Billiards

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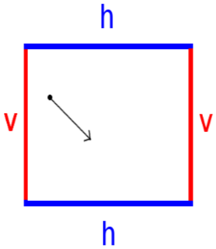
Introduction

- Billiard ball bouncing in a square
- Assume no gravity or friction

Basic Notation

Definition

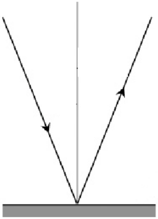
A table $T \subset \mathbb{R}^2$ is the unit square. Vertical sides are labelled with a v. Horizontal sides are labelled with an h.



Notation

Definition

A ball $p \in T$ begins at position $\vec{r}_0 \in T$ with initial velocity $\vec{v}_0 \neq 0$. When the ball collides with an edge of the table, it reflects its angle with the table edge.



Example

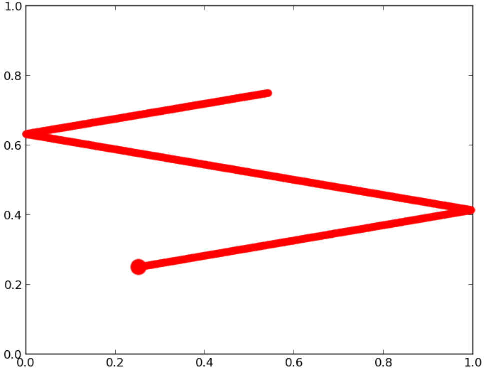
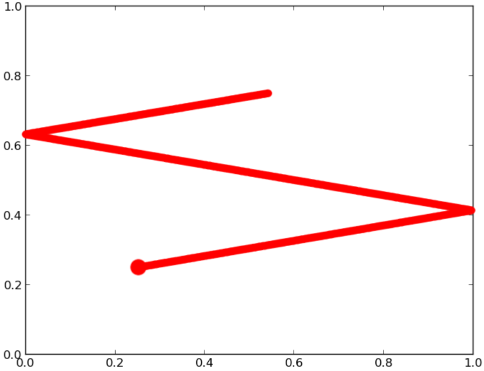


Figure: $\vec{r}_0 = (1/4, 1/4)$ and $\vec{v}_0 = (23, 5)$.

Example



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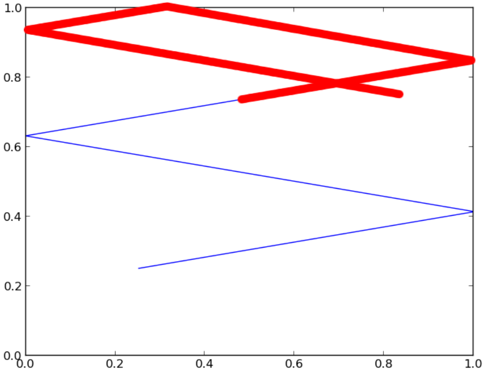

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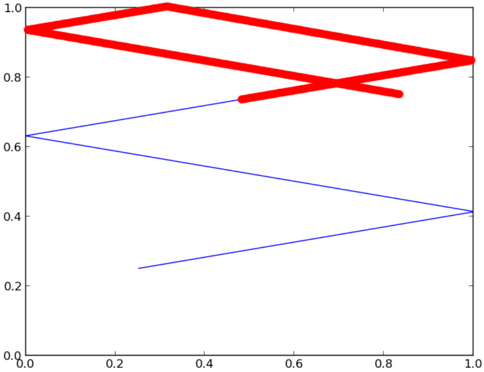

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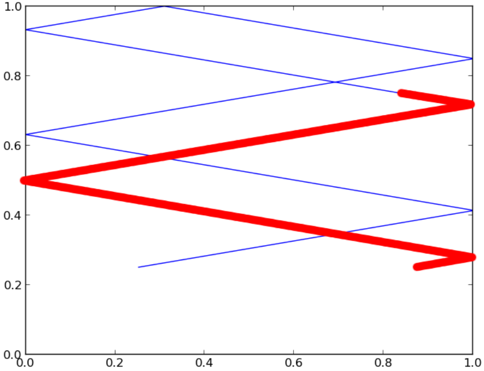
$$vvvhv \tag{2}$$

Example



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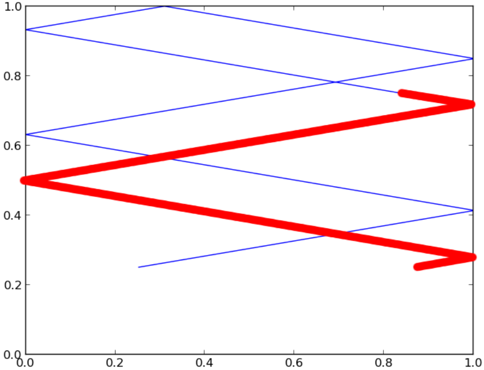

