Functional Mechanical Design

Linkage Mechanisms (2/3)

Simone Cinquemani

- **Kinematic analysis:** A mechanism is investigated to understand its behaviour in terms of kinematic reationships betwen links (eg. transmission ratio), knowing the mechanism geometry and kinematic quantities as angular velocity and acceleration, etc.

- **Kinematic synthesis:** It is the process of designing a mechanism to accomplish a desired task.

The first step for both kinematic analysis and synthesis is the drawing of kinematic diagrams and determining the number of the degrees of freedom of the under study mechanism.

Kinematic analysis

Introduction

Linkage machanisms are generally exploited to:

- transmit the motion
- generate a trajectory

Through the kinematic analysis we should be able to investigate the quality of the transmission of the motion or the real trajectory generated.

=> the transmission angle: decribes how good is the transmission of motion

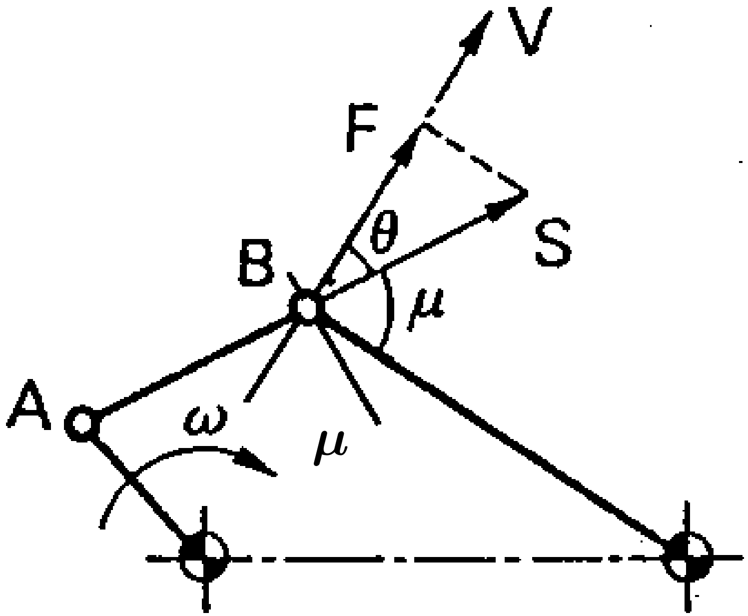
=> the generalized transmission ratio: describes the kinematic relationship between input/output velocity

=> complex number representation to calculate kinematic relationships

The transmission angle

The transmission angle

As the aim of a mechanism is to transmit a motion, it is useful to define a parameter that allows to evaluate the quality of the transmission as a function of the configuration assumed.

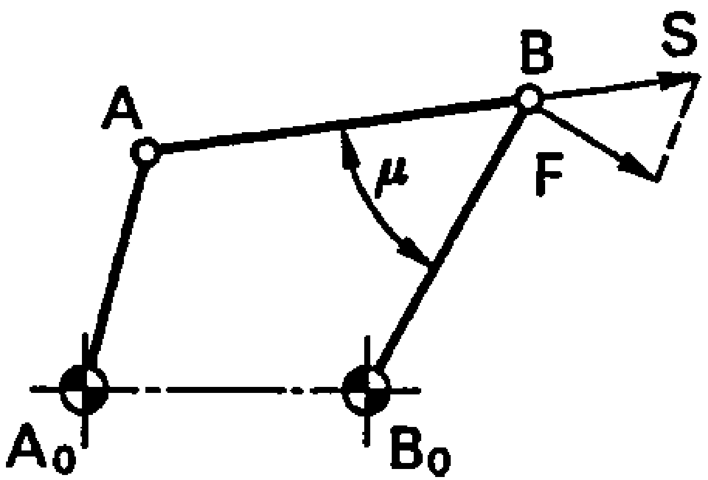


Considering the figure on the left, it shows two angles: the transmission angle μ and the pressure angle θ. The two angles are complementary: $\theta + \mu = \pi/2$.

The **transmission angle** μ describes in which way the coupler link transmits the motion between the crank and the rocker.

It is defined as the complimentary of the pressure angle θ.

The transmission angle

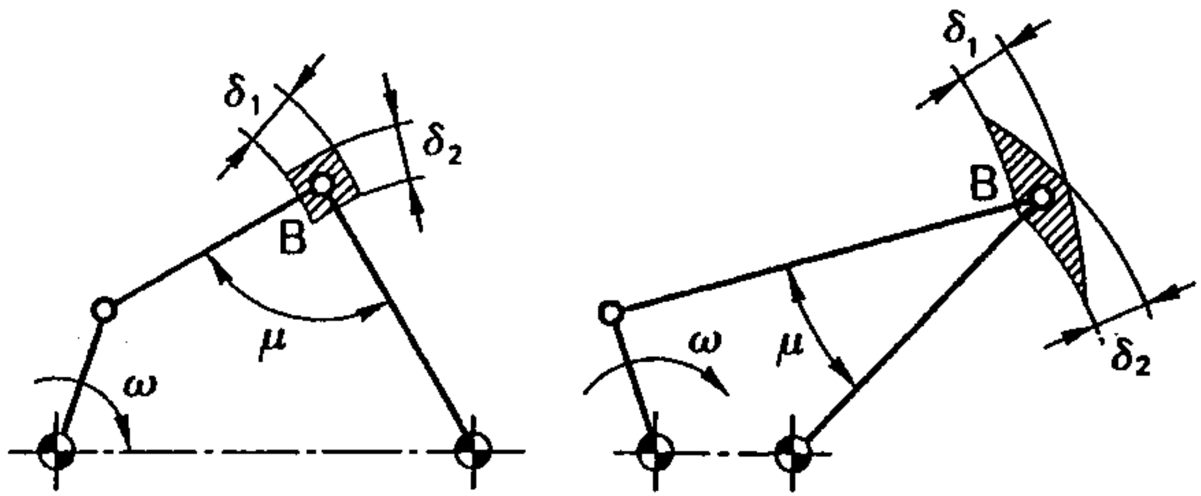


For four bar mechanisms the transmission angle μ is the lower of the angles between the directions of the rod connection and the follower.

If S is the force along the rod connection (the force direction is AB) and F is the useful component on the follower (the direction of this component is the same of the B velocity), we have:

$$S = \frac{F}{\sin \mu} \quad \text{when } \mu = 0 \text{ then } S \to \infty$$

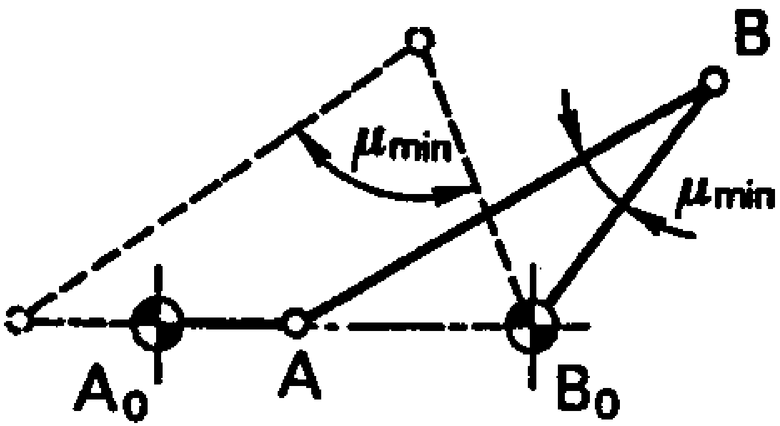
The transmission angle



Note that for small values of μ the backlash effects are amplified and the precision of the movement decreased.
With reference to the figure, δ_1 is the characteristic length of the backlash in the pin B along the direction of the rod connection and δ_2 the one along the follower. It is simple to understand that B can move inside the hatched area.

