# Billiards

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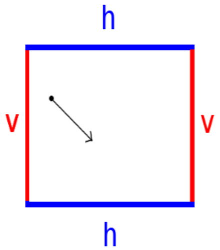
# Introduction

- Billiard ball bouncing in a square
- Assume no gravity or friction

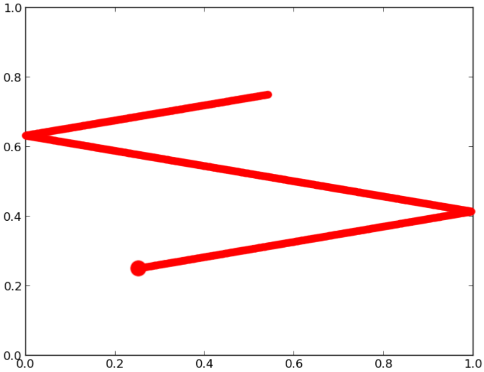
# Basic Notation

**Definition**

A table $T \subset \mathbb{R}^2$ is the unit square. Vertical sides are labelled with a $v$. Horizontal sides are labelled with an $h$.

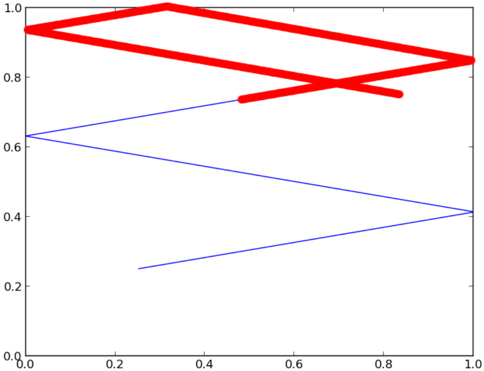


# Example



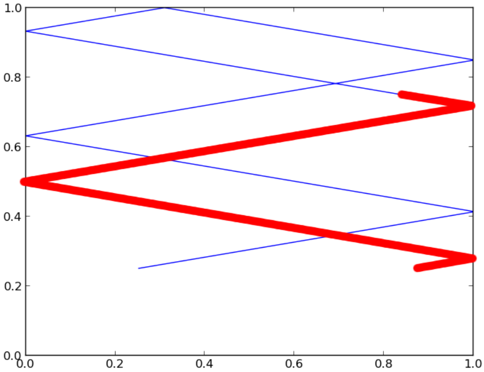

$$vv \tag{1}$$

# Example



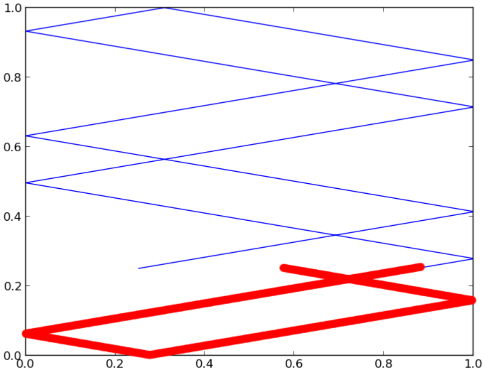

$$vv\color{red}{vhv} \tag{2}$$

# Example



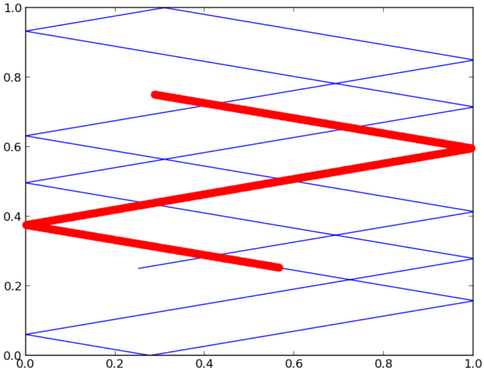

$$\mathit{vvvhv}\textcolor{red}{\mathit{vvv}} \tag{3}$$

# Example



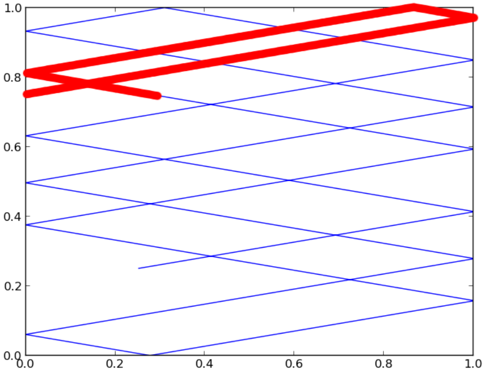

$$vvvhvvvv\textcolor{red}{vhv} \tag{4}$$

# Example




$$vvvhvvvvvhv\textcolor{red}{vv} \tag{5}$$

# Example




$$vvvhvvvvvhvvv\textcolor{red}{vhv} \tag{6}$$

# Resulting Sequence

$$vvvhvvvvvhvvvvhv \tag{7}$$

# Presentation Outline

# Problem Statement

Problem: Given a sequence of $v$ and $h$ collisions, determine if it is a valid collision sequence.

