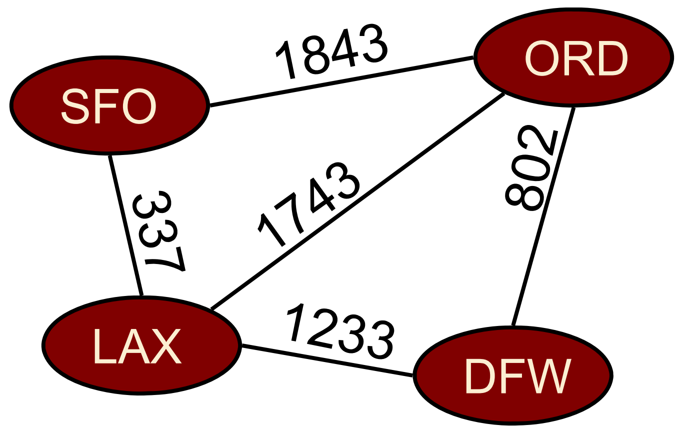
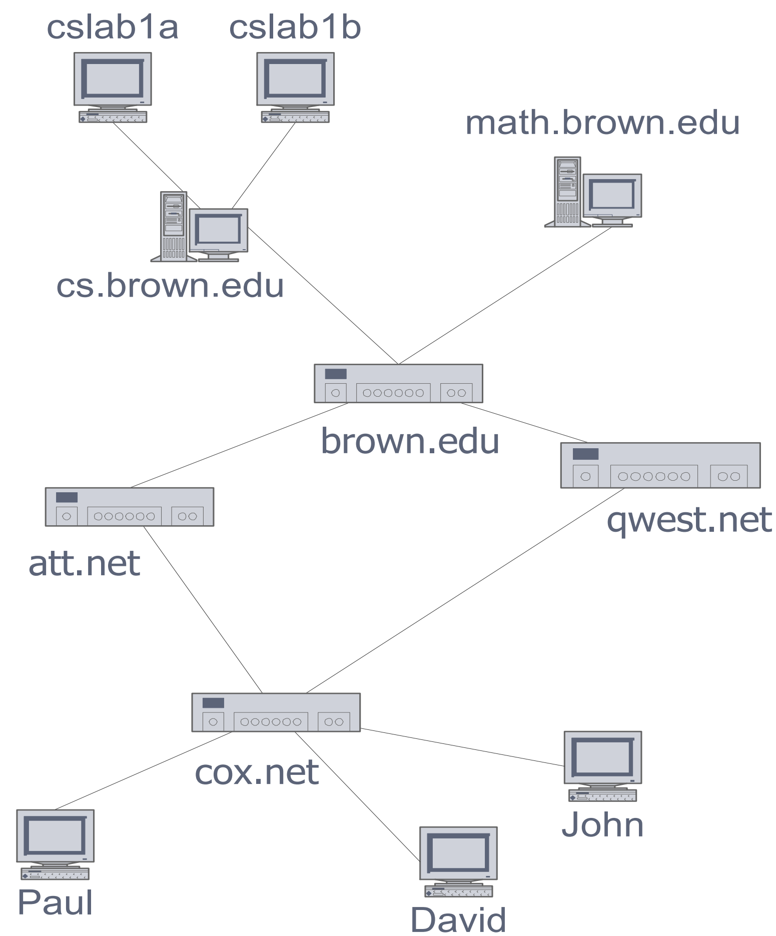
Graphs – ADTs and Implementations



Applications of Graphs

- Electronic circuits
  - Printed circuit board
  - Integrated circuit
- Transportation networks
  - Highway network
  - Flight network
- Computer networks
  - Local area network
  - Internet
  - Web
- Databases
  - Entity-relationship diagram



Outline

- Definitions
- Graph ADT
- Implementations

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- **Definitions**
- Graph ADT
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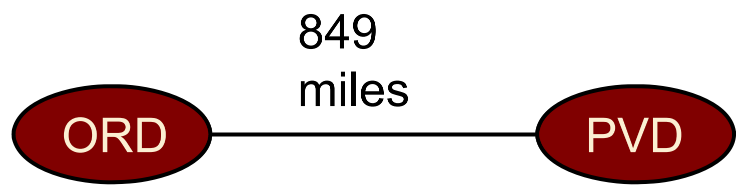
Edge Types

- Directed edge
  - ordered pair of vertices (u,v)
  - first vertex u is the origin
  - second vertex v is the destination
  - e.g., a flight

