Billiards

Jonathan Allen, John Wang

Massachusetts Institute of Technology

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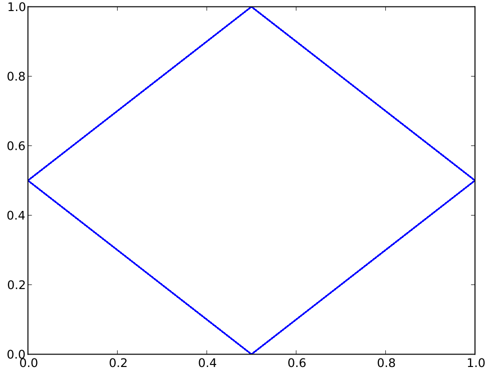
Introduction

- Billiard ball bouncing in a square
- Assume no gravity or friction
- Examine sequence of side collisions

Example

Example

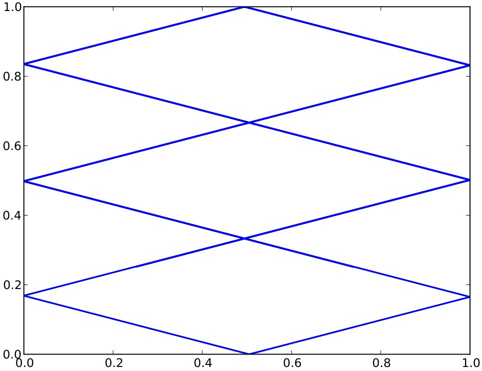
Examine the periodic sequence: ‘abab‘



Another Example

Example

Examine the periodic sequence: ‘aaabaaab‘

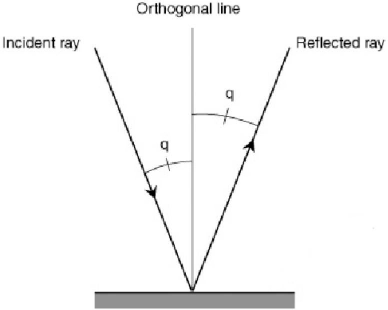


Presentation Outline

Notation

Definition

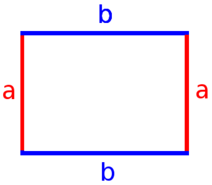
A table T is the unit square in $\mathbb{R}^2$. A particle $p \in T$ begins at position $\bar{x}_0 \in T$ with velocity $\bar{v}$. When the particle reaches an edge of the table, velocity is reflected about the line perpendicular to the table's edge.



Notation

Definition

Opposite sides of the table are named a and b. **Primary side** (most collisions) is a, **secondary side** is b.



Notation

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Collision string consists of the sides of the table that have been collided with for a given starting position and velocity.

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Primary substring is a subsequence from the collision string which contains the primary side collisions that occur between consecutive secondary side collisions.

Example

Collision string: ‘aabaaabaabaaab‘, **Primary substrings**: ‘aa‘, ‘aaa‘

Problem Statement

Problem: Characterize the properties of collision sequences.

- Given a sequence of a's and b's, determine if it is a valid collision sequence.
- Given a valid collision sequence, determine a possible starting position and velocity.

1. Introduction
   - Examples
   - Outline
   - Notation and Problem Statement
2. Lemmas
   - Tiling
   - 1-dimensional Problem

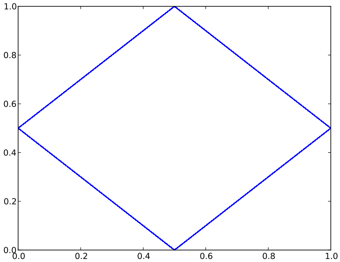
Representing Collision Strings

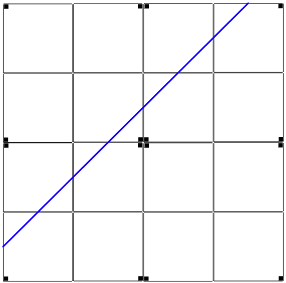
- Reflect squares about each side to create a tiling
- Solutions become lines in the plane
- Intersections become places where collisions occur

Representing Collision Strings

Example

Tiling of $\vec{x}_0 = (0, 0.5)$ and $\vec{v} = (0.25, 0.25)$.