Note that as a function of the value of μ, the hatched area changes dimensions.

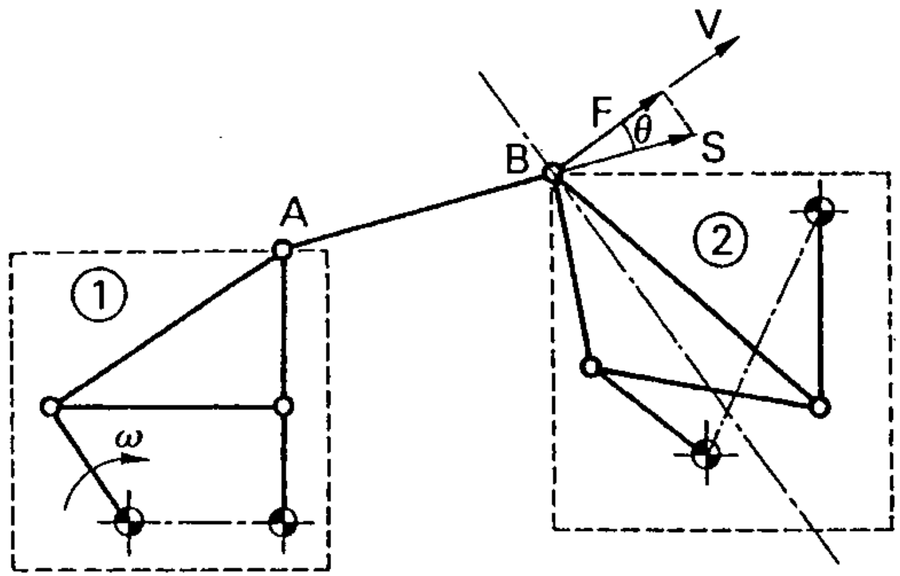
The transmission angle



In order to reduce these phenomenona (forces to high and backlash) the transmission angle must be greater than $40 \div 45°$.

Note that the transmission angle μ for a four bar mechanism is minimum when the distance AB_0 becomes minimum or maximum. In others words when the mover link AA_0 is aligned with the frame.

The transmission angle



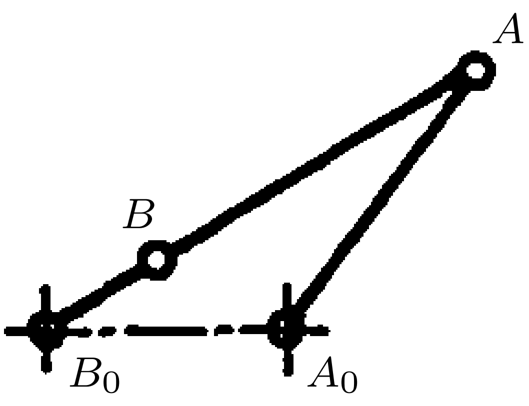
If the linkage mechanism is complex and has more than four links it is difficult to define the transmission angle.

Generally the meaning of the transmission angle (and the pressure angle too) can be extended to complex mechanisms considering them as made up of several simple four bar linkages. The transmission of motion can be evaluated by checking all the transmission angles involved.

The generalized transmission ratio

The generalized transmission ratio

The overall behaviour of a mechanism can be described by its **transmission ratio**. If α describes the movement of the mover link and y the movement of the follower, we can express one as a function of the other, using the transmission ratio $\tau(\alpha)$ exactly as we do for speed reducers. In this case the value of τ is not constant and it is called generalized transmission ratio and coincides with the geometrical speed of the output link (y'):



$$\frac{\mathrm{d}\alpha}{\mathrm{d}t}\tau(\alpha) = \frac{\mathrm{d}y}{\mathrm{d}t} \quad \Rightarrow \quad \tau(\alpha) = \frac{\mathrm{d}y/\mathrm{d}t}{\mathrm{d}\alpha/\mathrm{d}t} = \frac{\mathrm{d}y}{\mathrm{d}\alpha} = y'$$

According to this equation, when the mechanism reaches a dead center, its generalized transmission ratio y' tends to zero.

The generalized transmission ratio

The generalized transmission ratio relates not just the speeds of input and ouput links, but also the forces/torques applied on them in kinetostatic conditions. Under this hypothesis, being C_M the torque applied on the input link and C_R the one on the output, we have

$$C_M \cdot \dot{\alpha} - C_R \cdot \dot{\beta} = 0 \qquad \text{being} \quad \dot{\alpha}\tau = \dot{\beta} \qquad \text{we have:} \qquad C_M \frac{1}{\tau} = C_R$$

Substituting $\tau = y'$ into the previous equation, we obtain:

$$C_R = C_M \frac{1}{y'}$$

Note that <u>it is possible have force amplification when the geometrical velocity is small</u>. That means the mechanism behaviour, in terms of force amplification, is complementary with respect to the one in terms of velocity, when a singularity configuration is reached (y'=0).

The generalized transmission ratio

The generalized transmission ratio is usable also for mechanisms with more than one degree of freedom. By defining $\overline{S}$ the coordinate vector of outputs (the working space) and $\overline{Q}$ the coordinate vector of inputs (driven joints) one has:

$$\overline{S} = [s_1, s_2, ..., s_n]; \qquad \overline{Q} = [q_1, q_2, ..., q_m]$$

We can write kinematic relationships to have $\overline{S} = F(\overline{Q})$, where F is the function that links the joint and working space:

$$\begin{cases} s_1 = f_1(\overline{Q}) \\ s_2 = f_2(\overline{Q}) \\ ... \\ s_n = f_n(\overline{Q}) \end{cases}$$

and deriving with respect to time we obtain the relationship between outputs/inputs velocities:

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\overline{S}\right) = \dot{\overline{S}} = \frac{\mathrm{d}}{\mathrm{d}t}\left(F(\overline{Q})\right) \quad \Rightarrow \quad \dot{\overline{S}} = J\dot{\overline{Q}}$$

The generalized transmission ratio

$\dot{\overline{S}} = J\dot{\overline{Q}}$ where J is the **Jacobian matrix**:

$$J = \begin{bmatrix} \frac{\partial f_1}{\partial q_1} & \frac{\partial f_1}{\partial q_2} & \dots & \frac{\partial f_1}{\partial q_n} \\ \frac{\partial f_2}{\partial q_1} & \frac{\partial f_2}{\partial q_2} & \dots & \frac{\partial f_2}{\partial q_n} \\ \dots & \dots & \dots & \dots \\ \frac{\partial f_n}{\partial q_1} & \frac{\partial f_n}{\partial q_2} & \dots & \frac{\partial f_n}{\partial q_n} \end{bmatrix}$$

The obtained relationship is equal to the one expressed for one degree of freedom ($\mathrm{d}y/\mathrm{d}t = \tau \mathrm{d}\alpha/\mathrm{d}t$, $\dot{y} = \tau\dot{\alpha}$), in fact the Jacobian matrix is the generalized transmission ratio for more then one degree of freedom mechanisms.

