Functional Mechanical Design

Cam Mechanisms (2/2)

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Introduction

Now that main features of cams have been introduced (classification, pro&cons., limitations due to pressure angle and undercut) we can focus on kinematic analysis and synthesis:

Kinematic analysis: to evaluate the motion law designed into the cam profile.

Generally the kinematic analysis for cam mechanisms is performed when:

- we have realized the cam with simple machining operations and then we need to understand what kind of motion law we are using.
- we have to find out what motion law is realized by a cam that we haven't produced (reverse engineering)

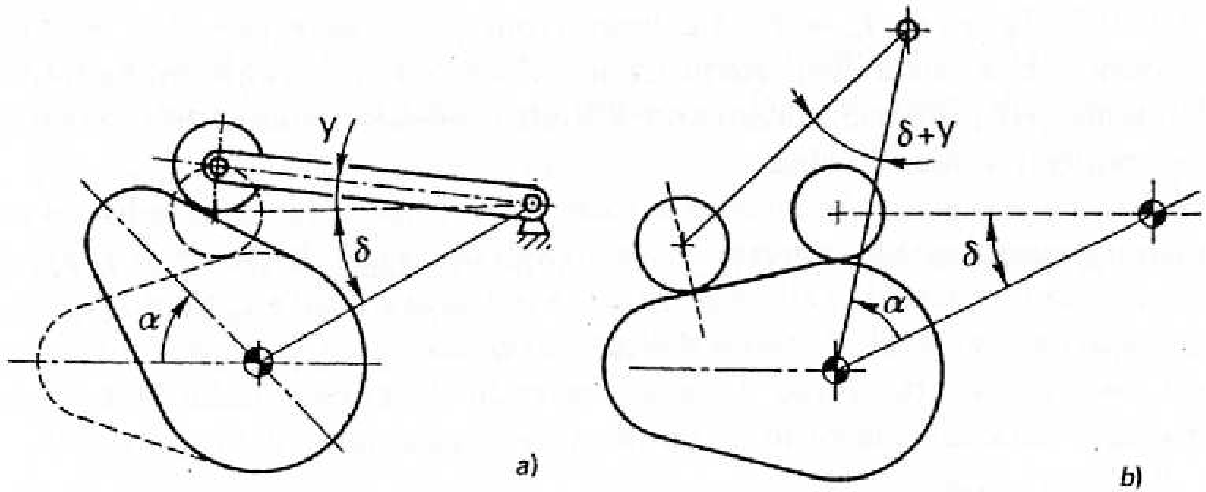
Kinematic synthesis: to calculate the cam profile to get the desired motion law.

During the synthesis some quantities like the pressure angle and the curvature radius can be easily calculated.

note: equations are sometimes very complex. It's not important to remember them by heart, it's important to understand which are the effects of main factors.

Kinematic analysis

Kinematic inversion method



Kinematic analysis is usually performed through the inversion of the motion. For this method, the cam is maintained fixed while the ground rotates in the opposite direction with respect to the rotating direction of the cam. During the motion the follower is in contact with the cam.

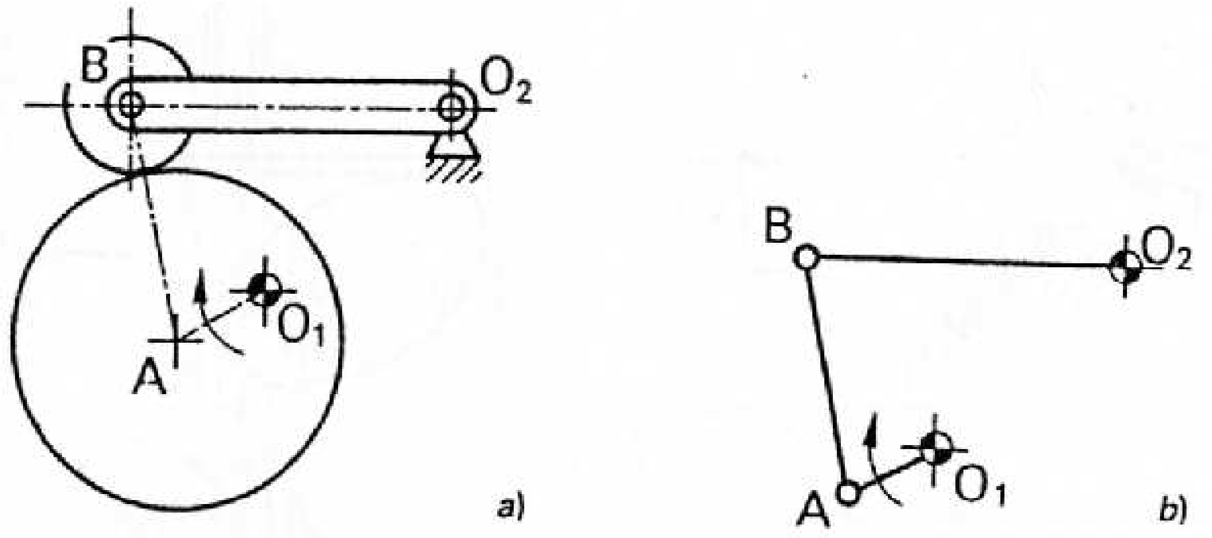
For each position we can relates the cam angle α and the displacement of the follower $y = y(\alpha)$.

Note that the relationship $y = y(\alpha)$ is **geometrical**. To obtain $y'(\alpha)$ and $y''(\alpha)$ we can derive the expression of y with respect to α, but the quality of the results strongly depends on the number of point where y has been evaluated.

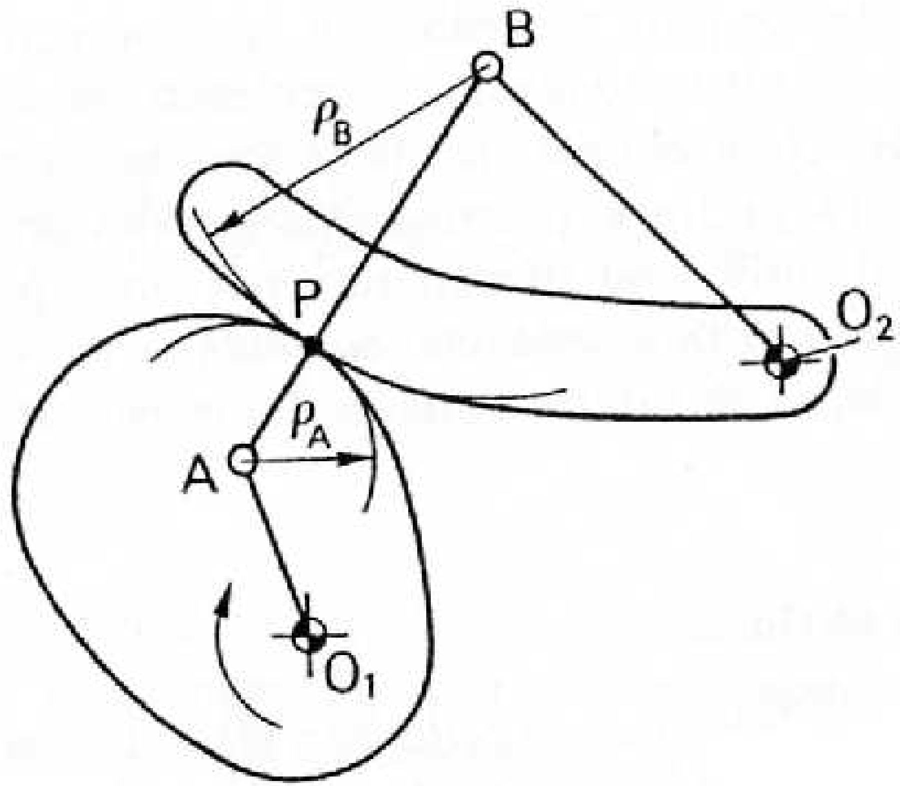
To obtain more suitable results it is better to use a different approach based on the analysis of the **relative kinematics**. With this approach it is possible to evaluate all the kinematic quantities of the follower. This approach can be graphical or analytical.

