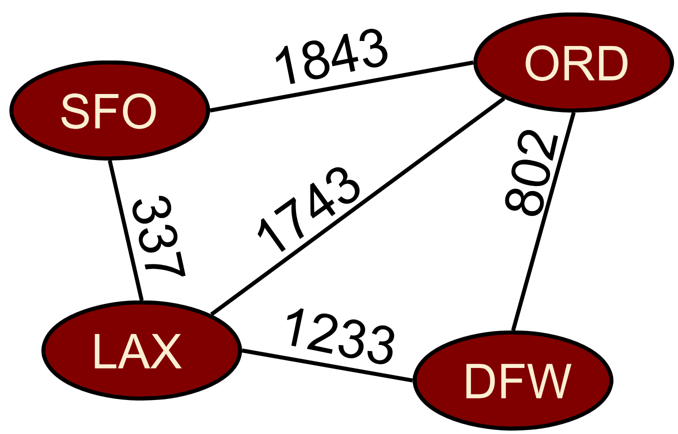
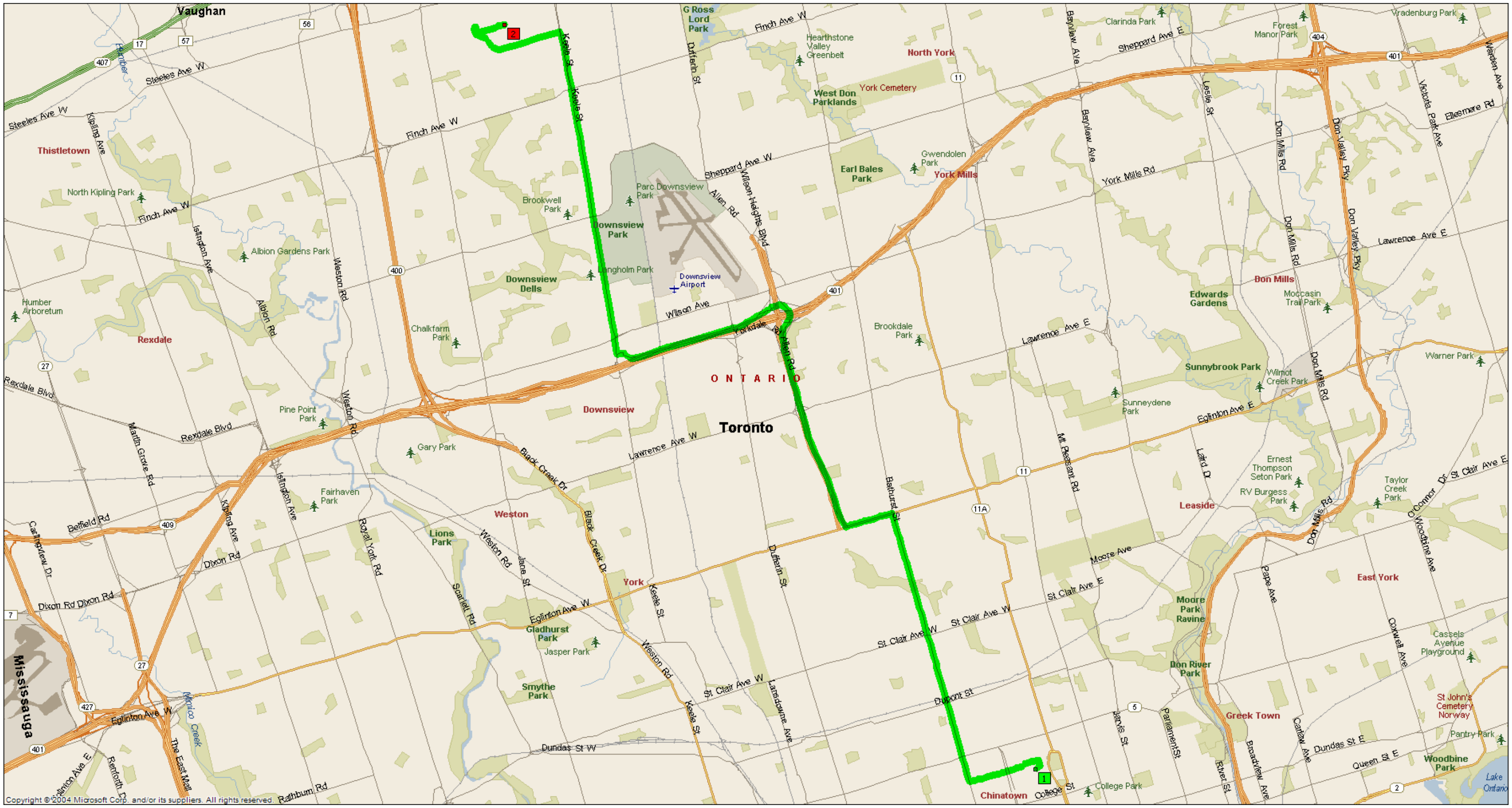
Graphs – Depth First Search



Graph Search Algorithms



Outline

- DFS Algorithm
- DFS Example
- DFS Applications

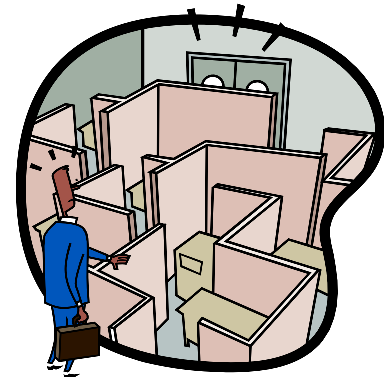
Outline

- **DFS Algorithm**
- DFS Example
- DFS Applications

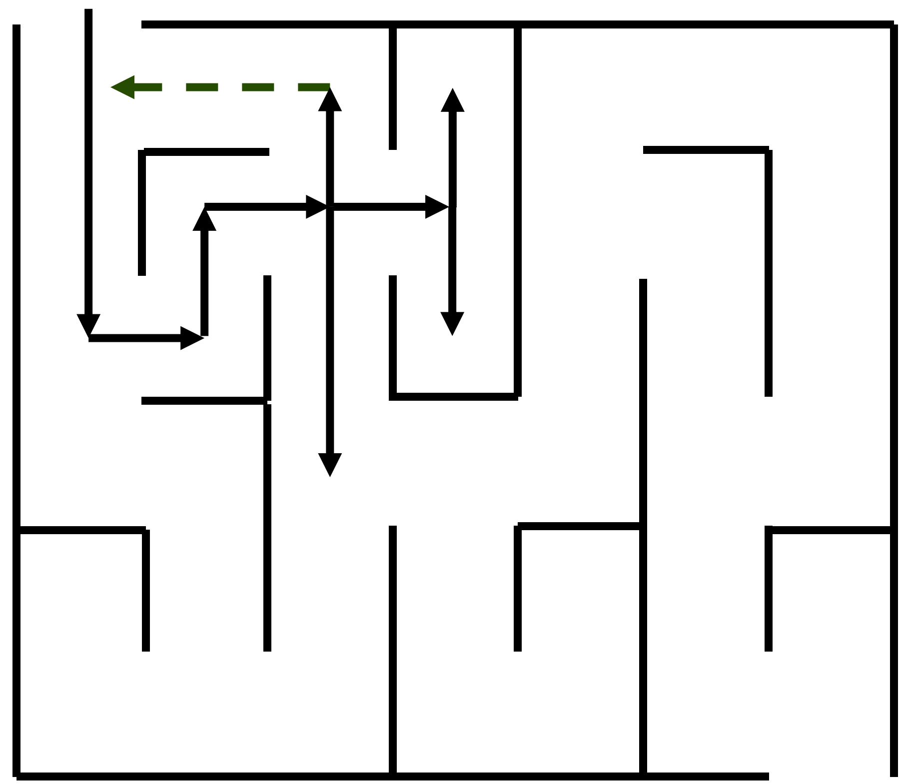
Depth First Search (DFS)

- Idea:
  - Continue searching "deeper" into the graph, until we get stuck.
  - If all the edges leaving *v* have been explored we "backtrack" to the vertex from which *v* was discovered.
  - Analogous to Euler tour for trees
- Used to help solve many graph problems, including
  - Nodes that are reachable from a specific node *v*
  - Detection of cycles
  - Extraction of strongly connected components
  - Topological sorts

Depth-First Search



- The DFS algorithm is similar to a classic strategy for exploring a maze
  - We mark each intersection, corner and dead end (vertex) visited
  - We mark each corridor (edge) traversed
  - We keep track of the path back to the entrance (start vertex) by means of a rope (recursion stack)

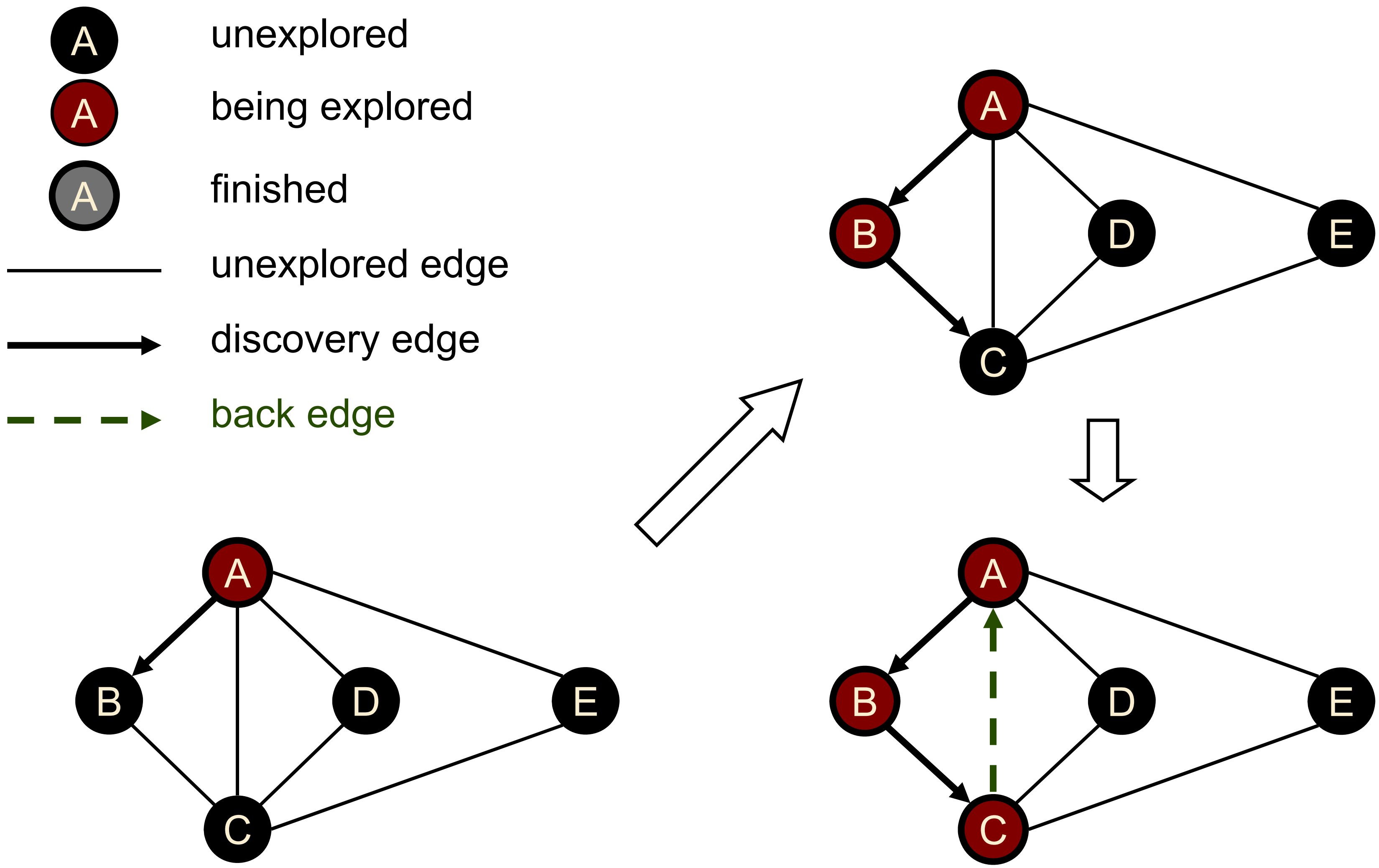


Depth-First Search

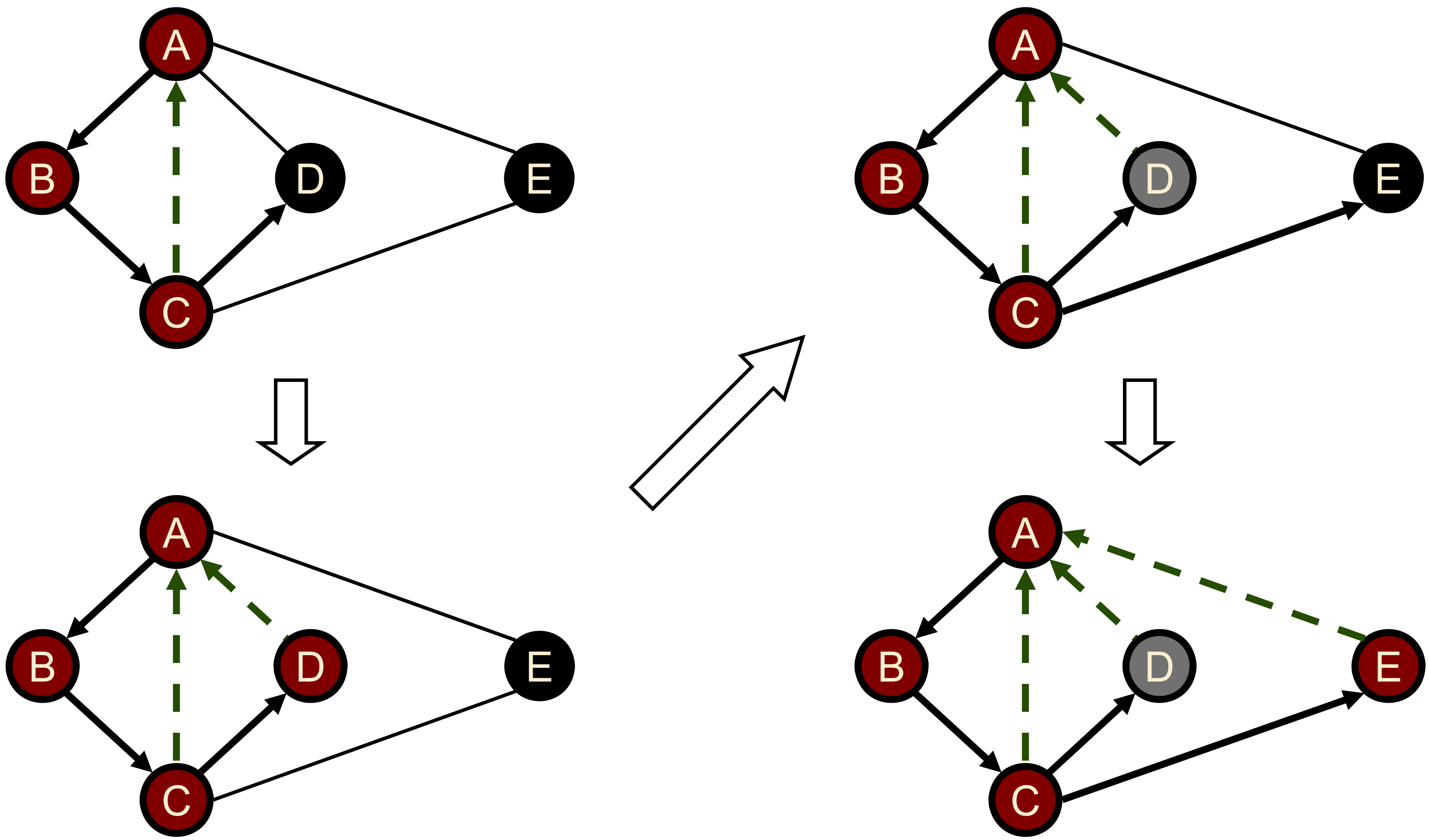
Input: Graph $G = (V, E)$ (directed or undirected)

- Explore *every* edge, starting from different vertices if necessary.
- As soon as vertex discovered, explore from it.
- Keep track of progress by colouring vertices:
  - Black: undiscovered vertices
  - Red: discovered, but not finished (still exploring from it)
  - Gray: finished (found everything reachable from it).

DFS Example on Undirected Graph



Example (cont.)



DFS Algorithm Pattern

DFS(G)

Precondition: G is a graph

Postcondition: all vertices in G have been visited

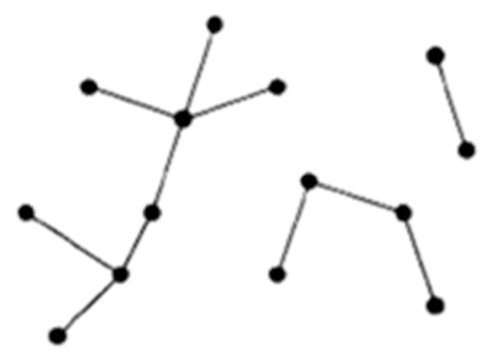
for each vertex u $\in V[G]$

color[u] = BLACK //initialize vertex

for each vertex u $\in V[G]$

if color[u] = BLACK //as yet unexplored

DFS-Visit(u)



DFS Algorithm Pattern

DFS-Visit (u)

Precondition: vertex u is undiscovered

Postcondition: all vertices reachable from u have been processed

```
    colour[u] ← RED
    for each v ∈ Adj[u] //explore edge (u,v)
        if color[v] = BLACK
            DFS-Visit(v)
    colour[u] ← GRAY
```

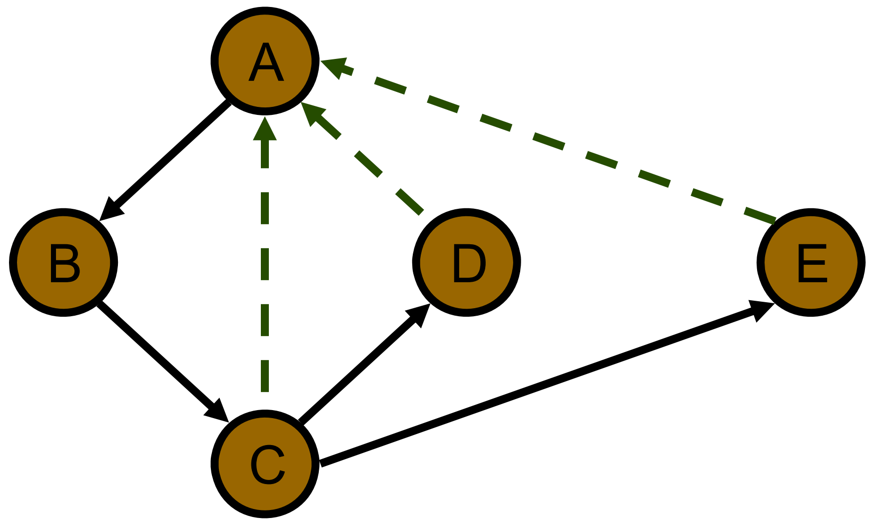
Properties of DFS

Property 1

DFS-Visit*(*u*)** visits all the vertices and edges in the connected component of ***u

Property 2

The discovery edges labeled by ***DFS-Visit*(*u*)** form a spanning tree of the connected component of ***u***



DFS Algorithm Pattern

DFS(G)

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for each vertex u $\in V[G]$

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for each vertex u $\in V[G]$

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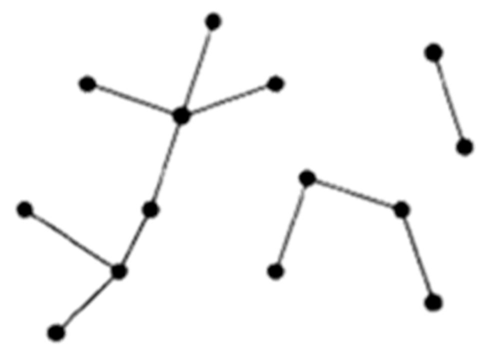
DFS-Visit(u)

total work $= \theta(V)$



DFS Algorithm Pattern

DFS-Visit (u)

Precondition: vertex u is undiscovered

Postcondition: all vertices reachable from u have been processed

colour[u] $\leftarrow$ RED

for each $v \in$ Adj[u] //explore edge (u,v)

if color[v] = BLACK

DFS-Visit(v)

$colour[u] \leftarrow GRAY$

total work $= \sum_{v \in V} |Adj[v]| = \theta(E)$

Thus running time = $\theta(V + E)$

(assuming adjacency list structure)

Variants of Depth-First Search

- In addition to, or instead of labeling vertices with colours, they can be labeled with **discovery** and **finishing** times.
- 'Time' is an integer that is incremented whenever a vertex changes state
  - from **unexplored** to **discovered**
  - from **discovered** to **finished**
- These **discovery** and **finishing** times can then be used to solve other graph problems (e.g., computing strongly-connected components)

Input: Graph $G = (V, E)$ (directed or undirected)

Output: 2 timestamps on each vertex:

$d[v] =$ discovery time.

$f[v] =$ finishing time.

$$1 \le d[v] < f[v] \le 2|V|$$

DFS Algorithm with Discovery and Finish Times

DFS(G)

Precondition: G is a graph

Postcondition: all vertices in G have been visited

for each vertex u $\in V[G]$

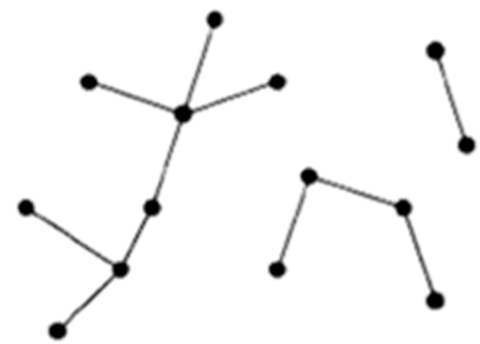
color[u] = BLACK //initialize vertex

time $\leftarrow$ 0

for each vertex u $\in V[G]$

if color[u] = BLACK //as yet unexplored

DFS-Visit(u)



DFS Algorithm with Discovery and Finish Times

DFS-Visit (u)

Precondition: vertex u is undiscovered

Postcondition: all vertices reachable from u have been processed

colour[u] $\leftarrow$ RED

$\text{time} \leftarrow \text{time} + 1$

$\text{d[u]} \leftarrow \text{time}$

for each $v \in$ Adj[u] //explore edge (u,v)

if color[v] = BLACK

DFS-Visit(v)

$colour[u] \leftarrow GRAY$

$\text{time} \leftarrow \text{time} + 1$

$f[u] \leftarrow \text{time}$

Other Variants of Depth-First Search

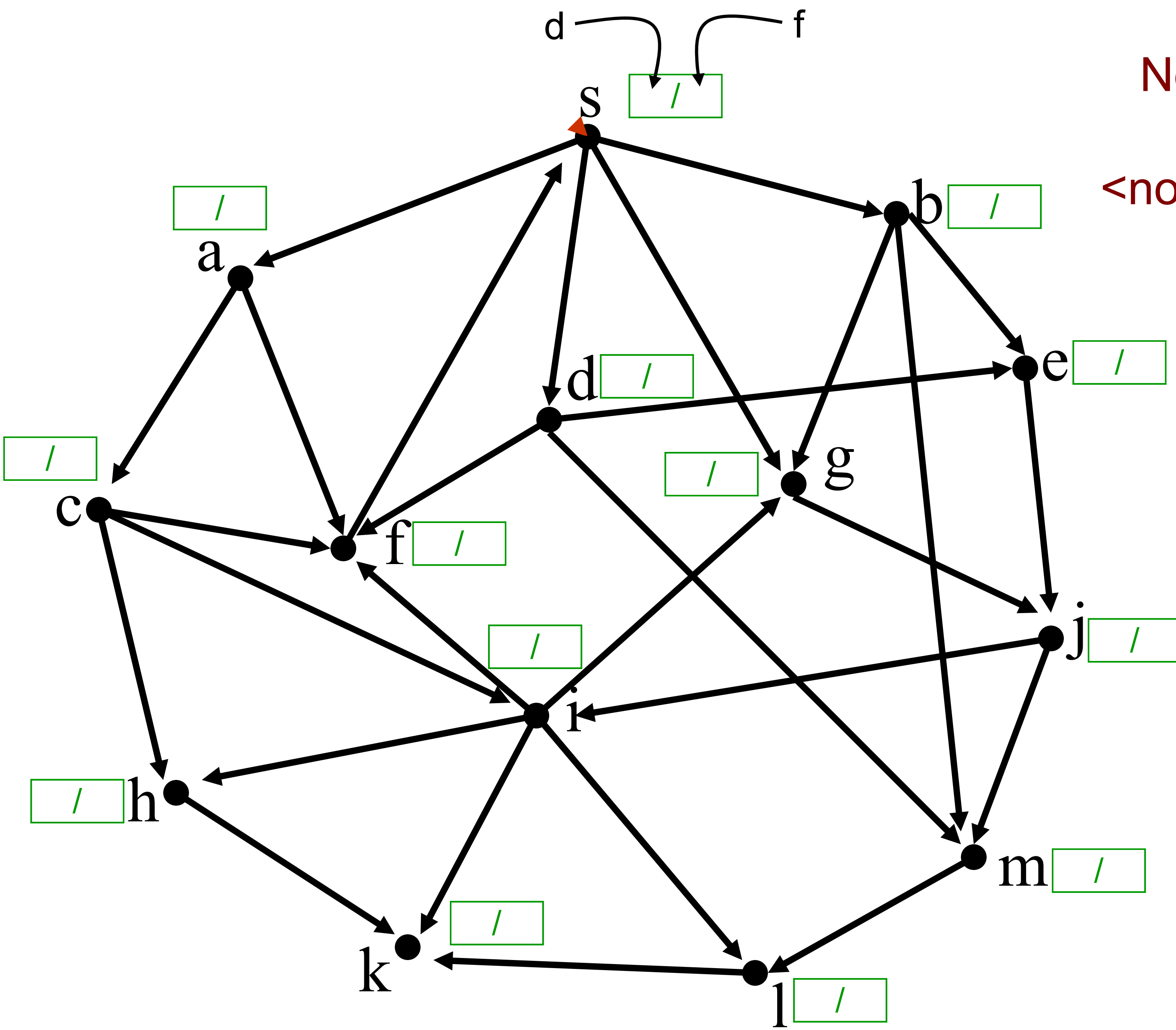
- The DFS Pattern can also be used to
  - Compute a forest of spanning trees (one for each call to DFS-visit) encoded in a predecessor list π[u]
  - Label edges in the graph according to their role in the search (see textbook)
    - **Tree edges**, traversed to an undiscovered vertex
    - **Forward edges**, traversed to a descendent vertex on the current spanning tree
    - **Back edges**, traversed to an ancestor vertex on the current spanning tree
    - **Cross edges**, traversed to a vertex that has already been discovered, but is not an ancestor or a descendent

Outline

- DFS Algorithm
- **DFS Example**
- DFS Applications

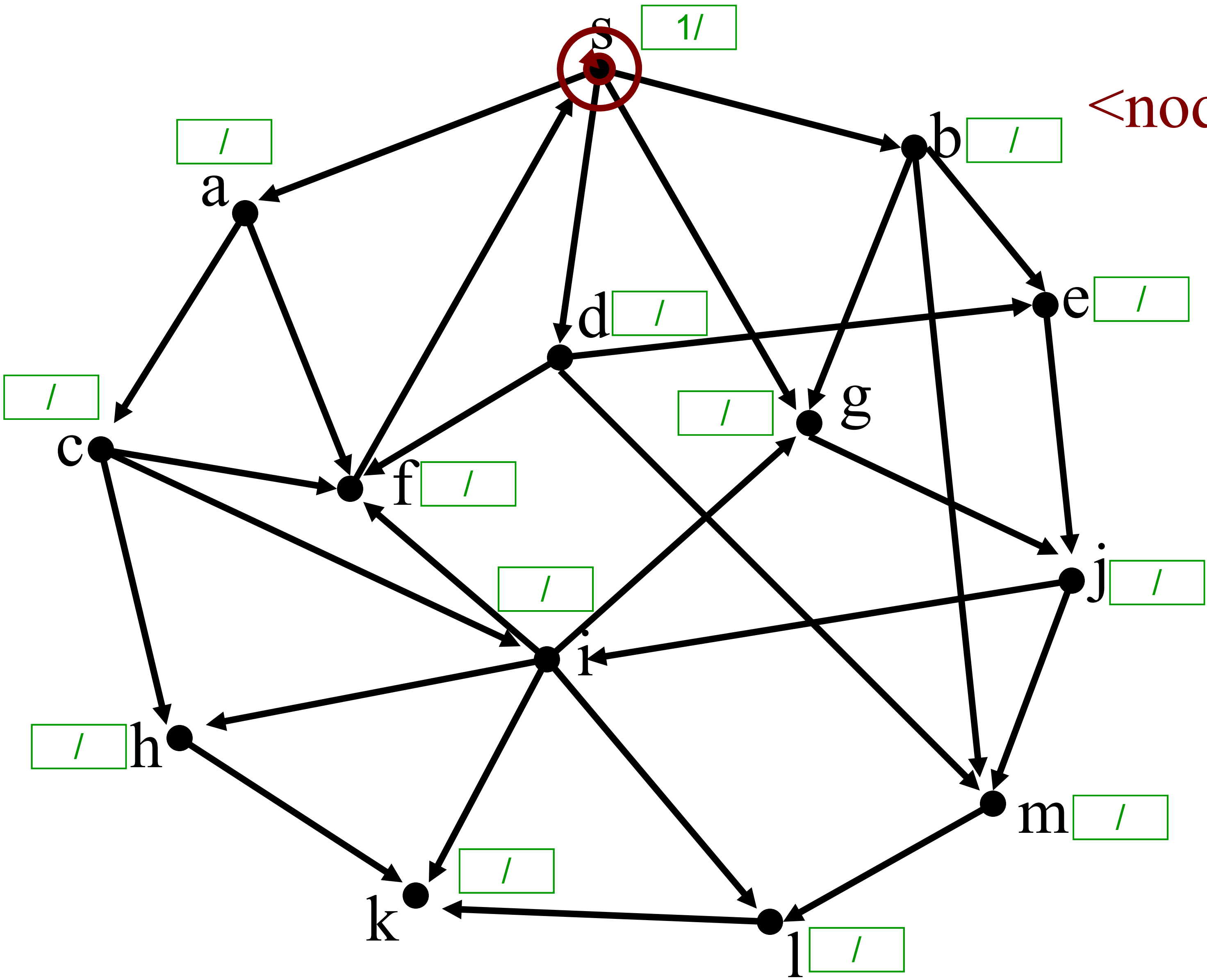
DFS

Note: Stack is Last-In First-Out (LIFO)



