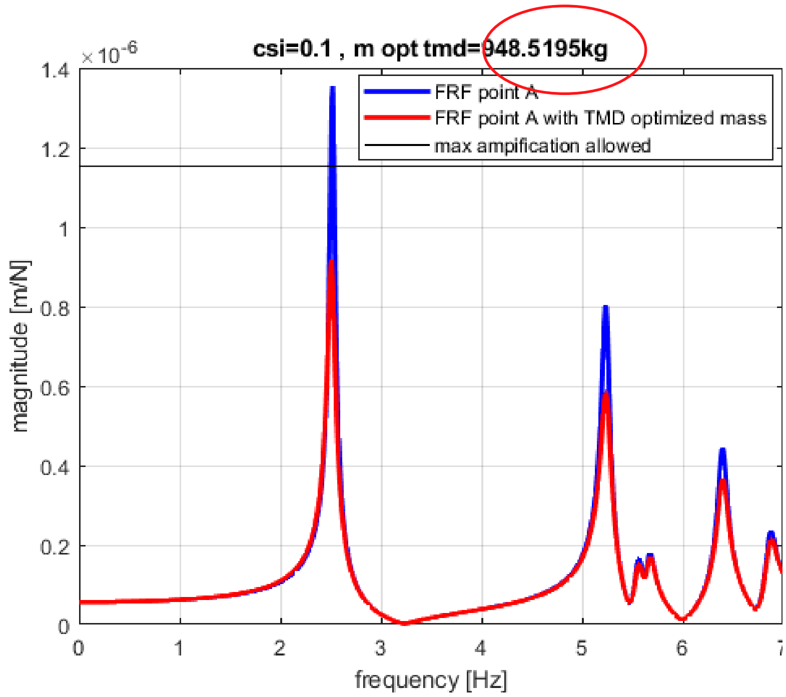


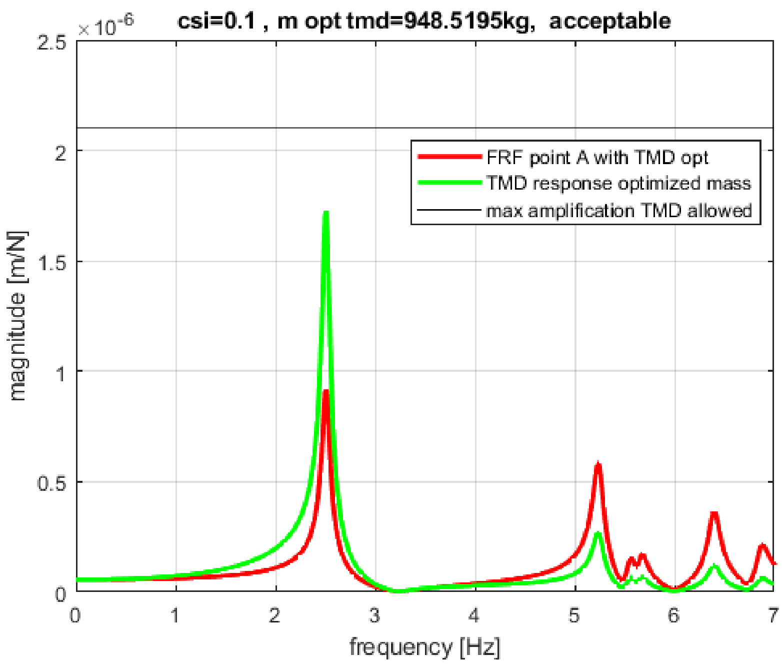
This solution with $\xi_{TMD} = 0$ is not correct: it reduces to zero the amplification in correspondence of ω_1 but in the two peaks on the left and on the right the amplification is really high. In this case the TMD just shift the resonance peaks without damping the response. In addition, the TMD amplification at resonance is too high since it is not damped.



