Poincaré inequality on convex domains: the optimal constant

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Outline

1. Weighted Korn inequality.
2. What type of decompositions are we interested in?
3. Decomposition $\Rightarrow$ Korn.
4. A certain covering of $\Omega \Rightarrow$ Decomposition.
5. Find an appropriate covering for John domains.
6. Another application: Solvability of divergence problem.
7. Current work: Generalized Korn inequality.

Korn inequality

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain, with $n \geq 2$, and $1 < p < \infty$.

Sobolev space $W^{1,p}(\Omega) := \{u \in L^p(\Omega) : \frac{\partial u}{\partial x_j} \in L^p(\Omega) \text{ for all } 1 \leq j \leq n\}$
We denote by $D(\mathbf{u})$ the differential matrix of $\mathbf{u}$, by $\varepsilon(\mathbf{u})$ the symmetric part of $D(\mathbf{u})$

$$\varepsilon_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial \mathbf{u}_i}{\partial x_j} + \frac{\partial \mathbf{u}_j}{\partial x_i}\right)$$

and by $\eta(\mathbf{u})$ the skew-symmetric part of $D(\mathbf{u})$

$$\eta_{ij}(\mathbf{u}) = \frac{1}{2}\left(\frac{\partial \mathbf{u}_i}{\partial x_j} - \frac{\partial \mathbf{u}_j}{\partial x_i}\right)$$

Korn inequality

The classical Korn inequality states (Korn 1906, 1909):

$$\|D\mathbf{u}\|_{L^p(\Omega)} \leq C\|\varepsilon(\mathbf{u})\|_{L^p(\Omega)},$$

for any $\mathbf{u} \in W^{1,p}(\Omega)^n$, with $\int_\Omega \frac{\partial \mathbf{u}_i}{\partial x_j} - \frac{\partial \mathbf{u}_j}{\partial x_i} = 0$.

Equivalently,

$$\inf_{\varepsilon(\mathbf{w})=0} \|D\mathbf{v} - D\mathbf{w}\|_{L^p(\Omega)} \leq C\|\varepsilon(\mathbf{v})\|_{L^p(\Omega)}$$

for all $\mathbf{v} \in W^{1,p}(\Omega)^n$.

Note: $\varepsilon(\mathbf{w}) = 0$ iff $\mathbf{w}(x) = Ax + b$, where $A \in \mathbb{R}^n$ is skew-symmetric and $b \in \mathbb{R}^n$.
Note: This inequality is basic on linear elasticity equations. $\mathbf{u}$ plays the role of the displacement of an elastic body, $\varepsilon(\mathbf{u})$ is called the linearized strain tensor and $\mathbf{w}$ with $\varepsilon(\mathbf{w}) = 0$ is the infinitesimal rigid motions.

Weighted Korn inequality

Let $\omega : \Omega \to \mathbb{R}_{>0}$ be a weight, with $\int_\Omega \omega^p < \infty$. Example: $\omega(x) = \rho(x)^\beta$ where ρ is the distance to $\partial\Omega$ and $\beta \geq 0$.
We consider $L^p(\Omega, \omega)$ equipped with the norm

$$\|u\|_{L^p(\Omega,\omega)} := \left(\int_\Omega |u(x)|^p \omega^p(x) \, \mathrm{d}x \right)^{1/p}$$

Weighted Korn inequality:

$$\|D\mathbf{u}\|_{L^p(\Omega,\omega)} \leq C \|\varepsilon(\mathbf{u})\|_{L^p(\Omega,\omega)},$$

for any $\mathbf{u} \in W^{1,p}(\Omega, \omega)^n$, with $\int_\Omega \eta_{ij}(\mathbf{u})\omega^p = 0$ for $1 \leq i < j \leq n$.

Decomposition of integrable functions

Covering of Ω: Collection of subdomains $\{\Omega_t\}_{t\in\Gamma}$ s.t. $\Omega = \bigcup_{t\in\Gamma}\Omega_t$ with $\sum_t \chi_{\Omega_t} \leq N\chi_\Omega$ a.e..

Take $1 < q < \infty$ with $\frac{1}{p} + \frac{1}{q} = 1$.

