Billiards

Jonathan Allen, John Wang

Massachusetts Institute of Technology

November 22nd, 2013

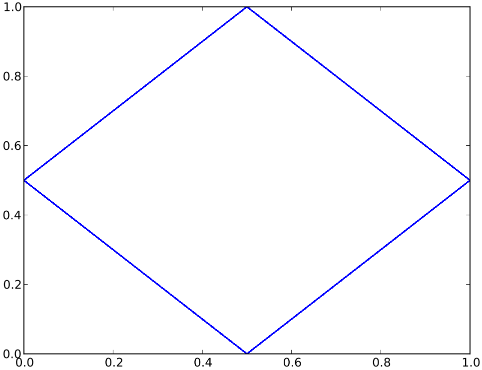
Introduction

- Billiard ball bouncing in a square
- Assume no gravity or friction
- Examine sequence of side collisions

Example

Example

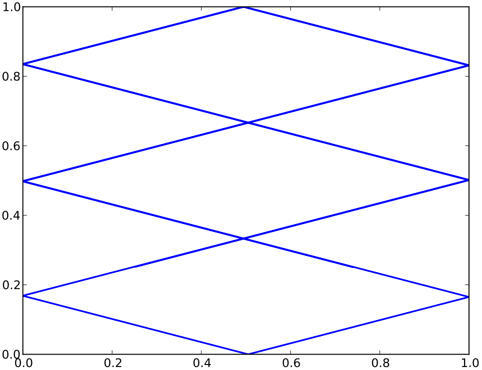
Examine the sequence: ‘abab‘



Another Example

Example

Examine the sequence: ‘aaabaaab‘

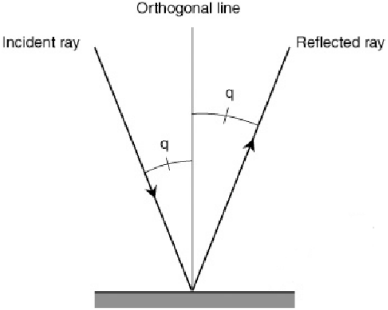


Presentation Outline

Notation

Definition

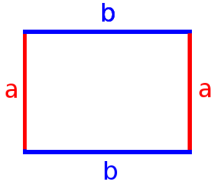
A table T is the unit square in $\mathbb{R}^2$. A particle $p \in T$ begins at position $\bar{x}_0 \in T$ with velocity $\bar{v}$. When the particle reaches an edge of the table, velocity is reflected about the line perpendicular to the table's edge.



Notation

Definition

Opposite sides of the table are named a and b. **Primary side** (most collisions) is a, **secondary side** is b.



Notation

Definition

Collision string consists of the sides of the table that have been collided with for a given starting position and velocity.

Notation

Definition

Collision string consists of the sides of the table that have been collided with for a given starting position and velocity.

Definition

Primary substring is a subsequence from the collision string which contains the primary side collisions that occur between consecutive secondary side collisions.

Notation

Definition

Collision string consists of the sides of the table that have been collided with for a given starting position and velocity.

Definition

Primary substring is a subsequence from the collision string which contains the primary side collisions that occur between consecutive secondary side collisions.

Example

Collision string: ‘aabaaabaabaaab‘, **Primary substrings**: ‘aa‘, ‘aaa‘

Problem Statement

Problem: Characterize the properties of collision sequences.

- Given a sequence of a's and b's, determine if it is a valid collision sequence.
- Given a valid collision sequence, determine a possible starting position and velocity.

