University of Tehran
School of Industrial Engineering

An Introduction to
Statistical Quality Control

Spring 2025

Chapter 0

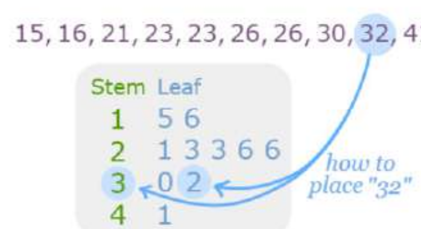
Statistics and Probability Background

Describing Variation

- *There are many ways to describe the variation of a random process but most applied ones are:*
  - *Stem-and-Leaf Plot (Visual)*
  - *Histogram (Visual)*
  - *Box Plot (Visual)*
  - *Numerical Summary of Data (Analytical)*
  - *Probability Distributions (Analytical)*

Describing Variation

- *Stem-and-Leaf Plot*
  - *One of the most useful graphical techniques*
  - *Suppose that the data are represented by $x_1, x_2, \ldots, x_n$ and that each number x_i consists of at least two digits*
  - *Divide each number x_i into two parts:*
    *a **stem**, consisting of one or more of the leading digits*
    *a **leaf**, consisting of the remaining digits*
  - *It is usually best to choose between 5 and 20 stems*



Describing Variation

- *Histogram*
  - *A more compact summary of data than a stem-and-leaf plot*
  - *Divide the range of the data into intervals, usually called class intervals, cells, or bins*
  - *A histogram that uses either too few or too many bins will not be informative, usually between **5 and 20 bins** is satisfactory in most cases and that the number of bins should increase with n.*
  - *Use the horizontal axis to represent the measurement scale for the data and the vertical scale to represent the counts, or frequencies.*
    *Sometimes the frequencies in each bin are divided by the total number of observations (n), and then the vertical scale of the histogram represents **relative frequencies**.*
  - *Rectangles are drawn over each bin, and the height of each rectangle is proportional to frequency (or relative frequency).*

Describing Variation

- *Example. Table 3.2 presents the thickness of a metal layer on 100 silicon wafers resulting from a chemical vapor deposition (CVD) process in a semiconductor plant. Construct a histogram for these data.*

TABLE 3.2
Layer Thickness (Å) on Semiconductor Wafers

438	450	487	451	452	441	444	461	432	471
413	450	430	437	465	444	471	453	431	458
444	450	446	444	466	458	471	452	455	445
468	459	450	453	473	454	458	438	447	463
445	466	456	434	471	437	459	445	454	423
472	470	433	454	464	443	449	435	435	451
474	457	455	448	478	465	462	454	425	440
454	441	459	435	446	435	460	428	449	442
455	450	423	432	459	444	445	454	449	441
449	445	455	441	464	457	437	434	452	439

Describing Variation

- *Answer. Because the data set contains 100 observations and sqrt(100)=10, we suspect that about 10 bins will provide a satisfactory histogram.*

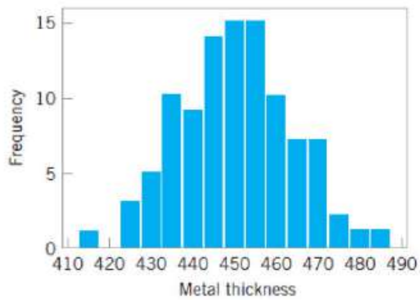


FIGURE 3.4 Minitab histogram with 15 bins for the metal layer thickness data.

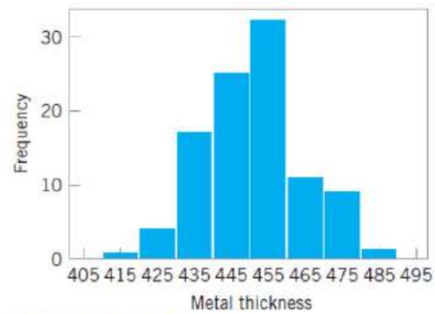


FIGURE 3.3 Minitab histogram for the metal layer thickness data in Table 3.2.

Describing Variation

- *Box Plot*
  - *The box plot is a graphical display that simultaneously displays several important features of the data, such as location or **central tendency**, spread or **variability**, **departure from symmetry**, and identification of observations that lie unusually far from the bulk of the data (**outliers**)*
  - *A box plot displays the three quartiles, the **minimum**, and the **maximum** of the data on a rectangular box, aligned either horizontally or vertically. The box encloses the interquartile range with the left (or lower) line at the first quartile Q1 and the right (or upper) line at the third quartile Q3. A line is drawn through the box at the second quartile (which is the fiftieth percentile or the **median**)*

Describing Variation

- *Example. The data in Table 3.4 are diameters (in mm) of holes in a group of 12 wing leading edge ribs for a commercial transport airplane. Construct and interpret a box plot of those data.*

TABLE 3.4
Hole Diameters (in mm) in Wing Leading Edge Ribs

120.5	120.4	120.7
120.9	120.2	121.1
120.3	120.1	120.9
121.3	120.5	120.8

Describing Variation

- *Answer. The box plot indicates that the hole diameter distribution is not exactly symmetric around a central value, because the left and right whiskers and the left and right boxes around the median are not the same lengths.*

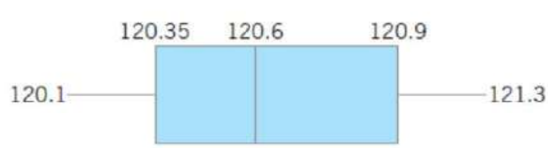


FIGURE 3.7 Box plot for the aircraft wing leading edge hole diameter data in Table 3.4.

Describing Variation

- *Numerical Summary of Data*
  - *The stem-and-leaf plot and the histogram provide a visual display of three properties of sample data: the shape of the distribution of the data, the central tendency in the data, and the scatter or variability in the data.*
  - *Suppose that $x_1, x_2, \ldots, x_n$ are the observations in a sample.*
  - *The most important measure of central tendency in the sample is the sample average,*

$$\bar{x} = \frac{x_1 + x_2 + \cdots + x_n}{n} = \frac{\sum_{i=1}^{n} x_i}{n} \tag{3.1}$$

Describing Variation

- *Numerical Summary of Data*
  - *The variability in the sample data is measured by the **sample variance**:*

$$s^2 = \frac{\sum_{i=1}^{n} (x_i - \bar{x})^2}{n-1} \tag{3.2}$$

  - *Note that the sample variance is simply the sum of the squared deviations of each observation from the sample average divided by the sample size minus 1.*
  - *If there is no variability in the sample, then each sample observation and the sample variance is zero.*
  - *Generally, the larger the sample variance s^2 is, the greater is the variability in the sample data.*