Given $g \in L^q(\Omega, \omega^{-1}) \subset L^1(\Omega)$ with $\int_\Omega g = 0$, a $\mathcal{P}_0$-orthogonal decomposition subordinate to $\{\Omega_t\}_{t\in\Gamma}$ is a collection of functions $\{g_t\}_{t\in\Gamma}$ s.t.

1. $g = \sum_{t\in\Gamma} g_t$
2. $supp(g_t) \subset \Omega_t$
3. $\int_{\Omega_t} g_t = 0$

Continuity property:

$$\sum_{t\in\Gamma} \|g_t\|^q_{L^q(\Omega_t,\omega^{-1})} \leq C_0{}^q \|g\|^q_{L^q(\Omega,\omega^{-1})},$$

$\mathcal{P}_0$-decomposition + Cont. + Local weighted Korn $\Rightarrow$ Weighted Korn inequality

$V_0(\Omega, \omega^{-1}) := \{g \in L^q(\Omega, \omega^{-1}) \, : \, \int_\Omega g = 0\}$.

Theorem

$\{\Omega_t\}_{t\in\Gamma}$ is a covering of Ω s.t. weighted Korn ineq. is valid in Ω_t with uniform constant C_1. Moreover, for any g in V_0 there is a $\mathcal{P}_0$-decomposition with $\sum_{t\in\Gamma} \|g_t\|^q_{L^q(\Omega_t,\omega^{-1})} \leq C_0^q \|g\|^q_{L^q(\Omega,\omega^{-1})}$. Then,

$$\|D\mathbf{u}\|_{L^p(\Omega,\omega)} \leq C\|\varepsilon(\mathbf{u})\|_{L^p(\Omega,\omega)}$$

for any $\mathbf{u} \in W^{1,p}(\Omega,\omega)^n$, with $\int_\Omega \eta_{ij}(\mathbf{u})\,\omega^p = 0$ for $1 \leq i < j \leq n$. In addition,

$$C = 1 + 2n^{2/p} N^{1/p}\, C_0\, C_1$$

$L^q(\Omega, \omega^{-1}) = \{g + \omega^p\psi \, : \, g \in V_0 \text{ and } \psi \text{ is constant}\}$. Moreover, $\|g\|_{L^q(\Omega,\omega^{-1})} \leq 2\|g + \omega^p\psi\|_{L^q(\Omega,\omega^{-1})}$

$\mathcal{P}_0$-decomposition + Cont. + Local weighted Korn $\Rightarrow$ Weighted Korn inequality

Proof: Let us take the sup over $\|g+\omega^p\psi\|_{L^q(\Omega,\omega^{-1})} \le 1$.

$$
\begin{aligned}
\int_\Omega \eta_{ij}(\mathbf{u})(g+\omega^p\psi) &= \int_\Omega \eta_{ij}(\mathbf{u}) g = \int_\Omega \eta_{ij}(\mathbf{u}) \sum_{t\in\Gamma} g_t \\
&= \sum_{t\in\Gamma} \int_{\Omega_t} \eta_{ij}(\mathbf{u}) g_t = \sum_{t\in\Gamma} \int_{\Omega_t} (\eta_{ij}(\mathbf{u})-\alpha) g_t \\
&\le \sum_{t\in\Gamma} \inf_{\alpha\in\mathcal{P}_0} \|(\eta_{ij}(\mathbf{u})-\alpha)\|_{L^p(\Omega_t,\omega)} \|g_t\|_{L^q(\Omega_t,\omega^{-1})} \\
&\le \sum_{t\in\Gamma} C_1 \|\varepsilon(\mathbf{u})\|_{L^p(\Omega_t,\omega)} \|g_t\|_{L^q(\Omega_t,\omega^{-1})} \\
&\le C_1 \left(\sum_{t\in\Gamma} \int_{\Omega_t} |\varepsilon(\mathbf{u})|^p \omega^p \right)^{1/p} \left(\sum_{t\in\Gamma} \|g_t\|^q_{L^q(\Omega_t,\omega^{-1})} \right)^{1/q} \\
&\le C_1 N^{1/p}\, C_0 \, \|\varepsilon(\mathbf{u})\|_{L^p(\Omega,\omega)} \|g\|_{L^q(\Omega,\omega^{-1})} \\
&\le 2C_1 N^{1/p}\, C_0 \, \|\varepsilon(\mathbf{u})\|_{L^p(\Omega,\omega)}.
\end{aligned}
$$

$\mathcal{P}_0$-decomposition and tree coverings

A rooted tree (or simply a tree) is a connected graph $G = (\Gamma, E)$ in which any vertex t is connected to a distinguished vertex a (the root) by exactly one path.

