Mini Project 1 : Random Walk on a Lonely Island

FM122 — Mathematics of Uncertainty

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I would also like to thank some of my friends who have been essential during the completion of this process. Kaustubh Singh and I took the time to build a movie that live updates the probability of Squeaky being at a certain position, including the pits. Ankit Kumar gave me a very important incite about the relation between the probability of the left fall and the right fall, analytically. And Arnab Dey and I together extended the random walk in 1D to 2 Dimensions.

I am indebted and happy for such gratitude from the afore mentioned parties.

-Vishal Paudel

Chapter 1

Setting the Premise

1.1 Introduction

Squeaky the squirrel is stranded on a plateaued island. The island is finite and deadly nonetheless. There is a cliff at position 0 and n, n being the length of the island.

Squeaky is a relatively gullible squirrel and always hops either forward or backward, with equal probability of 0.5. Alas, she is destined to die, since at any point of time there is always a finite probability of a "run" to death. And suppose if she continually survives then there is something suspicious going on.[1]

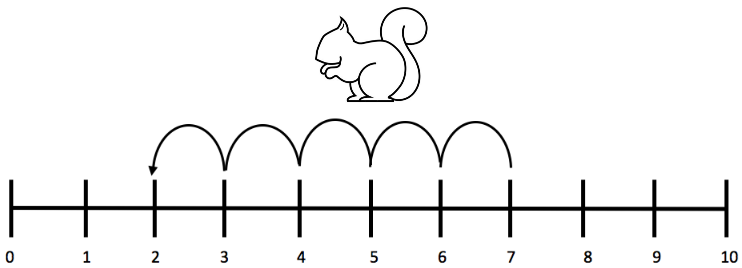


Figure 1.1: In middle of a run: m = 7, length = 10

The author's endeavour is to busy himself with exploration of the avid dynamics of Squeaky's so non-squirrel behaviour.

1.2 A Note on Death Symmetry for $\mathbf{P_{left}}$ =$\mathbf{P_{right}}$ =0.5

Right off the bat, there are spatial symmetries, like if Squeaky is 3 units away from the left edge then all of the probabilistic calculations about the deaths we expect intuitively to be equivalent to if she was 3 units away from the right edge. Off course symmetry is broken when we talk of dying from a specific cliff, either right or left, but overall things like the expected number of steps before dying from any hole are to be symmetrical.

[1] A simple cause may be that Squeaky has memory, or an eye to detect the cliff that is to say $P_{left}(1) = 0, P_{right}(n-1) = 0$

Chapter 2

Diving into the simulations

2.1 Assumptions, parameters, and quick inferences

In this section we only point out the inferences from the simulated behaviour of squeaky. The link for the MATLAB^{TM} project is given in the appendix.

Some of the assumptions we are making, A for the simulations and B for inferences, are really important to be pointed out. The reason for which only will dawn later on.

We have assumed the following:

1. The computer gives **sufficiently** random numbers, particularly the RAND command in MATLAB^{TM}
2. **The average is the representative of the general**. That is if squeaky dies 300 times in repeated simulations then she must also be dying in the 301st repetition, no longer how long it takes for the simulation to complete.

The parameters are:

1. **Island Size** : The size of the island, i.e n
2. **Initial Position** : The initial position of Squeaky.
3. $\mathbf{P_{left}}$ **and** $\mathbf{P_{right}}$: The probabilities or tendency of Squeaky to go left or right.[1]

Moving on with the inferences of the simulation

We very quickly infer the following:

1. The closer the initial position to a cliff, the more likely to die through that cliff
2. The longer the island size, the longer the lifespan.

Following are some graphs supporting the claims (1) and (2)

Clearly, the graph on the left, represents that more the closer is Squeaky's initial position to a particular cliff, more likely she'll die through that cliff.
And the graph on the left proves, at least suggests, that to prolong Squeaky's life, a reasonable starting position is somewhere near the middle. As so incidentally happens poor Squeaky doesn't get to decide that.

[1]These are 0.5 each only for now, but we will change them too as we progress on wards.