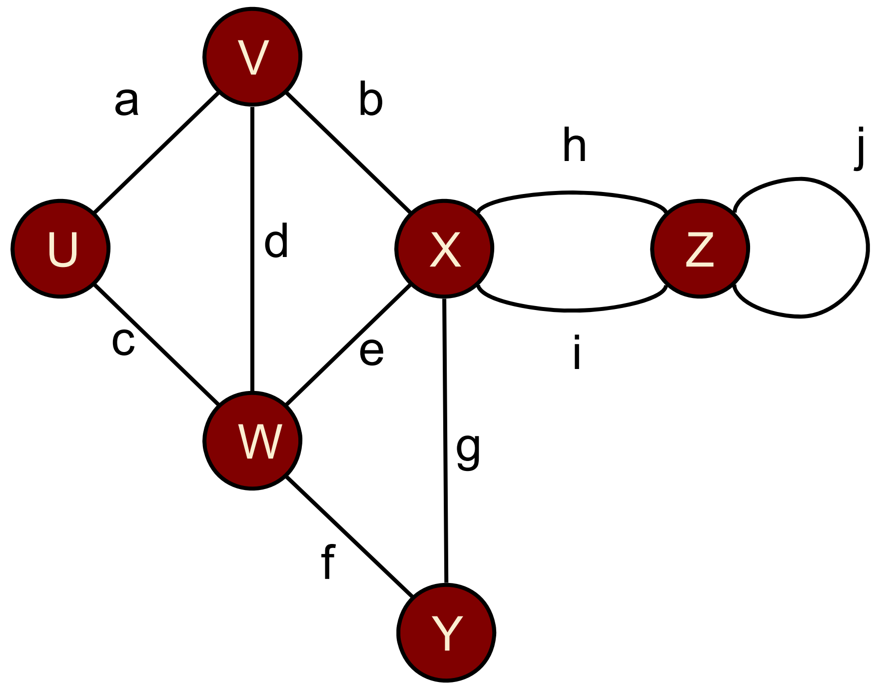
- Undirected edge
  - unordered pair of vertices (u,v)
  - e.g., a flight route



- Directed graph (Digraph)
  - all the edges are directed
  - e.g., route network
- Undirected graph
  - all the edges are undirected
  - e.g., flight network

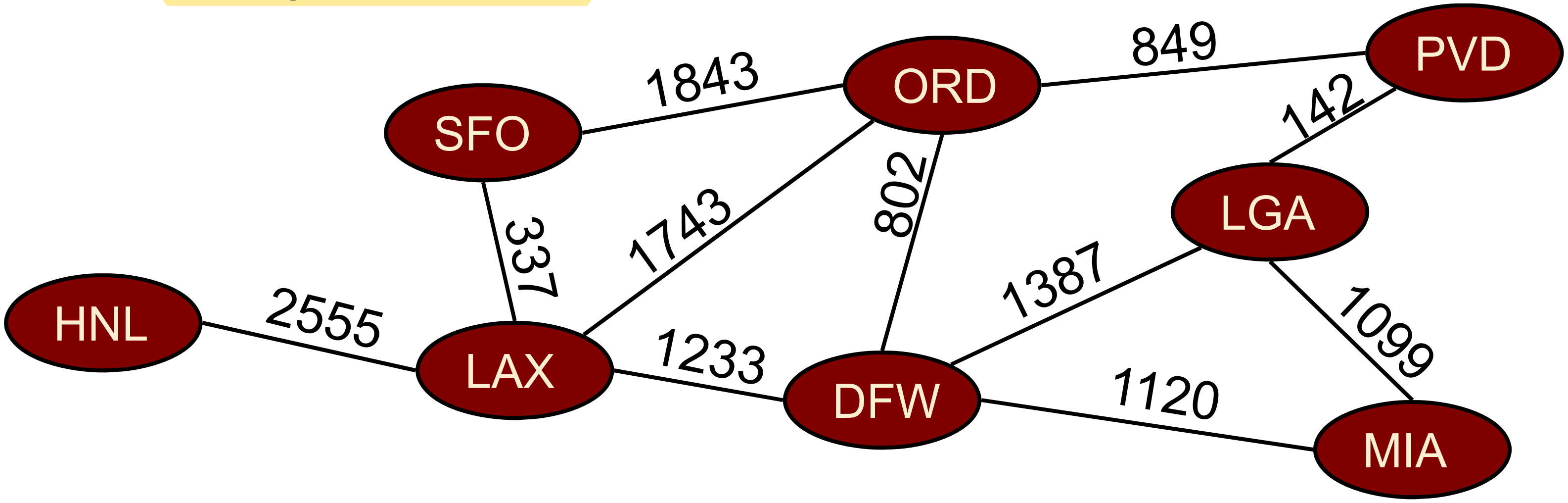
Vertices and Edges

- End vertices (or endpoints) of an edge
  - U and V are the endpoints of a
- Edges incident on a vertex
  - a, d, and b are incident on V
- Adjacent vertices
  - U and V are adjacent
- Degree of a vertex
  - X has degree 5
- Parallel edges
  - h and i are parallel edges
- Self-loop
  - j is a self-loop