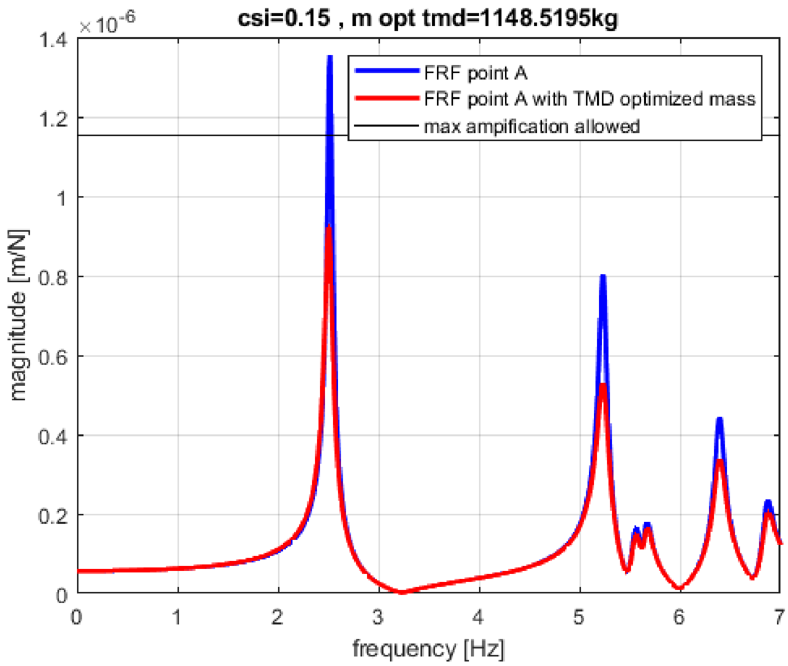


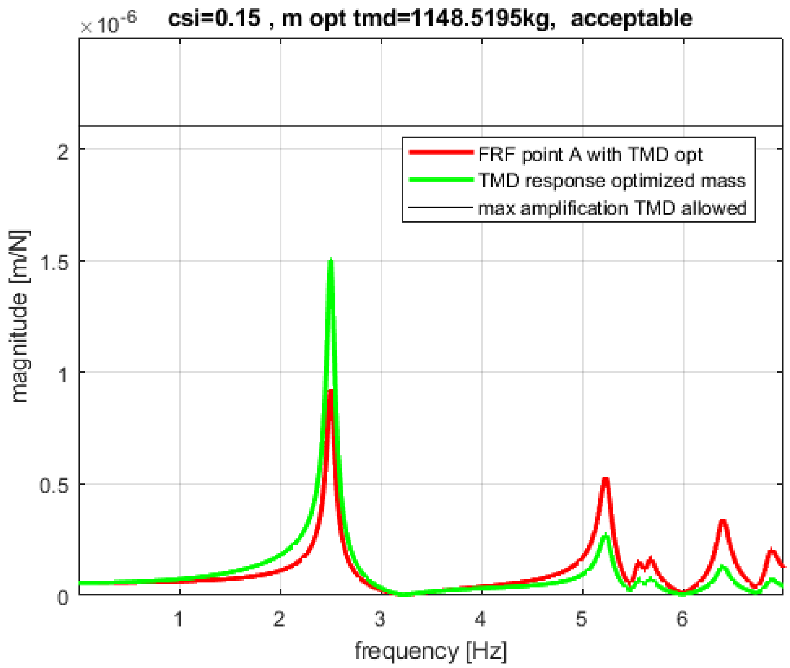
If $\xi = 0.05$ the TMD is too lowly damped and although the dynamic amplifications in point A are well damped, a too low ξ causes huge amplifications of the TMD itself: in order to reduce the TMD amplification a big mass of the TMD is needed.

