Billiards

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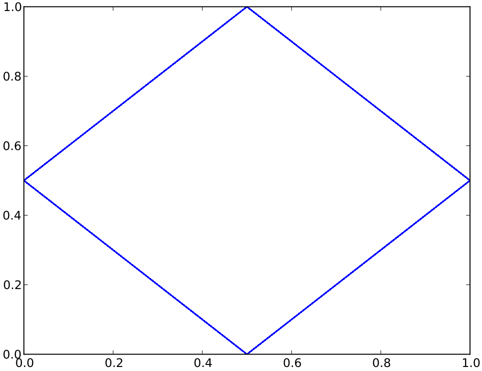
Introduction

- Billiard ball bouncing in a square
- Assume no gravity or friction
- Examine sequence of side collisions

Example

Example

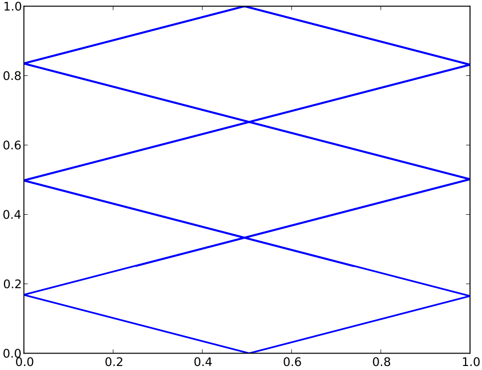
Examine the periodic sequence: ‘abab‘



Another Example

Example

Examine the periodic sequence: ‘aaabaaab‘

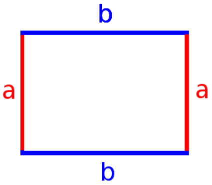


Presentation Outline

Notation

Definition

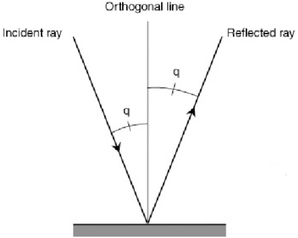
A table $T \in \mathbb{R}^2$ is the unit square. Opposite sides of the table are labeled a and b.



Notation

Definition

A ball $p \in T$ begins at position $\vec{r}_0 \in T$ with initial velocity $\vec{v}_0 \neq 0$. When the ball collides with an edge of the table, it reflects such that the component of its velocity normal to the edge is negated after the collision, and its component tangent to the edge is unchanged.



Notation

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Collision string: list of the sides of the table that a ball collides with, ordered by collision time. e.g. ‘abaaabaaaab’.
Primary side: side appearing most often in a collision string.
Secondary side the other side.

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Example

Collision string: ‘aabaaabaabaaab‘, **Primary substrings**: ‘aa‘, ‘aaa‘

Problem Statement

Problem: Characterize the properties of collision sequences.

- Given a sequence of a's and b's, determine if it is a valid collision sequence.
- Given a valid collision sequence, determine a possible starting position and velocity.

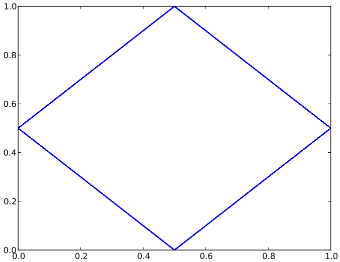
Representing Collision Strings

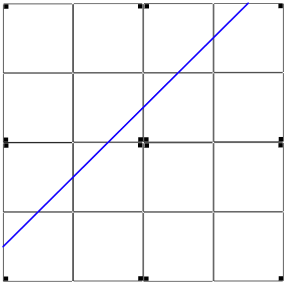
- Reflect squares about each side to create a tiling
- Solutions become lines in the plane
- Intersections become places where collisions occur

Representing Collision Strings

Example

Tiling of $\vec{x}_0 = (0, 0.5)$ and $\vec{v} = (0.25, 0.25)$.





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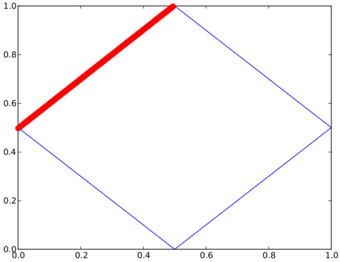

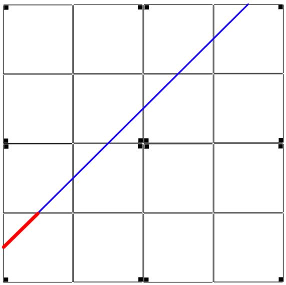

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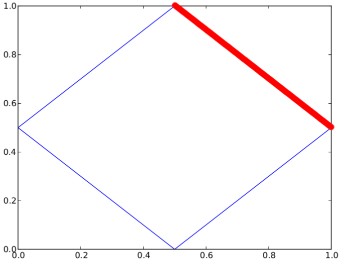

