Billiards

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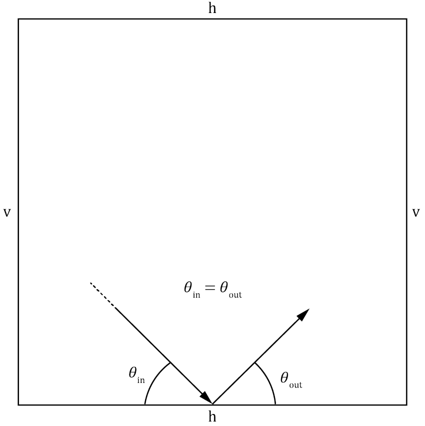
November 22nd, 2013

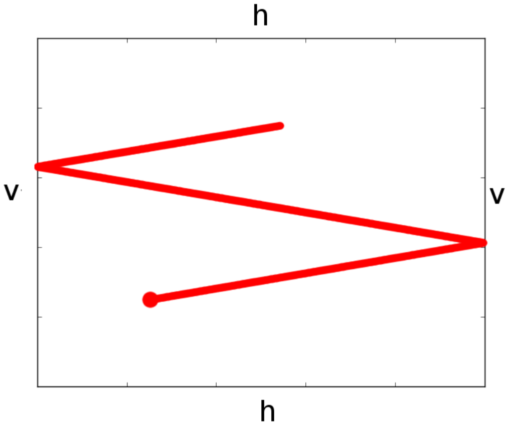
Introduction

Frictionless, massless, point-sized billiard ball bouncing in a square.



