* 1. Higher order derivatives and applications *

* Set :-

⇒ A set is a collection of well defined objects or elements.

* Function :-

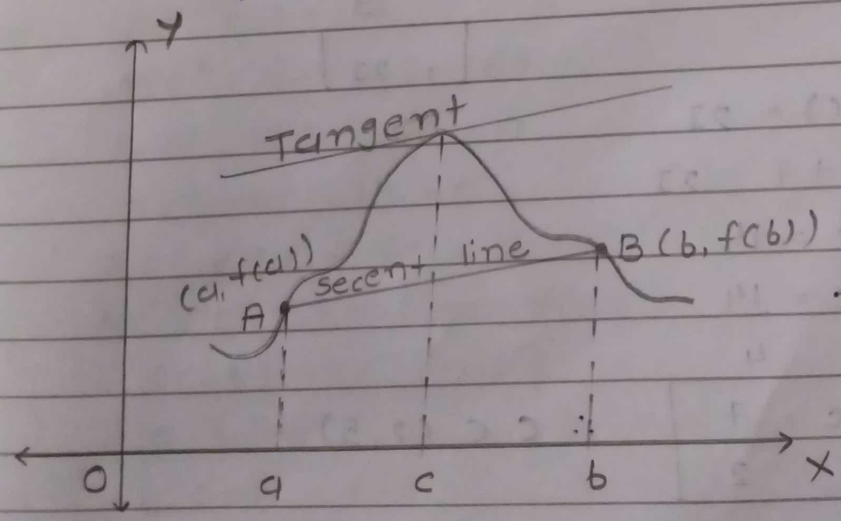
⇒ let $A, B \neq \phi$. If each element of set A is uniquely associates with some element of set B Than a relation between A and B is said to be a function.

* Lagrange's mean value theorem :-

⇒ statement : let f be a continues function on $[a, b]$ and differentiable in (a, b). Then there exits c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

⇒ The Mean value theorem says that the slope of tangent line to a curve is same as the line joining 2 points A and B.



* Classwork Examples :

1. Check whether the mean value Theorem can be applied to the function $2x^2 + 7x + 10$ on the closed interval $[2, 5]$. If so, find a value of c which satisfies the Mean value theorem in $(2, 5)$

solution: let $f(x) = 2x^2 + 7x + 10$

here, f is defined for all values of x in $[2, 5]$
clearly, f is continues for all values of x in $[2, 5]$.
Then, $f'(x) = 4x + 7$, which is defined for all values of x in $(2, 5)$.
$\therefore$ f is diffrentiable in $(2, 5)$.

now, by mean value theorem, that exist c in $(a, b) = (2, 5)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(5) - f(2)}{5 - 2}$$

$$= \frac{95 - 32}{3}$$

$$= \frac{63}{3}$$

$$\boxed{= 21}$$

$\therefore f'(c) = 21$

$4c + 7 = 21$

$4c = 14$

$c = \frac{14}{4}$

$\boxed{c = \frac{7}{2}}$ $\therefore c \in (2, 5)$

2. check whether the mean value Theorem can be applied to the fuction $4t^3 - 8t^2 + 7t - 2$ on the closed interval $[0, 5]$. If so, find a value of c which satisfies the Mean value theorem in $(0, 5)$.

Solution: let $f(x) = 4t^3 - 8t^2 + 7t - 2$

hear, f is defined for all values of x in $[0, 5]$

clearly, f is continues for all values of x in $[0, 5]$.

Then, $f'(x) = 12t^2 - 16t + 7$, which is defined for all values of x in $(0, 5)$.

$\therefore$ f is diffrendiable in $(0, 5)$.

now, by mean value theorem, that exist c in $(a, b) = (0, 5)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a} = \frac{f(5) - f(0)}{5 - 0}$$

$$= \frac{333 + 2}{5}$$

$$= \frac{335}{5}$$

$$\boxed{= 67}$$

Now, $\therefore f'(c) = 67$

$12c^2 - 16c + 7 = 67$

$12c^2 - 16c - 60 = 0$

$3c^2 - 4c - 15 = 0$

$3c^2 - 9c + 5c - 15 = 0$

$3c(c - 3) + 5(c - 3) = 0$

$(c - 3)(3c + 5) = 0$

$c - 3 = 0$ or $3c = -5$

$\boxed{c = 3}$ $\quad c = \frac{-5}{3}$

$(c \in (0, 5))$

3. Check whether the Mean value value Theorem can be applied to the function $f(x) = \log x$ on the closed interval $[1, e]$. If so, find a value of c which satisfies the Mean value theorem in $(1, e)$.

Solution:

S

4. using Mean value Theorem for the function $f(x) = \tan^{-1} x$, $x \in R$, prove that $\frac{x}{1+x^2} < \tan^{-1} x < x$.

Solution: we are given that $f(x) = \tan^{-1} x$, $x \in R$.

clearly f is continues function in $[0, x]$ and diffrentiable in $(0, x)$.

Now by mean value theorem there exist c in $(a, b) = (0, x)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b-a} = \frac{f(x) - f(0)}{x - 0}$$

$$= \frac{\tan^{-1} x - 0}{x}$$

$$= \frac{\tan^{-1} x}{x}$$

hear from f, $f'(x) = \frac{1}{1+x^2}$

$\therefore$ from above we have

$$\frac{1}{1+c^2} = \frac{\tan^{-1} x}{x}$$

$$\therefore \tan^{-1} x = \frac{x}{1+c^2} \quad \text{---} (1)$$

here $0 < c < x$

$0 < c^2 < x^2$

$1 < c^2 + 1 < 1 + x^2$

$1 > \frac{1}{1+c^2} > \frac{1}{1+x^2}$

$\frac{1}{1+x^2} < \frac{1}{1+c^2} < 1$

$$\therefore \frac{x}{1+x^2} < \frac{x}{1+x^2} < x$$

$$\therefore \frac{x}{1+x^2} < \tan^{-1}x < x \quad (\because eq.1)$$

5. Check whether the Mean value Theorem can be applied to the function $f(x) = \sin x$ on the clos interval $[0, \pi/2]$. If so, find a value of c which satisfies the Mean value theorem in $(0, \pi/2)$.

6. Check whether the Mean value Theorem can be applied to the function $f(x) = 6/x - 3$ on the closed interval $[-1, 2]$. If so, find a value of c which satisfies the Mean value theorem in $(-1, 2)$.

Solution: we are given that $f(x) = \frac{6}{x} - 3$, $[-1, 2]$

clearly f is not defined at zero $(x = 0)$
therefor f is not continues in $[-1, 2]$.
Then Mean value theorem is not applicable.

7. A truck travels on a toll road with a speed limit of 80 km/hr. The truck completes a 164 km journey in 2 hours. At the end of the toll road the trucker is issued with a speed violation notice. Justify this using the Mean value Theorem.

Solution: let $f(t)$ be the distance travelled by the trucker in t times (in hr).

Clearly the function f is continues in $[0, 2]$ and the diffrentiable in $(0, 2)$.

then $f(0) = 0$ and $f(2) = 164$
now I mean value theorem, there is some c in $(0, 2)$ such that

$$f'(c) = \frac{f(2) - f(0)}{2 - 0} = \frac{164 - 0}{2} = 82 > 80.$$

$\therefore$ at some point of time during the journey of two hours the trucker must have travelled with a speed more than 80 km/hr which justifies the issume of a speed violation notice.

8. A thermometer was taken from a freezer and placed in a boiling water. It took 22 seconds for the thermometer to raise from $-10^\circ C$ to $100^\circ C$. Show that the rate of change of temperature at some time t is $5^\circ C$ per second.

* Local maxima and minima of a function in one variable :-

⇒ maximum value of a function : If the value of a function $f(x)$ is largest at $x = a$ in some interval $(a-h, a+h)$ then we say that $f(x)$ is maximum at $x = a$.

→ The following two conditions must satisfied by $f(x)$ to be maximum at $x = a$:

$f'(x) = 0$ (nessesary condⁿ)

$f''(a) < 0$ (sufficient condⁿ)

⇒ minimum value of a function : If the value of a function $f(x)$ is smallest at $x = a$ in some interval $(a-h, a+h)$ then we say that $f(x)$ is minimum at $x = a$.

→ The following two conditions must satisfied by $f(x)$ to be minimum at $x = a$:

$f'(x) = 0$

$f''(a) > 0$

→ Note : 1) The point at which function attains is maximum or minimum value is known as stationary points.

