classmate

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* 1. Matrix *

* Types of matrix :

(1) Row
(2) collum
(3) square
(4) Diagnol
(5) scalar
(6) Identity
(7) Null
(8) upper diagonl
(9) lower diagnol
(10) Real
(11) complex
(12) symmetric
(13) skew symmetric
(14) orthogonal $[A^T = A^{-1}]$ OR $AA^{-1} = A^T A = I$
(15) conjugate matrix

$$A = \begin{bmatrix} 1+i & 2 & 3 \\ 0 & 1 & 2 \\ 2i & 0 & 0 \end{bmatrix} \quad \bar{A} = A^c = \begin{bmatrix} 1-i & 2 & 3 \\ 0 & 1 & 2 \\ -2i & 0 & 0 \end{bmatrix}$$

(16) Hermition matrix
(17) skew Hermition
(18) unitory matrix
(19) Idenpodent matrix
(20) Involutory matrix
(21) Nilpotent matrix

* Definitions :-

(1) complex number : The matrix $A = [a_{ij}]_{m \times n}$ is called complex matrix is if elleast one entry of complex number.

(2) conjugate matrix : The matrix obtained from any given matrix $A = [a_{ij}]_{m \times n}$ on replacing it's eliments by the corresponding conjugate complex number is called the conjugate matrix and it is demoted by $\bar{A}$ or A^c.

eg: $A = \begin{bmatrix} -i & 3i & 7+4i \\ 2 & 4i & 10 \end{bmatrix}$ $\quad A^c = \begin{bmatrix} i & -3i & 7-4i \\ 2 & -4i & 10 \end{bmatrix}$

(3) Hermitian matrix : A matrix $A = [a_{ij}]_{m \times n}$ is called Hermitian matrix if

$$(A^c)^T = (A^T)^c = A = A^\theta = A^H$$

(4) Skew Hermitian matrix : A matrix $A = [a_{ij}]_{m \times n}$ is called skew Hermitian matrix if

$$(A^c)^T = (A^T)^c = -A = A^\theta = A^H$$

(5) unitary matrix : A matrix $A = [a_{ij}]_{m \times n}$ is called unitary matrix if

$$(A^c)^T = (A^T)^c = A^\theta = A^H = A^{-1}$$

$$A^H A = A A^H = I$$

(6) Idenpodent matrix : if a matrix $A = [a_{ij}]_{m \times n}$ is a square matrix such that $A^2 = A$ then the given matrix is called idenpodent matrix.

(7) Involutory matrix : if a matrix $A = [a_{ij}]_{m \times n}$ is a square matrix such that $A^2 = I$ then the given matrix is called Involutory matrix.

(8) nilpotent matrix : if a matrix $A = [a_{ij}]_{m \times n}$ is a square matrix such that $A^m = 0$, where m is positive integer, than the given matrix is called nilpotant matrix.

* Classwork Examples

1. Find the determinant of $\begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}$ $= 0+6$ $\boxed{=6}$

2. Find the determinant of $\begin{pmatrix} 1 & -2 & 3 \\ 1 & -1 & 2 \\ 2 & 1 & 3 \end{pmatrix}$

$= 1(-3-2) + 2(3-4) + 3(1+2)$
$= -5 - 2 + 9$
$\boxed{=2}$

3. Find the determinant of $\begin{pmatrix} 2 & 0 & 0 & 3 \\ 4 & -3 & 1 & 2 \\ 3 & 1 & 2 & 1 \\ 0 & -4 & 0 & 7 \end{pmatrix}$

$= 2\begin{pmatrix} -3 & 1 & 2 \\ 1 & 2 & 1 \\ \cancel{0}\,-4 & 0 & 7 \end{pmatrix} - 3\begin{pmatrix} 4 & -3 & 1 \\ 3 & 1 & 2 \\ 0 & -4 & 0 \end{pmatrix}$

$= 2\left[(-3)(14) - 1(7+4) + 2(8)\right] - 3\left[(4)(8) + 3(0) + 1(-12)\right]$
$= 2(-42 - 11 + 16) - 3(32 - 12)$
$= -74 - 60$
$\boxed{=-134}$

4. Prove that the det $\begin{pmatrix} a^2 & a & bc \\ b^2 & b & ca \\ c^2 & c & ab \end{pmatrix} = -\det\begin{pmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{pmatrix}$

LHS $= \det\begin{pmatrix} a^2 & a & bc \\ b^2 & b & ca \\ c^2 & c & ab \end{pmatrix}$

$$= \det \frac{1}{abc} \begin{pmatrix} a^3 & a^2 & abc \\ b^3 & b^2 & abc \\ c^3 & c^2 & abc \end{pmatrix} \quad \left(\because R_1 \to aR_1,\ R_2 \to bR_2,\ R_3 \to cR_3\right)$$

$$= \det \frac{abc}{abc} \begin{pmatrix} a^3 & a^2 & 1 \\ b^3 & b^2 & 1 \\ c^3 & c^2 & 1 \end{pmatrix} \quad \left(\because C_3 \to \frac{1}{abc} C_3\right)$$

$$= \det \begin{pmatrix} a^3 & b^3 & c^3 \\ a^2 & b^2 & c^2 \\ 1 & 1 & 1 \end{pmatrix} \quad \left(\because (A \Rightarrow A^T)\right)$$

$$= -\det \begin{pmatrix} 1 & 1 & 1 \\ a^2 & b^2 & c^2 \\ a^3 & b^3 & c^3 \end{pmatrix} \quad \left(R_1 \Rightarrow R_3,\ R_3 \Rightarrow R_1\right)$$

$= RHS$

5. Find the determinant of $\begin{pmatrix} 2 & 0 & -1 & 3 \\ 0 & -3 & 0 & 2 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$

$$= 2 \begin{pmatrix} -3 & 0 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{pmatrix} + 1 \begin{pmatrix} 0 & -3 & 2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} - 3 \begin{pmatrix} 0 & -3 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$$

$= 2(0+0+0) + 1(0) - 3(0)$

$= 0 + 0 + 0$

$\boxed{= 0}$

6. Find the determinant of $\begin{pmatrix} 9 & 9 & 12 \\ 1 & 3 & -4 \\ 1 & 9 & 12 \end{pmatrix}$.

$= 9(36+36) - 9(12+4) + 12(9-3)$

$= 648 - 144 + 72 \quad \boxed{= 576}$

* Practice examples

(1) Find the determinant of $\begin{pmatrix} 0 & 2 & 7 \\ -1 & 5 & 0 \\ 8 & -3 & 2 \end{pmatrix}$.

$= -2(-2-0) + 7(3-40)$

$= 4 - 259$

$\boxed{= -255}$

(2) Find the determinant of $\begin{pmatrix} \sin\theta & -\cos\theta \\ \cos\theta & -\sin\theta \end{pmatrix}$

$= -\sin^2\theta + \cos^2\theta$

$= \cos^2\theta - \sin^2\theta$

$\boxed{= \cos 2\theta}$

(3) Find the determinant of $\begin{pmatrix} 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 7 \\ -2 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \end{pmatrix}$.