The approach is based on the idea that, for all the angular positions α, the cam mechanism can be approximated by an **equivalent linkage mechanism**.

Equivalent mechanisms



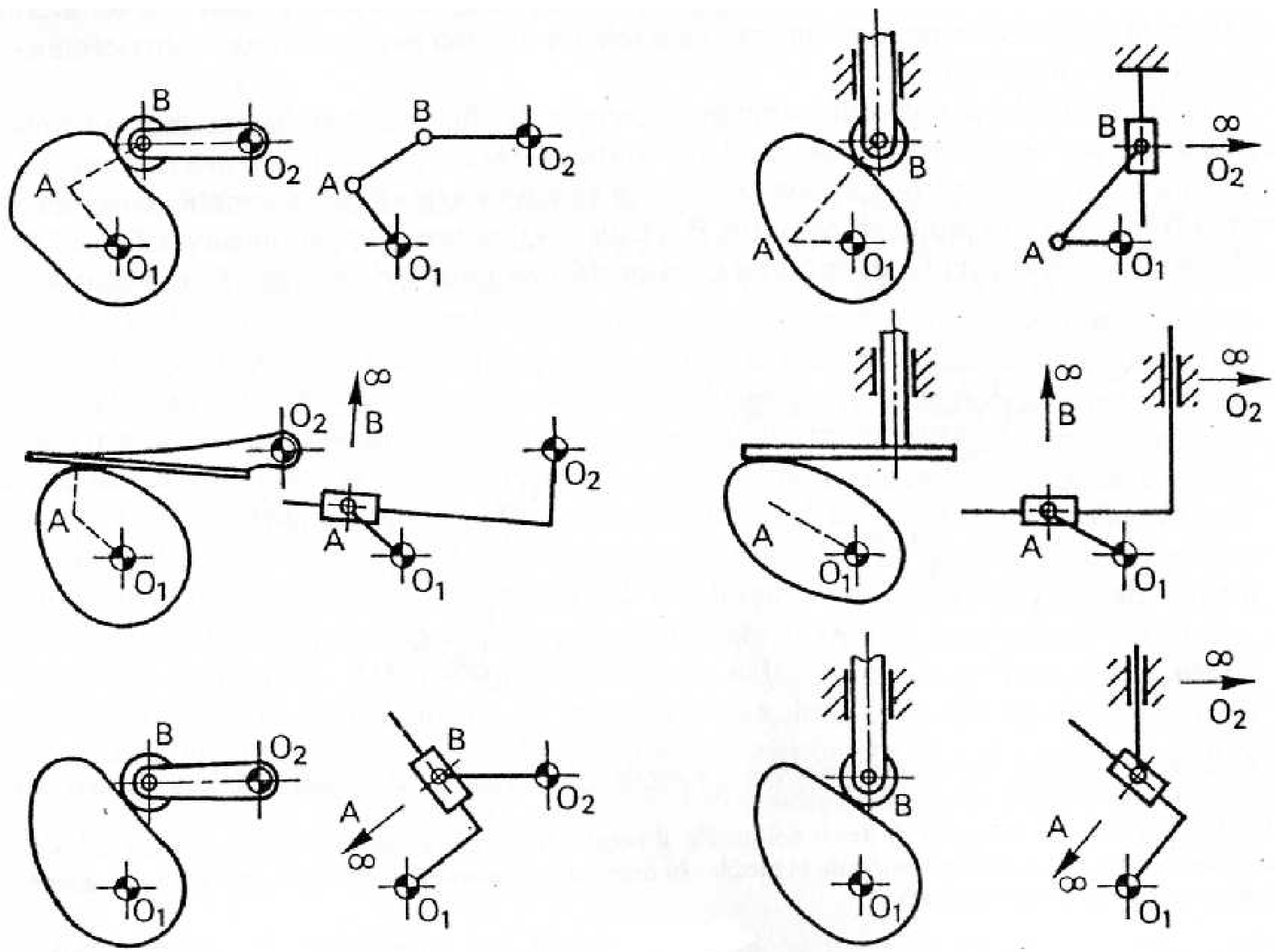
The link BO_2 of the four bar mechanism is moving as the follower BO_2 of the cam mechanism: the four bar mechanism is the system that, for a kinematic point of view, has the same behaviour of the cam mechanism shown in figure a.



Usually the equivalent mechanism changes for every position of the contact point between cam and follower: from a kinematic point of view, the cam profile can be substituted with the osculating circle for every single pose of the cam device. The curvature radius ρ of the cam profile represents the radius of the osculating circle. The kinematic equivalence is true till the acceleration.

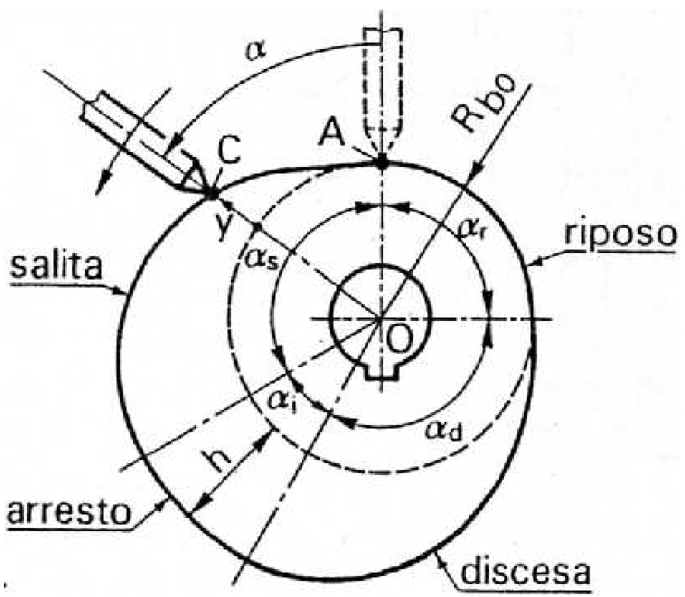
The four bar mechanism O_1ABO_2 is the **instantaneous equivalent mechanism** for the cam device shown in the figure.