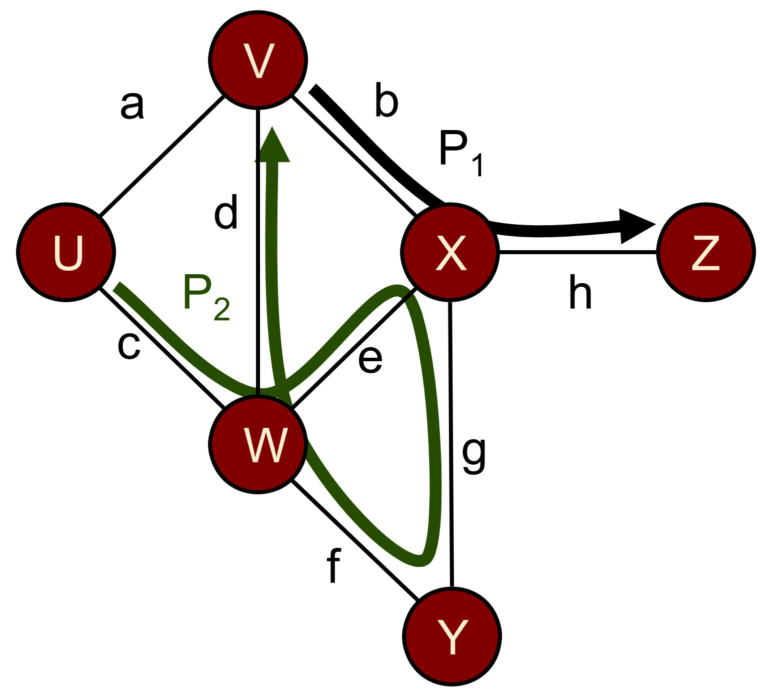
Graphs

- A graph is a pair (V, E), where
  - V is a set of nodes, called vertices
  - E is a collection of pairs of vertices, called edges
  - Vertices and edges are positions and store elements
- Example:
  - A vertex represents an airport and stores the three-letter airport code
  - An edge represents a flight route between two airports and stores the mileage of the route



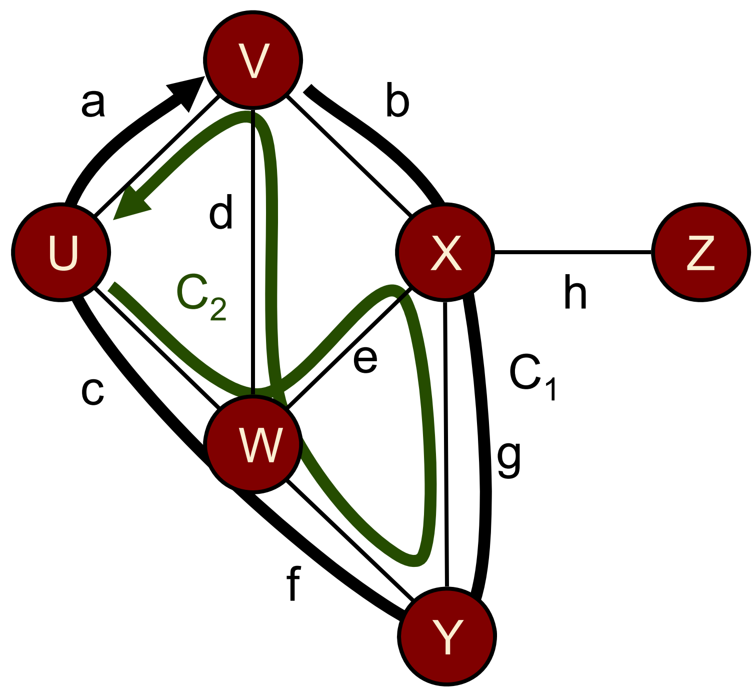
Paths

- Path
  - sequence of alternating vertices and edges
  - begins with a vertex
  - ends with a vertex
  - each edge is preceded and followed by its endpoints
- Simple path
  - path such that all its vertices and edges are distinct
- Examples
  - P_1=(V,b,X,h,Z) is a simple path
  - P_2=(U,c,W,e,X,g,Y,f,W,d,V) is a path that is not simple