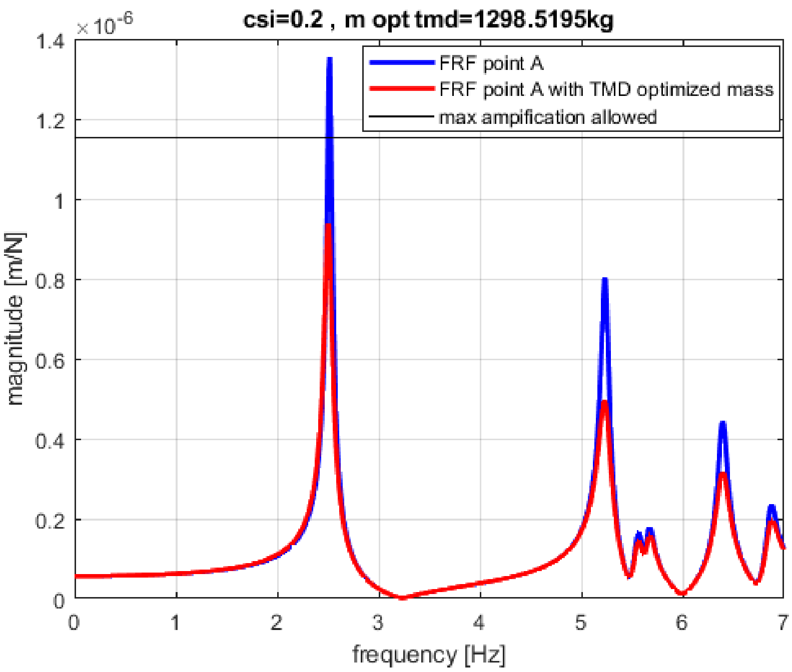


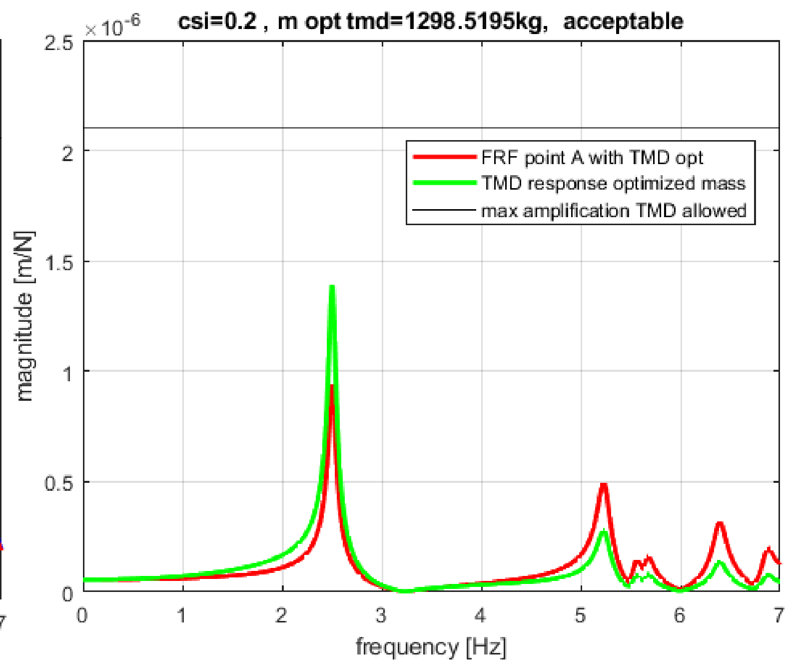


The solution with $\xi = 0.1$ is the best solution: that value of ξ is the best trade off to obtain low dynamic amplifications of point A (that usually requires relatively low TMD damping) and acceptable amplifications of the TMD (that usually requires relatively high TMD damping or as alternative high TMD mass).









By increasing ξ , the mass required to keep the dynamic response of point A sufficiently damped is higher. The amplitude of the TMD alone decrease, so the TMD with a higher ξ can bee a good solution if its dynamic amplitude is needed to be kept lower.

