Functional Mechanical Design
Motion Law (1/2)

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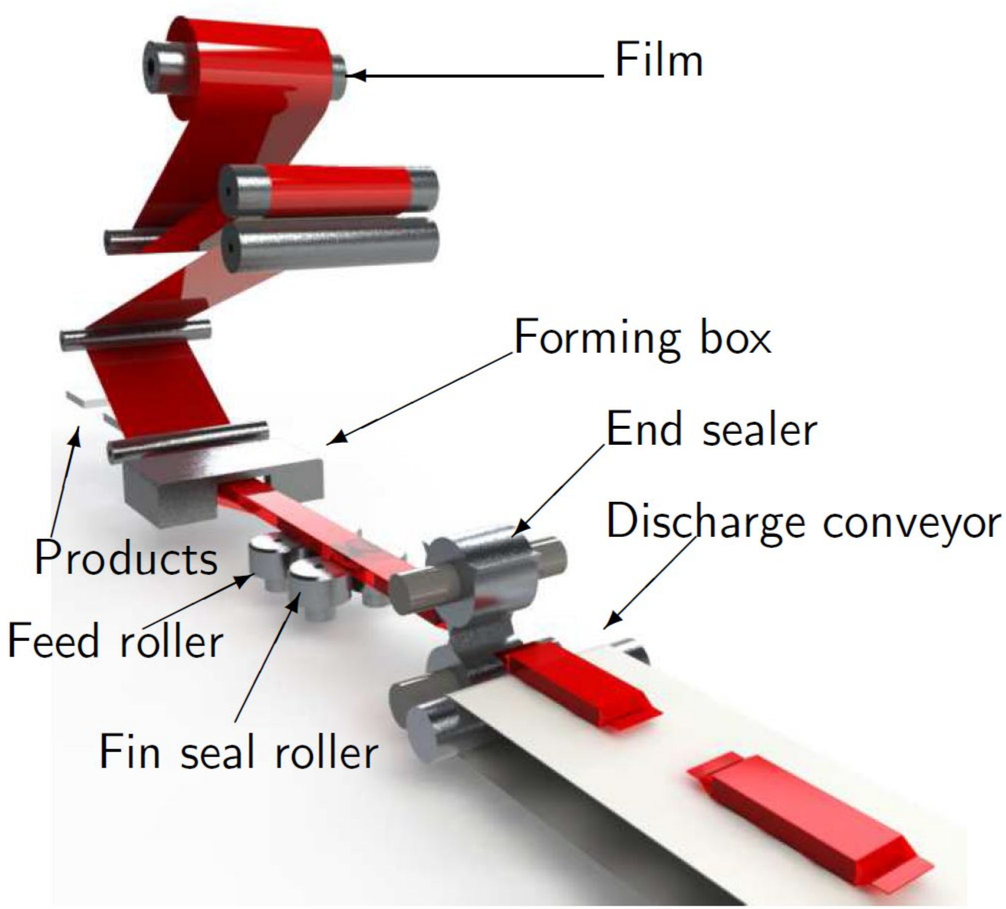
References:

1. P.L. Magnani, G. Ruggieri, “Meccanismi per Macchine Automatiche”, ed. UTET, Torino, Italia, 1986, ISBN:88-02-03934-8.

2. Luigi Biagiotti, Claudio Melchiorri, “Trajectory Planning For Automatic Machines and Robots” Springer-Verlag, 2008 ed, ISBN: 978-3-540-85629-0. (http://www.springer.com/us/book/9783540856283)

Introduction



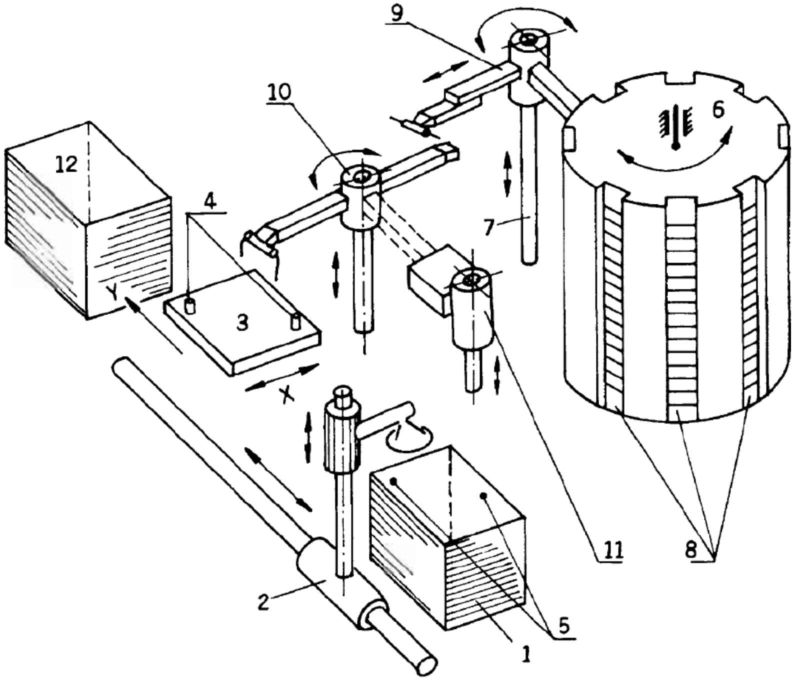
An automatic machine is generally a very complex electromechanical system made up of mechanical parts, electric and electronic devices interacting to accomplish a task.

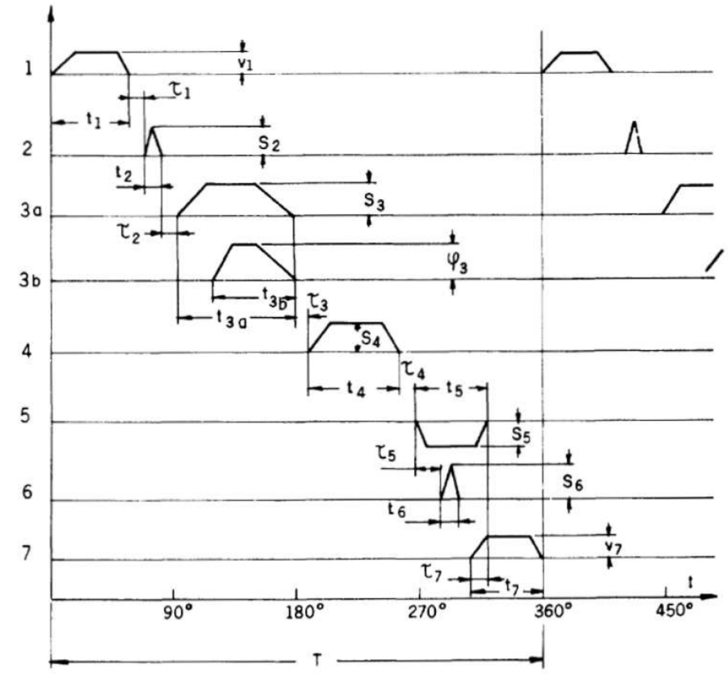
In an automatic machine, it is possible to distinguish between several different subsystems according to their function. These are named **functional groups**.

Introduction

The design of an automatic machine **starts from** the definition of the **movements** that every single functional group is supposed to carry out.