Note that the equivalent four bar mechanisms can degenerate:



Kinematic synthesis

Radial cam with translating roller follower



* in the synthesis we always start designing the pitch profile (trajectory followed by the trace point) and then we calculate the cam profile by considering the roller radius or the flat contact

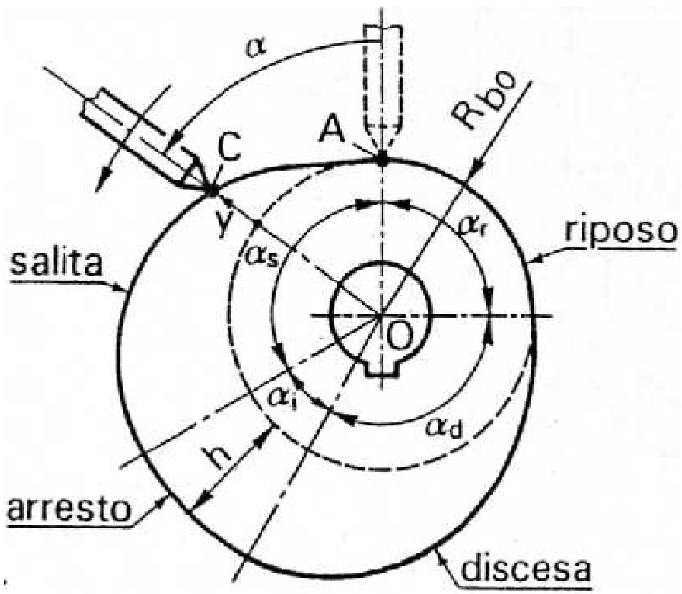
The **kinematic inversion** is used also for the analytical synthesis of a cam mechanism and it represents the starting point in the definition of the pitch profile*.

At the beginning we can consider that $\alpha = 0$ is the position in which the follower begins the rise phase and $y(\alpha)$ is the follower motion curve.

To obtain the **pitch profile** we just need to add $y(\alpha)$, along the radial direction, to the base circle. In polar coordinates the pitch profile is expressed by the following relationship:

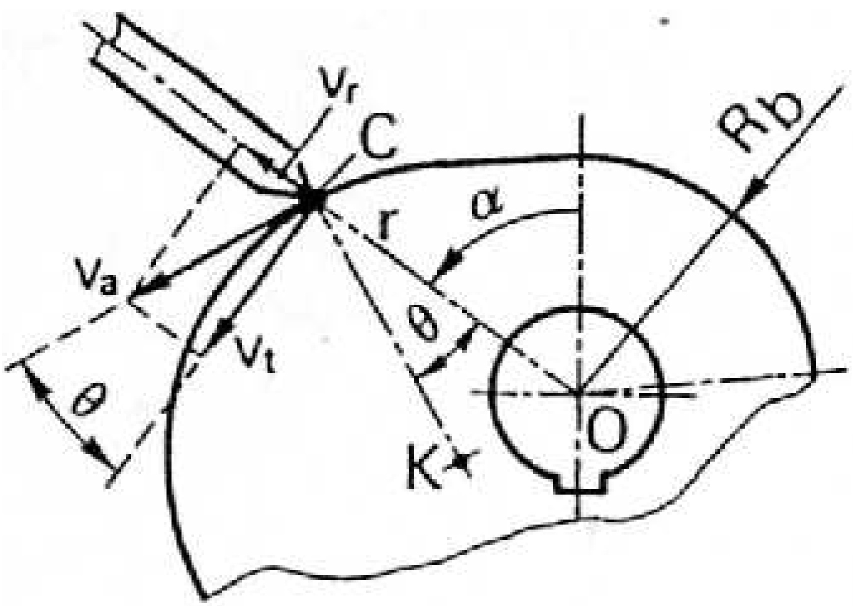
$$r_0(\alpha) = R_{b0} + y(\alpha)$$

Radial cam with translating roller follower



The profile is made up of:

- a rise segment α_s;
- a dwell segment α_i; the profile is a circle with radius equal to $R_{b0} + h$;
- a return segment α_d;
- a dwell segment α_r, the profile is a circle with radius equal to R_{b0}.



Let's consider the generic position of the follower as defined by the cam angle α

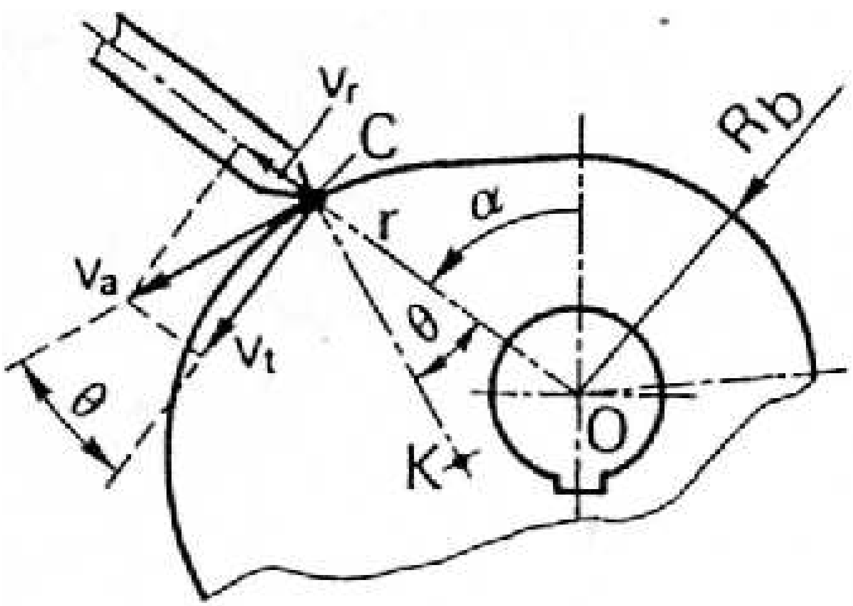
Considering the pitch profile, C is the center of the roller (trace point) and K is the center of curvature of the pitch profile (It is along the direction orthogonal to the profile in the contact point).

The **pressure angle** is:

$$\theta = K\widehat{C}O$$

(being OC oriented as the velocity and CK as the contact force).
To calculate θ we use an approach based on relative kinematics.

Radial cam with translating roller follower



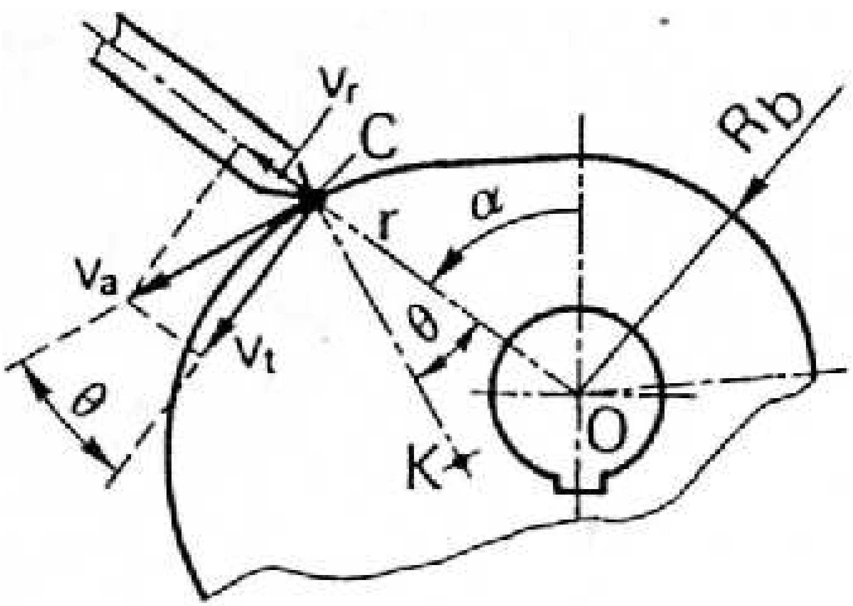
The motion of C can be described through a relative rotating frame in O as the sum of:

- **relative motion**, in which C is moving along OC:
$$v_r = \dot{y} = \frac{\partial y}{\partial \alpha} \cdot \frac{\partial \alpha}{\partial t} = y'\omega$$
- **drag motion**, in which C is moving along an arch of circle with radius OC$=R_{b0}+y$:
$$v_t = \omega \bar{OC} = \omega(R_{b0}+y)$$

