SEQUENCES OF BILLIARD BALL COLLISIONS

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Abstract.

1. Introduction

In this paper we shall explore sequences of billiard ball collisions. In particular, we look at the sequence of sides that a billiard ball collides with under perfect, frictionless conditions. We will show how a square billiards table can be analyzed by tiling the table in the plane, and prove a number of properties that billiard ball sequences must satisfy.

This introduction will explain the general setup of the problem and will give an example of how the definitions relate to a billiard ball with particular initial conditions.

1.1. **Setup.**

We will imagine an infinitesimally small billiard ball on a square table. For simplicity, we will assume that the square table is defined on the unit square, i.e. $[0,1]^2$. The ball will start at some initial position $\mathbf{x}_0$ inside of the table and with some velocity $\mathbf{v}_0$. We will assume that the ball is massless and frictionless, and that there is no gravity.

We will assume ideal, elastic collisions so that the ball conserves both kinetic energy and momentum when it hits a wall. To be more precise, when the ball collides with an edge of the table, the ball's velocity will be reflected across the line perpendicular to the edge of the table at the point of collision. In other words, the angle of the incoming velocity will be the same as the outgoing velocity. Figure 1 shows the general mechanics of a collision.

Now, we shall label the horizontal sides of the table h and the vertical sides of the table v. Whenever the billiard ball collides with a horizontal side (labelled h), we shall call the resulting collision an h-collision. Whenever the ball collides with a vertical side (labelled v), we shall call the resulting collision a v-collision.

We shall now define what this paper will be primarily interested in, a sequence of collisions:

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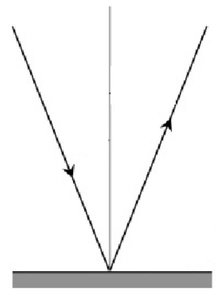

FIGURE 1. Mechanics of a Billiard Ball Colliding with a Table Edge

Definition 1.1. *A* collision sequence *(α) is a sequence of v and h collisions which starts and ends with an h collision for some ball b with initial position $\mathbf{x}_0$ and intial velocity $\mathbf{v}_0$.*

Notice that all non-trivial initial conditions for a billiard ball will result in infinitely many h-collisions. The only initial conditions for which this is not true are when the initial velocity is parallel to the horizontal ($\mathbf{v}_0 = (1, 0)$) so that the ball bounces infinitely between the two vertical sides. The proof of this is trivial and should become clear once the tiling representation is presented, so we will omit it.

Thus, it is perfectly reasonable to constrain a ball's collision sequence to begin and end with an h collision, since one simply needs to extend the number of collisions one watches until the sequence of collisions begins and ends with an h-collision. This constraint will later make it easier to reason about properties of sequences.

1.2. **Example Collision Sequence.** To understand collision sequences better, we shall provide an example. Consider a billiard ball with intial position $\mathbf{x}_0 = (0.75, 0.75)$ and initial velocity $\mathbf{v}_0 = (0.23, 0.05)$.

We can see from the figure that the collisions that it makes, denoting a v-collision with a v and an h-collision with an h, are $\{v, h, v, v, v, v, v, h, v, v, v, v, v, h, v, v, v\}$. A valid collision sequence given these initial conditions would start and end with an h collision. A possible example would be:

$$\alpha = hvvvvvhvvvvvh \tag{1}$$

Notice that the collisions will continue infinitely, so one could imagine extending this collision sequence to include more v and h collisions. Although not shown in the picture, the following collision sequence would also be valid if one continued showing the trajectory of the ball in future collisions:

$$\alpha = hvvvvvhvvvvvhvvvvvh \tag{2}$$

2. Tiling Representation

We shall now present a representation of the problem which will greatly simplify the analysis of v and h collisions for some ball b, called the tiling representation.

To understand the basics of how it works, imagine placing a square billiard table on the xy plane. The billiard table (as mentioned in the introduction) will be a unit square, so it will contain $[0, 1]^2$. The table's edges will be the four line segments bordering the unit square. A ball will start with some initial position $\mathbf{x}_0 \in [0, 1]^2$ and velocity $\mathbf{v}_0$. After some time, the ball will collide with an edge e_0 of the table. However, instead of thinking of the trajectory of the ball as being reflected across the line perpendicular to e at the point of collision, we will instead reflect the original unit square s_0 across the edge e_0 to create a new square s_1. Now, the trajectory of the ball after the first collision will be presented in the new square s_1.

In other words, the trajectory of b before the first collision will be confined to the original square s_0, and the trajectory after the first collision will be confined to the new reflected square s_1. We can continue the process for each new collision. Suppose the ball collides with edge e_1 in square s_1. Then, we shall create a new square s_2 which is a reflection of square s_1 across the edge e_1. The trajectory of the ball b after the second collision will be confined to the newest reflected square s_2. This process will continue on indefinitely.

3. Simple Properties

4. 1-Dimensional Representation

Rather than looking at an explicit representation of lines in the plane, we can gain much more insight from looking at a parametric representation. To simplify our analysis, we will choose our time parameter such that v collisions occur every $\Delta t = 1$ and h collisions occur every $\Delta t = \frac{1}{m}$. The equation for a line $y(x) = m\,x + b$ is equivalent to the following parametric system

$$x(t) = t + x_0 \tag{3}$$

$$y(t) = m\,t \tag{4}$$

Now our v and h collisions in the 2-dimensional plane can be projected onto the 1-dimensional parametric representation.