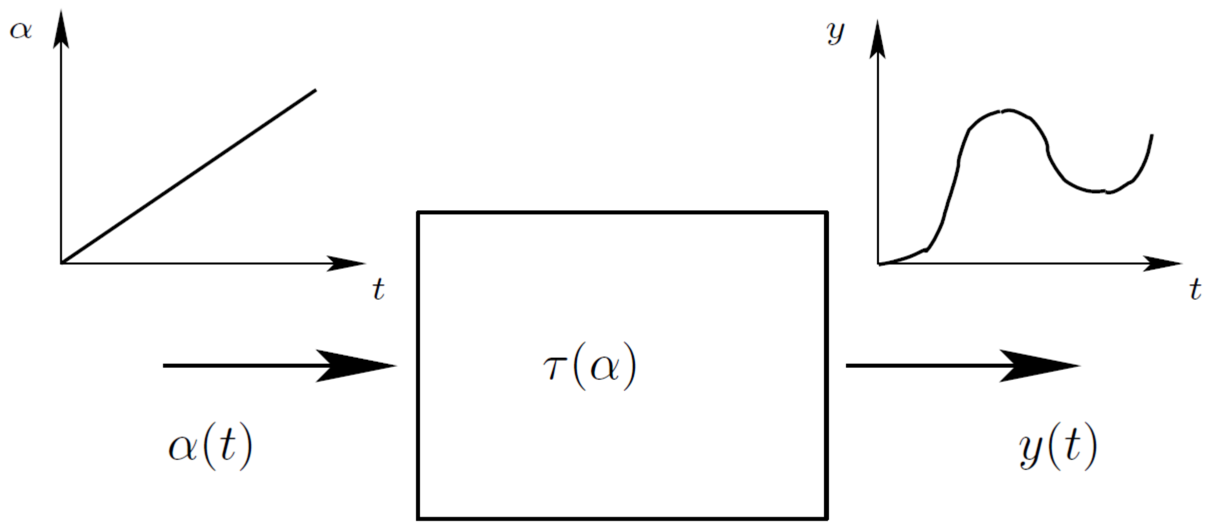
Then movements will be synchronized to perform the entire task.





There are two approaches to generate motion laws:

1. take movement from a motor with constant angular velocity and transform it by means of **mechanisms** (mechanical transmission with variable transmission ratio $\tau = \tau(\alpha)$). At the output point of such mechanisms (output organ) we have the desired motion. The output organ is usually called "follower".
2. take movement from a motor with variable angular velocity (i.e. controlled position) without a variable transmission ratio between motor and follower. In this case we are talking about **"electronic cams"**.

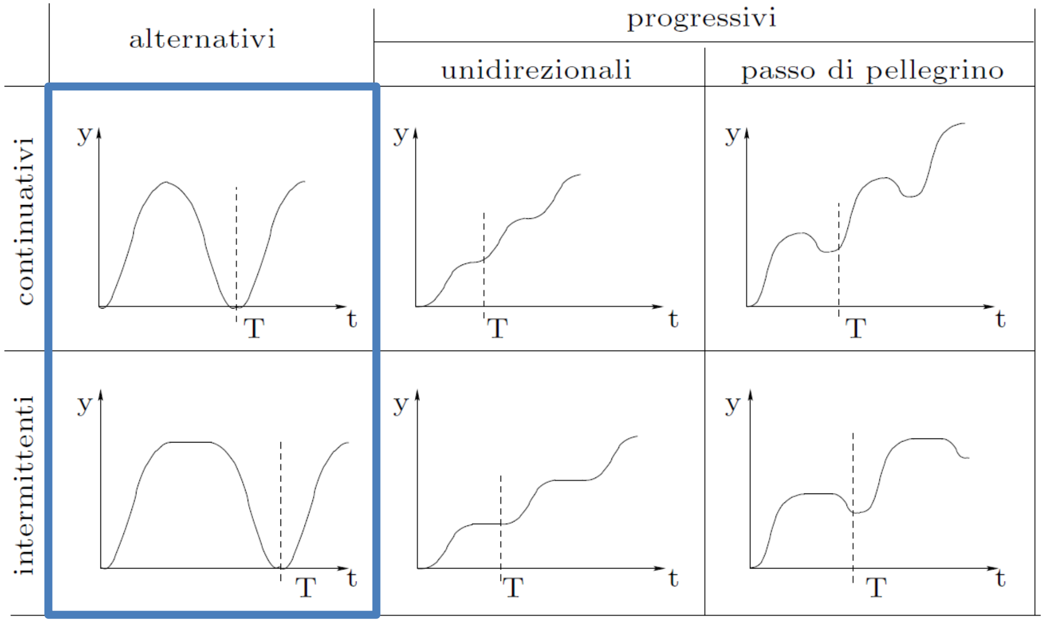


Mechanisms can be considered as machines whose function is to generate as output the desired motion law.

This point of view can be described with the block scheme in the figure. It can also be used to represent a functional group or a single degree of freedom machine.

Introduction

There are several kinds of movements….



…even if, for automatic machines, we will focus only on cyclical ones

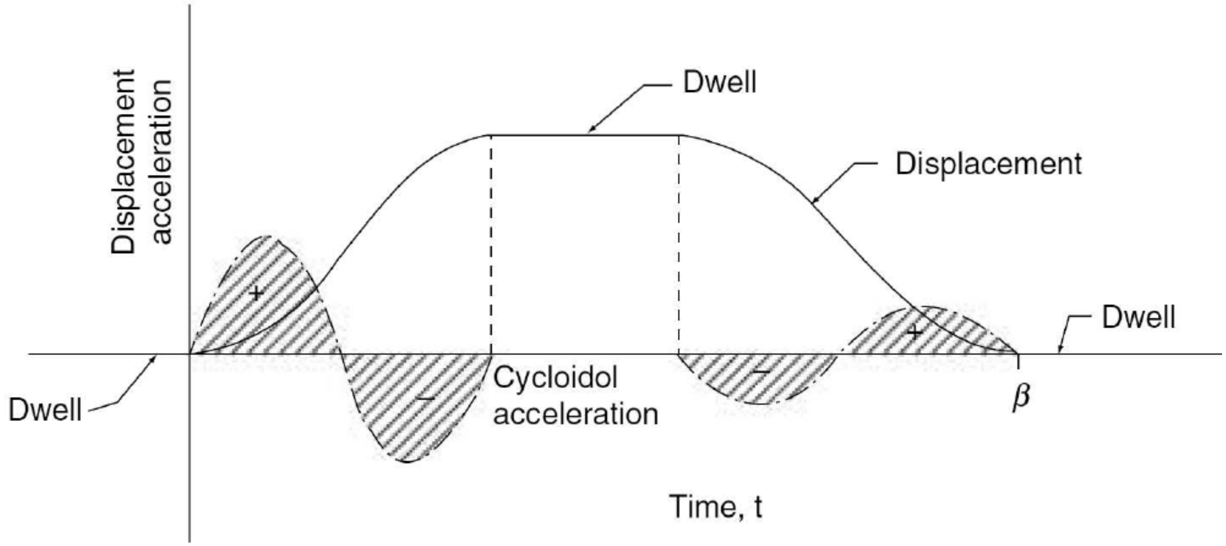
Motion law for manufacturing

Usually, the required follower displacement can be divided into different segments called **rise** (lift) and **return** (fall). Eventually between them there are some **dwell** phases (rest).

Rise and fall are characterized by two parameters: *h* total rise (lift or fall) and t_a time to reach the total rise.

For automatic machines, repeating the same movement every cycle of production, boundary conditions of the follower motion law *y = y(t)* are:

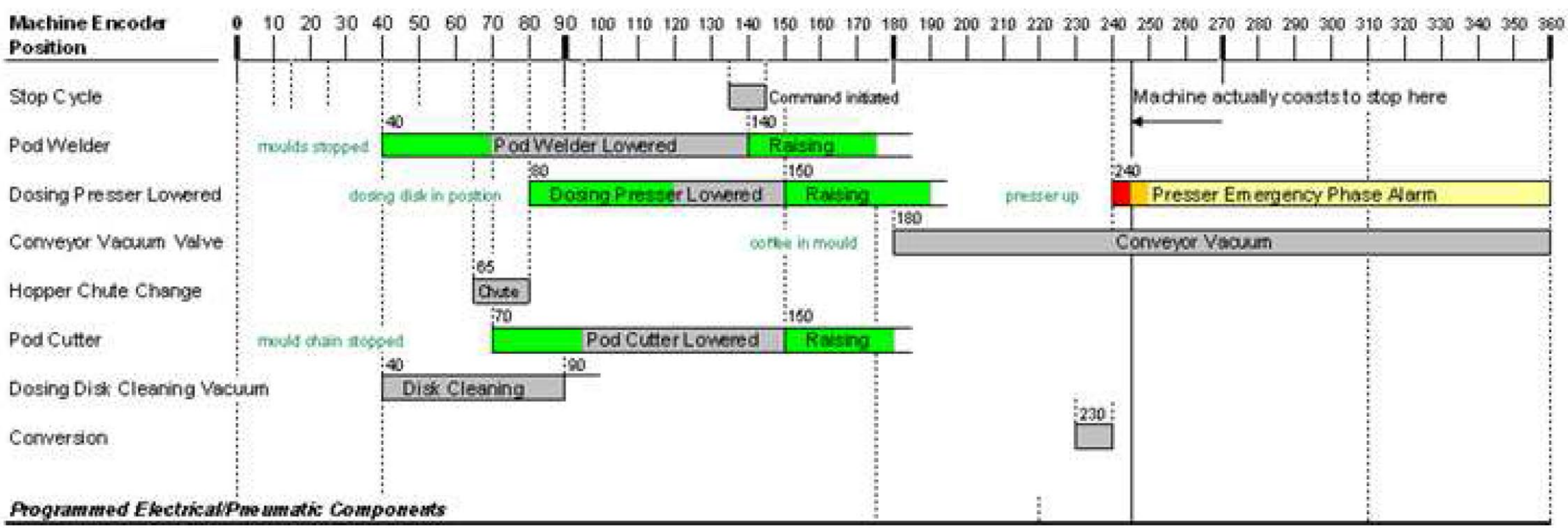
$$\begin{cases} y(0) = 0 & \dot{y}(0) = 0 \\ y(t_a) = h & \dot{y}(t_a) = 0 \end{cases}$$