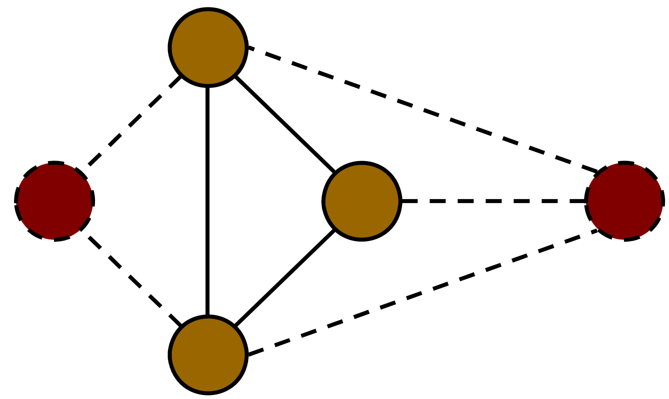
Cycles

- Cycle
  - circular sequence of alternating vertices and edges
  - each edge is preceded and followed by its endpoints
- Simple cycle
  - cycle such that all its vertices and edges are distinct
- Examples
  - C_1=(V,b,X,g,Y,f,W,c,U,a,V) is a simple cycle
  - C_2=(U,c,W,e,X,g,Y,f,W,d,V,a,U) is a cycle that is not simple

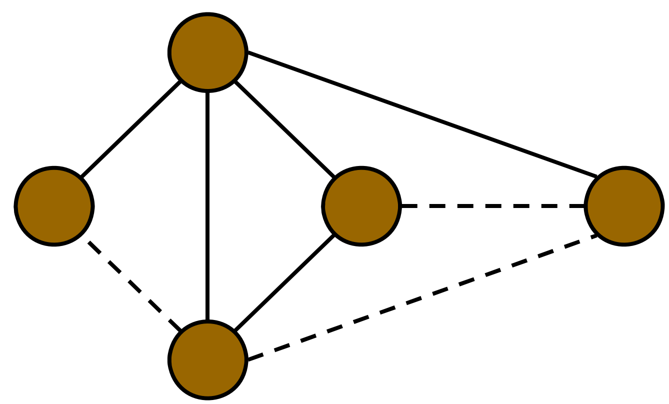


Subgraphs

- A subgraph S of a graph G is a graph such that
  - The vertices of S are a subset of the vertices of G
  - The edges of S are a subset of the edges of G
- A spanning subgraph of G is a subgraph that contains all the vertices of G



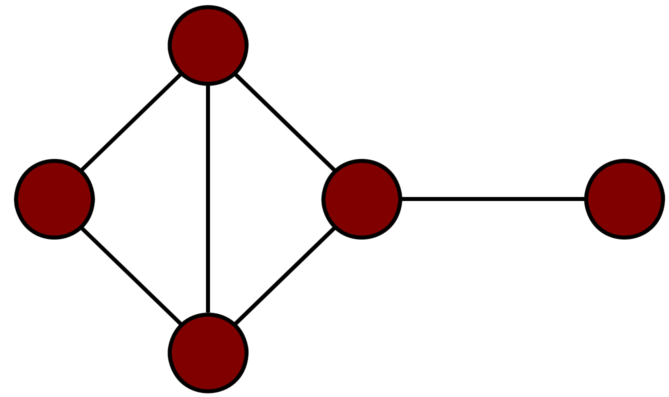
Subgraph



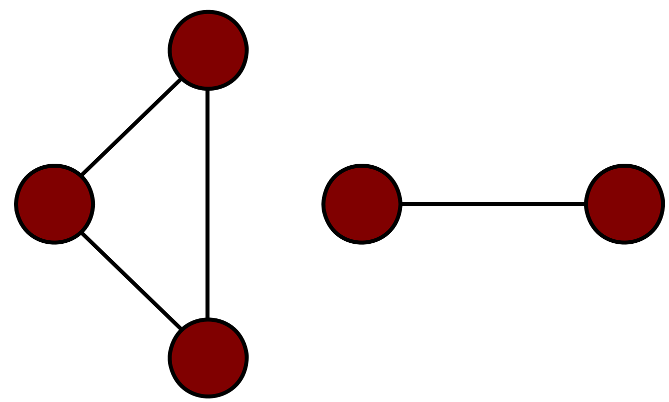
Spanning subgraph

Connectivity

- A graph is connected if there is a path between every pair of vertices
- A connected component of a graph G is a maximal connected subgraph of G