# Secondary Side Theorem

### Theorem

*At least one side will never have more than one consecutive occurrence in a valid collision string.*

# Secondary Side Theorem Examples

**Example**

Valid: *vhhhvhhhv*

**Example**

Valid: *vhvhvhv*

**Example**

Valid: *vvvvvhvvvvhvvvvvhvvvv*

**Example**

Invalid: *vvhhhvvvhhhvvhhh*

**Example**

Invalid: *vhhhvvhvh*

# Secondary Side Theorem Proof

- A billiard ball trajectory must be a line in the tiled grid with slope $m$.
- Case 1: $m = 1$.
- Case 2: $m < 1$ or $m > 1$.

# Secondary Side Theorem Proof

If $m = 1$, $v$ and $h$ alternate.

# Secondary Side Theorem Proof

If $m < 1$, there must exist an $h$ between each $v$.
If $m > 1$, similar argument holds.

# Notation

### Definition

**Secondary side**: a side which never has more than one consecutive occurrences. **Primary side**: a side which is not a secondary side.

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**Primary substring:** a subsequence from the collision string which contains a consecutive sequence of primary sides.

### Example

**Collision string**: $vvhvvvhvvhvvvh$ **Secondary Side**: $h$ **Primary Side**: $v$
**Primary substrings**: $vv$, $vvv$

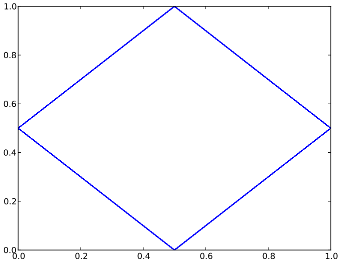
# Representing Collision Strings

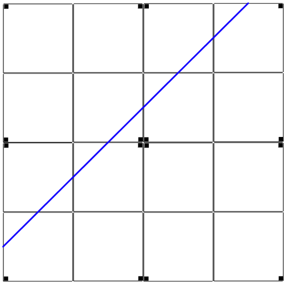
- Reflect squares about each side to create a tiling
- Solutions become lines in the plane
- Intersections become places where collisions occur

# Representing Collision Strings

**Example**

Tiling of $\vec{x}_0 = (0, 0.5)$ and $\vec{v} = (0.25, 0.25)$.



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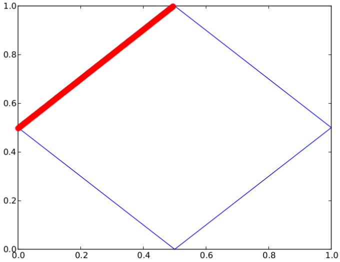


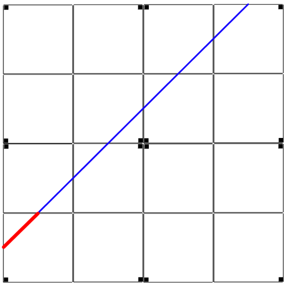

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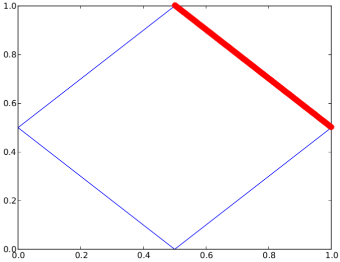


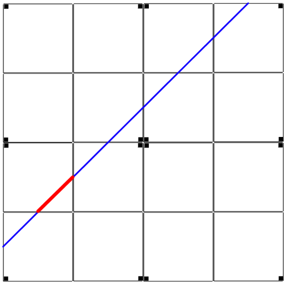

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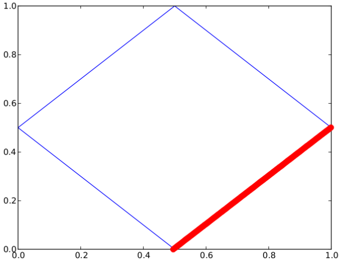


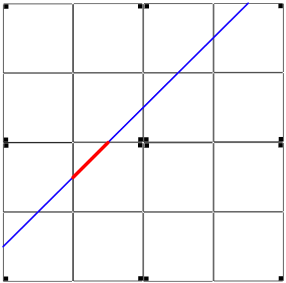

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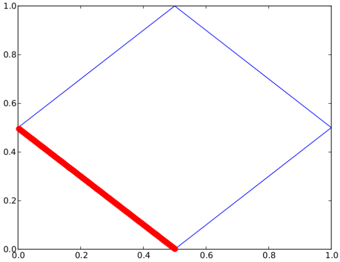