Found
Not Handled
Stack
<node,# edges>

DFS



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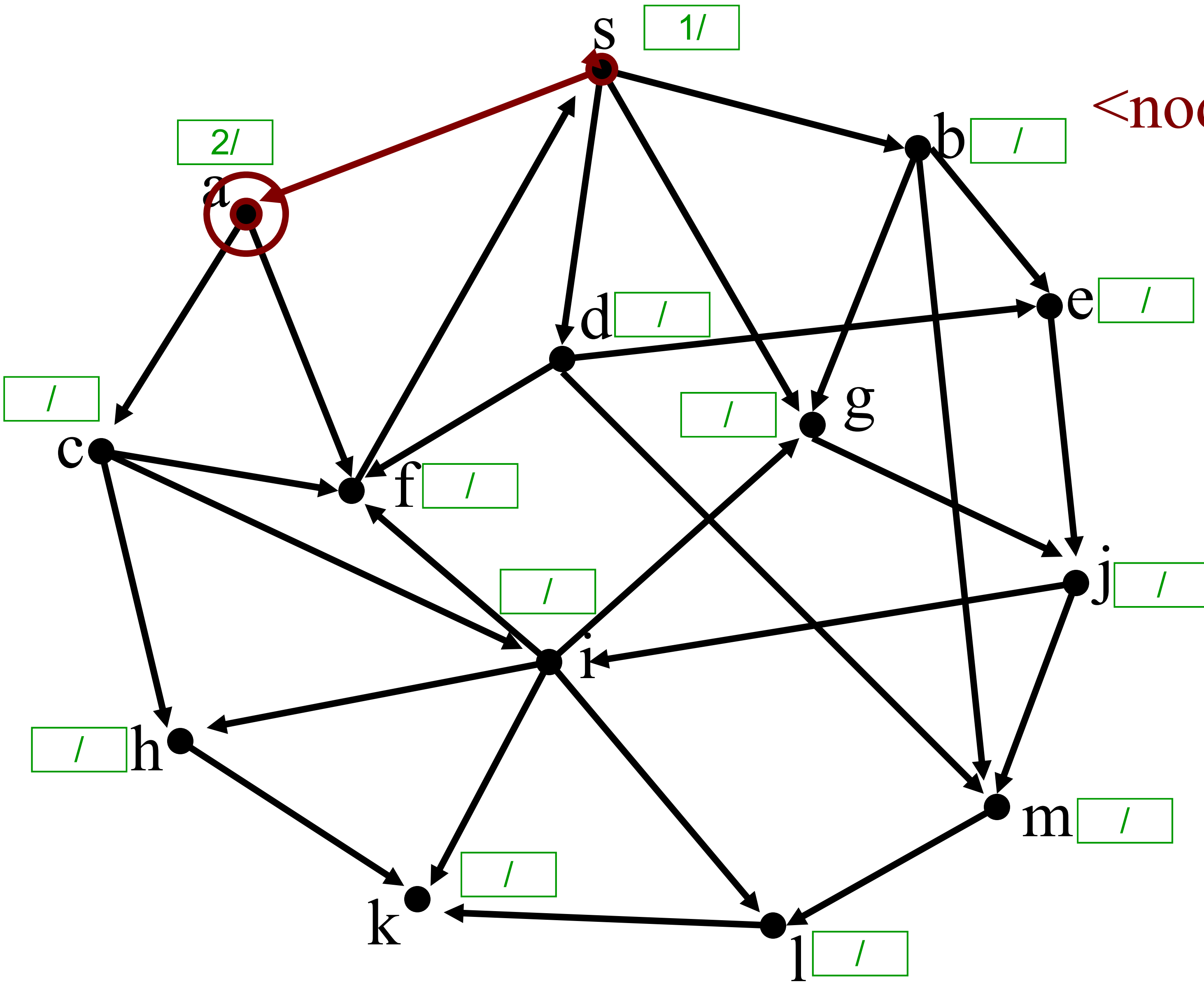
s,0

DFS



Found
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Stack
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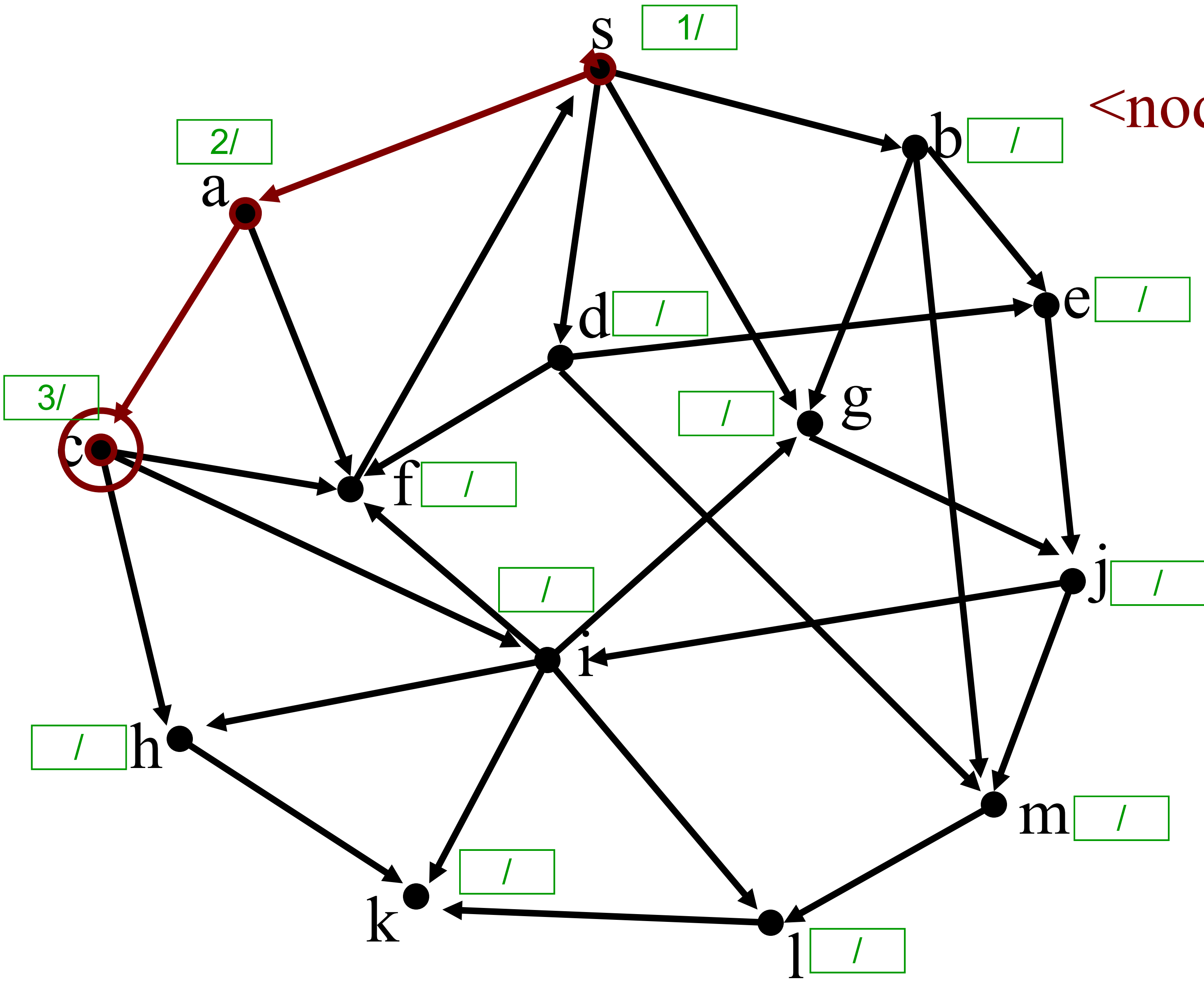
a,0
s,1

DFS



Found
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Stack
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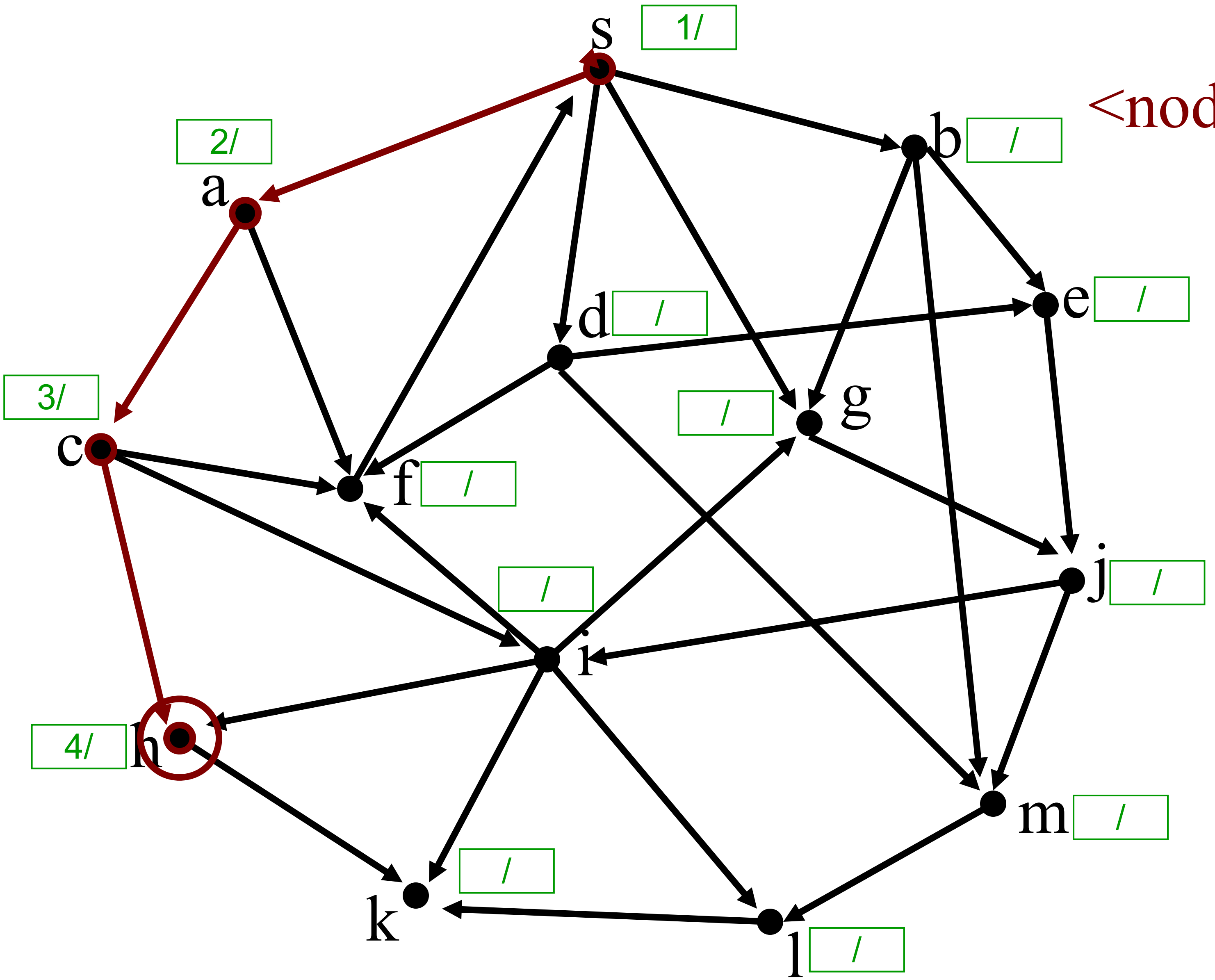
c,0
a,1
s,1

DFS



Found
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Stack
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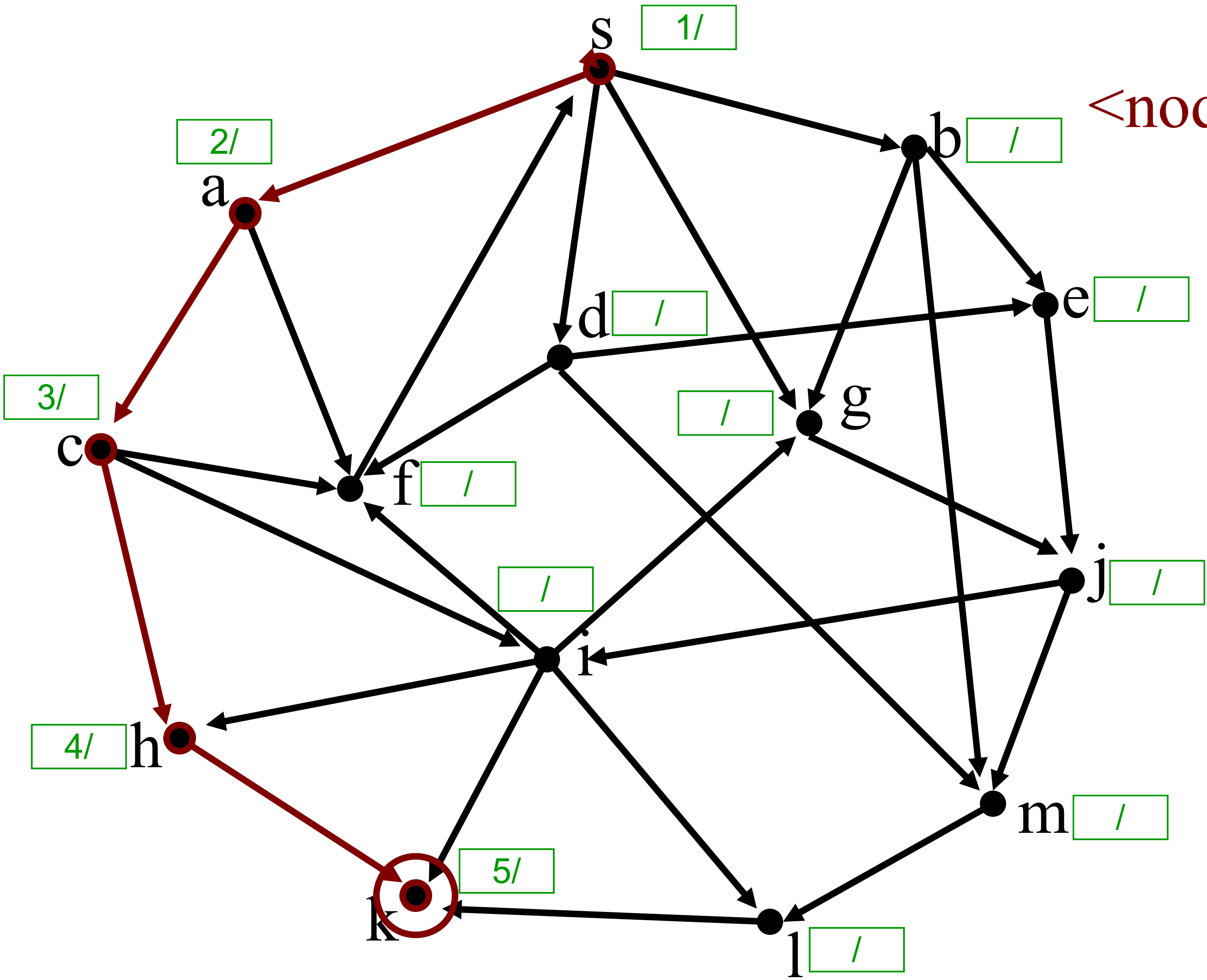
h,0
c,1
a,1
s,1

DFS



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Stack
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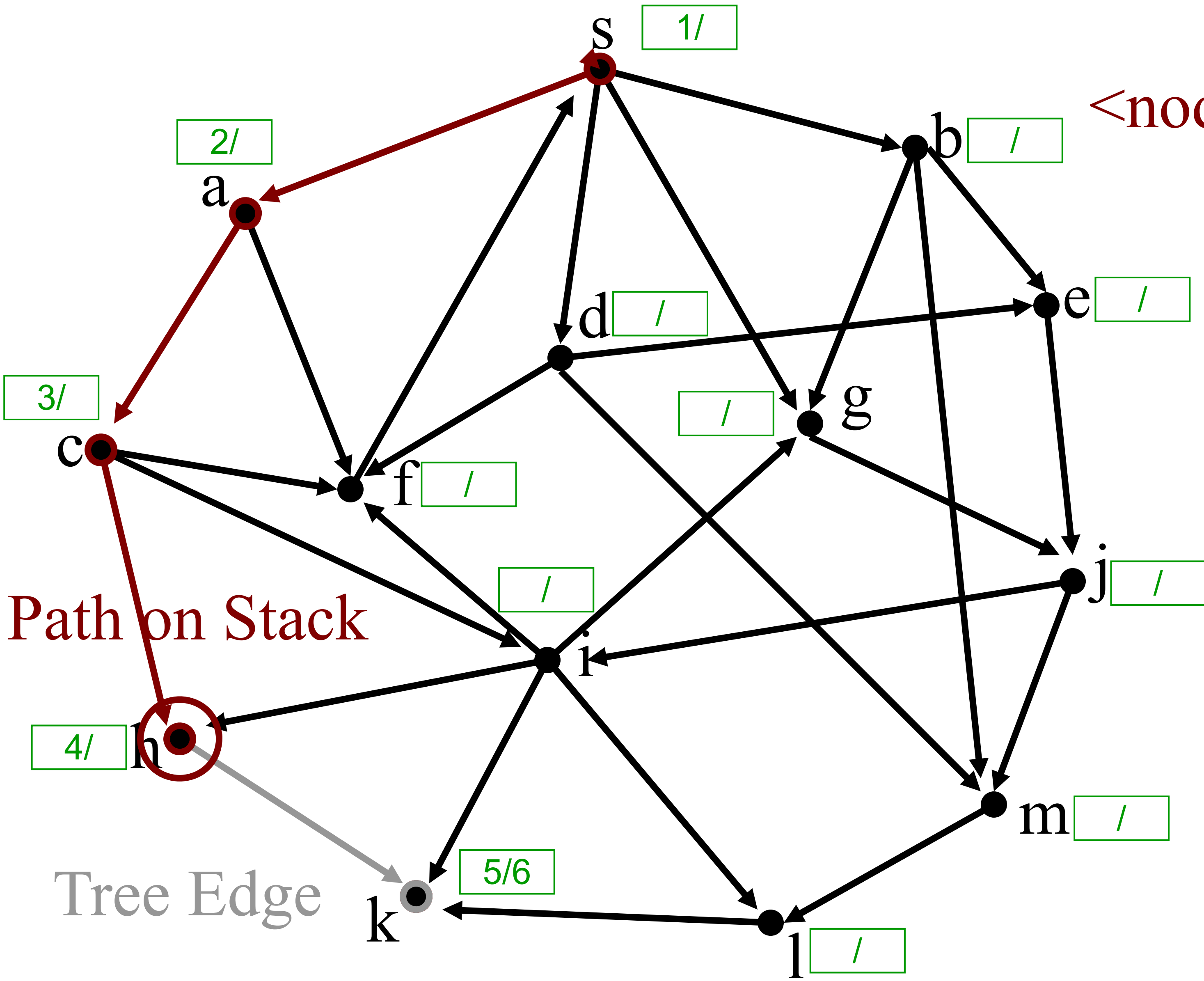
k,0
h,1
c,1
a,1
s,1

DFS



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Stack
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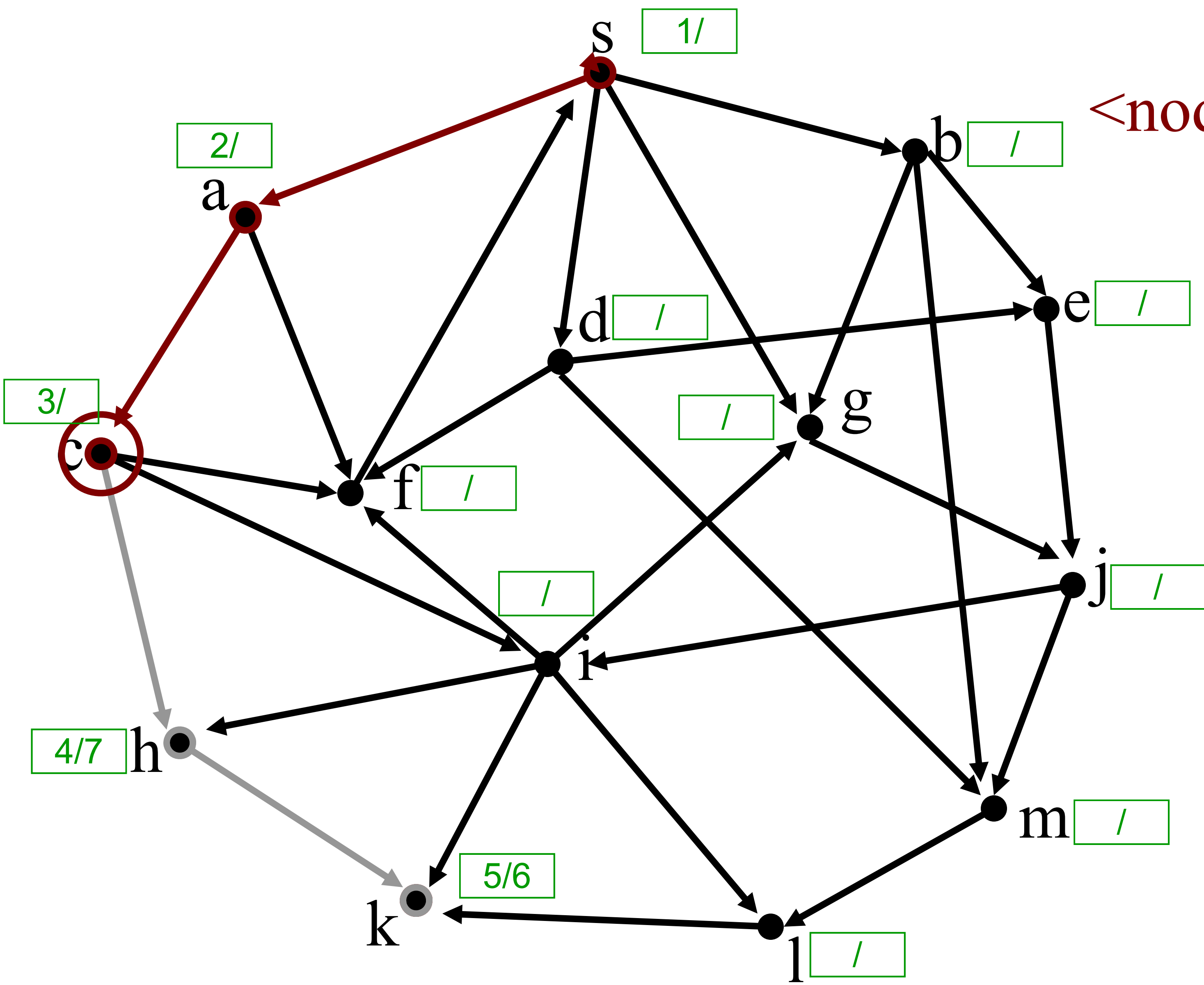
h,1
c,1
a,1
s,1

DFS



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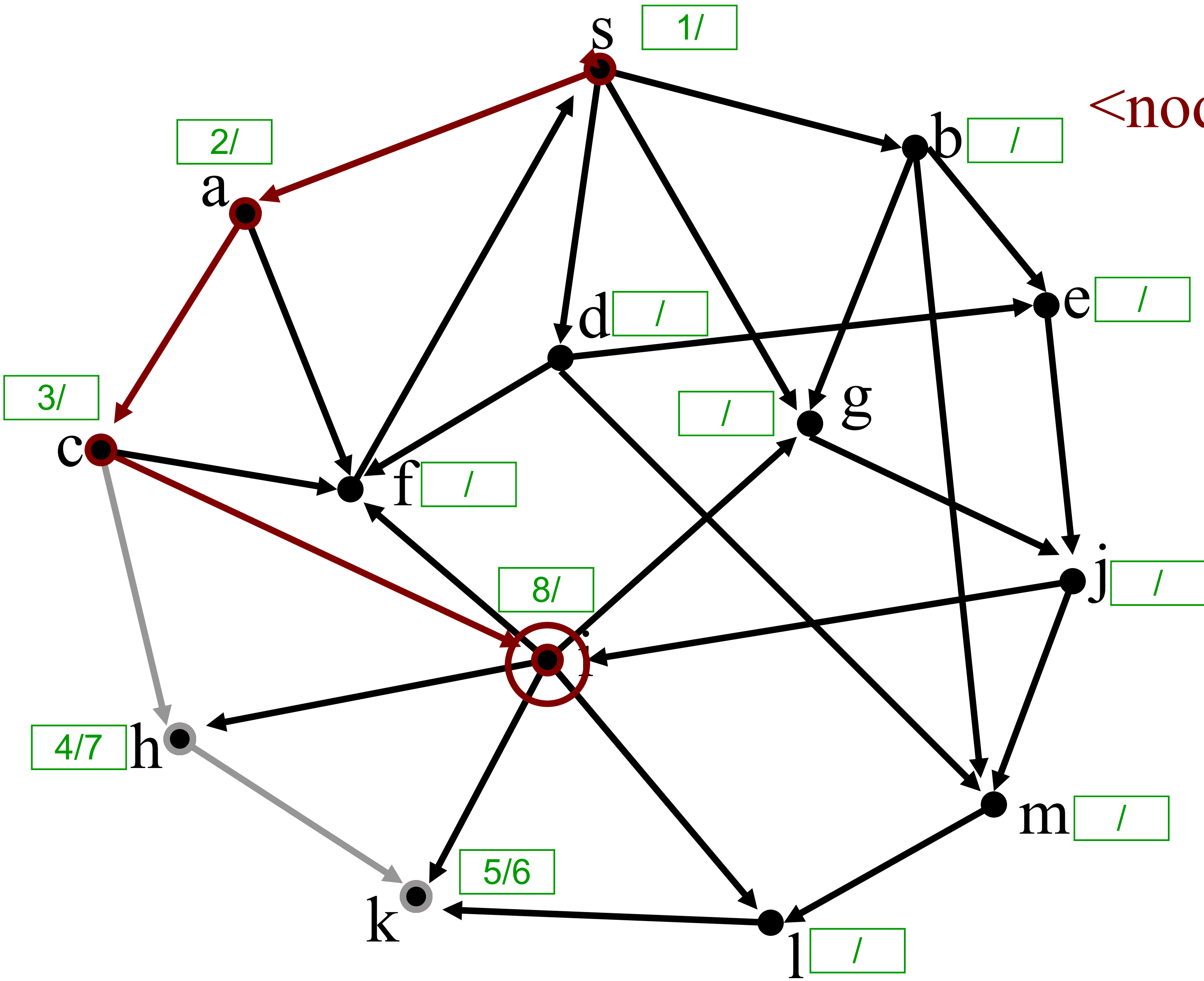
c, 1
a, 1
s, 1

DFS



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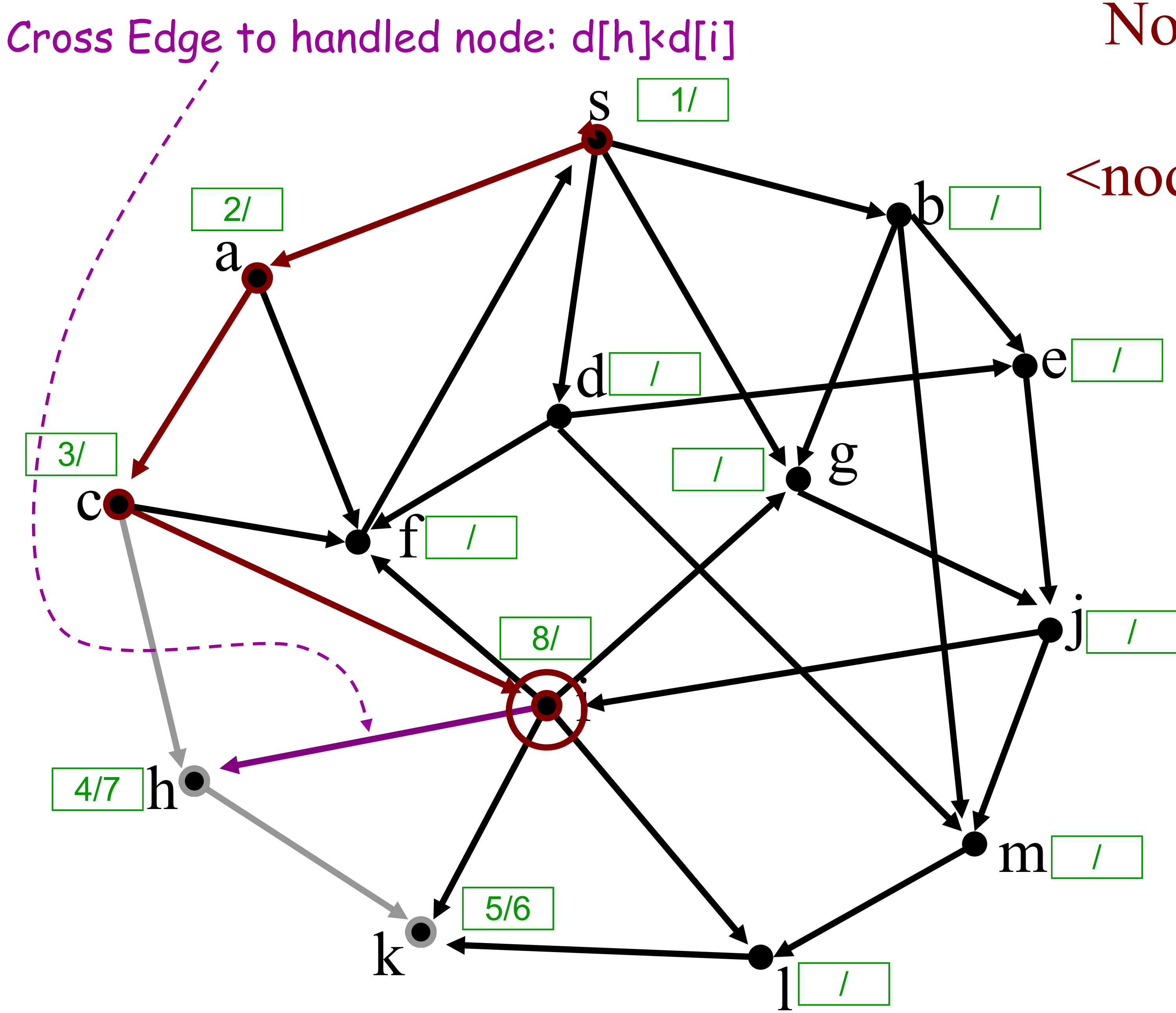
i,0
c,2
a,1
s,1