2) If $f''(a) = 0$ (a is stationary point) then we can not say anything about maximum and minimum value of a function f at $x = a$.

* classwork examples :

1. Show that the function $y = x^x$ is minimum at $x = e^{-1}$. $(x = \frac{1}{e})$.

olution : we are given that $y = x^x$
take log on both the sides we get

$$\log y = \log x^x$$

$$\log y = x \log x$$

diffrentiating with respect to 'x' we get

$$\frac{1}{y}\frac{dy}{dx} = x\frac{1}{x} + \log x$$

$$\frac{dy}{dx} = y(1 + \log x)$$

$$\frac{dy}{dx} = x^x(1 + \log x) \quad \text{---} (1)$$

now double diffretiating the function.

$$\frac{d^2y}{dx^2} = x^x\left(\frac{1}{x}\right) + (1 + \log x)\, x^x (1 + \log x)$$

$$\frac{d^2y}{dx^2} = \frac{x^x}{x} + x^x(1 + \log x)^2$$

Now at $x = \frac{1}{e}$ we have

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{1}{e}} = \left(\frac{1}{e}\right)^{1/e}\left[e + \left(1 + \log\frac{1}{e}\right)^2\right]$$

$$= \left(\frac{1}{e}\right)^{1/e}\left[e + (1 + \log e^{-1})^2\right]$$

$$= \left(\frac{1}{e}\right)^{1/e}\left[e + (1 + (-1)\log e)^2\right]$$

$$= \left(\frac{1}{e}\right)^{1/e}(e) > 0$$

Hence, the given function is minimum at $x = \frac{1}{e}$.

2. Show that the function $y = \sin x\,(1 + \cos x)$ is maximum when $x = \frac{\pi}{3}$.

ution: we are given that $y = \sin x\,(1 + \cos x)$

$$\frac{dy}{dx} = \cos x\,(1 + \cos x) + \sin x\,(-\sin x)$$

$$= \cos x\,(1 + \cos x) - \sin^2 x$$

$$= \cos x + \cos^2 x - \sin^2 x$$

$$= \cos x + \cos 2x \quad \text{---} (1)$$

Now, again diffrendiating (1) with respect to x,

$$\frac{d^2y}{dx^2} = -\sin x - 2\sin 2x$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=\frac{\pi}{3}} = -\sin\left(\frac{\pi}{3}\right) - 2\sin\left(\frac{2\pi}{3}\right)$$

$$= -\frac{\sqrt{3}}{2} - 2\left(\frac{\sqrt{3}}{2}\right)$$

$$= \frac{-\sqrt{3}}{2} - \sqrt{3} < 0.$$

Hence, the given function is maximum at $x = \frac{\pi}{3}$.

3. Investigate for maxima and minima of the function given by $f(x) = \frac{\log x}{x}$ in $(0, \infty)$.

4. Find the extreme values of the function $f(x) = 2x^3 - 9x^2 + 12x + 1$.

We are given a function;

$$f(x) = 2x^3 - 9x^2 + 12x + 1.$$

Now, differentiating the function with respect to x,

$$f'(x) = 6x^2 - 18x + 12 = 0$$
$$f'(x) = x^2 - 3x + 2 = 0$$
$$x^2 - 2x - x + 2 = 0$$
$$x(x-2) - 1(x-2) = 0$$

$(x-2)(x-1)=0$

$x=2$ or $x=1$

Now, double differentiating the function,

$f''(x) = 12x - 18 = 0$

$|f''(x)|_{x=2} = 24 - 18 > 0$ min.

$|f''(x)|_{x=1} = 12 - 18 < 0$ max.

Now, extreme values of $f(x)$,

Maximum, $x = 1$,

$f(1) = 2(1)^3 - 9(1) + 12(1) + 1$

$= 2 - 9 + 12 + 1$

$= 6$

Minimum, $x = 2$.

$f(2) = 2(2)^3 - 9(2)^2 + 12(2) + 1$

$= 2(8) - 9(4) + 24 + 1$

$= 16 - 36 + 25 = -20 + 25 = 5$

5. The reaction of the body to a doze of medicine can sometimes be represented by an equation of the form $R = M^2\left(\frac{C}{2} - \frac{M}{3}\right)$ where C is a positive constant and M is the amount of medicine absorbed in the blood and R is the amount of reaction (either blood pressure or temperature in their respective units). Determine the amount of the medicine to which the body is most sensitive. (rate of change in R with respect to M).

ution : We are given $R = M^2\left(\frac{C}{2} - \frac{M}{3}\right) = \frac{CM^2}{2} - \frac{M^3}{3}$

here, the sensitivity of body is given by

$$\frac{dR}{dM} = CM - M^2 = f \quad \text{--- (1)}$$

therefore $\frac{df}{dM} = C - 2M$ & $\frac{d^2f}{dM^2} = 0 - 2(1) = -2 < 0$

$\therefore$ to find the amount of medicine we take $\frac{df}{dM} = 0$

$$\therefore C - 2M = 0$$

$$C = 2M$$

$$\boxed{M = \frac{C}{2}}$$

Hence, the sensitivity of the body is maximum when $m = \frac{C}{2}$.

6. P is the perimeter of a rectangle, show that its area is maximum when it is a square.

Solution: let P be the perimeter of a rectangle with length x and y.

then $P = 2x + 2y$ and

Area $A = xy = f$

here from P we have

$$P - 2x = 2y$$

$$y = \frac{P}{2} - \frac{2x}{2}$$

$$y = \frac{P}{2} - x$$

then $A = xy$

$$= x\left(\frac{P}{2} - x\right)$$

$$A = \frac{Px}{2} - x^2 \quad \text{---} (1)$$

Now, diffrenciating with respect to x we get

$$\frac{dA}{dx} = \frac{P}{2} - 2x$$

Again differentiate with respect to x we get

$$\frac{d^2A}{dx^2} = -2$$

take $\frac{dA}{dx} = 0 \Rightarrow \frac{P}{2} - 2x = 0$

$$\frac{P}{2} = 2x$$

$$P = 4x$$

$$\boxed{x = \frac{P}{4}}$$

$$\left.\frac{d^2A}{dx^2}\right|_{x=\frac{P}{4}} = -2 < 0$$

Hence, the area is maximum at $x = \frac{P}{4}$.

from 1 the maximum value is

$$A = \frac{P}{2}\left(\frac{P}{4}\right) - \frac{P^2}{16}$$

$$= \frac{P^2}{8} - \frac{P^2}{16}$$

$$= \frac{2P^2 - P^2}{16}$$

$$A = \frac{P^2}{16}$$

from $y = \frac{P}{2} - x$

$$= \frac{P}{2} - \frac{P}{4}$$

$$y = \frac{P}{4}$$

$\therefore$ here, the length and height of a ractangle =

$\therefore$ It is square.

7. A steel plant is capable of producing x tonnes per day of a low-grade steel and y tonnes per day of a high-grade steel, where $y = \frac{40-5x}{10-x}$.

If the fixed market price of low-grade steel is half that of high-grade steel, then what should be optimal productions in low-grade steel and high-grade steel in order to have maximum receipts.

* Define :-

↳ Derivative or differentiation of a fuction :-

→ A function (f) is said to be differentiable at $x = a$ in (a, b). If $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exist.

If we take,
at $x = a + h$ then the above limit becomes,

$$\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

→ The first order derivative of a function $y = f(x)$ is denoted by $\frac{dy}{dx}$ or $\frac{df}{dx}$ or $f'(x)$ or $y'(x)$ or y_1.

→ Similarly, the second order derivative of a function is denoted by,

$\frac{d^2y}{dx^2}$ or $\frac{d^2f}{dx^2}$ or $f''(x)$ or $y''(x)$ or y_2.

→ Hence, in general the nth order derivative of a function $y = f(x)$ is denoted by,

$\frac{d^ny}{dx^n}$ or $\frac{d^nf}{dx^n}$ or $f^n(x)$ or y_n or $y^{(n)}$

* Successive differentiation :-

The successive differentiation is the process of differentiating a given function successive times and the results of such differentiation are called as successive derivatives.

* The nth order derivatives of some elementary functions :-

(1) $y = e^{ax+b}$, where $a, b \in R$.