$= 5\begin{pmatrix} 0 & 0 & 7 \\ -2 & 0 & 0 \\ 0 & 3 & 0 \end{pmatrix}$

$= 5\,[7\,(-6)]$

$= 5\,(-42)$

$\boxed{= -210}$

(4) Show that the det $\begin{pmatrix} 1 & 1 & 1 \\ a & b & c \\ a^2 & b^2 & c^2 \end{pmatrix} = (a-b)\cdot(b-c)\,(c-a)$.

$= \begin{vmatrix} 0 & 1 & 0 \\ a-b & b & c-b \\ a^2-b^2 & b^2 & c^2-b^2 \end{vmatrix}$ $\quad (\because C_1 \to C_1 - C_2$
$\quad C_3 \to C_3 - C_2)$

$$= \begin{vmatrix} 0 & 1 & 0 \\ a-b & b & c-b \\ (a-b)(a+b) & b^2 & (c-b)(c+b) \end{vmatrix}$$

$$= (a-b)(c-b) \begin{vmatrix} 0 & 1 & 0 \\ 1 & b & 1 \\ a+b & b^2 & c+b \end{vmatrix} \quad \left(\because C_1 \to \frac{1}{a-b} C_1 \quad C_3 \to \frac{1}{c-b} C_3\right)$$

$$= (a-b)(c-b)\,[-1\,(c+b-a-b)]$$

$$= (a-b)(c-b)(a-c)$$

$$= (a-b)(b-c)(c-a)$$

$= RHS$

(5) prove that the det $\begin{pmatrix} -a^2 & ab & ac \\ ba & -b^2 & bc \\ ac & bc & -c^2 \end{pmatrix} = 4a^2b^2c^2.$

$$= abc \begin{vmatrix} -a & b & c \\ a & -b & c \\ a & b & -c \end{vmatrix} \quad \left(\because R_1 = \tfrac{1}{a} R_1 \quad R_2 = \tfrac{1}{b} R_2 \quad R_3 = \tfrac{1}{c} R_3\right)$$

$$= a^2b^2c^2 \begin{vmatrix} -1 & 1 & 1 \\ 1 & -1 & 1 \\ 1 & 1 & -1 \end{vmatrix} \quad \left(\because C_1 \to \tfrac{1}{a} C_1 \quad C_2 \to \tfrac{1}{b} C_2 \quad C_3 \to \tfrac{1}{c} C_3\right)$$

$$= a^2b^2c^2\,[-1\,(1-1) - 1\,(-1-1) + 1\,(1+1)]$$

$$= a^2b^2c^2\,(2+2)$$

$$= 4a^2b^2c^2$$

$= RHS$

(6) Show that the det $\begin{pmatrix} 3a & b-a & c-a \\ a-b & 3b & c-b \\ a-c & b-c & 3c \end{pmatrix} = 3(a+b+c)(ab+bc+ca).$

$$LHS = \begin{vmatrix} 3a & b-a & c-a \\ a-b & 3b & c-b \\ a-c & b-c & 3c \end{vmatrix}$$

$$= \begin{vmatrix} a+b+c & b-a & c-a \\ a+b+c & 3b & c-b \\ a+b+c & b-c & 3c \end{vmatrix} \quad (\because C_1 \to C_1 + C_2 + C_3)$$

$$= (a+b+c)\begin{vmatrix} 1 & b-a & c-a \\ 1 & 3b & c-b \\ 1 & b-c & 3c \end{vmatrix} \quad \left(\because C_1 \to \frac{1}{a+b+c} C_1\right)$$

$$= (a+b+c)\begin{vmatrix} 0 & c-a & -2c-a \\ 0 & 2b+c & -2c-b \\ 1 & b-c & 3c \end{vmatrix}$$

$$= (a+b+c)\left[1\,(c-a)(-2c-b) + (2c+a)(2b+c)\right]$$

$$= (a+b+c)(-2c^2 - bc + 2ac + ab + 4bc + 2c^2 + 2ab + ac)$$

$$= (a+b+c)(3bc + 3ac + 3ab)$$

$$= 3(a+b+c)(ab+bc+ca)$$

$= RHS$

* Rank of a matrix :

- a number 'r' is said to be the rank of a matrix 'A' if

(i) there is at least one non-zero minor of order r.

(ii) all the minors of order greater than r are zero.

Rank of a matrix is dinoted by

$r(A)$ or $\gamma(A)$ or $\rho(A)$

* Property :-

(1) A matrix A is a null matrix or zero matrix if and only if rank of $A = 0$.

(2) let A be an $m \times n$ matrix then rank of A $\rho(A) \leq \min\{m, n\}$

(3) $\rho(A) = \rho(A^T)$

* classwork Examples :

(1) find the rank of $\begin{pmatrix} 2 & 3 & 4 \\ 3 & 5 & 7 \end{pmatrix}$ using minors.

given matrix is of order 2×3

From the property,

$\rho(A) \leq \min\{2,3\}$

$\rho(A) \leq 2$

all possible submatrix of order 2

$\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}, \begin{bmatrix} 3 & 4 \\ 5 & 7 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 3 & 7 \end{bmatrix}$

det of $\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix}$ is non zero

$\therefore$ rank of the given matrix is 2.

$\rho(A) = 2.$

2. Find the rank of $\begin{pmatrix} 1 & 3 & -4 \\ -1 & -3 & 4 \\ 2 & 6 & -8 \end{pmatrix}$ using minors.

given matrix is order of 3×3

from the property

$\rho(A) \leq \{\min\{3 \times 3\}$

$\rho(A) \leq 3$

all possible submatrix of order 3

here det $A = 0$

$\therefore$ $\rho(A)$ is not 3.

now, all possible submatrix of order 2.

$\begin{bmatrix} -3 & 4 \\ 6 & -8 \end{bmatrix}; \begin{bmatrix} -1 & 4 \\ 2 & -8 \end{bmatrix}; \begin{bmatrix} -1 & -3 \\ 2 & 6 \end{bmatrix}; \begin{bmatrix} 3 & -4 \\ 6 & -8 \end{bmatrix}, \begin{bmatrix} 1 & -4 \\ 2 & -8 \end{bmatrix},$

$\begin{bmatrix} 1 & 3 \\ 2 & 6 \end{bmatrix}, \begin{bmatrix} 3 & -4 \\ -3 & 4 \end{bmatrix}, \begin{bmatrix} 1 & -4 \\ -1 & 4 \end{bmatrix}, \begin{bmatrix} 1 & 3 \\ -1 & -3 \end{bmatrix}$

Here all submatrix of order 2 has det 0

$\therefore$ $\rho(A) \neq 2.$

Now, all possible submatrix of order 1.