The absolute velocity is:

$$v_a = \sqrt{v_r^2 + v_t^2} = \omega\sqrt{y'^2 + (R_{b0}+y)^2}$$

Radial cam with translating roller follower



The absolute velocity is tangent to the pitch profile and normal to CK. Therefore the pressure angle θ is equal to the angle between the absolute velocity and the drag velocity (perpendicular to OC)

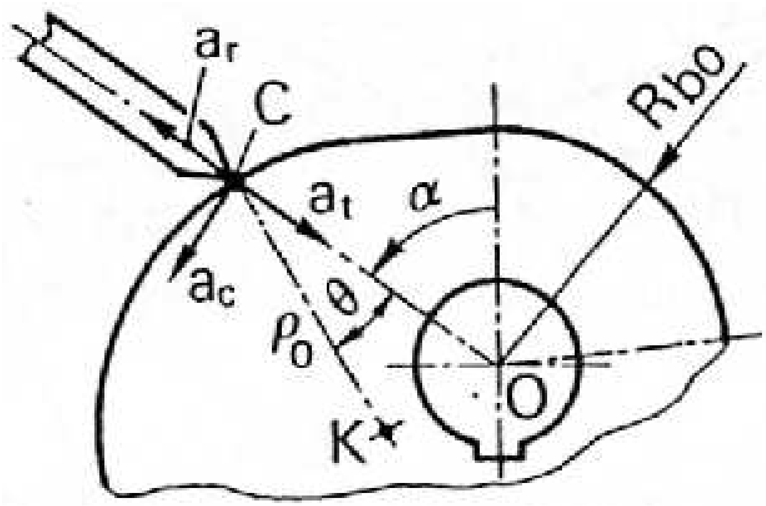
$$\begin{cases} v_a \sin\theta = v_r \\ v_a \cos\theta = v_t \end{cases}$$

$$\Downarrow$$

$$\tan\theta = \frac{v_r}{v_t} = \frac{y'}{R_{b0} + y}$$

To keep θ limited we can act on the maximum speed (c_v) and the base radius R_{b0}

Radial cam with translating roller follower



To evaluate the curvature radius of the pitch profile we can use the Coriolis theorem and calculate the acceleration of the point C:

$$\underline{a} = \underline{a}_r + \underline{a}_{tr} + \underline{a}_c$$

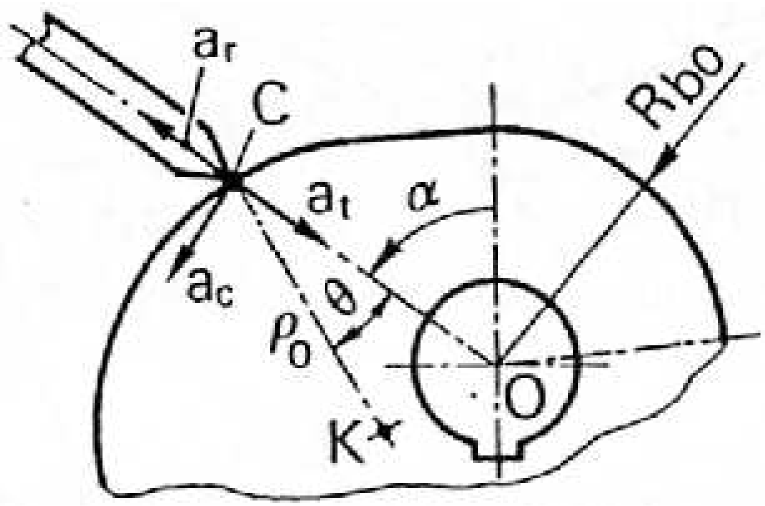
where:

$$\begin{cases} a_r = \ddot{y} = \omega^2 y'' & \text{(relative acc.)} \\ a_t = \omega^2 OC = \omega^2 (R_{b0} + y) & \text{(drag acc.)} \\ a_c = 2\omega \dot{y} = 2\omega^2 y' & \text{(coriolis acc.)} \end{cases}$$

The absolute acceleration $\underline{a}$ can be decomposed into the components (tangential and normal):

$$\underline{a} = \underline{a}_{\mathrm{T}} + \underline{a}_{\mathrm{N}}$$

Radial cam with translating roller follower



To get the curvature radius ρ_0 it only remains to project the absolute acceleration along the normal direction to the profile:

$$\begin{cases} a_N = (a_t - a_r)\cos\theta + a_c \sin\theta \\ \cos\theta = v_t / v_a \\ \sin\theta = v_r / v_a \end{cases}$$

$$a_N = \frac{(a_{tr} - a_r)v_t + a_c v_r}{v_a} = \frac{v_a^2}{\rho_0}$$

$$\Downarrow$$

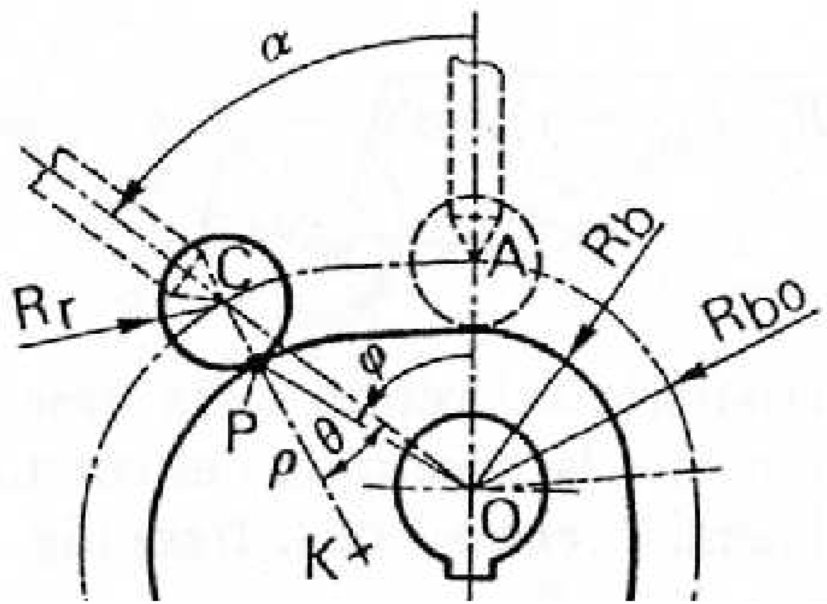
$$\rho_0 = \frac{v_a^3}{a_t v_t - a_r v_t + a_c v_r}$$

$$\rho_0 = \frac{[y'^2 + (R_{b0} + y)^2]^{3/2}}{(R_{b0} + y)^2 - (R_{b0} + y)y'' + 2y'^2}$$

If R_{b0} is small enough, ρ_0 can be negative and the profile is concave.