Describing Variation

- *Numerical Summary of Data*
  - *The units of the sample variance s^2 are the square of the original units of the data.*
  - *This is often inconvenient to interpret, and so we usually prefer to use the square root of s^2, called the **sample standard deviation s**, as a measure of variability.*

$$s = \sqrt{\frac{\sum_{i=1}^{n} (x_i - x)^2}{n-1}} \qquad (3.3)$$

Describing Variation

- *Numerical Summary of Data*
  - *Note that equations 3.2 and 3.3 are not very efficient computationally, because every number must be entered into the calculator twice. A more efficient formula is*

$$s = \sqrt{\frac{\sum_{i=1}^{n} x_i^2 - \frac{\left(\sum_{i=1}^{n} x_i\right)^2}{n}}{n-1}} \qquad (3.4)$$

Describing Variation

- *Probability Distributions.*
  - *A probability distribution is a mathematical model that relates the value of the variable with the probability of occurrence of that value in the population.*
  - *In other words, we might visualize layer thickness as a random variable because it takes on different values in the population according to some random mechanism, and then the probability distribution of layer thickness describes the probability of occurrence of any value of layer thickness in the population.*

Describing Variation

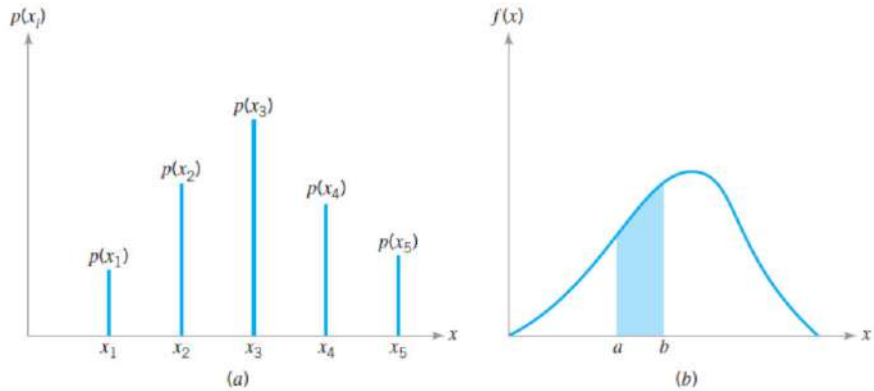
- *Probability Distributions.*
  - *There are two types of probability distributions.*

Definition

1. **Continuous distributions.** When the variable being measured is expressed on a continuous scale, its probability distribution is called a *continuous distribution.* The probability distribution of metal layer thickness is continuous.
2. **Discrete distributions.** When the parameter being measured can only take on certain values, such as the integers 0, 1, 2, . . . , the probability distribution is called a *discrete distribution.* For example, the distribution of the number of nonconformities or defects in printed circuit boards would be a discrete distribution.

Describing Variation

- *Probability Distributions.*
  - *Probability distributions. (a) Discrete case. (b) Continuous case.*



Describing Variation

- *Probability Distributions.*
  - *The appearance of a discrete distribution is that of a series of vertical "spikes," with the height of each spike proportional to the probability.*
  - *We write the probability that the random variable x takes on the specific value x_i as*

$$P\{x = x_i\} = p(x_i)$$

  - *The appearance of a continuous di ɔoth curve, with the area under the curve equal to probability, so that the probability that x lies in the interval from a to b is written as*

$$P\{a \le x \le b\} = \int_a^b f(x)dx$$

Probability Distributions

- *Example. A manufacturing process produces thousands of semiconductor chips per day. On the average, 1% of these chips do not conform to specifications. Every hour, an inspector selects a random sample of 25 chips and classifies each chip in the sample as conforming or nonconforming.*
  If we let x be the random variable representing the number of nonconforming chips in the sample, then the probability distribution of x is

$$p(x) = \binom{25}{x}(0.01)^x(0.99)^{25-x} \qquad x = 0,1,2,\ldots,25$$

Probability Distributions

- *Answer. This is a discrete distribution, since the observed number of nonconformances is x = 0, 1, 2, . . . , 25, and is called the binomial distribution. We may calculate the probability of finding one or fewer nonconforming parts in the sample as*

$$\begin{aligned} P(x \le 1) &= P(x=0) + P(x=1) \\ &= p(0) + p(1) \\ &= \sum_{x=0}^{1} \binom{25}{x}(0.01)^x(0.99)^{25-x} \\ &= \frac{25!}{0!\,25!}(0.99)^{25}(0.01)^0 + \frac{25!}{1!\,24!}(0.99)^{24}(0.01)^1 \\ &= 0.7778 + 0.1964 = 0.9742 \end{aligned}$$

$$p(x) = \binom{25}{x}(0.01)^x(0.99)^{25-x}$$

$$x = 0,1,2,\ldots,25$$

Probability Distributions

- *Example. Suppose that x is a random variable that represents the actual contents in ounces of a 1-pound bag of coffee beans. The probability distribution of x is assumed to be*

$$f(x) = \frac{1}{1.5} \qquad 15.5 \le x \le 17.0$$

This is a continuous distribution, since the range of x is the interval [15.5, 17.0]. This distribution is called the uniform distribution, and it is shown graphically in Figure 3.10. What is the probability of a bag containing less than 16.0 oz?

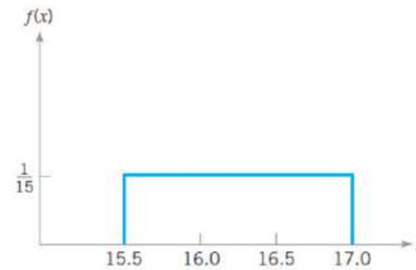


FIGURE 3.10 The uniform distribution for Example 3.6.

Probability Distributions

- *Answer. Note that the area under the function f (x) corresponds to probability, so that the probability of a bag containing less than 16.0 oz is*

$$P\{x \le 16.0\} = \int_{15.5}^{16.0} f(x)dx = \int_{15.5}^{16.0} \frac{1}{1.5} dx$$

$$= \frac{x}{1.5}\Big|_{15.5}^{16.0} = \frac{16.0 - 15.5}{1.5} = 0.3333$$

Describing Variation

- *Probability Distributions.*
  - *The mean* **μ** *of a probability distribution is a measure of the central tendency in the distribution, or its location. The mean is defined as*

$$\mu = \begin{cases} \int_{-\infty}^{\infty} x f(x)\, dx, x \text{ continuous} & (3.5a) \\ \sum_{i=1}^{\infty} x_i p(x_i), x \text{ discrete} & (3.5b) \end{cases}$$

