DIPARTIMENTO DI MECCANICA

POLITECNICO MILANO 1863

MECHANICAL SYSTEM DYNAMICS

Numerical computation of natural frequencies and mode shapes of a discrete system

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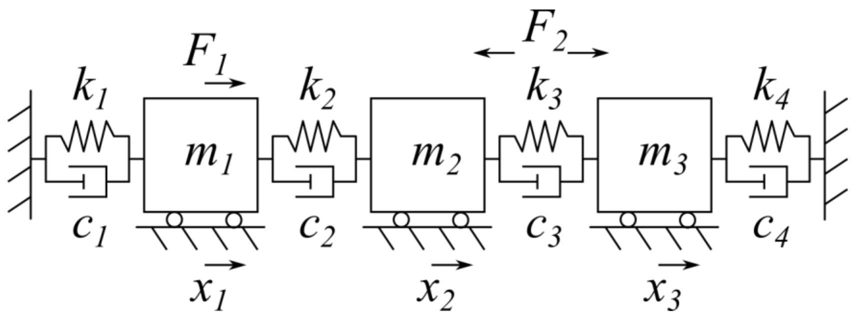
Natural frequencies and mode shapes of a discrete system

We want to identify the **natural frequencies** and **mode shapes** of a discrete system.



$$\boldsymbol{x} = \begin{Bmatrix} x_1 \\ x_2 \\ x_3 \end{Bmatrix}$$

$$\begin{bmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{bmatrix} \ddot{\boldsymbol{x}} + \begin{bmatrix} c_1 + c_2 & -c_2 & 0 \\ -c_2 & c_2 + c_3 & -c_3 \\ 0 & -c_3 & c_3 + c_4 \end{bmatrix} \dot{\boldsymbol{x}} + \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 + k_4 \end{bmatrix} \boldsymbol{x} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & -1 \end{bmatrix} \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

$$[M]\ddot{\boldsymbol{x}} + [C]\dot{\boldsymbol{x}} + [K]\boldsymbol{x} = [B]\boldsymbol{F}$$

Natural frequencies and mode shapes of a discrete system

Natural frequencies ω_0 (undamped system)

$$[M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} = 0$$

$$\boldsymbol{x} = \underline{\hat{X}} e^{j\omega_0 t}$$

$$\left(-\omega_0^2[M] + [K]\right)\underline{\hat{X}} e^{j\omega_0 t} = 0$$

$$\det\left(\omega_0^2[I] - [M]^{-1}[K]\right) = 0 \quad ? \quad \rightarrow \quad \omega_{0i}$$

Or, solving the eigen value problem,

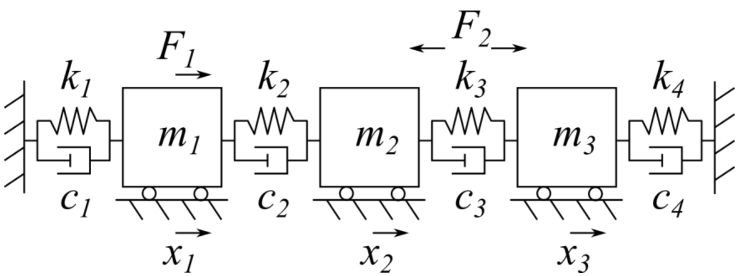
$$\boxed{\omega_0 = \sqrt{\lambda} \qquad \lambda = eig([M]^{-1}[K])}$$

Going back to the example

```
M = [m1 0 0; 0 m2 0; 0 0 m3];
K = [k1+k2 -k2 0; -k2 k2+k3 -k3; 0 -k3 k3+k4];
C = [c1+c2 -c2 0; -c2 c2+c3 -c3; 0 -c3 c3+c4];

% matricies
MK    = inv(M)*K;
% since M is diagonal
invM = diag(1./diag(M)); ?
MK    = invM*K;
% more efficient formulation for full matricies
MK    = M\K;
```

inv of M




```
m1  = 1;
m2  = 2;
m3  = 1;
k1  = 1000;
k2  = 3000;
k3  = 3000;
k4  = 1000;
c1  = 1;
c2  = 3;
c3  = 3;
c4  = 1;
N   = 3;
```