DFS



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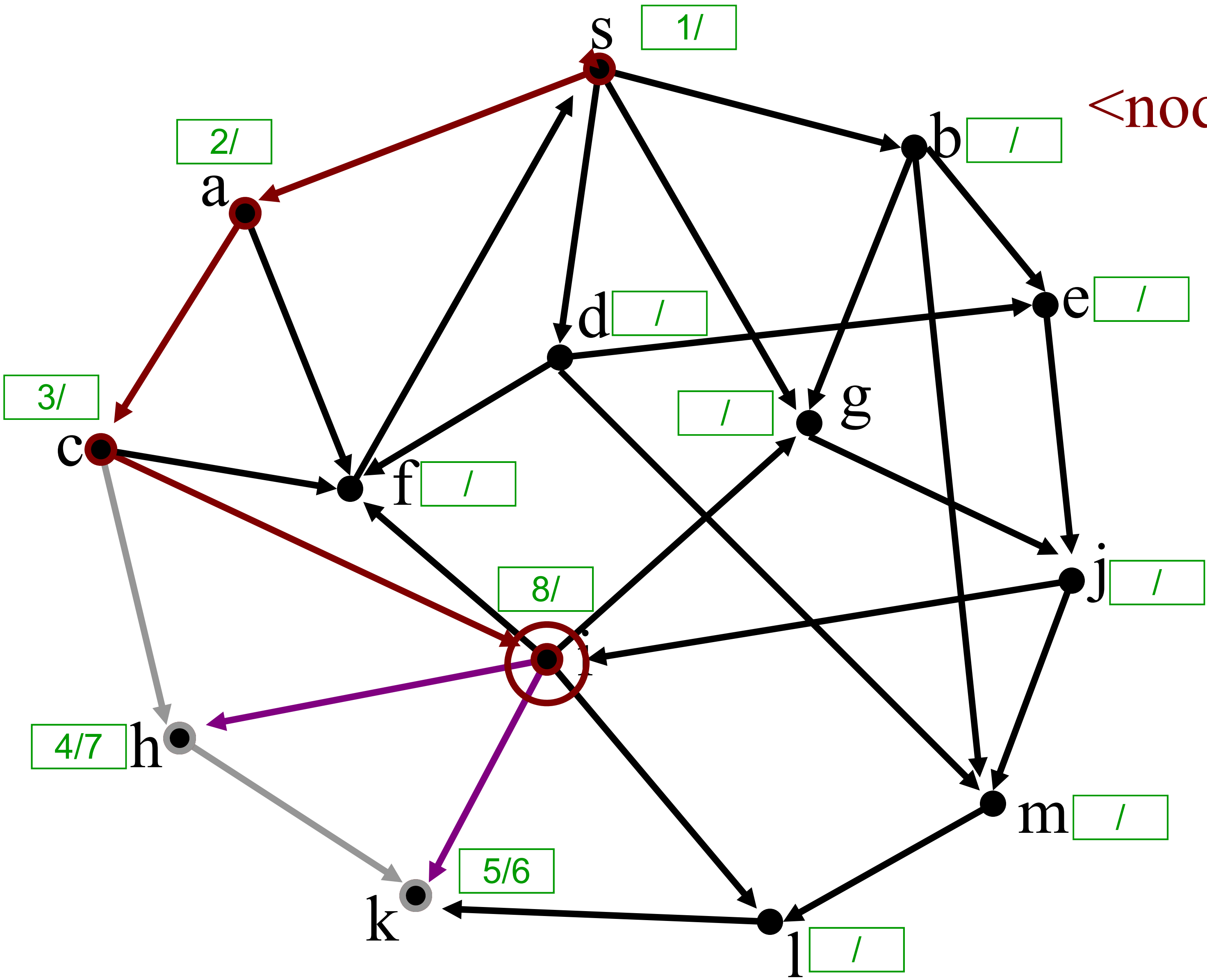
c,2
a,1
s,1

DFS



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i,2
c,2
a,1
s,1

DFS

Found
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Stack
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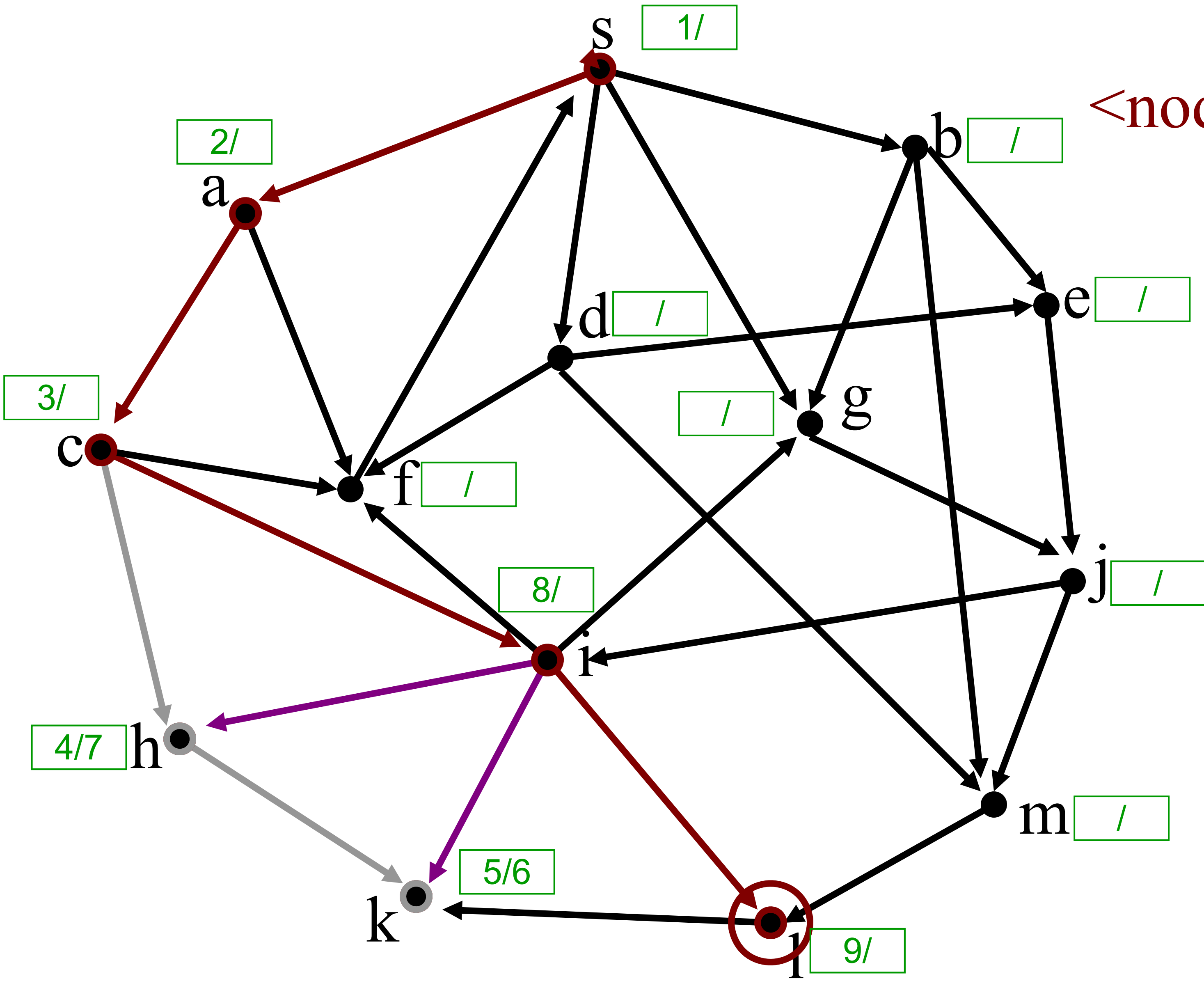
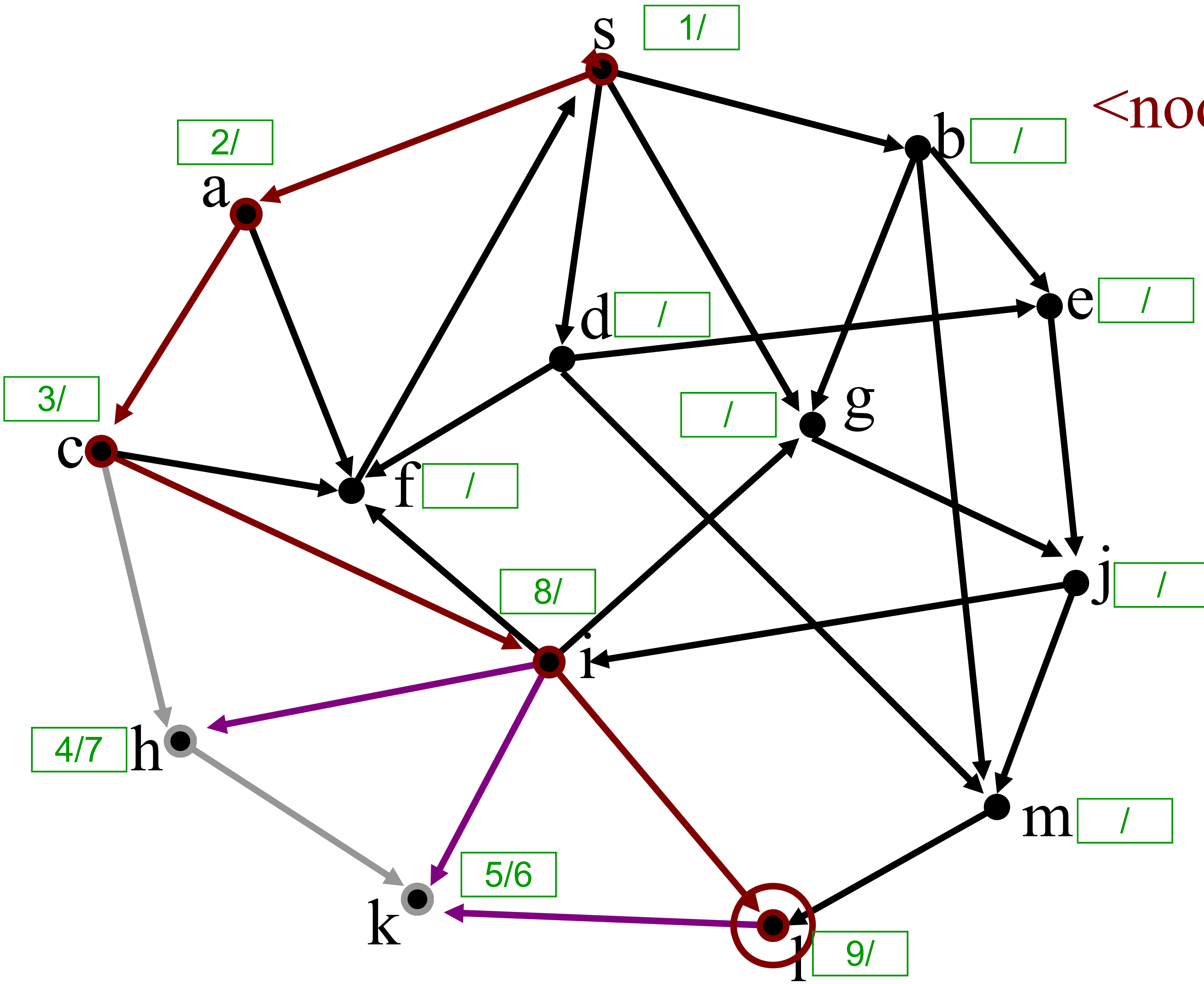


l,0
i,3
c,2
a,1
s,1

DFS



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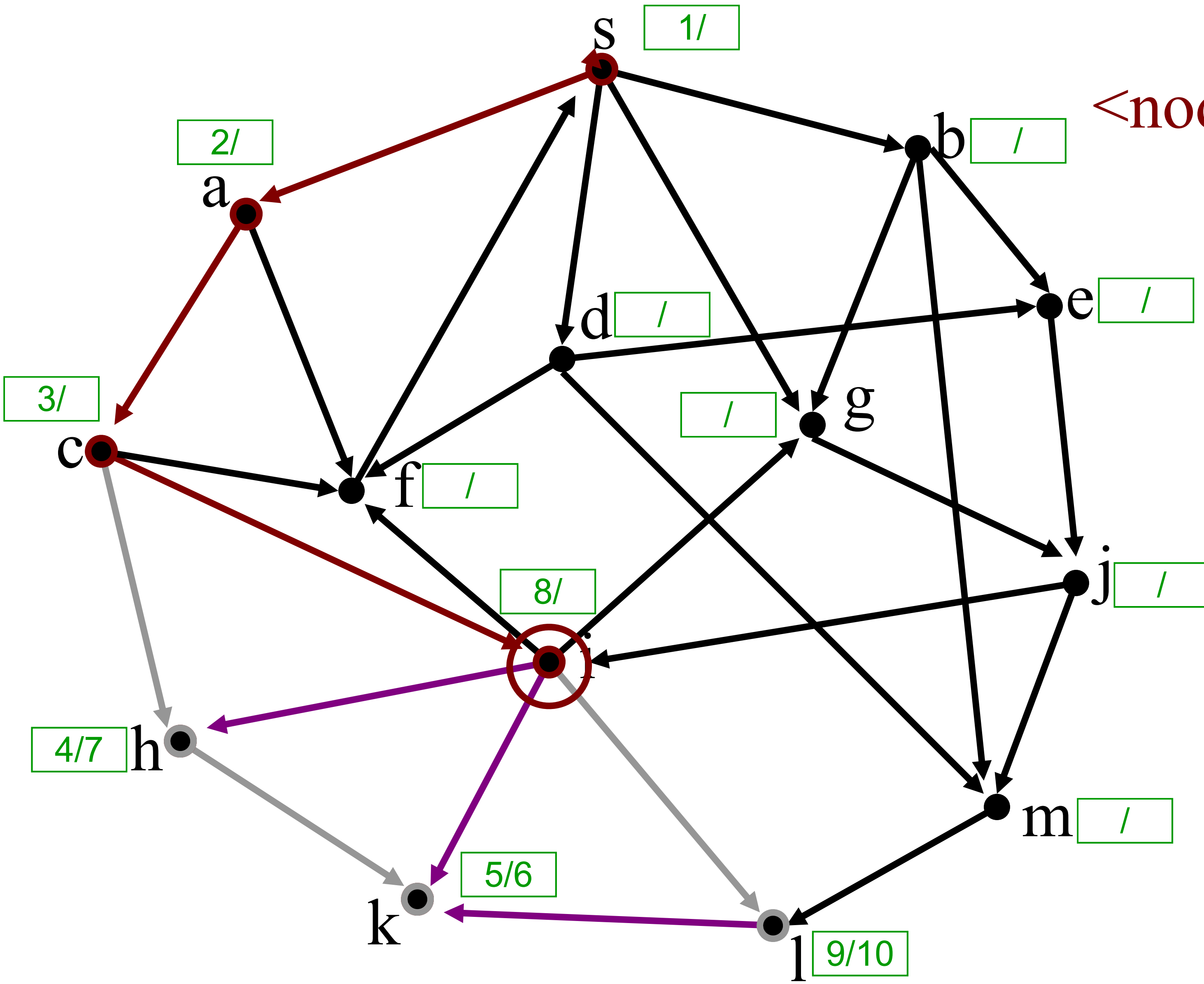
Stack
<node,# edges>

l,1
i,3
c,2
a,1
s,1

DFS



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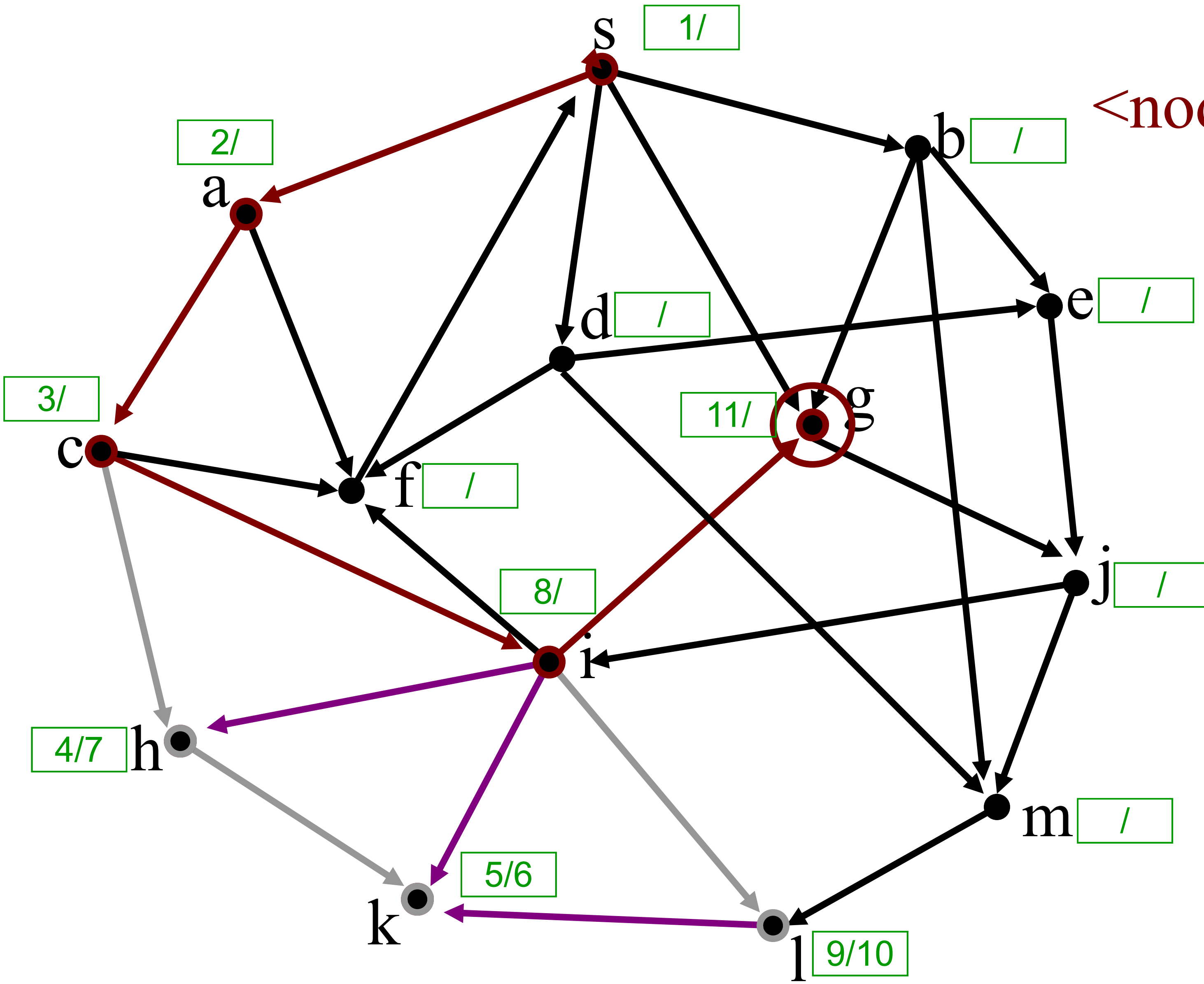
Stack
<node,# edges>

i,3
c,2
a,1
s,1

DFS



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Not Handled
Stack
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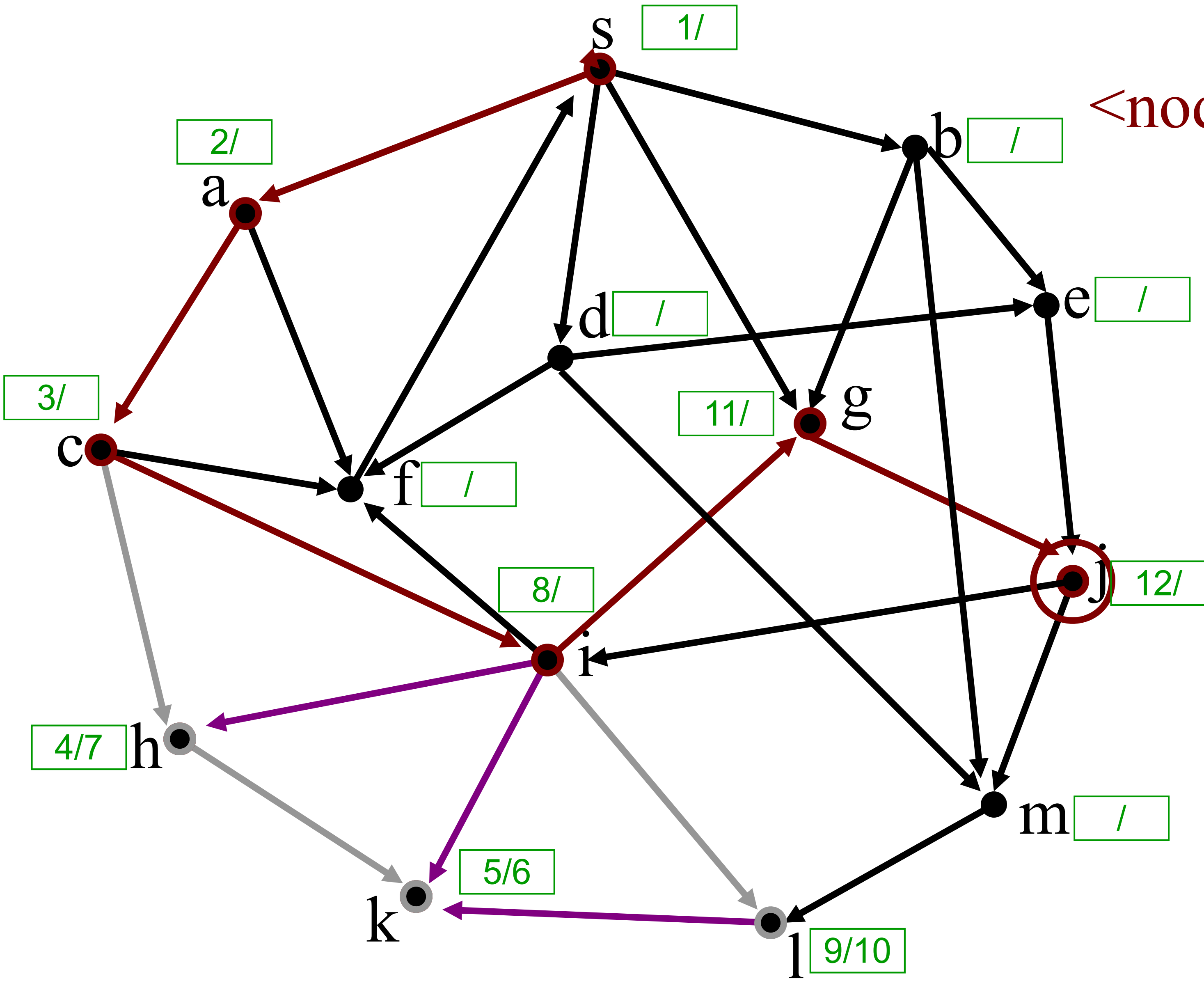
g,0
i,4
c,2
a,1
s,1

DFS

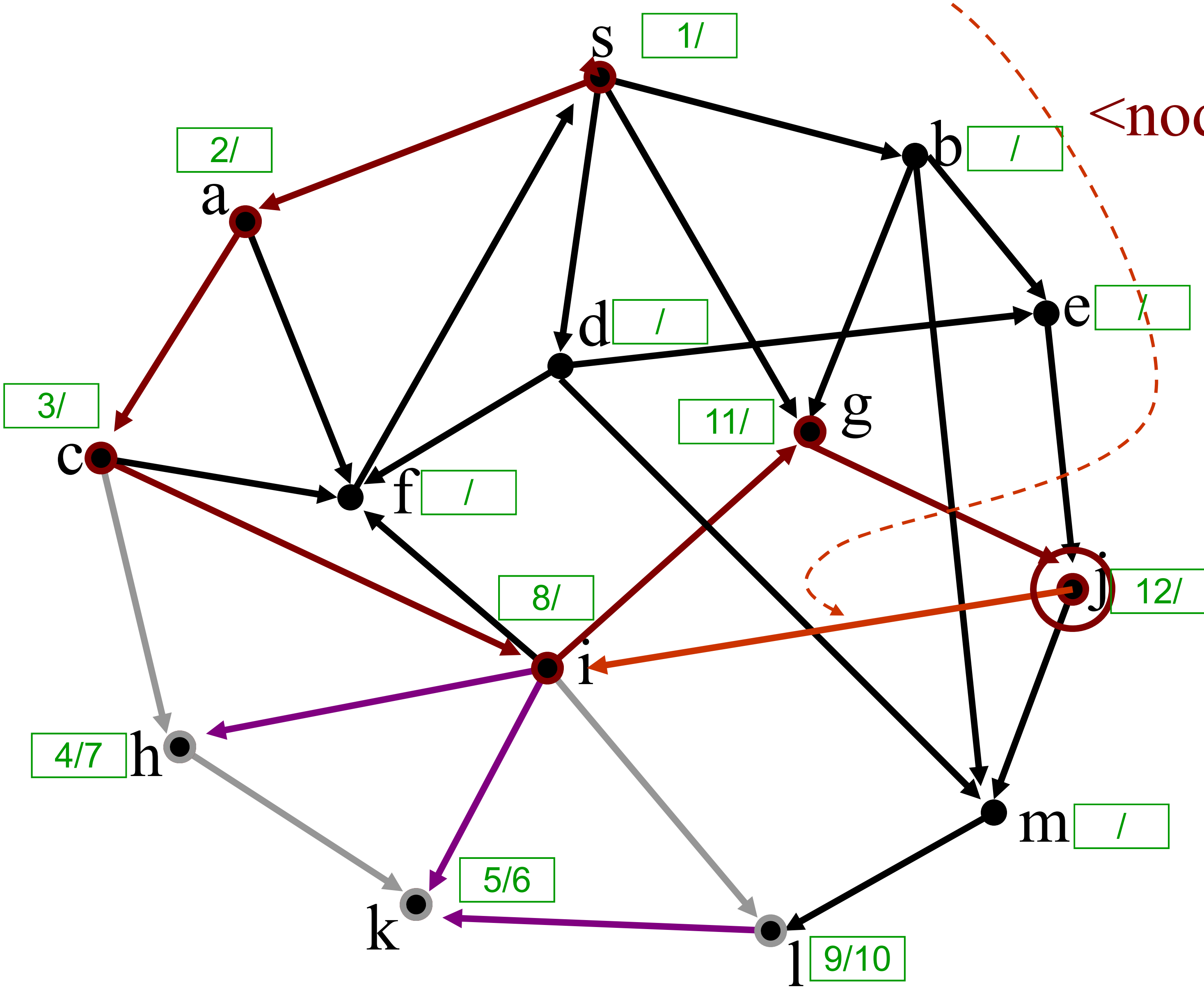


Found
Not Handled
Stack
<node,# edges>

j,0
g,1
i,4
c,2
a,1
s,1

DFS

Back Edge to node on Stack:



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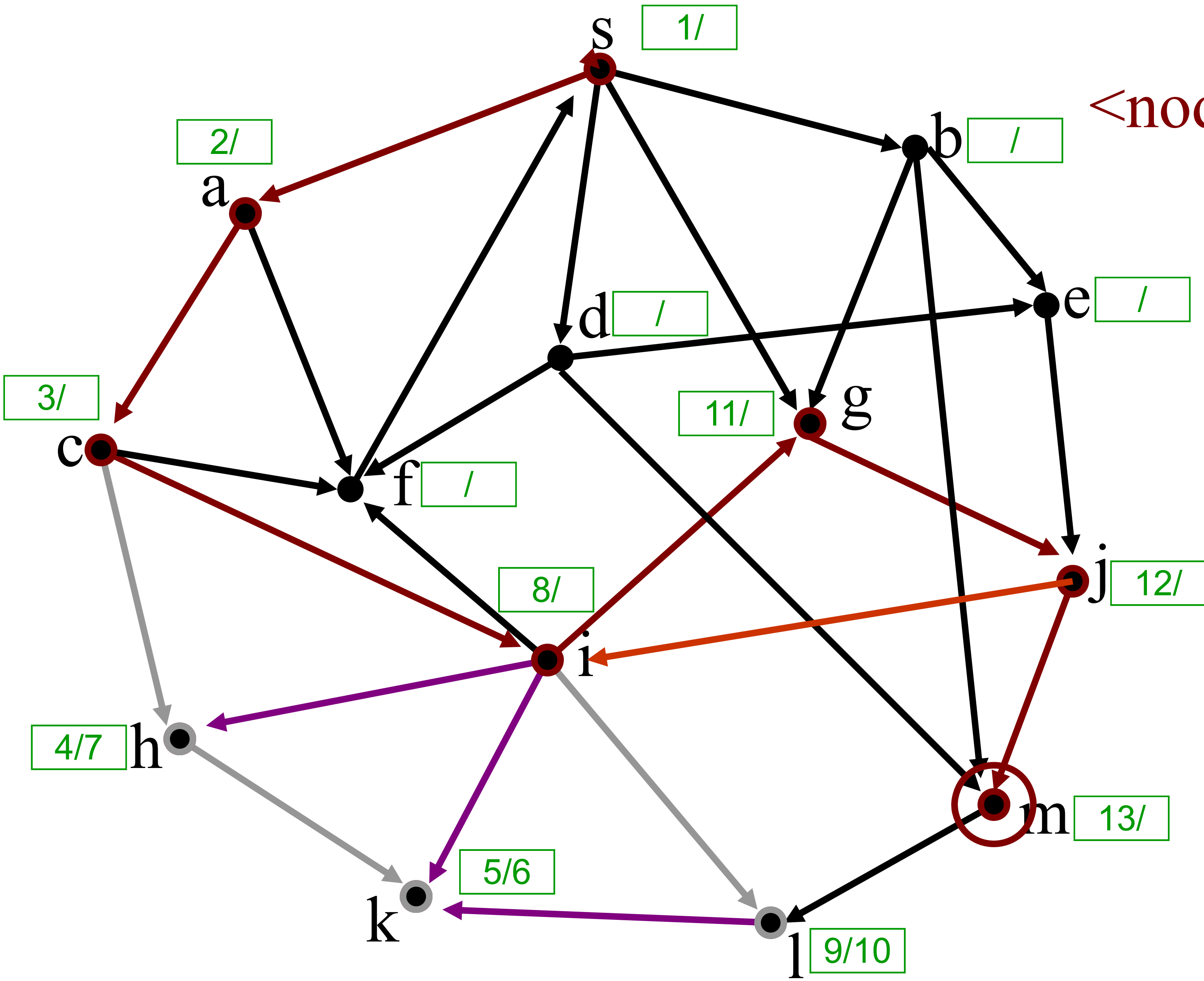
j,1
g,1
i,4
c,2
a,1
s,1