  - *Fc* … *$(x_i) = 1/N]$, then equation 3.5b reduces to*

$$\mu = \frac{\sum_{i=1}^{N} x_i}{N}$$

Describing Variation

- *Probability Distributions.*
  - *The scatter, spread, or variability in a distribution is expressed by the variance. The definition of the variance is*

$$\sigma^2 = \begin{cases} \int_{-\infty}^{\infty} (x-\mu)^2 f(x)dx, x \text{ continuous} & (3.6a) \\ \sum_{i=1}^{\infty} (x_i-\mu)^2 p(x_i), x \text{ discrete} & (3.6b) \end{cases}$$

  - *wher* … *nes*

$$\sigma^2 = \frac{\sum_{i=1}^{N} (x_i-\mu)^2}{N}$$

Describing Variation

- *Probability Distributions.*
  - *The variance is expressed in the square of the units of the original variable.*
  - *For example, if we are measuring voltages, the units of the variance are (volts)^2. Thus, it is customary to work with the square root of the variance, called the standard deviation* **σ**. *It follows that*

$$\sigma = \sqrt{\sigma^2} = \sqrt{\frac{\sum_{i=1}^{N}(x_i - \mu)^2}{N}} \qquad (3.7)$$

Important Discrete Distributions

- *Several discrete probability distributions arise frequently in statistical quality control. In this section, we discuss:*
  - *Hypergeometric distribution*
  - *Binomial distribution*
  - *Negative binomial*
  - *Geometric distributions*
  - *Poisson distribution*

Hypergeometric Distribution

- *Suppose that there is a finite population consisting of N items.*
- *Some number, say D (D ≤ N), of these items fall into a class of interest.*
- *A random sample of n items is selected from the population* ***without replacement****, and the number of items in the sample that fall into the class of interest, say x, is observed.*
- *Then x is a hypergeometric random variable with the probability distribution defined as follows:*

Hypergeometric Distribution

Definition

The **hypergeometric probability distribution** is

$$p(x)=\frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}} \qquad x=0,1,2,\ldots,\min(n,D) \tag{3.8}$$

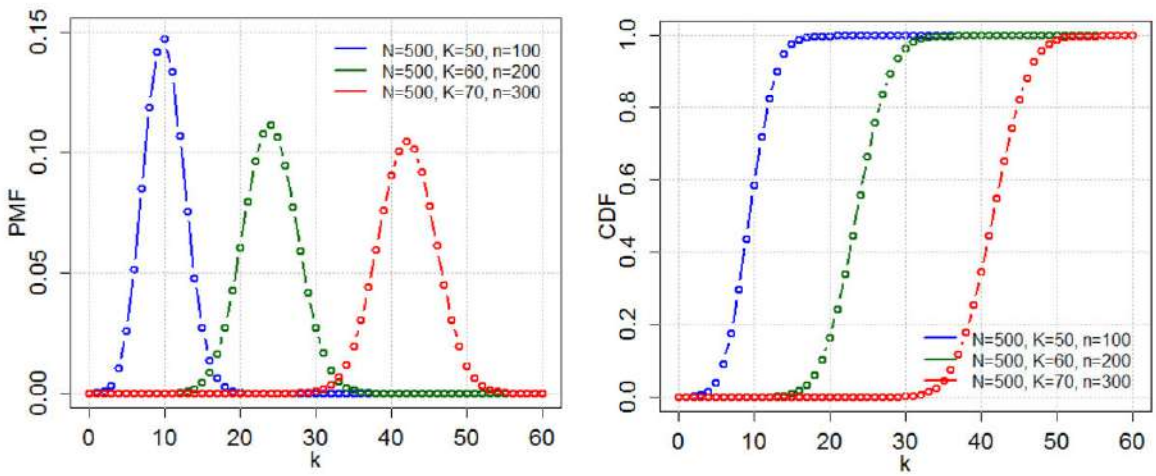
The mean and variance of the distribution are

$$\mu=\frac{nD}{N} \tag{3.9}$$

and

$$\sigma^2=\frac{nD}{N}\left(1-\frac{D}{N}\right)\left(\frac{N-n}{N-1}\right) \tag{3.10}$$

Hypergeometric Distribution



Binomial Distribution

- *Consider a process that consists of a sequence of n independent trials.*
- *By independent trials, we mean that the outcome of each trial does not depend in any way on the outcome of previous trials.*
- *When the outcome of each trial is either a "success" or a "failure," the trials are called Bernoulli trials.*
- *If the probability of "success" on any trial, say p, is constant, then the number of "successes" x in n Bernoulli trials has the binomial distribution with parameters n and p, defined as follows:*

Binomial Distribution

Definition

The **binomial distribution** with parameters $n \geq 0$ and $0 < p < 1$ is

$$p(x) = \binom{n}{x} p^x (1-p)^{n-x} \qquad x = 0, 1, \ldots, n \qquad (3.11)$$

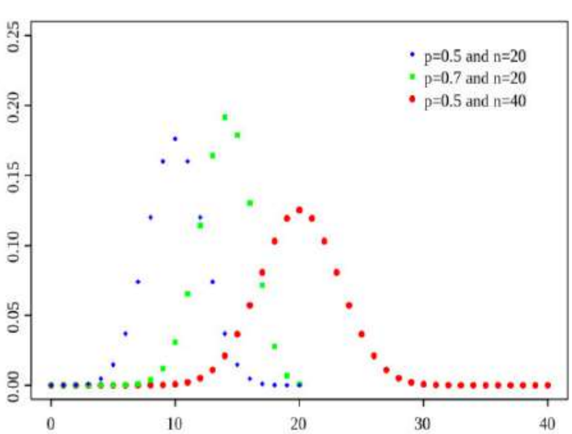
The mean and variance of the binomial distribution are

$$\mu = np \qquad (3.12)$$

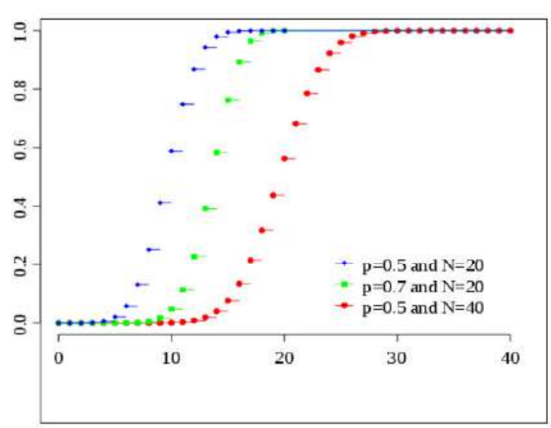
and

$$\sigma^2 = np(1-p) \qquad (3.13)$$

Binomial Distribution



PMF



CDF

Binomial Distribution

- *A random variable that arises frequently in statistical quality control is*

$$\hat{p} = \frac{x}{n} \qquad (3.14)$$

- *where x has a binomial distribution with parameters n and p. Often $\hat{p}$ is the ratio of the observed number of defective or nonconforming items in a sample (x) to the sample size (n), and this is usually called the sample fraction defective or sample fraction nonconforming.*