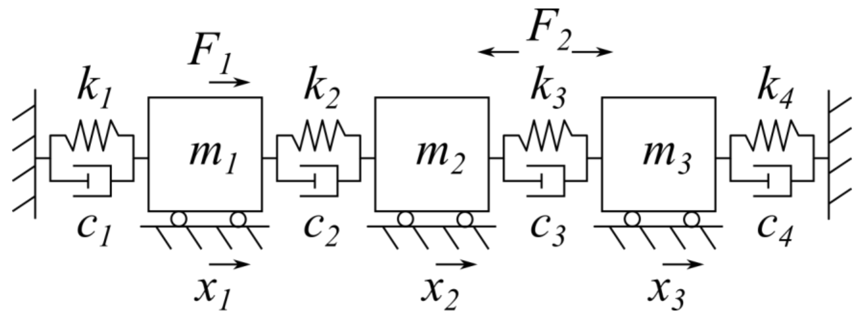
Now we use the **eig** function of Matlab

```
[V,D] = eig(A)

returns diagonal matrix D of eigenvalues
and matrix V whose columns are the
corresponding right eigenvectors, so
that A*V = V*D.
```

```
% natural frequencies
[X0,l0] = eig(MK);
w0 = sqrt(diag(l0));
```

Natural frequencies and mode shapes of a discrete system



Results (undamped system)

Matrix of the frequencies (w=2*pi/f)

$$f_0 = \{3.41 \quad 10.07 \quad 12.87\}$$

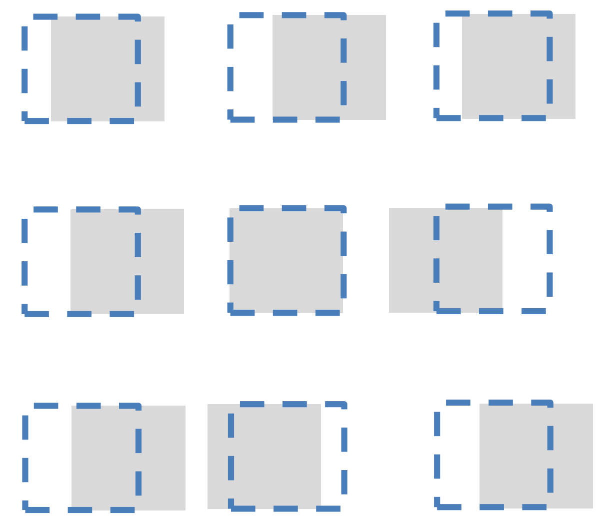
$$\omega_0 = \{21.42 \quad 63.25 \quad 80.88\}$$

$$\begin{array}{cc} & \begin{array}{ccc} \underline{\hat{X}}^{(1)} & \underline{\hat{X}}^{(2)} & \underline{\hat{X}}^{(3)} \end{array} \\ [\phi] = & \begin{bmatrix} 0.5428 & 0.7071 & 0.6066 \\ 0.6408 & -0.0000 & -0.5139 \\ 0.5428 & -0.7071 & 0.6066 \end{bmatrix} \end{array}$$

$\underline{\hat{X}}^{(1)}$

$\underline{\hat{X}}^{(2)}$

$\underline{\hat{X}}^{(3)}$



Natural frequencies and mode shapes of a discrete system

Damped frequencies ω **eigenvalue approach (damped case)**

$$[M]\ddot{\boldsymbol{x}} + [C]\dot{\boldsymbol{x}} + [K]\boldsymbol{x} = 0$$

State space representation

$$\begin{cases} [M]\ddot{\boldsymbol{x}} + [C]\dot{\boldsymbol{x}} + [K]\boldsymbol{x} = 0 \\ [I]\dot{\boldsymbol{x}} = [I]\dot{\boldsymbol{x}} \end{cases}$$

$$\begin{Bmatrix} \ddot{\boldsymbol{x}} \\ \dot{\boldsymbol{x}} \end{Bmatrix} = \begin{bmatrix} -[M]^{-1}[C] & -[M]^{-1}[K] \\ [I] & [0] \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{x}} \\ \boldsymbol{x} \end{Bmatrix}$$