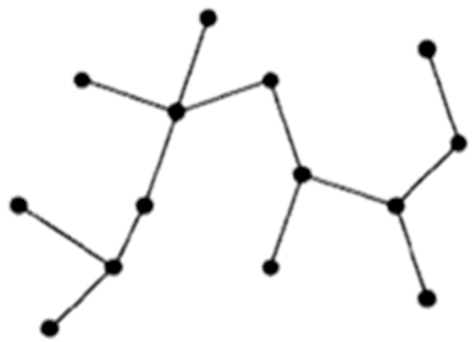


Connected graph



Non connected graph with two connected components

Trees



Tree

Forest

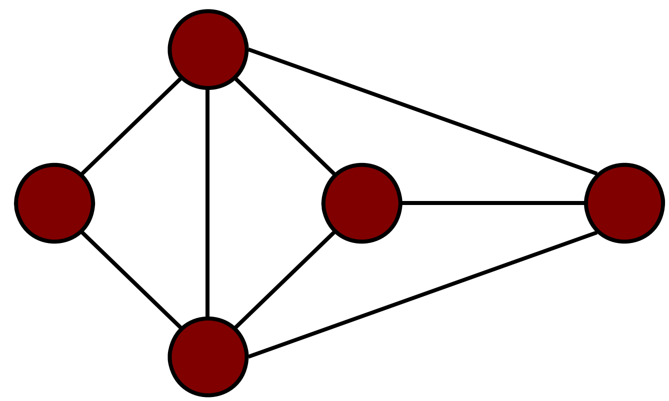
Graph with Cycle

A tree is a connected, acyclic, undirected graph.

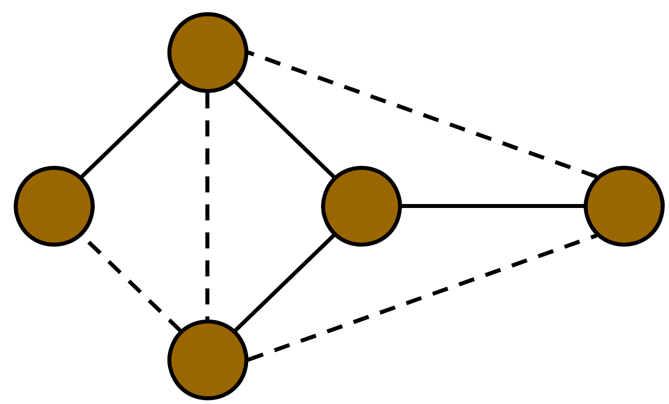
A forest is a set of trees (not necessarily connected)

Spanning Trees

- A spanning tree of a connected graph is a spanning subgraph that is a tree
- A spanning tree is not unique unless the graph is a tree
- Spanning trees have applications to the design of communication networks
- A spanning forest of a graph is a spanning subgraph that is a forest



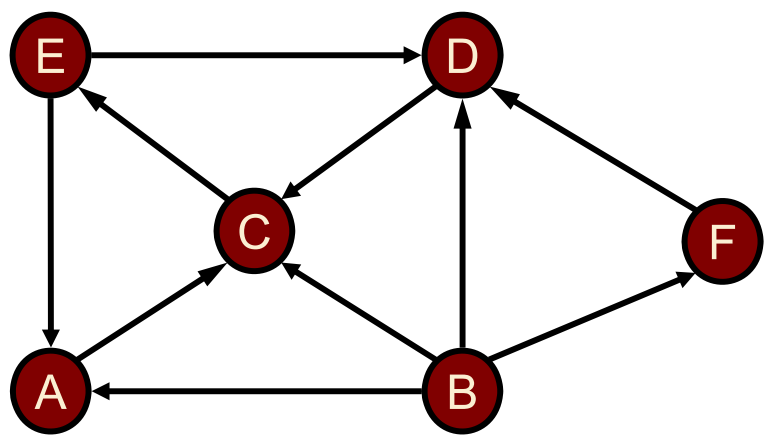
Graph



Spanning tree

Reachability in Directed Graphs

- A node w is ***reachable*** from v if there is a directed path originating at v and terminating at w.
  - E is reachable from B
  - B is not reachable from E



Properties

Property 1

$\Sigma_v \deg(v) = 2|E|$

Proof: each edge is counted twice

Property 2

In an undirected graph with no self-loops and no multiple edges

$|E| \leq |V| (|V| - 1)/2$

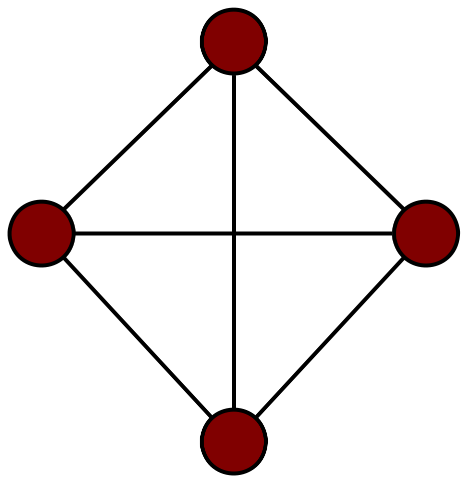
Proof: each vertex has degree at most $(|V| - 1)$

Q: What is the bound for a digraph?