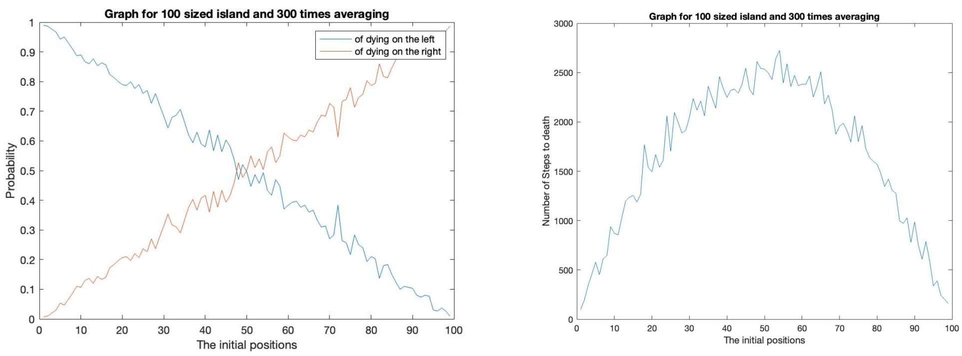


Figure 2.1:
Left: The probability distribution of dying on the left (blue) vs dying on the right (red) against possible initial positions.
Right: The average number of steps to die, either left or right, against possible initial positions

2.2 An Exhaustive Approach and Signs of Diffusion

A. Introducing the exhaustive tree approach

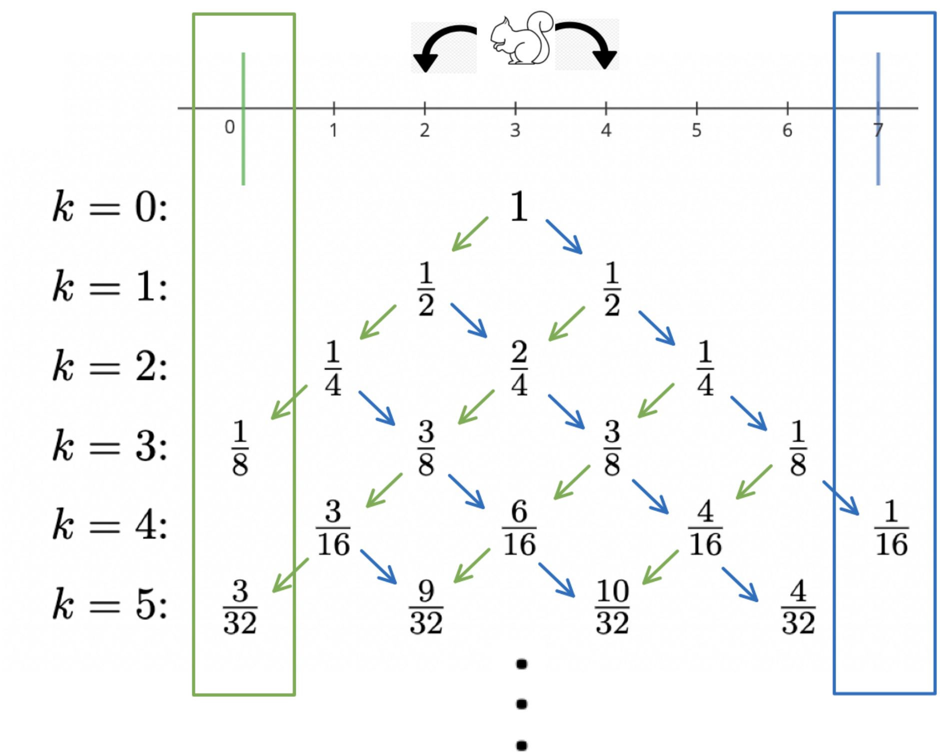


Figure 2.2: The death menu for Squeaky. (for m=3, n=7)

In what follows, the author has tried to look at Squeaky's situation from an exhaustive perspective, that is, to list down all the possible possibilities. Figure 2.2, which the author calls "Squeaky's Death menu", reflects the various choices that squeaky has at a given point on the island. In the figure specifically, she starts from the point 3, with island size equal to 7, and continues to hop left or right with equal bias. The at a certain k, that is after k number of hops, the row depicts the probability that she lands on a particular place. The left pit is marked with green, and the right pit marked with blue. Once crossed the cliff she dies and cannot come back, this is accounted in the tree above.