Proof :- Here, $y = e^{ax+b}$

Differentiating y with respect to x we get.

$$y_1 = a \cdot e^{ax+b}$$

Differentiating y_1 with respect to x,

$$y_2 = a^2 e^{ax+b}$$

Similarly, $y_3 \Rightarrow y_3 = a^3 e^{ax+b}$

Hence, in general,

$$\boxed{y_n = a^n e^{ax+b}}$$

(2) $y = a^{bx}$, where $a, b \in R$.

Proof: Here, $y = a^{bx}$

Differentiating y with respect to x, we get

$$y_1 = b(\log a)a^{bx}$$

Differentiating y_1 w.R.T. x we get,

$$y_2 = b(\log a)\left[b(\log a)a^{bx}\right]$$

$$= (b \cdot \log a)^2 a^{bx}$$

similarly $y_3 = (b \cdot \log a)^3 a^{bx}$.

Hence, in general,

$$\boxed{y_n = (b \cdot \log a)^n a^{bx}}$$

(3) $y = (ax+b)^m$, where $a, b \in R$

Proof: Hence, $y = (ax+b)^m$

Differentiating y w.R.T. x we get,

$$y_1 = am(ax+b)^{m-1}$$

Differentiating y_1 w.R.T. x we get,

$$y_2 = a^2 m(m-1)(ax+b)^{m-2}$$

Differentiating y_2 w.R.T. x we get,

$$y_3 = a^3 m(m-1)(m-2)(ax+b)^{m-3}$$

similarly $y_4 = a^4 m(m-1)(m-2)(m-3)(ax+b)^{m-}$

Hence in general,

$$y_n = a^n m(m-1)(m-2)\ldots(m-(n-1))(ax+b)^{m-n} \quad (1)$$

* Case 1 : $m > n$

from (1), we have

$$\boxed{y_n = a^n m(m-1)(m-2)\ldots(m-(n-1))(ax+b)^{m-n}}$$

$$\rightarrow y_n = a^n m(m-1)(m-2)\ldots(m-(n-1)) \frac{(m-n)!}{(m-n)!}(ax+b)^{m}$$

$$\boxed{y_n = \frac{a^n m!}{(m-n)!}(ax+b)^{m-n}}$$

* case 2 :- $m < n$.

If $m < n$, then one of the factor of $m(m-1)\ldots(m-(n-1))$ will be 0.

Therefore, for $m < n$, $\boxed{y_n = 0}$

For eg :- $y = 4x^3$ find y_5

$m = 3$ $m < n$

$n = 5$

$y_1 = 12x^2$

$y_2 = 24x$

$y_3 = 24$

$y_4 = 0$

$y_5 = 0.$

* case 3 : $m = n$.

from (1), we have

$$y_n = a^n m(m-1)\dots(m-(n-1))(ax+b)^{m-n}$$

$$= a^n n(n-1)\dots(n-(n-1))(ax+b)^{n-n}$$

$$= a^n n(n-1)\dots(1)(ax+b)^0$$

$$y_n = a^n n(n-1)\dots(1)$$

$$\boxed{y_n = a^n n!}$$

* Case 4 :- $m = -1$

from ①, we have

$$y_n = a^n m(m-1)(m-2)\dots(m-(n-1))(ax+b)^{m-n}$$

$$= a^n(-1)(-1-1)(-1-2)\dots(-1-(n-1))(ax+b)^{-1-n}$$

$$= a^n(-1)(-2)(-3)\dots(-n)(ax+b)^{-(n+1)}$$

$$= a^n(-1)^n(1\ 2\ 3\ \dots\ n)(ax+b)^{-(n+1)}$$

$$\boxed{y_n = \frac{a^n(-1)^n n!}{(ax+b)^{n+1}}}$$

* NOTE :-

1. If $y = (ax+b)^m$, then

$$y_n = \frac{a^n m!}{(m-n)!}(ax+b)^{m-n}, \quad m > n$$

$$= 0, \quad m < n$$

$$= a^n m!, \quad m = n$$

$$= \frac{a^n(-1)^n n!}{(ax+b)^{n+1}}, \quad m = -1$$

④ $y = \log(ax + b)$, where $a, b \in R$.

Proof: Here, $y = \log(ax+b)$

Differentiating with respect to x we get,

$$y_1 = \frac{a}{ax+b}$$

Again Differentiating y_1 w.R.T x we get,

$$y_2 = a \cdot \frac{-a}{(ax+b)^2} = \frac{-a^2}{(ax+b)^2}.$$

similarly,

$$y_3 = \frac{a^3(-1)(-2)}{(ax+b)^3}$$

$$= \frac{a^3(-1)^2(1 \cdot 2)}{(ax+b)^3}$$

$$y_3 = \frac{a^3(-1)^2 \cdot 2!}{(ax+b)^3}$$

$$y_4 = \frac{a^4(-1)(-2)(-3)}{(ax+b)^4}$$

$$= \frac{a^4(-1)^3(1 \cdot 2 \cdot 3)}{(ax+b)^4}$$

$$y_4 = \frac{a^4(-1)^3 3!}{(ax+b)^4}$$

Hence, in genral

$$\boxed{y_n = \frac{a^n (-1)^{n-1} (n-1)!}{(ax+b)^n}}$$

⑤ $y = \sin(ax+b)$, where $a, b \in R$.

proof: Here, $y = \sin(ax+b)$

Diffrentiating y w.r.t. x we get

$$y_1 = a\cos(ax+b) = a\sin\left(\frac{\pi}{2} + (ax+b)\right)$$

Diffrentiating y_1 w.r.t. x we get

$$y_2 = a^2 \cos\left(\frac{\pi}{2} + (ax+b)\right)$$

$$= a^2 \sin\left(\frac{\pi}{2} + \left(\frac{\pi}{2} + (ax+b)\right)\right)$$

$$y_2 = a^2 \sin\left(\frac{2\pi}{2} + (ax+b)\right)$$

Diffrentiating y_2 w.r.t. x we get

$$y_3 = a^3 \sin\left(\frac{3\pi}{2} + (ax+b)\right)$$

Hence, in genral,

$$\boxed{y_n = a^n \sin\left(\frac{n\pi}{2} + (ax+b)\right)}$$

Moto.

⑥ $y = \cos(ax+b)$, where $a, b \in R$.

proof: Here, $y = \cos(ax+b)$

Differentiating y w.r.t. x we get

$$y_1 = -a\sin(ax+b) = a\cos\left(\frac{\pi}{2} + (ax+b)\right)$$

Differentiating y_1 w.r.t. x we get

$$y_2 = -a^2 \sin\left(\frac{\pi}{2} + (ax+b)\right)$$

$$= a^2 \cos\left(\frac{\pi}{2} + \left(\frac{\pi}{2} + (ax+b)\right)\right)$$

$$y_2 = a^2 \cos\left(\frac{2\pi}{2} + (ax+b)\right)$$

similarly,

$$y_3 = a^3 \cos\left(\frac{3\pi}{2} + (ax+b)\right)$$

Hence, In genral,

$$\boxed{y_n = a^n \cos\left(\frac{n\pi}{2} + (ax+b)\right)}$$

⑦ $y = e^{ax}\sin(bx+c)$, where $a, b, c \in R$.

proof: Here, $y = e^{ax}\sin(bx+c)$

Differentiating y w.r.t. x we get

$$y_1 = [ae^{ax}](\sin(bx+c) + e^{ax}[b\cos(bx+c)]$$

$$y_1 = e^{ax}[a\sin(bx+c) + b\cos(bx+c)] \quad \text{--- (2)}$$

take $a = r\cos\theta$ and $b = r\sin\theta$

Then $a^2+b^2 = r^2\cos^2\theta + r^2\sin^2\theta$

$= r^2(\cos^2\theta + \sin^2\theta)$

$a^2+b^2 = r^2(1)$

$\boxed{r = \sqrt{a^2+b^2}}$ and $\frac{b}{a} = \frac{r\sin\theta}{r\cos\theta} = \tan\theta$

$$\boxed{\theta = \tan^{-1}\left(\frac{b}{a}\right)}$$

Subtitude a and b in eq (1) we have,

$$y_1 = e^{ax}[a\sin(bx+c) + b\cos(bx+c)]$$

$$= e^{ax}[r\cos\theta \sin(bx+c) + r\sin\theta \cos\theta(bx+c)$$

$$= re^{ax}[\cos\theta\sin(bx+c) + \sin\theta\cos(bx+c)]$$

$$y_1 = re^{ax}\sin((bx+c)+\theta)$$

diffrentiating y_1 w.r.t: x we get.

$$y_2 = r[ae^{ax}\sin((bx+c)+\theta) + be^{ax}\cos(bx+c)+\theta$$

$$y_2 = re^{ax}[a\sin(bx+c)+\theta] + b\cos(bx+c)+\theta]$$

take $a = r\cos\theta$ and $b = r\sin\theta$

we get

$$y_2 = re^{ax}[r\cos\theta \cdot \sin(bx+c)+\theta) + r\cdot\sin\theta \cos(bx+c)+\theta]$$

$$y_2 = r^2 e^{ax}[\cos\theta \sin(bx+c)+\theta) + \sin\theta\cos(bx+c)+\theta]$$

$$y_2 = r^2 e^{ax} \sin((bx+c+\theta)+\theta)$$

$$y_2 = r^2 e^{ax} \sin((bx+c)+2\theta)$$

similarly,

$$y_3 = r^3 e^{ax} \sin((bx+c)+3\theta)$$

Hence In general,

$$\boxed{y_n = r^n e^{ax}[\sin(bx+c)+n\theta]}$$

where r is equal to $\sqrt{a^2+b^2}$
and $\theta = \tan^{-1}\left(\frac{b}{a}\right)$.