$A: \ |E| \leq |V| (|V| - 1)$

Notation

$|V|$	number of vertices
$|E|$	number of edges
$\deg(v)$	degree of vertex v



Example

- $|V| = 4$
- $|E| = 6$
- $\deg(v) = 3$

Outline

- Definitions
- **Graph ADT**
- Implementations

Main Methods of the (Undirected) Graph ADT

- Vertices and edges
  - are positions
  - store elements
- Accessor methods
  - endVertices(e): an array of the two endvertices of e
  - opposite(v, e): the vertex opposite to v on e
  - areAdjacent(v, w): true iff v and w are adjacent
  - replace(v, x): replace element at vertex v with x
  - replace(e, x): replace element at edge e with x
- Update methods
  - insertVertex(o): insert a vertex storing element o
  - insertEdge(v, w, o): insert an edge (v,w) storing element o
  - removeVertex(v): remove vertex v (and its incident edges)
  - removeEdge(e): remove edge e
- Iterator methods
  - incidentEdges(v): edges incident to v
  - vertices(): all vertices in the graph
  - edges(): all edges in the graph

Directed Graph ADT

- Additional methods:
  - isDirected(e): return true if e is a directed edge
  - insertDirectedEdge(v, w, o): insert and return a new directed edge with origin *v* and destination *w*, storing element *o*

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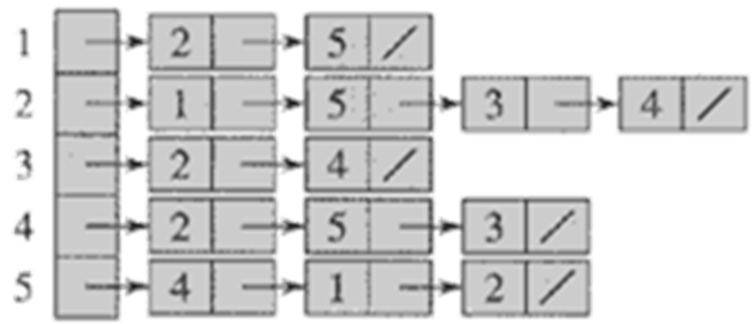
Running Time of Graph Algorithms

- Running time often a function of both |V| and |E|.
- For convenience, we sometimes drop the | … | in asymptotic notation, e.g. $O(V+E)$.

Implementing a Graph (Simplified)



$$\begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ 1 & 0 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 1 & 1 & 1 \\ 3 & 0 & 1 & 0 & 1 & 0 \\ 4 & 0 & 1 & 1 & 0 & 1 \\ 5 & 1 & 1 & 0 & 1 & 0 \end{array}$$

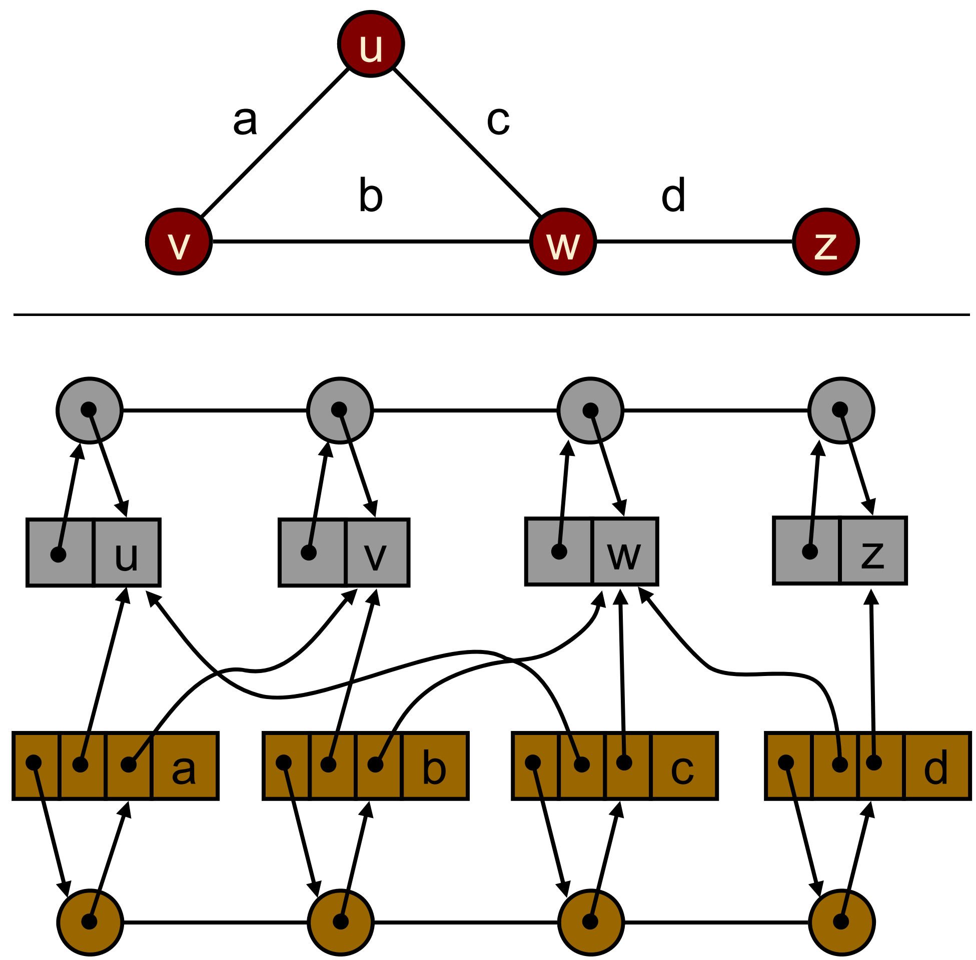
	Adjacency List	Adjacency Matrix
Space complexity:	$\theta(V+E)$	$\theta(V^2)$
Time to find all neighbours of vertex u :	$\theta(\text{degree}(u))$	$\theta(V)$
Time to determine if $(u,v) \in E$:	$\theta(\text{degree}(u))$	$\theta(1)$