It is possible to define an order "$\preceq$" in Γ by $s \preceq t$ if and only if the the path connecting t with the root a passes through s.

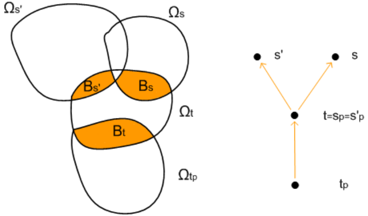
The parent of a vertex $t \in \Gamma$ is the adjacent vertex t_p that satisfies $t_p \preceq t$.

Given $\Omega = \bigcup_{t \in \Gamma} \Omega_t$, we are interested in finding an appropriate tree structure on Γ "consistent" with the geometry of Ω.

$\mathcal{P}_0$-decomposition and tree coverings

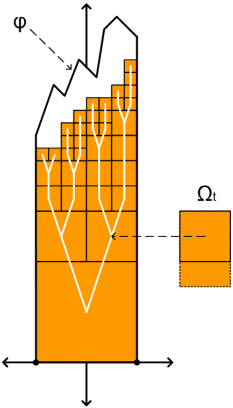
A covering $\{\Omega_t\}_{t\in\Gamma}$ of Ω is a tree covering if Γ is the set of vertices of a tree, with root a, that verifies that for any $t \in \Gamma$, with $t \neq a$, there exists a open cube $B_t \subseteq \Omega_t \cap \Omega_{t_p}$ such that the collection $\{B_t\}_{t\neq a}$ is pairwise disjoint.

Example 1:



$\mathcal{P}_0$-decomposition and tree coverings

Example 2: φ is a Hölder-α function (L'14), $|\varphi(x) = \varphi(y)| \leq C|x - y|^{\alpha}$.



$\mathcal{P}_0$-decomposition and tree coverings

Given a tree covering $\{\Omega_t\}_{t\in\Gamma}$ of Ω we define a Hardy type operator in the following way:

$$Tg(x) := \sum_{a\neq t\in\Gamma} \frac{\chi_t(x)}{|W_t|} \int_{W_t} |g|,$$

where $W_t = \bigcup_{s\succeq t} \Omega_s$ (the shadow of Ω_t) and χ_t is the characteristic function of B_t for all $t \neq a$.

$\mathcal{P}_0$-decomposition and tree coverings

Theorem (L'15)

Let $\Omega \subset \mathbb{R}^n$ be a bounded domain with tree covering $\{\Omega_t\}_{t\in\Gamma}$. Given $g \in L^1(\Omega)$, with $\int_\Omega g = 0$, there exists a $\mathcal{P}_0$-decomposition $\{g_t\}_{t\in\Gamma}$ of g such that

$$|g_t(x)| \leq |g(x)| + \frac{|W_t|}{|B_t|} Tg(x).$$

Now, if $\frac{|W_t|}{|B_t|} \leq M$ for all $t \neq a$, then

$$\begin{aligned}\sum_{t\in\Gamma} \|g_t\|^q_{L^q(\Omega_t,\omega^{-1})} &= \sum_{t\in\Gamma} \int_{\Omega_t} |g_t|^q \omega^{-q} \\ &\leq 2^{q-1} N \left(\|g\|^q_{L^q(\Omega,\omega^{-1})} + M^q \|Tg\|^q_{L^q(\Omega,\omega^{-1})} \right)\end{aligned}$$

$\mathcal{P}_0$-decomposition and tree coverings

Lemma ('15)

$T : L^q(\Omega) \to L^q(\Omega)$ is continuous for $1 < q \leq \infty$.

Proof: T is an average of f or zero, thus $\|T\|_{L^\infty \to L^\infty} \leq 1$.