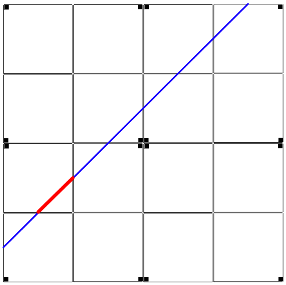

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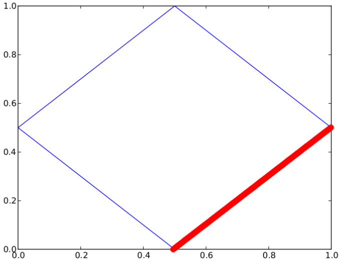

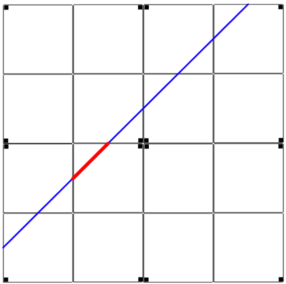

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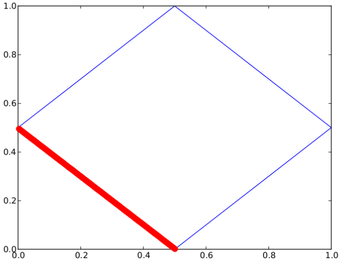

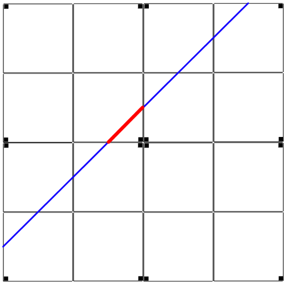

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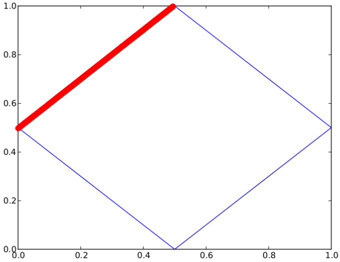

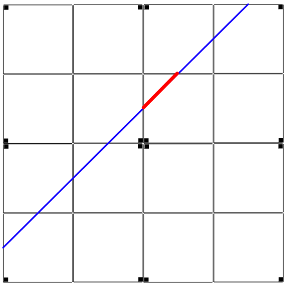

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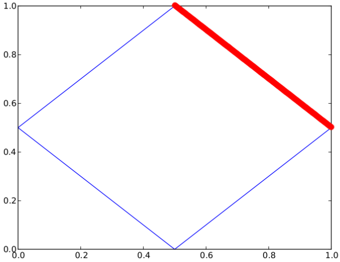

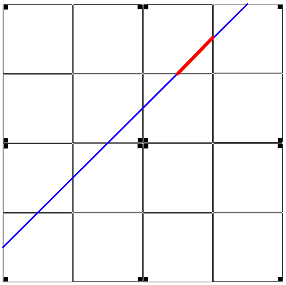

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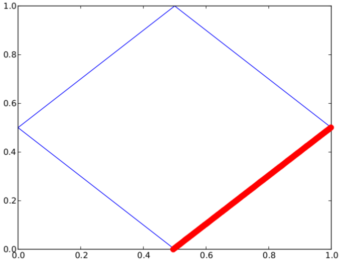

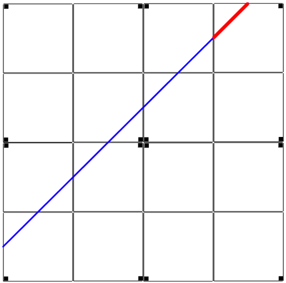

Periodicity of Rationals

We can represent a particle as a line in the plane $y = mx + y_0$ in the tiling. Periodicity occurs if $y_0 2k = mx + y_0$ for some $k \in \mathbb{N}$.

- If $m, y_0 \in \mathbb{Q}$, then all sequences are periodic.
- If m or y_0 are irrational, then the sequences are not periodic.

Maximum Differences between Primary Substring Lengths

Given a collision string, how different can primary substrings be?

Example

Is *abaaab* possible?