Binomial Distribution

- *The " ^ " symbol is used to indicate that $\hat{p}$ is an estimate of the true, unknown value of the binomial parameter p. The probability distribution of $\hat{p}$ is obtained from the binomial, since*

$$P\{\hat{p} \le a\} = P\left\{\frac{x}{n} \le a\right\} = p\{x \le na\} = \sum_{x=0}^{[na]} \binom{n}{x} p^x (1-p)^{n-x}$$

- *where [na] denotes the largest integer less than or equal to na.*
- *It is easy to show that the mean of $\hat{p}$ is p and that the variance $\hat{p}$ of is* $\sigma_{\hat{p}}^2 = \frac{p(1-p)}{n}$

Poisson Distribution

Definition

The **Poisson distribution** is

$$p(x) = \frac{e^{-\lambda}\lambda^x}{x!} \qquad x = 0, 1, \ldots \tag{3.15}$$

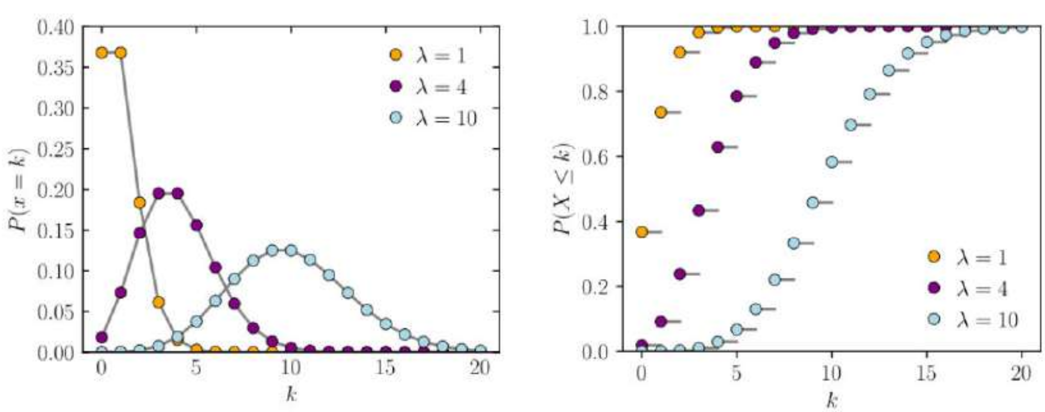
where the parameter $\lambda > 0$. The **mean** and **variance** of the Poisson distribution are

$$\mu = \lambda \tag{3.16}$$

and

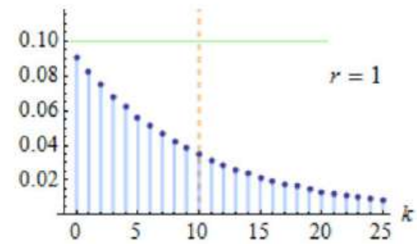
$$\sigma^2 = \lambda \tag{3.17}$$

Poisson Distribution



Negative Binomial and Geometric Distributions

- *The negative binomial distribution, like the binomial distribution, has its basis in Bernoulli trials.*
- *Consider a sequence of independent trials, each with probability of success p, and let x denote the trial on which the r^{th} success occurs.*
- *Then x is a negative binomial random variable with probability distribution defined as follows.*



Negative Binomial

Definition

The **negative binomial distribution** is

$$p(x) = \binom{x-1}{r-1} p^r (1-p)^{x-r} \qquad x = r, r+1, r+2, \ldots \qquad (3.18)$$

where $r \geq 1$ is an integer. The ***mean*** and ***variance*** of the negative binomial distribution are

$$\mu = \frac{r}{p} \qquad (3.19)$$

and

$$\sigma^2 = \frac{r(1-p)}{p^2} \qquad (3.20)$$

respectively.

Negative Binomial and Geometric Distributions

- *The negative binomial distribution is also called the Pascal distribution.*
- *A useful special case of the negative binomial distribution is if r = 1, in which case we have the geometric distribution.*
- *It is the distribution of the number of Bernoulli trials until the first success. The geometric distribution is*

$$p(x) = (1-p)^{x-1}p, \quad x = 1, 2, \ldots$$

The mean and variance of the geometric distribution are

$$\mu = \frac{1}{p} \quad \text{and} \quad \sigma^2 = \frac{1-p}{p^2}$$

Negative Binomial and Geometric Distributions

- *Some important point:*
  - *The geometric distribution has lack of memory property.*
  - *The negative binomial random variable can be defined as the sum of geometric random variables. That is, the sum of r geometric random variables each with parameter p is a negative binomial random variable with parameters p and r.*

Important Continuous Distributions

- *In this section we discuss several continuous distributions that are important in statistical quality control. These include:*
  - *Normal distribution*
  - *Lognormal distribution*
  - *Exponential distribution*
  - *Gamma distribution*
  - *Weibull distribution*

Normal Distribution

- *The normal distribution is probably the most important distribution in both the theory and application of statistics.*
- *If x is a normal random variable, then the probability distribution of x is defined as follows:*

Definition

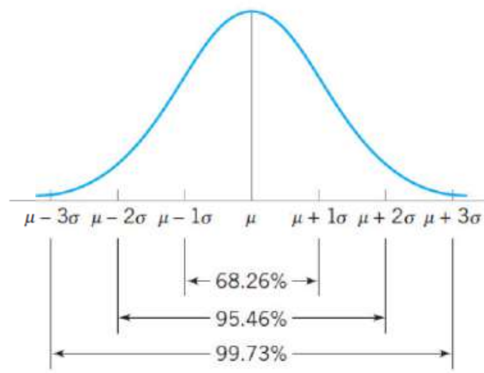
The **normal distribution** is

$$f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \qquad -\infty < x < \infty \qquad (3.21)$$

The mean of the normal distribution is μ $(-\infty < \mu < \infty)$ and the variance is $\sigma^2 > 0$.

Normal Distribution

- *The normal distribution is used so much that we frequently employ a special notation,* $x \sim N(\mu,\sigma^2)$
- *The visual appearance of the normal distribution is a symmetric, unimodal or bell-shaped curve and is shown in below.*



Normal Distribution

- *The cumulative normal distribution is defined as the probability that the normal random variable x is less than or equal to some value a, or*

$$P\{x \le a\} = F(a) = \int_{-\infty}^{a} \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} dx \qquad (3.22)$$

- *This integral cannot be evaluated in closed form. However, by using the change of variable* $z = \frac{x-\mu}{\sigma}$

$$P\{x \le a\} = P\left\{z \le \frac{a-\mu}{\sigma}\right\} \equiv \Phi\left(\frac{a-\mu}{\sigma}\right)$$

Normal Distribution

- *Example. The time to resolve customer complaints is a critical quality characteristic for many organizations. Suppose that this time in a financial organization, say, x—is normally distributed with mean μ=40 hours and standard deviation σ=2 hours denoted $x \sim N(40, 2^2)$.*
  What is the probability that a customer complaint will be resolved in less than 35 hours?