[1], [3], [-4], [-1], [-3], [4], [2], [6], [-8]

Det of above submatrix are non-zero rank of the given matrix is 1.

$\rho(A) = 1$.

4. Find the rank of $\begin{pmatrix} 2 & 1 & 5 & -1 \\ -1 & 2 & 5 & 3 \\ 3 & 2 & 9 & -1 \end{pmatrix}$ using minors

given matrix is of order 3x4.

from the property.

$\rho(A) \leq \min\{3, 4\}$

$\rho(A) \leq 3$

Now all the possible submatrix of order 3,

$\begin{bmatrix} 2 & 1 & 5 \\ -1 & 2 & 5 \\ 3 & 2 & 9 \end{bmatrix}$, $\begin{bmatrix} 2 & 1 & -1 \\ -1 & 2 & 3 \\ 3 & 2 & -1 \end{bmatrix}$, $\begin{bmatrix} 2 & 5 & -1 \\ -1 & 5 & 3 \\ 3 & 9 & -1 \end{bmatrix}$, $\begin{bmatrix} 1 & 5 & \\ 2 & 5 & \\ 2 & 9 & -1 \end{bmatrix}$

Here all submatrix of order 3 has det 0.

$\therefore \rho(A) \neq 3$.

Now all possible submatrix of order 2.

$\begin{bmatrix} 2 & 5 \\ 2 & 9 \end{bmatrix}$, $\begin{bmatrix} 5 & 3 \\ 9 & -1 \end{bmatrix}$, $\begin{bmatrix} 2 & 3 \\ 2 & -1 \end{bmatrix}$, and so on.

Det of above submatrix are non-zero, rank of the given matrix is 2.

$\rho(A) = 2$.

Find the rank of $\begin{pmatrix} 2 & 0 & 0 & 3 \\ 0 & -3 & 0 & 2 \\ 0 & 2 & 7 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$ using minors.

given matrix is of order 4 x 4.

from the property.

$\rho(A) \leq \min\{4, 4\}$

$\rho(A) \leq 4$

Now all the possible submatrix of order 4.

Here det $A = 0$

$\therefore \rho(A) \neq 4.$

Now all the possible submatrix of order 3.

$\begin{bmatrix} -3 & 0 & 2 \\ 2 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 2 \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -3 & 2 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & -3 & 0 \\ 0 & 2 & 7 \\ 0 & 0 & 0 \end{bmatrix},$

$\begin{bmatrix} 0 & 0 & 3 \\ 2 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 3 \\ 0 & 7 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 3 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 7 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 3 \\ -3 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 3 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

$\begin{bmatrix} 0 & 0 & 3 \\ -3 & 0 & 2 \\ 2 & 7 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & 7 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 3 \\ 0 & -3 & 2 \\ 0 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 2 & 0 & 0 \\ 0 & -3 & 0 \\ 0 & 2 & 7 \end{bmatrix}$

$$\text{det of } \begin{bmatrix} 2 & 0 & 3 \\ 0 & 0 & 2 \\ 0 & 7 & 0 \end{bmatrix} = 2(-14) - 3(0) = -28 \neq 0$$

$\therefore$ rank of the given matrix is 3.

$\rho(A) = 3.$

6. Find the rank of $\begin{pmatrix} 1 & -1 & 2 \\ 2 & 2 & -4 \\ -5 & 5 & -10 \\ 3 & -3 & 6 \end{pmatrix}$ using minors.

given matrix is of order 4×3.

from the property

$\rho(A) \le \min\{4, 3\}$

$\rho(A) \le 3$

Now all possible submatrix of order 3.

$\begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & -4 \\ -5 & 5 & -10 \end{bmatrix}$, $\begin{bmatrix} 2 & 2 & -4 \\ -5 & 5 & -10 \\ 3 & -3 & 6 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 & 2 \\ -5 & 5 & -10 \\ 3 & -3 & 6 \end{bmatrix}$, $\begin{bmatrix} 1 & -1 & 2 \\ 2 & 2 & -4 \\ 3 & -3 & 6 \end{bmatrix}$

here det of matrix is 0

$\therefore \rho(A) \ne 3$.

Now all possible submatrix of order 2.

$\begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix}$ $\begin{bmatrix} -1 & 2 \\ 2 & -4 \end{bmatrix}$ $\begin{bmatrix} 2 & 2 \\ -5 & 5 \end{bmatrix}$ $\begin{bmatrix} 2 & -4 \\ 5 & -10 \end{bmatrix}$ $\begin{bmatrix} -5 & 5 \\ 3 & -3 \end{bmatrix}$

$\begin{bmatrix} 5 & -10 \\ -3 & 6 \end{bmatrix}$ and so on.

here det of matrix is 0.

$\therefore \rho(A) \ne 2$.

Now all possible submatrix of order 1.

$[1]$, $[-1]$, $[2]$, $[2]$, $[2]$, $[-4]$, $[-5]$, $[5]$, $[-10]$,

$[3]$, $[-3]$, $[6]$

$\therefore$ here det of matrix is non-zero.

$\therefore \rho(A) = 1$.

3. Find the rank of $\begin{pmatrix} 1 & 1 & a \\ 1 & a & 1 \\ a & 1 & 1 \end{pmatrix}$ using minors.

$\therefore \det A = 1(a-1) - 1(1-a) + a(1-a^2)$

$= a-1-1+a+a-a^3$

$= -a^3+3a-2$

$= a^3-3a+2$

$$\begin{array}{r|l} & a^2+a-2 \\ \hline a-1 & a^3-3a+2 \\ & a^3-a^2 \\ & -\quad+ \\ \hline & a^2-3a+2 \\ & a^2-a \\ & -\quad+ \\ \hline & -2a+2 \\ & -2a+2 \\ & +\quad- \\ \hline & 0 \end{array}$$

$a^2+a-2 = a^2+2a-a-2$

$= a(a+2)-(a+2)$

$= (a-1)(a+2)$

if $a=1$ $A = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 1(1-1)-1(1-1)+1(1-1)$
$= 0$

$\rho(A) \neq 3$

Now all submatrix of order 2 $\det A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = 0$ $\rho(A) \neq 2$

$\therefore$ Now all submatrix of order 1, $\therefore \rho(A) = 1$.

if $a=-2$ $A = \begin{pmatrix} 1 & 1 & -2 \\ 1 & -2 & 1 \\ -2 & 1 & 1 \end{pmatrix} = 1(-2-1)-1(1+2)-2(1-4)$
$= -3-3+6$
$= 0.$ $\rho(A) \neq 3$

submatrix of order 2 $\det\begin{pmatrix} 1 & 1 \\ 1 & -2 \end{pmatrix} \neq 0$ $\rho(A) = 2$

* Practice examples :-

(1) Determine the rank of $\begin{pmatrix} 3 & 5 & 1 \\ 2 & -2 & 4 \\ 7 & 1 & 9 \end{pmatrix}$ using minors.

given matrix is of order 3×3.

from the property

$\rho(A) \leq \min \{3, 3\}$

$\rho(A) \leq 3$

Now all possible submatrix of order 3.