(9) $y = \sinh(ax)$, where $a \in R$.

Proof: Here, $y = \sinh(ax)$
we know that, $\sinh(ax) = \frac{e^{ax} - e^{-ax}}{2}$ and

$$\cosh(ax) = \frac{e^{ax} + e^{-ax}}{2}.$$

Therefore, $y = \sinh(ax) = \frac{e^{ax} - e^{-ax}}{2}$

Now, differentiating y w.R.T. x we get,

$$y_1 = \frac{ae^{ax} + ae^{-ax}}{2} = a\left[\frac{e^{ax} + e^{-ax}}{2}\right]$$

$$y_1 = a \cdot \cosh(ax)$$

Differentiating y_1 w.R.T. x we get,

$$y_1 = a\left[\frac{e^{ax} + e^{-ax}}{2}\right]$$

$$y_2 = \frac{a}{2}\left[ae^{ax} - ae^{-ax}\right]$$

$$y_2 = a^2\left[\frac{e^{ax} - e^{-ax}}{2}\right]$$

$$y_2 = a^2 \sinh(ax)$$

Similarly,

$$y_3 = a^3 \cosh(ax).$$

$$y_4 = a^4 \sinh(ax).$$

Hence in general,

$$\boxed{\begin{aligned} y_n &= a^n \cosh(ax) \quad ; \text{ if } n = \text{odd} \\ &= a^n \sinh(ax) \quad ; \text{ if } n = \text{even.} \end{aligned}}$$

(10) $y = \cosh(ax)$, where $a \in R$.

Proof: Here, $y = \cosh(ax)$.

we know that, $\cosh(ax) = \left(\frac{e^{ax} + e^{-ax}}{2}\right)$

and $\sinh(ax) = \left(\frac{e^{ax} - e^{-ax}}{2}\right)$.

Therefore, $y = \cosh(ax) = \frac{e^{ax} + e^{-ax}}{2}$

Now, differentiating y w.R.T. x we get;

$$y_1 = \frac{ae^{ax} - ae^{-ax}}{2} = a\left[\frac{e^{ax} - e^{-ax}}{2}\right]$$

$$y_1 = a\sinh(ax).$$

Differentiating y_1 w.R.T. x we get,

$$y_1 = a\left[\frac{e^{ax} - e^{-ax}}{2}\right]$$

$$y_2 = a\left[\frac{ae^{ax} + ae^{-ax}}{2}\right]$$

$$y_2 = a^2\left[\frac{e^{ax} + e^{-ax}}{2}\right]$$

$$y_2 = a^2\cosh(ax)$$

similarly,

$$y_3 = a^3 \sinh(ax).$$

Hence in general,

$$\boxed{\begin{aligned} y_n &= a^n \cosh(ax), \quad n = \text{even} \\ &= a^n \sinh(ax), \quad n = \text{odd} \end{aligned}}$$

* classwork examples :-

1. Find the nth order derivative of the function $y = e^{2x+4} + 6^{2x+4}$.

Here, $y = e^{2x+4} + 6^{2x+4}$

we know that n^{th} derivative of $y = e^{ax+b}$ & $y = a^{bx+c}$ are;

$$y_n = a^n e^{ax+b} \quad \& \quad y_n = (b \log a)^n \cdot a^{bx+c}.$$

respectively,

Therefore, $y_n = 2^n e^{2x+4} + (2^n \log 6^n) \cdot 6^{2x+4}$

$$\boxed{y_n = 2^n \left[e^{2x+4} + (\log 6)^n \cdot 6^{2x+4} \right]}$$

2. Find the nth order derivative of the function $y = \log(4x+3) \cdot e^{6x+7}$.

Here, $y = \log\left[(4x+3)\cdot e^{6x+7}\right]$

$$y = \log(4x+3) + \log(e^{6x+7})$$

$$y = \log(4x+3) + (6x+7)$$

we know that, the nth derivative of $y = \log(ax+b)$ and $y = (ax+b)^m$, $m<n$ are $y_n = \dfrac{(-1)^{n-1}(n-1)!\,a^n}{(ax+b)^n}$ and $y_n = 0$.

respectively,

therefore,

$$\boxed{y_n = \frac{(-1)^{n-1}(n-1)!\,4^n}{(4x+3)^n} + 0}$$

3. Find the nth order derivative of the function $y = \dfrac{x}{2x+5}$.

Here, $y = \dfrac{x}{2x+5}$

$$y = \frac{1}{2}\left(\frac{2x+5-5}{2x+5}\right)$$

$$= \frac{1}{2}\left(\frac{2x+5}{2x+5} - \frac{5}{2x+5}\right)$$

$$y = \frac{1}{2}\left(1 - 5(2x+5)^{-1}\right)$$

we know that the nth derivation of $y = (ax+b)^{-1}$ and constant are

$$y_n = \frac{(-1)^n n! a^n}{(ax+b)^{n+1}}$$

and zero respectively

therefore,

$$y_n = \frac{1}{2}\left(\frac{-5(-1)^n n! 2^n}{(2x+5)^{n+1}}\right)$$

$$\boxed{y_n = \frac{(-5)(-1)^{n+1} n! 2^{n-1}}{(2x+5)^{n+1}}}$$

6. Find the nth order derivative of the function $y = e^x \sin x \cos 2x$.

Here, $y = e^x \sin x \cos 2x$.

therefore,

$$y = \frac{1}{2}\left[e^x 2\sin x \cdot \cos 2x\right]$$

$$\left[\begin{array}{ll} 2sc = s+s & 2cc = c+c \\ 2cs = s-s & 2ss = c-c \end{array}\right]$$

$$y = \frac{e^x}{2}\left[\sin(x+2x) + \sin(x-2x)\right]$$

$$y = \frac{e^x}{2}\left[\sin 3x + \sin(-x)\right]$$

$$y = \frac{e^x}{2}\left[\sin 3x - \sin x\right]$$

$$y = \frac{e^x \sin 3x}{2} - \frac{e^x \sin x}{2} \quad \text{--- (1)}$$

we know that, the nth derevative of
$y = e^{ax} \sin(bx+c)$ is

$$y_n = r^n e^{ax} \sin[(bx+c) + n\theta], \quad r = \sqrt{a^2+b^2}$$
$$\theta = \tan^{-1}\frac{b}{a}.$$

$\rightarrow$ $r_1 = \sqrt{(1)^2+(3)^2} = \sqrt{1+9} = \sqrt{10} = 10^{1/2}$

$\theta_1 = \tan^{-1} 3$

$\rightarrow$ $r_2 = \sqrt{(1)^2+(1)^2} = \sqrt{1+1} = \sqrt{2} = 2^{1/2}$

$\theta_2 = \tan^{-1} 1 = \frac{\pi}{4}$

Now, from ①, we have,

$$y_n = \frac{1}{2}\left[10^{n/2} e^{x} \sin(3x) + n\tan^{-1}3\right] - \frac{1}{2}\left[2^{n/2} e^{x} \sin x + \frac{n\pi}{4}\right]$$

$$y_n = \frac{1}{2}\left[(10^{n/2} e^{x} \sin(3x + n\tan^{-1}3)\right] - \left(2^{n/2} e^{x} \sin\left(x+\frac{n\pi}{4}\right)\right)$$

$$\boxed{y_n = \frac{e^{x}}{2}\left[10^{n/2} \sin(3x + n\tan^{-1}3) - 2^{n/2} \sin\left(x+\frac{n\pi}{4}\right)\right]}$$