State vector $\boldsymbol{z} = \begin{Bmatrix} \dot{\boldsymbol{x}} \\ \boldsymbol{x} \end{Bmatrix}$ State matrix $[A] = \begin{bmatrix} -[M]^{-1}[C] & -[M]^{-1}[K] \\ [I] & [0] \end{bmatrix}$

$\dot{\boldsymbol{z}} = [A]\boldsymbol{z}$ equation of free motion in state space representation

? R

$$([A] - \lambda[I])\,\boldsymbol{Z_0} = 0 \qquad \omega = \Im(\lambda) \qquad \xi = \frac{\Re(\lambda)}{\omega_0}$$

the damping coefficient

I

Natural frequencies and mode shapes of a discrete system

Natural frequencies ω_0 **eigenvalue approach (undamped case [C] = [0])**

$$[M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} = 0$$

State space representation

$$\begin{cases} [M]\ddot{\boldsymbol{x}} + [K]\boldsymbol{x} = 0 \\ [I]\dot{\boldsymbol{x}} = [I]\dot{\boldsymbol{x}} \end{cases}$$

$$\begin{Bmatrix} \ddot{\boldsymbol{x}} \\ \dot{\boldsymbol{x}} \end{Bmatrix} = \begin{bmatrix} [0] & -[M]^{-1}[K] \\ [I] & [0] \end{bmatrix} \begin{Bmatrix} \dot{\boldsymbol{x}} \\ \boldsymbol{x} \end{Bmatrix}$$

State vector $\boldsymbol{z} = \begin{Bmatrix} \dot{\boldsymbol{x}} \\ \boldsymbol{x} \end{Bmatrix}$ State matrix $[A] = \begin{bmatrix} [0] & -[M]^{-1}[K] \\ [I] & [0] \end{bmatrix}$

$\dot{\boldsymbol{z}} = [A]\boldsymbol{z}$ equation of free motion in state space representation

$$([A] - \lambda[I])\,\boldsymbol{Z_0} = 0 \qquad \omega_0 = \Im(\lambda)$$

Natural frequencies (eigenvalues)

$$([A] - \lambda[I])\,\boldsymbol{Z_0} = 0$$

There are 2N eigenvalues (N = number of dofs)

$$\boldsymbol{\lambda} = \begin{Bmatrix} \lambda_1 \\ \lambda_2 \\ \vdots \\ \lambda_k \\ \vdots \\ \lambda_{2N} \end{Bmatrix} = \begin{Bmatrix} \omega_1 j \\ -\omega_1 j \\ \omega_2 j \\ -\omega_2 j \\ \vdots \\ \omega_N j \\ -\omega_N j \end{Bmatrix}$$

$$k = 1, \ldots, 2N$$

N couples of conjugate eigenvalues

One couple for each natural frequency

i-th natural frequency corresponds to *2k-1* eigenvalue

Mode shapes (eigen vectors)

$$([A] - \lambda[I])\,\boldsymbol{Z_0} = 0$$

In state space representation, eigenvectors cointan speed and position

$$\boldsymbol{Z}_0^{(k)} = \left\{ \begin{matrix} \dot{x}_1 \\ \dot{x}_2 \\ \vdots \\ \dot{x}_N \\ x_1 \\ x_2 \\ \vdots \\ x_N \end{matrix} \right\} \quad \begin{matrix} \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \text{speed} \\ \left.\begin{matrix} \\ \\ \\ \\ \end{matrix}\right\} \text{position} \end{matrix}$$