Representing Graphs (Details)

- Three basic methods
  - Edge List
  - Adjacency List
  - Adjacency Matrix

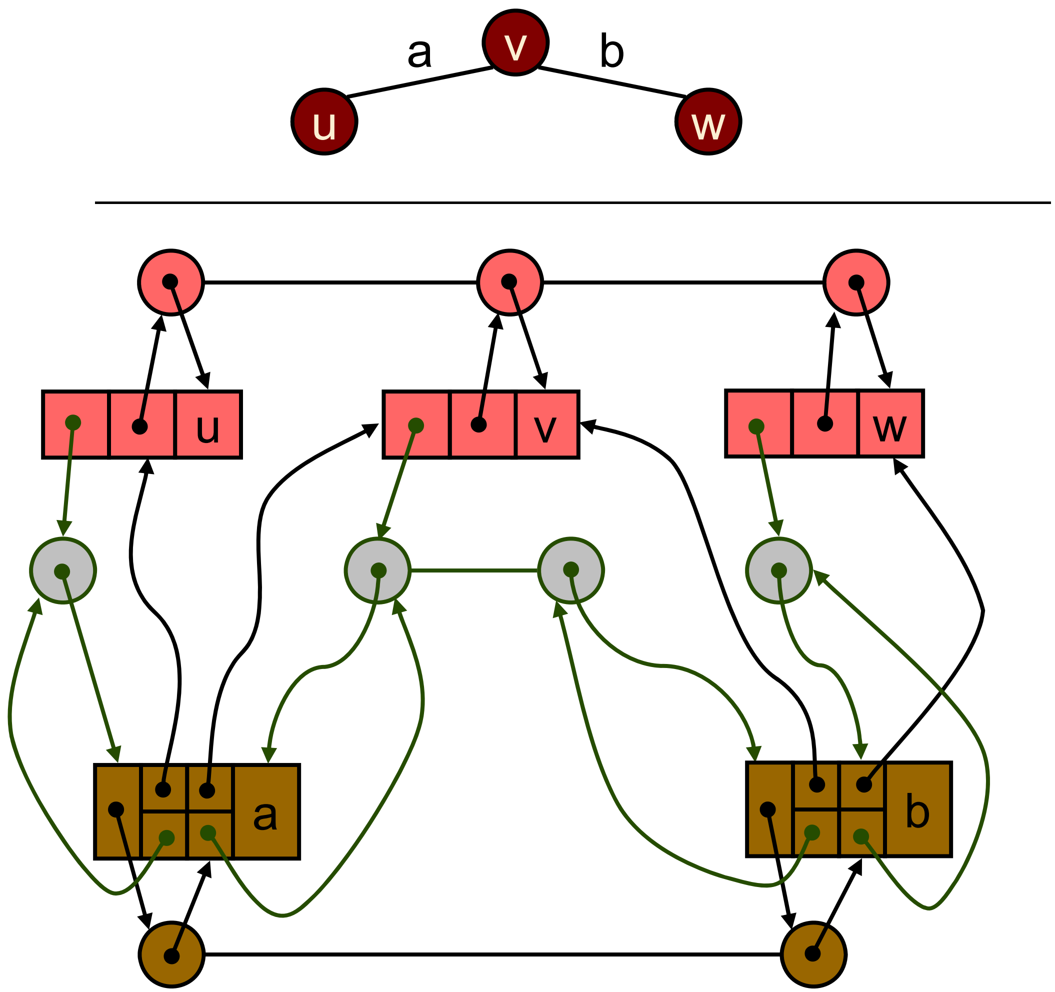
Edge List Structure

- Vertex object
  - element
  - reference to position in vertex sequence
- Edge object
  - element
  - origin vertex object
  - destination vertex object
  - reference to position in edge sequence
- Vertex sequence
  - sequence of vertex objects
- Edge sequence
  - sequence of edge objects