Kinetostatic analysis

The same for mechanisms with more than one degrees of freedom:

$$F_s = [F_{s1}, F_{s2}, ..., F_{sn}]^T \quad \Rightarrow \quad \text{External forces vector applied to the follower}$$

$$F_q = [F_{q1}, F_{q2}, ..., F_{qn}]^T \quad \Rightarrow \quad \text{Actuating forces vector applied to the mover}$$

It is possible to write the virtual power principle:

$$\begin{cases} \dot{S}^T F_s + \dot{Q}^T F_q = 0 \\ \dot{S} = J\dot{Q} \end{cases} \quad \Rightarrow \quad \dot{S}^T = J^T \dot{Q}^T$$

from which we can obtain:

$$(J^T \dot{Q}^T) F_s + \dot{Q}^T F_q = 0 \quad \Rightarrow \quad J^T F_s + F_q = 0$$

Kinetostatic analysis

That is:

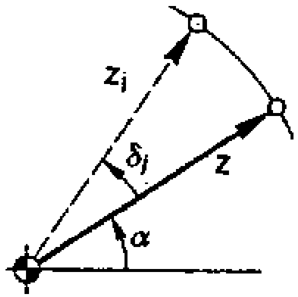
$$\begin{cases} F_q = -J^T F_s \\ F_s = -J^{-T} F_q \qquad \Leftarrow \end{cases}$$

When the mechanism is in a singularity configuration, the follower can produce high forces in some direction with respect to low forces required to the mover. At limit some elements of the Jacobian matrix J^{-T} tend to infinity and the mechanism is capable of producing infinite forces in the working space with nil forces in the joint space.

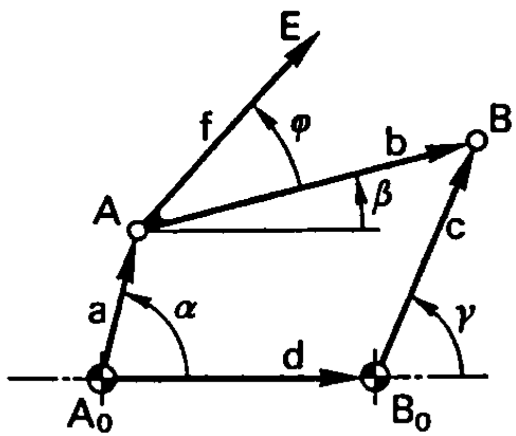
This behaviour is called **kinetostatic duality**

Kinematic relationships

Analysis of a linkage mechanism



The analysis of a linkage mechanism by an analytical method is carried out using a **complex number representation**. In this complex plane the generic link a, which is oriented with an angle α with respect to the real axis, is described by: $z = ae^{i\alpha}$



The closing vectorial equation of the mechanism is:

$$z_1 + z_2 - z_3 - z_4 = 0$$

whereas its projection on the real and imaginary axes, provides the equations system:

$$\begin{cases} a\cos\alpha + b\cos\beta - c\cos\gamma - d = 0 \\ a\sin\alpha + b\sin\beta - c\sin\gamma = 0 \end{cases} \Rightarrow \begin{cases} b\mathsf{c}\beta = c\mathsf{c}\gamma + d - a\mathsf{c}\alpha \\ b\mathsf{s}\beta = c\mathsf{s}\gamma - a\mathsf{s}\alpha \end{cases}$$

Analysis of a 4 bar mechanism: analytical solution

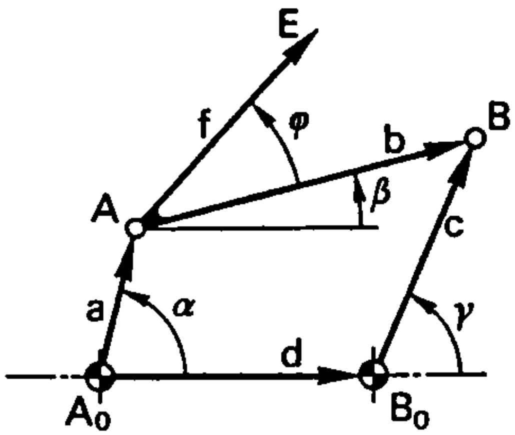
By doing some mathematical steps, we can obtain the angular velocity of the output links with respect to $\dot{\alpha}$:

$$\begin{cases} \dot{\gamma} = \dot{\alpha} \dfrac{a \sin(\alpha - \beta)}{c \sin(\gamma - \beta)} \\ \dot{\beta} = \dot{\alpha} \dfrac{a \sin(\alpha - \gamma)}{b \sin(\gamma - \beta)} \end{cases}$$

as well as the angular accelerations

$$\begin{cases} \ddot{\gamma} = \dfrac{a\ddot{\alpha} \sin(\alpha - \beta) + a\dot{\alpha}^2 \cos(\alpha - \beta) + b\dot{\beta}^2 - c\dot{\gamma}^2 \cos(\gamma - \beta)}{c \sin(\gamma - \beta)} \\ \ddot{\beta} = \dfrac{a\ddot{\alpha} \sin(\alpha - \gamma) + a\dot{\alpha}^2 \cos(\alpha - \gamma) + b\dot{\beta}^2 \cos(\beta - \gamma) - c\dot{\gamma}^2}{b \sin(\gamma - \beta)} \end{cases}$$

Analysis of a 4 bar mechanism: analytical solution



If we were interested in knowing the real trajectory followed by a generic point E belonging to the coupler link AB, we can define its position, with respect to AB, through the vector AE, whose length is f and anomaly φ (constant). We have:

$$A_0E = ae^{i\alpha} + fe^{i(\beta+\varphi)} \quad \Rightarrow \quad \begin{cases} x = a\cos\alpha + f\cos(\beta+\varphi) \\ y = a\sin\alpha + f\sin(\beta+\varphi) \end{cases}$$

while for speed and acceleration can be easily obtained once $\dot{\beta}$, $\ddot{\beta}$ are known.

Synthesis of linkage mechanisms

Introduction

Kinematic synthesis deals with the systematic design of mechanisms for a given performance.

Kinematic synthesis is performed in 2 steps:

- **Type synthesis**: Identifies the best kind of mechanism for a given task
- **Dimensional synthesis**: Defines the position of joints, the link lengths and the main geometrical features of the mechanism.