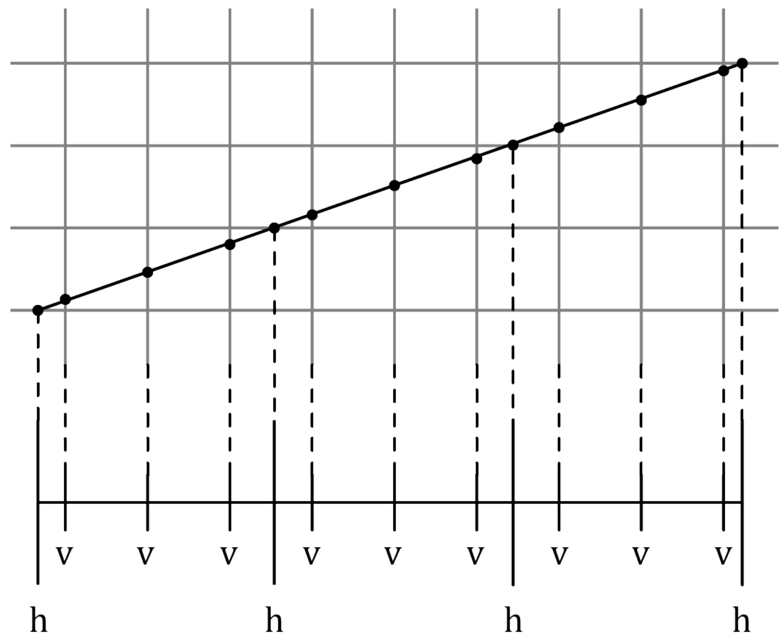

FIGURE 3. Projecting onto the parametric representation.

We will occasionally have to deal with some boundary conditions, and we will introduce some intermediate sequences which we will label as 'augmented' and mark with a tilde. The boundary treatments are fairly pedantic and can be ignored if the reader is just interested in getting a general understanding of our solutions.

All of our theorems in this section will rely on the lengths of various patterns in the original collision sequence. We will start off by looking at the lengths of subsequences of v collisions in the collision sequence. To make this accounting simpler, we need to define an augmented collision sequence.

Definition 4.1. ***augmented collision sequence*** **($\tilde{\alpha}$)** *which consists of the original collision sequence with one h collision added to both ends of the sequence.*

Now we can count lengths of v collision substrings

Definition 4.2. ***Augmented*** β_i***:*** *number of v collisions between* i^{th} *and (i+1)*th *h collisions in the augmented collision sequence*

Still working on this part, not sure how best to deal with the boundary conditions...

The first and last numbers in the β sequence were artificially created by augmenting our original collision sequence. These two numbers only give us a lower bound on the number of v collisions between h collisions, so we can safely discard them if

Definition 4.3.

$$\beta_{min} := \max_i \beta_i$$
$$\beta_{max} := \min_i \beta_i$$

The β sequence is much simpler to think of geometrically in terms of our parametric representation shown in Figure 3. β_i represents the number of v collision tick marks in between each h collision tick mark.

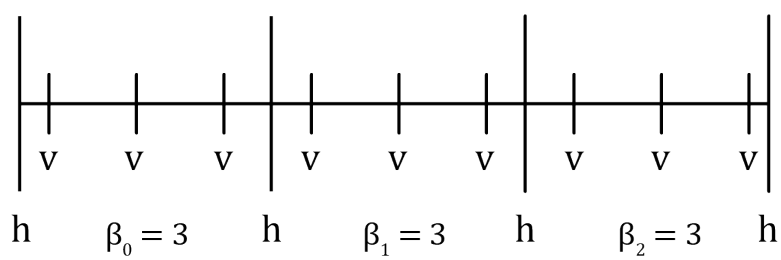

FIGURE 4. The β sequence.

Lemma 4.4.

$$\beta_{min} > 0$$

Proof. TODO □

Theorem 4.5. *For every valid collision sequence, the following must be true*

$$\beta_{max} - \beta_{min} \leq 1$$

Proof. From Equation 3, v collisions occur every $\Delta t = 1$ and h collisions occur every $\Delta t = \frac{1}{m}$. Thus, the following must be true

$$\beta_i \in \left(\left\lfloor \frac{1}{m} \right\rfloor, \left\lceil \frac{1}{m} \right\rceil \right)$$

For an m to exist that satisfies the above constraints, all numbers in the β sequence can only differ by 1.

$\square$

Definition 4.6. ***Augmented*** $C_i^{(0)}$***:*** *1 more than the number of occurrences of β_{max} between i^{th} and (i+1)th occurrence of β_{min} in the β sequence.*

Theorem 4.7. *Define*

$$\delta_i \coloneqq \begin{cases} x_0 & \textit{if} \quad i = 0 \\ i(\lceil \frac{1}{m} \rceil - \frac{1}{m}) & \textit{otherwise} \end{cases}$$

Then the following is true for all valid collision sequences

$$\beta_i = \lfloor \delta_i \rfloor + \beta_{max} - \lfloor \delta_{i+1} \rfloor$$

Proof. TODO... $\square$

5. Conclusion

TODO