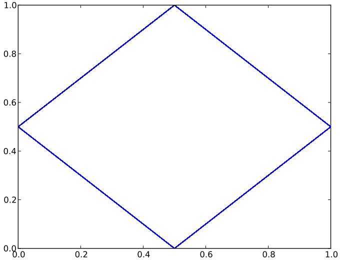
Representing Collision Strings

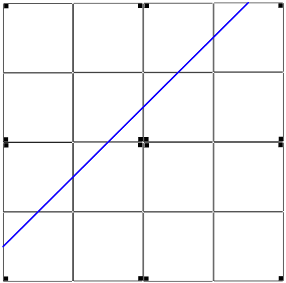
- Reflect squares about each side to create a tiling
- Solutions become lines in the plane
- Intersections become places where collisions occur

Representing Collision Strings

Example

Tiling of $\vec{x}_0 = (0, 0.5)$ and $\vec{v} = (0.25, 0.25)$.

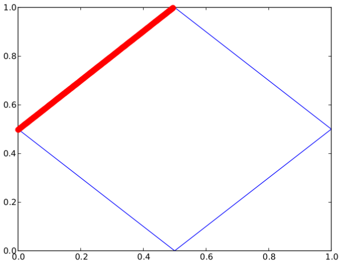


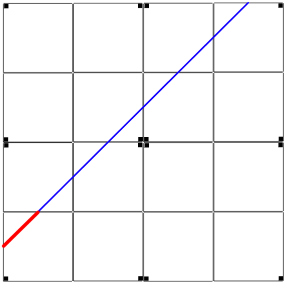


Representing Collision Strings

Example

Tiling of $\vec{x}_0 = (0, 0.5)$ and $\vec{v} = (0.25, 0.25)$.

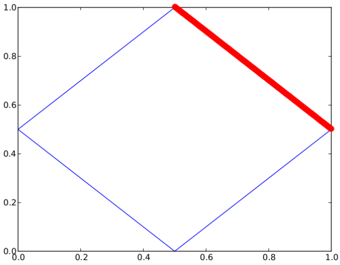


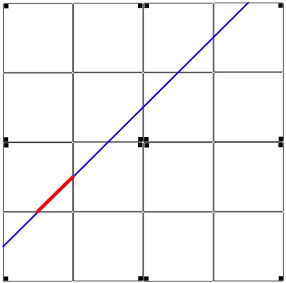


Representing Collision Strings

Example

Tiling of $\vec{x}_0 = (0, 0.5)$ and $\vec{v} = (0.25, 0.25)$.

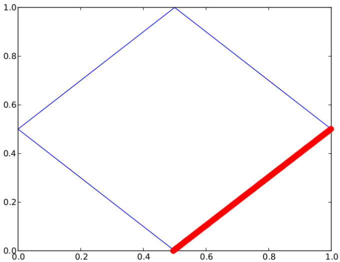


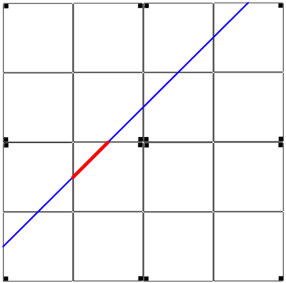


Representing Collision Strings

Example

Tiling of $\vec{x}_0 = (0, 0.5)$ and $\vec{v} = (0.25, 0.25)$.

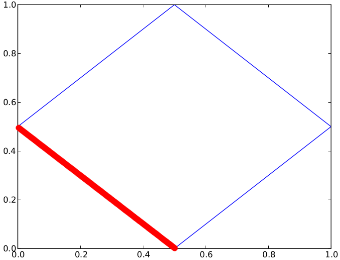


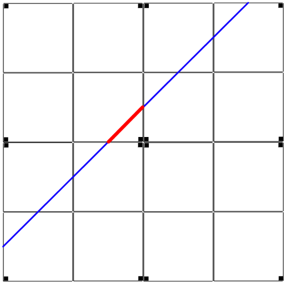


Representing Collision Strings

Example

Tiling of $\vec{x}_0 = (0, 0.5)$ and $\vec{v} = (0.25, 0.25)$.