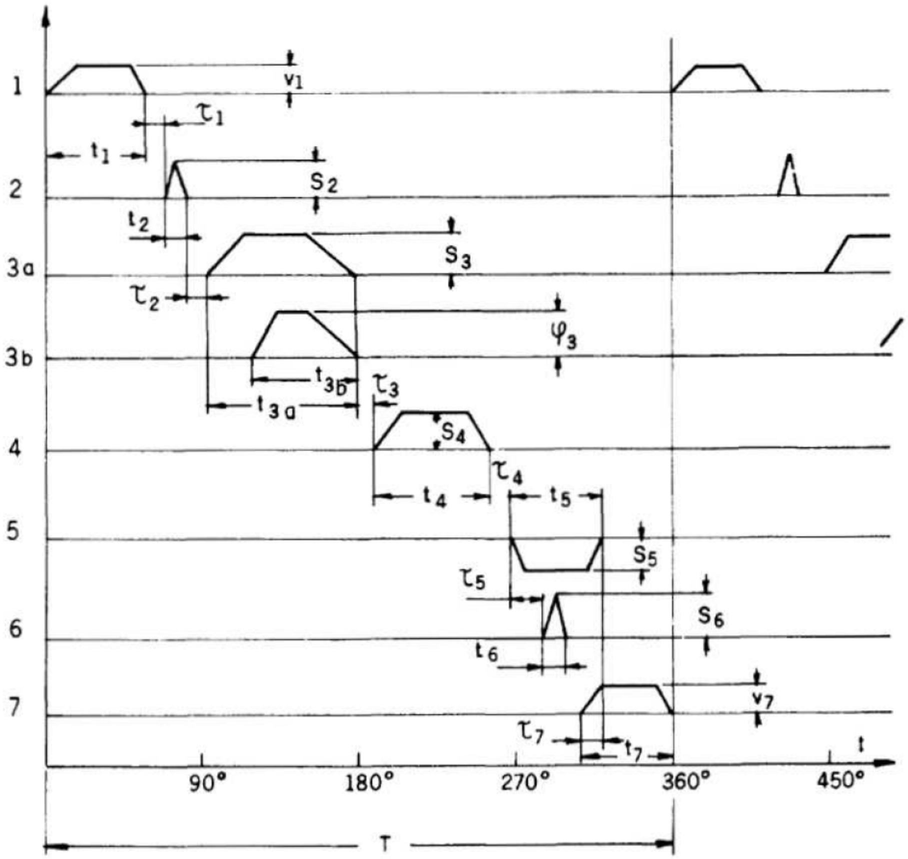
Master angle

The motion law can be described in terms of the rotational angle α of a **master shaft** that rotates at constant angular speed and performs a complete rotation in a period T. Diagram $y(\alpha)$ for $0 \le \alpha \le 2\pi$ takes the name of "displacements diagram".

$T=2\pi/\omega$



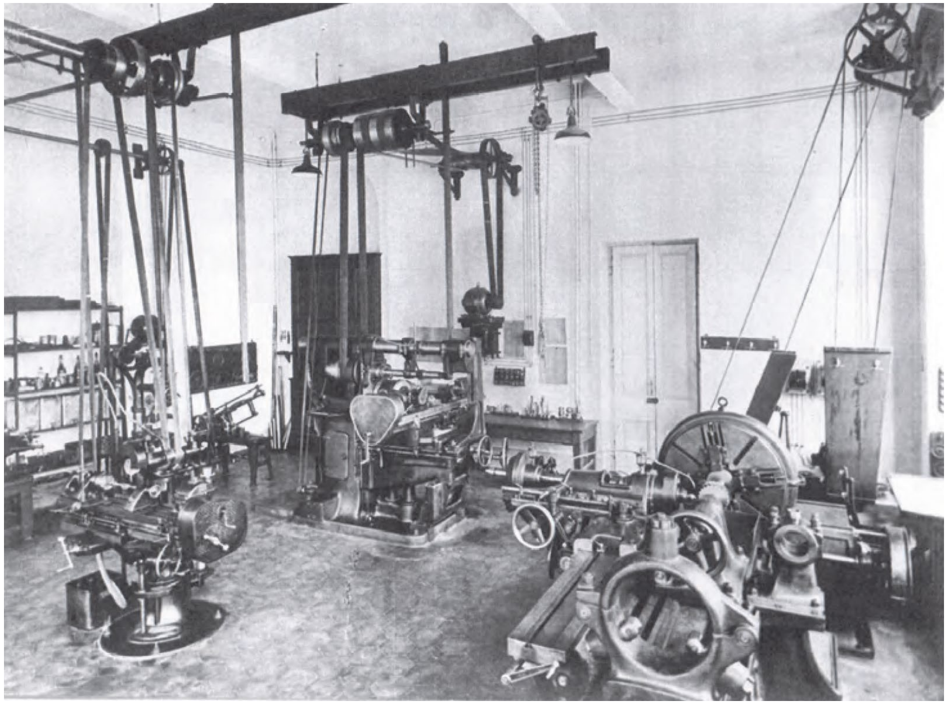
Displacement diagram



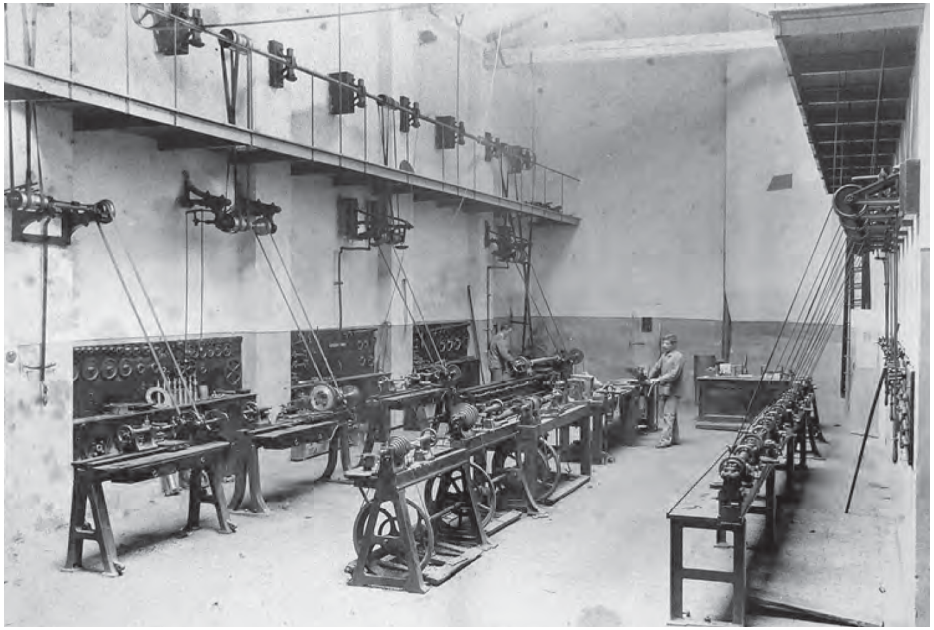
Master angle is usually preferred instead of time for two main reasons:

1. automatic machines perform the same operation in a **cyclical** way. It means the behavior of a machine can be described and analyzed in a period $T=2\pi$

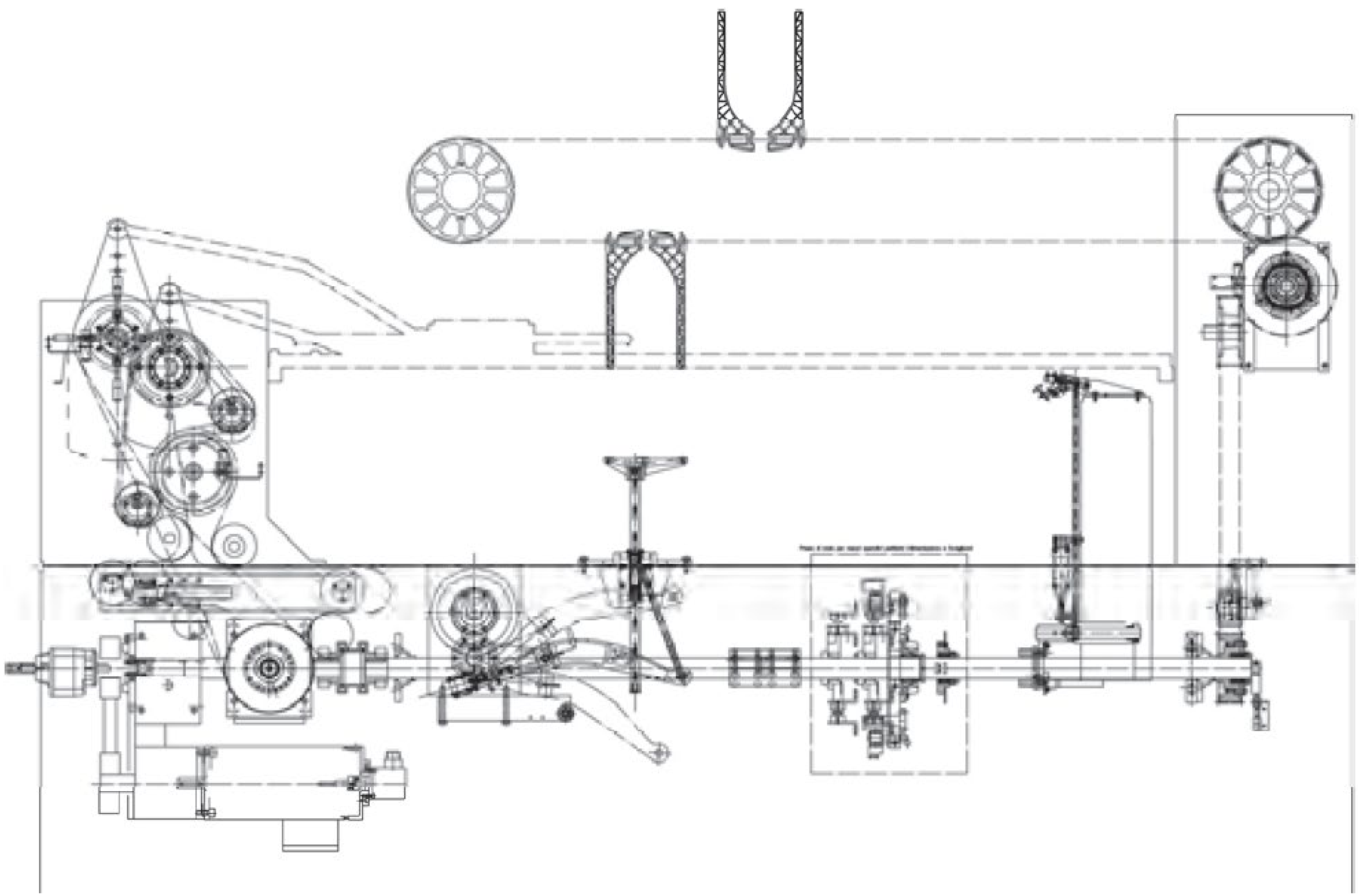
Master angle



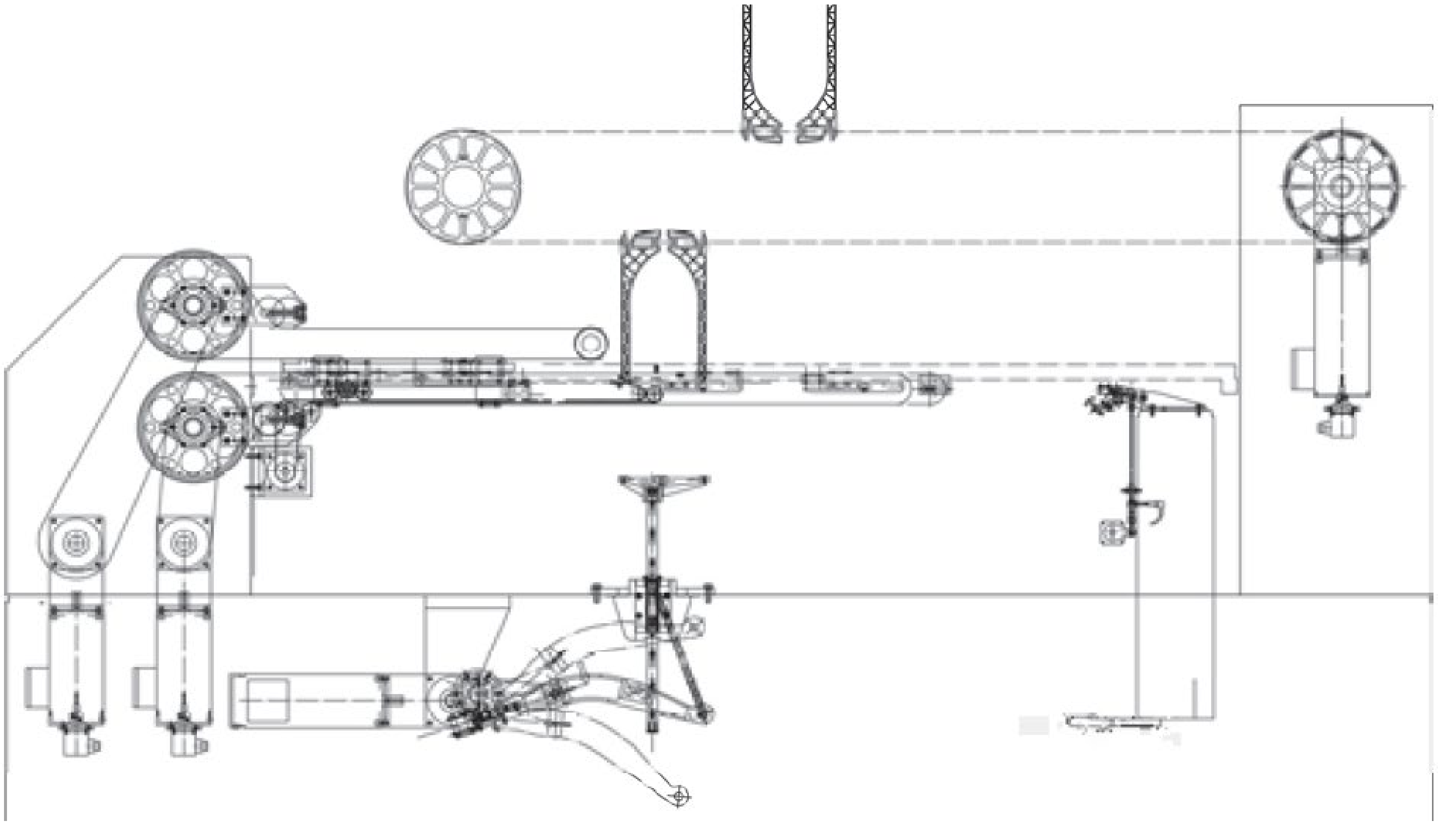
2. In the past, all the subsystems spilled the motion from a **main shaft** (master) that powered all the machines of the factory.