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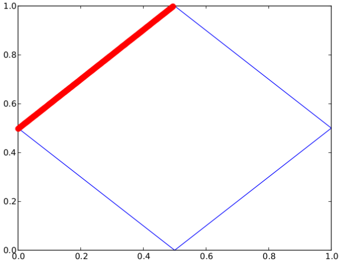

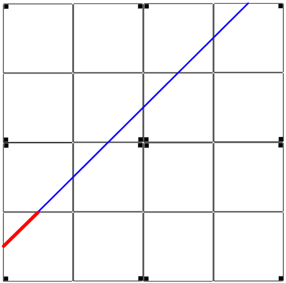

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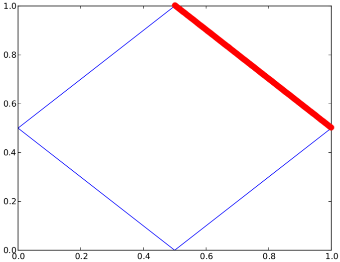

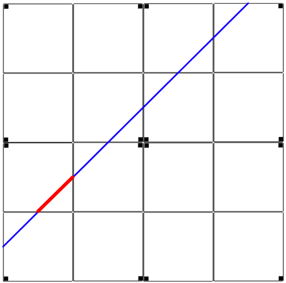

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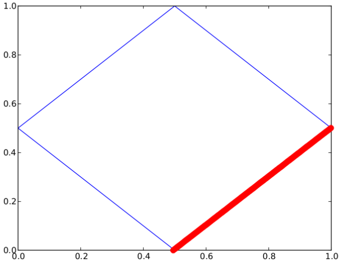

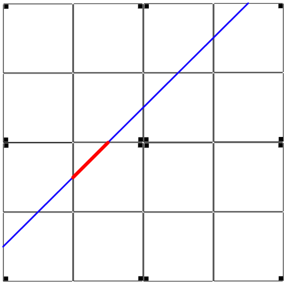

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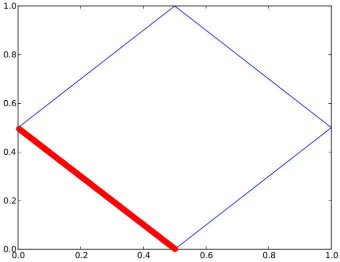

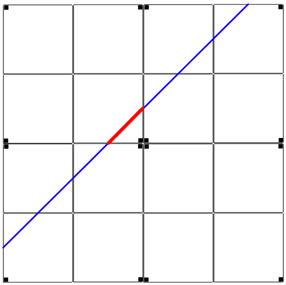

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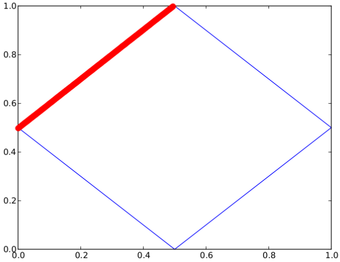

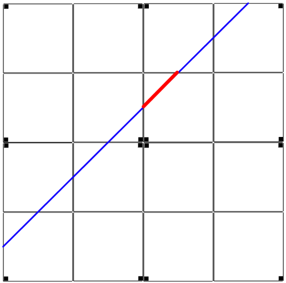

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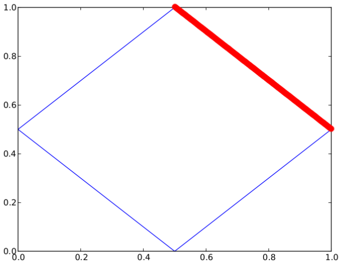

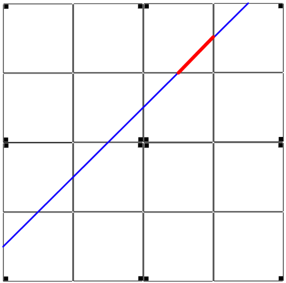

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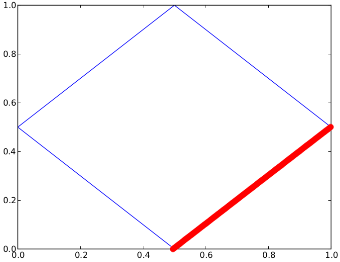

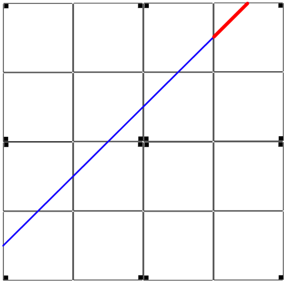

Periodicity of Rationals

We can represent a particle as a line in the plane $y = mx + y_0$ in the tiling. Periodicity occurs if $y_0 2k = mx + y_0$ for some $k \in \mathbb{N}$.

- If $m, y_0 \in \mathbb{Q}$, then all sequences are periodic.
- If m or y_0 are irrational, then the sequences are not periodic.

Maximum Differences between Primary Substring Lengths

Given a collision string, how different can primary substrings be?

Example

Is *abaaab* possible?