DFS



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Stack
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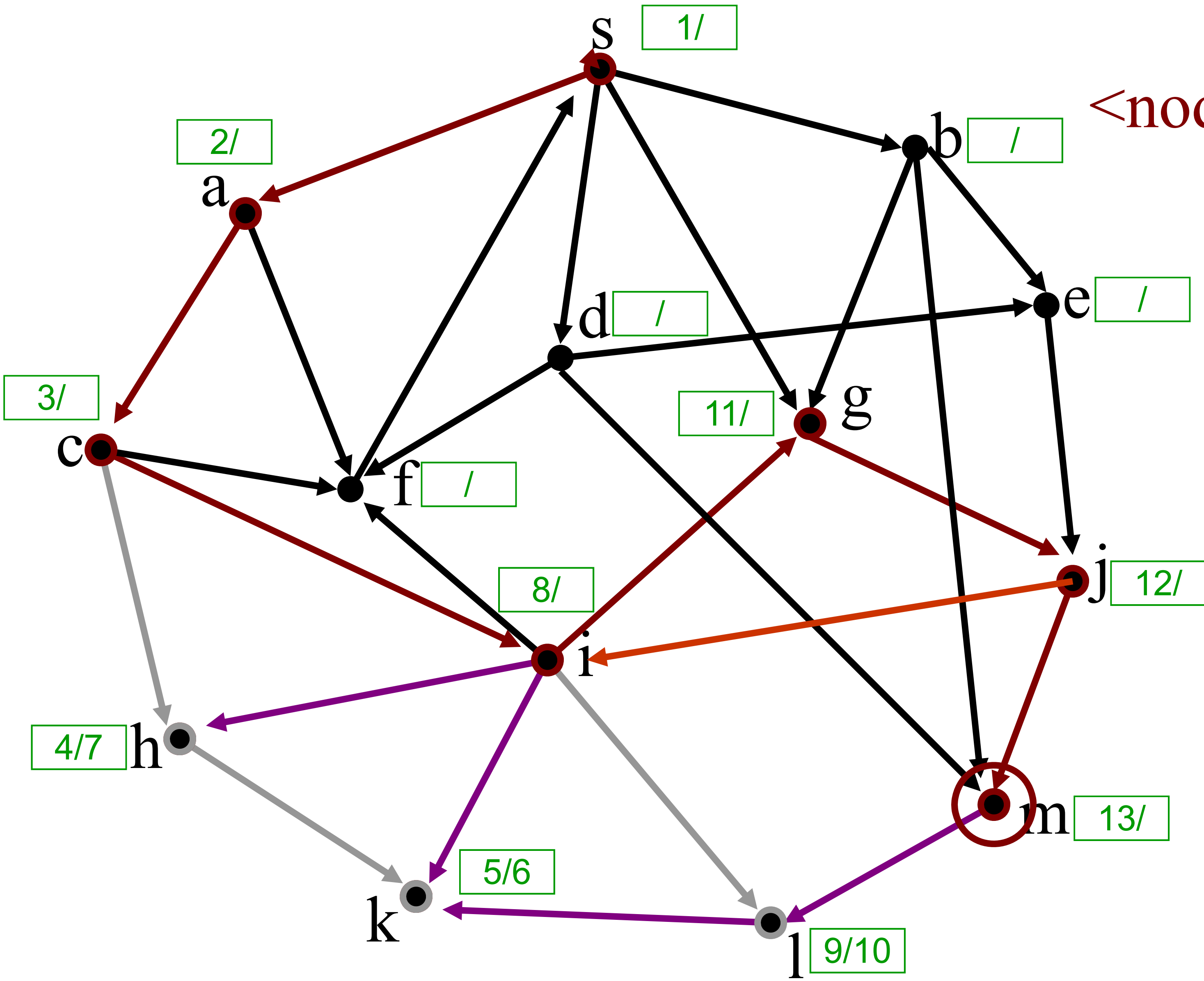
m,0
j,2
g,1
i,4
c,2
a,1
s,1

DFS



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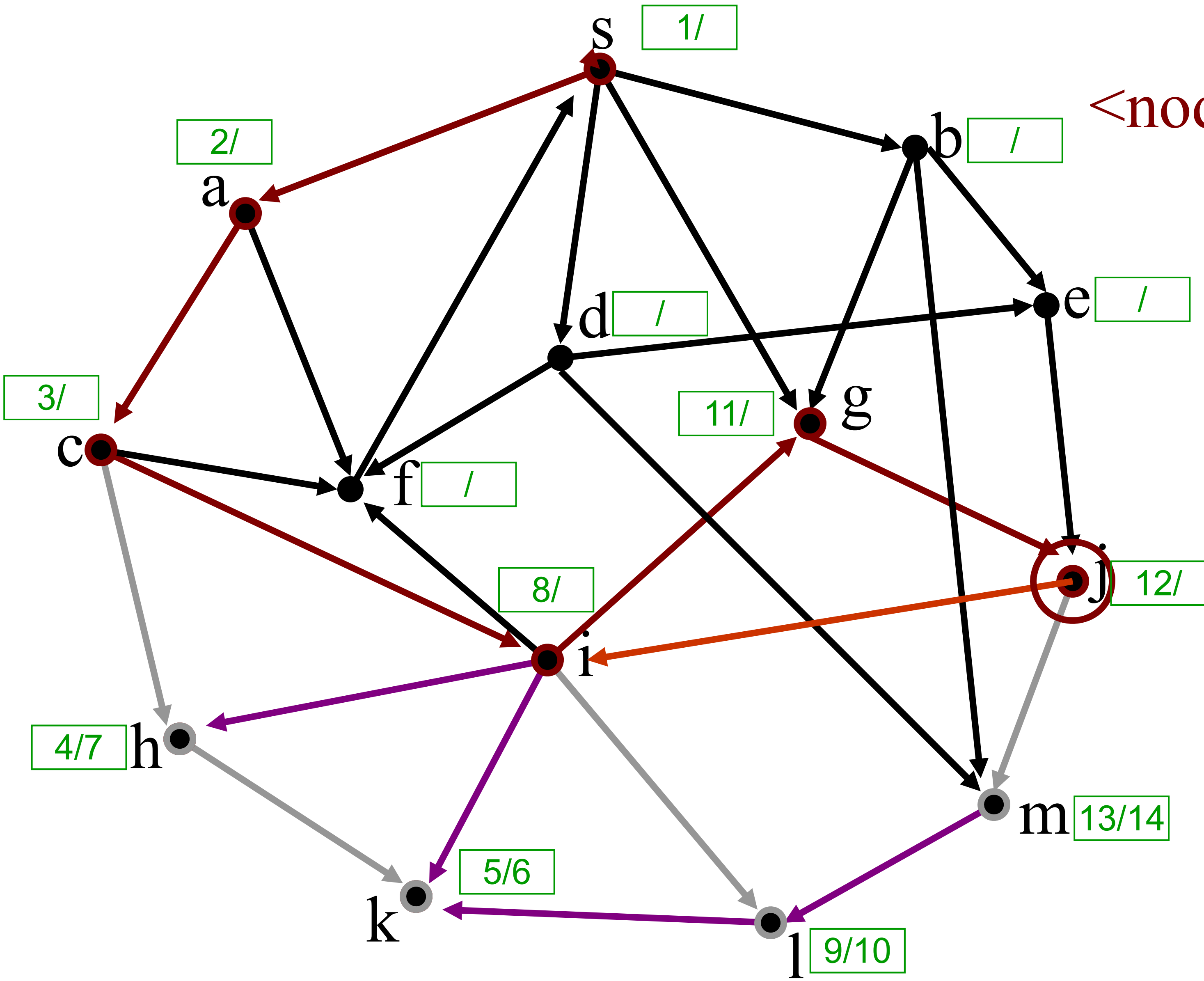
Stack
<node,# edges>

m,1
j,2
g,1
i,4
c,2
a,1
s,1

DFS



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Stack
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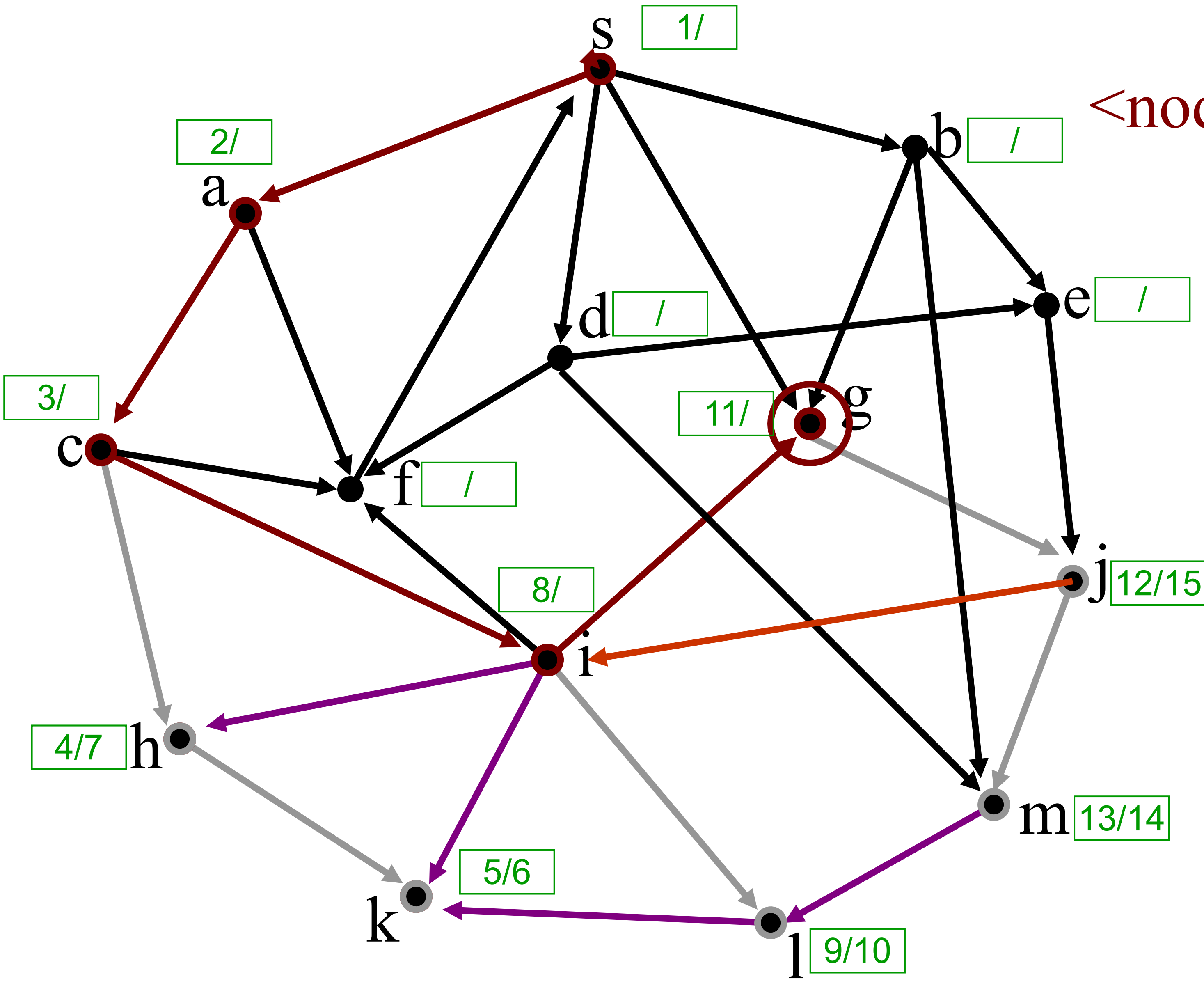
j,2
g,1
i,4
c,2
a,1
s,1

DFS



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Stack
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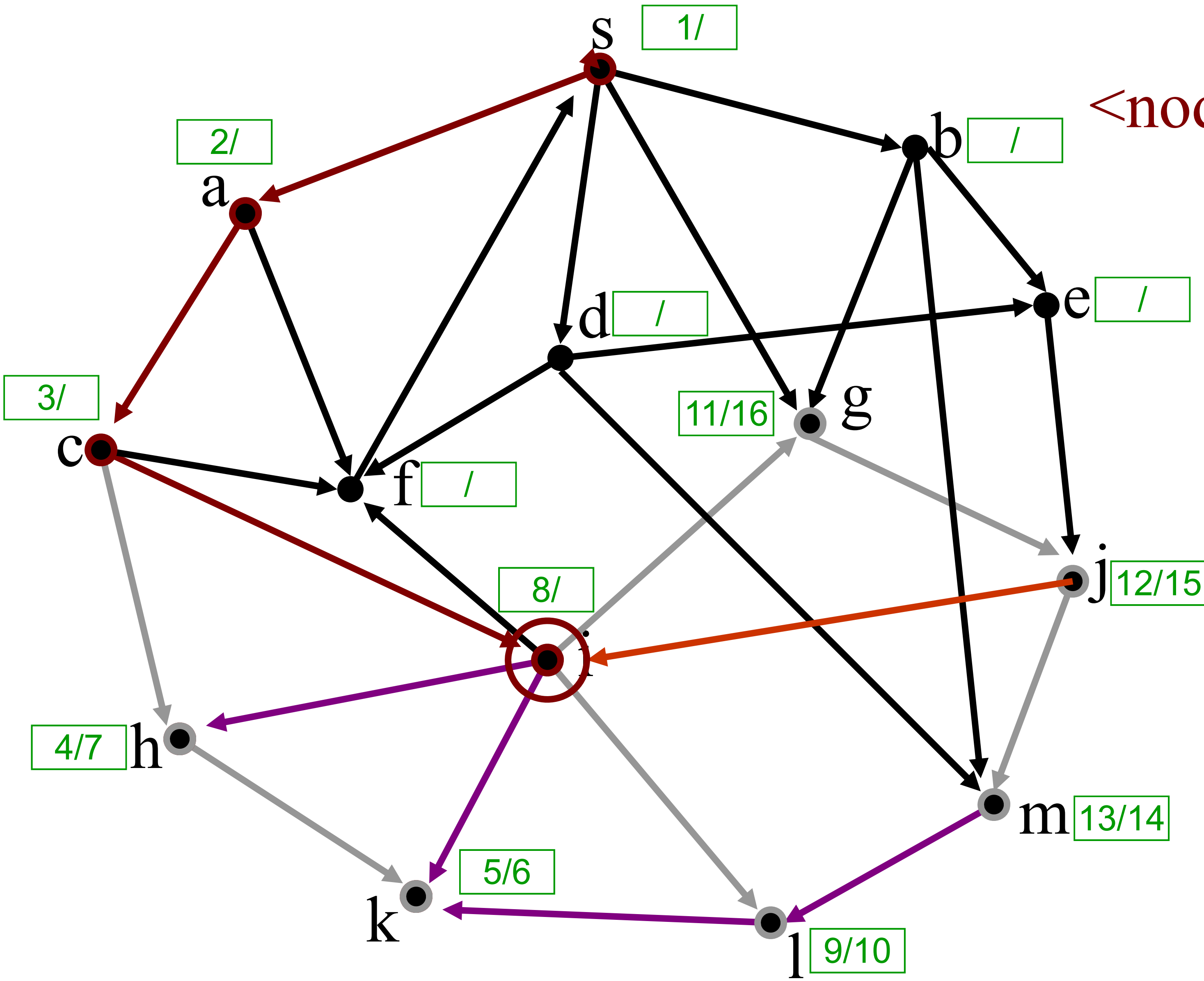
g,1
i,4
c,2
a,1
s,1

DFS



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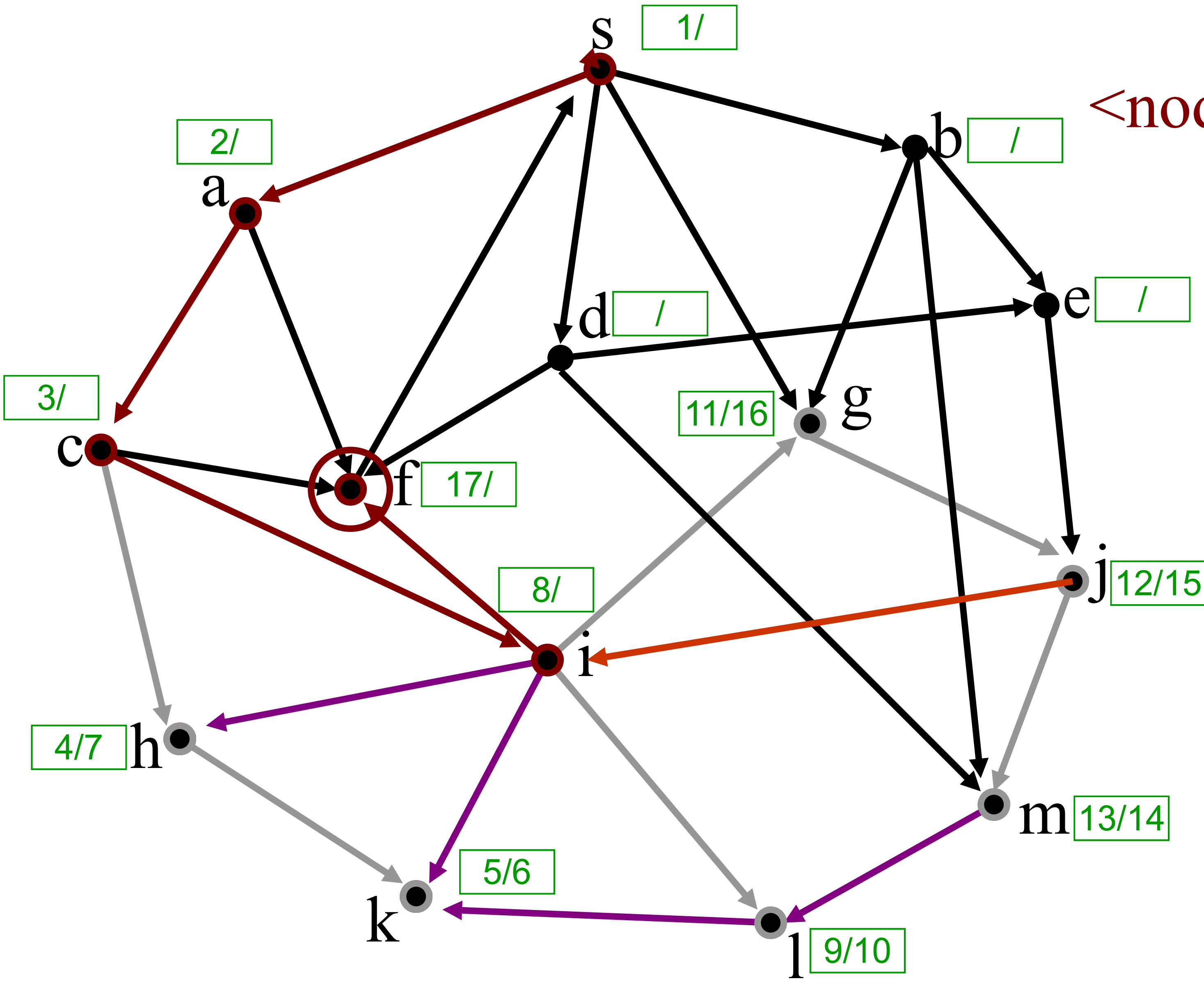
i,4
c,2
a,1
s,1

DFS



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Stack
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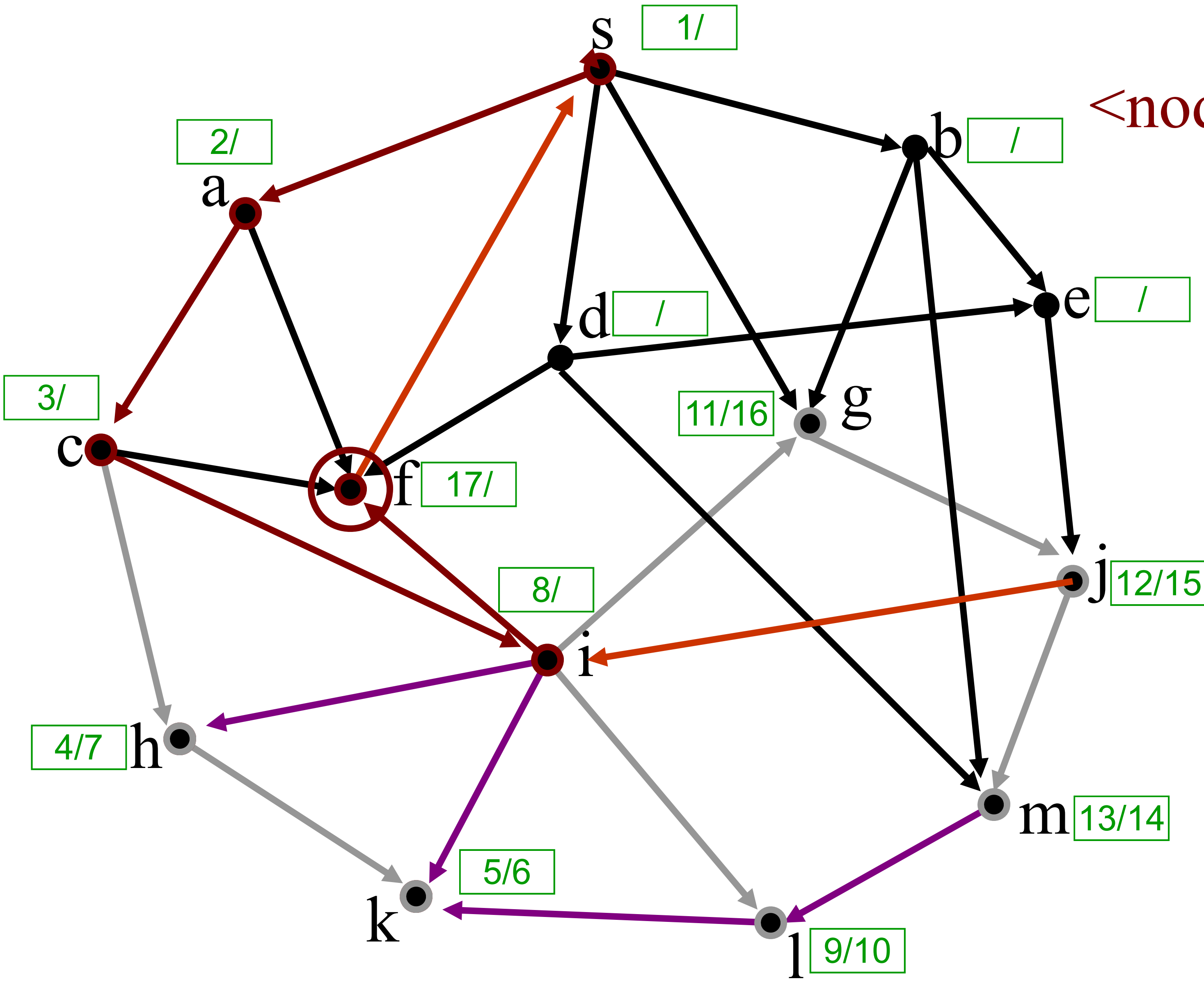
f,0
i,5
c,2
a,1
s,1

DFS



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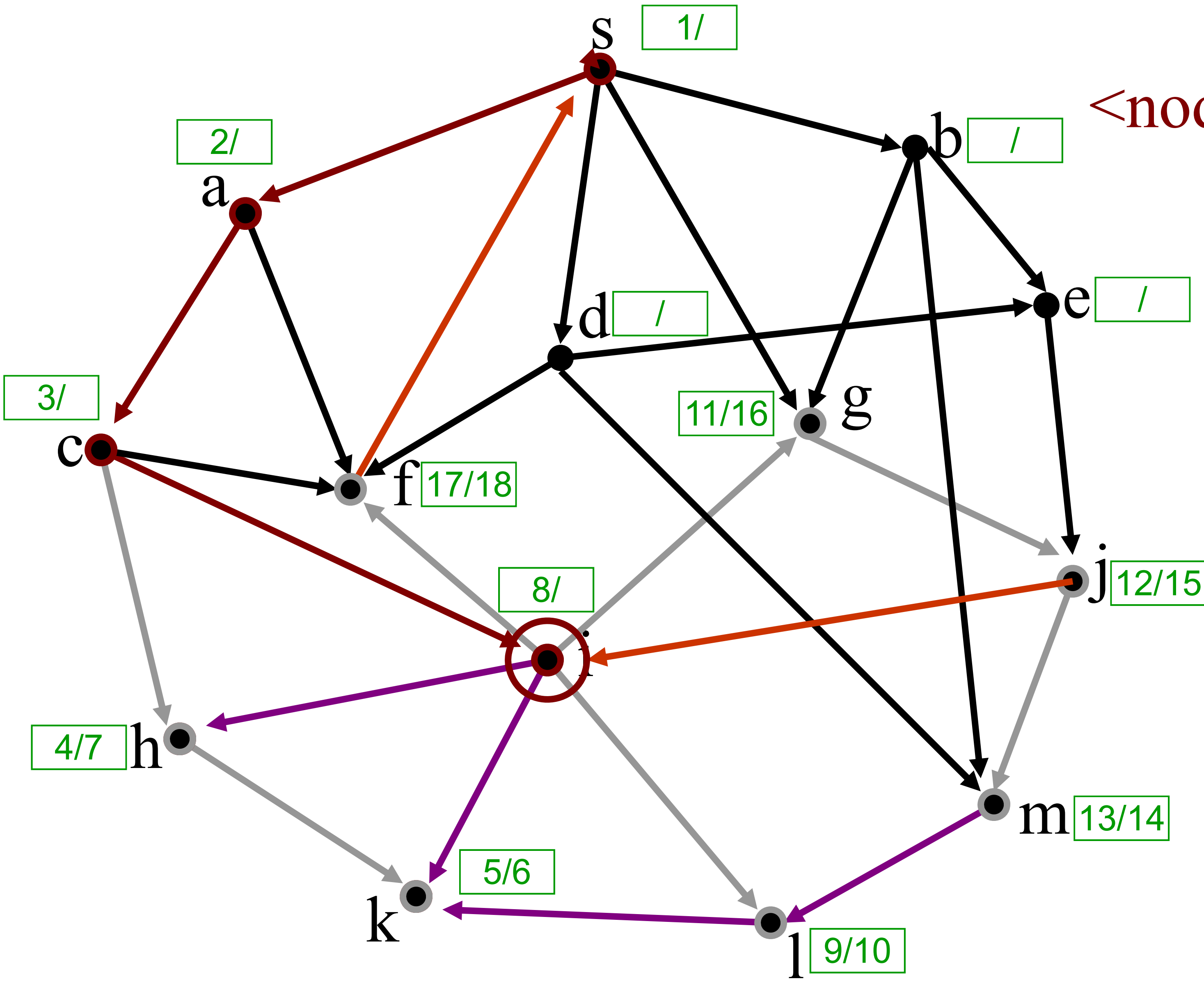
f,1
i,5
c,2
a,1
s,1

DFS



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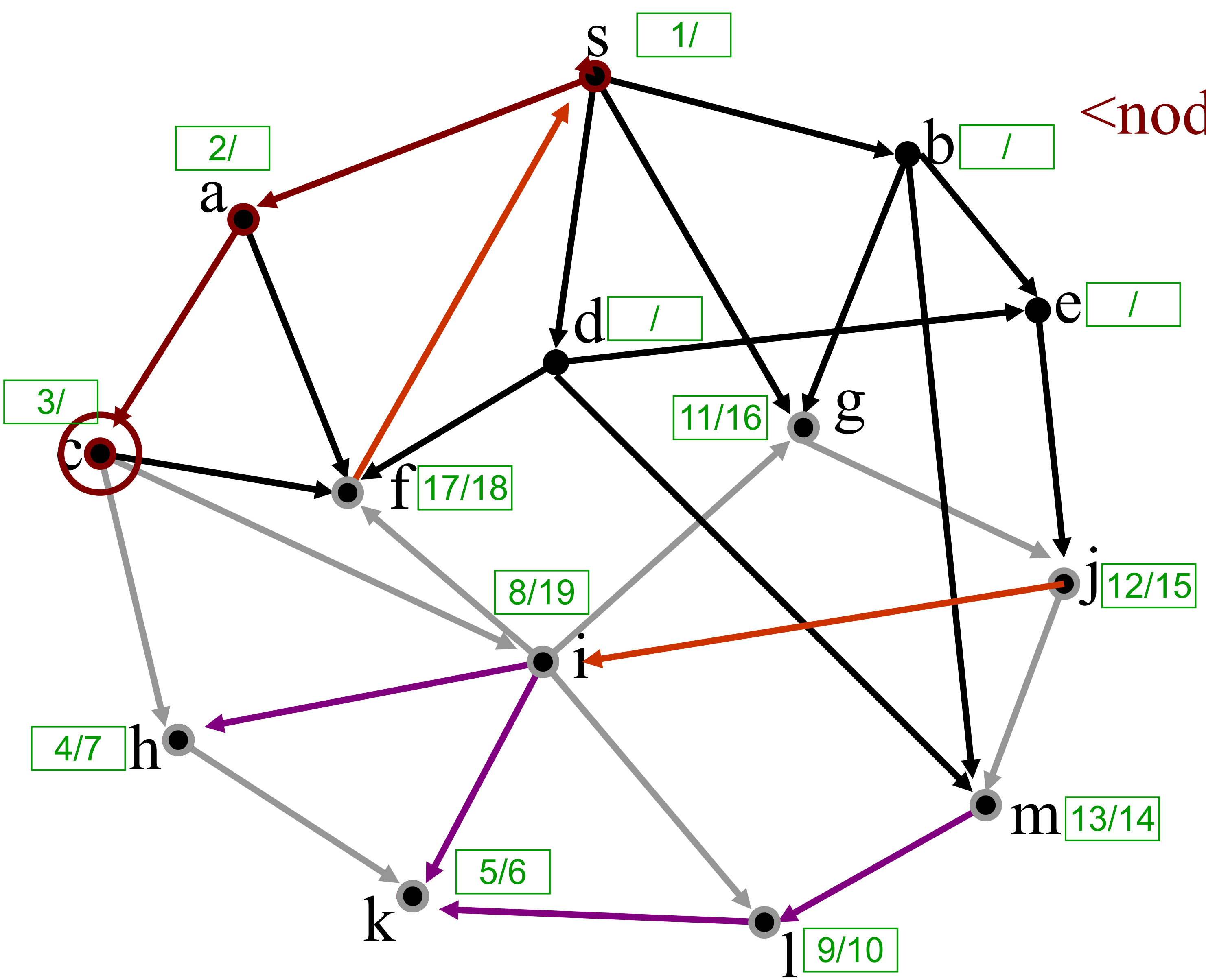
i,5
c,2
a,1
s,1