B. The extended simulation for the Probability distribution

To draw meaningful conclusions with this, we may want to see the change in distribution of these probabilities for different positions as the allowed number of steps in the future increase.

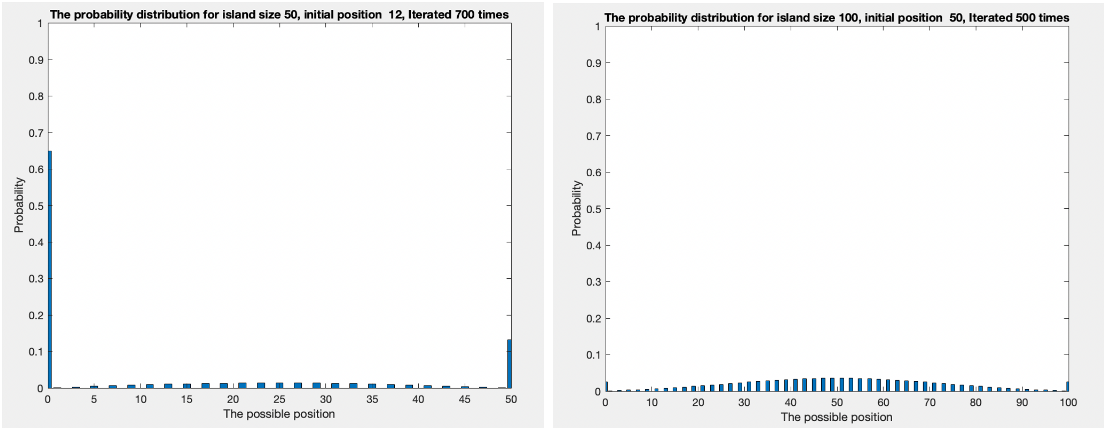


Figure 2.3: The probability distribution after a finite number of allowed jumps.

The links to the code and simulation are provided.[2]

C. Observing diffusion in Random Walk

Einstein published a paper where he modeled the motion of the pollen particles as being moved by individual water molecules, making one of his first major scientific contributions. Here in our project too we see elements of this.

The above simulation is phenomenologically related to diffusion, and to Brownian motion. In other words if we were to unleash a myriad Squeakys from a common starting position m, then as the simulation goes on step after step, the Squeakys will start dying and there density on the island will start decreasing.

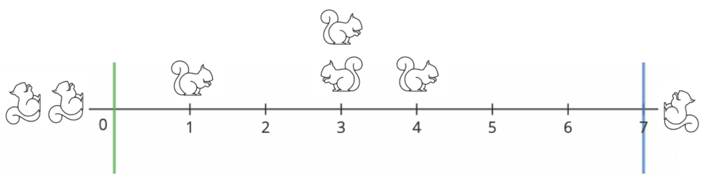
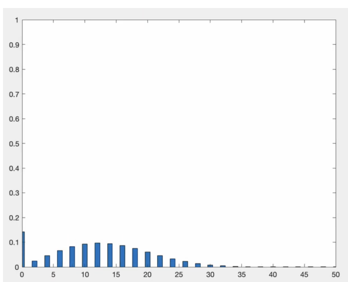


Figure 2.4: Squeaky's Multiverse



[2] The simulation video : https://bit.ly/3uaF8KG
The MATLAB^{TM} code : https://bit.ly/3JtQuQq

Chapter 3

Rubrics questions

3.1 Question 1

The completed simulation code is available at The author's GitHub page.[1] A short simulation video is available at The author's YouTube.[2]

A. Generate a computer simulation which...

...can produce the output of a squirrel jumping randomly and dying when reaching one end.
— Again, the simulation's code is provided.[1]

31 lines (19 sloc) | 421 Bytes

```

n_i = 0;
m = 15;
n_f = 30;

current_index = m + 1;
num_hop = 1;

while current_index > n_i && current_index < n_f

    move = 2 * rand(1);

    if move < 1
        current_index = current_index - 1;

    elseif move >= 1
        current_index = current_index + 1;
    end

    num_hop = num_hop + 1;

    stem(current_index, 1)

    xlim([n_i, n_f - n_i + 1])

    M(num_hop) = getframe;

end

close()
disp(num_hop)
```

Figure 3.1: Screenshot of the simulation code.