Master angle



Master angle



Nowaday, even if the use of electric motors allows to have independent subsystems, it is still convenient to refer to a master angle to synchronize the motions.

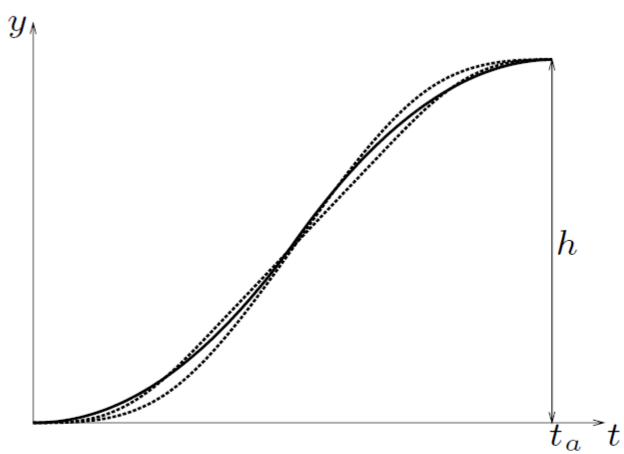
Determining of displacement diagram

There are two approaches to determine the shape of a motion law $y = y(t)$ (or of the displacement diagram $y = y(\alpha)$) with the requirement of reaching in a time t_a the rise h:

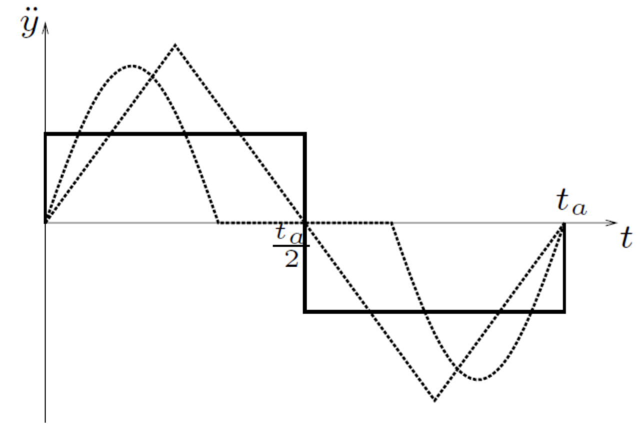
- Assign the trend of $\ddot{y}(t)$ and to obtain, by means of a double integration the motion law $y = y(t)$,
- Assign the trend of $y = y(t)$ in function to some constant parameters that must be determined basing on the boundary conditions.

From a mathematical point of view the two approaches are substantially the same. However in terms of control of the dynamics of the motion law it is fundamental to develop such motion law starting from the definition of the acceleration trend.

Motion law for manufacturing



(c) Displacements



(d) Accelerations

Knowing the total rise h and the time t_a to reach the total rise the motion law is not completely defined but there are several different trends that can be followed.

In terms of displacement these trends are not so different but in terms of acceleration they are.

Motion law for manufacturing

Very similar motion law displacements $y(t)$ could have very different acceleration trends and values $\ddot{y}(t)$.
As accelerations are directly related to inertia forces, it's important to keep that values under control to:

- decrease inertial loads
- improve performance of the system
- limit vibrational problems

In other words we are talking about a general concept called "**softness/ smoothness of motion**".

Proprieties of acceleration diagram

Can we really design a **generic motion curve without constraints**?

Obviously no!

To extricate the enormous number of different curves, it is useful to highlight some proprieties of the motion law with the previous boundary conditions (velocity nil at the beginning and at the end of the movement, starting from zero and arriving to rise *h*). A function $\ddot{y}(t)$ is feasible if:

<u>Condition 1</u>

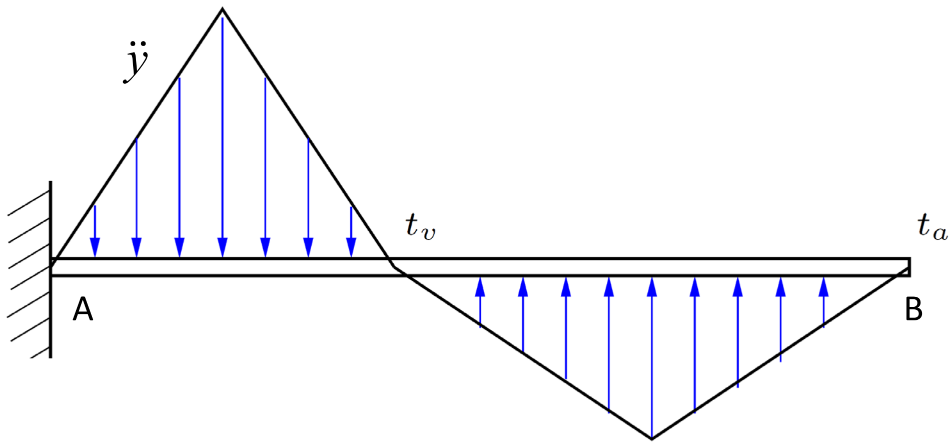
$$\int_0^{t_a} \ddot{y}\,\mathrm{d}t = \left|\,\dot{y}\,\right|_0^{t_a} = 0$$

Meaning that the function $\ddot{y}(t)$ has a null average value.

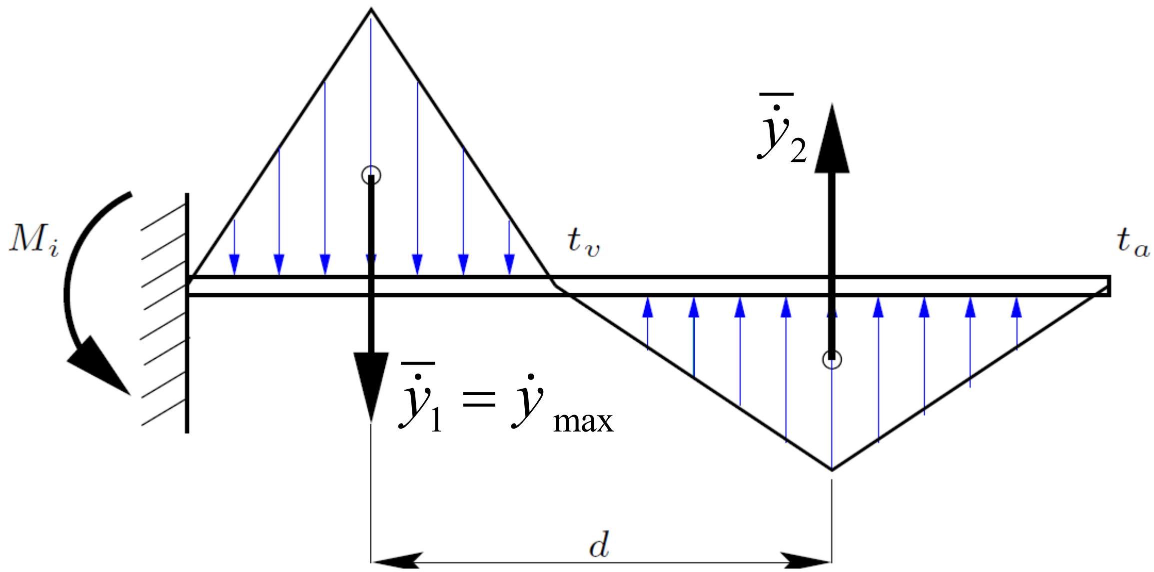
<u>Condition 2</u>

$$\int_0^{t_a} \ddot{y}\,t\,\mathrm{d}t = \left|\,\dot{y}t\,\right|_0^{t_a} - \int_0^{t_a} \dot{y}\,\mathrm{d}t = 0 - \left|\,y\,\right|_0^{t_a} = -h$$

To easily manage the two properties, a mechanical analogy can be done. Consider a cantilever beam, whose length is l, that has a static load distribution q. Now suppose that $\ddot{y} = q$ and $l = t_a$