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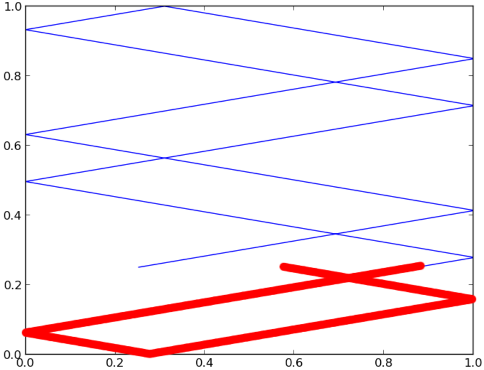
$$vvvhvvvv \tag{3}$$

Example



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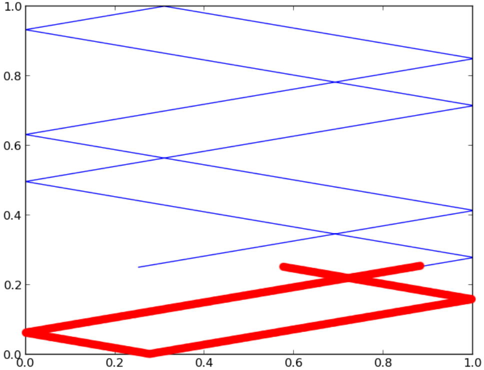

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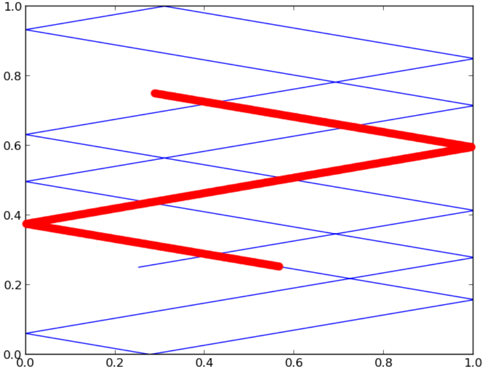
$$vvvhvvvvvhv \tag{4}$$

Example



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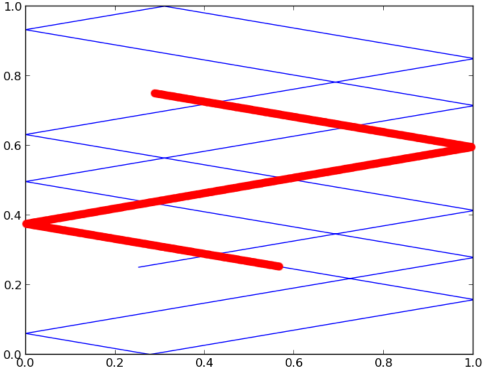

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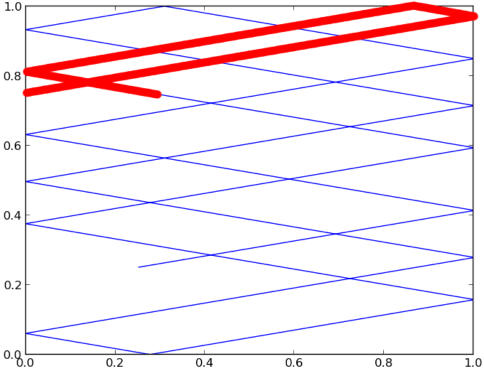
$$vvvhvvvvvhvvv \tag{5}$$

Example



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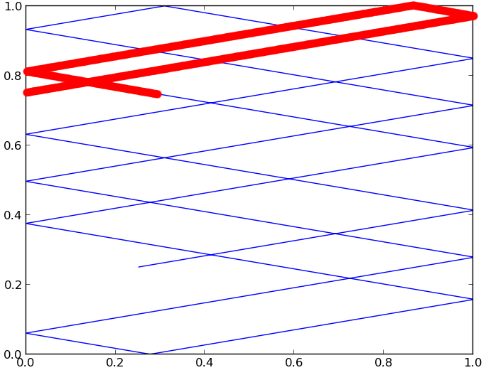

Figure: $\vec{r}_0 = (1/4, 1/4)$ and $\vec{v}_0 = (23, 5)$.

$$vvvhvvvvvhvvvvhv \tag{6}$$

Resulting Sequence

$$vvvhvvvvvhvvvvhv \tag{7}$$

Problem Statement

Problem: Characterize the properties of collision sequences.