Undercut can be evaluated comparing ρ_0 and R_r

Radial cam with translating roller follower

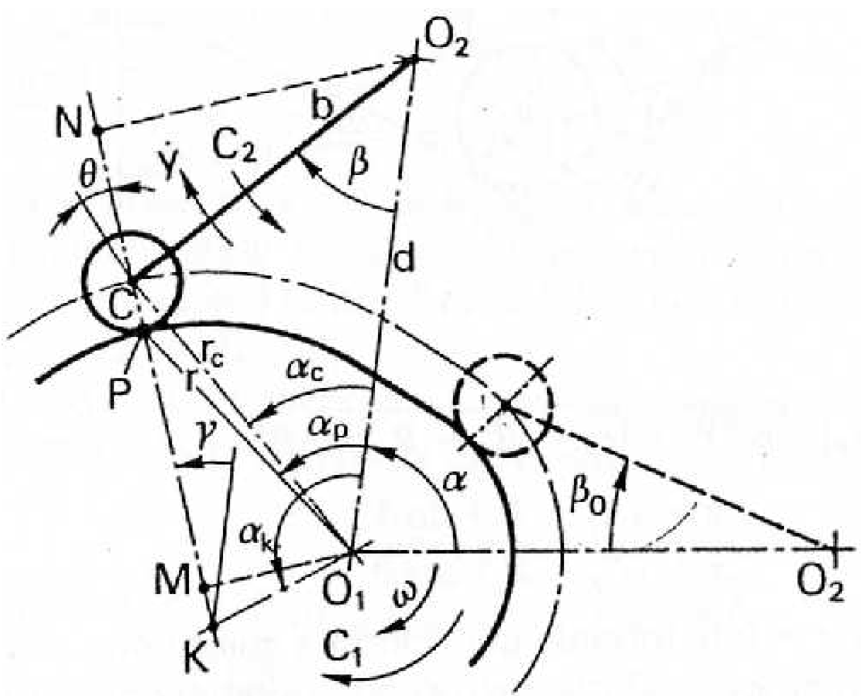


Even if $R_r \neq 0$, the curvature center of the cam profile is always K:

$$\bar{CK} = R_r + \bar{PK} \quad \Rightarrow \quad \rho = \rho_0 - R_r$$

where P is the contact point between cam and roller and $\bar{PK} = \rho$ is the curvature radius of the cam profile. Note that P is not placed along OC.
From the triangle CPO it is possible to obtain the polar coordinates of the cam profile, as follows:

$$\textit{Th. Carnot} \begin{cases} r(\alpha) = \bar{OP} = \\ \sqrt{R_r^2 + (R_{b0} + y)^2 - 2R_r(R_{b0} + y)\cos\theta} \\ \varphi(\alpha) = P\widehat{O}A = \\ \alpha + \arcsin\left(\dfrac{R_r \sin\theta}{r}\right) \end{cases}$$



A similar approach can be used for oscillating roller followers. Being β_0 the angle between follower and ground when the roller is on the base circle R_b, we have:

$$\beta = \beta_0 + y(\alpha)$$

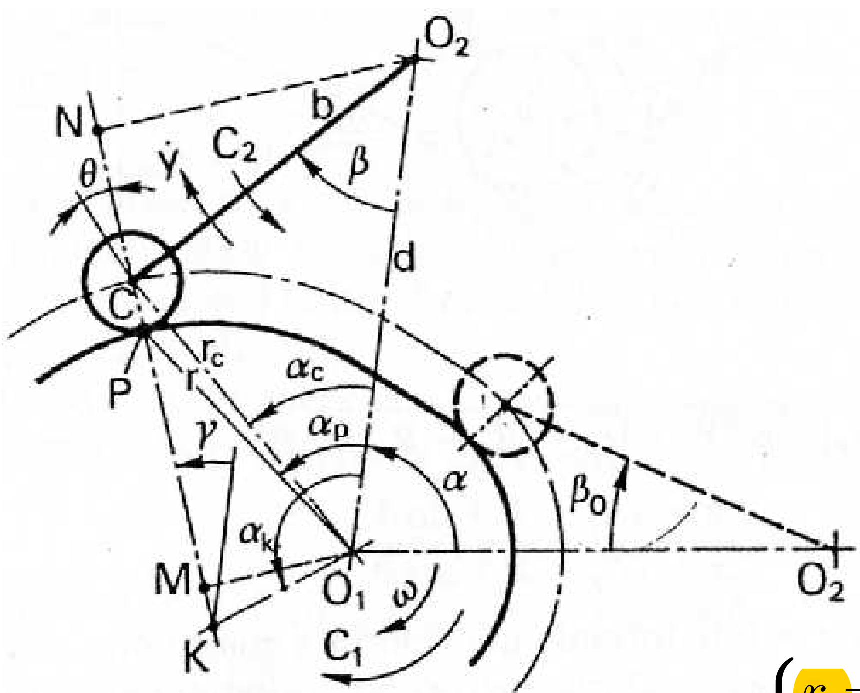
The polar coordinates $(r_c(\alpha), \varphi_c(\alpha))$ of the pitch profile are:

$$\begin{cases} r_c = \sqrt{b^2 + d^2 - 2bd\cos\beta} = \\ = \sqrt{(d - b\cos\beta)^2 + b^2\sin^2\beta} \\ \varphi_c = \alpha + \alpha_c = \alpha + \arctan\left(\dfrac{b\sin\beta}{d - b\cos\beta}\right) \end{cases}$$

where:

$$\begin{cases} r_c \sin\alpha_c = b\sin\beta \\ r_c\cos\alpha_c + b\cos\beta = d \end{cases} \Rightarrow \quad \tan\alpha_c = \frac{b\sin\beta}{d - b\cos\beta}$$

Radial cam with oscillating roller follower



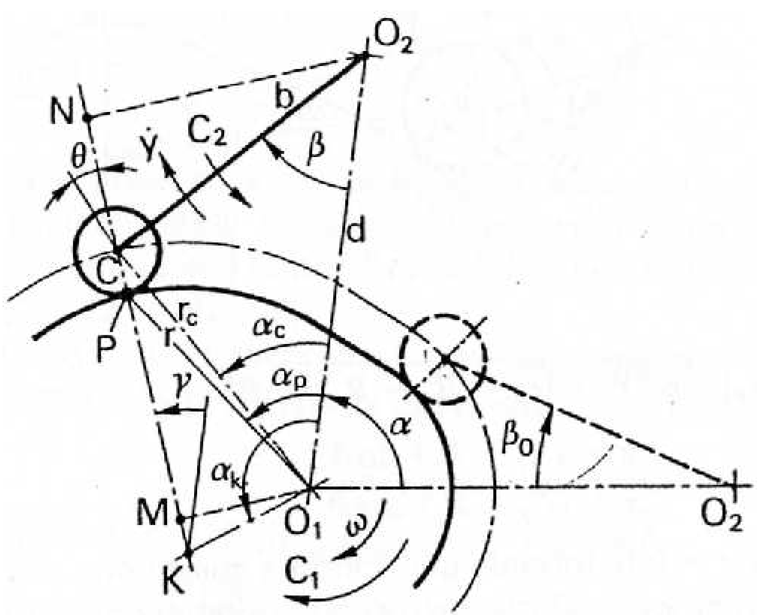
Considering the triangle O_2NC, the pressure angle is:

$$\theta = \frac{\pi}{2} - \beta - \gamma = \frac{\pi}{2} - \beta_0 - y(\alpha) - \gamma$$

While with reference to the right triangle O_1PZ, where Z is the projection of P on the ground, it is possible to obtain the polar coordinates of the cam profile:

$$\begin{cases} r = \sqrt{(d - R_r\cos\gamma - b\cos\beta)^2 + (b\sin\beta - R_r\sin\gamma)^2} \\ \varphi = \alpha + \alpha_p = \alpha + \arctan\dfrac{b\sin\beta - R_r\sin\gamma}{d - R_r\cos\gamma - b\cos\beta} \end{cases}$$