Generally $\ddot{y}$ has two characteristic segments: one with positive acceleration (or negative) between $0 < t < t_v$ and the other with negative acceleration (or positive) between $t_v < t < t_a$.



$$\bar{\dot{y}}_1 = \int_0^{t_v} \ddot{y}dt$$

$$\bar{\dot{y}}_2 = \int_{t_v}^{t_a} \ddot{y}dt$$

Form condition 1 it has to be: $\bar{\dot{y}}_1 = \bar{\dot{y}}_2 = \bar{\dot{y}}$ (the two areas has to be equal)

while condition 2 requires that: $M_i = \bar{\dot{y}}\, d = h \quad \rightarrow \quad h = \dot{y}_{\max}\, d$

Normalization

To better understand the behaviour of a motion law it is useful to highlight the influence of its shape by means of normalized diagrams (dimensionless diagrams).
To this end it is possible to divide abscissa and ordinate by some quantities, that are in the case of the abscissa the time t_a and for the ordinate the total rise h.
In this way it is possible to refer to a normalized diagram $s(\xi)$ joined to the effective motion law $y(t)$ as follows:

$$y = hs(\xi) \qquad \text{where}\,, \qquad \xi = \frac{t}{t_a}\,.$$

$s = s(\xi)$ is a function defined between $0 < \xi < 1$ and is equal at its extremities to 0 and 1.

Normalization

Summarizing we have:

$$y(t) = hs\left(\frac{t}{t_a}\right) = hs(\xi)$$

The derivative with respect to time of allows us to obtain velocity and acceleration:

$$\dot{y}(t) = \frac{h}{t_a} v\left(\frac{t}{t_a}\right) = \frac{h}{t_a} v(\xi) \qquad \ddot{y}(t) = \frac{h}{t_a^2} a\left(\frac{t}{t_a}\right) = \ddot{y}(t) = \frac{h}{t_a^2} a(\xi)$$

where $v(\xi) = \mathrm{d}s/\mathrm{d}t$ and $a(\xi) = \mathrm{d}^2 s/\mathrm{d}t^2$

The proprieties of the acceleration diagram are also valid for the dimensionless motion law:

$$\int_0^1 a(\xi)\,\mathrm{d}\xi = 0\,, \qquad \int_0^1 a(\xi)\xi\,\mathrm{d}\xi = -1$$

Coefficients

There are several elementary motion laws available and we can design several others. However, when we choose a motion law we should always be able to answer to some main questions:

- Is this motion law appropriate for the application?
- How far is the selected motion law from the optimum?
- Does exist a better motion law? / Can I improve the selected motion law?

To answer these questions we introduce some **coefficients**, which only depend on the shape of the motion law, useful to:

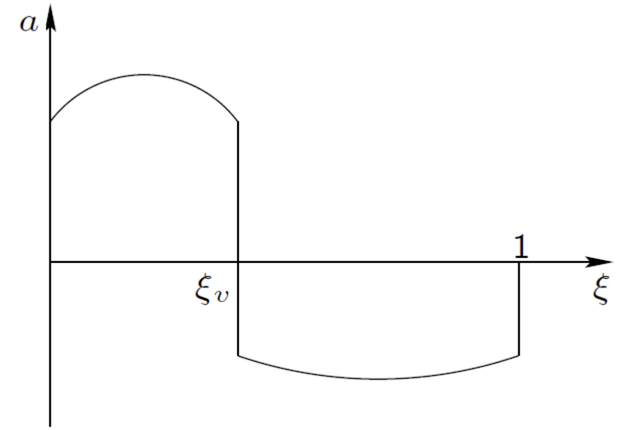
- compare different motion laws
- provide an absolute measure of the quality of the selected law of motion

Coefficients

Usually the trend of $a(\xi)$ shows only two characteristic segments:

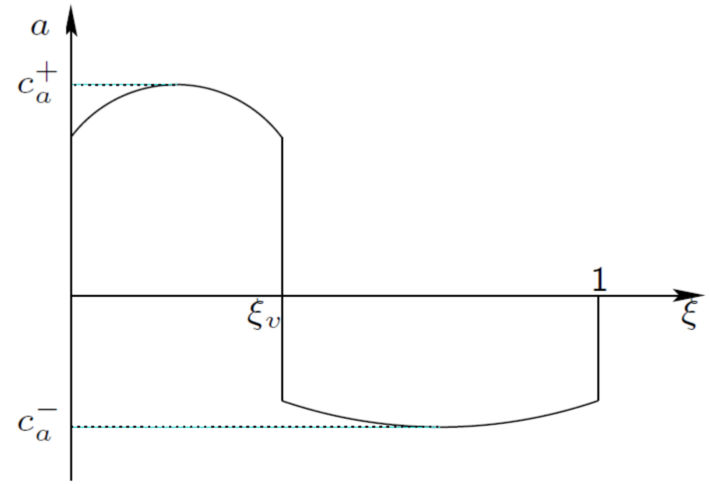
$$\begin{cases} 0 < t < t_v \\ 0 < \xi < \xi_v \end{cases}$$ Positive or negative acceleration

$$\begin{cases} t_v < t < t_a \\ \xi_v < \xi < 1 \end{cases}$$ Negative or positive acceleration



Where the area under the positive acceleration is equal to the one under negative acceleration. **Note:** the presence of only two segments is the 5^{th} condition

Coefficients



To compare different acceleration shapes specific values of $a(\xi)$ called acceleration coefficients are used:

- c_a^+: maximum positive value of $a(\xi)$
- c_a^-: maximum negative absolute value of $a(\xi)$
- c_a maximum between the two

Coefficients

Knowing the acceleration coefficients for a motion law, it is possible to evaluate simply:

$$\ddot{y}^{+}_{max} = \frac{h}{t_a^2} c_a^{+} \quad \text{Maximum positive acceleration}$$

$$\ddot{y}^{-}_{max} = \frac{h}{t_a^2} c_a^{-} \quad \text{Maximum negative acceleration}$$

$$\boxed{\ddot{y}_{max} = \frac{h}{t_a^2} c_a \quad \text{Maximum acceleration}}$$

Coefficients

In the same way the velocity coefficient is defined:

- c_v: maximum value of $v(\xi)$

Through the velocity coefficient c_v it is possible to evaluate the maximum velocity as follows:

$$\boxed{\dot{y}_{max} = \frac{h}{t_a} c_v \quad \text{Maximum velocity}}$$

Coefficients can be defined according to specific needs (eg. related to mechanical power, jerk, etc.) .

Elementary motion law

After defining the principal aspects of the motion curve treatment we have to go into more in detail from the design point of view. Independently of the application, when you are planning a motion curve, there are some design requirements widespread, for example:

- to limit the maximum acceleration
- to limit the maximum velocity
- to limit the mechanical power
- to limit the vibrations

To limit the maximum acceleration

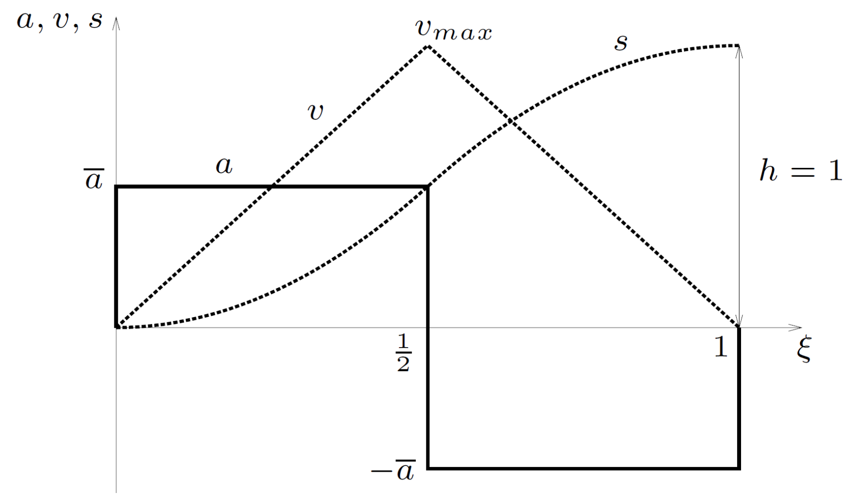
Firstly we consider the problem of decreasing the maximum acceleration:

$$\ddot{y}_{max} = c_a \frac{h}{t_a^2} \quad \begin{array}{lll} \Rightarrow & h & \text{better small} \\ \Rightarrow & t_a & \text{better great} \end{array}$$

Note: h and t_a work with different rates in decreasing the maximum acceleration. It is better to act on the time than on the total rise value.