DFS



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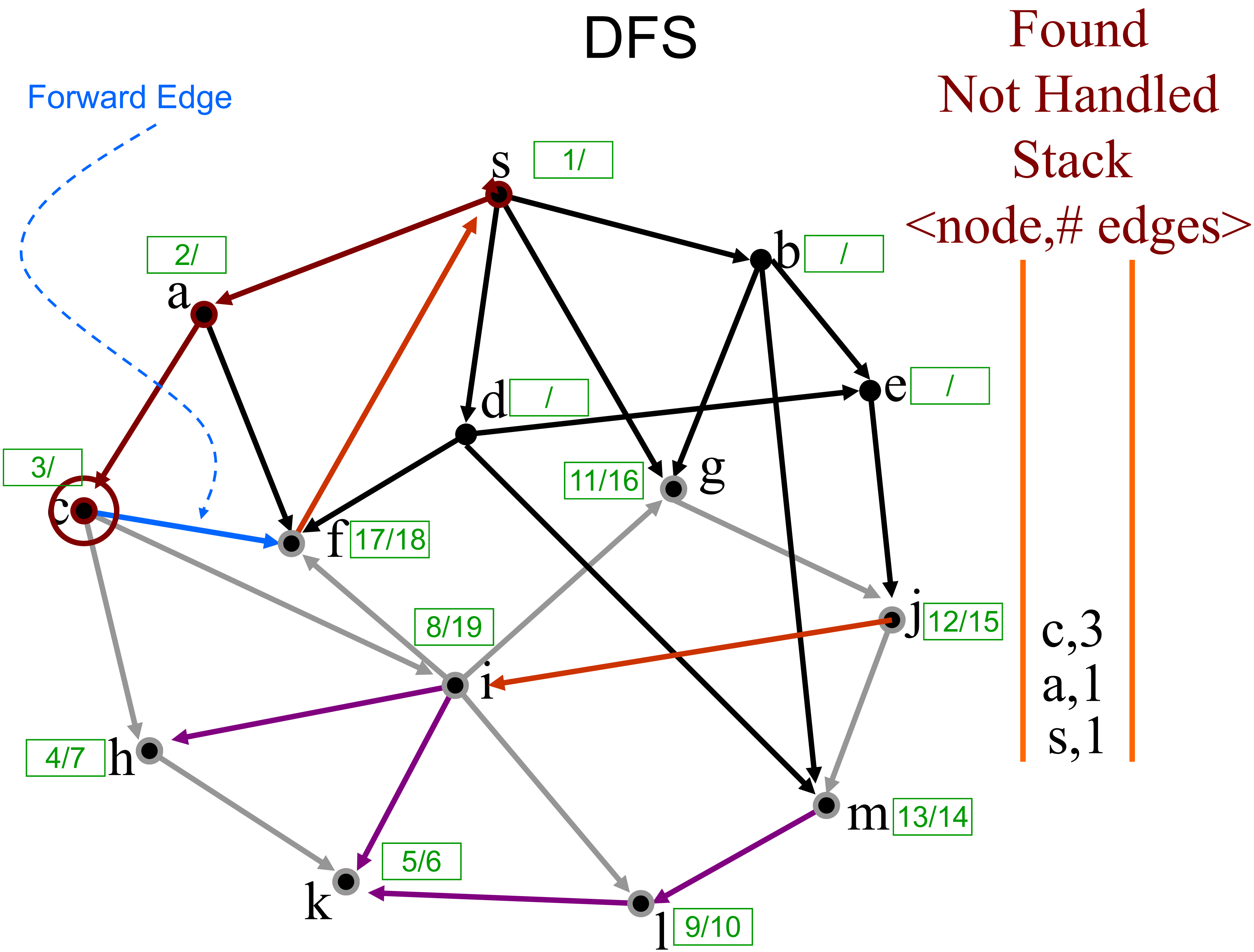
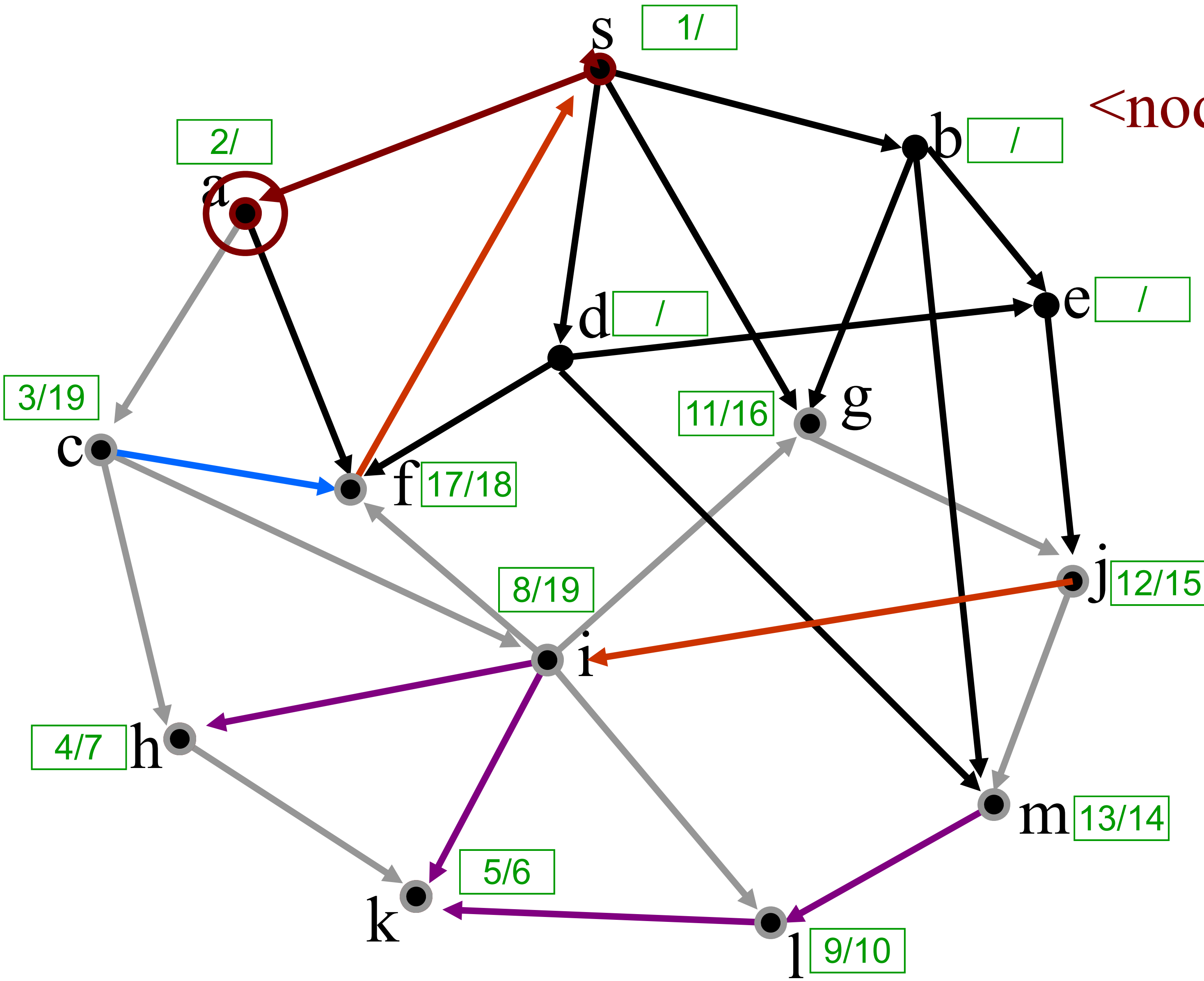
c,2
a,1
s,1

DFS



Found
Not Handled
Stack
<node,# edges>
Forward Edge
s
1/
a
2/
b
/
d
/
e
/
c
3/
f
17/18
g
11/16
j
12/15
i
8/19
h
4/7
m
13/14
k
5/6
l
9/10
c,3
a,1
s,1

DFS



Found
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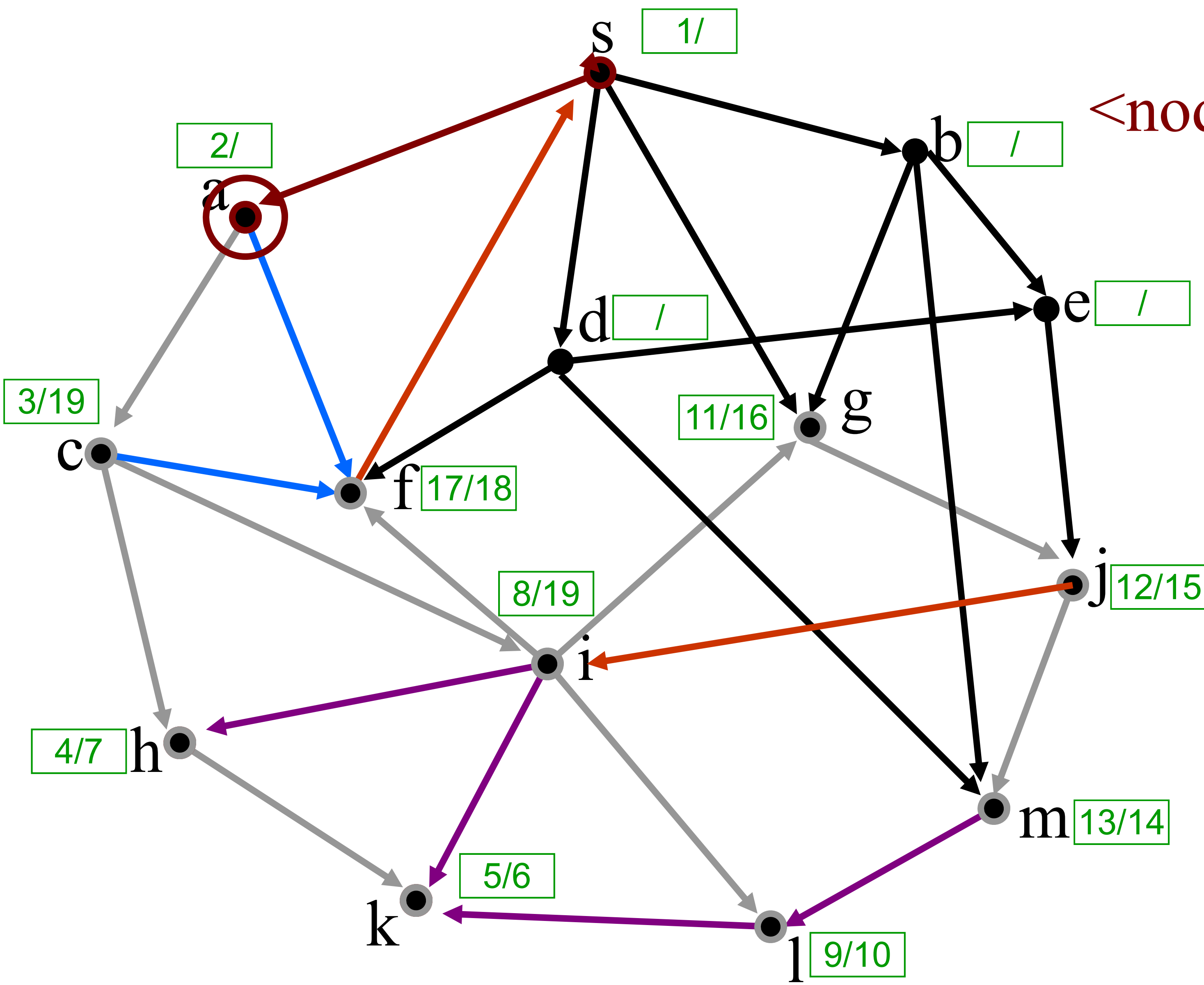
a,1
s,1

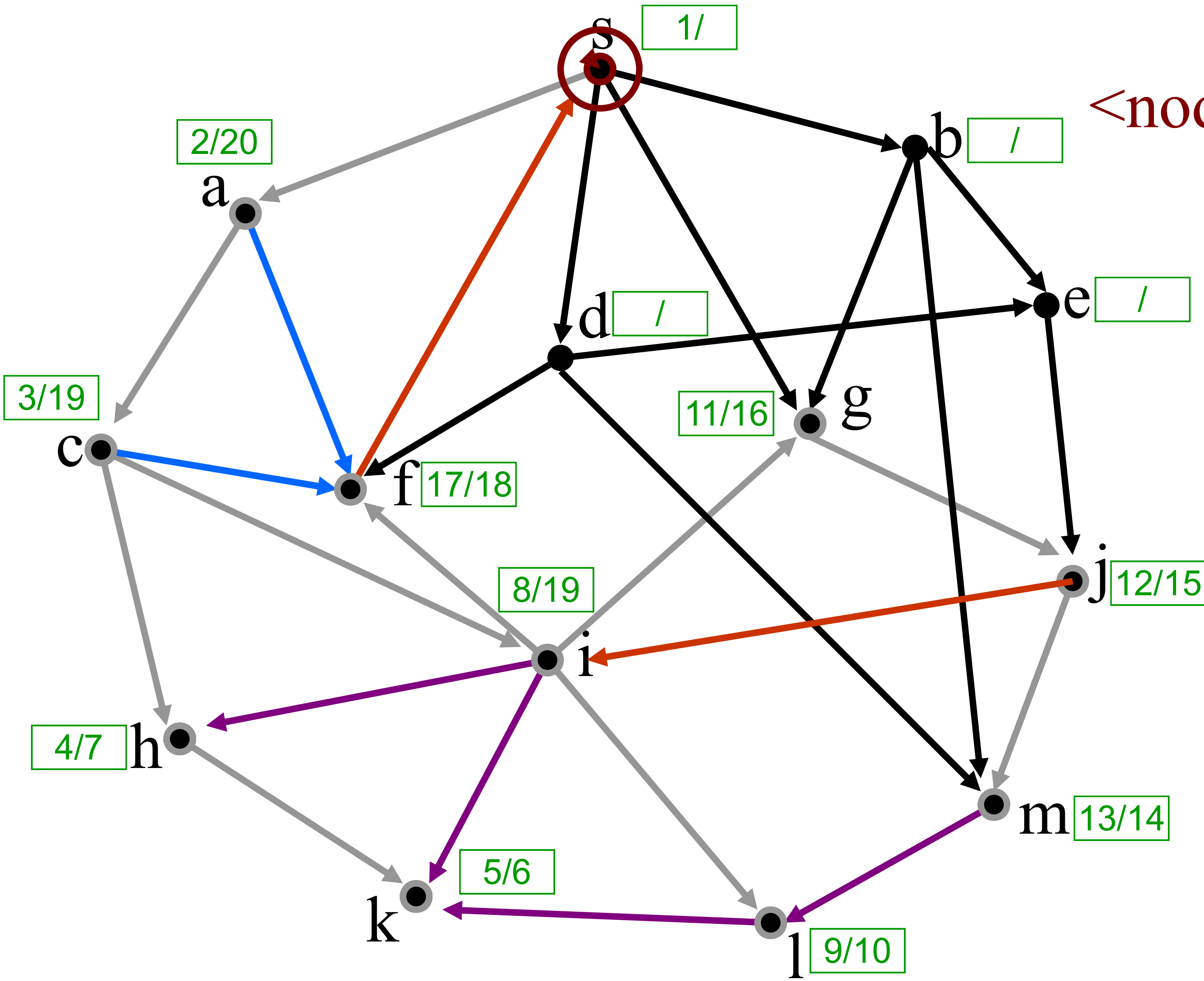
DFS

Found
Not Handled
Stack
<node,# edges>



a,2
s,1

DFS



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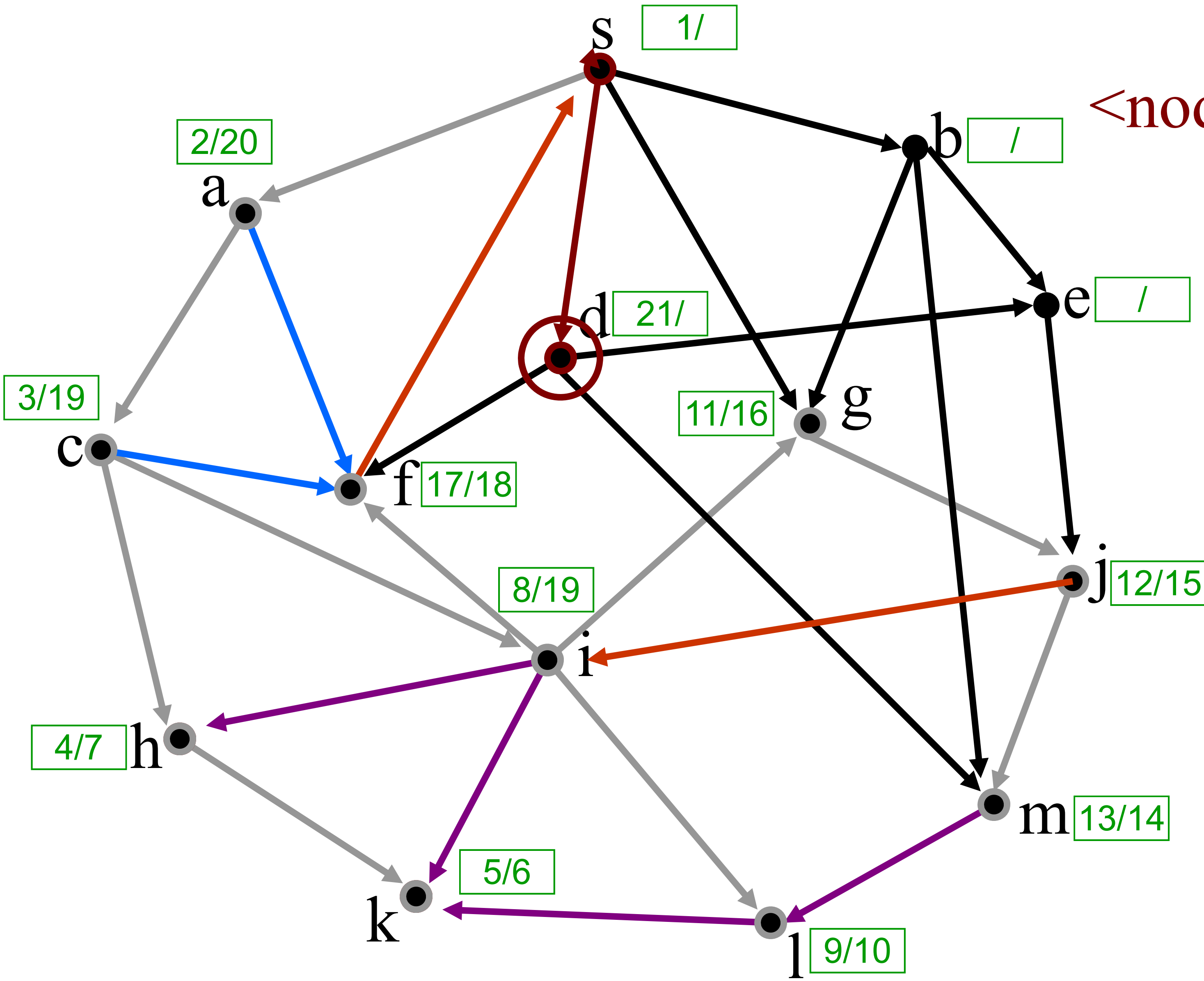
s, 1

DFS



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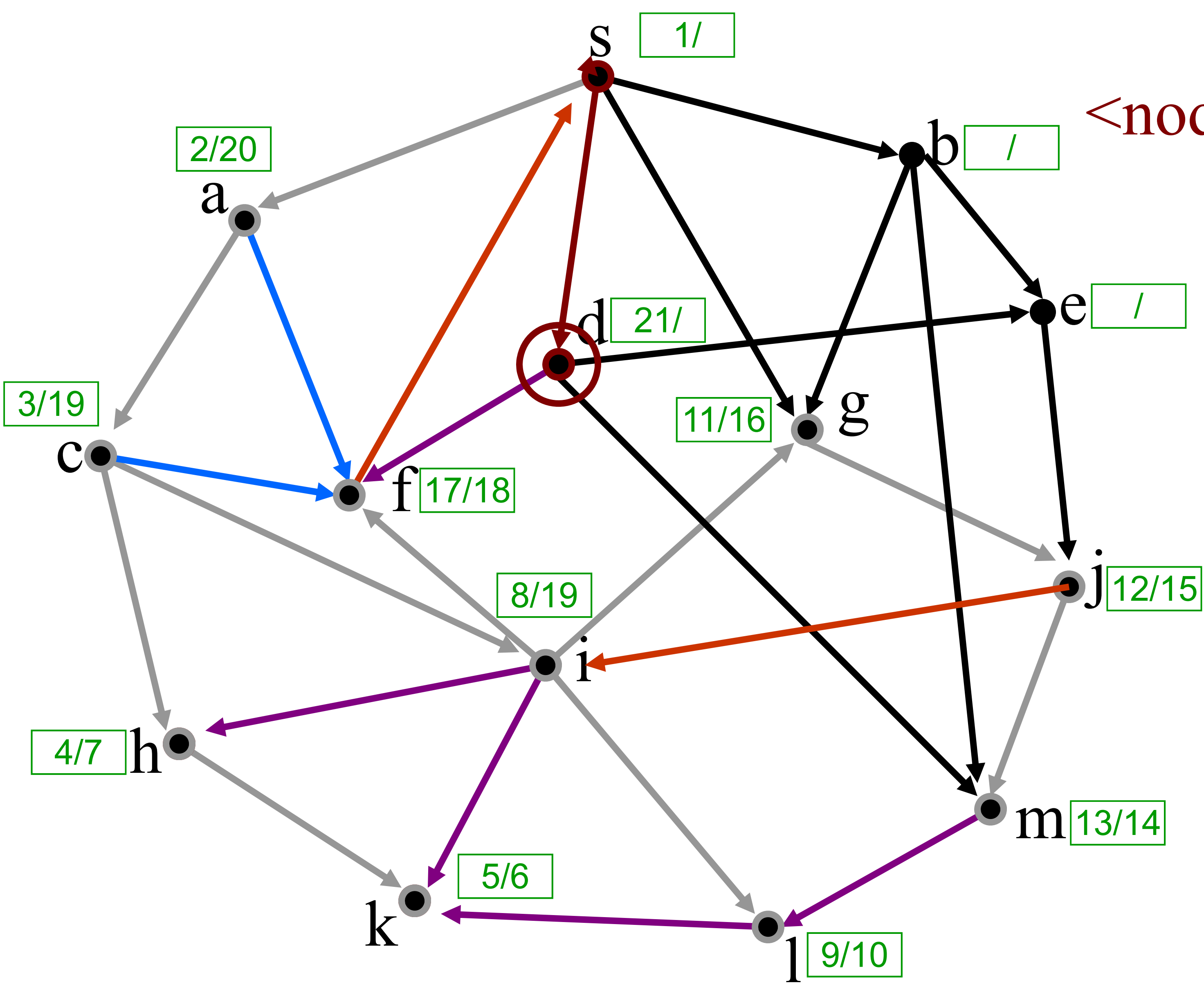
d,0
s,2

DFS



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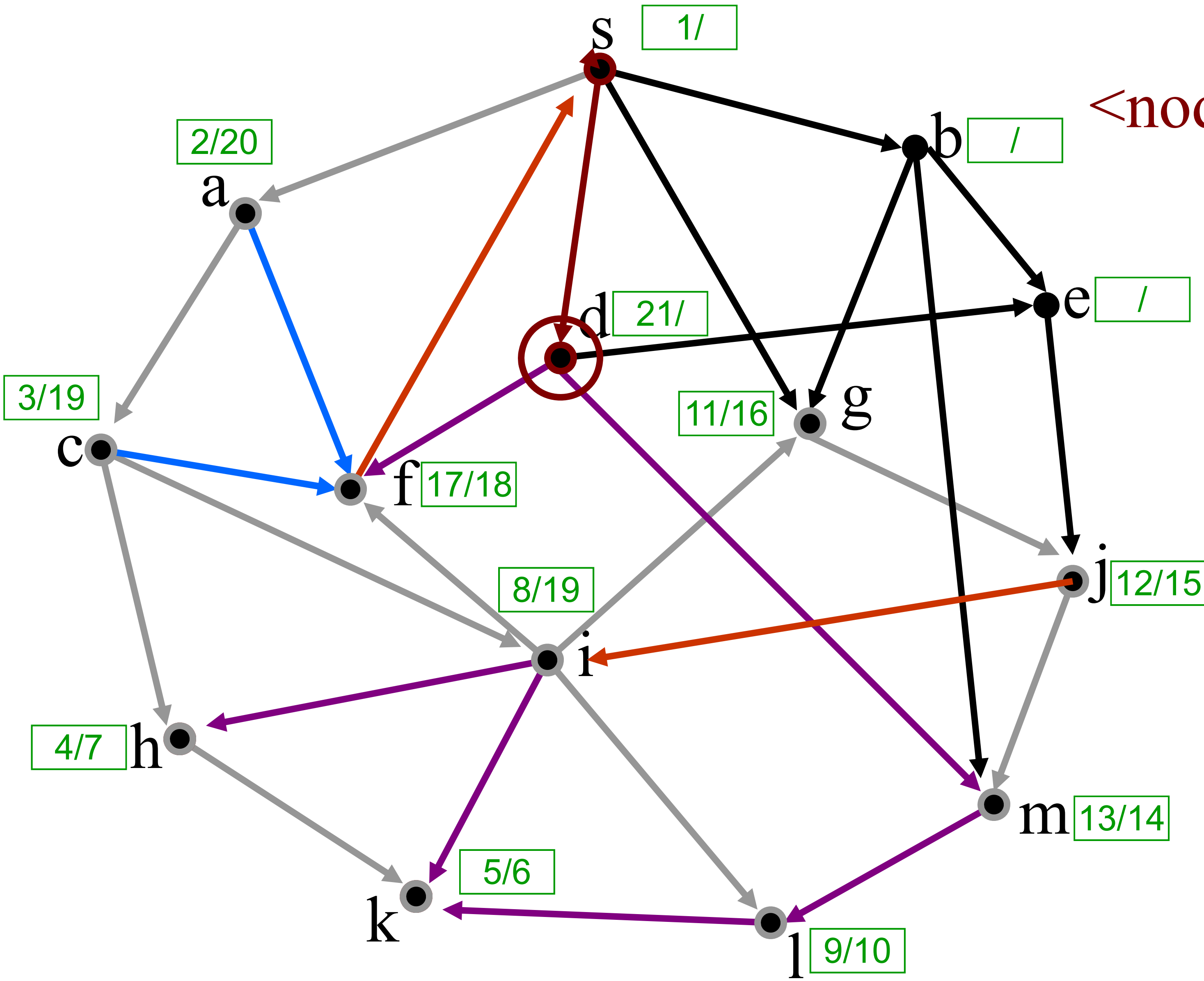
d,1
s,2

DFS



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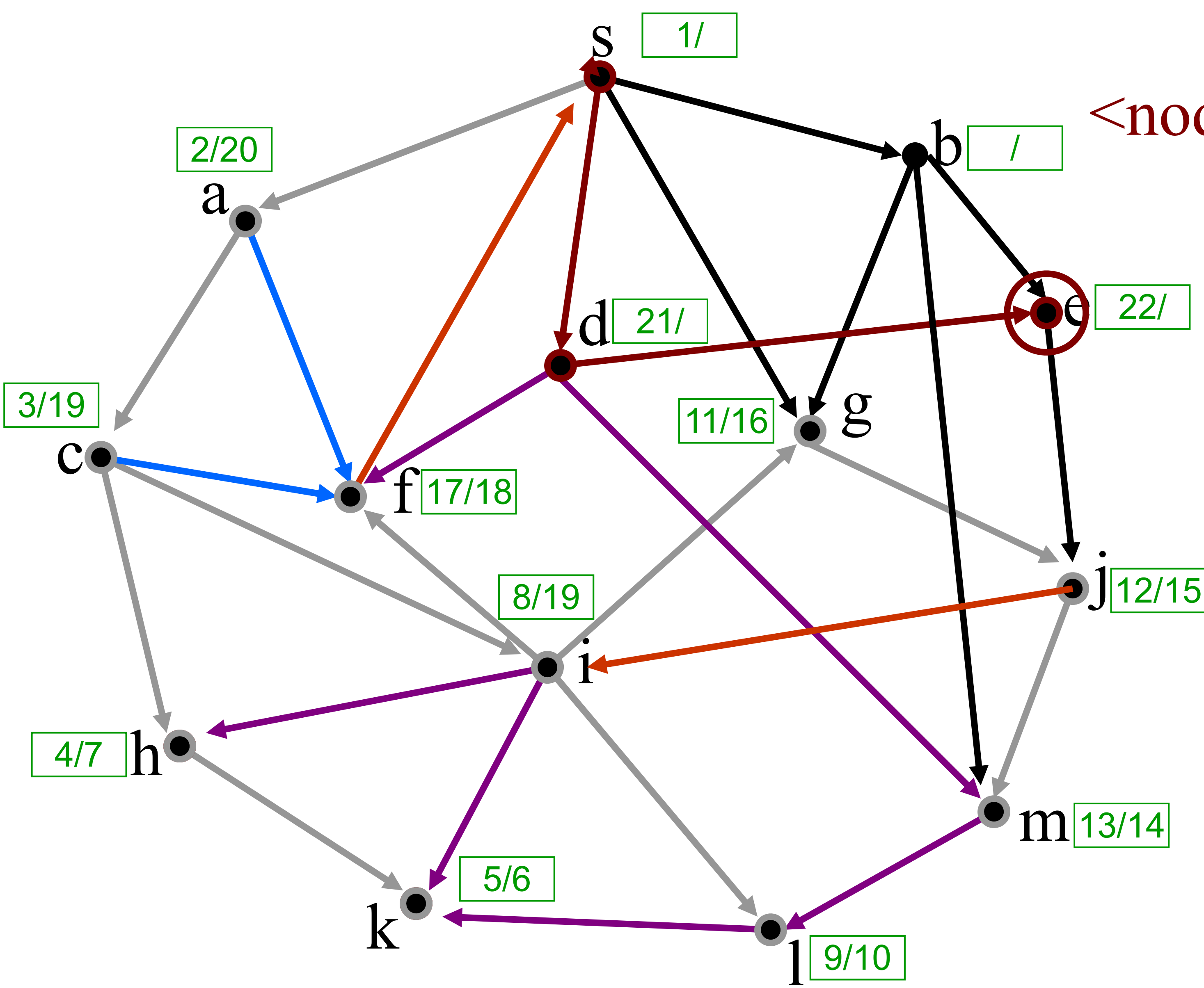
d,2
s,2