[1]https://bit.ly/3qieC0w
[2]https://bit.ly/3tWBDr3

B. Plot each step of this simulation...

...with x axis as the current position. Display this data in a movie (use commands getframe and movie in matlab)

— the simulation's video is provided.[2]

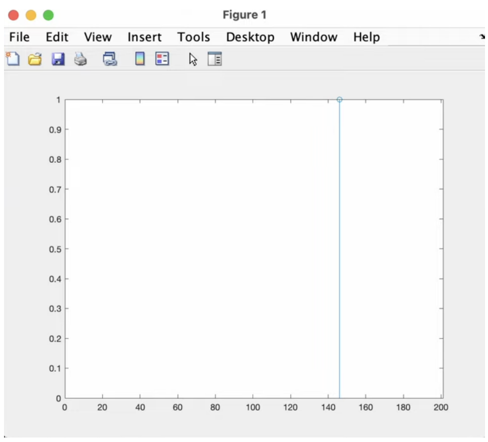


Figure 3.2: A frame of the simulation, n=200; initial_position=120

3.2 Question 2

A. For fixed grid length (=10) and...

...start position (=5), comment out the movie portion and repeat the previous simulation 30 times

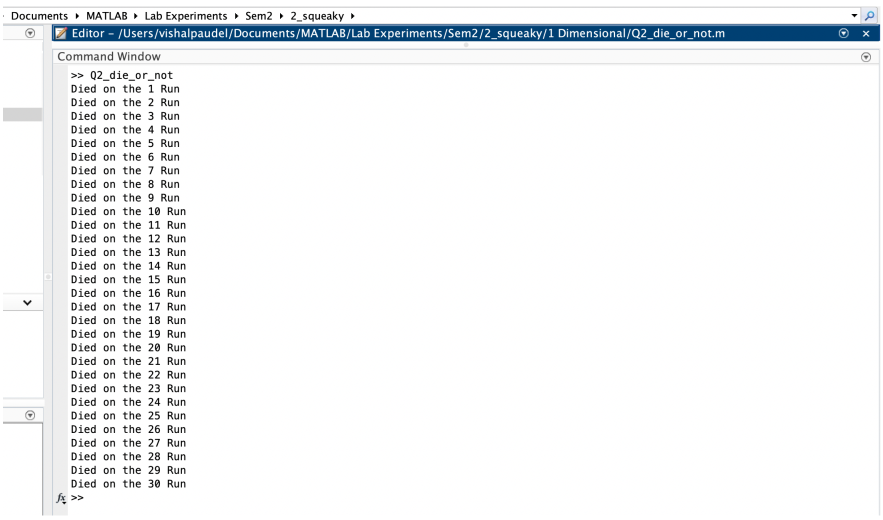


Figure 3.3: Simulation, Repeated 30 times

B. From this data, report if you...

...believe Squeaky will die or not die if you run the simulation the 31st time

Squeaky dies each of the 30 times, hence we are motivated to believe that Squeaky is destined to die on the 31st run also.

3.3 Question 3

A. Fix the grid length to be 15 and...

...vary the start position from 0 to 15 in steps of length 1. Plot a graph of current position vs average number of hops required to die (take at least 30 observations). Report your conclusion on relationship between start position and life expectancy for Squeaky