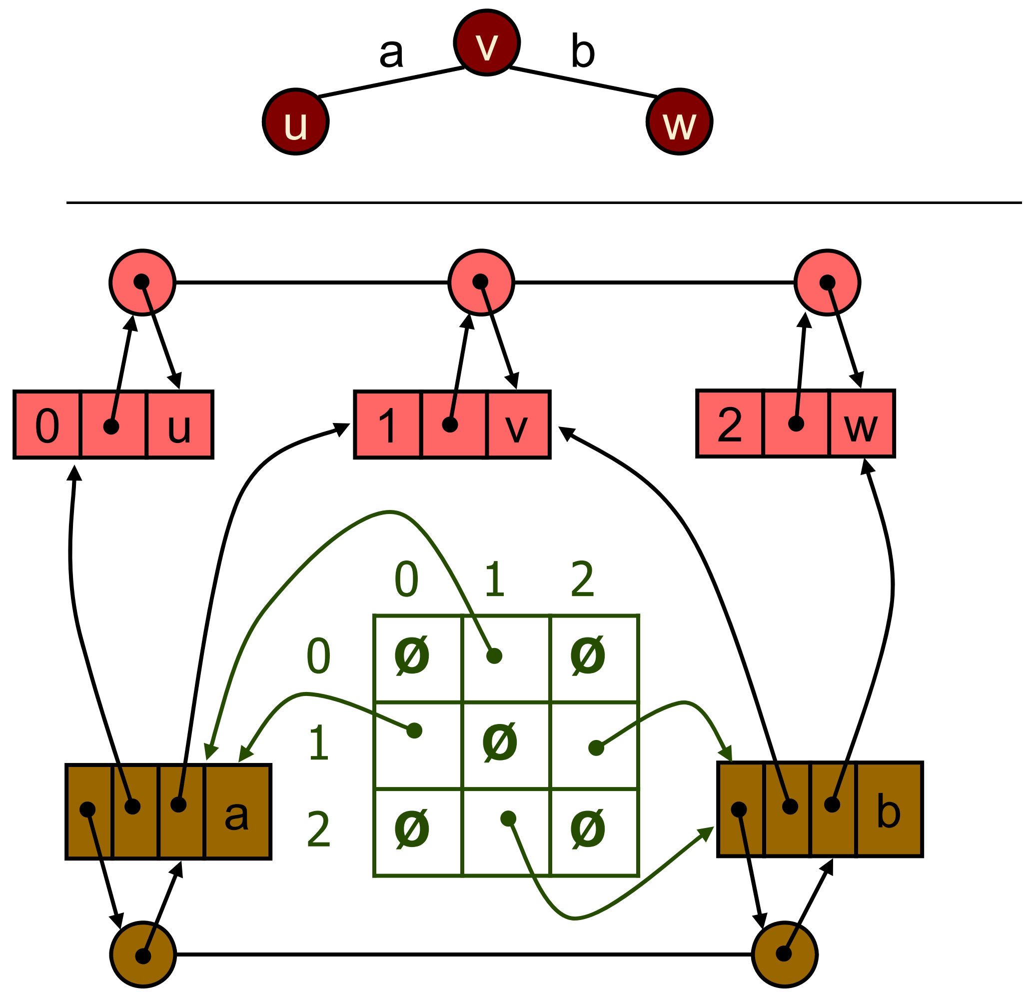
Adjacency List Structure

- Edge list structure
- Incidence sequence for each vertex
  - sequence of references to edge objects of incident edges
- Augmented edge objects
  - references to associated positions in incidence sequences of end vertices



Adjacency Matrix Structure

- Edge list structure
- Augmented vertex objects
  - Integer key (index) associated with vertex
- 2D-array adjacency array
  - Reference to edge object for adjacent vertices
  - Null for non-nonadjacent vertices



Asymptotic Performance (assuming collections V and E represented as doubly-linked lists)

◆ $\lvert V\rvert$ vertices, $\lvert E\rvert$ edges ◆ no parallel edges ◆ no self-loops ◆ Bounds are "big-Oh"	Edge List	Adjacency List	Adjacency Matrix
Space	$\lvert V\rvert+\lvert E\rvert$	$\lvert V\rvert+\lvert E\rvert$	$\lvert V\rvert^2$
incidentEdges(v)	$\lvert E\rvert$	$\deg(v)$	$\lvert V\rvert$
areAdjacent (v, w)	$\lvert E\rvert$	$\min(\deg(v), \deg(w))$	1
insertVertex(o)	1	1	$\lvert V\rvert^2$
insertEdge(v, w, o)	1	1	1
removeVertex(v)	$\lvert E\rvert$	$\deg(v)$	$\lvert V\rvert^2$
removeEdge(e)	1	1	1

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