8) $y = e^{ax} \cos(bx + c)$

$y_1 = ae^{ax} \cos(bx+c) - be^{ax} \sin(bx+c)$

$= e^{ax} [a\cos(bx+c) - b\sin(bx+c)]$

Put $a = r\cos\theta$, $b = r\sin\theta$, $r = \sqrt{a^2+b^2}$ $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

$y_1 = e^{ax} [r\cos\theta \cos(bx+c) - r\sin\theta \sin(bx+c)]$

$= re^{ax} (\cos\theta \cos(bx+c) - \sin\theta \sin(bx+c)]$

$= re^{ax} \cos(bx+c+\theta)$

finding y_2

$y_2 = ra\, e^{ax} \cos(bx+c+\theta) - rbe^{ax} \sin(bx+c+\theta)$

$= re^{ax} [a\cos(bx+c+\theta) - b\sin(bx+c+\theta)]$

$= re^{ax} [r\cos\theta \cos(bx+c+\theta) - r\sin\theta \sin(bx+c+\theta)]$

$= r^2 e^{ax} \cos(bx+c+2\theta)$

$y_3 = r^3 e^{ax} \cos(bx+c+3\theta)$

$\boxed{\text{similarly } y_n = r^n e^{ax} \cos(bx+c+n\theta).}$

H.W

① $y = e^{2x+5} \cos(2x)\cos(x)$

② $y = \sin(2x)\cos(2x)$

* $y = \frac{P(x)}{q(x)} \rightarrow \deg(P(x)) < \deg(q(x))$

* Method of partial fraction:

→ consider $y = \frac{P(x)}{q(x)}$, where $P(x)$ and $q(x)$ are the polinomiear in x.

If deg $(p(x)) < \deg(q(x))$ then only the method of partial frention is applicable.

case 1: If $q(x)$ can be factorised into non repeted linear factors, say $(x-a), (x-b), (x-c), \ldots$ then

$$y = \frac{P(x)}{q(x)} = \frac{P(x)}{(x-a)(x-b)(x-c)\ldots}$$

$$= \frac{A_1}{x-a} + \frac{A_2}{x-b} + \frac{A_3}{x-c} + \ldots$$

case 2: If $q(x)$ can be factorised into repeated linear factors, say $(x-a)^n$, then

$$y = \frac{P(x)}{q(x)} = \frac{P(x)}{(x-a)^n}$$

$$= \frac{A_1}{x-a} + \frac{A_2}{(x-a)^2} + \frac{A_3}{(x-a)^3} + \ldots$$

case 3: If $q(x)$ can be factorised into non-repeated non-linear factors, say $(a_1x^2 + b_1x + c_1)$ & $(a_2x^2 + b_2x + c_2)$, then

$$y = \frac{P(x)}{q(x)} = \frac{P(x)}{(a_1x^2 + b_1x + c_1)(a_2x^2 + b_2x + c_2)}$$

$$= \frac{A_1x + A_2}{a_1x^2 + b_1x + c_1} + \frac{A_3x + A_4}{a_2x^2 + b_2x + c_2}$$

case 4: If $q(x)$ can be factorised into repeated non-linear factors, say $(a_1x^2 + b_1x + c_1)^2$, then

$$y = \frac{P(x)}{q(x)} = \frac{P(x)}{(a_1x^2+b_1x+c_1)^2}$$

$$= \frac{A_1x+A_2}{a_1x^2+b_1x+c_1} + \frac{A_3x+A_4}{(a_1x^2+b_1x+c_1)^2}$$

(1) $y = e^{2x+5} \cos(2x) \cos(x)$

here, $y = e^{2x+5} \cos(2x) \cos(x)$.

therefore,

$$y = \frac{1}{2} e^{2x+5} [2\cos(2x) \cos(x)]$$

$$= \frac{1}{2} e^{2x+5} [\cos 3x + \cos x]$$

$$= \frac{e^{2x+5} \cos 3x}{2} + \frac{e^{2x+5} \cos x}{2} \quad \text{----} (1)$$

we know that the nth derevative of $y = e^{ax} \cos(bx+c)$ is $y_n = r^n e^{ax} \cos[(bx+c)+n\theta$

$$r = \sqrt{a^2+b^2}, \quad \theta = \tan^{-1} \frac{b}{a}$$

$$r_1 = \sqrt{(2)^2+(3)^2} = \sqrt{4+9} = \sqrt{13} = 13^{1/2}$$

$$\theta_1 = \tan^{-1}\left(\frac{3}{2}\right)$$

$$r_2 = \sqrt{(2)^2+(1)^2} = \sqrt{4+1} = \sqrt{5} = 5^{1/2}$$

$$\theta_2 = \tan^{-1}\left(\frac{1}{2}\right)$$

Now from ① we have

$$y = \frac{1}{2}\left[e^{2x+5}\cos 3x + e^{2x+5}\cos x\right]$$

$$y_n = \frac{1}{2}\left[13^{n/2} e^{2x+5}\cos(3x) + n\tan^{-1}\left(\frac{3}{2}\right)\right] - \frac{1}{2}\left[5^{n/2} e^{2x+5}\cos(x) + n\tan^{-1}\left(\frac{1}{2}\right)\right]$$

$$= \frac{1}{2}\left[13^{n/2} e^{2x+5}\cos(3x) + n\tan^{-1}\frac{3}{2}\right] - \left[5^{n/2} e^{2x+5}\cos(x) + n\tan^{-1}\frac{1}{2}\right]$$

$$= \frac{e^{2x+5}}{2}\left[13^{n/2}\cos(3x) + n\tan^{-1}\frac{3}{2}\right] - 5^{n/2}\cos(x) + n\tan^{-1}\frac{1}{2}\Big]$$

(2) $y = \sin(2x)\cos(2x)$

here $y = \sin(2x)\cos(2x)$

therefore,

$$y = \frac{1}{2}\, 2\sin(2x)\cos(2x)$$

$$= \frac{1}{2}\left[\sin 4x + \sin 0\right]$$

$$y = \frac{1}{2}\sin 4x \quad \text{---①}$$

we know that the nth derivative of $y = \sin(ax+b)$ is $y_n = a^n \sin\left(\frac{n\pi}{2} + (ax+b)\right)$.

Now from ① we have

$$y_n = \frac{1}{2}\left[4^n \sin\left(\frac{n\pi}{2} + 4x\right)\right]$$

$$y_n = \frac{4^n}{2} \sin\left(4x + \frac{n\pi}{2}\right).$$

4. Find the nth derivative of $y = \frac{x}{(x-1)(2x+3)}$ is.

Here, $y = \frac{x}{(x-1)(2x+3)}$

Now, by method of partial fraction:

$$y = \frac{x}{(x-1)(2x+3)} = \frac{A}{x-1} + \frac{B}{2x+3} \quad \text{—— ①}$$

$$\therefore x = A(2x+3) + B(x-1) \quad \text{—— ②}$$

Put $x = \frac{-3}{2}$ in ② we get

$$\frac{-3}{2} = 0 + B\left(\frac{-3}{2} - 1\right)$$

$$\frac{-3}{2} = B\left(\frac{-5}{2}\right)$$

$$\boxed{B = \frac{3}{5}}$$

Also put $x = 1$ in ② we get

$$1 = A(2+3) + 0$$

$$1 = 5A$$

$$\boxed{A = \frac{1}{5}}$$

therefore from (1) we have,

$$y = \frac{1/5}{x-1} + \frac{3/5}{2x+3}$$

$$y = \frac{1}{5}(x-1)^{-1} + \frac{3}{5}(2x+3)^{-1}$$

we know that the nth derevative of $(ax+b)^{-1}$ is $y_n = \frac{a^n(-1)^n n!}{(ax+b)^{n+1}}$.