Note: after having fixed the values of h and t_a, in order to decrease the maximum acceleration it is only possible to set in the shape of the motion law, or in other words to change the acceleration coefficient c_a.

To this end it is interesting to ask which is the motion law with the minimum value of the acceleration coefficient c_a.



The motion curve with the minimum value of the maximum acceleration is the **constant acceleration symmetric curve**

Let's calculate coefficients c_a and c_v . From condition 2 we have:

$$\left|v_{\max} \cdot d\right| = h \quad \Rightarrow \quad \left(\overline{a} \cdot \frac{1}{2}\right) \cdot \frac{1}{2} = 1 \quad \Rightarrow \quad \overline{a} = 4$$

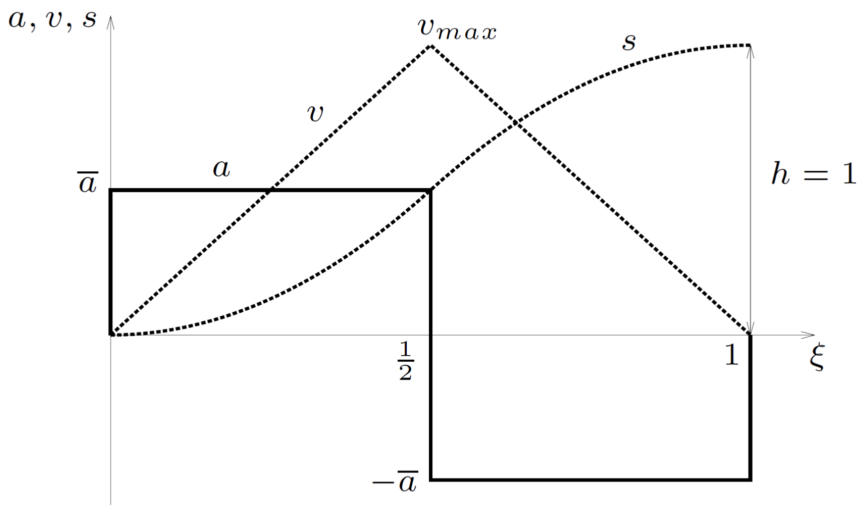
and then:

$$\ddot{y}_{\max} = c_a \frac{h}{t_a^2} \quad \Rightarrow \quad 4 = c_a \frac{1}{1} \quad \Rightarrow \quad \boxed{c_a = 4}$$

and:

$$\dot{y}_{\max} = c_v \frac{h}{t_a} \quad \Rightarrow \quad 4 \cdot \frac{1}{2} = c_v \frac{1}{1} \quad \Rightarrow \quad \boxed{c_v = 2}$$

Minimum value of the maximum acceleration

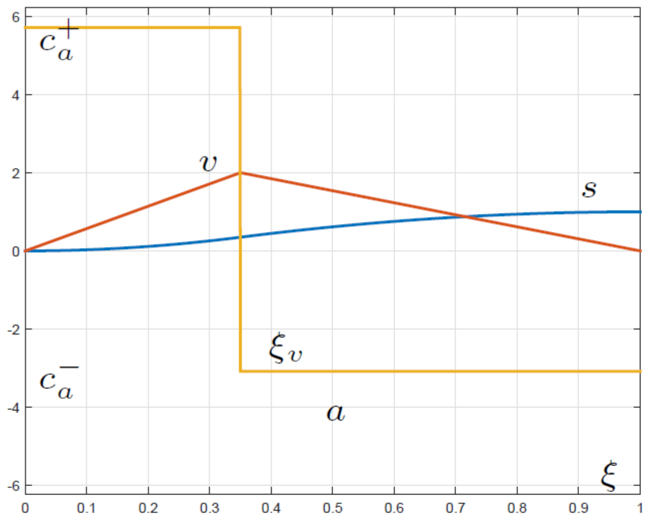


the coefficient values are $c_v = 2$ and $c_a = c_a^+ = c_a^- = 4$

There is no motion curve with $c_a < 4$ i.e. there is no motion curve capable of reaching h in a time t_a with maximum acceleration lower than:

$$\ddot{y}_{max} = 4\frac{h}{t_a^2}$$

Minimum value of the maximum acceleration



$$\ddot{y}^+_{max} t_v = \dot{y}_{max} \quad \Rightarrow \quad c_a^+ \xi_v = c_v$$

$$\ddot{y}^-_{max} (1 - t_v) = \dot{y}_{max} \quad \Rightarrow \quad c_a^- (1 - \xi_v) = c_v$$

The velocity coefficient for this kind of motion curve is:

$$\frac{1}{2} c_v \xi_v + \frac{1}{2} c_v (1 - \xi_v) = 1 \quad \Rightarrow \quad c_v = 2$$

Summarizing, we have:

$$\boxed{c_v = 2 \qquad c_a^+ = \frac{2}{\xi_v} \qquad c_a^- = \frac{2}{1 - \xi_v} \qquad c_a^+ = \frac{2 c_a^-}{c_a^- - 2}}$$

Constant acceleration curve

$$c_v = 2 \qquad c_a^+ = \frac{2}{\xi_v} \qquad c_a^- = \frac{2}{1-\xi_v}$$

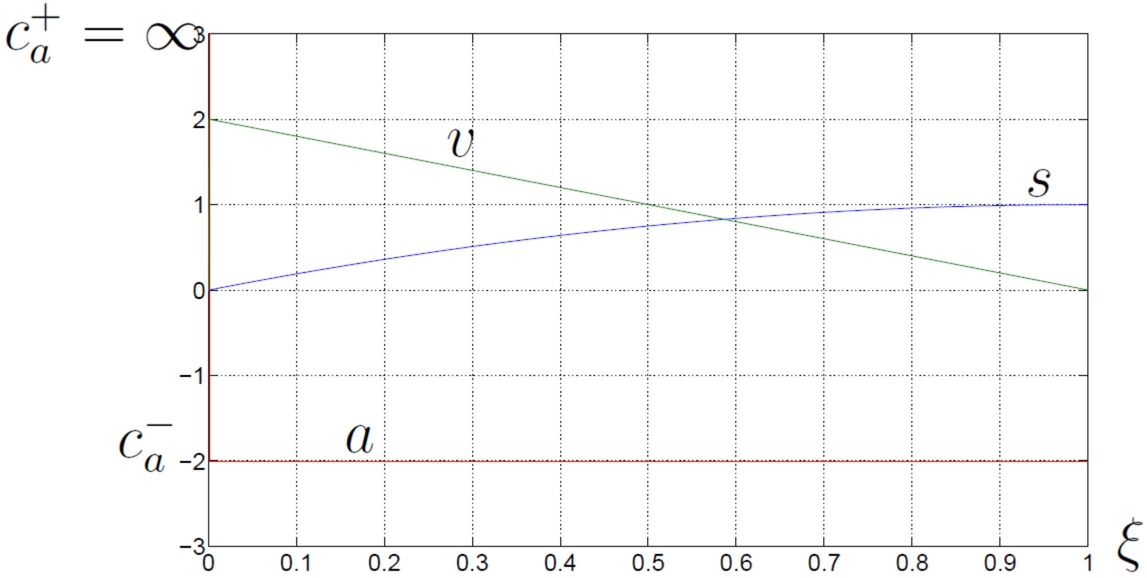
To obtain the equations of displacement diagram and its derivative $v(\xi)$ and $a(\xi)$, it is possible to use the methods described before:

$$0 < \xi < \xi_v \qquad\qquad \xi_v < \xi < 1$$

$$\begin{cases} a(\xi) = \dfrac{2}{\xi_v} \\ v(\xi) = \left(\dfrac{2}{\xi_v}\right)\xi \\ s(\xi) = \dfrac{1}{2}\left(\dfrac{2}{\xi_v}\right)\xi^2 \end{cases} \qquad \begin{cases} a(\xi) = -\dfrac{2}{1-\xi_v} \\ v(\xi) = \left(\dfrac{2}{1-\xi_v}\right)(\xi - 1) \\ s(\xi) = \left(\dfrac{2}{1-\xi_v}\right)\left(\xi - \dfrac{\xi^2}{2} - \dfrac{\xi_v}{2}\right) \end{cases}$$