- Given a sequence of a's and b's, determine if it is a valid collision sequence.
- Given a valid collision sequence, determine a possible starting position and velocity.

Presentation Outline

Secondary Side Theorem

Theorem

At least one side will never have more than one consecutive occurrence in a valid collision string.

Secondary Side Theorem Examples

Example

Valid: $vhhhvhhhv$

Example

Valid: $vhvhvhv$

Example

Valid: $vvvvvhvvvvhvvvvvhvvvv$

Example

Invalid: $vvhhhvvvhhhvvhhh$

Example

Invalid: $vhhhvvhvh$

Secondary Side Theorem Proof

- A billiard ball trajectory must be a line in the tiled grid with slope m.
- Case 1: $m = 1$.
- Case 2: $m < 1$ or $m > 1$.

Secondary Side Theorem Proof

If $m = 1$, v and h alternate.

Secondary Side Theorem Proof

If $m < 1$, there must exist an h between each v.
If $m > 1$, similar argument holds.

Notation

Definition

Secondary side: a side which never has more than one consecutive occurrences. **Primary side**: a side which is not a secondary side.

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Primary substring: a subsequence from the collision string which contains a consecutive sequence of primary sides.

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Example

Collision string: *vvhvvvhvvhvvvh* **Secondary Side**: *h* **Primary Side**: *v*
Primary substrings: *vv*, *vvv*

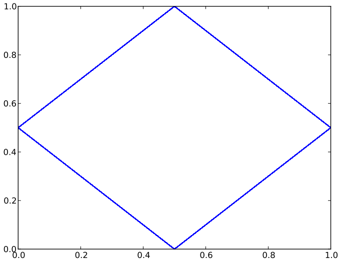
Representing Collision Strings

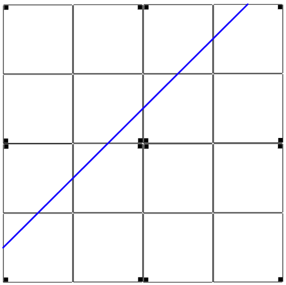
- Reflect squares about each side to create a tiling
- Solutions become lines in the plane
- Intersections become places where collisions occur

Representing Collision Strings

Example

Tiling of $\vec{x}_0 = (0, 0.5)$ and $\vec{v} = (0.25, 0.25)$.





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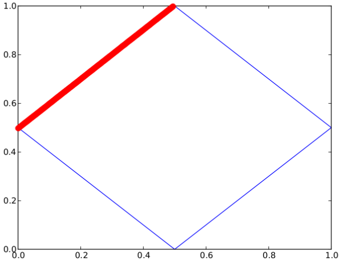

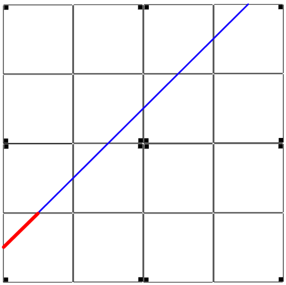

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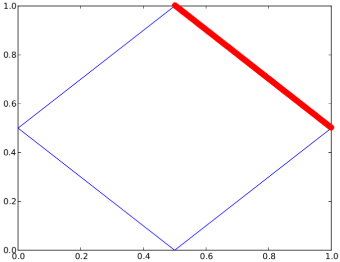

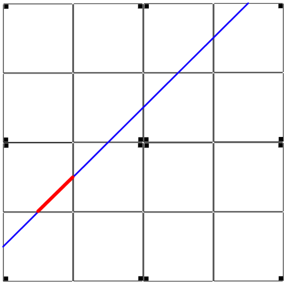

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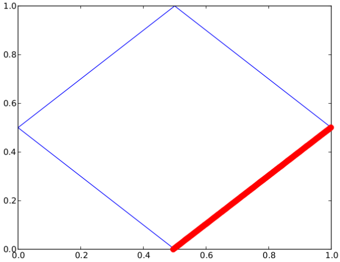

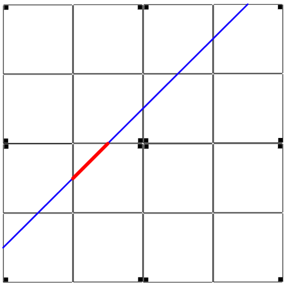

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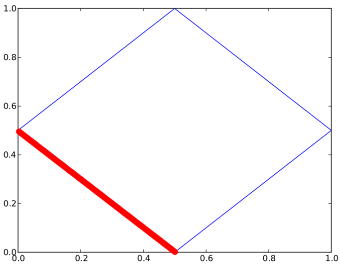