The parameters on the TMD chosen are:

$$\begin{cases} \xi_{TMD}=10\% \\ m_{TMD_{opt}}=949\ \mathrm{kg} \\ k_{TMD}=238000\ \mathrm{N/m} \end{cases}$$

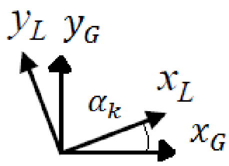
Static deflection of the structure due to his own weight (only deck and total)

In static conditions, the equation of motion is simplified to:

$$[K_{FF}]\underline{x_F} = \underline{F^G} \quad since \quad \underline{\dot{x}_F} = \underline{\ddot{x}_F} = \underline{0} \;\; ; \qquad \underline{F^G} = \left(F_1^G, F_2^G, \dots, F_i^G, \dots, F_{ndof}^G\right)^T$$

$\underline{F^G}$ is the vector containing the sum of the equivalent weight forces of each beam element k concentrated at the nodes. These forces are equivalent in terms of energy introduced inside the system. In reality the weight is a distributed load, and for each beam element k it can computed as:

$$\overrightarrow{p_k^G} = -g m_k \vec{j} = p_{y,k}^G \vec{j} \qquad p_k \left[\frac{\mathrm{N}}{\mathrm{m}}\right] \qquad k = 1 : N_{elements}$$



In global coordinates: $\underline{p_k^G} = \left(0 \,,\, p_{y,k}^G\right)^T$;

In local coordinates: $\underline{p_k^L} = \left(p_{y,k}^G \sin(\alpha_k) \;,\; p_{y,k}^G \cos(\alpha_k)\right)^T = \left(p_x^L \,, p_{y_k}^L\right)^T$

Starting from the definition of virtual work:

$$\delta W_{ext,k} = \int_0^{L_k} p_{x,k}^L \, \delta u(\xi, t) d\xi + \int_0^{L_k} p_{y,k}^L \, \delta w(\xi, t) d\xi$$

$$\begin{cases} \delta u(\xi,t) = \underline{f_u}(\xi)^T \delta \underline{x_k^L}(t) = (\delta \underline{x_k^L}(t))^T \underline{f_u}(\xi) \\ \delta w(\xi,t) = \underline{f_w}(\xi)^T \delta \underline{x_k^L}(t) = (\delta \underline{x_k^L}(t))^T \underline{f_w}(\xi) \end{cases} \qquad \underline{x_k^L}(t) = \left(x_i^L,\; y_i^L,\; \theta_i^L,\; x_j^L,\; y_j^L,\; \theta_j^L\right)^T$$

$$\delta W_{ext,k} = (\delta \underline{x}_k(t))^T \int_0^{L_k} [p_{x,k}^L \, \underline{f_u}(\xi) d\xi + p_{y,k}^L \underline{f_w}(\xi)] d\xi = (\delta \underline{x}_k(t))^T \underline{P_k^L}$$

Switching back to global coordinates:

$$\underline{P_k^G}_{6x1} = [\Lambda_k]_{GL_{6x6}} \underline{P_k^L}_{6x1} \quad \text{with} \quad [\Lambda_k]_{GL} = \begin{bmatrix} [\lambda_k]_{GL} & [0] \\ [0] & [\lambda_k]_{GL} \end{bmatrix} \qquad [\lambda_k]_{GL} = \begin{bmatrix} \cos(\alpha_k) & -\sin(\alpha_k) & 0 \\ \sin(\alpha_k) & \cos(\alpha_k) & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

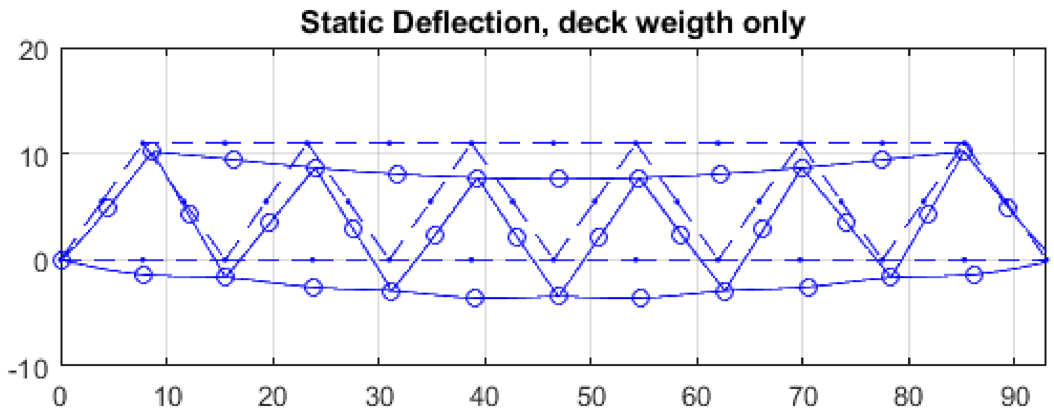
By means of the extraction matrices $[\mathrm{E}_k]$ it is possible to perform the final assembly of $\underline{F}^G$:

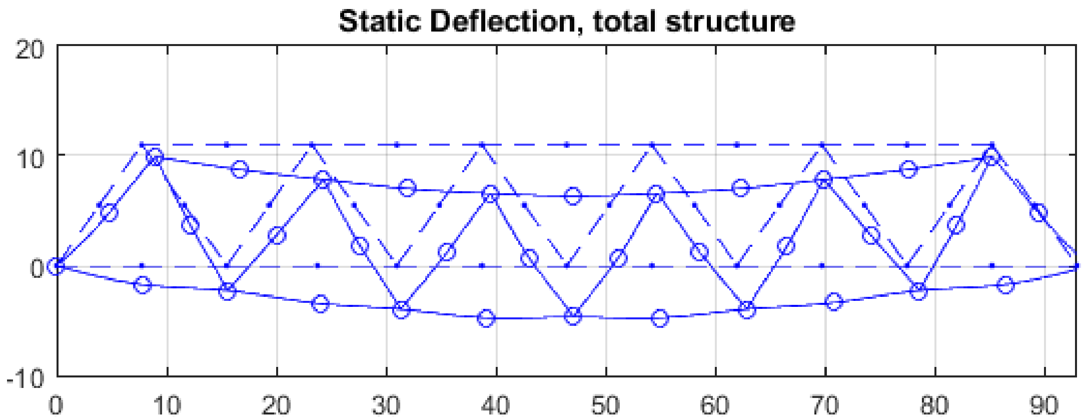
$$\underline{F_k^G}_{ndofx1} = [\mathrm{E}_k]^T{}_{ndofx6} \underline{P_k^G}_{6x1} \qquad [\mathrm{E}_k]_{ndofx6} = \begin{array}{c} \begin{array}{cccccc} & & \underline{n}_{k_j} & & \underline{n}_{k_i} & \end{array} \\ \begin{bmatrix} [0]_{3x3} & \cdots & [I]_{3x3} & \cdots & [0]_{3x3} & \cdots & [0]_{3x3} \\ [0]_{3x3} & \cdots & [0]_{3x3} & \cdots & [I]_{3x3} & \cdots & [0]_{3x3} \end{bmatrix} \end{array} \begin{array}{c} \underline{x}_{k_j} \\ \underline{x}_{k_i} \end{array}$$

And finally, by summing all the contributes: $\underline{F_{ndofx1}^G} = \sum_{k=1}^{n_k} \underline{F_k^G}_{ndofx1}$

The static nodal displacements will be: $\underline{x_F} = [K_{FF}]^{-1} \underline{F}^G$.