Therefore,

$$y_n = \frac{1}{5}\left[\frac{(-1)^n n!\,(1)^n}{(x-1)^{n+1}}\right] + \frac{3}{5}\left[\frac{(-1)^n n!\,(2)^n}{(2x+3)^{n+1}}\right]$$

$$y_n = \frac{(-1)^n n!}{5}\left[\frac{1}{(x-1)^{n+1}} + \frac{3\cdot(2)^n}{(2x+3)^{n+1}}\right]$$

5. Find the nth derivative of $y = \frac{x^4}{(x-1)(x-2)}$.

here, $y = \frac{x^4}{(x-1)(x-2)}$

$$= \frac{x^4}{x^2-3x+2}$$

Now divide x^4 by x^2-3x+2 we get

$$
\begin{array}{r|l}
 & x^2 + 3x + 7 \\
\hline
x^2 - 3x + 2 & x^4 \\
 & x^4 - 3x^3 + 2x^2 \\
 & - \quad + \quad - \\
\hline
 & 3x^3 - 2x^2 \\
 & 3x^3 - 9x^2 + 6x \\
 & - \quad + \quad - \\
\hline
 & 7x^2 - 6x \\
 & 7x^2 - 21x + 14 \\
 & - \quad + \quad - \\
\hline
 & 15x - 14
\end{array}
$$

now by division algorithm we have

$$x^4 = (x^2 - 3x + 2)(x^2 + 3x + 7) + (15x - 14)$$

$$\therefore \frac{x^4}{x^2 - 3x + 2} = (x^2 + 3x + 7) + \frac{(15x - 14)}{x^2 - 3x + 2}$$

$$\therefore \quad y = (x^2 + 3x + 7) + \frac{(15x - 14)}{(x-1)(x-2)} \quad \text{---- (1)}$$

now for $\frac{15x - 14}{(x-1)(x-2)}$ we use method of Partial function,

$$\frac{15x - 14}{(x-1)(x-2)} = \frac{A}{x-1} + \frac{B}{x-2}$$

$$15x - 14 = A(x-2) + B(x-1) \quad \text{--- (2)}$$

for $x = 2$ we get

$30 - 14 = B$

$\boxed{B = 16}$

for $x = 1$ we get

$15 - 14 = A(-1)$

$\boxed{A = -1}$

Now from ① we have

$$y = x^2 + (3x+7) + \left[\frac{-1}{x-1} + \frac{16}{x-2}\right]$$

$$y = x^2 + (3x+7) + \left[(-1)(x-1)^{-1} + 16(x-2)^{-1}\right]$$

$$\therefore y_n = 0 + 0 + \left[\frac{(-1)^n n!(1)^n(-1)}{(x-1)^{n+1}} + \frac{16(-1)^n n!(1)^n}{(x-2)^{n+1}}\right]$$

$$\therefore y_n = (-1)^n n!(1)^n \left[\frac{-1}{(x-1)^{n+1}} + \frac{16}{(x-2)^{n+1}}\right]$$

5. $y = \dfrac{x^n - 1}{x-1}$, $n \in N$, then find y_n.

we have given $y = \dfrac{x^n - 1}{x-1}$

divide $x^n - 1$ by $x - 1$ we get

$$\begin{array}{r|l} & x^{n-1} + x^{n-2} + x^{n-3} + \dots + 1 \\ \hline x-1 & x^n - 1 \\ & x^n - x^{n-1} \\ & - \quad + \\ \hline & x^{n-1} - 1 \\ & x^{n-1} - x^{n-2} \\ & - \quad + \\ \hline & x^{n-2} - 1 \\ & \vdots \\ \hline \end{array}$$

Now by divison algorithm we have

$$(x^n - 1) = (x-1)(x^{n-1} + x^{n-2} + x^{n-3} + \dots + 1)$$

$$\frac{x^n - 1}{(x-1)} = (x^{n-1} + x^{n-2} + x^{n-3} + \dots + 1)$$

$$y = (x^{n-1} + x^{n-2} + x^{n-3} + \dots + 1)$$

$$\boxed{\therefore y_n = 0}$$

* Leibnitz's Rule for the nth derivative of product of two functions.

Statement : If u and v are two functions of x whose n^{th} derivative exist then

* $$(uv)_n = \binom{n}{0} u_n v + \binom{n}{1} u_{n-1} v_1 + \binom{n}{2} u_{n-2} v_2 + \dots + \binom{n}{n} u \cdot v_n$$

* NOTE :

$$\binom{n}{r} = \frac{n!}{r!\,(n-r)!}$$

$$\binom{n}{0} = 1 \qquad \binom{n}{3} = \frac{n(n-2)(n-1)}{6}$$

$$\binom{n}{1} = n \qquad \binom{n}{n} = 1$$

$$\binom{n}{2} = \frac{n(n-1)}{2}$$

1. find the nth derivative of a function $y = x^3 \log(2x+1)$

we are given, $y = x^3 \log(2x+1)$

take $u = \log(2x+1)$ and $v = x^3$

Now by Leibnit's rule,

$$(uv)_n = \binom{n}{0} u_n v + \binom{n}{1} u_{n-1} v_1 + \dots + \binom{n}{n} u v_n \quad \text{--- (1)}$$

therefore,

$$(\log(2x+1) \cdot x^3)_n = (1)\left[\frac{(-1)^{n-1}(n-1)!\,2^n}{(2x+1)^n}\right] \cdot (x^3) +$$

$$(n)\left[\frac{(-1)^{n-2}(n-2)!\,2^{n-1}}{(2x+1)^{n-1}}\right](3x^2) +$$

$$\frac{n(n-1)}{2}\left[\frac{(-1)^{n-3}(n-3)!\,2^{n-2}}{(2x+1)^{n-2}}\right](6x) +$$

$$\frac{n(n-1)(n-2)}{6}\left[\frac{(-1)^{n-4}(n-4)!\,2^{n-3}}{(2x+1)^{n-3}}\right](6) + 0$$

$$= \frac{(-1)^{n-1}(n-1)!\,2^n}{(2x+1)^n}(x^3) + 3n\frac{(-1)^{n-2}(n-2)!\,2^{n-1}}{(2x+1)^{n-1}}(x^2) +$$

$$n(n-1)\frac{(-1)^{n-3}(n-3)!\,2^{n-2}}{(2x+1)^{n-2}}(x) + \frac{n(n-1)(n-2)}{}$$

$$\left[\frac{(-1)^{n-4}(n-4)!\,2^{n-3}}{(2x+1)^{n-3}}\right]$$

ex. If $y = \tan^{-1}x$, then prove that $(1+x^2)\,y_{n+1} + 2nx\,y_n + n(n-1)\,y_{n-1} = 0$.

Here $y = \tan^{-1} x$

differentiating y w.R.T x we get,

$$y_1 = \frac{1}{1+x^2}$$

$\therefore (1+x^2) y_1 = 1$

by Libnitz's rule

now, take $u = y_1$ and $v = 1+x^2$

$$(uv)_n = \binom{n}{0} u_n v + \binom{n}{1} u_{n-1} v_1 + \binom{n}{2} u_{n-2} v_2 + \cdots + \binom{n}{n} u . v_n \quad \text{—(1)}$$

from (1) we have

$$((1+x^2) y_1)_n = (1)_n$$

take $u = y_1$ and $v = 1+x^2$

$\therefore = (1)\, y_{n+1} (1+x^2) + (n)\, y_n (2x) + \frac{n(n-1)}{2} y_{n-1} (2) + 0 = 0$

$= (1+x^2) y_{n+1} + 2(n) x\, y_n + n(n-1) y_{n-1} = 0$

ex. If $y = \sin(\log(x^2+2x+1))$ then prove that $(x+1)^2 y_{n+2} + (2n+1)(x+1) y_{n+1} + (n^2+4) y_n = 0$

→ we are given,

$$y = \sin(\log(x^2+2x+1)).$$
$$= \sin(\log(x+1)^2)$$
$$y = \sin(2\log(x+1)) \quad \text{---} (1)$$

differetiating 1 w.r.t. x we get,

$$y_1 = \cos(2\log(x+1) \times 2 \frac{1}{x+1} \times (1)$$

$$(x+1) y_1 = 2\cos(2\log(x+1))$$

differentiating w.r.t. x we get,

$$(x+1) y_2 + (1) y_1 = -2\sin(2\log(x+1)) \times 2 \times \frac{1}{x+1}$$

$$(x+1)^2 y_2 + (x+1) y_1 = -4\sin(2\log(x+1))$$

$$(x+1)^2 y_2 + (x+1) y_1 = -4y$$

$$(x+1)^2 y_2 + (x+1) y_1 + 4y = 0$$

Now by Leibnitz's rule we have,

$$(uv)_n = \binom{n}{0} u_n v + \binom{n}{1} u_{n-1} v_1 + \dots + \binom{n}{n} u v_n$$

therefore,

$$[(x+1)^2 y_2]_n + [(x+1) y_1]_n + [4y]_n = 0$$

$$[(1) y_{n+2} (x+1)^2 + (n) y_{n+1} (2(x+1)) + \frac{n(n-1)}{2} y_n (2(1)) + 0$$

$+[(1)\, y_{n+1}\,(x+1) + (n)\, y_n\,(1) + 0] + 4\, y_n = 0$

$(x+1)^2 y_{n+2} + 2n\,(x+1)\, y_{n+1} + (n^2-n)\, y_n + (x+1)\, y_{n+1}$
$4 y_n = 0$

$(x+1)^2\, y_{n+2} + [2n+1]\,(x+1)\, y_{n+1} + (n^2+4)\, y_n = 0.$

Exer. If $y = (\sin^{-1}x)^2$ then prove that $[(1-x^2)\, y_{n+2} - (2n+1)$
$x\, y_{n+1} - n^2 y_n = 0.$

differentiating w.r.t. x.

$$y_1 = 2\sin^{-1}x \times \frac{1}{\sqrt{1-x^2}}$$

$(\sqrt{1-x^2})\, y_1 = 2\sin^{-1}x$

$(1-x^2)\, y_1^2 = 4\,(\sin^{-1}x)^2 = 4y$

differentiating w.r.t. x.