Normal Distribution

- *Answer. The desired probability is*

$$P\{x \leq 35\}$$

- *To evaluate this probability from the standard normal tables, we standardize the point 35 and find*

$$P\{x \leq 35\} = P\left\{z \leq \frac{35-40}{2}\right\} =$$

$$P\{z \leq -2.5\} = \Phi(-2.5) = 0.0062$$

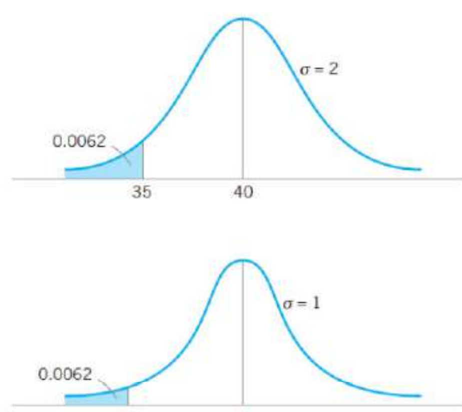


FIGURE 3.18 Calculation of $P\{x \leq 35\}$ in Example 3.7.

Normal Distribution

- *The normal distribution has many useful properties. One of these is relative to linear combinations of normally and independently distributed random variables.*
- *If $x_1, x_2 \ldots, x_n$ are normally and independently distributed random variables with means $\mu_1, \mu_2, \ldots, \mu_n$ and variances $\sigma_1, \sigma_2, \ldots, \sigma_n$ respectively, then the distribution of the linear combination*

$$y = a_1x_1 + a_2x_2 + \cdots + a_nx_n$$

is normal with mean

$$\mu_y = a_1\mu_1 + a_2\mu_2 + \cdots + a_n\mu_n \tag{3.27}$$

and variance

$$\sigma_y^2 = a_1^2\sigma_1^2 + a_2^2\sigma_2^2 + \cdots + a_n^2\sigma_n^2 \tag{3.28}$$

where $a_1, a_2, \ldots, a_n$ are constants.

Normal Distribution

- *The Central Limit Theorem*

Definition

The Central Limit Theorem If $x_1, x_2, \ldots, x_n$ are independent random variables with mean μ_i and variance σ_i^2, and if $y = x_1 + x_2 + \cdots + x_n$, then the distribution of

$$\frac{y - \sum_{i=1}^{n} \mu_i}{\sqrt{\sum_{i=1}^{n} \sigma_i^2}}$$

approaches the $N(0, 1)$ distribution as n approaches infinity.

Lognormal Distribution

- *Variables in a system sometimes follow an exponential relationship, say x = exp(w).*
- *If the exponent w is a random variable, then x = exp(w) is a random variable and the distribution of x is of interest.*
- *An important special case occurs when w has a normal distribution.*
- *In that case, the distribution of x is called a lognormal distribution.*
- *The name follows from the transformation ln (x) = w.*
- *That is, the natural logarithm of x is normally distributed.*

Lognormal Distribution

Definition

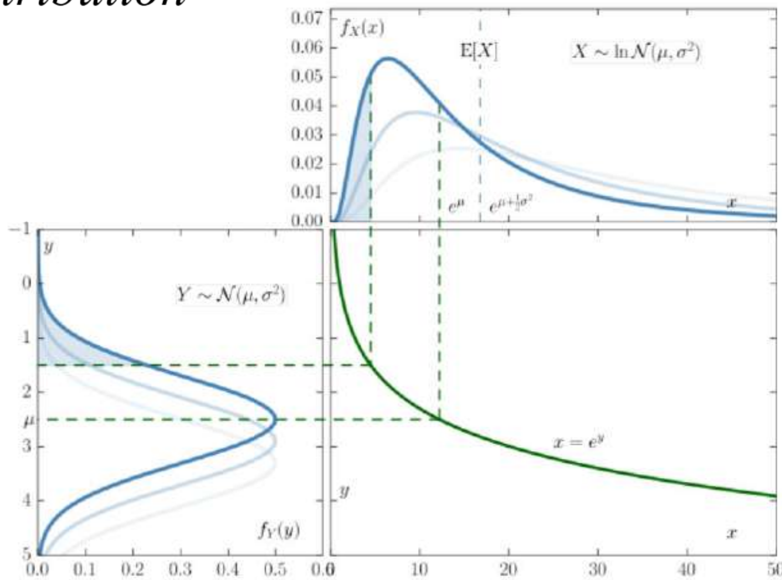
Let w have a normal distribution mean θ and variance ω^2; then $x = \exp(w)$ is a **lognormal random variable,** and the lognormal distribution is

$$f(x) = \frac{1}{x\omega\sqrt{2\pi}} \exp\left[-\frac{\left(\ln(x)-\theta\right)^2}{2\omega^2}\right] \quad 0 < x < \infty \tag{3.29}$$

The mean and variance of x are

$$\mu = e^{\theta+\omega^2/2} \quad \text{and} \quad \sigma^2 = e^{2\theta+\omega^2}\left(e^{\omega^2}-1\right) \tag{3.30}$$

Lognormal Distribution



Lognormal Distribution

- *Example. The lifetime of a medical laser used in eye surgery has a lognormal distribution with θ=6 and w=1.2 hours*
  What is the probability that the lifetime exceeds 500 hours?
- *Answer. From the cumulative distribution function for the lognormal random variable*

$$P(x > 500) = 1 - P[\exp(w) \le 500] = 1 - P[w \le \ln(500)]$$

$$= \Phi\left(\frac{\ln(500) - 6}{1.2}\right) = 1 - \Phi(0.1788)$$

$$= 1 - 0.5710 = 0.4290$$

Exponential Distribution

Definition

The **exponential distribution** is

$$f(x) = \lambda e^{-\lambda x} \qquad x \geq 0 \tag{3.31}$$

where $\lambda > 0$ is a constant. The **mean** and **variance** of the exponential distribution are

$$\mu = \frac{1}{\lambda} \tag{3.32}$$

and

$$\sigma^2 = \frac{1}{\lambda^2} \tag{3.33}$$

respectively.