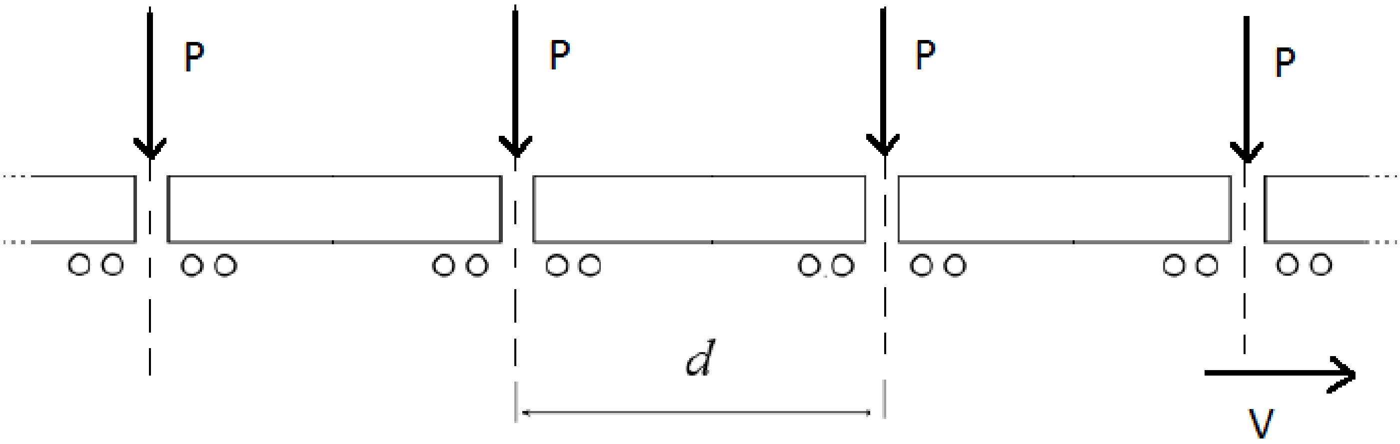
If only the deck weight is considered: k = beam elements composing the deck.
If the total structure weight is considered: k = beam elements composing the entire structure





Computation of the critical speeds

If the bridge is crossed by a long train, the weight forces of the train's wagons are like an infinite sequence of moving loads acting on the whole bridge.



Defining:

$T = L/V$ time needed for a single wagon of weight P to cross the bridge.

$\tau = d/V$ periodicity of the infinite sequence of loads P acting on the bridge.

In order to find the response of the system due to this loading condition is necessary to switch to modal coordinates: $[\widetilde{M}_{FF}]\underline{\ddot{q}} + [\tilde{C}_{FF}]\underline{\dot{q}} + [\widetilde{K}_{FF}]\underline{q} = \underline{Q}$.

In modal coordinates the system is completely decoupled since all the matrices are diagonal, so:

$$[\tilde{M}_{FF}]\ddot{q}_i + [\tilde{C}_{FF}]\dot{q}_i + [\tilde{K}_{FF}]q_i = Q_i \qquad i = 1{:}n_{dof}$$

Q_i is the generalized force resulted by the sum of the infinite sequence of loads and it is a generic periodic function with a fundamental frequency equal to $f_0 = 1/\tau \rightarrow \Omega_0 = 2\pi/\tau$. A generic periodic function can be developed in Fourier series as follow:

$$Q_i = \sum_{k=1}^{\infty} Q_{ik} \cos(k\Omega_o t + \varphi_{ik}) \qquad \Omega_0 = \frac{2\pi}{\tau}\ ; \quad f_0 = \frac{1}{\tau}$$

If $k\Omega_o = \omega_i$ (i^{th} natural frequency of the system), the bridge can be excited in resonance.

$$kf_0 = f_i \quad \rightarrow \quad \frac{k}{\tau} = f_i \quad \rightarrow \quad k\frac{V}{d} = f_i \quad \rightarrow \quad V_{cr} = \frac{f_i d}{k}$$

Typically the most dangerous situation corresponds to the first vibration modes and for values of $k \leq 3$ in which the amplitude of the harmonic components Q_{ik} are larger.

In the specific case of the truss bridge analyzed, considering the first 2 vibration modes and $k \leq 3$, the following critical speeds are computed:

Critical speeds [km/h]		
k \ f_i	2.52 Hz	5.24 Hz
1	249.25	518.49
2	124.62	259.24
3	83.08	172.83

A dangerous condition can be represented by the following critical speeds:

$$\begin{cases} V_{cr_{1,1}} = 249.25 \text{ km/h} \\ V_{cr_{2,2}} = 259.24 \text{ km/h} \end{cases} \quad \text{for high speed trains}$$

$$\begin{cases} V_{cr_{2,1}} = 124.62 \text{ km/h} \\ V_{cr_{3,2}} = 172.83 \text{ km/h} \end{cases} \quad \text{for common trains}$$