Maximum Differences between Primary Substring Lengths

Length of primary substring i is given by:

$$L(i, m, y_0) = \lfloor \frac{i - y_0}{m} \rfloor - \lfloor \frac{i - 1 - y_0}{m} \rfloor \tag{1}$$

Maximum Differences between Primary Substring Lengths

For $i, j \in \mathbb{N}$, $m \in [0,1]$, $y_0 \in [0,1]$, we will show:

$$\max_{i>j} L(i, m, y_0) - L(j, m, y_0) \leq 2 \tag{2}$$

Maximum Differences between Primary Substring Lengths

Lemma

$$\lfloor a-x \rfloor + \lfloor b-x \rfloor = \lfloor a \rfloor - \lfloor b \rfloor + (\lfloor \{a\} - \{x\} \rfloor - \lfloor \{b\} - \{x\} \rfloor) \tag{3}$$

$$\begin{aligned} L(i,m,y_0) - L(j,m,y_0) &= \left(\lfloor \frac{i}{m} \rfloor - \lfloor \frac{i-1}{m} \rfloor\right) + \left(\lfloor \frac{j}{m} \rfloor - \lfloor \frac{j-1}{m} \rfloor\right) && (4) \\ &+ \left(\lfloor \{\frac{i}{m}\} - \{\frac{y_0}{m}\} \rfloor - \lfloor \{\frac{i-1}{m}\} - \{\frac{y_0}{m}\} \rfloor\right) && (5) \\ &- \left(\lfloor \{\frac{j}{m}\} - \{\frac{y_0}{m}\} \rfloor - \lfloor \{\frac{j-1}{m}\} - \{\frac{y_0}{m}\} \rfloor\right) && (6) \end{aligned}$$

Maximum Differences between Primary Substring Lengths

Let $g(x, m, y_0) = \lfloor \{\frac{x}{m}\} - \{\frac{y_0}{m}\} \rfloor - \lfloor \{\frac{x-1}{m}\} - \{\frac{y_0}{m}\} \rfloor$.

$$\begin{aligned} L(i, m, y_0) - L(j, m, y_0) &= \{\frac{i-1}{m}\} - \{\frac{i}{m}\} - \left(\{\frac{j-1}{m}\} - \{\frac{j}{m}\}\right) \quad (7) \\ &+ g(i, m, y_0) - g(j, m, y_0) \quad (8) \end{aligned}$$

Maximum Differences between Primary Substring Lengths

$$L(i,m,y_0) - L(j,m,y_0) = \{\frac{i-1}{m}\} - \{\frac{i}{m}\} - \left(\{\frac{j-1}{m}\} - \{\frac{j}{m}\}\right) \quad (9)$$
$$+ \; g(i,m,y_0) - g(j,m,y_0) \quad (10)$$

- If $g(x,m,y_0) = 1$, then:

$$\{\frac{x-1}{m}\} < \{\frac{y_0}{m}\} < \{\frac{x}{m}\} \quad (11)$$

  Which implies $0 < L(x,m,y_0) < 1$.

- If $g(x,m,y_0) = -1$, then:

$$\{\frac{x}{m}\} < \{\frac{y_0}{m}\} < \{\frac{x-1}{m}\} \quad (12)$$

  Which implies $-1 < L(x,m,y_0) < 0$.

- If $g(x,m,y_0) = 0$, then: $-1 < L(x,m,y_0) < 1$.

Maximum Differences between Primary Substring Lengths

$$\max_{i>j} L(i, m, y_0) - L(j, m, y_0) \leq 2 \quad (13)$$

Sequence Characterization

$$\underbrace{aaa}_{3}\quad b\quad \underbrace{aa}_{2}\quad b\quad \underbrace{aa}_{2}\quad b\quad \underbrace{aaa}_{3}\quad b\quad \underbrace{aa}_{2}\quad b$$

$$\underbrace{aa}_{2}\quad b\quad \underbrace{aaa}_{3}\quad b\quad \underbrace{aa}_{2}\quad b\quad \underbrace{aaa}_{3}\quad \underset{\dots}{\dots}$$

Sequence Characterization

$$3 \quad \underbrace{22}_{2} \quad 3 \quad \underbrace{22}_{2} \quad 3 \quad \underbrace{2}_{1} \quad \underset{\ldots}{3} \quad \ldots$$

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Extensions to Tileable Polygons

Other Tileable Polygons:

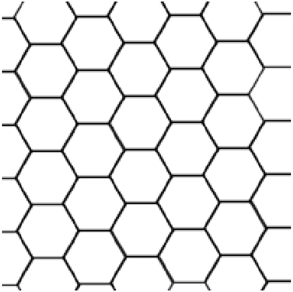

Figure: Regular Hexagons

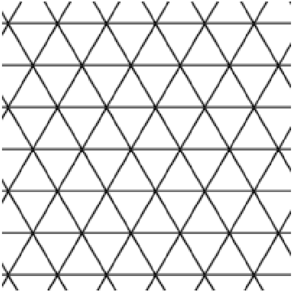

Figure: Equilateral Triangles

Extensions to Non-Tileable Polygons

- Irregular triangles
- Pentagons
- Octagons

Extensions to Circles

- Characterize how particle bounces around circle
- Analog to *a*, *b* might be sequence of collision points as you move around circle.