Exponential Distribution

- *The cumulative exponential distribution is*

$$\begin{aligned} F(a) &= P\{x \leq a\} \\ &= \int_0^a \lambda e^{-\lambda t} dt \\ &= 1 - e^{-\lambda a} \qquad a \geq 0 \end{aligned} \tag{3.34}$$

Exponential Distribution

- *The exponential distribution is widely used in the field of reliability engineering as a model of the time to failure of a component or system.*
- *In these applications, the parameter* $\boldsymbol{\lambda}$ *is called the* ***failure rate*** *of the system, and the mean of the distribution $1/\lambda$ is called the* ***mean time to failure***.
- *The exponential distribution has a* ***lack of memory*** *property.*

Gamma Distribution

$$\Gamma(z) = \int_0^\infty t^{z-1} e^{-t}\, dt, \qquad \Re(z) > 0 .$$

Definition

The **gamma distribution** is

$$f(x) = \frac{\lambda}{\Gamma(r)} (\lambda x)^{r-1} e^{-\lambda x} \qquad x \geq 0 \tag{3.36}$$

with **shape parameter** $r > 0$ and **scale parameter** $\lambda > 0$. The **mean** and **variance** of the gamma distribution are

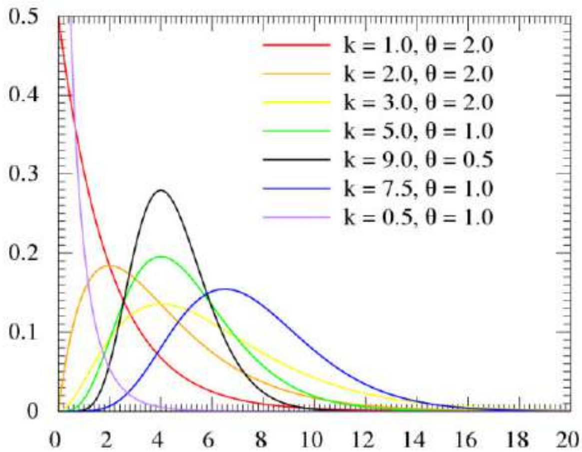
$$\mu = \frac{r}{\lambda} \tag{3.37}$$

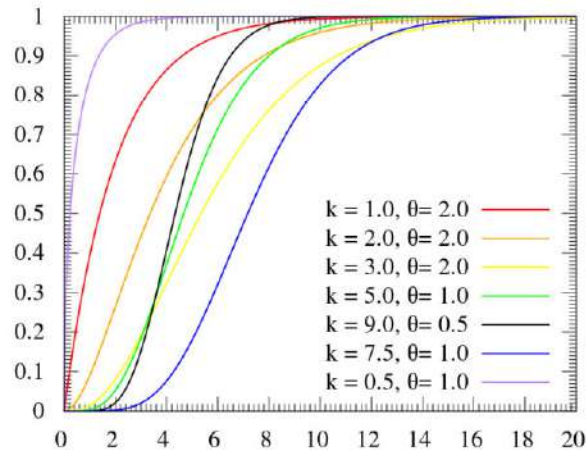
and

$$\sigma^2 = \frac{r}{\lambda^2} \tag{3.38}$$

respectively.[3]

Gamma Distribution





Gamma Distribution

- *Note that if $r=1$, the gamma distribution reduces to the exponential distribution with parameter λ.*
- *If the parameter r is an integer, then the gamma distribution is the sum of r independently and identically distributed exponential distributions, each with parameter λ.*
- *That is, if $x_1, x_2, \ldots, x_r$ are exponential with parameter and independent, then*

$$y = x_1 + x_2 + \cdots + x_r$$

is distributed as gamma with parameters r and λ.

Gamma Distribution

- *Example. Consider the system shown in Figure 3.24. This is called a standby redundant system, because while component 1 is on, component 2 is off, and when component 1 fails, the switch automatically turns component 2 on.*
  If each component has a life described by an exponential distribution with $\lambda=10^{-4}$, say, then the system life is gamma distributed with parameters $r = 2$
  Thus, the mean time to failure is
  $\mu = r/\lambda = 2/10^{-4} = 2 \times 10^4$ *h.*

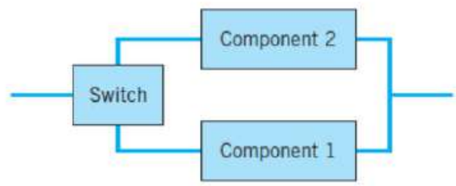


FIGURE 3.24 The standby redundant system for Example 3.11.

Gamma Distribution

- *The cumulative gamma distribution is*

$$F(a) = 1 - \int_a^{\infty} \frac{\lambda}{\Gamma(r)} (\lambda t)^{r-1} e^{-\lambda t} \, dt \qquad (3.39)$$

- *If r is an integer, then equation (3.39) becomes*

$$F(a) = 1 - \sum_{k=0}^{r-1} e^{-\lambda a} \frac{(\lambda a)^k}{k!} \qquad (3.40)$$

Weibull Distribution

Definition

The **Weibull distribution** is

$$f(x) = \frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} \exp\left[-\left(\frac{x}{\theta}\right)^{\beta}\right] \qquad x \geq 0 \tag{3.41}$$

where $\theta > 0$ is the **scale parameter** and $\beta > 0$ is the **shape parameter.** The **mean** and **variance** of the Weibull distribution are

$$\mu = \theta\Gamma\left(1+\frac{1}{\beta}\right) \tag{3.42}$$

and

$$\sigma^2 = \theta^2\left[\Gamma\left(1+\frac{2}{\beta}\right) - \left\{\Gamma\left(1+\frac{1}{\beta}\right)\right\}^2\right] \tag{3.43}$$

respectively.

Weibull Distribution

- *Note that when β=1 the Weibull distribution reduces to the exponential distribution with mean θ.*
- *The cumulative Weibull distribution is* $F(a) = 1 - \exp\left[-\left(\frac{a}{\theta}\right)^{\beta}\right]$ (3.44)

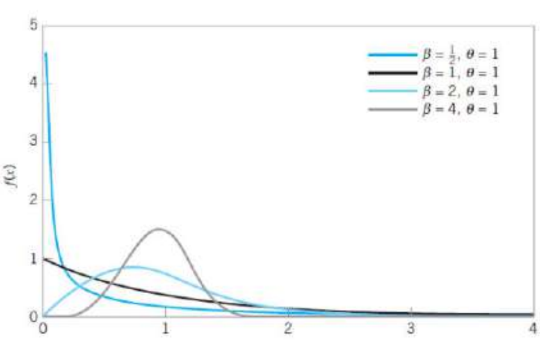


FIGURE 3.25 Weibull distributions for selected values of the shape parameter β and scale parameter $\theta = 1$.

