Computer Algebra
Lecture 9

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Beyond Rational Functions — algebraic Definitions

The expression θ is an *elementary generator* over K if one of the following is satisfied:

(a) θ is algebraic over K, i.e. θ satisfies a polynomial equation with coefficients in K;

(b) nonzero θ is an *exponential* over K, i.e. there is an η in K such that $\theta' = \eta'\theta$, which is only an algebraic way of saying that $\theta = \exp \eta$;

(c) θ is a *logarithm* over K, i.e. there is an η in K such that $\theta' = \eta'/\eta$, which is only an algebraic way of saying that $\theta = \log \eta$.

If θ is a logarithm of η, then so is $\theta + c$ for any constant c. Similarly, if θ is an exponential of η, so is $c\theta$ for any constant c, including the case $c = 0$, which explains the stipulation of nonzeroness in (b).

Algebraic consequences I

A consequence of these definitions is that log and exp satisfy the usual laws "up to constants".

log Suppose θ_i is a logarithm of η_i. Then

$$\begin{aligned}
(\theta_1+\theta_2)' &= \theta_1'+\theta_2' \\
&= \frac{\eta_1'}{\eta_1}+\frac{\eta_2'}{\eta_2} \\
&= \frac{\eta_1'\eta_2+\eta_1\eta_2'}{\eta_1\eta_2} \\
&= \frac{(\eta_1\eta_2)'}{\eta_1\eta_2},
\end{aligned}$$

and hence $\theta_1+\theta_2$ is *a* logarithm of $\eta_1\eta_2$, a rule normally expressed as

$$\log\eta_1+\log\eta_2=\log(\eta_1\eta_2). \tag{1}$$

Algebraic consequences II

exp Suppose now that θ_i is an exponential of η_i. Then

$$\begin{aligned}(\theta_1\theta_2)' &= \theta_1'\theta_2 + \theta_1\theta_2' \\ &= \eta_1'\theta_1\theta_2 + \theta_1\eta_2'\theta_2 \\ &= (\eta_1 + \eta_2)'(\theta_1\theta_2)\end{aligned}$$

and hence $\theta_1\theta_2$ is *an* exponential of $\eta_1 + \eta_2$, a rule normally expressed as

$$\exp\eta_1 \exp\eta_2 = \exp(\eta_1 + \eta_2). \tag{2}$$

Algebraic consequences III

(1) Suppose θ is a logarithm of η, and ϕ is an exponential of θ. Then

$$\begin{aligned} \phi' &= \theta'\phi = \frac{\eta'}{\eta}\phi, \quad \text{so} \\ \frac{\phi'}{\phi} &= \frac{\eta'}{\eta} = \theta', \end{aligned}$$

and θ is *a* logarithm of ϕ, as well as ϕ being an exponential of θ.

This is as close as we get to saying "log and exp are inverses of each other".

Algorithm (Integration Paradigm)

Input *an elementary expression f in x*
Output *An elementary g with $g' = f$, or failure*

Find fields $\mathcal{C}$ of constants,
L of elementary expressions over $\mathcal{C}(x)$ with $f \in L$
If this fails
Then error `"integral not elementary"`
Find an elementary extension M of L, and $g \in M$ with $g' = f$
If this fails
Then error `"integral not elementary"`
Else return g

Looks pretty open-ended

Less Openended than it looks

Theorem (Liouville's Principle)

Let f be a expression from some expression field L. If f has an elementary integral over L, it has an integral of the following form:

$$\int f = v_0 + \sum_{i=1}^{n} c_i \log v_i, \tag{3}$$

where v_0 belongs to L, the v_i belong to $\hat{L}$, an extension of L by a finite number of constants algebraic over L_{const}, and the c_i belong to $\hat{L}$ and are constant.

The proof of this theorem (see, for example, [Rit48]), while quite subtle in places, is basically a statement that the only new expression in $g := \int f$ which can disappear on differentiation is a logarithm with a constant coefficient.

“e^{-x^2} has no elementary integral” I

Following our Paradigm, we choose $\mathbf{Q}$ as our field of constants, and L as $\mathbf{Q}(x,\theta)$ with $\theta' = -2x\theta$. The integrand is then just θ. Then Liouville's Principle states

$$\theta = v_0' + \sum_{i=1}^{n} \frac{c_i v_i'}{v_i}, \tag{4}$$

where v_0 belongs to L, the v_i belong to $\hat{L}$, an extension of L by a finite number of constants algebraic over L_{const}, and the c_i belong to $\hat{L}$ and are constant.
Furthermore, we can assume that v_i ($i > 0$) are actually in $\mathbf{Q}(x)[\theta]$ and are square-free and relatively prime (making use of algebraic properties of log).

