Computer Algebra
Lecture 4

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6 September 2018

How much mathematics should a system know?

Nothing: I tell the system `x-x` and it replies $x - x$. **Not very useful!**

Minimal: Probably associative, commutative, combining. We want

- `x*y+y*x` $\rightarrow 2xy$
- `x+(y+x)` $\rightarrow 2x + y$

Everything: I tell it $n, x, y, z \in \mathbf{Z} \& n > 2 \& x^n + y^n = z^n$ and it returns $xyz = 0$ [Fermat's Last Theorem]. **Not feasible**

Should we expand: distributive law?

`a*(b+c)` $\rightarrow ab + ac$. Possibly not: $(a_1 + b_1)(a_2 + b_2)\cdots(a_n + b_n)$ has length $2n$ like that, but $n2^n$ if expanded.

Reduce expands by default, Maple does not.

Reduce

```
1: off exp;
2: f:=(x-1)*(x^2-4);
```

$$f := \left(x^2 - 4\right)(x - 1)$$

```
3: g:=(x^2+x-2)*(x-2);
```

$$g := \left(x^2 + x - 2\right)(x - 2)$$

```
4: f-g;
```

$$0$$

```
5: f+g;
```

$$2\left(x^3 - x^2 - 4x + 4\right)$$

```
6:
```

Maple

$f := (x^2 - 4)\cdot(x - 1);$

$$f := \left(x^2 - 4\right)(x - 1)$$

$g := (x^2 + x - 2)\cdot(x - 2);$

$$g := \left(x^2 + x - 2\right)(x - 2)$$

$h := f - g;$

$$h := \left(x^2 - 4\right)(x - 1) - \left(x^2 + x - 2\right)(x - 2)$$

$expand(h);$

$$0$$

$f + g;$

$$\left(x^2 - 4\right)(x - 1) + \left(x^2 + x - 2\right)(x - 2)$$

But look at $f - g$ in the two cases.

Some definitions: Normal

Definition

A representation is said to be *normal* if the only representation of the object 0 is `0`.

So Maple's representation, even of polynomials, is not normal, but Reduce's might be. [It is, but we haven't proved it.] Normal representations are very important in practice, since many algorithms contain tests for zero/non-zero of elements. Sometimes these are explicit, as in Gaussian elimination of a matrix, but often they are implicit, as in Euclid's Algorithm, where we take the remainder after dividing one polynomial by another, which in turn requires that we know the leading *non-zero* coefficient of the divisor.

Some definitions: Canonical

Definition

A representation is said to be *canonical* if every object (not just zero) has only one representation.

With a canonical representation, we can say that two objects "*are* the same if, and only if, they *look* the same".
We cannot have *both* `(x+1)^2` and `x^2+2x+1`, since $(x+1)^2 = x^2 + 2x + 1$.

Definition

A representation is said to be *locally canonical* (with respect to a certain context) if every object whose introduction does not change the context has only one representation

A typical context-changer is Reduce's `korder` command for changing the order of variables (`order` just changes the order as printed)

Candid expressions

Definition ([Sto11, Definition 3])

A *candid* expression is one that is not equivalent to an expression that visibly manifests a simpler expression class.

In particular, if the "simpler class" is $\{0\}$, "candid" means the same as normal, but the concept is much more general, and useful.
In particular, if a candid expressions contains a variable v, then it really depends on v.
If it looks like it has a certain degree, it has that degree.
If an expression looks like a fraction, it really has to be a fraction, etc.

Some systems: polynomials

System	Normal	Canonical (Locally)	Candid
Maple	X	X	X
Maple `expand`	$\surd$	$\surd$	$\surd$
Reduce	$\surd$	$\surd$	$\surd$
Reduce `off exp`	$\surd$	X	??

I have no examples of Reduce's `off exp` not being candid, but that's not a proof.

Some systems: rational functions (Maple)

System	Normal	Canonical (Locally)	Candid
Maple	X	X	X
Maple `expand`	X	X	X
Maple `normal`	$\surd$	$\surd$	$\surd$

Maple's `expand` is not very helpful when it comes to fractions: I think you need to use `normal` (or `simplify`).

$$expand\left(\frac{(x-1)}{x^2-1}-\frac{1}{x+1}\right);$$

$$\frac{x}{x^2-1}-\frac{1}{x^2-1}-\frac{1}{x+1}$$

$$simplify(\%);$$

$$0$$

Some systems: rational functions (Reduce)

exp	mcd	gcd	Normal	Canonical (Locally)	Candid	Comments
$\surd$	$\surd$	X	$\surd$	X	(1)	Default
$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	$\surd$	My choice
–	X	–	X(2)	X	X	
X	$\surd$	X				
X	$\surd$	X				

MCD =Make Common Denominators [HS18, p. 130]

(1) Even with `off gcd`, it still checks to see if the denominator divides the numerator. So a rational function printed really is a rational function, not a polynomial in disguise.

(2) In the lecture I got myself confused here

$$\frac{2}{x^2-1} + \frac{-1}{x-1} + \frac{1}{x+1}$$

doesn't simplify to 0.

A.C. Hearn and R. Schöpf.
REDUCE User's Manual (Free Version; June 8, 2018).
`http://reduce-algebra.sourceforge.net/`, 2018.

D. Stoutemyer.
Ten commandments for good default expression simplification.
J. Symbolic Comp., 46:859–887, 2011.