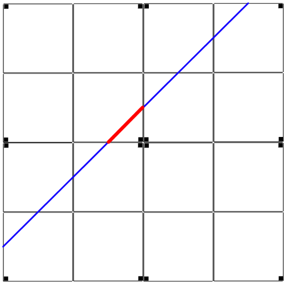

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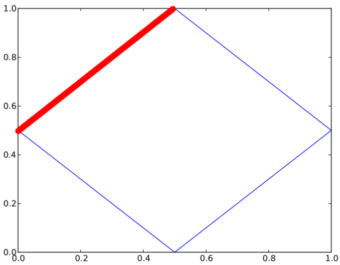


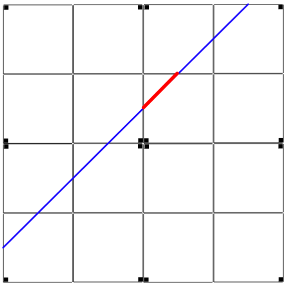

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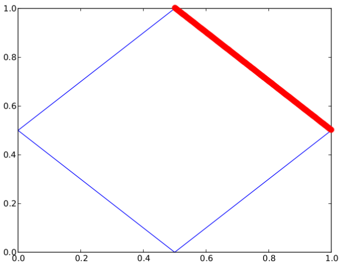


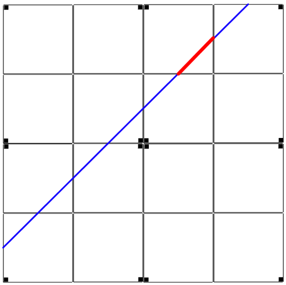

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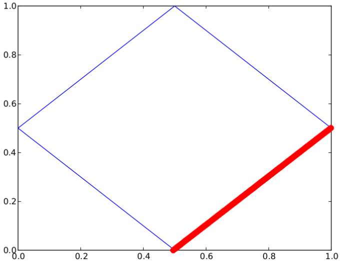


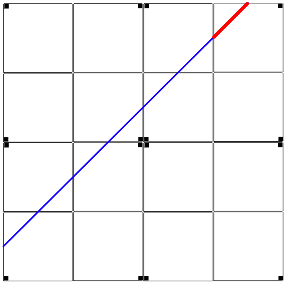

## Sequence Characterization

$$\underbrace{aaa}_{3} \quad b \quad \underbrace{aa}_{2} \quad b \quad \underbrace{aa}_{2} \quad b \quad \underbrace{aaa}_{3} \quad b \quad \underbrace{aa}_{2} \quad b$$

$$\underbrace{aa}_{2} \quad b \quad \underbrace{aaa}_{3} \quad b \quad \underbrace{aa}_{2} \quad b \quad \underbrace{aaa}_{3} \quad \underset{\dots}{\dots}$$

# Sequence Characterization

$$3 \quad \underbrace{22}_{2} \quad 3 \quad \underbrace{22}_{2} \quad 3 \quad \underbrace{2}_{1} \quad \underset{\dots}{3} \quad \dots$$

1. Introduction
2. Lemmas
3. 1-dimensional Problem
4. Future Research

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$$3 \quad \underbrace{22}_{2} \quad 3 \quad \underbrace{22}_{2} \quad 3 \quad \underbrace{2}_{1} \quad \underset{\dots}{3} \quad \dots$$

# Algorithm

$$dx_n = \bigcap_{i=0}^{n} \left( \frac{i}{1 + \sum_{j=0}^{i} n_j}, \frac{1}{-1 + \sum_{j=0}^{i} n_j} \right)$$

$$\delta_n = \bigcap_{i=0}^{n} \left( i - dx_{n,max}(1 + \sum_{j=0}^{i} n_j), i - dx_{n,min}(1 + \sum_{j=0}^{i} n_j) \right)$$

# Extensions to Tileable Polygons

Other Tileable Polygons:

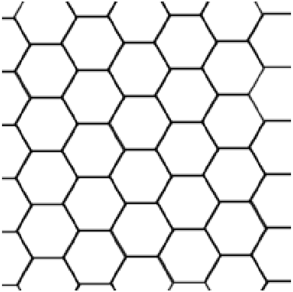

Figure: Regular Hexagons

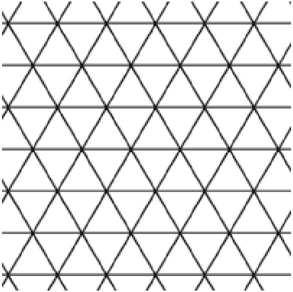

Figure: Equilateral Triangles

# Extensions to Non-Tileable Polygons

- Irregular triangles
- Pentagons
- Octagons

# Extensions to Circles

- Characterize how particle bounces around circle
- Analog to $a$, $b$ might be sequence of collision points as you move around circle.