Maximum Differences between Primary Substring Lengths

Length of primary substring i is given by:

$$L(i, m, y_0) = \left\lfloor \frac{i - y_0}{m} \right\rfloor - \left\lfloor \frac{i - 1 - y_0}{m} \right\rfloor$$

Maximum Differences between Primary Substring Lengths

For $i, j \in \mathrm{N}$, $m \in [0,1]$, $y_0 \in [0,1]$, we will show:

$$\max_{i>j} L(i, m, y_0) - L(j, m, y_0) \leq 2$$

Maximum Differences between Primary Substring Lengths

Lemma

$$\lfloor a - x \rfloor + \lfloor b - x \rfloor = \lfloor a \rfloor - \lfloor b \rfloor + (\lfloor \{a\} - \{x\} \rfloor - \lfloor \{b\} - \{x\} \rfloor)$$

$$\begin{aligned} L(i, m, y_0) - L(j, m, y_0) &= \left(\left\lfloor \frac{i}{m} \right\rfloor - \left\lfloor \frac{i-1}{m} \right\rfloor \right) + \left(\left\lfloor \frac{j}{m} \right\rfloor - \left\lfloor \frac{j-1}{m} \right\rfloor \right) \\ &+ \left(\left\lfloor \left\{ \frac{i}{m} \right\} - \left\{ \frac{y_0}{m} \right\} \right\rfloor - \left\lfloor \left\{ \frac{i-1}{m} \right\} - \left\{ \frac{y_0}{m} \right\} \right\rfloor \right) \\ &- \left(\left\lfloor \left\{ \frac{j}{m} \right\} - \left\{ \frac{y_0}{m} \right\} \right\rfloor - \left\lfloor \left\{ \frac{j-1}{m} \right\} - \left\{ \frac{y_0}{m} \right\} \right\rfloor \right) \end{aligned}$$

Maximum Differences between Primary Substring Lengths

Let $g(x,m,y_0) = \left\lfloor \left\{\frac{x}{m}\right\} - \left\{\frac{y_0}{m}\right\} \right\rfloor - \left\lfloor \left\{\frac{x-1}{m}\right\} - \left\{\frac{y_0}{m}\right\} \right\rfloor$.

$$\begin{aligned} L(i,m,y_0) - L(j,m,y_0) &= \left\{\frac{i-1}{m}\right\} - \left\{\frac{i}{m}\right\} - \left(\left\{\frac{j-1}{m}\right\} - \left\{\frac{j}{m}\right\}\right) \\ &+ g(i,m,y_0) - g(j,m,y_0) \end{aligned}$$

Maximum Differences between Primary Substring Lengths

- If $g(x, m, y_0) = 1$, then:

$$\left\{\frac{x-1}{m}\right\} < \left\{\frac{y_0}{m}\right\} < \left\{\frac{x}{m}\right\}$$

  Which implies $0 < L(x, m, y_0) < 1$.

- If $g(x, m, y_0) = -1$, then:

$$\left\{\frac{x}{m}\right\} < \left\{\frac{y_0}{m}\right\} < \left\{\frac{x-1}{m}\right\}$$

  Which implies $-1 < L(x, m, y_0) < 0$.

- If $g(x, m, y_0) = 0$, then: $-1 < L(x, m, y_0) < 1$.

Maximum Differences between Primary Substring Lengths

$$\max_{i>j} L(i, m, y_0) - L(j, m, y_0) \leq 2$$

Sequence Characterization

$$\underbrace{aaa}_{3}\quad b\quad \underbrace{aa}_{2}\quad b\quad \underbrace{aa}_{2}\quad b\quad \underbrace{aaa}_{3}\quad b\quad \underbrace{aa}_{2}\quad b$$

$$\underbrace{aa}_{2}\quad b\quad \underbrace{aaa}_{3}\quad b\quad \underbrace{aa}_{2}\quad b\quad \underbrace{aaa}_{3}\quad \underset{\ldots}{\ldots}$$

Sequence Characterization

$$3 \quad \underbrace{22}_{2} \quad 3 \quad \underbrace{22}_{2} \quad 3 \quad \underbrace{2}_{1} \quad \underset{\ldots}{3} \quad \ldots$$

Extensions to Tileable Polygons

Other Tileable Polygons:

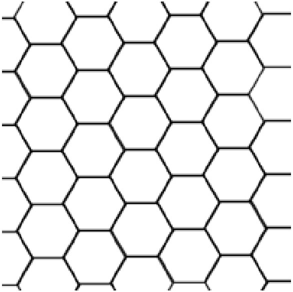

Figure: Regular Hexagons

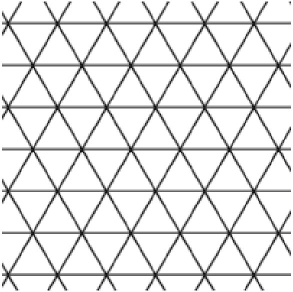

Figure: Equilateral Triangles

Extensions to Non-Tileable Polygons

- Irregular triangles
- Pentagons
- Octagons

Extensions to Circles

- Characterize how particle bounces around circle
- Analog to a, b might be sequence of collision points as you move around circle.