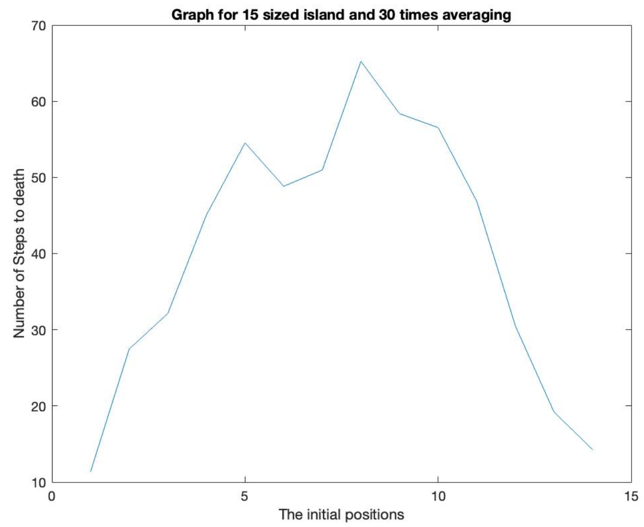


Figure 3.4: Average numbers of steps to death vs the initial positions

We observe from the above image that the life expectancy is low near the cliff and is maximum around the middle of the island, for constant island size.

B. Fix the current position to 7 from...

the left and vary the grid length from 15 to 30. Plot a graph of grid length vs average number of hops required to die (take at least 30 observations). Report your conclusion on relationship between grid length and life expectancy for Squeaky.

The life expectancy increases as the island size increase, citrus paribus.

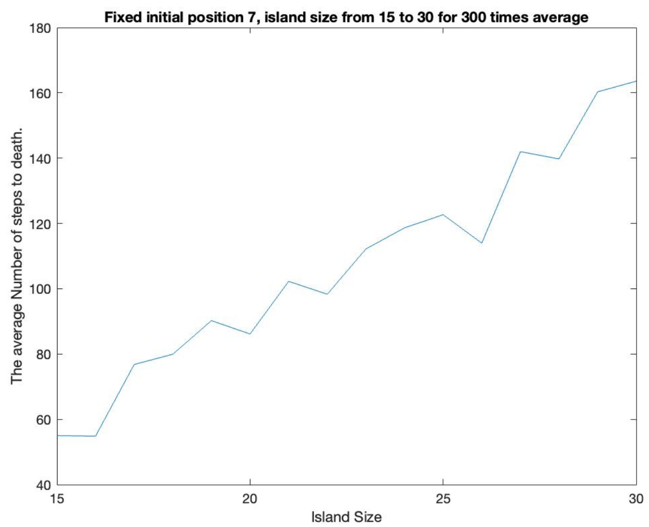


Figure 3.5: Life expectancy as Island Size changes

3.4 Question 4

Now it is time we Analytically dissect the plot and fate of poor Squeaky.

A. Clearly state and derive the formula...

...for life expectancy in terms of starting position and grid length. Calculate the theoretical life expectancy for grid length 15 and starting position varying from 0 to 15.

Here we extend the previous arguments analytically by deriving the expected number of steps to death on either side.

Let $\mathrm{E_m}$ represent the expectation of the random variable $\mathcal{X}$, that squeaky dies in D number of hops given that he starts from the m position in an island of n size.

From the above description it is clear that

$$E_0 = E_\mathrm{n} = 0$$

Now, using **Law of Total Expectation**:

$$\begin{aligned} E_m = &\ \mathrm{P(she\ hops\ to\ (m + 1))} \cdot E(\mathcal{X} \mid \mathrm{she\ hops\ to\ (m + 1)}) \\ &+ \mathrm{P(she\ hops\ to\ (m - 1))} \cdot E(\mathcal{X} \mid \mathrm{she\ hops\ to\ (m - 1)}) \end{aligned}$$

$$E_m = P_{left} \cdot (1 + E_{m-1}) + P_{right} \cdot (1 + E_{m+1})$$

$$E_m = P_{left} \cdot E_{m-1} + P_{right} \cdot E_{m+1} + P_{left} + P_{right}$$

for the case $P_{left} = P_{right} = 0.5$ we get the following non-homogeneous recurrence equation:

$$E_m = 1 + 0.5 \cdot E_{m-1} + 0.5 \cdot E_{m+1}$$

$$E_{m+1} - 2 \cdot E_m + E_{m-1} = -2 \tag{3.1}$$

for the homogeneous part (${E_\mathrm{m}}^\mathrm{h}$):