Radial cam with oscillating roller follower

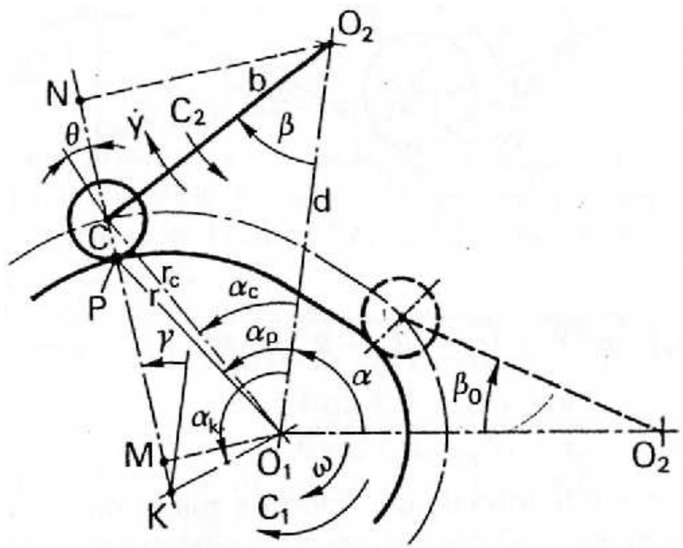


Indicating with γ the angle between CK (normal to the pitch profile in the contact point) and O^1O^2 (ground), with C^1 and C^2 the torques acting on cam and follower, we can write the power balance:

$$C_1\omega = C_2\dot{y}$$
$$\Downarrow$$
$$S \cdot O_1M \cdot \omega = S \cdot O_2N \cdot \dot{y}$$
$$\Downarrow$$
$$O_1M = O_2Ny'$$

$$\begin{cases} O_2N = b\cos[90 - (\gamma + \beta)] = b\sin(\gamma + \beta) \\ O_1M = O_2N - d\cos(90 - \gamma) = O_2N - d\sin\gamma \end{cases}$$

$$b\sin(\gamma + \beta) - d\sin(\gamma) = b\sin(\gamma + \beta)y' \quad \ldots \quad \tan\gamma = \frac{b\sin\beta(1 - y')}{d - b\cos\beta(1 - y')}$$



To evaluate the curvature radius of the pitch profile $\rho_0 = CK$ we can write the vector equation:

$$\overrightarrow{O_1K} + \overrightarrow{KC} + \overrightarrow{CO_2} + \overrightarrow{O_1O_2} = 0$$

$$\Downarrow$$

$$\overrightarrow{r_k} + \overrightarrow{\rho_0} + \overrightarrow{b} + \overrightarrow{d} = 0$$

Projecting the vector equation along the ground and perpendicular to this, we obtain two scalar equations:

$$\begin{cases} r_k \cos\alpha_k + \rho_0 \cos\gamma + b\cos\beta - d = 0 \\ r_k \sin\alpha_k + \rho_0 \sin\gamma - b\sin\beta = 0 \end{cases}$$

Radial cam with oscillating roller follower

Taking into account the two scalar equations:

$$\begin{cases} r_k \cos\alpha_k + \rho_0 \cos\gamma + b\cos\beta - d = 0 \\ r_k \sin\alpha_k + \rho_0 \sin\gamma - b\sin\beta = 0 \end{cases}$$

and deriving the second equation with respect to α we obtain:

$$\frac{\mathrm{d}}{\mathrm{d}\alpha}\left(r_k \sin\alpha_k + \rho_0 \sin\gamma - b\sin\beta\right) = 0$$

$$\frac{\mathrm{d}r_k}{\mathrm{d}\alpha}\sin\alpha_k + r_k\frac{\mathrm{d}}{\mathrm{d}\alpha}(\sin\alpha_k)\frac{\mathrm{d}\alpha_k}{\mathrm{d}\alpha} + \frac{\mathrm{d}\rho_0}{\mathrm{d}\alpha}\sin\gamma + \rho_0\frac{\mathrm{d}}{\mathrm{d}\alpha}(\sin\gamma)\frac{\mathrm{d}\gamma}{\mathrm{d}\alpha} - \\ + \frac{\mathrm{d}b}{\mathrm{d}\alpha}\sin\beta - b\frac{\mathrm{d}}{\mathrm{d}\alpha}(\sin\beta)\frac{\mathrm{d}\beta}{\mathrm{d}\alpha} = 0$$

ρ_0 and r_k (O_1K) for small displacement can be considered constant:

$$r_k\frac{\mathrm{d}}{\mathrm{d}\alpha}(\sin\alpha_k)\frac{\mathrm{d}\alpha_k}{\mathrm{d}\alpha} + \rho_0\frac{\mathrm{d}}{\mathrm{d}\alpha}(\sin\gamma)\frac{\mathrm{d}\gamma}{\mathrm{d}\alpha} - b\frac{\mathrm{d}}{\mathrm{d}\alpha}(\sin\beta)\frac{\mathrm{d}\beta}{\mathrm{d}\alpha} = 0$$

Radial cam with oscillating roller follower

When the ground rotates of an infinitesimal angle $\mathrm{d}\alpha$, the segment r_k (being K the curvature center) remains fix and then the angle $\mathrm{d}\alpha_k$ decreases by the quantity $\mathrm{d}\alpha$.

$$\frac{\mathrm{d}\alpha_k}{\mathrm{d}\alpha} = \frac{(\alpha_k - \mathrm{d}\alpha) - \alpha_k}{\mathrm{d}\alpha} = -1$$

from which:

$$r_k \cos\alpha_k = \rho\cos\gamma\gamma' - b\cos\beta y'$$

Substituting the expression of $r_k \cos\alpha_k$ into the projection along the ground of the vector equation, we obtain:

$$\rho_0 \cos\gamma\gamma' - b\cos\beta y' + \rho_0 \cos\gamma + b\cos\beta - d = 0$$

$$\Downarrow$$

$$\rho_0 = \frac{d - b\cos\beta(1 - y')}{\cos\gamma(1 + \gamma')}$$

Where

$$\gamma' = \frac{\mathrm{d}\gamma}{\mathrm{d}\alpha} = \frac{\mathrm{d}}{\mathrm{d}\alpha}\left\{\arctan\left[\frac{b\sin\beta(1-y')}{d-b\cos\beta(1-y')}\right]\right\} =$$

$$= \ldots = \frac{b(1-y')y'\cos(\beta+\gamma) - by''\sin(\beta+\gamma)}{d\cos\gamma - b(1-y')\cos(\beta+\gamma)}$$