$(1-x^2)\, 2y_1 y_2 + (-2x)\, y_1^2 = 4\, y_1$

divide if by $2y_1$

$(1-x^2)\, y_2 - x\, y_1 = 2$

by leibnitz's rule we have

$(1-x^2)\, y_2 - x\, y_1 = 2.$

$$[(1-x^2)y_2]_n - [xy_1]_n = [2]_n$$

$$\therefore \left[(1)\, y_{n+2}\,(1-x^2) + (n)\, y_{n+1}\,(-2x) + \frac{n(n+1)}{2}\, y_n\,(-2) + 0\right]$$

$$- \left[(1)\, y_{n+1}\, x + (n)\, y_n\,(1) + 0\right] = 0$$

$$\therefore (1-x^2)\, y_{n+2} - 2nx\, y_{n+1} - (n^2+n)\, y_n - x\, y_{n+1} - n\, y_n = 0$$

$$\therefore (1-x^2)\, y_{n+2} - (2n+1)\, x\, y_{n+1} - n^2 y_n = 0$$

Excl. If $y = \sqrt{\dfrac{1+x}{1-x}}$ then show that $(1-x^2)\, y_n - [2(n-1)x + 1]\, y_{n-1} - (n-1)(n-2)\, y_{n-2} = 0$

here, $y = \sqrt{\dfrac{1+x}{1-x}} = \left(\dfrac{1+x}{1-x}\right)^{1/2}$

take log both the sides. we get.

$$\log y = \log\left(\frac{1+x}{1-x}\right)^{1/2}$$

$$\log y = \frac{1}{2} \log\left(\frac{1+x}{1-x}\right)$$

$$\log y = \frac{1}{2}\left[\log(1+x) - \log(1-x)\right]$$

diffrentiating w.r.t. x we get

$$\frac{1}{y}\, y_1 = \frac{1}{2}\left[\frac{1}{1+x} - \frac{(-1)}{1-x}\right] = \frac{1}{2}\left[\frac{(1-x)+(1+x)}{(1+x)(1-x)}\right]$$

$$= \frac{1}{2}\left[\frac{2}{(1-x^2)}\right]$$

$$(1-x^2)\, y_1 = y$$

$$(1-x^2)\, y_1 - y = 0$$

Now by leibnitz's rule we have,

$$[(1-x^2)y_1]_n - [y]_n = 0$$

$$\left[(1)\, y_{n+1}(1-x^2) + (n)\, y_n (-2x) + \frac{n(n-1)}{2} y_{n-1}(-2) + 0\right]$$

$$- y_n = 0$$

$$(1-x^2)\, y_{n+1} - 2nx\, y_n - n(n-1)\, y_{n-1} - y_n = 0$$

$$(1-x^2)\, y_{n+1} - (2nx+1)\, y_n - n(n-1)\, y_{n-1} = 0$$

replacing n by $n-1$ we get,

$$(1-x^2)\, y_n - (2(n-1)x+1)\, y_{n-1} - (n-1)(n-2)\, y_{n-2} = 0$$

* Power series :

If $c_0, c_1, c_2, \ldots$ and a are constants, then an infinite series of the form

$$c_0 + c_1(x-a) + c_2(x-a)^2 + c_3(x-a)^3 + \ldots + c_n(x-a)^n + \ldots$$

$$= \sum_{n=0}^{\infty} c_n (x-a)^n \quad \text{--- (1)}$$

is called a power series in $(x-a)$ or at a

If $a=0$ in (1) then we have

... of matrix :-

$$c_0 + c_1 x + c_2 x^2 + c_3 x^3 + \dots + c_n x^n + \dots$$

$$= \sum_{n=0}^{\infty} c_n x^n$$

is called a power series in x or at 0 or around origin.

* Taylor's series :

If $f(x)$ having derivative of all orders at point a then

$$f(x) = f(a) + \frac{(x-a)^1}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots + \frac{(x-a)^3}{3!} f'''(a) + \dots + \frac{(x-a)^n}{n!} f^n(a) + \dots \quad (1)$$

is known as Taylor series of the function f at a.

if $a = 0$ in (1) then it becomes,

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots + \frac{x^n}{n!} f^n(0) + \dots$$

is known as Maclaurin's series of the function f at $a=0$.

ex. if we take $x = a + h$ in taylor series 1 it becomes.

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^n}{n!} f^n(a)$$

which is known as another form of taylor series expantion.

ex. 1 find the taylor series expansion of $x^4 - 2x^2 + 3x + 5$ at point 3.

solution: let $f(x) = x^4 - 2x^2 + 3x + 5$ and $a = 3$.

by taylor series of f at a, we have

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots \quad -①$$

here, $f(x) = x^4 - 2x^2 + 3x + 5$

$f(3) = (3)^4 - 2(3)^2 + 3(3) + 5$

$f(3) = 77$

$f'(x) = 4x^3 - 4x + 3$ $\quad f'(3) = 99$

$f''(x) = 12x^2 - 4$ $\quad f''(3) = 104$

$f'''(x) = 24x$ $\quad f'''(3) = 72$

$f''''(x) = 24$ $\quad f''''(3) = 24$

from ① we have

$$x^4 - 2x^2 + 3x + 5 = 77 + \frac{(x-3)}{1}(99) + (x-3)^2 (104) +$$

$$\frac{(x-3)^3}{6}(72) + \frac{(x-3)^4}{24}(24)$$

$$= 77 + 99(x-3) + 52(x-3)^2 + 12(x-3)^3 + (x-3)^4$$

ex. 2 use the series expansion to find the expansion of $\log_e x = \ln x$ in power of $(x-1)$. using it find the aproximate value of $\log 1.1$.

olution : let $f(x) = \log x$ and $a = 1$.

By Taylor series of f at a, we have

$$f(x) = f(a) + \frac{(x-a)}{1!} f'(a) + \frac{(x-a)^2}{2!} f''(a) + \dots \quad (1)$$

here, $f(x) = \log x$ $\quad f(1) = \log 1 = 0$

$f'(x) = \frac{1}{x}$ $\quad f'(1) = 1$

$f''(x) = \frac{-1}{x^2}$ $\quad f''(1) = -1$

$f'''(x) = \frac{2}{x^2}$ $\quad f'''(1) = 2$

$f^4(x) = \frac{-6}{x^4}$ $\quad f^4(1) = -6$

from (1) we have

$$\log x = 0 + \frac{(x-1)}{1}(1) + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{6}(2) + \frac{(x-1)^4}{24}(-6) + \dots$$

$$\log x = (x-1) - \frac{(x-1)^2}{2} + \frac{(x-1)^3}{3} - \frac{(x-1)^4}{4} + \frac{(x-1)^5}{5} + \dots \quad \text{--- (2)}$$

from ②, take $x = 1.1$ we get

$$\log 1.1 = (0.1) - \frac{(0.1)^2}{2} + \frac{(0.1)^3}{3} - \frac{(0.1)^4}{4} + \dots$$

$$= 0.1 - 0.005 + 0.0003 - 0.000025 + \dots$$

$$= 0.09531$$

ex. 3 expand the function $\log(\tan(\frac{\pi}{4} + x))$ in powers of x using the taylor series.