Introduction

The type synthesis is usually carried out according to the following table:

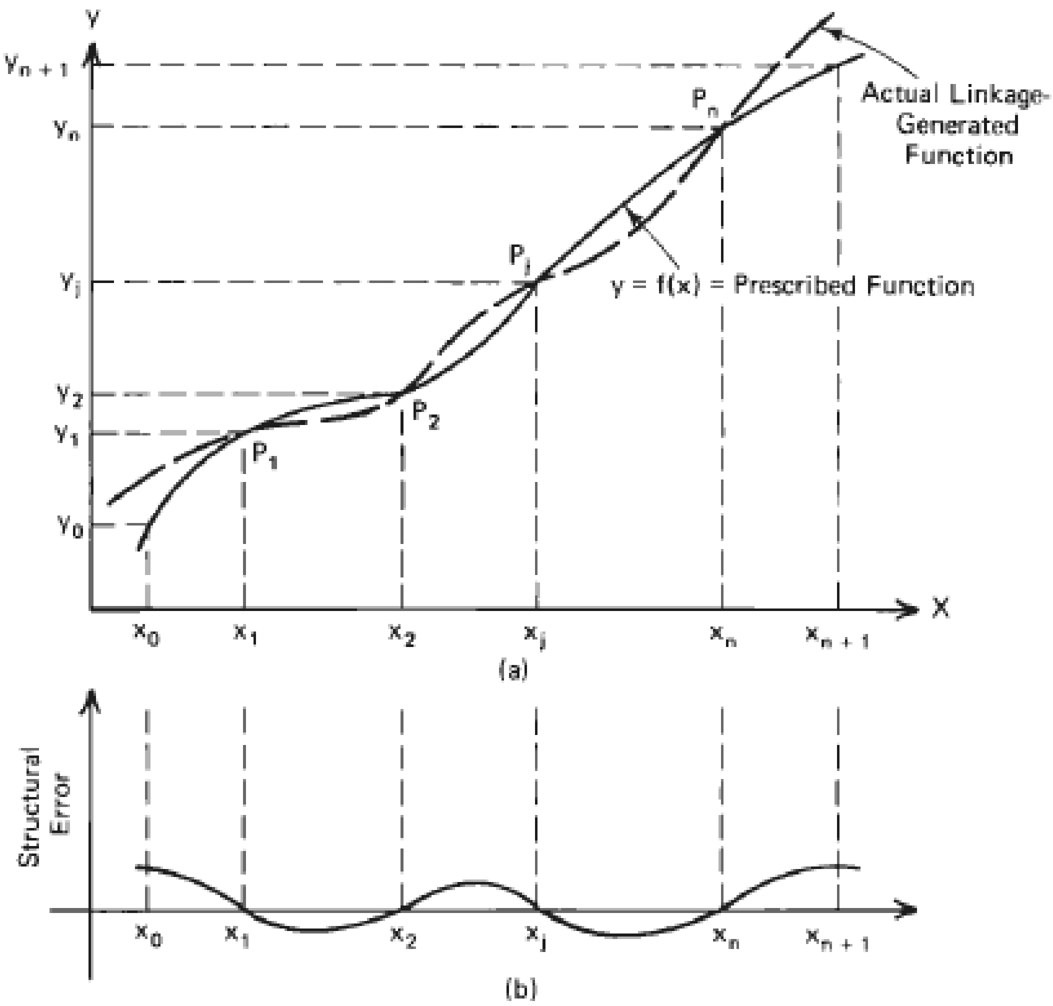
transmit circular motion	4 bar linkage (parallelogram)
unidirectional rotational motion to alternate rotational motion	4 bar linkage slider crank with yoke
unidirectional rotational motion to alternate linear motion	slider crank mechanism
follow a path	4 bar linkage (1dof), 5 bar linkage (2dof)
quick return mechanism	slider crank, 4/6 bar linkages
mechanism with delay	
force multiplier	

Introduction

In a linkage mechanism the number of parameters available for the kinematic synthesis is limited; it is thus not possible to assign a generic motion curve to the follower $y = f(\alpha)$ (where α is the coordinate that describes the follower position) but an approximation thereof e.g. $y = g(\alpha)$

The difference between the two motion curves is called **structural error**:

$$e(\alpha) = f(\alpha) - g(\alpha)$$



Introduction

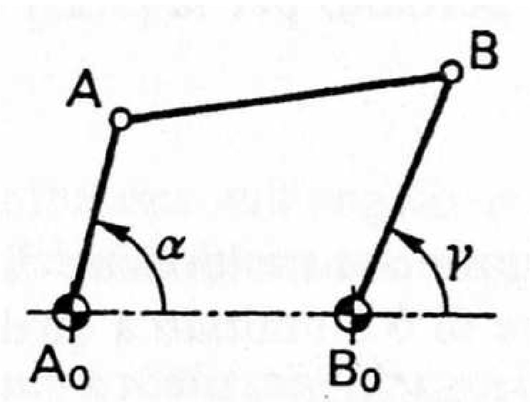
The kinematic synthesis methods reducing or nullifying $e(\alpha)$ can be divided into two groups:

- **Direct synthesis methods**: such method imposes the nullification of the structural error ($e(\alpha) = 0$) in some positions of the mechanism that are called *precision points*. The kinematic behaviour of the mechanism between two subsequent precision points is not controlled and a check is always necessary.
- **Indirect synthesis methods**: are based on repeated analysis of the mechanism behavior until a satisfactory outcome is obtained. These approaches are usually based on numerical optimization techniques.

The synthesis problems of a linkage mechanism can be grouped as follows:

- function generation
- trajectory (path) generation
- plane movement (motion) generation

Four bar mechanism as a function generator

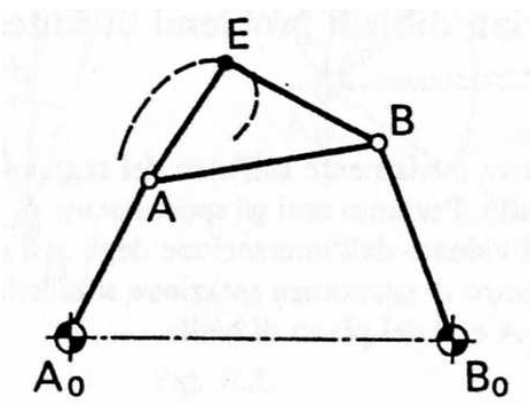


The problem: to synthesize the four bar mechanism so that the angular positions of the links A_0A and B_0B are correlated ($\gamma = \gamma(\alpha)$).

Unknown variable are a, b, c, d, α_0. Neglecting the overall dimensions variables are reduced to four : $a/d, b/d, c/d, \alpha_0$.

It is thus possible to take into account only four subsequent rotations of the two links.