T is weak $(1,1)$ continuous with norm lesser than or equal to N:
Given $\lambda > 0$, we define the subset of minimal vertices $\Gamma_0 \subseteq \Gamma$ as

$$\Gamma_0 := \left\{ t \in \Gamma \, : \, \frac{1}{|W_t|} \int_{W_t} |f| > \lambda \text{ and } \frac{1}{|W_s|} \int_{W_s} |f| \leq \lambda \text{ for all } s \prec t \right\}.$$

$$\begin{aligned} \text{Thus, } \quad |\{x \in \Omega \, : \, Tf(x) > \lambda\}| \quad &\leq \quad \sum_{t \in \Gamma_0} |W_t| \\ &< \quad \frac{1}{\lambda} \sum_{t \in \Gamma_0} \int_{W_t} |f| \leq \frac{N}{\lambda} \|f\|_{L^1(\Omega)}. \end{aligned}$$

By Marcinkiewicz interpolation, $\|T\|_{L^q \to L^q} \leq 2 \left(\dfrac{qN}{q-1} \right)^{1/q}$.

Finding an appropriate covering for John domains

Ω is called a John domain with parameter $\beta > 1$ if there exists a point $x_0 \in \Omega$ such that every $y \in \Omega$ has a rectifiable curve parameterized by arc length $\gamma : [0, l] \to \Omega$ such that $\gamma(0) = y$, $\gamma(l) = x_0$ and

$$\mathrm{dist}(\gamma(t), \partial\Omega) \geq \frac{1}{\beta} t$$

for all $t \in [0, l]$. Introduced by F. John '61. Named after him by Martio and Sarvas '79.

Examples: Convex, star-shaped domains with respect to a ball, Lipschitz, Kock snowflake.

Finding an appropriate covering for John domains

A Whitney decomposition of Ω is a collection $\{Q_t\}_{t\in\Gamma}$ of closed dyadic cubes whose interiors are pairwise disjoint, which verifies

1. $\Omega = \bigcup_{t\in\Gamma} Q_t$,
2. $\mathrm{diam}(Q_t) \leq \rho(Q_t, \partial\Omega) \leq 4\mathrm{diam}(Q_t)$,
3. $\frac{1}{4}\mathrm{diam}(Q_s) \leq \mathrm{diam}(Q_t) \leq 4\mathrm{diam}(Q_s)$, if $Q_s \cap Q_t \neq \emptyset$.

A domain Ω satisfies the Boman chain condition if for any cube Q_t in a Whitney decomp. there is a chain $Q_{t,0}$, $Q_{t,1}$, $\cdots$, $Q_{t,\kappa}$ s.t. $Q_{t,0} = Q_t$, $Q_{t,\kappa} = Q_a$ (a distinguished cube) and

$$Q_{t,i} \subset \lambda Q_{t,j},$$

for all $1 \leq i \leq j \leq \kappa$. The length of the chain depends on Q_t (Boman '82).

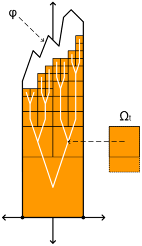
Finding an appropriate covering for John domains

Buckley, Koskela and Lu '96: Boman chain condition characterizes John domains.

Boman tree condition: Given a bounded John domain $\Omega \subset \mathbb{R}^n$, there exists a Whitney decomposition $\{Q_t\}_{t\in\Gamma}$ where Γ has a tree structure that satisfies

$$Q_s \subseteq KQ_t$$

for any $s, t \in \Gamma$, with $s \succeq t$. We use some ideas from Vasil'eva '13.



Finding an appropriate covering for John domains

Then, there exists a tree covering $\{\Omega_t\}_{t\in\Gamma}$ of Ω such that

$$|W_t| = |\bigcup_{s\succeq t} \Omega_s| \leq K^n |\Omega_t| \leq K^n c_n |B_t|.$$

Moreover, using that $\beta \geq 0$, we have $T : L^q(\Omega, \rho^{-\beta}) \to L^q(\Omega, \rho^{-\beta})$ is continuous with norm $\|T\| \leq C_{n,q,\beta} K^\beta$.