$\det A = 3(-18-4) - 5(18-28) + 1(2+14)$

$= -66 + 50 + 16$

$= 0$

$\rho(A) \neq 3$

Now all possible submatrix of order 2.

* $\begin{bmatrix} 3 & 5 \\ 2 & -2 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ -2 & 4 \end{bmatrix}, [$

$\begin{bmatrix} -2 & 4 \\ 1 & 9 \end{bmatrix}, \begin{bmatrix} 2 & 4 \\ 7 & 9 \end{bmatrix}. \begin{bmatrix} 2 & -2 \\ 7 & 1 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ 1 & 9 \end{bmatrix}, \begin{bmatrix} 3 & 1 \\ 7 & 9 \end{bmatrix}$

$\begin{bmatrix} 3 & 5 \\ 7 & 1 \end{bmatrix}, \begin{bmatrix} 5 & 1 \\ -2 & 4 \end{bmatrix}. \begin{bmatrix} 3 & 1 \\ 2 & 4 \end{bmatrix}, \begin{bmatrix} 3 & 5 \\ 2 & -2 \end{bmatrix}$

$\therefore \det A \neq 0$

$\therefore \rho(A) = 2$

2. Determine the rank of $\begin{pmatrix}1 & 1 & 1\\ a & b & c\\ a^3 & b^3 & c^3\end{pmatrix}$ using minors.

$$\det A = 1(bc^3 - cb^3) - 1(ac^3 - ca^3) + 1(ab^3 - ba^3)$$

$$= bc^3 - cb^3 - ac^3 + ca^3 + ab^3 - ba^3$$

$$= ca^3 - ba^3 - cb^3 + ab^3 + bc^3 - ac^3$$

$$= a^3(c-b) + b^3(a-c) + c^3(b-a)$$

3. Determine the rank of $\begin{pmatrix} 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -3 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$ using minors.

given matrix is of order 4×4
from the property
$\rho(A) \leq \min \{4 \times 4\}$
$\rho(A) \leq 4$

det of matrix $= 2 \begin{pmatrix} 0 & 0 & -1 \\ 0 & -3 & 0 \\ 1 & 0 & 0 \end{pmatrix}$

$= 2 [0 - 1(-3)$
$= 2(3)$
$= 6$
$\neq 0$

rank of the given matrix is 4.
$\rho(A) = 4.$

4. Determine the rank of $\begin{pmatrix} -1 & -2 & -1 \\ 6 & 12 & 6 \\ 5 & 10 & 5 \end{pmatrix}$ using minors

given matrix is of order 3×3.
from the property
$\rho(A) \leq \min \{3 \times 3\}$
$\rho(A) \leq 3$
Now, det A is 0
$\therefore \rho(A) \neq 3.$

Now all possible submatrix of order 2.
$\begin{bmatrix} -1 & -2 \\ 6 & 12 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 12 & 6 \end{bmatrix} \begin{bmatrix} 6 & 12 \\ 5 & 10 \end{bmatrix} \begin{bmatrix} 12 & 6 \\ 10 & 5 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 5 & 10 \end{bmatrix}$

$\begin{bmatrix} -2 & -1 \\ 10 & 5 \end{bmatrix}$ $\therefore$ here det of the matrix is 0.

$\therefore \rho(A) \neq 2$

now all possible submatrix of order 1.
[-1], [-2], [-1], [6], [12], [6], [5], [10], [5]
here det of matrix is non zero.
$\therefore$ rank of the matrix is 1.
$\therefore \rho(A) = 1$.

(5) Determine the rank of $\begin{pmatrix} 1 & 2 & -1 \\ 5 & 10 & -5 \end{pmatrix}$ using minors.

given matrix is of order 2 x 3
From the property
$\rho(A) \leq \min\{2 \times 3\}$
$\rho(A) \leq 2$
now, all possible submatrix of order 2
$\begin{bmatrix} 1 & 2 \\ 5 & 10 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ 10 & -5 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 5 & -5 \end{bmatrix}$
$\therefore$ here det of submatrix is 0.
$\therefore \rho(A) \neq 2$.
now, all possible submatrix of order 1.

[1], [2], [-1], [5], [10], [-5]
here det of submatrix is non zero.
$\therefore \rho(A) = 1$.

(6) Determine the rank of $\begin{pmatrix} 0 & 1 & -3 & -1 \\ 0 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & 2 & 0 \end{pmatrix}$ using minors

given matrix is of order 4 x 4.
from the property
$\rho(A) \leq \min\{4 \times 4\}$
$\rho(A) \leq 4$.

det of matrix $= -1\begin{pmatrix} 0 & 1 & 1 \\ 3 & 0 & 2 \\ 1 & 1 & 0 \end{pmatrix} + 3\begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 2 \\ 1 & 1 & 0 \end{pmatrix}$

$-1\begin{pmatrix} 0 & 0 & 1 \\ 3 & 1 & 0 \\ 1 & 1 & 2 \end{pmatrix}$

$= -1[-1(-2)+1(3)] + 3[1(3-1)] - 1[1(3-1)]$

$= -1[5] + 3[2] - 1[2]$

$= -5 + 6 - 2$

$= -1$

$\neq 0$

here det of matrix is non zero

$\therefore$ rank of the matrix is 4.

$\therefore \rho(A) = 4.$

* Nullity :

→ Nullity of an $m \times n$ matrix A is, $n - r(A)$.
where n is number of collum.

It is donoted by $n(A)$.

* Leading entry : The first nonzero entry of i^{th} row is known as the leading entry of i^{th} row. And the column containing the leading entry of i^{th} row is denoted by $l(i)$.

* Excemples :-

① $\begin{array}{ccc} l(1) & l(3) & \\ \end{array}$ $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 0 & 0 \\ 0 & 2 & 3 \end{bmatrix}$

② $\begin{array}{ccc} l(2) & l(1) & l(3) \\ \end{array}$ $\begin{bmatrix} 0 & 3 & 4 \\ 2 & 1 & 3 \\ 0 & 0 & 3 \end{bmatrix}$

* A matrix A is said to be in row-echelon form if
(i) all zero rows are at the bottom of the matrix.
(ii) all leading entries must be 1.
(iii) leading entries moves from left to right if we go down. i.e. $l(i) < l(i+1)$.

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$ → zero row at bottom.

all leading entries are 1.

* upper triangle matrix with leading entries 0 and 1 will be in row echelon form.

* classwork Examples :

1. Reduce the matrix $\begin{pmatrix} 1 & -1 & 2 & 3 & -2 \\ 0 & 1 & -1 & -6 & 6 \\ 0 & -2 & 2 & 4 & -4 \end{pmatrix}$ to row-echelon / reduced row-echelon form and hence determine the rank.