Four bar mechanism trajectory generator



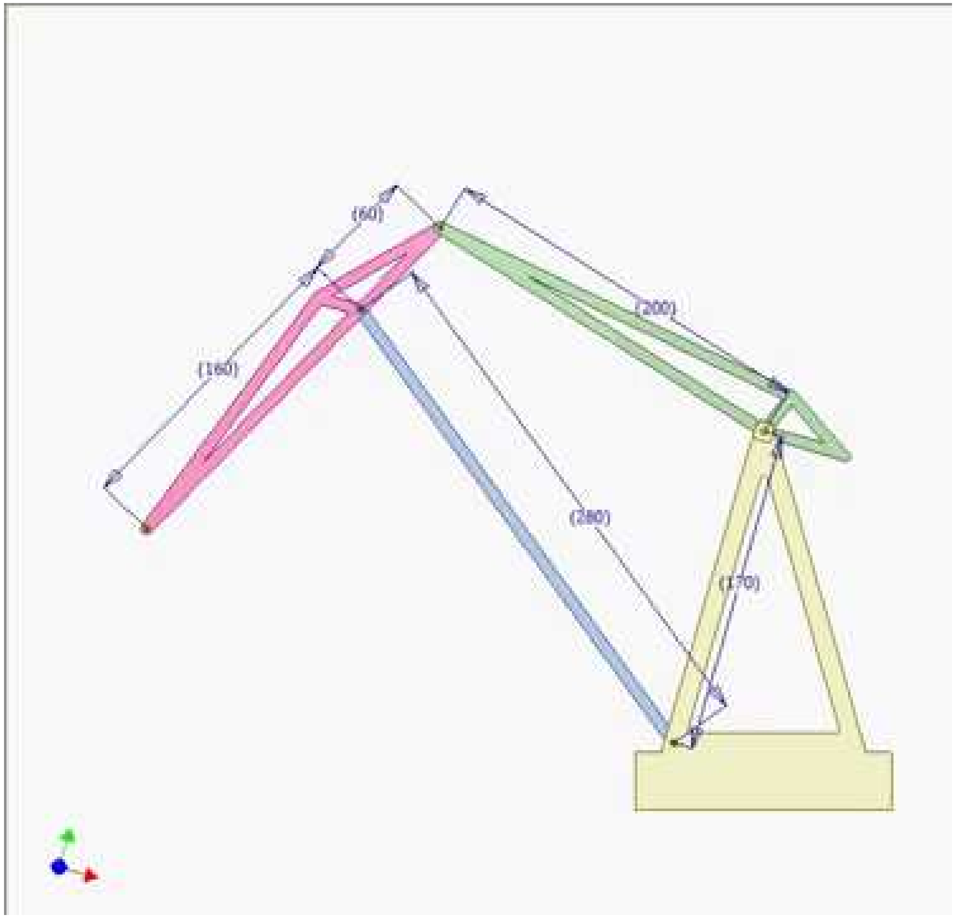
The problem: To synthesize the four bar mechanism in such a way a point *E* of a connecting rod traces an assigned trajectory.

The assigned trajectory is scaled with respect to coordinates system, thus the unknown variables are:

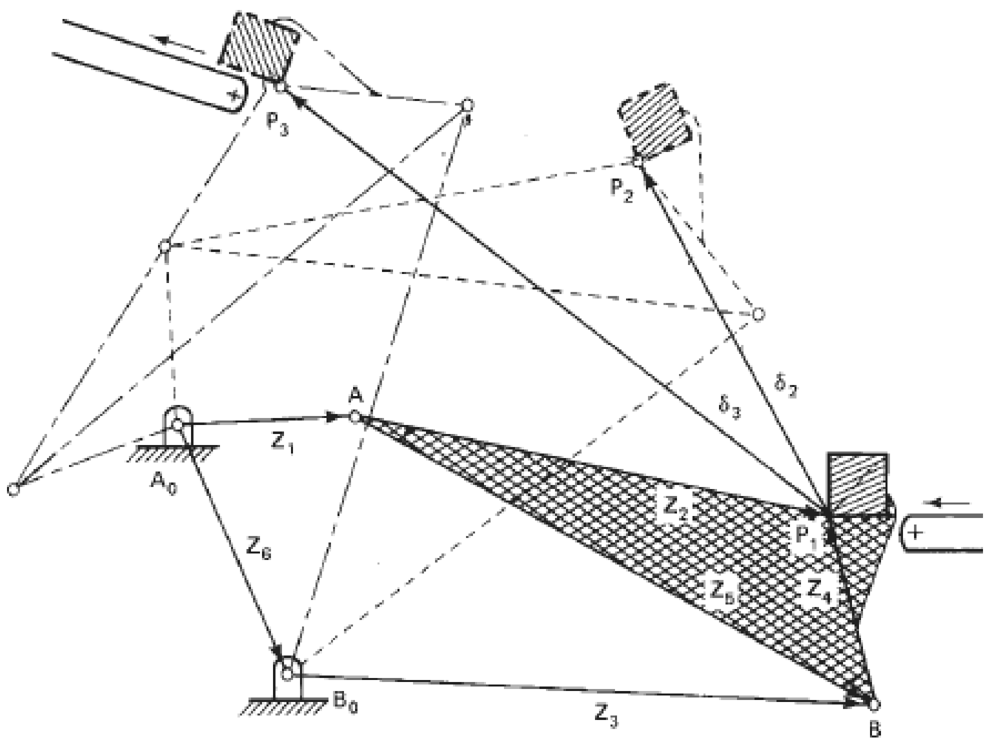
4 Position of A_0 and B_0
3 lengths of links A_0A, AB, BB_0
2 lengths of AE and EB

Thus it is possible to impose 9 precision points.
If we want to connect the positions of E to the angle α, then variables become 10 (α_0), but for every pose it is necessary to impose two conditions and the degrees of freedom go down to 5 (and then 5 precision points).