Solution: here $f(a+h) = \log(\tan(\frac{\pi}{4} + x))$

then the value of $f(x)$ is

$f(x) = \log(\tan x)$ and $a = \frac{\pi}{4}$, $h = x$.

therefore by Taylor series of f at point a, we have,

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \frac{h^3}{3!} f'''(a) + \dots \quad ①$$

here $f(x) = \log(\tan x)$ then $f(\frac{\pi}{4}) = \log(\tan\frac{\pi}{4})$

$$= \log(1) = 0$$

$$f'(x) = \frac{1}{\tan x} \times \sec^2 x = \frac{1 + \tan^2 x}{\tan x} = \cot x + \tan x$$

$$\therefore f'(\frac{\pi}{4}) = 1 + 1 = 2.$$

$$f''(x) = -\operatorname{cosec}^2 x + \sec^2 x$$

$$\therefore f''(\frac{\pi}{4}) = -(\sqrt{2})^2 + (\sqrt{2})^2 = 0$$

$$f'''(x) = -2\operatorname{cosec}x\,(-\operatorname{cosec}x\cdot\cot x) + 2\sec x\,(\sec x\tan x)$$
$$= 2\operatorname{cosec}^2 x\,\cot x + 2\sec^2 x\,\tan x$$
$$f'''\left(\frac{\pi}{4}\right) = 2\,(\sqrt{2})^2\,(1) + 2\,(\sqrt{2})^2(1)$$
$$= 8$$

$\therefore$ from ① we have

$$f\left(\frac{\pi}{4}+x\right) = f\left(\frac{\pi}{4}\right) + \frac{x}{1}f'\left(\frac{\pi}{4}\right) + \frac{x^2}{2}f''\left(\frac{\pi}{4}\right) + \dots$$

$$\log\left(\tan\left(\frac{\pi}{4}+x\right)\right) = 0 + x(2) + 0 + \frac{x^3}{6}(8) + \dots$$
$$= 2x + \frac{4x^3}{3} + \dots$$

ex. 4 find the Maclaurin's series expansion of following function :

(i) e^x

let $f(x) = e^x$ and $a=0$.

by Maclaurin's series of f at $a=0$, we have

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \dots \quad ①$$

here $f(x) = e^x$ $\quad f(0) = e^0 = 1$
$f'(x) = e^x$ $\quad f'(0) = 1$
$f''(x) = e^x$ $\quad f''(0) = 1$
$f'''(x) = e^x$ $\quad f'''(0) = 1$
$\vdots$ $\quad \vdots$

therefore from ① we have

$$e^{x} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots$$

H.W. e^{-x}

(ii) $\sin x$

let $f(x) = \sin x$ and $a = 0$.

by Maclaurin series of f at $a=0$.

we have

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots \quad (1)$$

here, $f(x) = \sin x$ $\quad f(0) = \sin 0 = 0$

$f'(x) = \cos x$ $\quad f'(0) = \cos 0 = 1$

$f''(x) = -\sin x$ $\quad f''(0) = 0$

$f'''(x) = -\cos x$ $\quad f'''(0) = -\cos 0 = -1$

$f^{4}(x) = \sin x$ $\quad f^{4}(0) = 0$

from (1) we have,

$$\sin x = 0 + \frac{x}{1!}(1) + 0 + \frac{x^3}{3!}(-1) + 0 \dots$$

$$\therefore \sin x = \frac{x}{1!} - \frac{x^3}{3!} + \frac{x^5}{5!} - \dots$$

H.W. $\cos x$

(iii) $\sinh x$

let $f(x) = \sinh x$ and $a = 0$

here, $f(x) = \sinh x$ $\quad f(0) = 0$

$f'(x) = \cosh x$ $\quad f'(0) = 1$

$f''(x) = \sinh x$ $\quad f''(0) = 0$

$f'''(x) = \cosh x$ $\quad f'''(0) = 1$

$$\sinh x = \frac{x}{1!} + \frac{x^3}{3!} + \frac{x^5}{5!} + \dots$$

H.W. $\cosh x$

(iv) $\frac{1}{1-x}$

let $f(x) = \frac{1}{1-x}$ and $a = 0$

here, $f(x) = \frac{1}{1-x}$ $\quad f(0) = 1$

$f'(x) = \frac{-1}{(1-x)^2}(-1)$ $\quad f'(0) = 1$

$= \frac{1}{(1-x)^2}$

$f''(x) = \frac{-2}{(1-x)^3} \cdot (-1)$ $\quad f''(0) = 2$

$= \frac{2}{(1-x)^3}$

$f'''(x) = \frac{6}{(1-x)^4}$ $\quad f'''(0) = 6$

$$\frac{1}{1-x} = 1 + \frac{x}{1!}(1) + \frac{x^2}{2!}(2) + \frac{x^3}{6}(6) + \dots$$

$$\frac{1}{1-x} = 1 + x + x^2 + x^3 + \dots$$

H.W. $f(x) = \log(1+x)$

(v) e^{-x}

let $f(x) = e^{-x}$ and $a = 0$

by Maclaurin's series of f at $a = 0$, we have

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots \quad (1)$$

here $f(x) = e^{-x}$ $\quad f(0) = e^{-0} = 1$

$f'(x) = -e^{-x}$ $\quad f'(0) = -1$

$f''(x) = e^{-x}$ $\quad f''(0) = 1$

$f'''(x) = -e^{-x}$ $\quad f'''(0) = -1$

$f^4(x) = e^{-x}$ $\quad f^4(0) = 1$

$\vdots$ $\quad \vdots$

therefore from (1) we have

$$e^{-x} = 1 + \frac{x}{1}(-1) + \frac{x^2}{2}(1) + \frac{x^3}{6}(-1) + \dots$$

$$e^{-x} = 1 - \frac{x}{1!} + \frac{x^2}{2!} - \frac{x^3}{3!} + \dots$$

(vi) $\cos x$

let $f(x) = \cos x$ and $a = 0$.

by Maclaurin's series of f at $a=0$, we have

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots \quad (1)$$

here $f(x) = \cos x$ $\quad f(0) = \cos 0 = 1$

$f'(x) = -\sin x$ $\quad f'(0) = -\sin 0 = 0$

$f''(x) = -\cos x$ $\quad f''(0) = -\cos 0 = -1$

$f'''(x) = +\sin x$ $\quad f'''(0) = \sin 0 = 0$

$f^4(x) = \cos x$ $\quad f^4(0) = \cos 0 = 1$

$\vdots$ $\quad \vdots$

therefore from (1) we have

$$\cos x = 1 + 0 + \frac{x^2}{2!}(-1) + \frac{x^3}{3!}(0) + \frac{x^4}{4!}(1) + \dots$$

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \dots$$

(vii) $\cos hx$

let $f(x) = \cos hx$ and $a=0$.

by Maclaurin's series of f at $a=0$, we have

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots \quad (1)$$

here, $f(x) = \cos hx$ $\quad f(0) = \cos 0 = 1$

$f'(x) = +\sin hx$ $\quad f'(0) = \sin 0 = 0$

$f''(x) = \cos hx$ $\quad f''(0) = \cos 0 = 1$

$f'''(x) = \sinh x \qquad f'''(0) = \sin 0 = 0$

$f^{4}(x) = \cosh x \qquad f^{4}(0) = \cos 0 = 1$

$\vdots \qquad\qquad \vdots$

therefore from ① we have

$$\cosh x = 1 + \frac{x}{1!}(0) + \frac{x^2}{2!}(1) + \frac{x^3}{3!}(0) + \dots$$

$$\cosh x = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \dots$$

(viii) $f(x) = \log(1+x)$

let $f(x) = \log(1+x)$ and $a = 0$.

by Maclaurin's series of f at $a=0$, we have

$$f(x) = f(0) + \frac{x}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots$$

here, $f(x) = \log(1+x) \qquad f(0) = \log 1 = 0$

$f'(x) = \dfrac{1}{1+x} \qquad f'(0) = 1$

$f''(x) = \dfrac{-1}{(1+x)^2} \qquad f''(0) = \dfrac{-1}{1} = -1$

$f'''(x) = \dfrac{2}{(1+x)^3} \qquad f'''(0) = 2$

$f^{4}(x) = \dfrac{-6}{(1+x)^4} \qquad f^{4}(0) = -6$

$\vdots \qquad\qquad \vdots$

therefore from ① we have

$$\log(1+x) = 0 + \frac{x}{1}(1) + \frac{x^2}{2}(-1) + \frac{x^3}{6}(2) + \frac{x^4}{24}(-6) + \dots$$

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$