Row echelon form :

$R_3 \rightarrow R_3 + 2R_2$

$$\begin{pmatrix} 1 & -1 & 2 & 3 & -2 \\ 0 & 1 & -1 & -6 & 6 \\ 0 & 0 & 0 & -8 & 8 \end{pmatrix}$$

$$R_3 \rightarrow \left(\frac{-1}{8}\right) R_3$$

$$\begin{pmatrix} 1 & -1 & 2 & -3 & -2 \\ 0 & 1 & -1 & -6 & 6 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

Reduced row echelon form :

$$\Rightarrow \begin{pmatrix} 1 & -1 & 2 & 3 & -2 \\ 0 & 1 & -1 & -6 & 6 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$R_1 \rightarrow R_1 + R_2$$

$$\begin{pmatrix} 1 & 0 & 1 & -3 & 4 \\ 0 & 1 & -1 & -6 & 6 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$$R_1 \rightarrow R_1 + 3R_3$$
$$R_2 \rightarrow R_2 + 6R_3$$

$$\begin{pmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 & -1 \end{pmatrix}$$

$\rho(A)$ = no. of non zero rows in reduced row-echelon form.

$\rho(A) = 3.$

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* Reduced row-echelon form : A matrix A is said to be in reduced row-echelon form if

(i) the matrix is in row-echelon form.
(ii) all the entries in a column containing the leading entry must be zero other than the leading entry.

e.g. $A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$

3. Reduce the matrix $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 5 \\ 1 & 5 & 5 & 7 \\ 8 & 1 & 14 & 17 \end{pmatrix}$ to row-echelon / reduced row-echelon form and hence determine the rank.

$R_2 \to R_2 - 2R_1$
$R_3 \to R_3 - R_1$
$R_4 \to R_4 - 8R_1$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & -3 & -2 & -3 \\ 0 & 3 & 2 & 3 \\ 0 & -15 & -10 & -15 \end{pmatrix}$$

$R_2 \to \frac{-1}{3} R_2$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2/3 & 1 \\ 0 & 3 & 2 & 3 \\ 0 & -15 & -10 & -15 \end{pmatrix}$$

$R_3 \to R_3 - 3R_2$

$R_4 \to R_4 + 15R_2$

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 0 & 1 & 2/3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

→ Reduced row echelon form :

$R_1 \to R_1 - 2R_2$

$$\begin{pmatrix} 1 & 0 & 5/3 & 2 \\ 0 & 1 & 2/3 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$\rho(A) = 2$.

4. Reduce the matrix $\begin{pmatrix} 3 & 1 & 7 \\ 1 & 2 & 4 \\ 4 & -1 & 7 \\ 4 & -1 & 5 \end{pmatrix}$ to row-echelon / reduced row-echelon form and hence determine the rank.

$R_1 \leftrightarrow R_2$, $R_3 \leftrightarrow R_4$

$$\begin{pmatrix} 1 & 2 & 4 \\ 3 & 1 & 7 \\ 4 & -1 & 5 \\ 4 & -1 & 7 \end{pmatrix}$$

$R_2 \to R_2 - 3R_1$

$R_3 \to R_3 - 4R_1$

$R_4 \to R_4 - 4R_1$

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & -5 & -5 \\ 0 & -9 & -11 \\ 0 & -9 & -9 \end{pmatrix}$$

$$R_2 \to \frac{1}{-5} R_2$$

$$\begin{pmatrix} 1 & 2 & 4 \\ 0 & 1 & 1 \\ 0 & -9 & -11 \\ 0 & -9 & -9 \end{pmatrix}$$

$R_1 \to R_1 - 2R_2$

$R_3 \to R_3 - R_4$

$R_4 \to R_4 + 9R_2$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & -2 \\ 0 & 0 & 0 \end{pmatrix}$$

$$R_3 \to \frac{1}{-2} R_3$$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$R_1 \to R_1 - 2R_3$

$R_2 \to R_2 - R_3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}$$

$\therefore \rho(A) = 3.$

2. Reduce the matrix $\begin{pmatrix} 1 & 3 & 5 \\ 2 & -1 & 4 \\ -2 & 8 & 2 \end{pmatrix}$ to row-echelon reduced row-echelon form and hence determine the rank.

$R_2 \to R_2 - 2R_1$

$R_3 \to R_2 + R_3$

$$\begin{pmatrix} 1 & 3 & 5 \\ 0 & -7 & -6 \\ 0 & 7 & 6 \end{pmatrix}$$

$R_2 \to \frac{1}{-7} R_2$

$$\begin{pmatrix} 1 & 3 & 5 \\ 0 & 1 & 6/7 \\ 0 & 7 & 6 \end{pmatrix}$$

$R_1 \to R_1 - 3R_2$

$R_3 \to R_3 - 7R_2$

$$\therefore \begin{pmatrix} 1 & 0 & 17/7 \\ 0 & 1 & 6/7 \\ 0 & 0 & 0 \end{pmatrix}$$

$\rho(A) = 2.$

$\therefore$

6. Reduce the matrix $\begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 2 \\ 2 & 2 & 3 \end{pmatrix}$ to row-echelon / reduced row-echelon form and hence determine the rank.

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - 2R_1$

$$\begin{pmatrix} 1 & 1 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$R_1 \to R_1 - R_2$

$$\begin{pmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

$R_3 \to \frac{1}{-1} R_3$

$R_1 \to R_1 - 2R_3$

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$\therefore \rho(A) = 3.$

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↳ Solution of a system of liner eq^n by Gauss elimination and Gauss Jordan Methods.

↳ System of L.E. :

$$a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \quad \text{--- (1)}$$
$$a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \quad \text{--- (2)}$$
$$\vdots$$
$$a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m$$

Matrix form

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

↳ HOMOGINIOUS

if $b_i = 0$ then its known as homogenious system.

↳ co-efficient matrix :

The matrix $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}$ is called

co-efficient matrix of eq^n --- (1)

↳ Argumented Matrix :

The matrix (A/b)

$$\left[\begin{array}{cccc} a_{11} & a_{12} & --- & a_{1n} \\ a_{21} & a_{22} & --- & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & --- & a_{mn} \end{array}\right] \left[\begin{array}{c} b_1 \\ b_2 \\ \vdots \\ b_m \end{array}\right]$$ is called Argumented matrix.

↳ solution of homogeneous eq^n.

- A homogeneous system always has atleast 1 solution namely which is known as a trivial solution.

- If rank of $A = n$ then trivial solution is the only solution of the system.

- If rank of $A < n$ then the system is infinetly many solution.

* Homogeneous system : $b = 0$

* Non-Homogeneous system : $b \neq 0$.

* Solution of non homogeneous eq^n.

- If $r(A/b) \neq r(A)$, then the system is said to be inconsistent. (i.e the system does not have any solution)

- If $r(A/b) = r(A) = n$, then the system has unique solution.

- If $r(A/b) = r(A) < n$, then the system has infinitely many solutions.