Four bar mechanism trajectory generator



Four bar mechanism motion generator

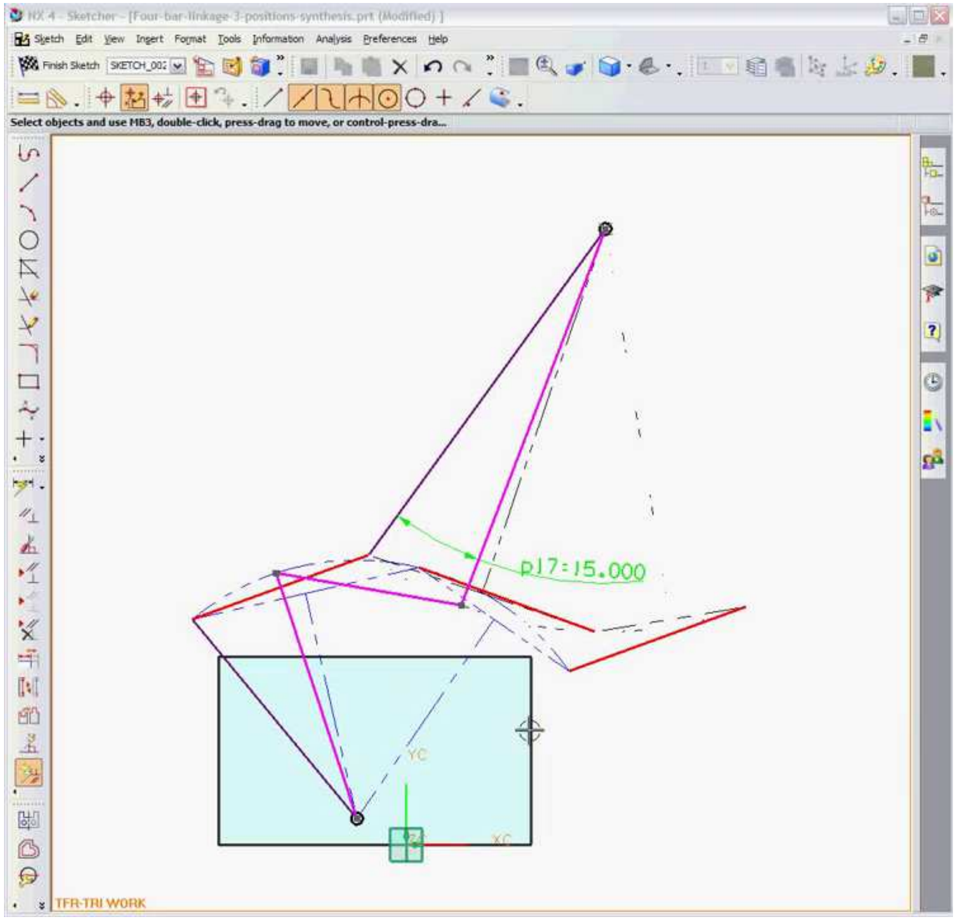


The problem: To synthesize the four bar mechanism to move a body, linked with the connecting rod AB, following assigned positions.

Variables are 10, the coordinates of the grounded joints (4), lengths of links (3) and the coordinates to define the position of the body with respect to the connecting rod (3).

To determine the body position in the plane, three conditions are necessary, so we can define three position (3 precision points). If we do not want to link the body positions to a specific angular position of the crank we can impose 5 precision points.

Four bar mechanism motion generator



Synthesis with optimization techniques

Optimisation techniques for kinematic synthesis allow to assign precision points and optimize an object function (usually the structural error).
For example, four bar mechanisms for trajectory generation have a point on the connecting rod that passes through a certain number of precision points with respect to some defined crank rotations.
The objective function that we can use in this kind of problem is the structural error, that is the sum of the squared distances between the desired points and the actual ones for every prescribed crank positions:

$$FO = \sum_{i=1}^{N} (x_i - x_{i_p})^2 + (y_i - y_{i_p})^2$$

where N is the number of the design points, $x_i,\ y_i$ the coordinates of the connecting rod point for the ith crank rotation, $x_{i_p},\ y_{i_p}$ the design point coordinates for the ith crank rotation.

Quick return mechanisms

Quick return mechanisms

Quick return mechanisms are capable of generating alternative movements (usually linear), with different times for the forward motion (t_s) and the backward one (t_d). Usually:

- the forward motion is slow and it is associated to some operations done by the machine
- the return phase is usually faster as it's just aimed to return the mechanism to the initial position.

These mechanisms are supposed to:

- transform the uniform motion into an alternative one,
- transform the rotary motion with the linear one,
- obtain the required time ratio t_s/t_d.

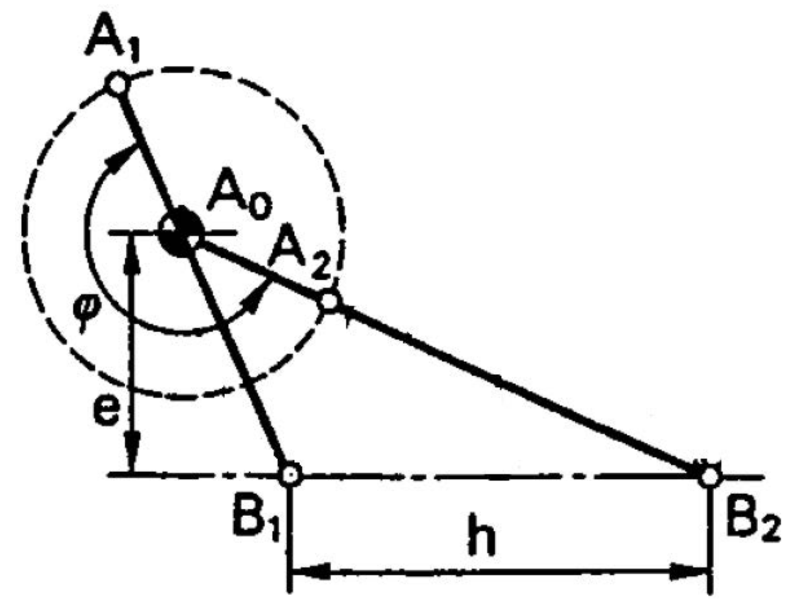
Quick return mechanisms are usually obtained with slider crank mechanisms, four / six bar mechanisms.

Slider crank mechanisms

The problem: to synthesize a slider crank mechanism in such a way that one end of the connecting rod B performs a linear stroke h in a time t_s for the forward motion and t_d for the return.
Time period T is obviously equal to $t_s + t_d$. If the crank rotates at a constant angular velocity ($\omega =$cost), it is possible to substitute times with the corresponding crank angles. Being φ crank angle corresponding to the forward motion, we have:

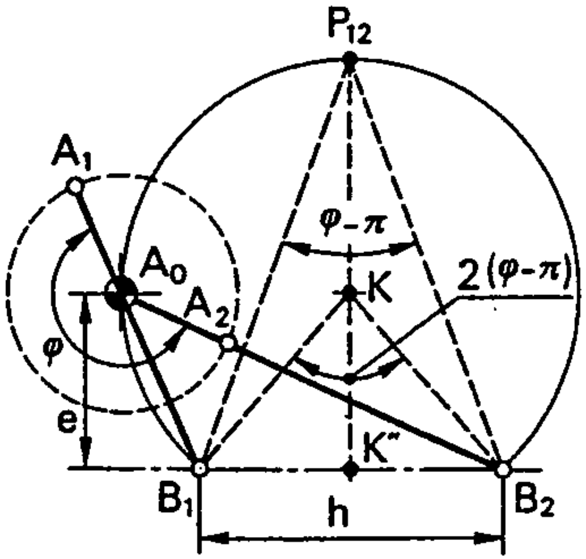
$$\begin{cases} \omega = \dfrac{2\pi}{T} \\ \varphi = \omega t_s \end{cases} \quad \Rightarrow \quad \varphi = \frac{2\pi}{t_s + t_d} t_s$$

then once t_s and t_d are known, φ is defined.



Simplest linkage mechanisms that allow a quick return are based on deviated slider crank ones.

Slider crank mechanisms

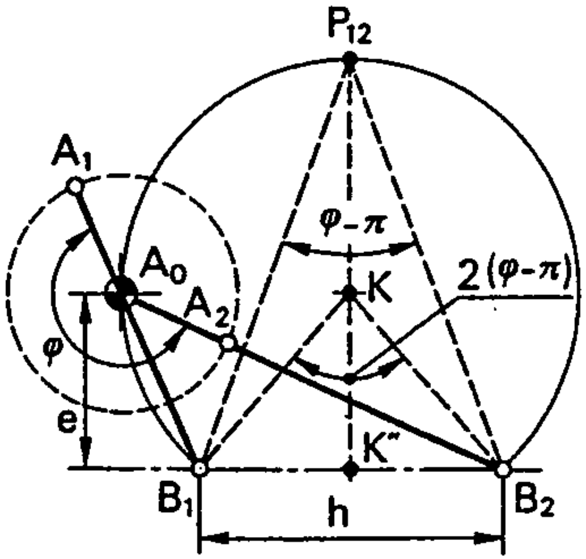


The figure shows a mechanism in its two dead center positions. Note that to go from B_1 to B_2, the crank rotates in φ and point A_0 sees the stroke h under an angle $\varphi - \pi$. This consideration is fundamental for the mechanism synthesis.