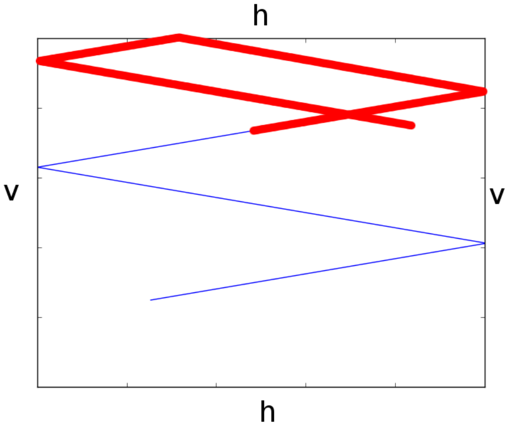
Example

vv



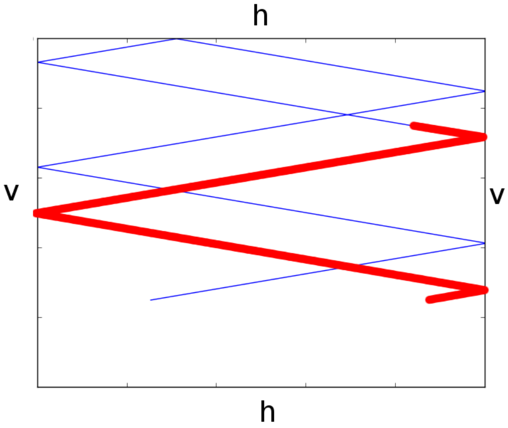
Example

vv *vhv*



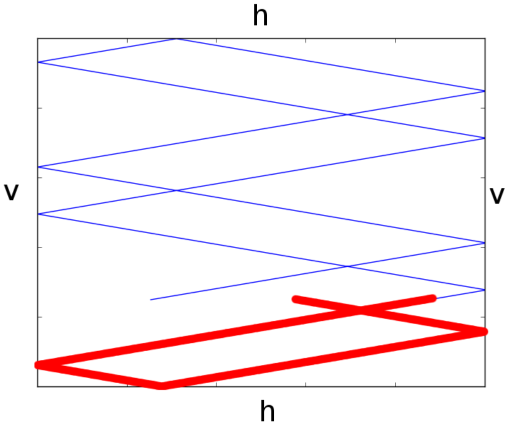
Example

vvvhvvvv



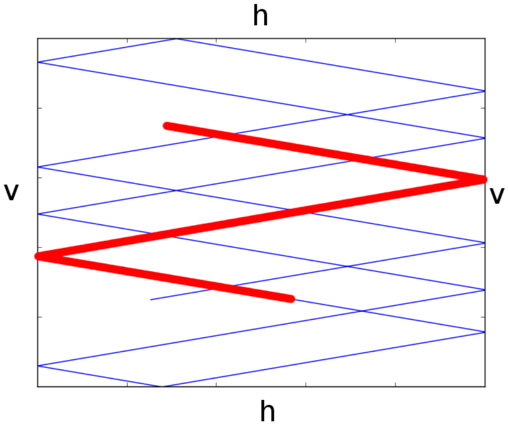
Example

vvvhvvvv *vhv*



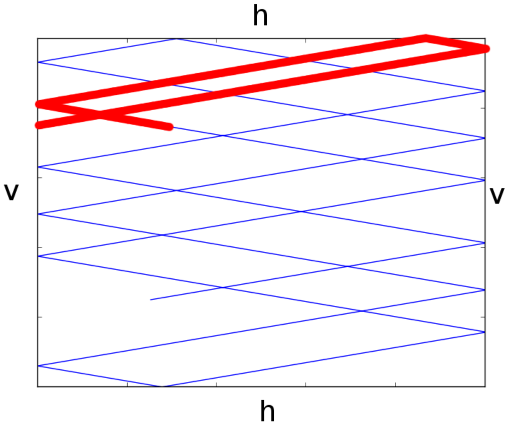
Example

vvvhvvvvvhvvv



Example

vvvhvvvvvhvvv *vhv*



Resulting Sequence

$vvvhvvvvvhvvvvhv$

Presentation Outline

Problem Statement

Problem: Given a sequence of v and h collisions, determine if it is a valid collision sequence.

Basic Notation

Definition

v collision: when the ball collides with a v side

Definition

h collision: when the ball collides with an h side

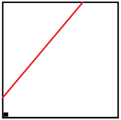
Definition

Collision sequence (α): a sequence of v and h collisions which starts and ends with an h collision.

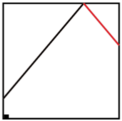
Tiling Representation

- Tile the table in the plane for a more powerful representation of the problem
- Tiling will reflect the table about each side
- After tiling, we only need to deal with straight line trajectories in a tiled plane

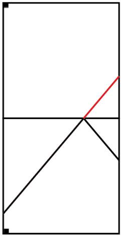
Tiling Tables



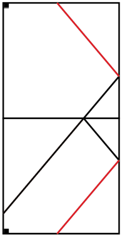
Tiling Tables



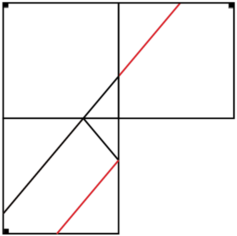
Tiling Tables



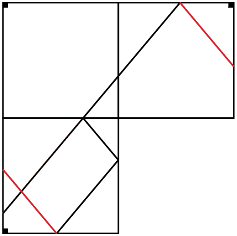
Tiling Tables



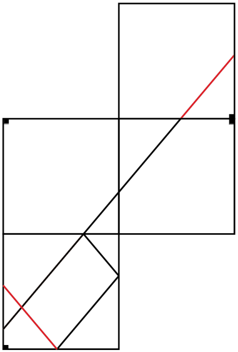
Tiling Tables



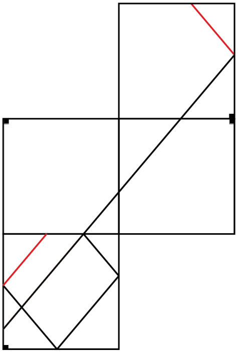
Tiling Tables



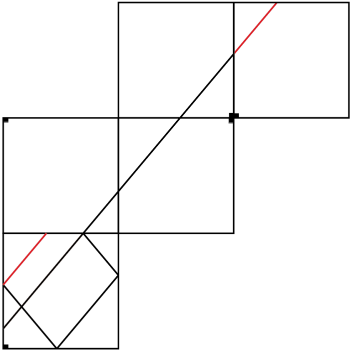
Tiling Tables



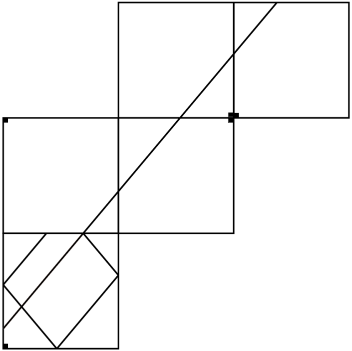
Tiling Tables



Tiling Tables



Tiling Tables



Indexing Definitions

$I(A, b)_k$: index of k$^{\text{th}}$ occurrence of b in A

β_i: # of v's between i$^{\text{th}}$ and (i+1)$^{\text{th}}$ h collisions: $[I(\alpha, h)]_{i+1} - [I(\alpha, h)]_i - 1$

