SEQUENCES OF BILLIARD BALL COLLISIONS

JONATHAN ALLEN, JOHN WANG

Abstract. In this paper we explore properties of sequences of billiard ball collisions. We present a tiling representation which is used to help elucidate some simple properties of these sequences. Then, we provide a one-dimensional representation which builds upon the tiling representation. Next, we provide a method which takes an arbitrary sequence and checks to see if it could have been created by a billiard ball colliding with a billiard table. Finally, we show how these sequences relate to continued fractions.

1. Introduction

In this paper we will explore sequences of billiard ball collisions. In particular, we look at the sequence of sides that a billiard ball collides with under perfect, frictionless conditions. We will show how a square billiard table can be analyzed by tiling the plane with the table, and prove a number of properties that billiard ball sequences must satisfy.

This introduction will explain the general setup of the problem and will give an example of how the definitions relate to a billiard ball with particular initial conditions.

1.1. Setup.

We will imagine an infinitesimally small billiard ball on a square table. For simplicity, we will assume that the square table is defined on the unit square $[0,1]^2$. The ball will start at some initial position $\mathbf{x}_0$ inside of the table and with some velocity $\mathbf{v}_0$. We will assume that the ball is massless and frictionless, and that there is no gravity.

We will assume ideal, elastic collisions. To be more precise, when the ball collides with an edge of the table, the ball's velocity will be reflected across the line perpendicular to the edge of the table at the point of collision. In other words, the angle of incidence is equal to the angle of reflection on all billiard ball collisions. Figure 1 shows the general mechanics of a collision.

Date: December 8, 2013.

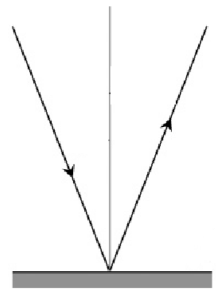

FIGURE 1. Mechanics of a Billiard Ball Colliding with a Table Edge

Now, we will label the horizontal sides of the table h and the vertical sides of the table v. Whenever the billiard ball collides with a horizontal side (labelled h), we will call the resulting collision an h-collision. Whenever the ball collides with a vertical side (labelled v), we will call the resulting collision a v-collision.

We will now define what this paper will be primarily interested in, a sequence of collisions:

Definition 1.1. *A* collision sequence *(α) for a ball B is the sequence of v's and h's which appear as the ball collides with the walls of the billiard table.*

Notice that all non-trivial initial conditions for a billiard ball will result in infinitely many h-collisions. The only initial conditions for which this is not true are when the initial velocity is parallel to the horizontal (for example when $\mathbf{v}_0 = (1, 0)$) so that the ball bounces infinitely between the two vertical sides. The proof of this is trivial and should become clear once the tiling representation is presented, so we will omit it.

Thus, it is perfectly reasonable to constrain a ball's collision sequence to begin and end with an h collision, since one simply needs to extend the number of collisions one watches until the sequence of collisions begins and ends with an h-collision. This constraint will later make it easier to reason about properties of sequences.

1.2. **Example Collision Sequence.** To understand collision sequences better, we will provide an example. Consider a billiard ball with intial position $\mathbf{x}_0 = (0.75, 0.75)$ and initial velocity $\mathbf{v}_0 = (0.23, 0.05)$.

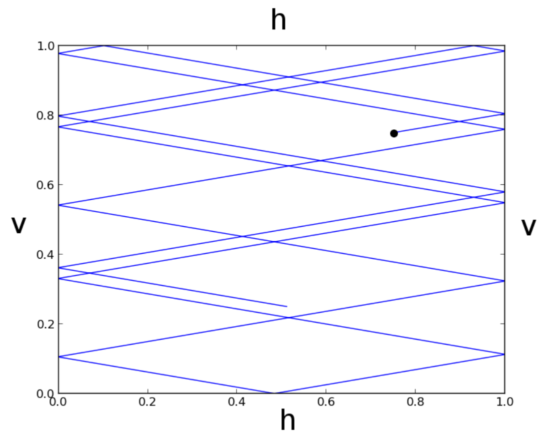

FIGURE 2. Example Billiard Ball Trajectory, $x_0 = (0.75, 0.75)$, $v_0 = (0.23, 0.05)$

We can see from the figure that the collisions that the ball makes, denoting a v-collision with a v and an h-collision with an h, are:

$$(v, h, v, v, v, v, v, v, h, v, v, v, v, v, h, v, v, v) \tag{1}$$

A valid collision sequence given these initial conditions would start and end with an h collision. A possible example would be:

$$\alpha = hvvvvvhvvvvvh \tag{2}$$

Notice that the collisions will continue infinitely, so one could imagine extending this collision sequence to include more v and h collisions. Although not shown in the picture, the following collision sequence would also be valid if one continued showing the trajectory of the ball from figure 2 in future collisions:

$$\alpha = hvvvvvhvvvvvhvvvvvh \tag{3}$$

1.3. **Simplifications.** There are some simplifications that we can make which will make it easier to talk about billiard balls, their initial conditions, and their collision sequences.