Thus, for any $g \in L^q(\Omega, \rho^{-\beta})$, with $\int_\Omega g = 0$, there exists a $\mathcal{P}_0$-decomposition $\{g_t\}_{t\in\Gamma}$ of g such that

$$|g_t(x)| \leq |g(x)| + \frac{|W_t|}{|B_t|} Tg(x) \leq |g(x)| + c_n K^n Tg(x)$$

Which implies

$$\sum_{t\in\Gamma} \|g_t\|^q_{L^q(\Omega_t,\rho^{-\beta})} \leq C_{n,q,\beta} K^{q(n+\beta)} \|g\|^q_{L^q(\Omega,\rho^{-\beta})}$$

Main Theorem on John domains

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded John domain with $n \geq 2$, $1 < p < \infty$ and $\beta \in \mathbb{R}_{\geq 0}$. Then, there exists a constant C depending only on n, p and βsuch that

$$\left(\int_\Omega |D\mathbf{u}|^p \rho^{p\beta} \,\mathrm{d}x \right)^{1/p} \leq C\, K^{n+\beta} \left(\int_\Omega |\varepsilon(\mathbf{u})|^p \rho^{p\beta} \,\mathrm{d}x \right)^{1/p}$$

for all vector field $\mathbf{u} \in W^{1,p}(\Omega, \rho^\beta)^n$ that satisfies that $\int_\Omega \eta_{ij}(\mathbf{u})\, \rho^{\beta p} = 0$, for $1 \leq i < j \leq n$. The function $\rho(x)$ is the distance to the boundary of Ω. The constant K appears in the Boman tree condition.

$\mathcal{P}_0$-decomposition + Cont. + Local weighted solutions $\Rightarrow$ Weighted solutions of $\operatorname{div}\mathbf{u}$=g

$W^{1,q}(\Omega) := \left\{ v \in L^q(\Omega) \,:\, \frac{\partial v}{\partial x_i} \in L^q(\Omega) \text{ for all } i \right\}.$

$W_0^{1,q}(\Omega) := \overline{C_0^\infty(\Omega)} \subset W^{1,q}(\Omega).$

Given $g \in L_0^q(\Omega) = \left\{ g \in L^p(\Omega) \,:\, \int_\Omega g = 0 \right\}$ there exists $\mathbf{u} \in W_0^{1,q}(\Omega)^n$ such that

$$\operatorname{div}\mathbf{u} = g$$

$$\|D\mathbf{u}\|_{L^q(\Omega)} \leq C \|g\|_{L^q(\Omega)},$$

where C depends only on Ω and q.

Note: The existence of a solution of $\operatorname{div}\mathbf{u} = g$ is basic on the variational analysis of the Stokes equations.

$\mathcal{P}_0$-decomposition + Cont. + Local weighted solutions $\Rightarrow$ Weighted solutions of $\operatorname{div}\mathbf{u}$=g

Theorem

Let $\Omega \subset \mathbb{R}^n$ be a bounded John domain with $n \geq 2$, $1 < q < \infty$ and $\beta \geq 0$. Given $g \in L^q(\Omega, \rho^{-\beta})$, with $\int_\Omega g = 0$, there exists a solution $\mathbf{u} \in W_0^{1,q}(\Omega, \rho^{-\beta})^n$ of $\operatorname{div}\mathbf{u} = g$ that satisfies

$$\|D\mathbf{u}\|_{L^q(\Omega,\rho^{-\beta})} \leq C_{n,q,\beta} K^{n+\beta} \|g\|_{L^q(\Omega,\rho^{-\beta})},$$

where $\rho(x)$ is the distance to $\partial\Omega$ and K appears in the Boman tree condition.

$\mathcal{P}_0$-decomposition + Cont. + Local weighted solutions $\Rightarrow$ Weighted solutions of $\operatorname{div}\mathbf{u}$=g

Proof: Given $g \in L^q(\Omega, \rho^{-\beta}) \subset L^1(\Omega)$, with $\int g = 0$, we take a $\mathcal{P}_0$-decomposition of g (i.e. $g = \sum_t g_t$, $\text{supp}(g_t) \subset \Omega_t$ and $\int_{\Omega_t} g_t = 0$) and

$$\sum_{t\in\Gamma} \|g_t\|^q_{L^q(\Omega_t,\rho^{-\beta})} \leq C_0^q \|g\|^q_{L^q(\Omega,\rho^{-\beta})}.$$