DFS



Found
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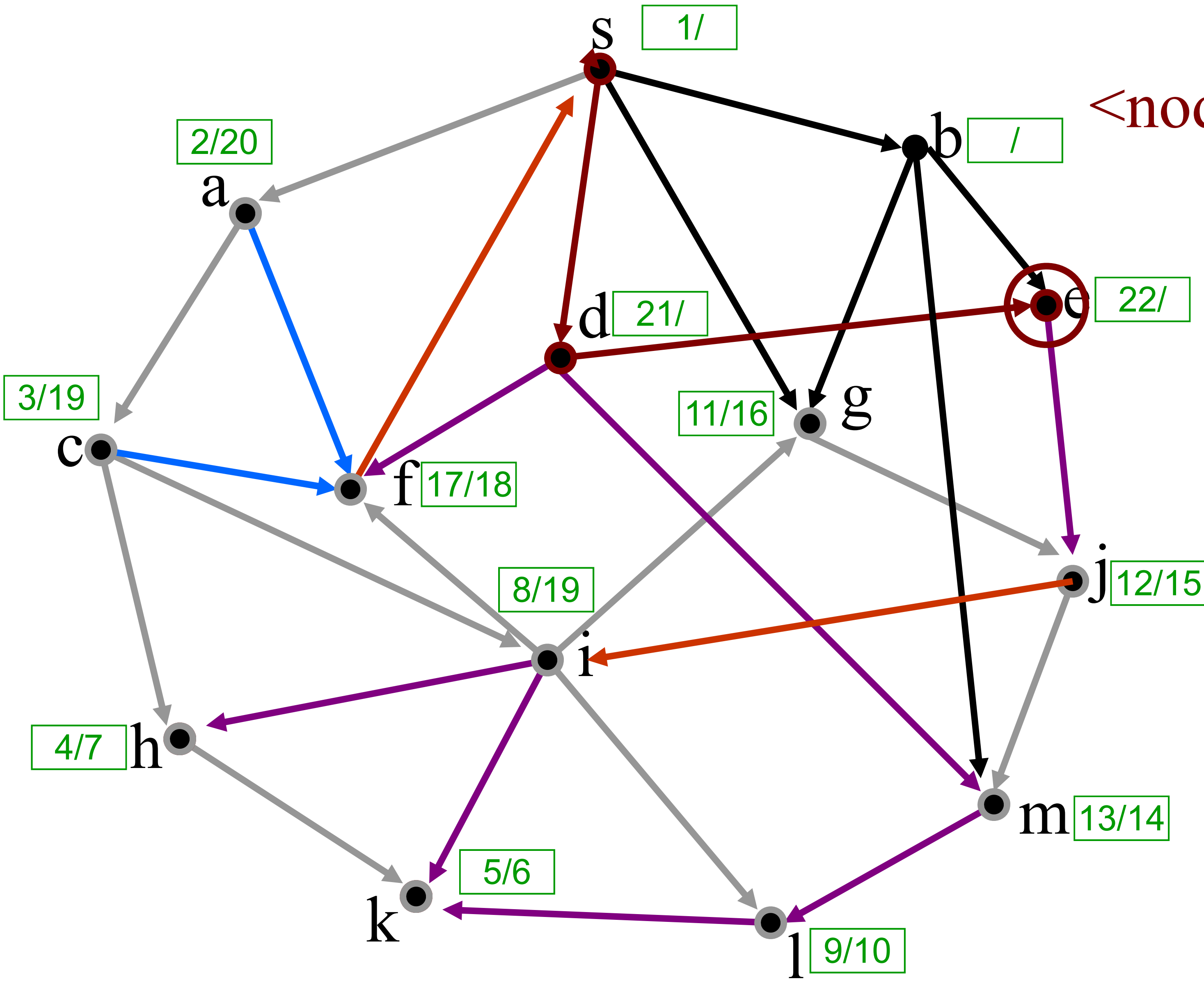
Stack
<node,# edges>

e,0
d,3
s,2

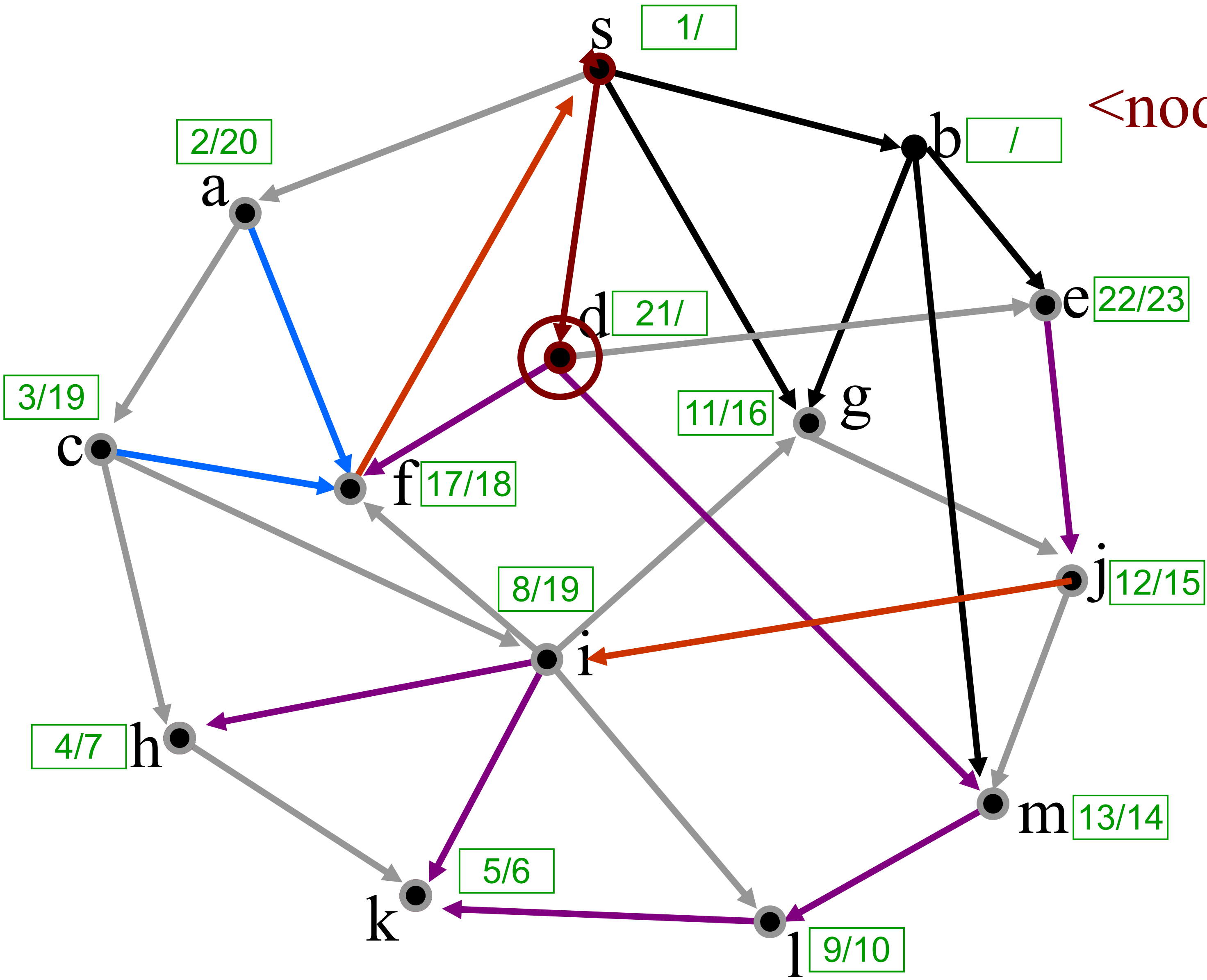
DFS

Found
Not Handled
Stack
<node,# edges>



e,1
d,3
s,2

DFS



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Stack
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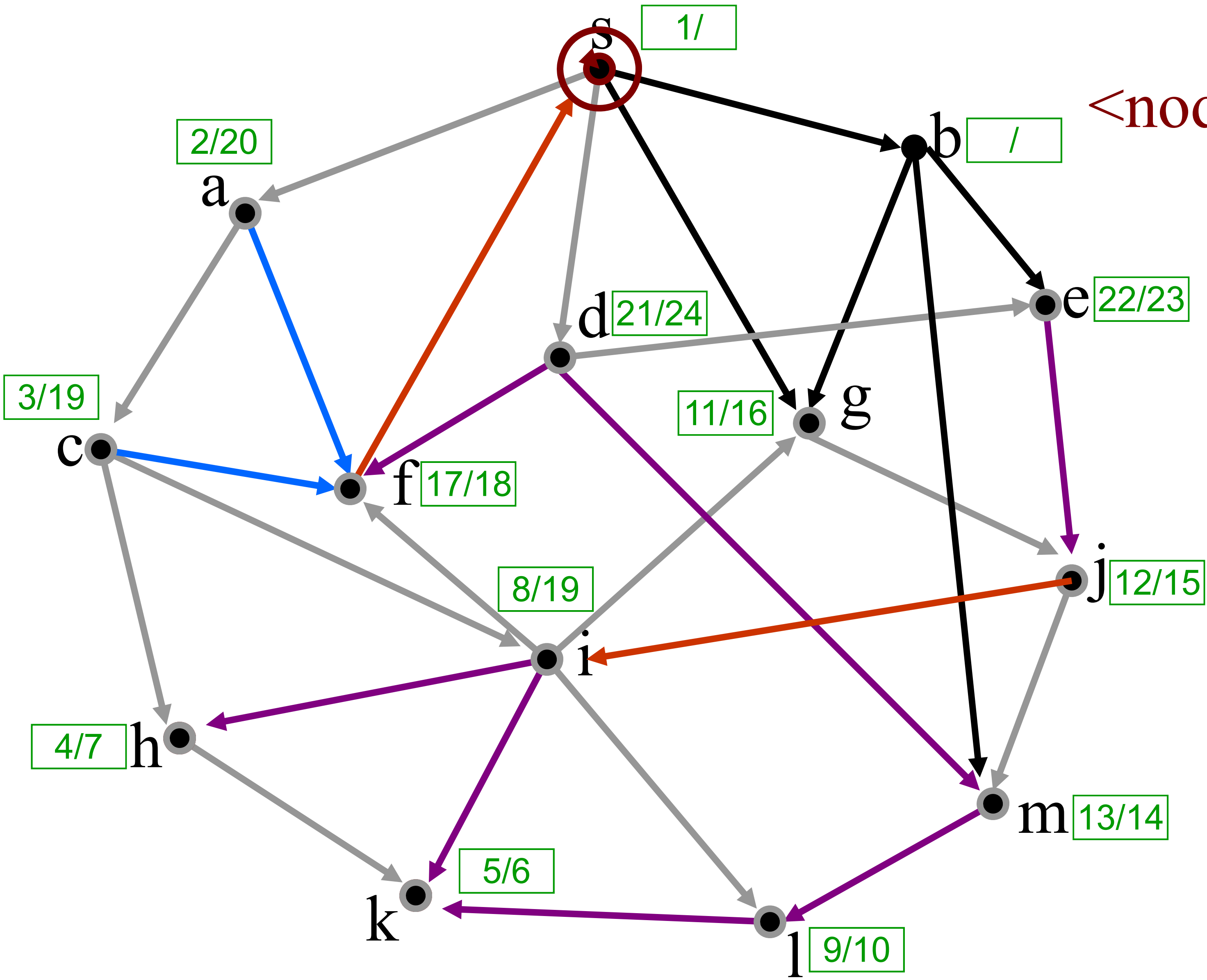
d,3
s,2

DFS



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Stack
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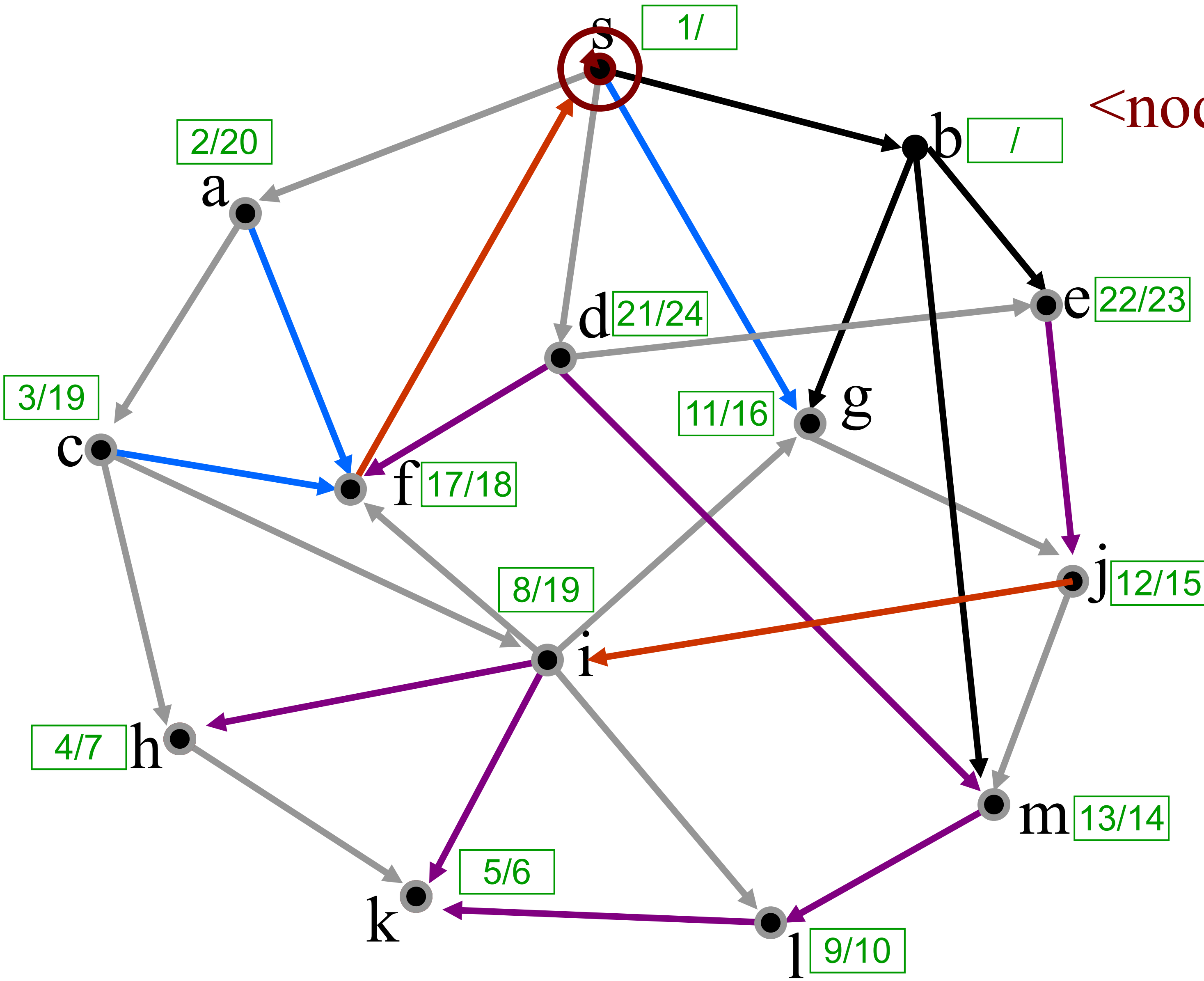
s,2