$$E_{m+1} + 2 \cdot E_m + E_{m-1} = 0 \tag{3.2}$$

Let,

$$E_m = r^m$$

Substituting in 3.2,

$$r^{\mathrm{m\ +\ 1}} - 2r^m + r^{\mathrm{m\ -\ 1}} = 0$$

Factorizing and using quadratic equation we get,

$$r^2 - 2r + 1 = 0$$

$$r_1 = r_2 = 1$$

The multiplicity of this root is 2, thus we need to extend our homogeneous part:

$${E_\mathrm{m}}^\mathrm{h} = (c_1 + c_2 m) r^m$$

For the particular part (${E_\mathrm{m}}^\mathrm{s}$):

$$E_{\mathrm{m}}{}^{\mathrm{s}} = c_3 m^2 r^m$$

Substituting in 3.1, r = 1 from homogeneous part,

$$c_3(m+1)^2 - 2c_3 m^2 + c_3(m-1)^2 + 2 = 0$$

$$c_3(m^2 + 2m + 1) - 2c_3 m^2 + c_3(m^2 - 2m + 1) + 2 = 0$$

$$2c_3 = -2$$

So, $c_3 = -1$, and $E_{\mathrm{m}}{}^{\mathrm{s}} = -m^2$
Now, the final solution as the linear combination of the

$$E_m = E_{\mathrm{m}}{}^{\mathrm{h}} + E_{\mathrm{m}}{}^{\mathrm{s}} = (c_1 + c_2 m) - m^2$$

substituting E_m in 3.1, we get $c_1 = 0, c_2 = n$

$$E_m = nm - m^2 = m(n - m) \tag{3.3}$$

As an immediate side note, the solution for expected number of steps to death is symmetric from each end.
Attaching the graph for this theoretical calculation, as position varies:

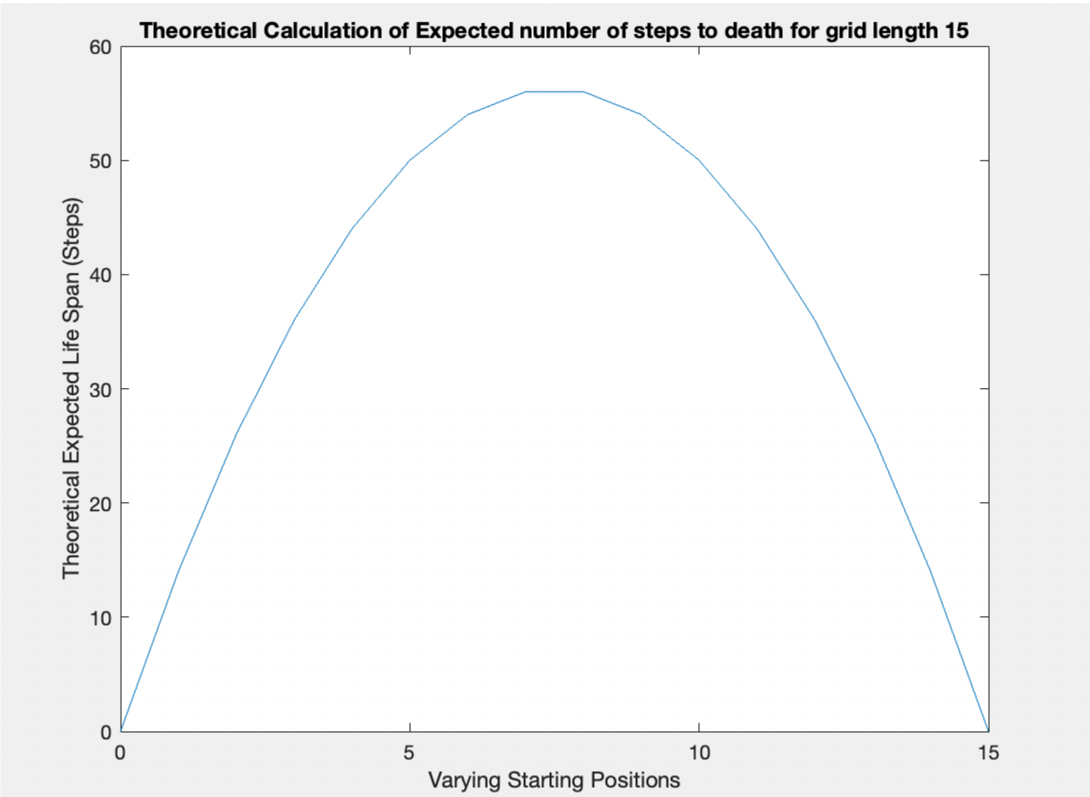


Figure 3.6: The theoretical expected number of steps to death. This graph in contrast to our earlier guestimations is pretty smooth.

This is inline with our earlier comments and guesses on the nature of Squeaky's number of steps to death.

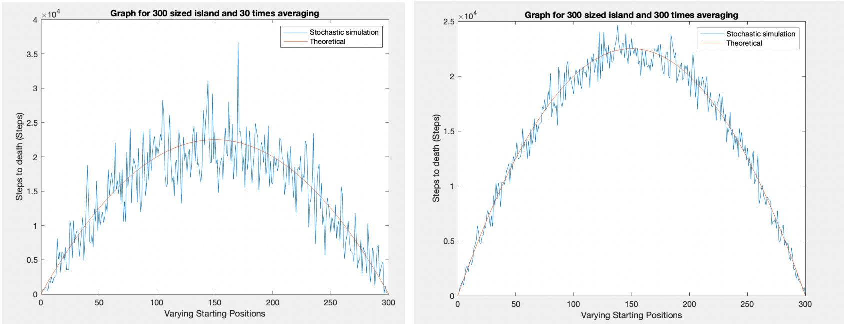


Figure 3.7: The simulation data compared with theoretical results, for island size of 300 and averaging times in the left and .

B. Calculate the discrepancy of this theoretical life expectancy...

from the life expectancy obtained from your simulation in question 3 part a. Make a table with simulated life expectancy, theoretical life expectancy and discrepancy as 3 columns.

We clearly observe from figure 3.8 that as we increase the number of times of averaging, the discrepancy reduce substantially. Besides this, from table **??**

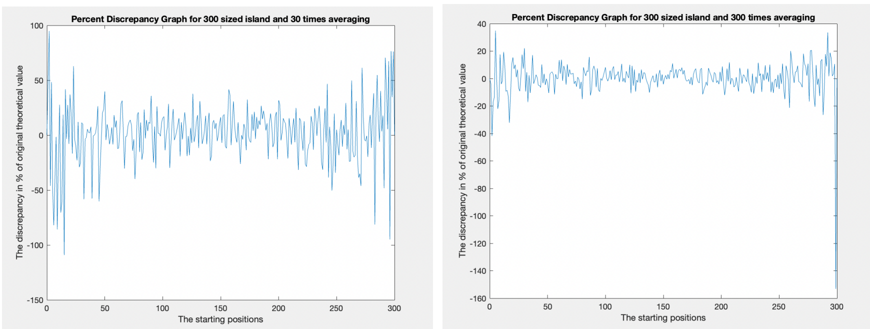


Figure 3.8: The discrepancy, left: 30 times averaging, right: 300 times averaging, for island size of 300.

Table 3.1: Discrepancy in Expected and Observed results, for island size of 15

Initial Position	Expected Results	Observed Results	Discrepancy (Abs. Difference)
0	0	0	0
1	14	10.4	3.6
2	26	31.067	5.067
3	36	27.867	8.133
4	44	51.73	7.73
5	50	61.33	11.33
6	54	59.967	5.967
7	56	67.767	11.767
8	56	59.933	3.933
9	54	41.767	12.233
10	50	43.567	6.433
11	44	53.567	9.567
12	36	32.3	3.7
13	26	16.733	9.267
14	14	12.4	1.6
15	0	0	0

C. Explain the discrepancy.

The result obtained by us in 3.3 was the theoretical average number of steps to death. This is bound to be different from the simulation as the it only happens for a finite number of repetitions to record the average number of steps to death.