The curvature radius of the cam profile is:

$$\rho = \rho_0 - R_r$$

Radial cam with oscillating roller follower

Same equations can be used to synthesize an **internal cam** with oscillating roller follower. The pitch profile remains the same, but now:

$$R_r = -R_r$$

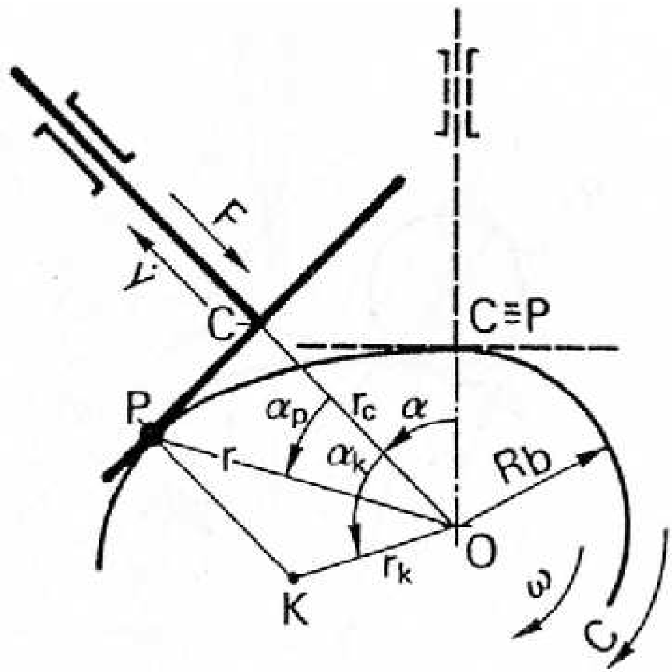
To realize a **negative cam** with oscillating roller follower, we have to change the following statements:

$$y = -y(\alpha); \qquad y' = -y'(\alpha); \qquad y'' = -y''(\alpha)$$

To realize a cam with oscillating roller follower, where the rotation direction of the cam is **counterclockwise**, we have to put:

$$\alpha = -\alpha; \qquad \frac{\mathrm{d}y}{\mathrm{d}\alpha} = -\frac{\mathrm{d}y}{\mathrm{d}\alpha} = -y'(\alpha)$$

Radial cam with translating flat-faced follower



With reference to the figure, the distance $CO = r_c$ is given by:

$$r_c = R_b + y(\alpha)$$

The contact point P between flat-faced follower and cam does not coincide with the follower center but it changes between minimum and maximum value.
To calculate the distance PC we can use the power budget approach:

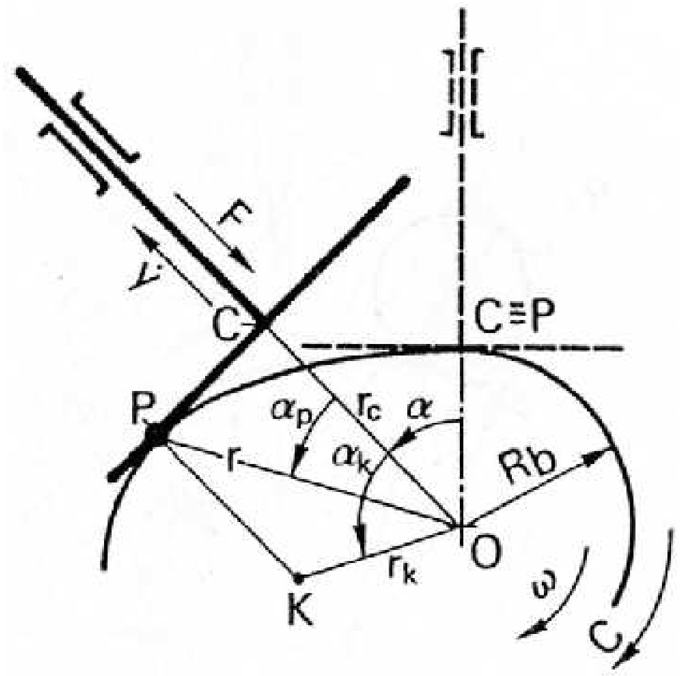
$$C\omega = F\dot{y}$$

Note that $PK \perp PC$ then we will have always $\theta = 0$, from which we obtain the value $F = S$:

$$C\omega = S \cdot PC\omega = F\dot{y} = Fy'\omega \quad \Rightarrow \quad PC = y'$$

that is $y'_{min} < PC < y'_{max}$

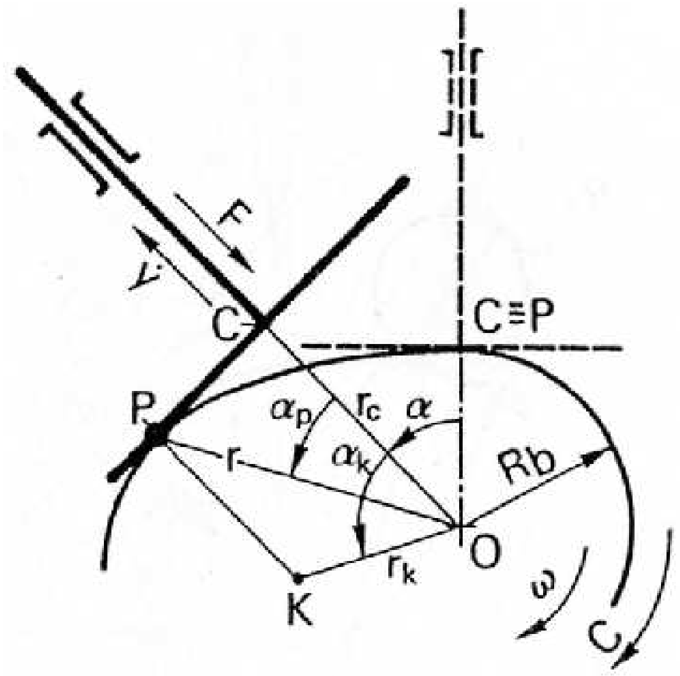
Radial cam with translating flat-faced follower



The polar coordinates of the cam profile are obtained considering the triangle OPC:

$$\begin{cases} r = \sqrt{r_c^2 + y'^2} = \sqrt{(R_b + y)^2 + y'^2} \\ \varphi = \alpha + \alpha_p = \alpha + \arctan\left(\dfrac{y'}{R_b + y}\right) \end{cases}$$

Radial cam with translating flat-faced follower



The curvature radius is obtained from a vectorial equation:

$$\overrightarrow{OK} + \overrightarrow{KP} + \overrightarrow{PC} + \overrightarrow{CO} = 0$$

$$\Downarrow$$

$$\overrightarrow{r_k} + \overrightarrow{\rho} + \overrightarrow{y'} + \overrightarrow{r_c} = 0$$

Projecting it along the movement direction of the follower and perpendicular to this, we obtain two scalar equations:

$$\begin{cases} r_k \cos\alpha_k + \rho = r_c \\ r_k \sin\alpha_k = y' \quad \Rightarrow \quad \dfrac{\mathrm{d}}{\mathrm{d}\alpha}(r_k \sin\alpha_k - y') = 0 \quad \Rightarrow \quad r_k \cos\alpha_k = -y'' \end{cases}$$

Substituting $\quad \rho = r_c + y'' = R_b + y + y''$

Cam size determination

Base circle determination

To design a cam mechanism it is necessary to choose:

- the **cam typology** having regard to the geometrical constrains
- the **motion curve** of the follower
- the radius of the **base circle**