Steps needed:

1. draw $B_1 B_2 = h$;
2. draw the circular arch passing through B_1 and B_2 so that $B_1 \widehat{P_{12}} B_2 = \varphi - \pi$;
3. choose the point A_0 on the circular arch and obtain e
4. calculate r and l ($l - r = A_0 B_1$, $l + r = A_0 B_2$).

Slider crank mechanisms

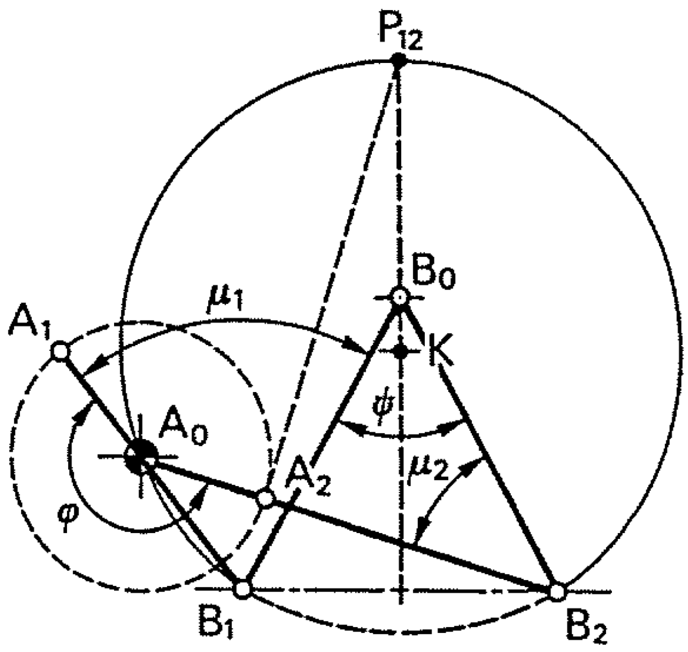


A limitation exists in the choice of A_0: all the points included in the circular arch between the perpendiculars to B_1B_2 traced by the points B_1 and B_2 should be excluded. In this configuration the mechanism wouldn't be able to reach B_1 from B_2 and viceversa.

Note that this limitation is greater the closer the angle $(\varphi - \pi)$ approaches $\pi/2$. As in this case there are no solutions, it should be

$\varphi \leq \frac{3\pi}{2}$ and then: $\varphi = 2\pi \frac{t_s}{t_s + t_d} \Rightarrow \frac{t_s}{t_d} = \frac{\varphi}{2\pi - \varphi} \Rightarrow \left(\frac{t_s}{t_d}\right)_{lim} = 3$

Four bar mechanisms



To design the system we choose arbitrarily B_1 and B_2, thus on the circular arc that sees $B_1 B_2$ under the angle $\varphi - \pi$, we take the point A_0 and we choose the lengths r and l as for the crank slider mechanism. The position B_0 of the rocker rotation center needs to be defined. Remembering that the angle $B_1 \widehat{B_0} B_2 = \psi$ is defined, the limitation on φ is:

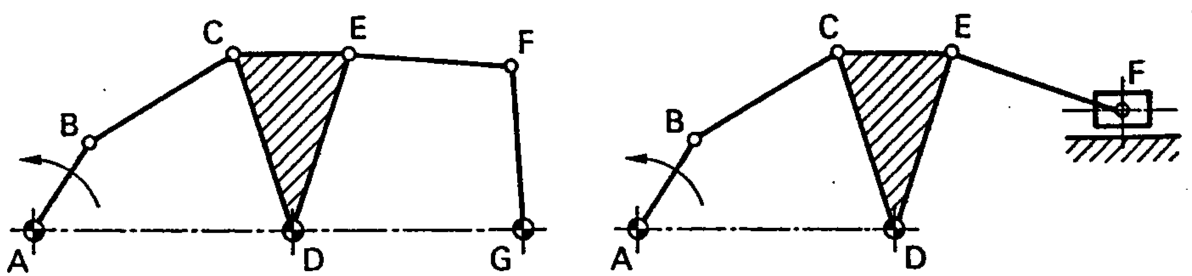
$$\varphi < \frac{3\pi}{2} + \frac{\psi}{2}.$$

Depending on design limitations on ψ, this mechanism can be profitably used for a quick return. considering as example $\psi = \pi/2$:

$$\varphi \leq \frac{3\pi}{2} + \frac{\pi}{4} \qquad \text{and then:} \qquad \varphi = 2\pi \frac{t_s}{t_s + t_d} \quad \Rightarrow \quad \frac{t_s}{t_d} = \frac{\varphi}{2\pi - \varphi} \quad \Rightarrow \quad \left(\frac{t_s}{t_d}\right)_{lim} = 7$$

Six bar mechanisms

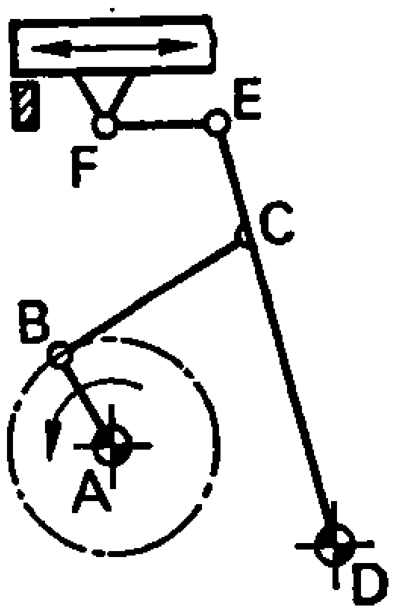
Six bar linkages can be profitably used as quick return mechanisms.



The figure shows the Watt six bar linkage of most common practical use: the AB mover link rotates with a constant angular velocity and the follower can be FG or the slider F. The mechanisms can be considered as a serie of 2 four-bar linkages or a 4bar linkage + a slider crank.

Six bar mechanisms

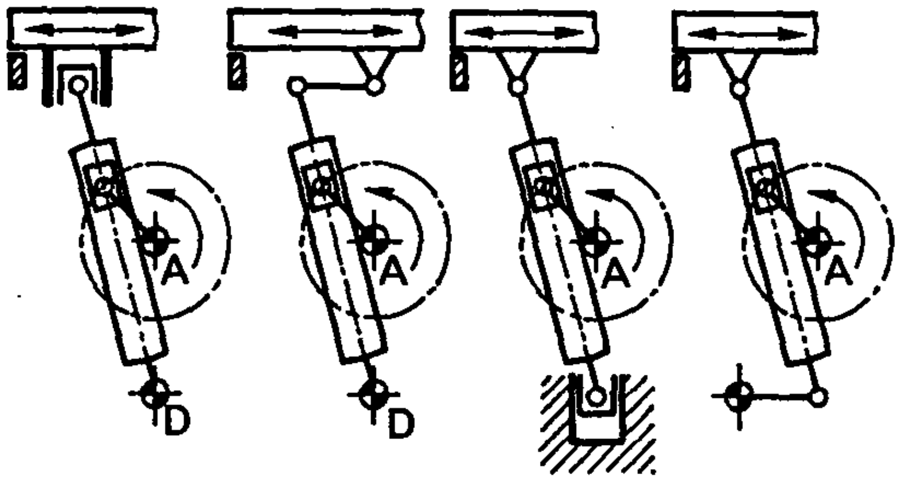
Let us see how it is possible to use a six bar linkage in order to realize rapid return mechanisms. We begin by considering a Watt six bar made up of a crank-rocker four bar linkage and a slider-crank.