However we do argue that as the number of averaging increases to infinity, the discrepancy will be lower and lower. There will always be an inherent difference in the two approaches as the whole simulation process is stochastic, and the theoretical values are deterministic as an overall outcome. The two main sources of the discrepancy are as follows:

1. The simulation runs only finite number of times for averaging
2. The random number generator of MATLAB$^{\text{TM}}$'s rand function may not be completely random

Appendix A

Links

1. Github Project Page: https://bit.ly/3iqSrAY

2. YouTube Playlist: https://bit.ly/3IFQlYZ

Appendix B

Random walk in 2 Dimensions — Author's exploration

Imagine that Squeaky can now actually move anywhere on the 2D plane surface of the Island, that is, up, down, left and right with equal probabilities.

This is exactly the intent of the following simulations,
First up is the usual depiction of Squeaky as a point on a 2-D plane.

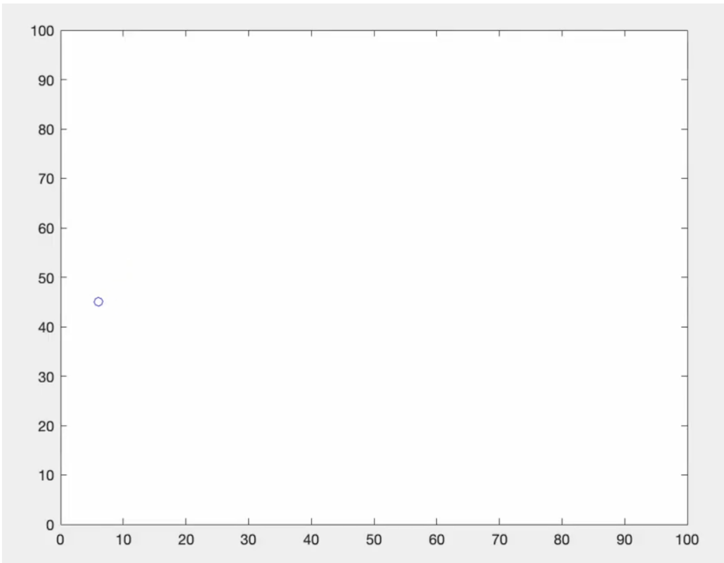


Figure B.1: Squeaky 2D, the circle represents the current position. y limit = 100, x limit = 100, initial position = (40, 50)

The video of the simulation: https://bit.ly/34Xvx12

Next we plan to plot the average number of steps to death with each grid point as the initial position:

We did not analyse this exploration analytically due to time constraints, but have a general sense of the approach. Also if our poor squeaky had digging abilities, we could have extended this to 3D.

With this the author concludes the project work.

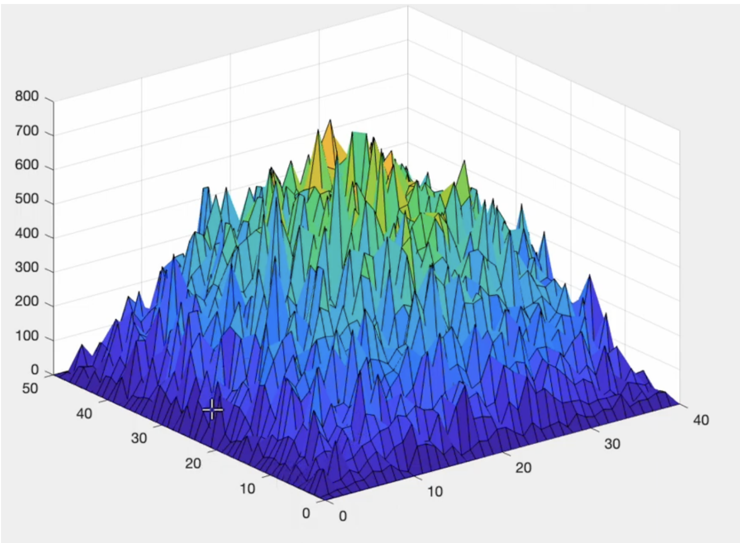


Figure B.2: The z value at each grid point represents the average number of steps to death. 30 times averaged. Island size = 100 X 100, starting position = (40, 50)