1. Solve the system:

$2x + y - z = 4$

$x - y + 2z = -2$

$-x + 2y - z = 2$

by Gauss elimination / Gauss-Jordan method, if is consistent.

$$(A|b) = \left(\begin{array}{ccc|c} 2 & 1 & -1 & 4 \\ 1 & -1 & 2 & -2 \\ -1 & 2 & -1 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 2 & 1 & -1 & 4 \\ -1 & 2 & -1 & 2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 0 & 3 & -5 & 8 \\ 0 & 1 & 1 & 0 \end{array}\right) \quad \begin{pmatrix} R_2 \to R_2 - 2R_1 \\ R_3 \to R_1 + R_3 \end{pmatrix}$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 0 & 1 & 1 & 0 \\ 0 & 3 & -5 & 8 \end{array}\right) \quad (R_2 \leftrightarrow R_3)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -8 & 8 \end{array}\right) \quad (R_3 \to R_3 - 3R_2)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 2 & -2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right) \quad (R_3 \to \frac{-1}{8} R_3)$$

Note that
$\rho(A) = \rho(A/B) = 3$

Gauss elimination method;

$x - y + 2z = -2$

$y + z = 0$

$\boxed{z = -1}$

$\boxed{y = 1}$

$x - 1 - 2 = -2$

$\boxed{x = 1}$

$(x, y, z) = (1, 1, -1)$

Gauss Jordan method

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 3 & -2 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & -1 \end{array}\right) \quad (R_1 \to R_1 + R_2)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -1 \end{array}\right) \quad \begin{pmatrix} R_2 \to R_2 - R_3 \\ R_1 \to R_1 - 3R_3 \end{pmatrix}$$

$(x, y, z) = (1, 1, -1)$

2. solve the system:

$2x + z = 3,$

$x - y + z = 1,$

$4x - 2y + 3z = 3.$

by Gauss elimination / Gauss - Jordan method, if it is consistent.

$$(A/b) = \left(\begin{array}{ccc|c} 2 & 0 & 1 & 3 \\ 1 & -1 & 1 & 1 \\ 4 & -2 & 3 & 3 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 2 & 0 & 1 & 3 \\ 4 & -2 & 3 & 3 \end{array}\right) \quad (R_1 \longleftrightarrow R_2)$$

$$\times \quad \sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 1 & 0 & 1/2 & 3/2 \\ 0 & & & \end{array}\right) \quad \left(\begin{array}{l} R_2 \to \frac{1}{2} R_2 \\ R_3 \to R_3 - 4R_1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 2 & -1 & 1 \\ 0 & 2 & -1 & -1 \end{array}\right) \quad \left(\begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 4R_1 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & -1 & 1 & 1 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 2 & -1 & -1 \end{array}\right) \quad (R_2 \to \frac{1}{2} R_2)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1/2 & 3/2 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 0 & -2 \end{array}\right) \quad \left(\begin{array}{l} R_1 \to R_1 + R_2 \\ R_3 \to R_3 - 2R_2 \end{array}\right)$$

$$\sim \left(\begin{array}{ccc|c} 1 & 0 & 1/2 & 3/2 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{array}\right) \quad (\because R_3 \to \frac{-1}{2} R_3)$$

Note that $\rho(A) = 2$: $\rho(A/b) = 3$

Gauss Jordan method :-

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 3/2 \\ 0 & 1 & -1/2 & 1/2 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad R_1 \to R_1 + R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1/2 & 0 \\ 0 & 1 & -1/2 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 - 3/2\, R_2 \\ R_2 \to R_2 - 1/2\, R_3 \end{array}$$

$\therefore r(A) = 2$

$\therefore r(A/b) = 3$

$\therefore r(A/b) \neq r(A)$

$\therefore$ The system is inconsistent.

3. Solve the system:

$4x - 3y - 9z + 6w = 0$

$2x + 3y + 3z + 6w = 6$

$4x - 21y - 39z - 6w = -24$

by Gauss elimination / Gauss-Jordan method, if is consistent.

$$(A/b) = \left[\begin{array}{cccc|c} 4 & -3 & -9 & 6 & 0 \\ 2 & 3 & 3 & 6 & 6 \\ 4 & -21 & -39 & -6 & -24 \end{array}\right]$$

$$\sim \left[\begin{array}{cccc|c} 1 & -3/4 & -9/4 & 6/4 & 0 \\ 2 & 3 & 3 & 6 & 6 \\ 4 & -21 & -39 & -6 & -24 \end{array}\right] \quad (R_1 \to \frac{1}{4} R_1)$$

$$\sim \left[\begin{array}{cccc|c} 1 & -3/4 & -9/4 & 6/4 & 0 \\ 0 & 9/2 & 15/2 & 3 & 6 \\ 0 & 18 & 30 & -12 & -24 \end{array}\right] \quad \begin{array}{l} (R_2 \to R_2 + (-2) R_1 \\ R_3 \to R_3 + (-4) R_1) \end{array}$$

$$\sim \left[\begin{array}{cccc|c} 1 & -3/4 & -9/4 & 6/4 & 0 \\ 0 & 1 & 5/3 & 2/3 & 4/3 \\ 0 & 18 & 30 & 12 & 24 \end{array}\right] \quad (R_2 \to (\frac{-2}{9}) R_2)$$

$$\sim \left[\begin{array}{cccc|c} 1 & 0 & -1 & 2 & 1 \\ 0 & 1 & 15/9 & 2/3 & 4/3 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} (R_1 \to R_1 + (3/4)R_2 \\ (R_3 \to R_3 + (-15)R_2 \end{array}$$

Note that $\rho(A) = \rho(A/b) = 2$

By Gauss Jordan Method.

$$x - z + 2w = 1$$

$$y + \frac{15}{9} z + \frac{2}{3} \omega = \frac{4}{3}$$

Take $z = k_1$ and $\omega = k_2$

$$x = 1 + z - 2w$$

$$= 1 + k_1 - 2k_2$$

and

$$y = \frac{4}{3} - \frac{15}{9} z - \frac{2}{3} \omega$$

$$= \frac{4}{3} - \frac{15}{9} k_1 - \frac{2}{3} k_2$$

4. $3x + y + 2z = 0$

$x - 2y + 3z = 0$

$x + 5y - 4z = 0$

$$(A/b) = \left[\begin{array}{ccc|c} 3 & 1 & 2 & 0 \\ 1 & -2 & 3 & 0 \\ 1 & 5 & -4 & 0 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 3 & 1 & 2 & 0 \\ 1 & 5 & -4 & 0 \end{array}\right] \quad (R_1 \leftrightarrow R_2)$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 7 & -7 & 0 \\ 0 & 7 & -7 & 0 \end{array}\right] \quad \begin{array}{l} (R_2 \to R_2 + (-3) R_1) \\ (R_3 \to R_3 + (-1) R_1) \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -2 & 3 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 7 & -7 & 0 \end{array}\right] \quad (R_2 \to (\tfrac{1}{7}) R_2)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad \begin{array}{l} (R_1 \to R_1 + (2) R_2) \\ (R_3 \to R_3 + (-7) R_2) \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \quad (R_3 \to (\tfrac{1}{8} R_3))$$

Note that $r(A) = r(A|b) = 2$

By Gauss Jordan Method.