Weibull Distribution

- *Example. The time to failure for an electronic component used in a flat panel display unit is satisfactorily modeled by a Weibull distribution with $\beta=1/2$ and $\theta=5000$. Find the mean time to failure and the fraction of components that are expected to survive beyond 20,000 hours.*

Weibull Distribution

- *Answer. The mean time to failure is*

$$\mu = \theta\Gamma\left(1+\frac{1}{\beta}\right) = 5000\Gamma\left(1+\frac{1}{\frac{1}{2}}\right)$$

$$= 5000\Gamma(3) = 10,000 \text{ hours}$$

Weibull Distribution

- *Answer. The fraction of components expected to survive a = 20,000 hours is:*

$$1 - F(a) = \exp\left[-\left(\frac{a}{\theta}\right)^{\beta}\right]$$

$$1 - F(20,000) = \exp\left[-\left(\frac{20,000}{5,000}\right)^{\frac{1}{2}}\right]$$

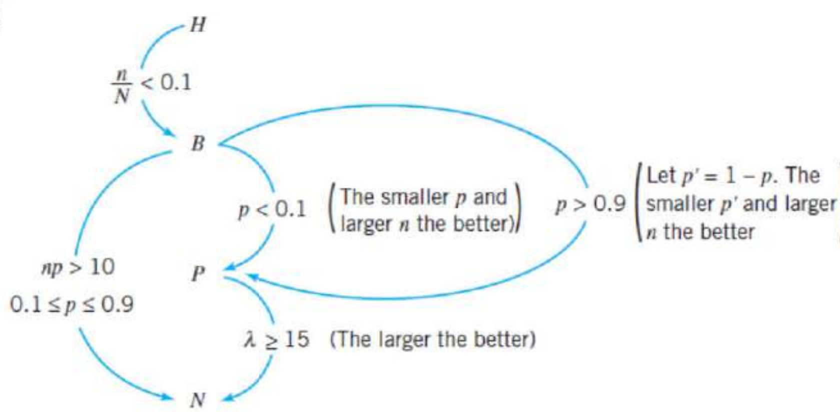
$$= e^{-2}$$

$$= 0.1353$$

That is, all but about 13.53% of the subassemblies will fail by 20,000 hours

Some Useful Approximations

- *We present three important approximations:*
  1. *Hypergeometric to Binomial*
  2. *Binomial to Poisson*
  3. *Binomial to Normal*
  4. *Poisson to Normal*



X^2 Distribution

- *An important sampling distribution defined in terms of the normal distribution is the* ***chi-square*** *or X^2.*
- *If $x_1, x_2, \ldots, x_n$ are normally and independently distributed random variables with* ***mean zero*** *and* ***variance one****, then the random variable*

$$y = x_1^2 + x_2^2 + \cdots + x_n^2$$

is distributed as ***chi-square*** *with* ***n*** *degrees of freedom.*

- *The chi-square probability distribution with n degrees of freedom is*

$$f(y) = \frac{1}{2^{n/2}\Gamma\left(\frac{n}{2}\right)} y^{(n/2)-1} e^{-y/2} \qquad y > 0 \qquad (4.4)$$

X^2 Distribution

- *The distribution is skewed with mean* $\boldsymbol{\mu = n}$ *and variance* $\boldsymbol{\sigma^2 = 2n}$.

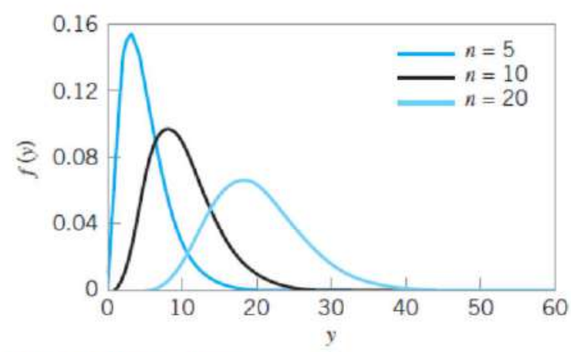


FIGURE 4.2 Chi-square distribution for selected values of n (number of degrees of freedom).

T-Student Distribution

- *Another useful sampling distribution is the* ***t distribution****.*
- *If x is a* ***standard normal*** *random variable and if y is a chi-square random variable with* ***k*** *degrees of freedom, and if x and y are independent, then the random variable*

$$t = \frac{x}{\sqrt{y/k}} \qquad (4.6)$$

is distributed as t with k degrees of freedom.

- *The probability distribution of t is*

$$f(t) = \frac{\Gamma[(k+1)/2]}{\sqrt{k\pi}\,\Gamma(k/2)}\left(\frac{t^2}{k}+1\right)^{-(k+1)/2} \qquad -\infty < t < \infty \qquad (4.7)$$

T-Student Distribution

- *The mean and variance of t are*
  mean $\boldsymbol{\mu=0}$
  and
  variance $\boldsymbol{\sigma^2=k/(k-2)}$ *for* $k>2$

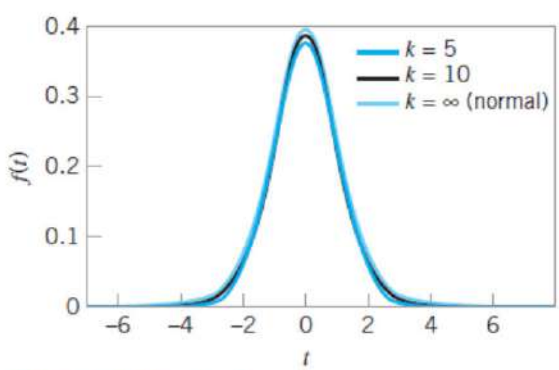


FIGURE 4.3 The t distribution for selected values of k (number of degrees of freedom).

Fisher Distribution

- *The last sampling distribution based on the normal process that we will consider is the F distribution.*
- *If w and y are two independent chi-square random variables with u and v degrees of freedom, respectively, then the ratio*

$$F_{u,v} = \frac{w/u}{y/v} \qquad (4.9)$$

the distribution function is

$$f(x) = \frac{\Gamma\left(\frac{u+v}{2}\right)\left(\frac{u}{v}\right)^{u/2}}{\Gamma\left(\frac{u}{2}\right)\Gamma\left(\frac{v}{2}\right)} \frac{x^{(u/2)-1}}{\left[\left(\frac{u}{v}\right)x+1\right]^{(u+v)/2}} \qquad 0 < x < \infty \qquad (4.10)$$

Fisher Distribution

- *As an example of a random variable that is distributed as F, suppose we have two independent normal processes $x_1 \sim N(\mu_1, \sigma_1^2)$ and $x_2 \sim N(\mu_2, \sigma_2^2)$ then the ratio*

$$\frac{s_1^2/\sigma_1^2}{s_2^2/\sigma_2^2} \sim F_{n_1-1,\, n_2-1}$$

follows from the F distribution

Sampling Distributions

- *If we know the probability distribution of the population from which the sample was taken, we can often determine the probability distribution of various statistics computed from the sample data.*
- *The probability distribution of a statistic is called a* **sampling distribution**.
- *We now present the sampling distributions associated with three common sampling situations:*
  1. *Sampling from a Normal Distribution*
  2. *Sampling from a Bernoulli Distribution*
  3. *Sampling from a Poisson Distribution*