N couples of conjugate eigenvectors

One couple for each natural frequency

Mode shapes

$$\underline{\hat{X}}^{(i)} = \boldsymbol{Z}_0^{(2k-1)}(N+1,\ldots,2N)$$

i-th mode shape corresponds to *2k-1* half eigenvector

Natural frequencies and mode shapes of a discrete system

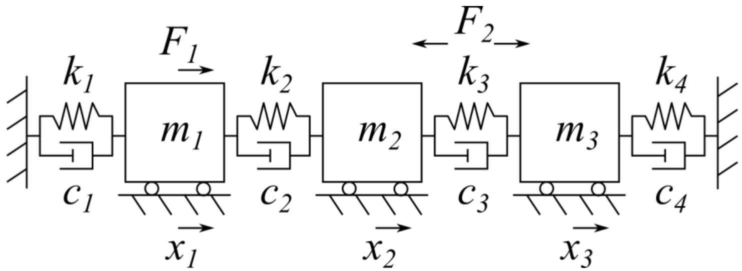
Going back to the example

```
M = [m1 0 0; 0 m2 0; 0 0 m3];
K = [k1+k2 -k2 0; -k2 k2+k3 -k3; 0 -k3 k3+k4];
C = [c1+c2 -c2 0; -c2 c2+c3 -c3; 0 -c3 c3+c4];

%% UNDAMPED SYSTEM
% state matricies
A0 = [ zeros(size(M)) -M\K; eye(size(M))
zeros(size(M))];

% eigen values and eigen vectors
[Z0,l0] = eig(A0);

% eigen frequencies and mode shapes (undamped)
w0  = imag(diag(l0));
w0  = w0(1:2:2*N-1);
X0 = Z0(N+1:2*N,:);
for ii=1:N
    X0(:,ii) = X0(:,ii)./max((X0(:,ii)));
    X0(:,ii) = X0(:,ii)/sqrt(X0(:,ii)'*X0(:,ii));
end
```

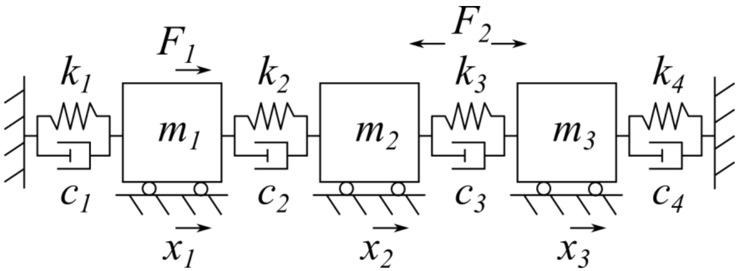


$$f_0 = \{3.41 \quad 10.07 \quad 12.87\}$$

$$\omega_0 = \{21.42 \quad 63.25 \quad 80.88\}$$

$$\begin{array}{cc} & \underline{\hat{X}}^{(1)} \qquad \underline{\hat{X}}^{(2)} \qquad \underline{\hat{X}}^{(3)} \\ [\phi] = & \begin{bmatrix} 0.5428 & 0.7071 & 0.6066 \\ 0.6408 & -0.0000 & -0.5139 \\ 0.5428 & -0.7071 & 0.6066 \end{bmatrix} \end{array}$$

Natural frequencies and mode shapes of a discrete system

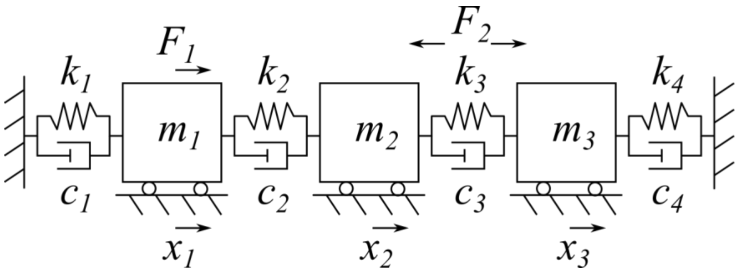


Going back to the example

```
%% DAMPED SYSTEM
% state matrices
A  = [ -M\C -M\K; eye(size(M)) zeros(size(M))];
% eigen value and eigen vectors
[Z,l] = eig(A);
w   = imag(diag(l));
w   = w(1:2:2*N-1);
f   = w/2/pi;
csi = -real(diag(l));
csi = csi(1:2:2*N-1)./w0;
X  = Z(N+1:2*N,1:2:2*N-1);
for ii=1:N
    X(:,ii) = X(:,ii)./max((X(:,ii)));
    X(:,ii) = X(:,ii)/sqrt(X(:,ii)'*X(:,ii));
end
```