Minimum value of the maximum acceleration

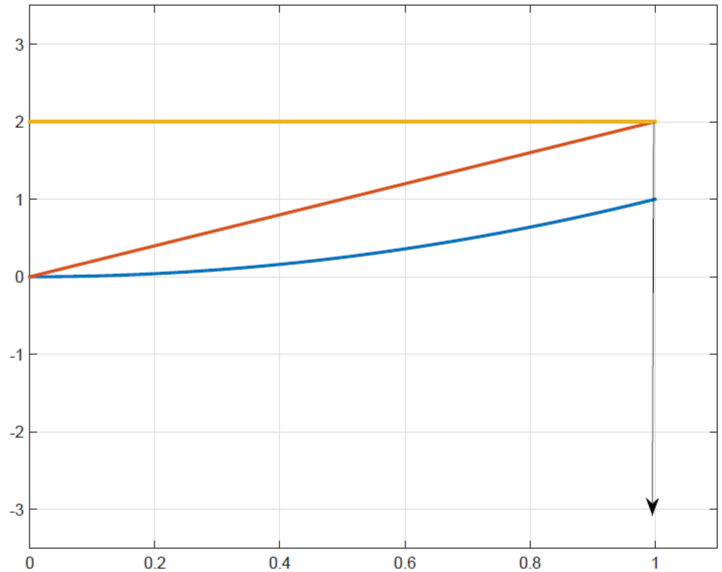
It is possible to think of a motion curve with c_a^+ or c_a^- lower than 4 but in that case c_a will necessary be greater than four.
There is a limit under which also c_a^+ or c_a^- cannot go



This value is equal to two and it is possible to reach if there is an infinite acceleration peak at the beginning or at the end of the motion curve.

Minimum value of the maximum acceleration

It is simple to calculate the coefficient value $c_a^{\pm}$.



With a uniformly accelerated motion from $\xi = 0$ to $\xi = 1$ the displacement is:

$$\frac{1}{2}a1^2 = 1 \quad \Rightarrow \quad a = 2$$

Then $c_a^+ = 2$ and for the acceleration diagram proprieties the other acceleration coefficient is $c_a^- = \infty$

To obtain a complete view of the behaviour of the constant acceleration curves (parabolic curves) it is worthwhile to parametrize the value of the acceleration coefficient in function of the parameter ξ_v corresponding to the inversion point of the acceleration diagram.

Determining of displacement diagram

Example 1: is $\ddot{y} = at + b$ the shape of the acceleration desired.

1. determining the integration constants:

$$\begin{cases} \dot{y} = \frac{1}{2}at^2 + bt + C_1 \\ y = \frac{1}{6}at^3 + \frac{1}{2}bt^2 + C_1 t + C_2 \end{cases} \qquad \begin{cases} \dot{y}(0) = 0 \\ y(0) = 0 \end{cases} \qquad \begin{cases} C_1 = 0 \\ C_2 = 0 \end{cases}$$

2. impose the mean value of the acceleration to be nil (that means $\dot{y}(t_a) = 0$):

$$\int_0^{t_a} (at + b)\mathrm{d}t \quad \Rightarrow \quad b = -\frac{1}{2}at_a$$

3. definition of the acceleration scale ($y(t_a) = h$):

$$\begin{cases} y = \frac{1}{6}at^3 - \frac{1}{4}at_a t^2 \\ y(t_a) = h \end{cases} \qquad \Rightarrow a = -\frac{12h}{t_a^3}$$

Example 2: Constant acceleration curve:

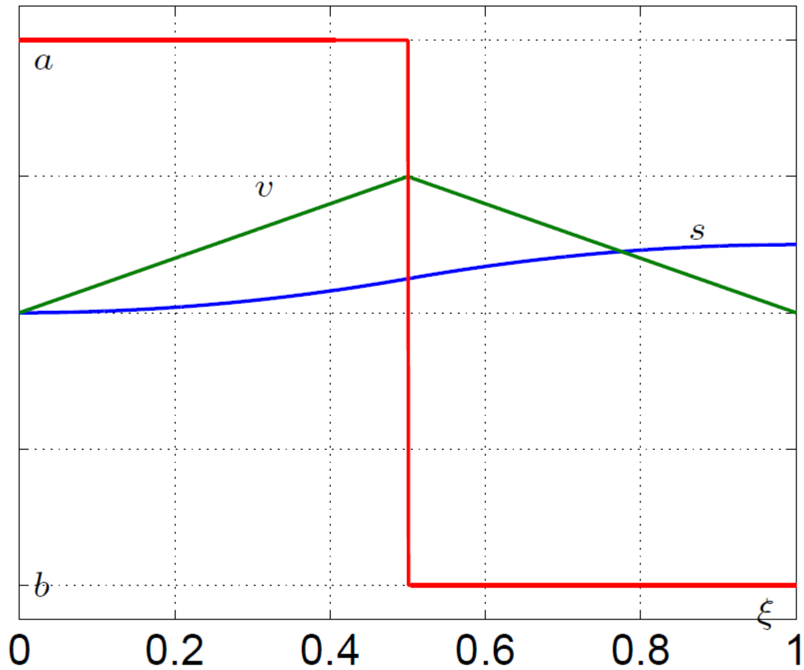


Figure shows the position, velocity and acceleration of the dimensionless constant acceleration curve ($h = 1$, $t_a = 1$).

Determining of displacement diagram

The equation of this motion curve is divided into two domains:

$$0 < \xi < \frac{1}{2} \quad \begin{cases} a_+(\xi) = a \\ v_+(\xi) = a\xi \\ s_+(\xi) = \frac{1}{2}a\xi^2 \end{cases} \qquad \frac{1}{2} < \xi < 1 \quad \begin{cases} a_-(\xi) = -a \\ v_-(\xi) = -a\xi + d_1 \\ s_-(\xi) = -\frac{1}{2}a\xi^2 + d_1\xi + d_2 \end{cases}$$

Where the integration constants of the first domain are easily obtained imposing $v(0) = 0$ and $s(0) = 0$.
The constants d_1 and d_2 are obtained through the conditions $s(1) = 1$ and $v(1) = 0$: $d_1 = a$, $d_2 = 1 - 0.5a$
To evaluate the value of a it is necessary add another condition in one of the two domains, for example at the end of the first domain the displacement will be equal to $h/2$:

$$\frac{1}{2} = a\frac{1}{2^2} = \frac{h}{2} \quad \rightarrow \quad a = 4$$

Determining of displacement diagram

The motion curve could be defined starting from the displacement and assigning an arbitrary shape through some constants which must be determined by means of the boundary conditions ($y = y(t, a, b, c, ...)$).
For example the constant acceleration curve (parabolic law) you will have:

$$0 < \xi < \frac{1}{2} \qquad \begin{cases} s(\xi) = a_0 + a_1\xi + a_2\xi^2 \\ v(\xi) = a_1 + 2a_2\xi \end{cases} \qquad \begin{cases} s(0) = 0 \\ s(.5) = \frac{1}{2} \\ v(0) = 0 \end{cases}$$

$$\frac{1}{2} < \xi < 1 \qquad \begin{cases} s(\xi) = b_0 + b_1\xi + b_2\xi^2 \\ v(\xi) = b_1 + 2b_2\xi \end{cases} \qquad \begin{cases} s(0) = 0 \\ s(.5) = \frac{1}{2} \\ v(0) = 0 \end{cases}$$

imposing the 6 boundary conditions shown, it is possible to obtain the values of the 6 constants: $a_0 = 0, a_1 = 0, a_2 = 2$ and $b_0 = -1, b_1 = 4, b_2 = -2$.