- We can change the magnitude of the initial velocity $\mathbf{v}_0$ without changing any collision sequences. This is because only the direction of $\mathbf{v}_0$ affects the points of collision. Thus, the only part of the initial velocity that we care about is the velocity's direction so we can characterize $\mathbf{v}_0$ by the angle γ that $\mathbf{v}_0$ makes with the positive horizontal line (just like in polar coordinates). From now on, we will use $\mathbf{v}_0$ and γ interchangeably.

- We can constrain the initial velocity's angle to the range $\gamma \in [0, \pi/2]$. This is because any intial velocity with an angle in the range $\gamma \in [\pi/2, 3\pi/2]$ will have equivalent collision sequences as $\pi + \gamma \in [-\pi/2, \pi/2]$ (this will be clear once we present the tiling representation later in the paper). Moreover, any angle in the range $\gamma \in [0, \pi/2]$ will create equivalent collision sequences as those in $[-\pi/2, 0]$ by simply reflecting the square about one of its horizontal sides. Thus, for the rest of the paper, we will assume $\gamma \in [0, \pi/2]$.

2. Tiling Representation

We will now present the tiling representation of the problem which will greatly simplify the analysis of v and h collisions for some ball B. First we will define some basic definitions.

Definition 2.1. *A billiard ball* trajectory $\tau(t_0, t_1)$ *is the curved traced by a billiard ball B between times t_0 and t_1.*

Definition 2.2. *A collision time κ_i for a billiard ball B is the time at which the ith collision occurs.*

To understand the basics of how the tiling representation works, imagine placing a unit square billiard table on the xy plane. The table's edges will be the four line segments bordering the unit square. A ball will start with some initial position $\mathbf{x}_0 \in [0, 1]^2$ and velocity $\mathbf{v}_0$. After some time, the ball will collide with an edge e_0 of the table at time κ_1. However, instead of thinking of the trajectory of the ball as being reflected across the line perpendicular to e_0 at the point of collision, we will instead reflect the original unit square s_0 across the edge e_0 to create a new square s_1. Now, the trajectory $\tau(\kappa_1, \kappa_2)$ of the ball after the first collision will be traced in the new square s_1.

In other words, the trajectory $\tau(0, \kappa_1)$ before the first collision will be confined to the original square s_0, and the trajectory$\tau(\kappa_1, \kappa_2)$ after the first collision will be confined to the new reflected square s_1. We can continue the process for each new collision. Suppose the ball collides with edge e_1 in square s_1. Then, we will create a new square s_2 which is a reflection of square s_1 across the edge e_1. The trajectory of the ball $\tau(\kappa_2, \kappa_3)$ after the second collision will be confined to the newest reflected square s_2. This process will continue on indefinitely so that the trajectory $\tau(\kappa_j, \kappa_{j+1})$ will be confined to the square s_j, where square

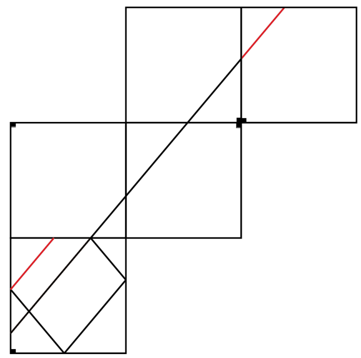

FIGURE 3. Example of Tiling Billiard Tables

s_{j+1} is generated by reflecting square s_j across the edge e_j which is collided with at time κ_j, as seen in figure 4.

In essence, the tiling representation reflects a table about each of its four sides. These reflections will perform the same process, eventually tiling and completely filling the xy plane. The combined trajectory of a particular billiard ball can then be traced through the tiling in the xy-plane, as seen in figure 3.

A couple of interesting observations can be made:

- The combined trajectory $T = \{\tau(0, \kappa_1), \tau(\kappa_1, \kappa_2), \tau(\kappa_2, \kappa_3), \ldots\}$ of a ball creates a ray in the xy plane.
- All v-collisions happen exactly when the combined trajectory T intersects with the integer vertical lines $x = k$ where $k \in \mathbb{Z}$. The same can be said of h-collisions and integer horizontal lines $y = k$ for $k \in \mathbb{Z}$.

We can formalize and prove each of these observations in turn:

Theorem 2.3. *The combined trajectory $T = \{\tau(0, \kappa_1), \tau(\kappa_1, \kappa_2), \ldots\}$ of a billiard ball is a ray in the xy plane under the tiling representation.*

Proof. We need to show that all trajectories $\tau(0, \kappa_1), \tau(\kappa_1, \kappa_2), \ldots$ which constitute the combined trajectory lie on a single line. We know that trajectories $\tau(\kappa_i, \kappa_{i+1})$ are line segments because the velocity of the billiard ball only changes during a collision. Thus, we just need to

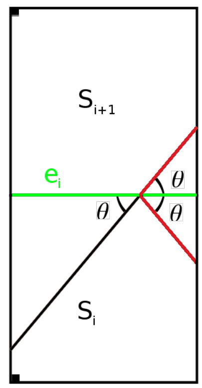

FIGURE 4. Trajectories $\tau(\kappa_{i-1}, \kappa_i)$ in s_i and $\tau(\kappa_i, \kappa_{i+1}$ s_{i+1} forming a line segment

show that each trajectory $\tau(\kappa_i, \kappa_{i+1})$ lies on the same line, i.e. that $\tau(\kappa_i, \kappa_{i+2})$ is a line segment for all $i \in \mathrm{Z}$.