Example

$$\alpha = hvvvvhvvvvhvvvh$$
$$I(\alpha, h) = (0, 5, 10, 14)$$
$$\beta = (4, 4, 3)$$

First Indexing Theorem

N_h: the number of h's in α: $length(I(\alpha, h))$

Theorem

Every valid collision sequence has the following property

$$\beta_i \geq 1 \qquad \forall\, i \in \{0, \ldots, N_h - 2\}$$

First Indexing Theorem

Example

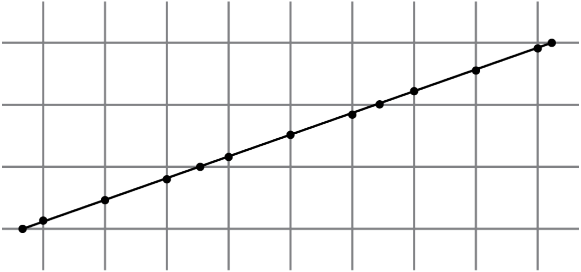
- $\alpha = hvvhhvvvhvvhhvhhvvvvhvvvh \qquad \beta = (2, 0, 3, 2, 0, 1, 2, 4, 3)$
- $\alpha = hvvvvhvvvvhvhvvvhvvvvvvvh \qquad \beta = (4, 4, 1, 3, 7)$
- $\alpha = hvvvvhvvvvhvvvhvvvvhvvvvh \qquad \beta = (4, 4, 3, 4, 4)$

First Indexing Theorem

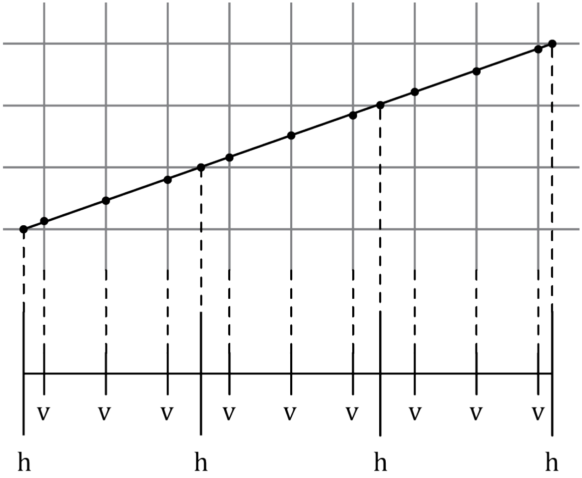
Example

- ~~$\alpha = hvvhhvvvhvvhhvhhvvvvhvvvh$ $\beta = (2, 0, 3, 2, 0, 1, 2, 4, 3)$~~
- $\alpha = hvvvvhvvvvhvhvvvhvvvvvvvh$ $\beta = (4, 4, 1, 3, 7)$
- $\alpha = hvvvvhvvvvhvvvhvvvvhvvvvh$ $\beta = (4, 4, 3, 4, 4)$

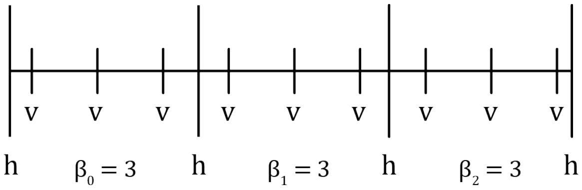
Parametric Representation



Parametric Representation



Parametric Representation



Second Indexing Theorem

β_{max}: maximum β value in α: $\max_i \beta_i$

β_{min}: minimum β value in α: $\min_i \beta_i$

Theorem

Every valid collision sequence has the following property

$$\beta_{max} - \beta_{min} \leq 1$$

Second Indexing Theorem

Example

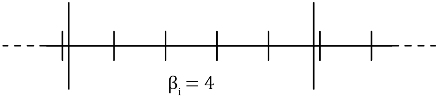
- $\alpha = hvvhhvvvhvvhhvhhvvvvhvvvh$ $\beta = (2, 0, 3, 2, 0, 1, 2, 4, 3)$
- $\alpha = hvvvvhvvvvhvhvvvhvvvvvvvh$ $\beta = (4, 4, 1, 3, 7)$
- $\alpha = hvvvvhvvvvhvvvhvvvvhvvvvh$ $\beta = (4, 4, 3, 4, 4)$