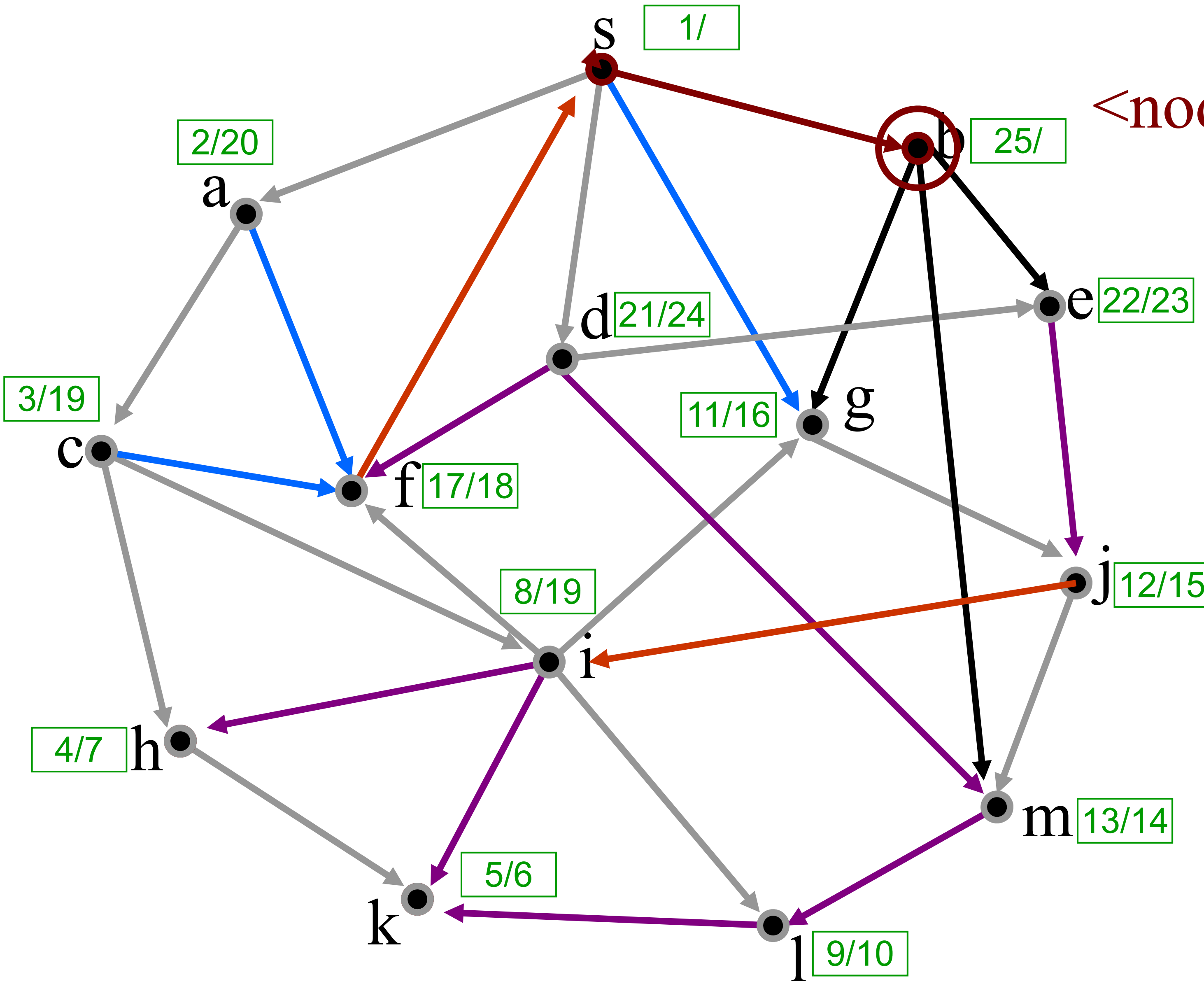
DFS



Found
Not Handled
Stack
<node,# edges>

s,3

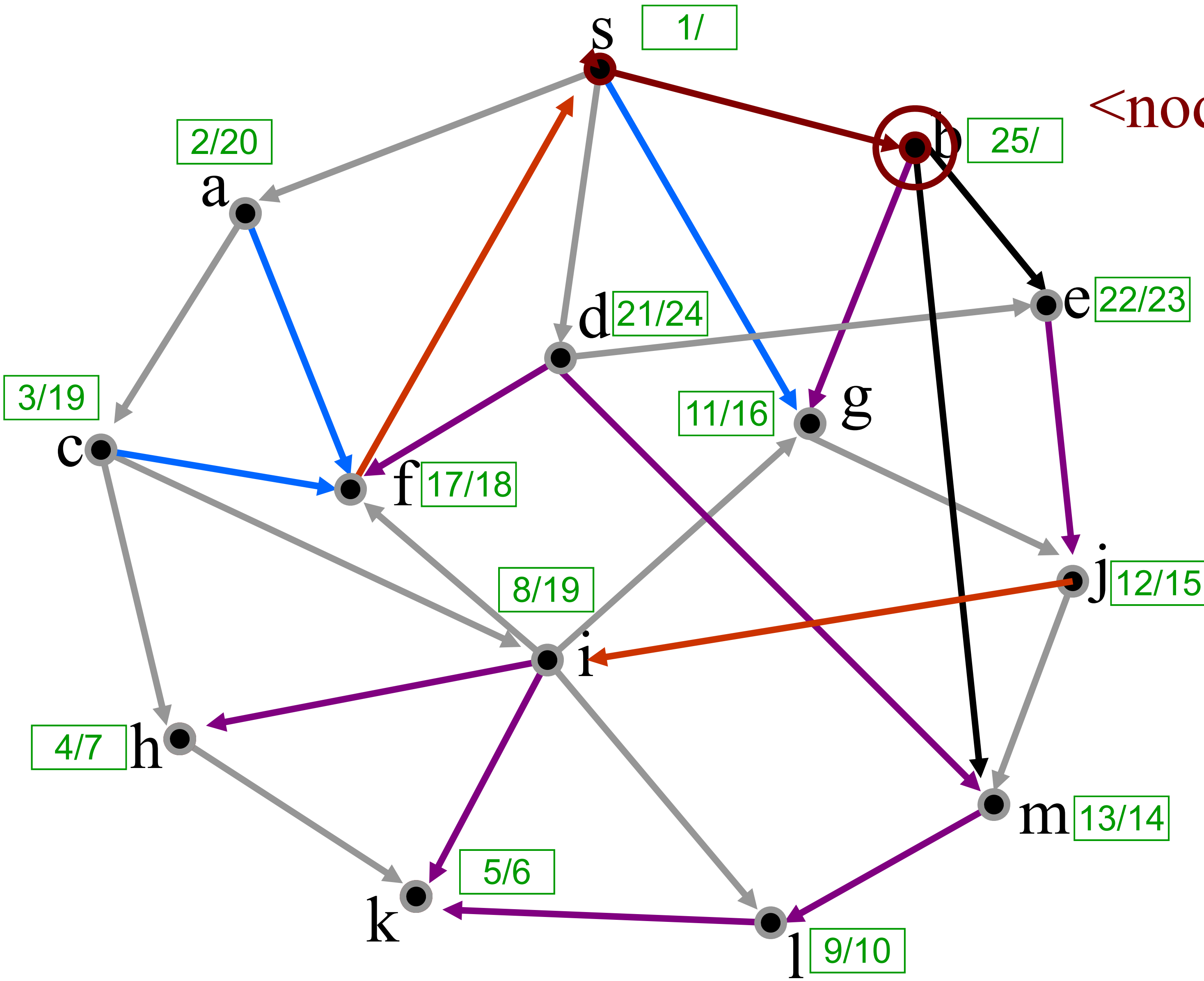
DFS



Found
Not Handled
Stack
<node,# edges>

b,0
s,4

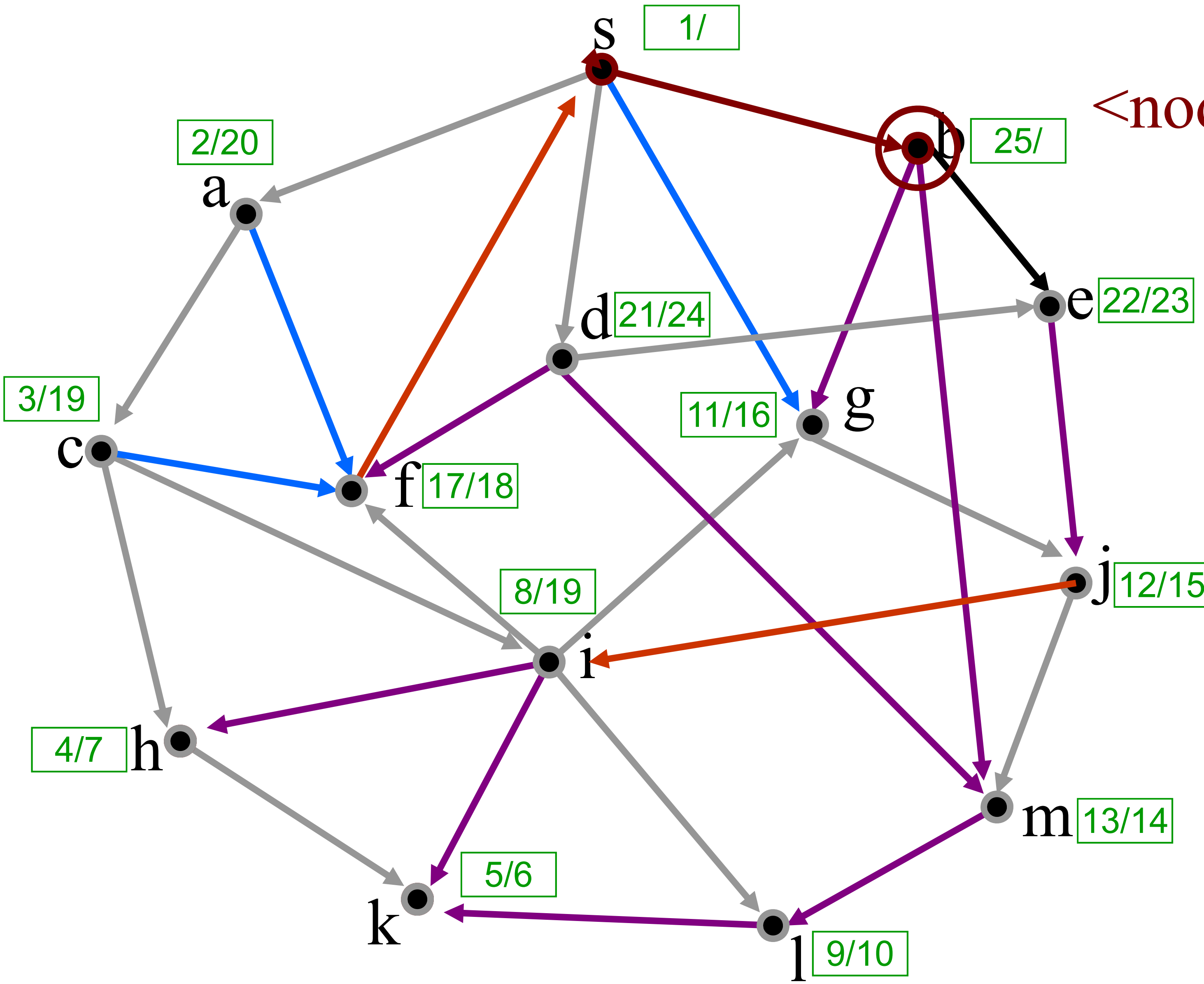
DFS



Found
Not Handled
Stack
<node,# edges>

b,1
s,4

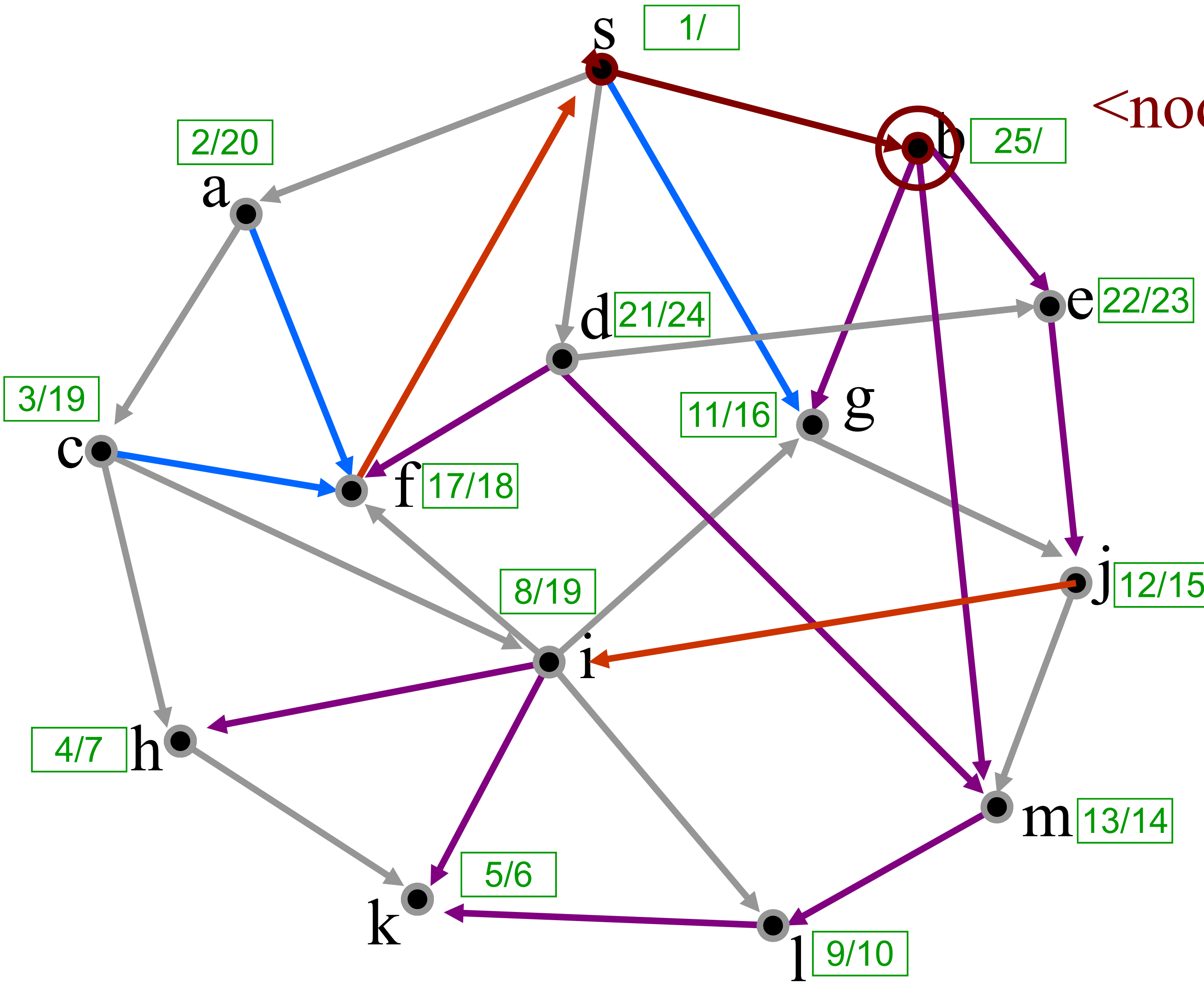
DFS



Found
Not Handled
Stack
<node,# edges>

b,2
s,4

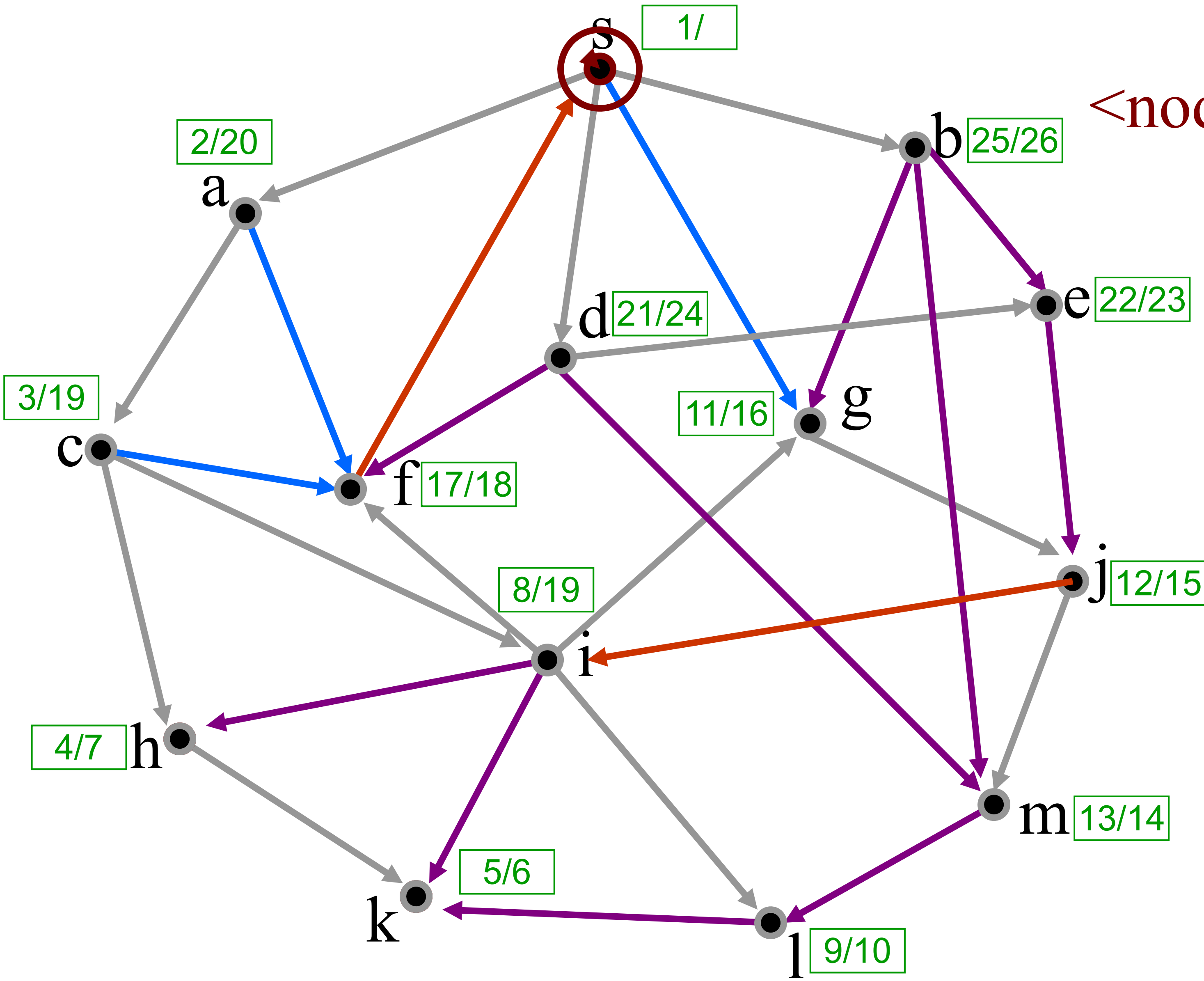
DFS



Found
Not Handled
Stack
<node,# edges>

b,3
s,4

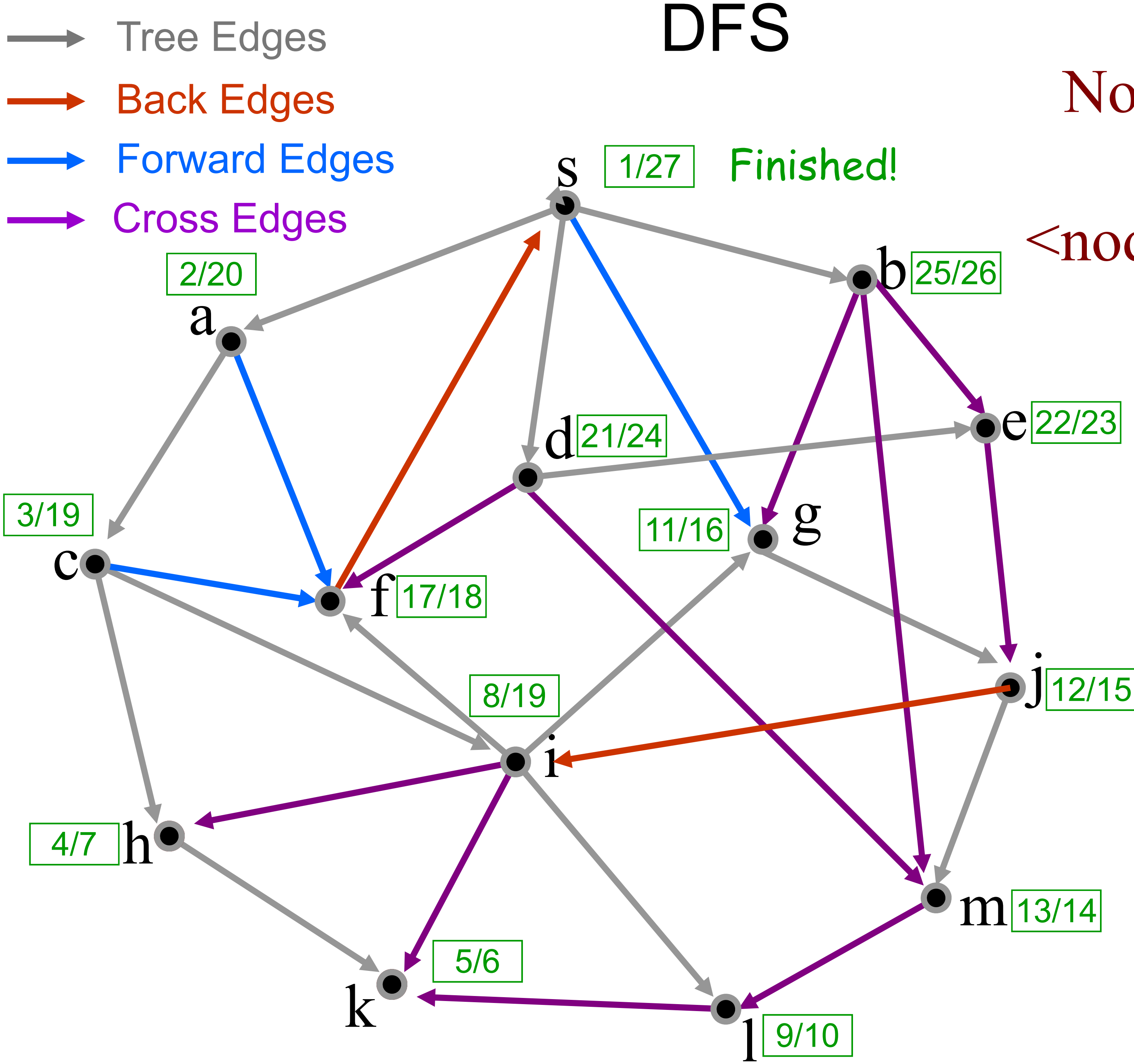
DFS



Found
Not Handled
Stack
<node,# edges>

s,4

DFS



Found
Not Handled
Stack
<node,# edges>

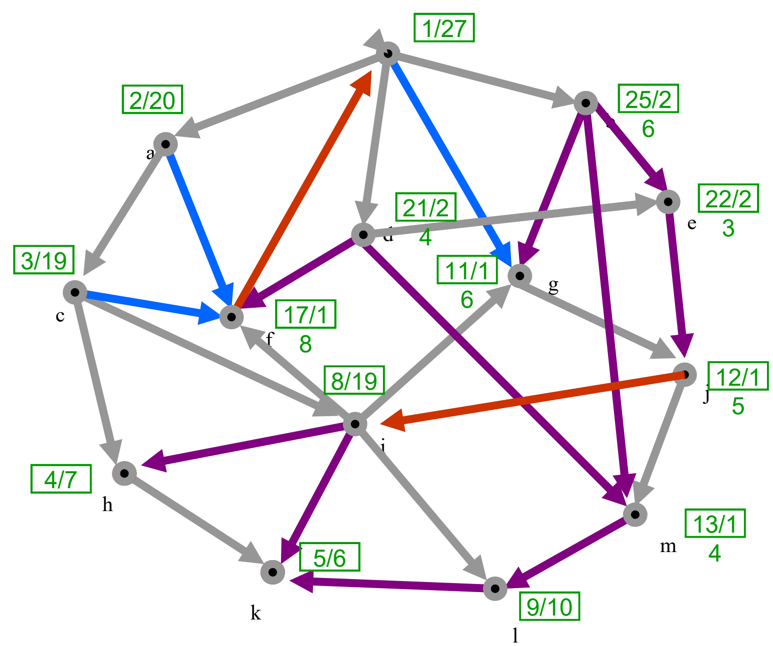
Classification of Edges in DFS

1. ***Tree edges*** are edges in the depth-first forest G_π. Edge (u, v) is a tree edge if v was first discovered by exploring edge (u, v).

2. ***Back edges*** are those edges (u, v) connecting a vertex u to an ancestor v in a depth-first tree.

3. ***Forward edges*** are non-tree edges (u, v) connecting a vertex u to a descendant v in a depth-first tree.

4. ***Cross edges*** are all other edges. They can go between vertices in the same depth-first tree, as long as one vertex is not an ancestor of the other.



Classification of Edges in DFS

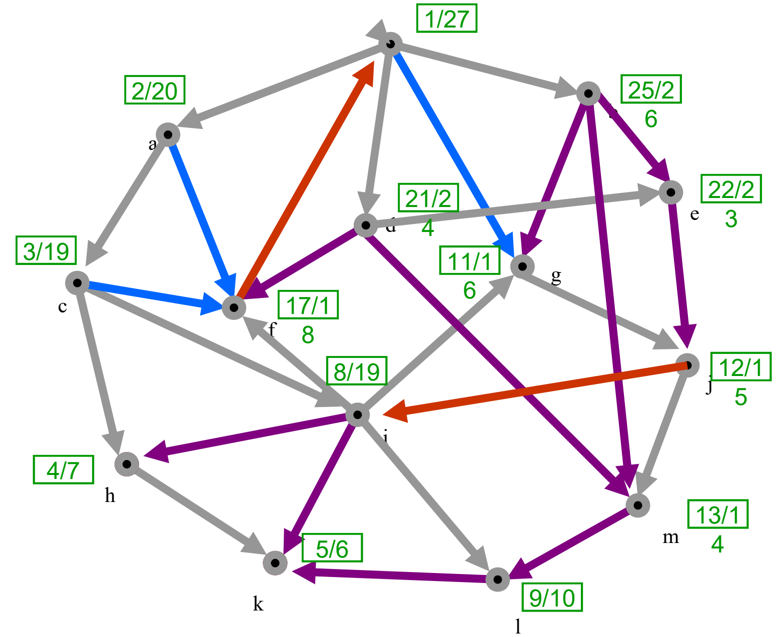
1. ***Tree edges***: Edge (*u*, *v*) is a **tree edge** if *v* was **black** when (*u*, *v*) traversed.

2. ***Back edges:*** (*u*, *v*) is a **back edge** if v was **red** when *(u, v)* traversed.

3. ***Forward edges***: *(u, v)* is a **forward edge** if v was **gray** when *(u, v)* traversed and $d[v] > d[u]$.

4. ***Cross edges*** *(u,v)* is a **cross edge** if v was **gray** when *(u, v)* traversed and $d[v] < d[u]$.

Classifying edges can help to identify properties of the graph, e.g., a graph is acyclic iff DFS yields no back edges.

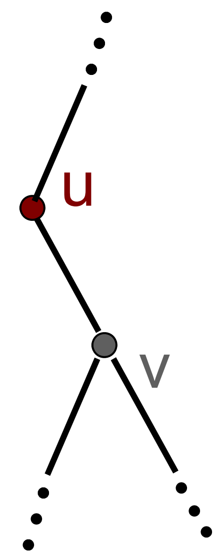


DFS on Undirected Graphs

- In a depth-first search of an ***undirected*** graph, every edge is either a **tree edge** or a **back edge**.
- **Why?**

DFS on Undirected Graphs

- **Suppose that (u,v)** is a **forward edge** or a **cross edge** in a DFS of an undirected graph.
- **(u,v)** is a **forward edge** or a **cross edge** when **v** is already **handled** (**grey**) when accessed from **u**.
- This means that all vertices reachable from **v** have been explored.
- Since we are currently handling **u**, **u** must be **red**.
- Clearly **v** is reachable from **u**.
- Since the graph is undirected, **u** must also be reachable from **v**.
- Thus **u** must already have been handled: **u** must be **grey**.
- **Contradiction!**



Outline

- DFS Algorithm
- DFS Example
- **DFS Applications**

DFS Application 1: Path Finding

- The DFS pattern can be used to find a path between two given vertices u and z, if one exists
- We use a stack to keep track of the current path
- If the destination vertex z is encountered, we return the path as the contents of the stack

```
DFS-Path (u,z,stack)
Precondition: u and z are vertices in a graph, stack contains current path
Postcondition: returns true if path from u to z exists, stack contains path
    colour[u] ← RED
    push u onto stack
    if u = z
        return TRUE
    for each v ∈ Adj[u] //explore edge (u,v)
        if color[v] = BLACK
            if DFS-Path(v,z,stack)
                return TRUE
    colour[u] ← GRAY
    pop u from stack
    return FALSE
```

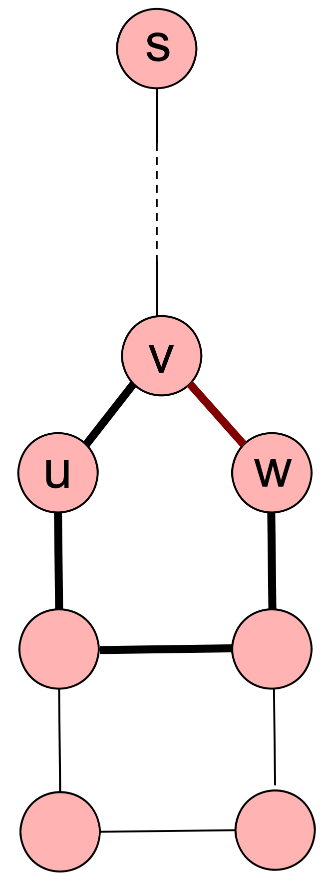
DFS Application 2: Cycle Finding

- The DFS pattern can be used to determine whether a graph is acyclic.
- If a back edge is encountered, we return true.

```
DFS-Cycle (u)
Precondition: u is a vertex in a graph G
Postcondition: returns true if there is a cycle reachable from u.
    colour[u] ← RED
    for each v ∈ Adj[u] //explore edge (u,v)
        if color[v] = RED //back edge
            return true
        else if color[v] = BLACK
            if DFS-Cycle(v)
                return true
    colour[u] ← GRAY
    return false
```

Why must DFS on a graph with a cycle generate a back edge?

- Suppose that vertex *s* is in a connected component *S* that contains a cycle *C*.
- Since all vertices in *S* are reachable from *s*, they will all be visited by a DFS from *s*.
- Let *v* be the first vertex in *C* reached by a DFS from *s*.
- There are two vertices *u* and *w* adjacent to *v* on the cycle *C*.
- wlog, suppose *u* is explored first.
- Since *w* is reachable from *u*, *w* will eventually be discovered.
- When exploring *w*'s adjacency list, the back-edge *(w, v)* will be discovered.



DFS Application 3. Topological Sorting (e.g., putting tasks in linear order)

Note: The textbook also describes a breadth-first TopologicalSort algorithm (Section 13.4.3)

DAGs and Topological Ordering

- A directed acyclic graph (DAG) is a digraph that has no directed cycles
- A topological ordering of a digraph is a numbering

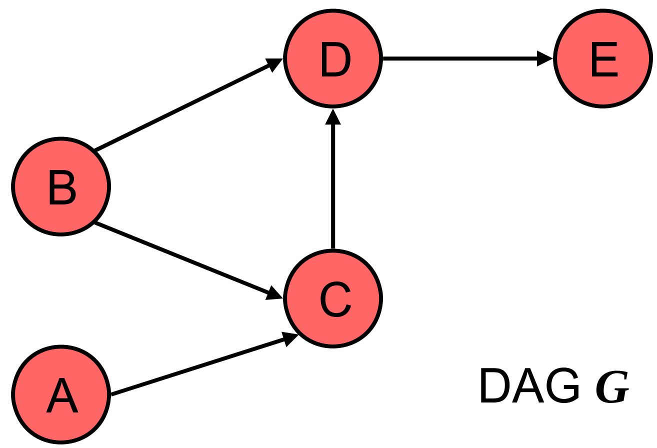
$$v_1, \ldots, v_n$$

of the vertices such that for every edge (v_i, v_j), we have $i < j$

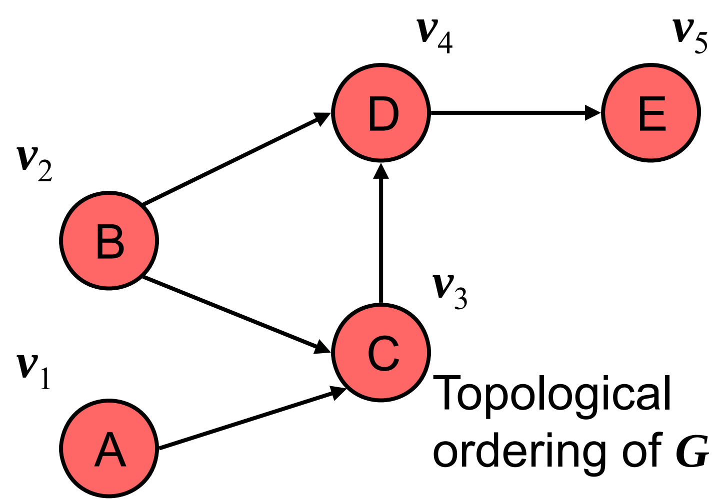
- Example: in a task scheduling digraph, a topological ordering is a task sequence that satisfies the precedence constraints

Theorem

A digraph admits a topological ordering if and only if it is a DAG

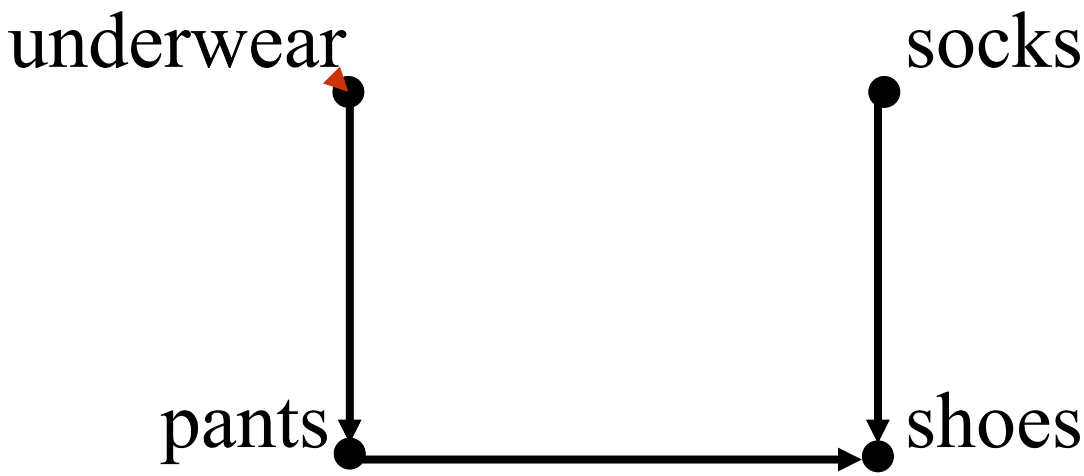


DAG G



Topological ordering of G

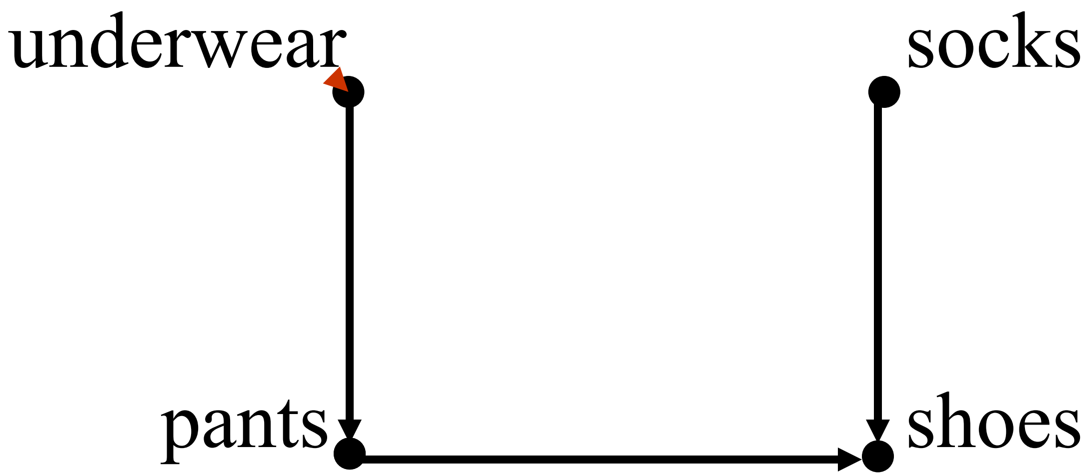
Topological (Linear) Order



underwear
pants
socks
shoes

socks
underwear
pants
shoes

Topological (Linear) Order



Invalid Order

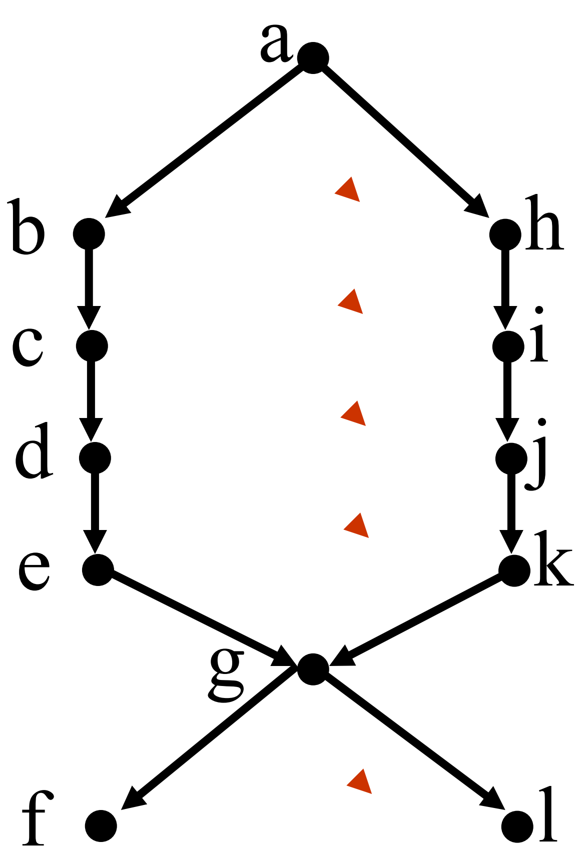
~~socks~~
~~shoes~~
~~pants~~
~~underwear~~

Algorithm for Topological Sorting

- Note: This algorithm is different than the one in Goodrich-Tamassia

```
Method TopologicalSort(G)
    H ← G   // Temporary copy of G
    n ← G.numVertices()
    while H is not empty do
        Let v be a vertex with no outgoing edges
        Label v ← n
        n ← n - 1
        Remove v from H //as well as edges involving v
```

Linear Order



Pre-Condition:
A Directed Acyclic Graph (DAG)

Post-Condition:
Find one valid linear order

Algorithm:
- Find a terminal node (sink).
- Put it last in sequence.
- Delete from graph & repeat

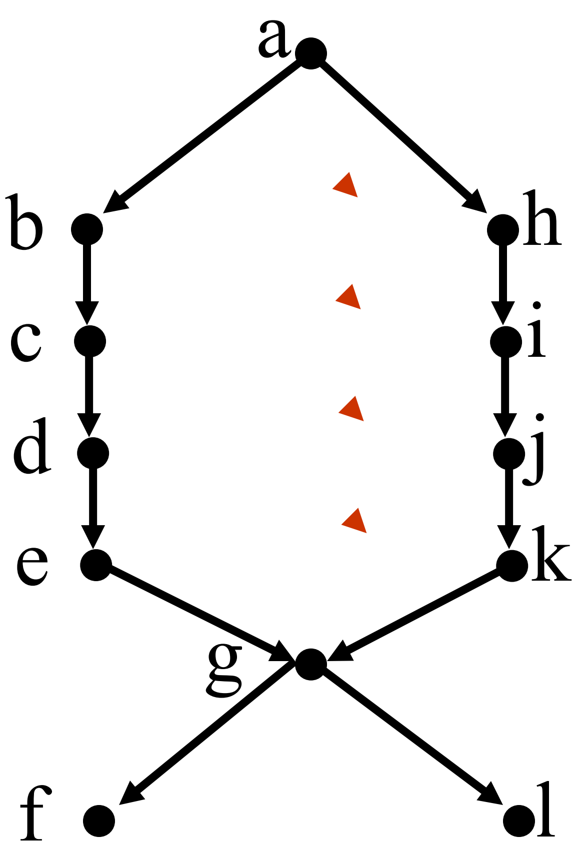
O(|V|)

Running time: $\sum_{i=1}^{|V|} i = O\left(|V|^2\right)$

….. 1 Can we do better?

Linear Order

Alg: DFS



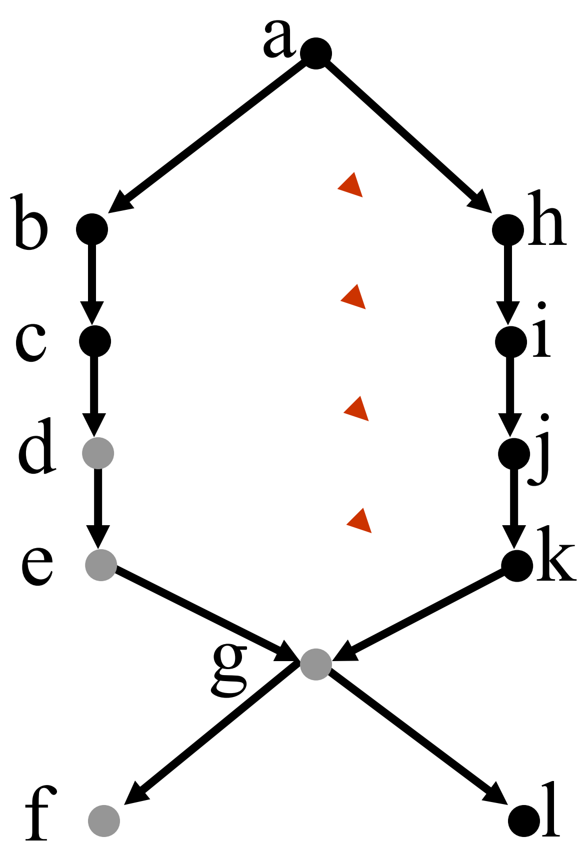
Found
Not Handled
Stack

f
g
e
d

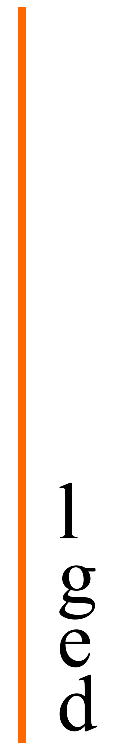
….. f

Linear Order

Alg: DFS



Found
Not Handled
Stack



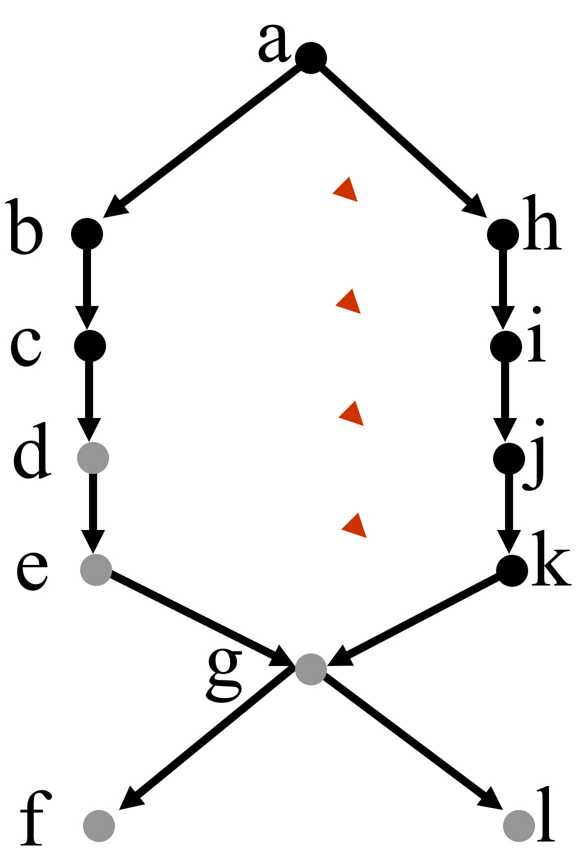
When node is popped off stack, insert at front of linearly-ordered “to do” list.