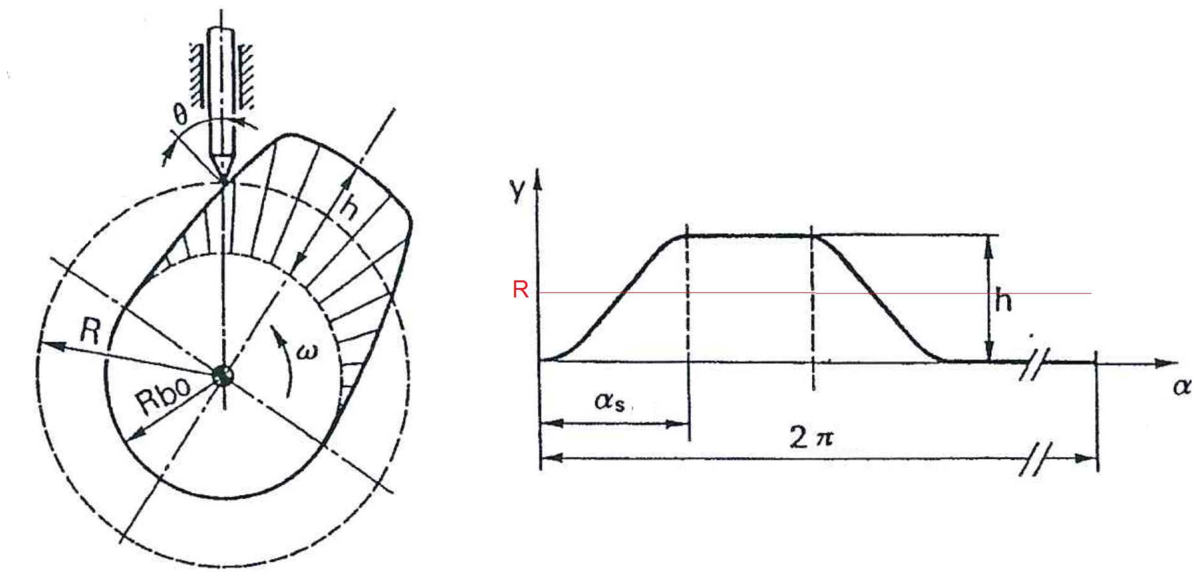
To properly size the base radius (and then the overall dimensions of the cam) there are some easy design formulas that depend on the cam type. We will consider only radial cam with translating roller follower, radial cam with oscillating roller follower and radial cam with translating flat-faced follower.

Remember that the **base radius** is fundamental in determining the cam size and it is the main parameter that can be changed to comply with the limits on pressure angle and curvature radius.

Base circle determination

Radial cam with translating roller follower

To design a radial cam with translating follower we have to draw a pitch circle with a radius R_{b0} and envelop the motion curve chosen around this circle.



The cam pitch curve is calculated as $r = R_{b0} + y(\alpha)$ and be $R \approx R_{b0} + \dfrac{h}{2}$

Base circle determination

Let's remember the pressure angle for this cam (slide 16) can be obtained as:

$$\tan\theta(\alpha) = \frac{y'(\alpha)}{R_{b0} + y(\alpha)}$$

whose maximum value θ_{max} is obtained when $y' = y'_{max}$. For the motion law depicted, this happens for $\alpha = \dfrac{\alpha_s}{2}$. In this condition we have $R_{b0} + y(\alpha) \approx R$. Then it is:

$$\tan\theta_{max} = \frac{y'_{max}}{R} = \frac{c_v \cdot h}{R \cdot \alpha_s}$$

Once the maximum value of the pressure angle is defined, it results:

$$R \geq \frac{c_v \cdot h}{\tan\theta_{max} \cdot \alpha_s} \qquad \text{and then} \qquad R_{b0} \approx R - h/2$$

The ratio R/h represents a **cam size index**: big cams have a ratio $R/h > 5$ (usually we have this value when $\alpha_s < 40°$).

A similar formula is used for the return segment of the motion curve, but in this case the pressure angle limit is less restrictive ($\theta = 45° \div 50°$).

In the case of roller follower could be necessary to choose a base radius bigger compare to the one obtained using this approach in order to avoid the **undercut** problem or to increase the curvature radius and decrease the contact pressure.

Base circle determination

The minimum curvature radius of the pitch profile can be approximate as follows:

$$\rho_{0_{min}} = \frac{R^2}{R + c_{a-} \dfrac{h}{\alpha_s^2}} \quad \overset{y'=0}{\Longleftarrow} \quad \rho_0 = \frac{[y'^2 + (R_{b0} + y)^2]^{3/2}}{(R_{b0} + y)^2 - (R_{b0} + y)y'' + 2y'^2}$$

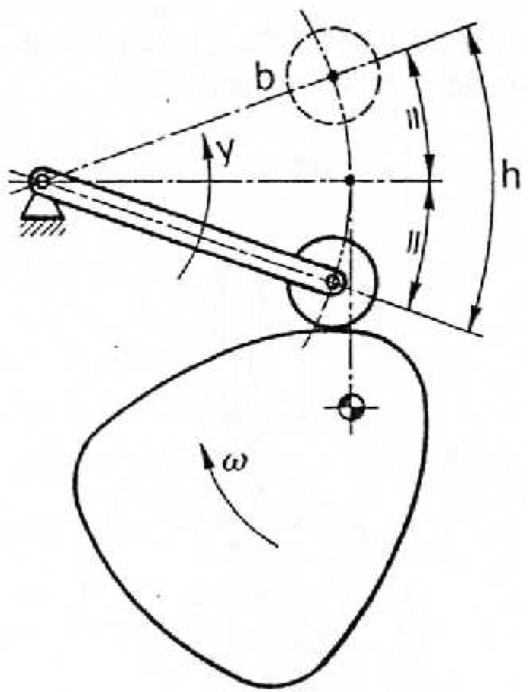
To not have undercut problems, it must be:

$$\rho_{0_{min}} > R_r$$

From which we obtain:

$$\begin{cases} R_r < \dfrac{R^2}{R + c_{a-} \dfrac{h}{\alpha_s^2}} & \Rightarrow \quad \text{Checking formula} \\ R > \dfrac{R_r}{2}\left(1 + \sqrt{1 + \dfrac{4h}{R_r} \dfrac{c_{a-}}{\alpha_s^2}}\right) & \Rightarrow \quad \text{Sizing formula} \end{cases}$$

Swinging follower



Sometime we can approximate the circular trajectory with a linear one. The configuration in which the line tangent to the circular trajectory of the follower in the $h/2$ position pass through the cam center is called "centred layout". For this configuration the design formula is:

$$R \geq \frac{c_v h b}{\alpha_s \tan\theta_{max}}$$

where b is the follower length

It is important to verify the value of the pressure angle for $h = 0$ and $h = h_{max}$. Usually the displacement diagram shows rises included between $30° \div 40°$ therefore the cam dimension increases in function of the follower length.

Base circle determination

flat-faced translating follower

It si well known that for this kind of cams there is no problem for what concerns the pressure angle value: R_b could be very small. Actually it is necessary to chose the value of R_b so that the curvature of the cam profile is greater than zero (convex profile) otherwise we have undercut problems

$$\rho = R_b + y + y''$$

Note that $\rho < 0$ only if $y'' < 0$. We can write:

$$R_b = \rho - y - y'' \quad \Rightarrow \quad R_b + y = \rho - y'' \quad \Rightarrow \quad R = \rho - y''$$

Where R is the average radius. Imposing the condition $\rho > 0$:

$$\rho > 0 \quad \Rightarrow \quad R + y'' > 0 \quad \Rightarrow \quad R > -y'' = c_{a-} \frac{h}{\alpha_s^2}$$

Base circle determination

The design formula for cam with flat-faced follower is:

$$\frac{R}{h} \geq \frac{c_{a-}}{\alpha_s^2}$$

It is posible to use the followimg equations to calculate the base circle radius:

$$\begin{cases} R_b = R - \dfrac{h}{2} & \\ R_b = R & \text{Precautionary} \end{cases}$$

When the ratio between R and h is $R/h > 3$ we have bulky cams (for example $c_{a-} = 4$, $\alpha_s < 65°$). Using the value α_d instead of α_s, it is possible to evaluate the return phase.
The maximum R_b obtained must be taken into account.
For what concern the length of the flat-faced follower we can use the easy to use relationship:

$$L_p = c_v \frac{h}{\alpha_s} + c_v \frac{h}{\alpha_d}$$