The four bar mechanism realizes the ratio t_s/t_d while the crank-slider linkage transforms the rotational motion into a linear one. For the design of the four bar linkage we adopt the method previously set out, while for the crank-slider linkage, it is usual to let it operate far from the dead center so as to keep as much as possible constant its transmission ratio.

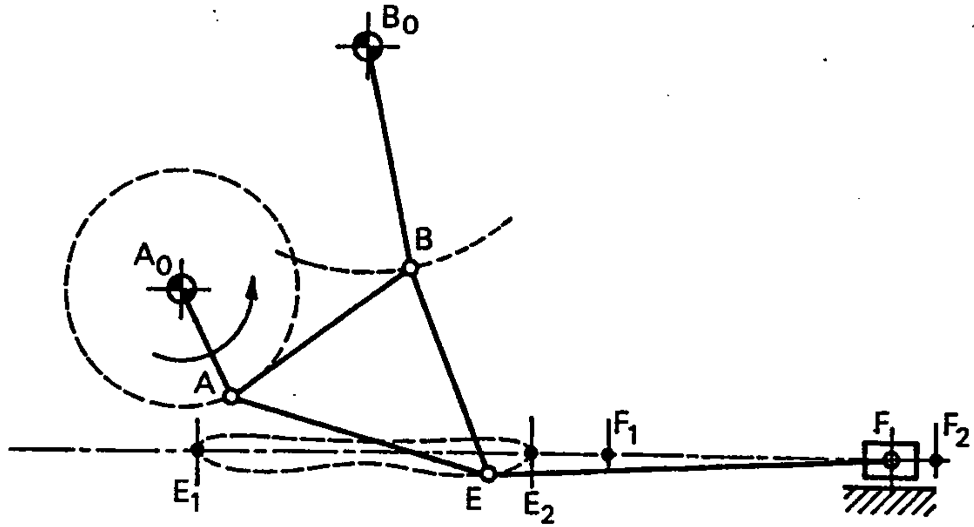
Six bar mechanisms

Analogues solutions can be obtained using degenerate four bar linkages:



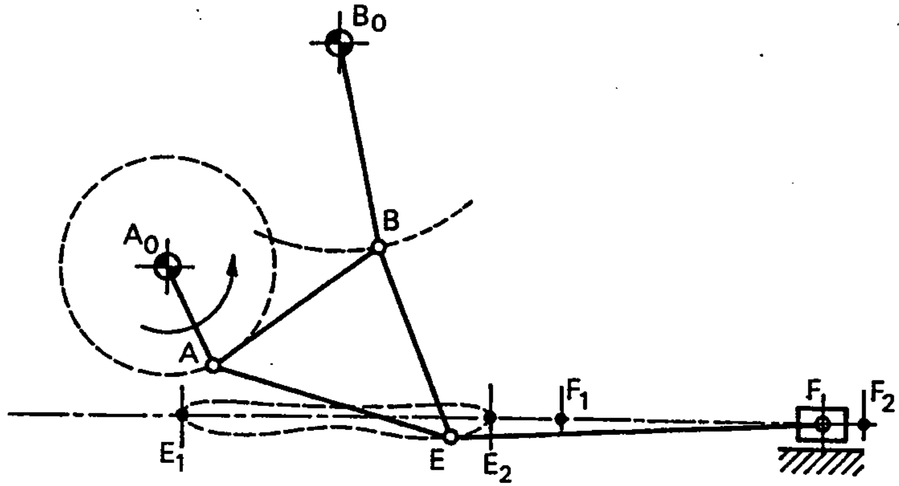
We can use the Watt kinematic chain in which the initial four bar mechanism is with double crank and realises the required ratio t_s/t_d, while the crank-slider linkage (usually centred) transforms the non uniform rotational movement into an alternative one with differing going and return times.

Six bar mechanisms



Taking advantage of the Stephenson chain, as set out in the figure, the F slider is actuated by means of the connecting rod point E. Everything depends on the trajectory of E and how the the direction of the slider movement is oriented.

Six bar mechanisms

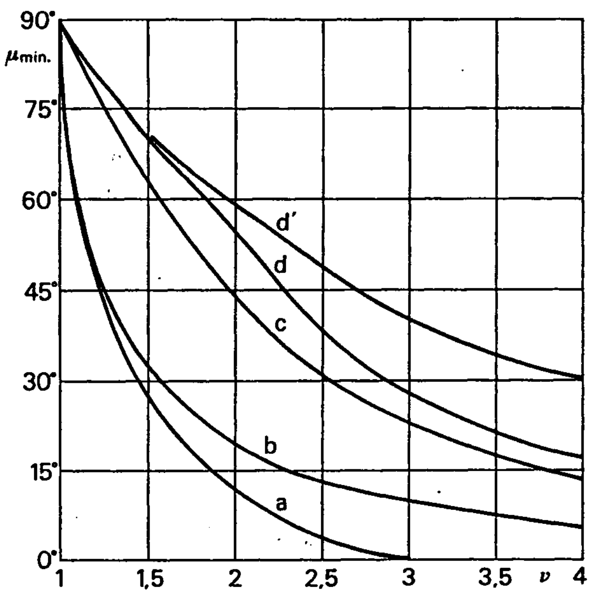


Having assigned the position F_1 and F_2 and the corresponding positions of EF, we find the points E_1, E_2 through which the trajectory of E must pass and the directions of the tangents in these points: usually we choose E_1 and E_2 on the line F_1F_2 (stroke $= F_1F_2 = E_1E_2$) so that the tangents in E_1 and E_2 are parallel between themselves and orthogonal to F_1F_2.

Having fixed φ, being the crank angle which moves the point E from E_1 to E_2, we can demonstrate that there exists a fourfold infinity of solutions to solve the synthesis mechanism problem.

Six bar mechanisms

It is possible to add further conditions which the mechanism must be capable of performing and there are numerous graphical and numerical methods supplying the desired solutions. Without going into the merits of the project of the single mechanism we want to furnish a framework which compares the minimum transmission angle, the maximum ratio value of $\nu = t_s/t_d$, for some kinematic solutions:



a) Offset slider crank linkage;
b) Watt six bar mechanism (crank-rocker and slider crank linkage);
c) Stephenson six bar mechanism (three grounded pins);
d) Watt six bar mechanism (double crank and slider crank linkage);
d') Watt six bar mechanism (double crank and and offset slider crank linkage);