SEQUENCES OF BILLIARD BALL COLLISIONS

JONATHAN ALLEN, JOHN WANG

1. 1-Dimensional Representation

Note: There are more h's than v's: $m \geq 1$

Rather than looking at an explicit representation of lines in the plane, we can gain much more insight from looking at a parametric representation. To simplify our analysis, we will choose our time parameter such that v collisions occur every $\Delta t = 1$ and h collisions occur every $\Delta t = m$. The equation for a line $y(x) = m\,x + b$ is equivalent to the following parametric system

$$x(t) = \frac{1}{m}\,t + x_0 \tag{1}$$

$$y(t) = t \tag{2}$$

Now our v and h collisions in the 2-dimensional plane can be projected onto the 1-dimensional t axis.

Date: December 5, 2013.

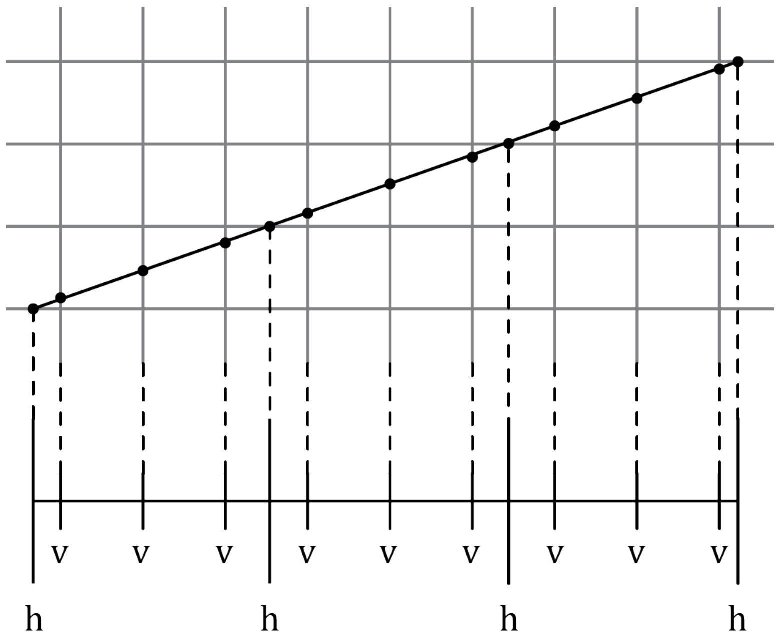

FIGURE 1. Projecting onto the parametric representation.

Lemma 1.1. *A sequence α is a valid collision sequence iff there exists at least one valid collision sequence containing α that starts and ends with an h.*

Proof. TODO □

Because of Lemma 1.1, without loss of generality we can confine ourselves to only look at collision sequences that start and end with an h.

Definition 1.2. *Given a collision sequence α, define a sequence β where each element $\beta_i^{(0)}$ is the number of v collisions between the i^{th} and $(i{+}1)^{th}$ h in α.*

The $\beta^{(0)}$ sequence is much simpler to think of geometrically: $\beta_i^{(0)}$ represents the number of v collision tick marks in between each h collision tick mark which is shown in Figure 2.

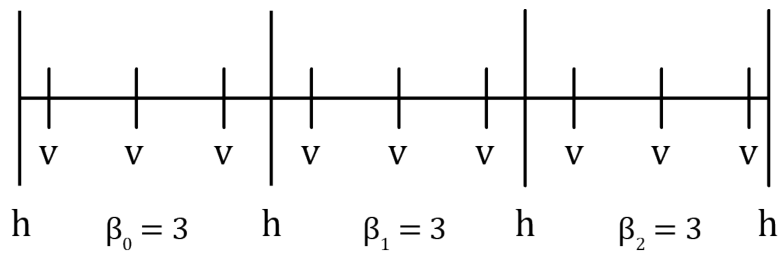

FIGURE 2. The $\beta^{(i)}$ sequence.

Lemma 1.3. *For every valid collision sequence, the following must be true*

$$\beta^{(0)}_{min} > 0 \tag{3}$$

Proof. TODO □

Theorem 1.4. *For every valid collision sequence, the following must be true*

$$\beta^{(0)}_{max} - \beta^{(0)}_{min} \leq 1 \tag{4}$$

Proof. From Equation 1, v collisions occur every $\Delta t = 1$ and h collisions occur every $\Delta t = m$. Thus, the following must be true

$$\beta^{(0)}_i \in (\lfloor m \rfloor, \lceil m \rceil) \tag{5}$$

For an m to exist that satisfies the above constraints, all numbers in the β sequence can only differ by 1.

□

Definition 1.5. *Given a valid collision sequence, define the sequence $\beta^{(j)}$ where each element $\beta^{(j)}_i$ is 1 more than the number of occurrences of $\beta^{(j-1)}_{max}$ between the i^{th} and $(i+1)^{th}$ occurrence of $\beta^{(j-1)}_{min}$ in the $\beta^{(j-1)}$ sequence.*

If, for some j_f, the length of $\beta^{(j_f-1)}$ is 1, then $\beta^{(j_f-1)}$ is the terminating m̈etas̈equence, and all subsequent $\beta^{(j)}$ for $j \geq j_f$ are undefined.

Definition 1.6. *Define the sequence a in the following manner*

$$a_j := \begin{cases} m & \textit{for} \quad j = 0 \\ 1 & \textit{for} \quad j = 1 \\ \beta^{(j-2)}_{max} a_{j-1} - a_{j-2} & \textit{for} \quad 2 \leq j < j_f \end{cases} \tag{6}$$

From now on we will only consider collision sequences, where each $\beta^{(j)}$ either starts and ends with $\beta^{(j)}_{min}$ or has length 1.

Theorem 1.7. *A collision sequence is valid iff the following is true for all j*

$$\beta_i^{(j)} \in \left\{ \left\lfloor \frac{a_{j-2}}{a_{j-1}} \right\rfloor, \left\lceil \frac{a_{j-2}}{a_{j-1}} \right\rceil \right\} \tag{7}$$

Proof. If we pick a $\beta_i^{(j)}$ and plot it in the following manner, we notice that

Define

$$\gamma = \left\lceil \frac{a_{j-1}}{a_{j-2}} \right\rceil \tag{8}$$

$$\geq \frac{a_{j-1}}{a_{j-2}} \tag{9}$$

and

$$\delta_i^{(j)} := \begin{cases} x_0 & \text{for} \quad i = 0 \\ i(\gamma * -a_{i-2}) + x_0 & \text{for} i \geq 1 \end{cases} \tag{10}$$

We can notice that

$$\beta_i^{(j)} = \left\lfloor \delta_i^{(j)} \right\rfloor + \beta^{(j)}_{max} - \left\lfloor \delta_{i+1}^{(j)} \right\rfloor \tag{11}$$

□

We can immediately notice that the $\delta^{(0)}$ sequence has the following features:

(1) The $\delta^{(0)}$ sequence is increasing, because $\beta^{(0)}_{max} \geq m$
(2) Combining Theorem 1.4 and Equation 11, we get the following:

$$\left\lfloor \delta_{i+1}^{(0)} \right\rfloor - \left\lfloor \delta_{i+1}^{(0)} \right\rfloor = \beta^{(0)}_{max} - \beta^{(0)}_{min} \tag{12}$$

$$\leq 1 \tag{13}$$

Thus, if we plot the values of the $\delta^{(0)}$ sequence on a line, we notice something interesting: the plot looks very similar to our original plot of the collision sequence parameterized by t.

We can continue in this fashion forming more m̈etas̆equences.

Theorem 1.8. *For every valid collision sequence, $a \to 0$*

Proof. □