Linear Order:

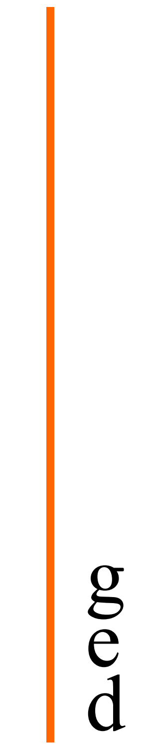
….. f

Linear Order

Alg: DFS



Found
Not Handled
Stack

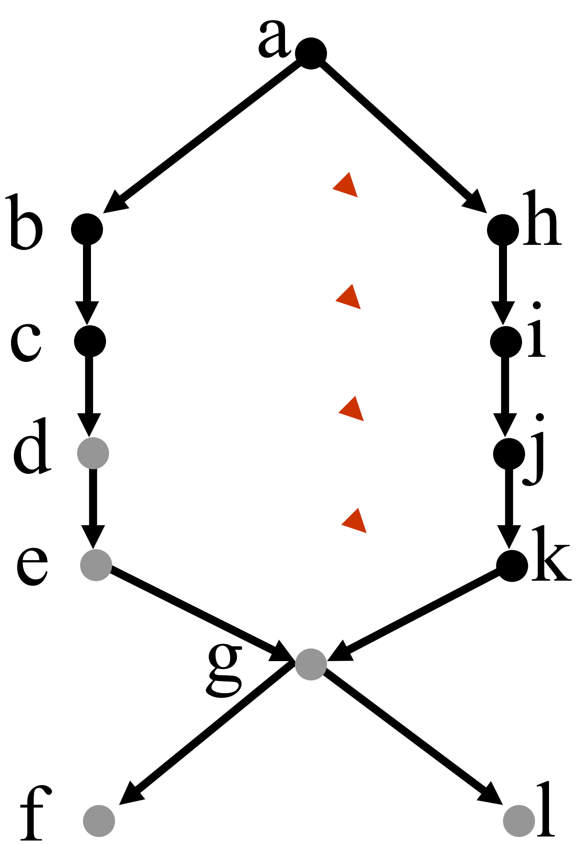


Linear Order:

l,f

Linear Order

Alg: DFS



Found
Not Handled
Stack

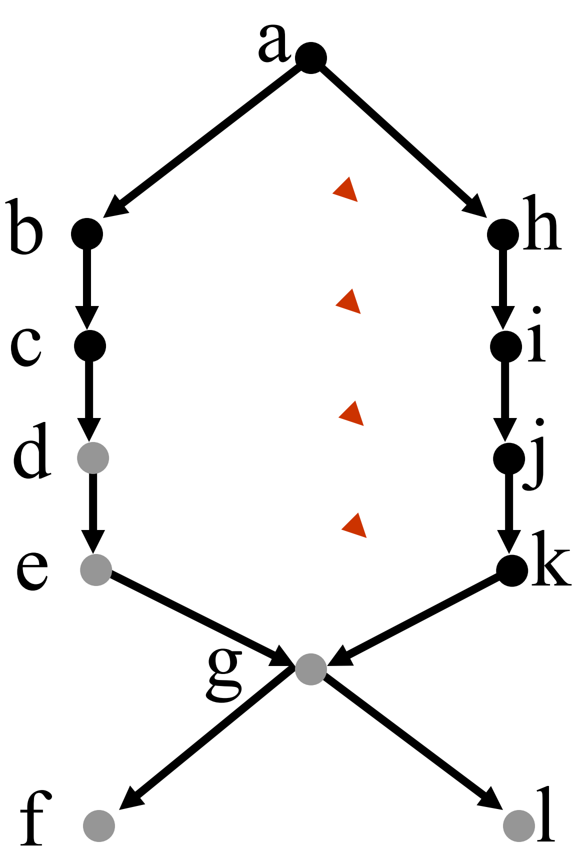
e
d

Linear Order:

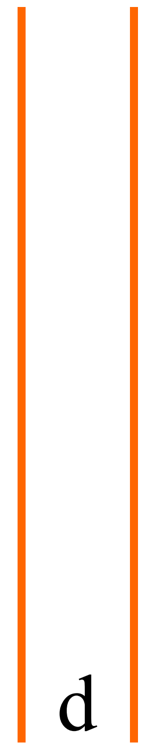
g,l,f

Linear Order

Alg: DFS



Found
Not Handled
Stack

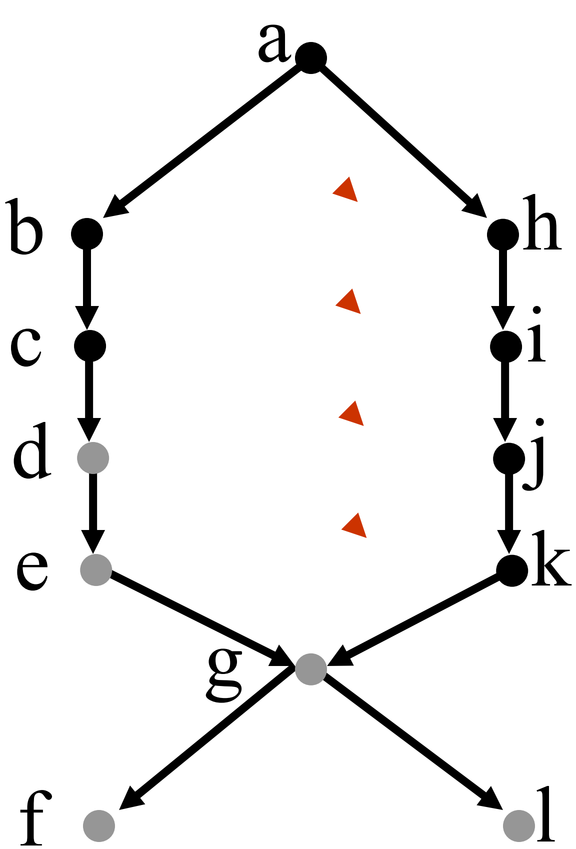


Linear Order:

e,g,l,f

Linear Order

Alg: DFS



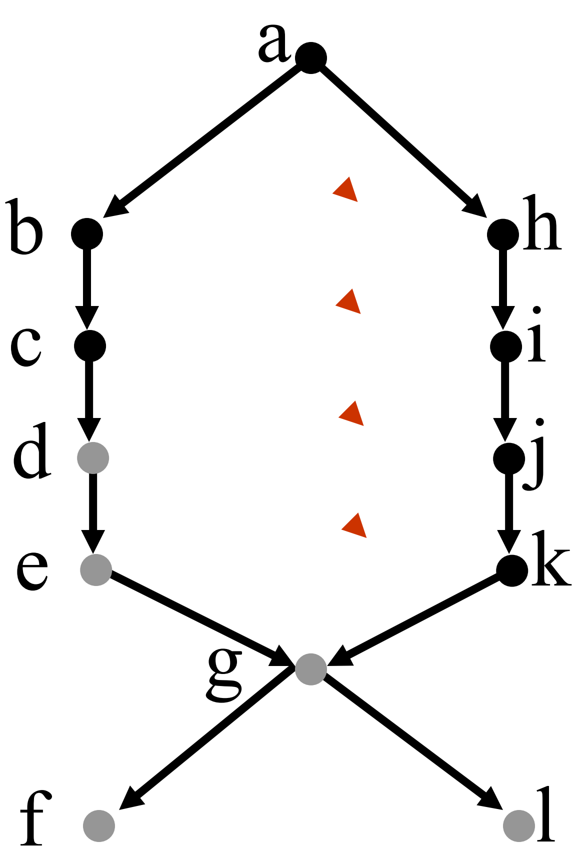
Found
Not Handled
Stack

Linear Order:

d,e,g,l,f

Linear Order

Alg: DFS



Found
Not Handled
Stack

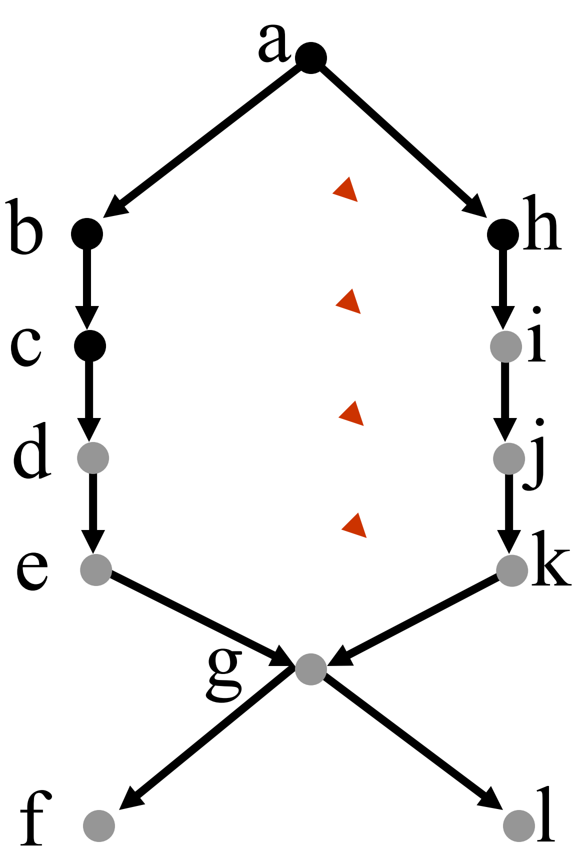
k
j
i

Linear Order:

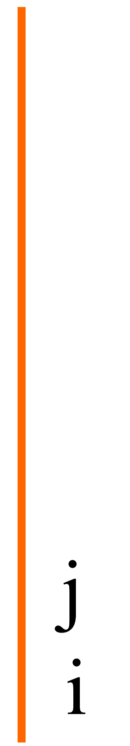
d,e,g,l,f

Linear Order

Alg: DFS



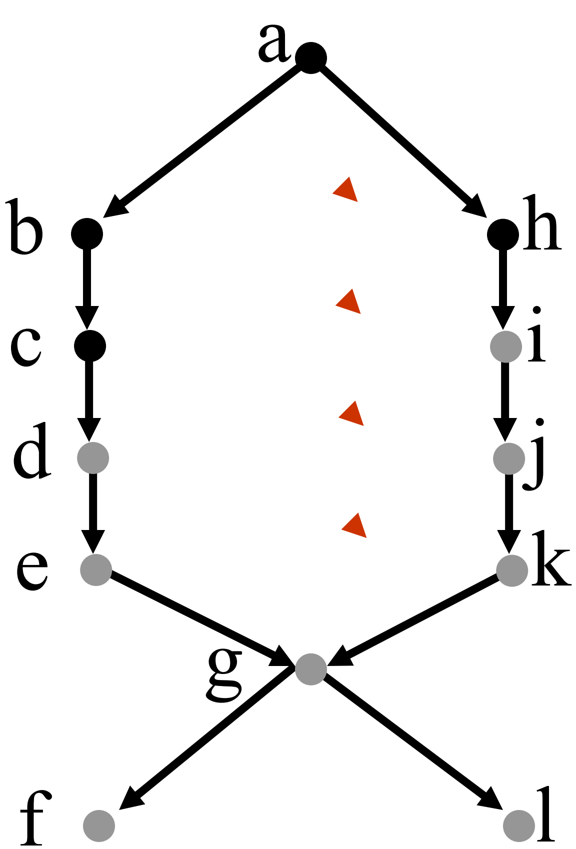
Found
Not Handled
Stack



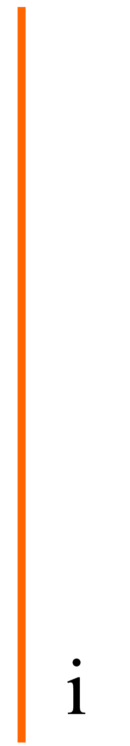
Linear Order: k,d,e,g,l,f

Linear Order

Alg: DFS



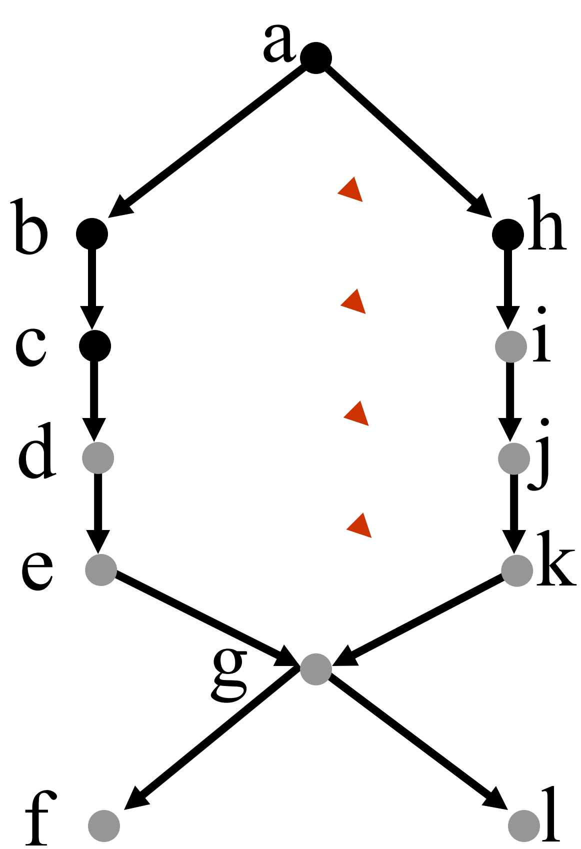
Found
Not Handled
Stack



Linear Order: j,k,d,e,g,l,f

Linear Order

Alg: DFS

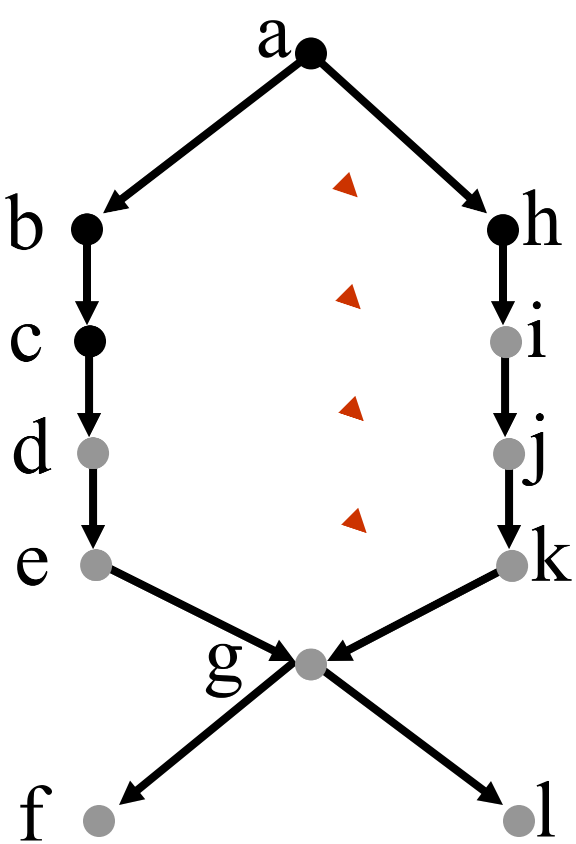


Found
Not Handled
Stack

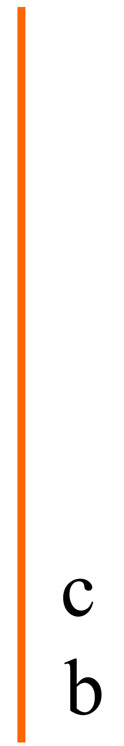
Linear Order: i,j,k,d,e,g,l,f

Linear Order

Alg: DFS



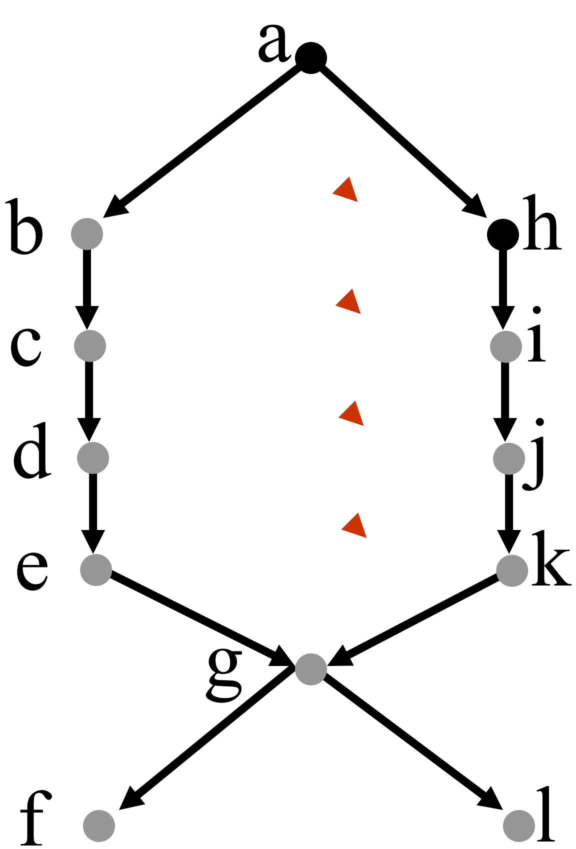
Found
Not Handled
Stack



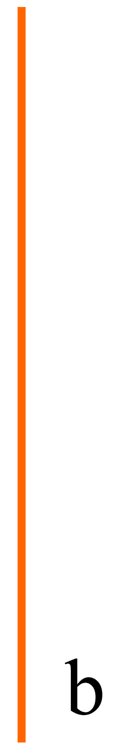
Linear Order: i,j,k,d,e,g,l,f

Linear Order

Alg: DFS



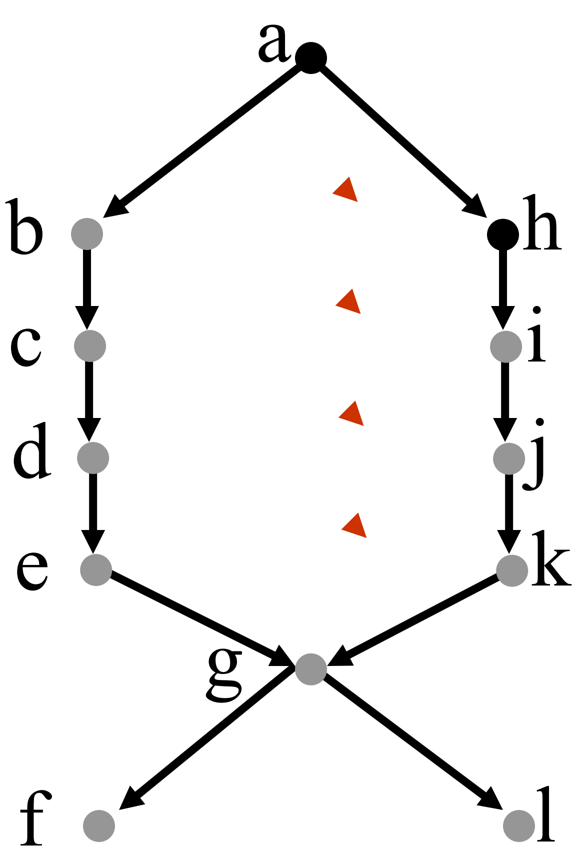
Found
Not Handled
Stack



Linear Order: c,i,j,k,d,e,g,l,f

Linear Order

Alg: DFS

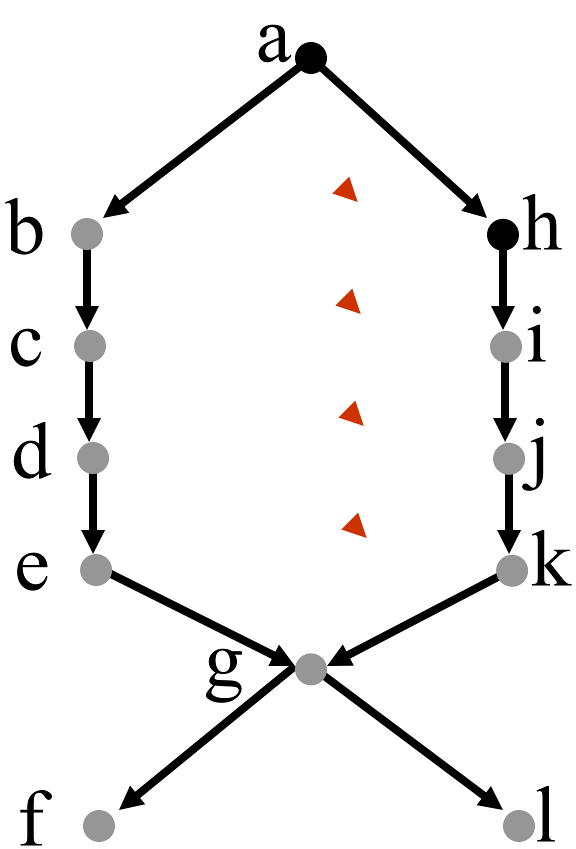


Found
Not Handled
Stack

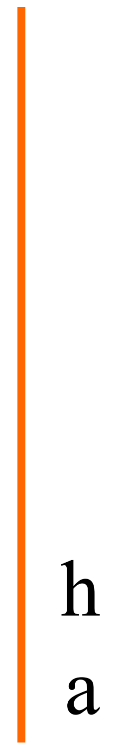
Linear Order: b,c,i,j,k,d,e,g,l,f

Linear Order

Alg: DFS



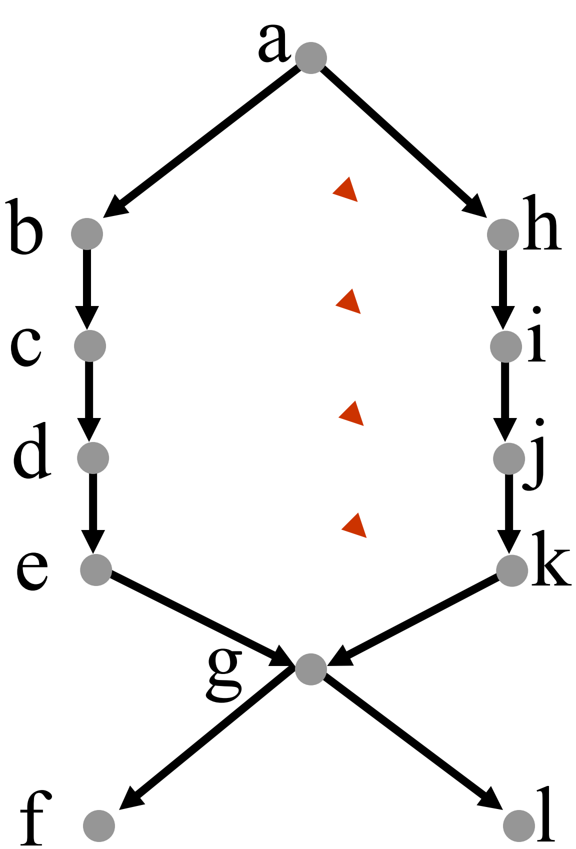
Found
Not Handled
Stack



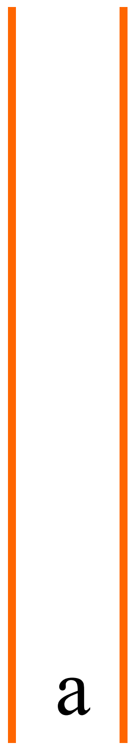
Linear Order: b,c,i,j,k,d,e,g,l,f

Linear Order

Alg: DFS



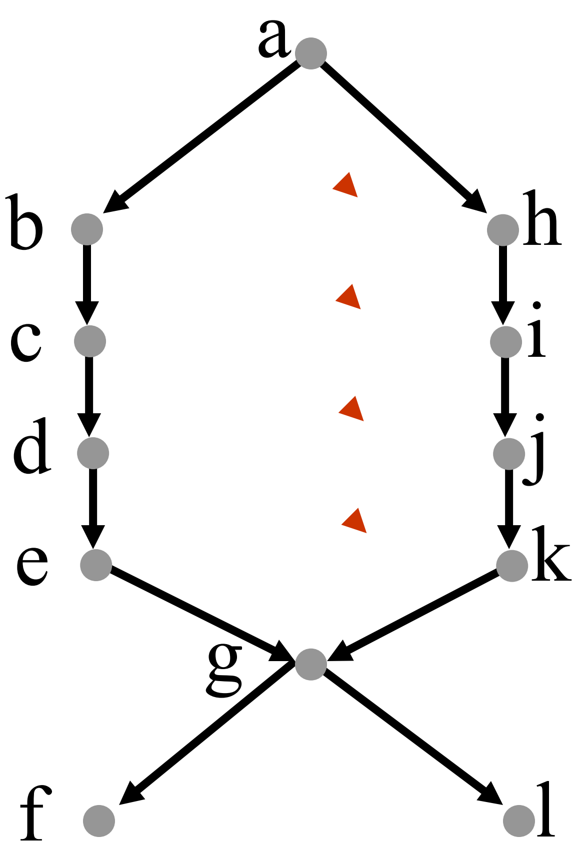
Found
Not Handled
Stack



Linear Order: h,b,c,i,j,k,d,e,g,l,f

Linear Order

Alg: DFS



Found
Not Handled
Stack

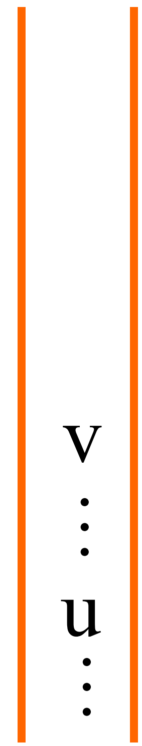
Linear Order: a,h,b,c,i,j,k,d,e,g,l,f Done!

DFS Algorithm for Topologial Sort

- Makes sense. But how do we prove that it works?

Linear Order

Found
Not Handled
Stack



Proof: Consider each edge

- Case 1: u goes on stack first before v.
  - Because of edge,
    v goes on before u comes off
  - v comes off before u comes off
  - v goes after u in order. ☺

$u \bullet\!\!\longrightarrow\!\!\bullet\ v$

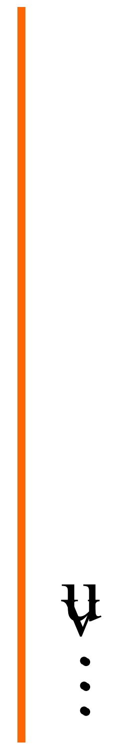
u…v…

Linear Order

Found
Not Handled
Stack

Proof: Consider each edge

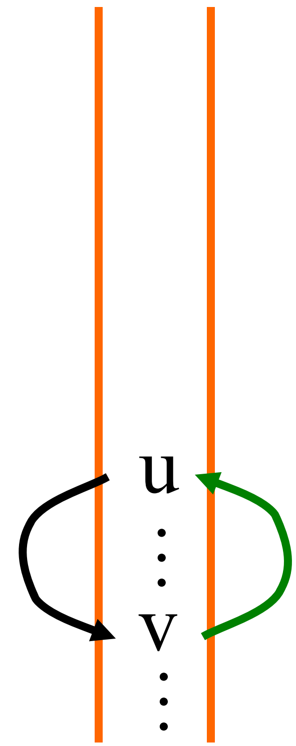
- Case 1: u goes on stack first before v.
- Case 2: v goes on stack first before u.
  v comes off before u goes on.
  - v goes after u in order. ☺



u…v…

Linear Order

Found
Not Handled
Stack



Proof: Consider each edge

- Case 1: u goes on stack first before v.
- Case 2: v goes on stack first before u.
  v comes off before u goes on.

Case 3: v goes on stack first before u.
u goes on before v comes off.

- Panic: u goes after v in order. ☹
- Cycle means linear order
  is impossible ☺

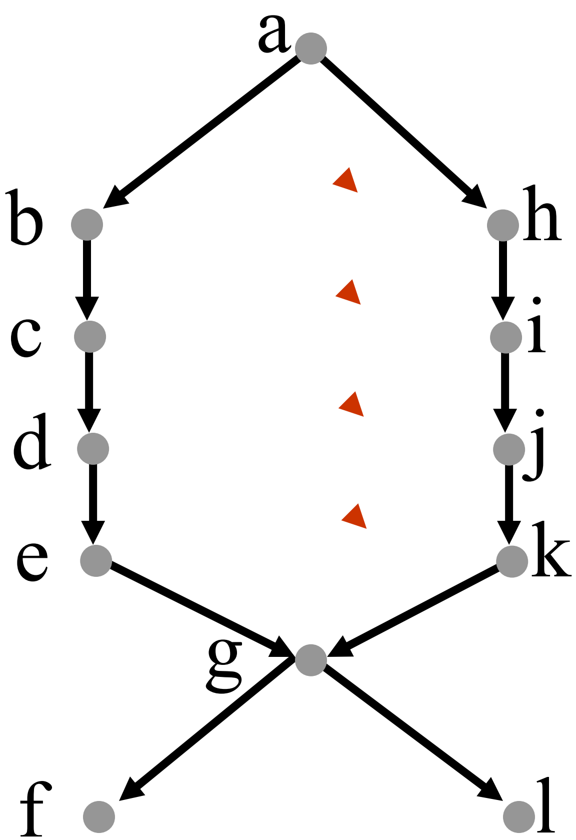
The nodes in the stack form a path starting at s.

$u \bullet\!\longrightarrow\!\bullet\ v$

v…u…

Linear Order

Alg: DFS



Found

Not Handled

Stack

Analysis: $\Theta(V+E)$

Linear Order: a,h,b,c,i,j,k,d,e,g,l,f Done!

DFS Application 3. Topological Sort

Topological-Sort(G)

Precondition: G is a graph

Postcondition: all vertices in G have been pushed onto stack in reverse linear order

```
for each vertex u ∈V[G]
    color[u] = BLACK //initialize vertex
for each vertex u ∈V[G]
    if color[u] = BLACK //as yet unexplored
        Topological-Sort-Visit(u)
```

DFS Application 3. Topological Sort

Topological-Sort-Visit (u)

Precondition: vertex u is undiscovered

Postcondition: u and all vertices reachable from u have been pushed onto stack in reverse linear order

```
colour[u] ← RED
for each v ∈ Adj[u] //explore edge (u,v)
        if color[v] = BLACK
                Topological-Sort-Visit(v)
push u onto stack
colour[u] ← GRAY
```

Outline

- DFS Algorithm
- DFS Example
- DFS Applications