“e^{-x^2} has no elementary integral” II

1. Write $v_0 = \sum_{i=-m}^{n} a_i\theta^i + \widehat{v}_0$, where $\widehat{v}_0$ is a proper rational fraction in $\mathbf{Q}(x)(\theta)$ whose denominator is not divisible by θ (as all such factors are covered in the first summation). We can then perform a square-free decomposition of the denominator of $\widehat{v}_0$, and, as in (12/Lecture 9), write

$$\widehat{v}_0 = \sum \frac{q_i}{r_i^i}, \tag{5}$$

where the r_i are square-free and relatively prime, and the q_i are relatively prime with the r_i. (4) then becomes

$$\theta = \sum_{i=-m}^{n} (a_i' - 2ixa_i)\theta^i + \sum \frac{q_i'}{r_i^i} + \sum \frac{-iq_i r_i'}{r_i^{i+1}} + \sum_{i=1}^{n} \frac{c_i v_i'}{v_i}. \tag{6}$$

But there is an r_i^{i+1} in the denominator of the right-hand side, and nothing elsewhere to cancel it. This is a contradiction unless there are no r_i, i.e. $\widehat{v}_0 = 0$.

"e^{-x^2} has no elementary integral" III

2. (6) then becomes

$$\theta = \sum_{i=-m}^{n} (a_i' - 2xia_i)\theta^i + \sum_{i=1}^{n} \frac{c_i v_i'}{v_i}. \qquad (7)$$

   We can now observe that there is a v_i in the denominator of the right-hand side, and nothing elsewhere to cancel it. This is a contradiction unless there are no v_i.

3. So $\theta = \sum_{i=-m}^{n}(a_i' - 2ixa_i)\theta^i$, and equating coefficients of θ shows that the only value of i is 1, and we have

$$1 = a_1' - 2xa_1, \qquad (8)$$

   with $a_1 \in \mathbf{Q}(x)$.

i.e. "the integral is $a_1(x)e^{-x^2}$"

4. Write $a_1 = p + \sum \frac{q_i}{r_i^i}$, where $p \in \mathbf{Q}[x]$, the r_i are square-free and relatively prime in $\mathbf{Q}[x]$, and the q_i are relatively prime with the r_i. (8) then becomes

$$1 = p' + \sum \frac{q_i'}{r_i^i} + \sum \frac{-ir_i'q_i}{r_i^{i+1}} - 2xp - \sum \frac{2xq_i}{r_i^i}. \qquad (9)$$

But there is an r_i^{i+1} in the denominator of the right-hand side, and nothing elsewhere to cancel it. This is a contradiction unless there are no r_i, i.e. $a_1 = p$.

5. Write $p = \sum_{i=0}^{n} c_i x^i$, with $c_i \in \mathbf{Q}$. Then

$$1 = p' - 2xp = \underbrace{\sum_{i=0}^{n} ic_i x^{i-1}}_{\text{degree } n-1} - 2\underbrace{\sum_{i=0}^{n} c_i x^{i+1}}_{\text{degree } n+1}, \qquad (10)$$

and this is impossible, hence θ has no elementary integral.

Nice hack, but not an algorithm

Indeed, and (8) isn't even an integral, it's $y' + fy = g$, which I call the Risch Differential Equation Problem (RDE) [Ris69, Dav86]. Furthermore, we want to solve the integration problem in any $\mathbf{Q}(x = \theta_0, \theta_1, \ldots, \theta_n)$. It turns out to be an induction on

Hypothesis

H_n: we can solve both integration and the RDE problem in $L_n := \mathbf{Q}(x = \theta_0, \theta_1, \ldots, \theta_n)$.

This will follow by induction from these six results.

Six theorems

1. Integration in $\mathbf{Q}(x)$ — lecture 9
2. RDE in $\mathbf{Q}(x)$ [Ris69, Dav86].
3. H_{n-1}, θ_n logarithmic has integration in L_n [Ris69]
4. H_{n-1}, θ_n exponential has integration in L_n [Ris69]
5. H_{n-1}, θ_n logarithmic has RDE in L_n [Ris69, Dav86].
6. H_{n-1}, θ_n exponential has RDE in L_n [Ris69, Dav86].

This last is particularly messy: see [Dav86, p. 915].

A word of warning

This is a completely algebraic theory. It works fine for indefinite integration, and looking for formulae.

But nothing really connects the θ_i we have used with actual functions $\mathbf{R} \mapsto \mathbf{R}$ (or $\mathbf{C} \mapsto \mathbf{C}$).

When this happens we can get nasty surprises.

Note that log is *not* a continuous function $\mathbf{C} \mapsto \mathbf{C}$, and complex calculus shows that this means arctan is not continuous at infinity, so $\arctan\left(\frac{1}{x^2-1}\right)$ is discontinuous at ± 1. But an integral has to be continuous.

The problem isn't the algebra, it's the interpretation as actual functions $\mathbf{R} \mapsto \mathbf{R}$ (or $\mathbf{C} \mapsto \mathbf{C}$).

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