Next, we take a solution $\mathbf{u}_t \in W_0^{1,q}(\Omega_t, \rho^{-\beta})^n$ of $\operatorname{div}\mathbf{u}_t = g_t$ in Ω_t with

$$\|D\mathbf{u}_t\|^q_{L^q(\Omega_t,\rho^{-\beta})} \leq C_1^q \|g_t\|^q_{L^q(\Omega_t,\rho^{-\beta})},$$

where C_1 is independent of t.
Then, $\mathbf{u}_t \in W_0^{1,q}(\Omega, \rho^{-\beta})^n$ by extending by zero. Thus, $\mathbf{u} = \sum_{t\in\Gamma} \mathbf{u}_t$ is a solution of $\operatorname{div}\mathbf{u} = g$ with

$$\|D\mathbf{u}\|^q_{L^q(\Omega,\rho^{-\beta})} \leq C \sum_{t\in\Gamma} \|D\mathbf{u}_t\|^q_{L^q(\Omega_t,\rho^{-\beta})} \leq C \sum_{t\in\Gamma} \|g_t\|^q_{L^q(\Omega_t,\rho^{-\beta})} \leq C\|g\|^q_{L^q(\Omega,\rho^{-\beta})}$$

Question: Upper bounds of the constants on star-shaped domains

A domain $\Omega \subset \mathbb{R}^n$ with diameter R is a star-shaped domain with respect to a ball $B := B(x_0, \rho)$ if the segment that connects any two arbitrary points $x \in B$ and $y \in \Omega$ is included in Ω.

The estimate of the constant C_Ω in the problem $\operatorname{div} \mathbf{u} = g$, where $\mathbf{u} \in W_0^{1,2}(\Omega)^n$, with

$$\|D\mathbf{u}\|_{L^2(\Omega)} \leq C_\Omega \|g\|_{L^2(\Omega)}.$$

Bogovski '79: Solvability on star-shaped domains.

Can we estimate the constant in terms of the ratio $\dfrac{R}{\rho}$?

Question: Upper bounds of the constants on star-shaped domains

Galdi's book '94: $C_\Omega \leq C_n \left(\frac{R}{\rho}\right)^{n+1}$.

Durán '12: $C_\Omega \leq C_n \frac{R}{\rho}\left(\frac{|\Omega|}{|B|}\right)^{\frac{n-2}{2(n-1)}}\left(\log\frac{|\Omega|}{|B|}\right)^{\frac{n}{2(n-1)}}$.

In particular, if $n=2$: $C_\Omega \leq C_n \frac{R}{\rho}\log\frac{|\Omega|}{|B|}$.

Costabel-Dauge '15: For $n=2$, $C_\Omega = C_n \frac{R}{\rho}$, which is optimal.

Question: Upper bounds of the constants on star-shaped domains

Question: Given a star-shaped domain $\Omega \subset \mathbb{R}^n$ as in the previous definition, is there a Whitney decomposition $\Omega = \bigcup_{t\in\Gamma} \Omega_t$, where Γ is a rooted tree, and a real number $m \geq 1$ such that

$$\frac{\left|\bigcup_{s\succeq t} \Omega_s\right|}{|\Omega_t|} \leq C_n \left(\frac{R}{\rho}\right)^m$$

for all $t \in \Gamma$?

If we have a positive answer to this question, then

$$C_\Omega \leq C_n \left(\frac{R}{\rho}\right)^m.$$

What is the infimum the those m's?

Current work: Generalized Korn inequality

Given $\Omega \subset \mathbb{R}^n$ with $n \geq 3$, generalized Korn inequality states:

$$\inf_{I(\mathbf{w})=0} \|D\mathbf{v} - D\mathbf{w}\|_{L^p(\Omega)} \leq C\|I(\mathbf{v})\|_{L^p(\Omega)}$$

for all $\mathbf{v} \in W^{1,p}(\Omega)^n$, where the operator I is the trace free part of ε. Indeed,

$$I(\mathbf{u}) := \varepsilon(\mathbf{u}) - \frac{\operatorname{div}\mathbf{u}}{n} I_n$$

$I(\mathbf{w}) = 0$ if and only if

$$\mathbf{w}(x) = a + Ax + \lambda x + \left\{ \langle b, x\rangle x - \frac{1}{2}|x|^2 b \right\},$$

where $a, b \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and $A \in \mathbb{R}^{n\times n}$ is skew-symmetric.

Note: We have to consider V-orthogonal decomposition for another vector space V different from $\mathcal{P}_0$.

Thanks for your attention!