Natural frequencies and mode shapes of a discrete system

Results (damped system)



$$f = \{3.41 \quad 10.06 \quad 12.86\}$$

$$\omega = \{21.41 \quad 63.21 \quad 80.81\}$$

$$\xi = \{0.0107 \quad 0.0404 \quad 0.0316\}$$

$$[\phi_d] = \begin{bmatrix} 0.5428 - 0.0025i & 0.6067 - 0.0000i & 0.7071 + 0.0000i \\ 0.6408 + 0.0000i & -0.5136 + 0.0091i & -0.0000 - 0.0000i \\ 0.5428 - 0.0025i & 0.6067 + 0.0000i & -0.7071 + 0.0000i \end{bmatrix}$$

Eigenvectors (position) of damped system are **complex**, it means that there is a time shift between mass displacements

Frequency Response (direct approach)

$$[M]\ddot{\boldsymbol{x}} + [C]\dot{\boldsymbol{x}} + [K]\boldsymbol{x} = [B]\boldsymbol{F}$$

$$\boldsymbol{F} = \boldsymbol{F}_0 \mathrm{e}^{j\Omega t}$$

$$\boldsymbol{x} = \boldsymbol{X} \mathrm{e}^{j\Omega t}$$

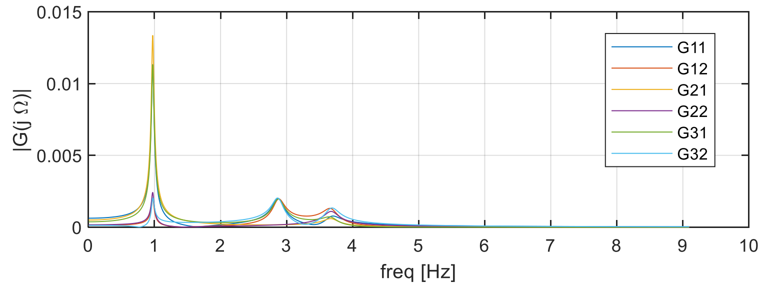
$$(-\Omega^2[M] + j\Omega[C] + [K])\boldsymbol{X}\mathrm{e}^{j\Omega t} = [B]\boldsymbol{F}_0\mathrm{e}^{j\Omega t}$$

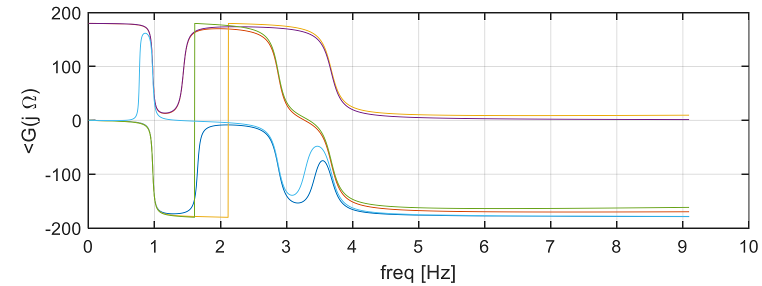
$$[D]\boldsymbol{X}\mathrm{e}^{j\Omega t} = [B]\boldsymbol{F}_0\mathrm{e}^{j\Omega t}$$

$$\boldsymbol{X} = [D]^{-1}[B]\boldsymbol{F}_0$$

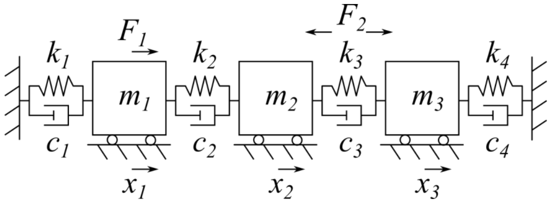
$$\boldsymbol{X} = [G(j\Omega)]\boldsymbol{F}_0$$

```
B = [1 0; 0 -1; 0 1];
O = 0.1:.1:200;
for ii=1:length(O)
    D = -O(ii)^2*M + 1i*O(ii)*C + K;
    G = D\B;
    G11(ii) = G(1,1);
    G12(ii) = G(1,2);
    G21(ii) = G(2,1);
    G22(ii) = G(2,2);
    G31(ii) = G(3,1);
    G32(ii) = G(3,2);
end
```