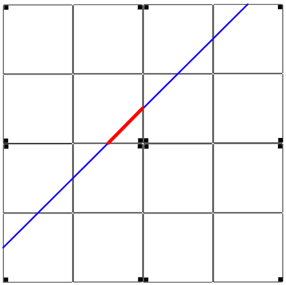

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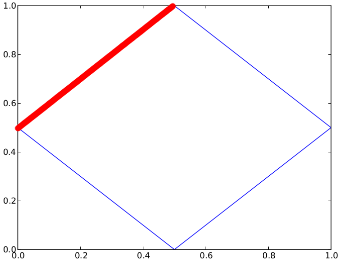

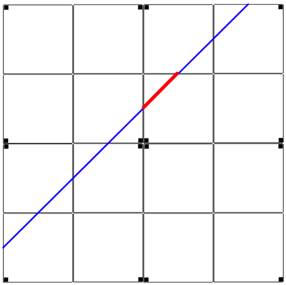

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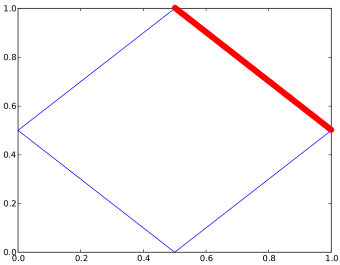

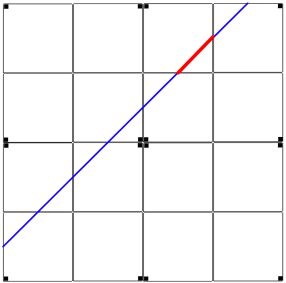

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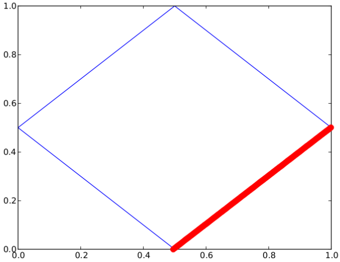

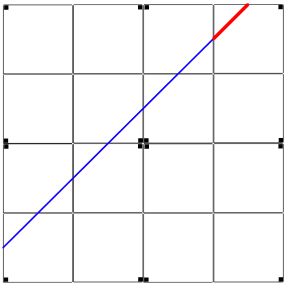

Sequence Characterization

$$\underbrace{aaa}_{3}\ b\ \underbrace{aa}_{2}\ b\ \underbrace{aa}_{2}\ b\ \underbrace{aaa}_{3}\ b\ \underbrace{aa}_{2}\ b$$

$$\underbrace{aa}_{2}\ b\ \underbrace{aaa}_{3}\ b\ \underbrace{aa}_{2}\ b\ \underbrace{aaa}_{3}\ \underbrace{\ldots}_{\ldots}$$

Sequence Characterization

$$3 \quad \underbrace{22}_{2} \quad 3 \quad \underbrace{22}_{2} \quad 3 \quad \underbrace{2}_{1} \quad \underset{\ldots}{3} \quad \ldots$$

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Sequence Characterization

$$3 \quad \underbrace{22}_{2} \quad 3 \quad \underbrace{22}_{2} \quad 3 \quad \underbrace{2}_{1} \quad \underset{\ldots}{3} \quad \ldots$$

Algorithm

$$dx_n = \bigcap_{i=0}^{n} \left(\frac{i}{1 + \sum_{j=0}^{i} n_j}, \frac{1}{-1 + \sum_{j=0}^{i} n_j} \right)$$

$$\delta_n = \bigcap_{i=0}^{n} \left(i - dx_{n,max}(1 + \sum_{j=0}^{i} n_j), i - dx_{n,min}(1 + \sum_{j=0}^{i} n_j) \right)$$

Extensions to Tileable Polygons

Other Tileable Polygons:

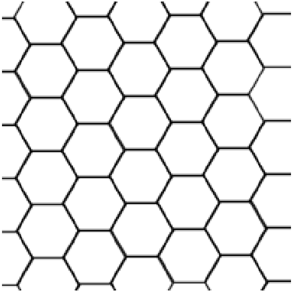

Figure: Regular Hexagons

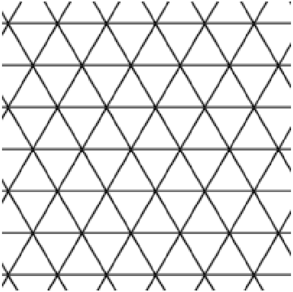

Figure: Equilateral Triangles

Extensions to Non-Tileable Polygons

- Irregular triangles
- Pentagons
- Octagons

Extensions to Circles

- Characterize how particle bounces around circle
- Analog to *a*, *b* might be sequence of collision points as you move around circle.