Second Indexing Theorem

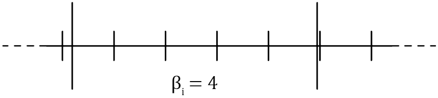
Example

- $\cancel{\alpha = hvvhhvvvhvvhhvhhvvvvhvvvh \qquad \beta = (2,0,3,2,0,1,2,4,3)}$
- $\cancel{\alpha = hvvvvhvvvvhvhvvvhvvvvvvvh \qquad \beta = (4,4,1,3,7)}$
- $\alpha = hvvvvhvvvvhvvvhvvvvhvvvvh \qquad \beta = (4,4,3,4,4)$

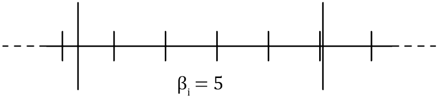
Windowing



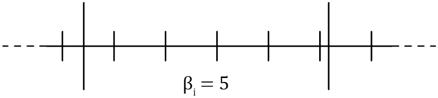
Windowing



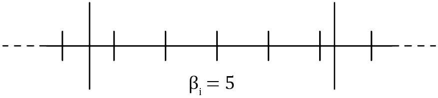
Windowing



Windowing



Windowing



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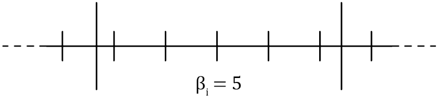

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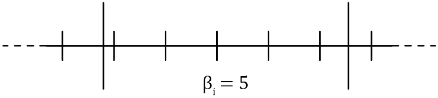

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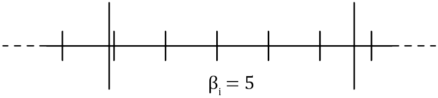

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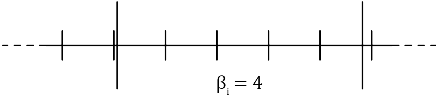
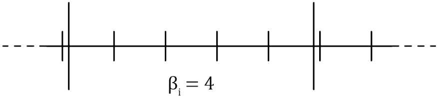


Windowing



More Sub-Sequences

$$C_i^{(0)} := [I(\beta, \beta_{min})]_{i+1} - [I(\beta, \beta_{min})]_i \qquad \forall\ i \in \{0, \ldots, length(I(\beta, \beta_{min}) - 2)\}$$

More Sub-Sequences

N_j: $length(I(C^{(j-1)}, C^{(j-1)}_{min}))$

$$C_i^{(j)} := [I(C_i^{(j-1)}, C_{min}^{(j-1)})]_{i+1} - [I(C_i^{(j-1)}, C_{min}^{(j-1)})]_i$$
$$\forall\ i \in \{0, \ldots, N_j - 2\},\ \forall\ j \in \{0, N_C - 1\}$$

where N_C is defined s.t.

$$\begin{cases} C^{(j)} \neq (1,) & \text{for } j < N_C - 1 \\ C^{(j)} = (1,) & \text{for } j = N_C - 1 \end{cases}$$

Satisfiability Test (Fractal Version)

Definition

$$C_{max}^{(j)} := \max_i C_i^{(j)} \tag{1}$$

$$C_{min}^{(j)} := \min_i C_i^{(j)} \tag{2}$$

Theorem

A collision sequence is valid iff the following is true:

$$\beta_{max} - \beta_{min} \leq 1 \text{ and } C_{max}^{(j)} - C_{min}^{(j)} \leq 1 \qquad \forall\ j \in \{0, \ldots, N_C - 1\}$$

Extensions to Cubes

- Assign x, y, and z as the opposite pairs of faces of a cube.
- Characterize sequences of x, y, and z collisions.

Intuition

- Examine collisions in xy, yz, and xz planes.
- Movement in each plane is independent.
- Combine xy, yz, and xz collision sequences to get final sequence.

Example

Example

xy sequence: $xxyyxx$

yz sequence: $zzyzyz$

xz sequence: $xxzzzx$