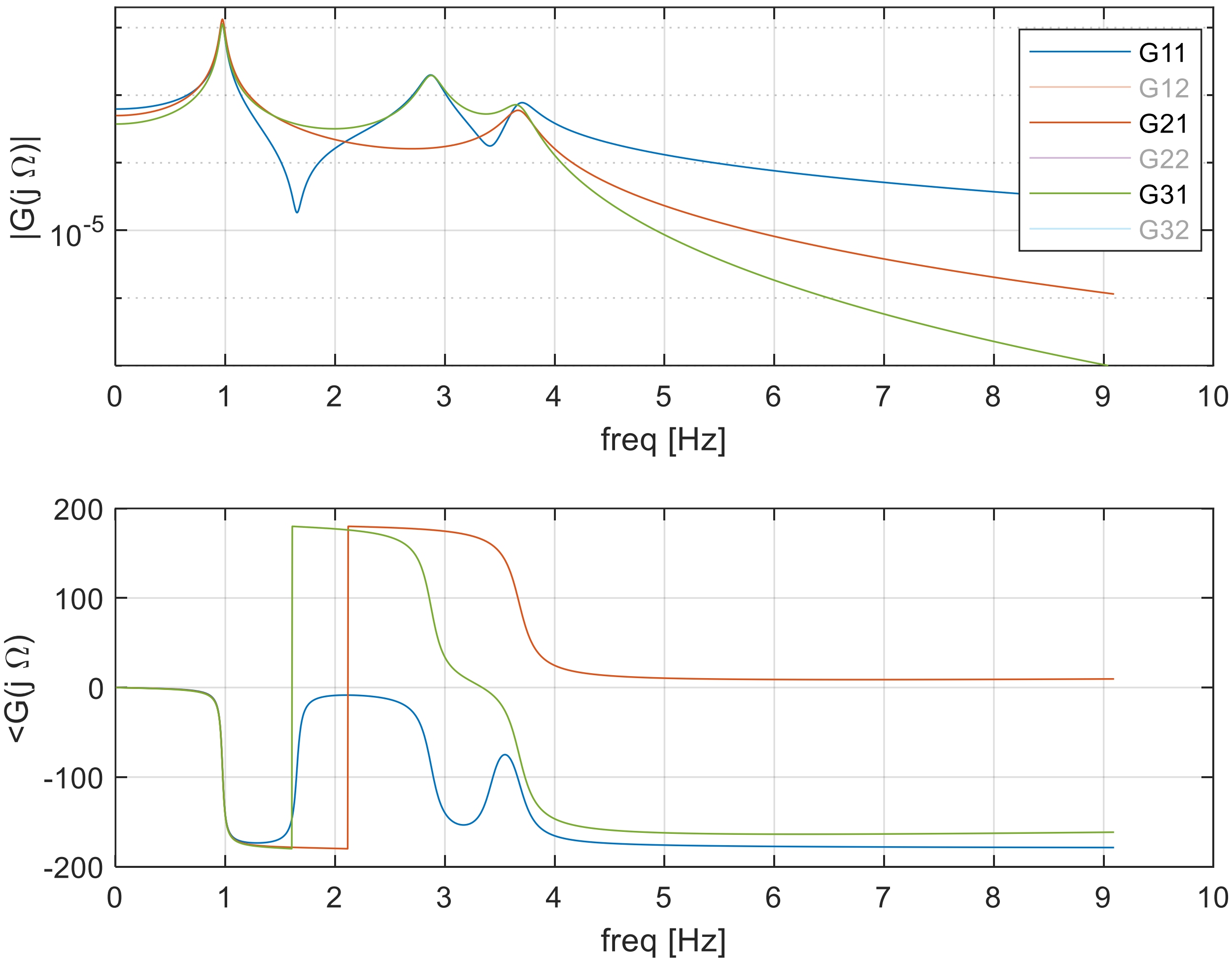




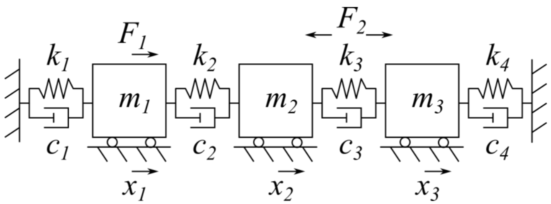
Frequency Response (direct approach)



Response of the system to F_1



Frequency Response (direct approach)



Response of the system to F_2