$x + z = 0$

$y - z = 0$

take $z = +k$

$x = -k$

$y = k$

$\{(-k, k) / k \in R\}$

$\{k(-1, 1, 1) / k \in R)\}$

5. Consider the system of Linear eqation

$x + y + z = 6$

$x + 2y + 3z = 10$

$x + 2y + \lambda z = \mu$

find the value of λ and μ so that the system (a) has a unique so^n (b) has infinite number of solution (c) is inconsistent.

$$(A/b) = \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 1 & 2 & \lambda & \mu \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 1 & 2 & 3 & 10 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right] \quad (R_3 \to R_3 - R_2)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1 & 1 & 6 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & \lambda-3 & \mu-10 \end{array}\right] \quad (R_2 \to R_2 - R_1)$$

→ If $\lambda = 3$ & $\mu \neq 10$ then $r(A) = 2$ & $r(A/b) = 3$
∴ The system is consistent.

→ If $\lambda \neq 3$ & $\mu \in R$ then $r(A) = r(A/b) = 3$
∴ The system is unique solⁿ

→ If $\lambda = 3$ & $\mu = 10$ then $r(A) = r(A/b) = 2 < 3$.
∴ The system infinite solution.

H.W. Investigate for what value of λ and μ

$$x + 2y + z = 8$$
$$2x + 2y + 2z = 13$$
$$3x + 4y + \lambda z = \mu.$$

have (i) uniqe solⁿ, (ii) infinite solⁿ (iii) inconsistent solⁿ.

6. find the value of k, so that the equation

$x + y + 3z = 0$

$4x + 3y + kz = 0$

$2x + y + 2z = 0$

have a non-trivial sol^n

$$A = \begin{pmatrix} 1 & 1 & 3 \\ 4 & 3 & k \\ 2 & 1 & 2 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & -1 & k-12 \\ 0 & -1 & -4 \end{pmatrix} \quad \begin{matrix} (R_2 \to R_2 - 4R_1) \\ (R_3 \to R_3 - 2R_1) \end{matrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 12-k \\ 0 & 1 & 4 \end{pmatrix}$$

$$\sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 1 & 12-k \end{pmatrix} \quad (R_2 \leftrightarrow R_3)$$

$$\sim \begin{pmatrix} 1 & 1 & 3 \\ 0 & 1 & 4 \\ 0 & 0 & 8-k \end{pmatrix} \quad (R_3 \to R_3 - R_2)$$

if $k = 8$ then the rank of $A = 2 < 3$

so the system has non trivial sol^n.

7. by applying kirchhoff's law to a circuit we obtain the following equations:

$$7i_1 + 9i_2 = 3$$
$$5i_1 + 7i_2 = 1$$

where i_1 and i_2 represents currents. Find the values of i_1 and i_2 by Gauss elimination Gauss-Jordan method. if it is consistent.

$$(A|b) = \left[\begin{array}{cc|c} 7 & 9 & 3 \\ 5 & 7 & 1 \end{array}\right]$$

$$\left(R_1 \to \frac{R_1}{7}\right)$$

$$\sim \left[\begin{array}{cc|c} 1 & 9/7 & 3/7 \\ 5 & 7 & 1 \end{array}\right]$$

$$(R_2 \to R_2 - 5R_1)$$

$$\sim \left[\begin{array}{cc|c} 1 & 9/7 & 3/7 \\ 0 & 4/7 & -8/7 \end{array}\right]$$

$$\left(R_2 \to \frac{R_2}{4/7}\right)$$

$$\sim \left[\begin{array}{cc|c} 1 & 9/7 & 3/7 \\ 0 & 1 & -2 \end{array}\right]$$

$$I_2 = -2$$

$$I_1 + \frac{9I_2}{7} = \frac{3}{7}$$

$$I_1 + \frac{9(-2)}{7} = \frac{3}{7}$$

$$7I_1 - 18 = 3$$

$$7I_1 = 21$$

$$\boxed{I_1 = 3}$$

$$\sim \left[\begin{array}{cc|c} 1 & 9/7 & 3/7 \\ 0 & 1 & -2 \end{array}\right]$$

$$\left(R_1 \to R_1 - \frac{9}{7}R_2\right)$$

$$\sim \left[\begin{array}{cc|c} 1 & 0 & 3 \\ 0 & 1 & -2 \end{array}\right]$$

$\therefore$ $I_1 = 3$

$I_2 = -2$.

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10. A pulley system gives the following equations

$$\ddot{x}_1 + \ddot{x}_2 = 0$$

$$2\ddot{x}_1 = 20 - T$$

$$5\ddot{x}_2 = 50 - T$$

where $\ddot{x}_1$, $\ddot{x}_2$ represent acceleration and T represents tension in the rope. Determine $\ddot{x}_1$, $\ddot{x}_2$ and T by Gauss elimination / Gauss - Jordan method. if it is consistent.

→ $$(A|b) = \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 2 & 0 & 1 & 20 \\ 0 & 5 & 1 & 50 \end{array}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & -2 & 1 & 20 \\ 0 & 5 & 1 & 50 \end{array}\right) \quad (R_2 \to R_2 - 2R_1)$$

$$= \left(\begin{array}{ccc|c} 1 & 1 & 0 & 0 \\ 0 & 1 & -1/2 & -10 \\ 0 & 5 & 1 & 50 \end{array}\right) \quad \left(R_2 \to \frac{R_2}{(-2)}\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & +1/2 & +10 \\ 0 & 1 & -1/2 & -10 \\ 0 & 0 & 7/2 & 100 \end{array}\right) \quad \begin{array}{l}(R_1 \to R_1 - R_2) \\ (R_3 \to R_3 - 5R_2)\end{array}$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & +1/2 & +10 \\ 0 & 1 & -1/2 & -10 \\ 0 & 0 & 1 & 200/7 \end{array}\right) \quad \left(R_3 \to \frac{2}{7} R_3\right)$$

$$= \left(\begin{array}{ccc|c} 1 & 0 & 0 & -30/7 \\ 0 & 1 & 0 & 30/7 \\ 0 & 0 & 1 & 200/7 \end{array}\right) \quad \begin{array}{l}\left(R_1 \to R_1 - \left(\frac{1}{2}\right) R_3\right) \\ \left(R_2 \to R_2 + \left(\frac{1}{2}\right) R_3\right)\end{array}$$

$\therefore (x, y, z) = \left(\frac{-30}{7}, \frac{30}{7}, \frac{200}{7} \right)$