Sampling from a Normal Distribution

- *Suppose that x is a normally distributed random variable with mean μ and variance σ^2.*
- *If $x_1, x_2, \ldots, x_n$ is a random sample of size n from this process, then the distribution of the sample mean $\bar{x}$ is $N(\mu,\sigma^2/n)$*
- *From the central limit theorem we know that regardless of the distribution of the population, the sampling distribution of the sample mean is approximately*

$$\bar{x} \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

Sampling from a Normal Distribution

- *One of the illustration form and important application of chi-square distribution is when $x_1, x_2, \ldots, x_n$ is a random sample from a $N(\mu,\sigma^2)$ normal distribution.*
- *Then the random variable is*

$$y = \frac{\sum_{i=1}^{n} \left(x_1 - \bar{x}\right)^2}{\sigma^2} \qquad (4.5)$$

has a ***chi-square*** *distribution with* ***n-1*** *degrees of freedom or* X_{n-1}^2

- *In another word,*

$$y = \frac{(n-1)s^2}{\sigma^2} \sim \chi_{n-1}^2$$

Sampling from a Normal Distribution

- *As an example of a random variable that is distributed as t, suppose that $x_1, x_2, \ldots, x_n$ is a random sample from the $N(\mu,\sigma^2)$ distribution.*
- *If $\bar{x}$ and s^2 are computed from this sample, then*

$$\frac{\bar{x} - \mu}{s/\sqrt{n}} = \frac{\dfrac{\bar{x} - \mu}{\sigma/\sqrt{n}}}{s/\sigma} \sim \frac{N(0,1)}{\sqrt{\chi_{n-1}^2/(n-1)}}$$

using the fact that $(n-1)s^2/\sigma^2 \sim \chi_{n-1}^2$

Sampling from a Normal Distribution

- *Now, $\bar{x}$ and s^2 are independent, so the random variable*

$$\frac{\bar{x}-\mu}{s/\sqrt{n}} \qquad (4.8)$$

has a t distribution with degrees n-1 of freedom.

Sampling from a Bernoulli Distribution

- *The random variable x with probability function*

$$p(x)=\begin{cases} p & x=1 \\ (1-p)=q & x=0 \end{cases}$$

- *is called a **Bernoulli random variable**.*
- *The sequence of Bernoulli trials $x_1, x_2, \ldots, x_n$ is a Bernoulli process. The outcome x=1 is often called "success," and the outcome x=0 is often called "failure"*

Sampling from a Bernoulli Distribution

- *Suppose that a random sample of n observations $x_1, x_2, \ldots, x_n$ is taken from a Bernoulli process with constant probability of success p.*
- *Then the sum of the sample observations*

$$x = x_1 + x_2 + \cdots + x_n \tag{4.11}$$

*has a **binomial** distribution with parameters **n** and **p***

- *The mean and variance of $\bar{x}$ are*

$$\mu_{\bar{x}} = p$$

$$\sigma_{\bar{x}}^2 = \frac{p(1-p)}{n}$$

Sampling from a Poisson Distribution

- *Consider a random sample of size n from a Poisson distribution with parameter λ, $x_1, x_2, \ldots, x_n$.*
- *The distribution of the sample sum*

$$x = x_1 + x_2 + \cdots + x_n \tag{4.13}$$

*is also **Poisson** with parameter **nλ***

Sampling from a Poisson Distribution

- *Now consider the distribution of the sample mean*

$$\overline{x} = \frac{1}{n}\sum_{i=1}^{n} x_i \tag{4.14}$$

- *The mean and variance of $\bar{x}$ are*

$$\mu_{\overline{x}} = \lambda$$

$$\sigma_{\overline{x}}^2 = \frac{\lambda}{n}$$

Summary of Common Probability Distributions Often Used in Statistical Quality Control

Name	Probability Distribution	Mean	Variance
Discrete			
Uniform	$\frac{1}{b-a},\ a \le b$	$\frac{(b+a)}{2}$	$\frac{(b-a+1)^2-1}{12}$
Binomial	$\binom{n}{x} p^x (1-p)^{n-x}$, $x = 0,1,\ldots,n, 0 \le p \le 1$	np	$np(1-p)$
Negative Binomial	$\binom{x-1}{r-1} p^r (1-p)^{x-r}$, $x = r, r+1, r+2, \ldots$	$\frac{r}{p}$	$\frac{r(1-p)}{p^2}$
Geometric	$(1-p)^{x-1} p$, $x = 1,2,\ldots, 0 \le p \le 1$	$\frac{1}{p}$	$\frac{(1-p)}{p^2}$
Hypergeometric	$\frac{\binom{D}{x}\binom{N-D}{n-x}}{\binom{N}{n}}$, $x = 0,1,\ldots,\min(D,n), D \le N, n \le N$	np, where $p = \frac{D}{N}$	$np(1-p)\left(\frac{N-n}{N-1}\right)$
Poisson	$\frac{e^{-\lambda}\lambda^x}{x!}, x = 0,1,2,\ldots, \lambda > 0$	λ	λ

Summary of Common Probability Distributions Often Used in Statistical Quality Control

Name	Probability Distribution	Mean	Variance
Uniform	$\frac{1}{b-a}, a \le x \le b$	$\frac{(b+a)}{2}$	$\frac{(b-a)^2}{12}$
Normal	$\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, $-\infty < x < \infty, -\infty < \mu < \infty, \sigma > 0$	μ	σ^2
Exponential	$\lambda e^{-\lambda x}, x \ge 0 \quad \lambda > 0$	$1/\lambda$	$1/\lambda^2$
Gamma	$\frac{\lambda x^{r-1} e^{-\lambda x}}{\Gamma(r)}, x > 0,\ r > 0,\ \lambda > 0$	r/λ	r/λ^2
Erlang	$\frac{\lambda^r x^{r-1} e^{-\lambda x}}{(r-1)!}, x > 0,\ r = 1, 2, \ldots$	r/λ	r/λ^2
Weibull	$\frac{\beta}{\theta}\left(\frac{x}{\theta}\right)^{\beta-1} e^{-(x/\theta)^\beta}$, $x > 0,\ \beta > 0,\ \theta > 0$	$\theta\Gamma\left(1+\frac{1}{\beta}\right)$	$\theta^2\Gamma\left(1+\frac{2}{\beta}\right) -\theta^2\left[\Gamma\left(1+\frac{1}{\beta}\right)\right]^2$