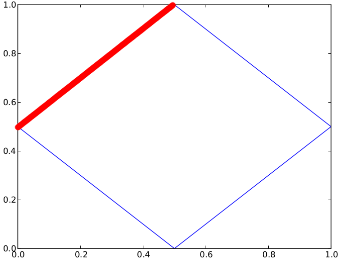


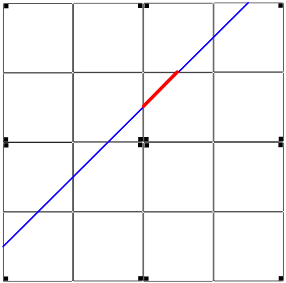


Representing Collision Strings

Example

Tiling of $\vec{x}_0 = (0, 0.5)$ and $\vec{v} = (0.25, 0.25)$.

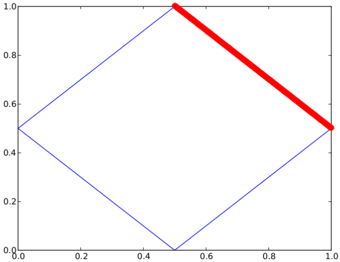


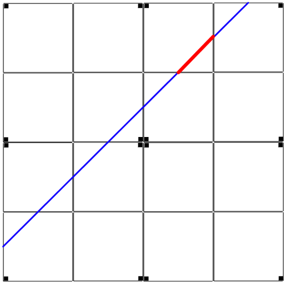


Representing Collision Strings

Example

Tiling of $\vec{x}_0 = (0, 0.5)$ and $\vec{v} = (0.25, 0.25)$.

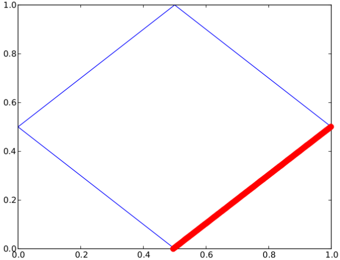


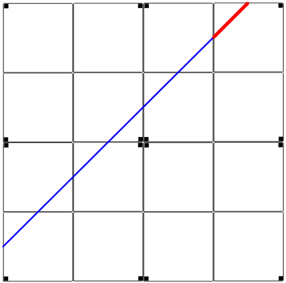


Representing Collision Strings

Example

Tiling of $\vec{x}_0 = (0, 0.5)$ and $\vec{v} = (0.25, 0.25)$.





Sequence Characterization

$$\underbrace{aaa}_{3} \quad b \quad \underbrace{aa}_{2} \quad b \quad \underbrace{aa}_{2} \quad b \quad \underbrace{aaa}_{3} \quad b \quad \underbrace{aa}_{2} \quad b$$

$$\underbrace{aa}_{2} \quad b \quad \underbrace{aaa}_{3} \quad b \quad \underbrace{aa}_{2} \quad b \quad \underbrace{aaa}_{3} \quad \underset{\ldots}{\ldots}$$

Sequence Characterization

$$3 \quad \underbrace{22}_{2} \quad 3 \quad \underbrace{22}_{2} \quad 3 \quad \underbrace{2}_{1} \quad \underset{\dots}{3} \quad \dots$$

Extensions to Tileable Polygons

Other Tileable Polygons:

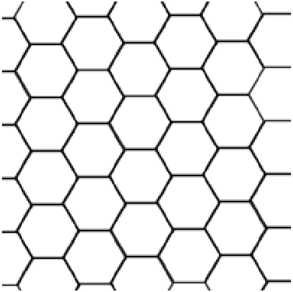


Figure : Regular Hexagons

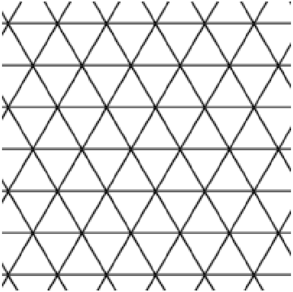


Figure : Equilateral Triangles

Extensions to Non-Tileable Polygons

- Irregular triangles
- Pentagons
- Octagons

Extensions to Circles

- Characterize how particle bounces around circle
- Analog to *a*, *b* might be sequence of collision points as you move around circle.