Ex. Investigate for what value of λ and μ

$$x + 2y + z = 8$$
$$2x + 2y + 2z = 13$$
$$3x + 4y + \lambda z = \mu$$

have (i) unique solⁿ (ii) infinite solⁿ (iii) inconsistent solⁿ.

$$(A/b) = \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 2 & 2 & 2 & 13 \\ 3 & 4 & \lambda & \mu \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -2 & 0 & -3 \\ 0 & -2 & \lambda-3 & \mu-24 \end{array}\right] \quad \begin{array}{l} R_2 \to R_2 - 2R_1 \\ R_3 \to R_3 - 3R_1 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & 3/2 \\ 0 & -2 & \lambda-3 & \mu-24 \end{array}\right] \quad R_2 \to \frac{1}{-2} R_2$$

$$\sim \left[\begin{array}{ccc|c} 1 & \cancel{0}2 & 1 & 8 \\ 0 & 1 & 0 & 3/2 \end{array}\right] \quad \begin{array}{l} \cancel{(R_1 \to R_1 + R_3)} \\ (R_3 \to R_3 + R_1) \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & -2 & 0 & -3 \\ 0 & 0 & \lambda-3 & \mu-21 \end{array}\right] \quad (R_3 \to R_3 - R_2)$$

$$R_2 \to \frac{R_2}{-2}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 2 & 1 & 8 \\ 0 & 1 & 0 & 3/2 \\ 0 & 0 & \lambda-3 & \mu-21 \end{array}\right]$$

→ $r(A) = 3$ if $\lambda \neq 3$ & $r(A/b) = 3$ if $\lambda \neq 3$ so for unique solⁿ we have $\lambda \neq 3$.

→ if $\lambda = 3$, $\mu = 21$ then solⁿ has infinitly many solⁿ.

→ If $\lambda = 3$, $\mu \neq 21$ then solⁿ has no solⁿ

11. $-4y + 2x + 5z = 15000$

$-2z + 3x + y = 1000$

$-x + 3y + z = 4000$

$$2x - 4y + 5z = 15000$$
$$3x + y - 2z = 1000$$
$$-x + 3y + z = 4000$$

$$(A/b) = \left[\begin{array}{ccc|c} 2 & -4 & 5 & 15000 \\ 3 & 1 & -2 & 1000 \\ -1 & 3 & 1 & 4000 \end{array}\right]$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 5/2 & 7500 \\ 3 & 1 & -2 & 1000 \\ -1 & 3 & 1 & 4000 \end{array}\right] \quad (R_1 \to \tfrac{1}{2} R_1)$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 5/2 & 7500 \\ 0 & 7 & -19/2 & -21500 \\ 0 & 1 & 7/2 & 11500 \end{array}\right] \quad \begin{array}{l} (R_2 \to R_2 - 3R_1) \\ (R_3 \to R_3 + R_1) \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & -2 & 5/2 & 7500 \\ 0 & 1 & -19/114 & -21500/7 \\ 0 & 1 & 7/2 & 11500 \end{array}\right] \quad (R_2 \to \frac{1}{7} R_2)$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -3/114 & \frac{95500}{7} \\ 0 & 1 & -19/114 & -19500/7 \\ 0 & 0 & 34/7 & 102000/7 \end{array}\right] \quad \begin{array}{l} (R_1 \to R_1 + 2R_2) \\ (R_3 \to R_3 - R_2) \end{array}$$

$$= \left[\begin{array}{ccc|c} 1 & 0 & -3/114 & 95500/7 \\ 0 & & & \end{array}\right] \quad (R_2 \to \frac{-14}{19} R_2)$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -1 & -4000 \\ 3 & 1 & -2 & 1000 \\ 2 & -4 & 5 & 15000 \end{array}\right] \quad (R_1 \to -R_1)$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -1 & -4000 \\ 0 & 10 & 1 & 13000 \\ 0 & 2 & 7 & 23000 \end{array}\right] \quad \begin{array}{l} (R_2 \to R_2 - 3R_1) \\ (R_3 \to R_3 - 2R_1) \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & -3 & -1 & -4000 \\ 0 & 1 & 1/10 & 1300 \\ 0 & 2 & 7 & 23000 \end{array}\right] \quad (R_2 \to \frac{R_2}{10})$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -7/10 & -100 \\ 0 & 1 & 1/10 & 1300 \\ 0 & 0 & 34/5 & 20400 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 + 3R_2 \\ R_3 \to R_3 - 2R_2 \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -7/10 & -100 \\ 0 & 1 & 1/10 & 1300 \\ 0 & 0 & 1 & 3000 \end{array}\right] \quad R_3 \to \frac{5}{34} R_3$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2000 \\ 0 & 1 & 0 & 1000 \\ 0 & 0 & 1 & 3000 \end{array}\right] \quad \begin{array}{l} R_1 \to R_1 + \frac{7}{10} R_3 \\ R_2 \to R_2 - \frac{1}{10} R_3 \end{array}$$

$\therefore (x, y, z) = (2000, 1000, 3000)$

12. $9a + 3b + c = 64$ $\quad v(t) = at^2 + bt + c \quad , 0 \le t \le 100$

$36a + 6b + c = 133$ $\quad v(3) = 64 \quad v(9) = 208$

$81a + 9b + c = 208$ $\quad v(6) = 133$

$$(A/b) = \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 36 & 6 & 1 & 133 \\ 81 & 9 & 1 & 208 \end{array}\right]$$

$$\sim \left[\begin{array}{ccc|c} 9 & 3 & 1 & 64 \\ 0 & -6 & -3 & -123 \\ 0 & -18 & -8 & -368 \end{array}\right] \quad \begin{array}{l} (R_3 \to R_3 - 9R_1) \\ (R_2 \to R_2 - 4R_1) \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1/3 & 1/9 & 64/9 \\ 0 & -6 & -3 & -123 \\ 0 & -18 & -8 & -368 \end{array}\right] \quad \left(R_1 \to \frac{R_1}{9}\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 1/3 & 1/9 & 64/9 \\ 0 & 1 & 1/2 & 123/6 \\ 0 & -18 & -8 & -368 \end{array}\right] \quad \left(R_2 \to \frac{R_2}{-6}\right)$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & -1/18 & 5/18 \\ 0 & 1 & 1/2 & 123/6 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} \left(R_1 \to R_1 - \frac{1}{3} R_2\right) \\ (R_3 \to R_3 + 18R_2) \end{array}$$

$$\sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1/3 \\ 0 & 1 & 0 & 20 \\ 0 & 0 & 1 & 1 \end{array}\right] \quad \begin{array}{l} \left(R_1 \to R_1 + \frac{1}{18} R_3\right) \\ (R_2 \to R_2 - 1/2 R_3) \end{array}$$

$(a, b, c) = (1/3, 20, 1)$

$$v(15) = at^2 + bt + c$$

$$= \left(\frac{1}{3}\right)(15)^2 + 20(15) + 1$$

$$= 75 + 300 + 1$$

$$v(15) = 376$$