We will show this by analyzing the change in trajectory at time κ_i. We know that at any time $t \in (\kappa_{i-1}, \kappa_i)$, the billiard ball will be on the line segment defined by the trajectory $\tau(\kappa_{i-1}, \kappa_i)$. This trajectory makes an angle θ with the edge e_i which the billiard ball collides with at time κ_i. Moreover, we know that after colliding with e_i, the angle of reflection of the trajectory is equal to the angle of incidence θ by our definition of collision.

The velocity is reflected about the line perpendicular to e_i at the point of collision, but the angle that the trajectory $\tau(\kappa_i, \kappa_{i+1})$ makes with e_i remains the same. Thus, when square s_i is reflected about e_i, the resulting trajectory in s_{i+1} makes an angle of θ with e_i. Figure 4 shows this process graphically.

It is clear that the angles that $\tau(\kappa_{i-1}, \kappa_i)$ and $\tau(\kappa_i, \kappa_{i+1})$ make with e_i are both θ. Since both of these trajectories are already line segments, we see that the union of the two trajectories is also a line segment. Thus, we see that all trajectories $\tau(\kappa_j, \kappa_{j+1})$ for all $j \in \mathrm{Z}$ lie on the same line, which completes the proof. □

Theorem 2.4. *Let $t_{v,k}$ be a time when the combined trajectory T for some ball B intersects with some integer vertical line $x = k$ where $k \in \mathrm{Z}$. We must have $\kappa_k^v = t_{v,k}$, where κ_k^v is the time at which the kth v-collision occurs.*

Similarly, let $t_{h,k}$ be a time when the combined trajectory T intersects with some horizontal vertical line $y = k$ for $k \in \mathbb{Z}$. Then, $\kappa_k^h = t_{h,k}$.

Proof. The proof of the first statement is exactly analogous to the proof of the second statement, so we will only prove the theorem for $\kappa_k^v = t_v$.

Recall that in the tiling representation, whenever the billiard ball collides with an edge e_j, a new square s_{j+1} gets reflected across e_j. It is clear that e_j, if it is a horizontal edge, lies on some integer horizontal line $y = c$ where $c \in \mathbb{Z}$. Alternatively, if e_j is a vertical edge then it lies on some integer vertical line $x = c$ where $c \in \mathbb{Z}$. Thus each κ_j corresponds to when the combined trajectory T intersects with some integer vertical or integer horizontal line.

It is also clear that the billiard ball cannot make a v-collision at time t unless the combined trajectory T intersects with an integer vertical line at time t as well. Therefore, we see that v-collisions occur exactly when T intersects with an integer vertical line.

We know that T traces a ray in the xy plane by theorem 2.3. By assumption, this ray starts in the unit square $[0, 1]^2$ and has an angle between 0 and $\pi/2$ with the horizontal. Thus, the first v-collision occurs when T intersects $x = 1$, the second v-collision occurs when T intersects $x = 2$, and the kth v-collision occurs when T intersects $x = k$.

However, we know that the kth v-collision occurs at time κ_k^v by definition, and that the $t_{v,k}$ is the time when T intersects with $x = k$. Therefore, we see that $\kappa_k^v = t_{v,k}$. □

We now see that we can represent the combined trajectory T of a billiard ball as a ray in the plane. We can define the line which the ray lies on as $y = mx + y_0$, where m is the slope of the line and is given by $m = \frac{\mathbf{v}_y}{\mathbf{v}_x}$, where $\mathbf{v}_y$ and $\mathbf{v}_x$ are the y and x components of $\mathbf{v}_0$ respectively. Also, y_0 is the y-intercept which can be determined using $\mathbf{x}_0$ by solving for $\mathbf{x}_y = m\mathbf{x}_x + y_0$, where $\mathbf{x}_y$ and $\mathbf{x}_x$ are the y and x components of $\mathbf{x}_0$ respectively. The resulting line $y = mx + y_0$ represents the entire combined trajectory, and also determines all possible collision sequences for a particular billiard ball, since v-collisions happen when x is an integer and h-collisions happen when y is an integer.

Thus, we see that the ith v-collision occurs when $y = mi + y_0$ and the ith h-collision occurs when $x = (i - y_0)/m$.

3. Simple Properties

We will now use the tiling representation that we have developed to discover properties of collision sequences. The first simple property is that collision sequences are periodic when the initial conditions are rational numbers. Now that we can define a billiard ball trajectory as a line, instead of giving initial conditions $\mathbf{x}_0$ and $\mathbf{v}_0$, we can give the slope m and the y-intercept y_0 of the complete trajectory's line.

Theorem 3.1 will show that if $m \in \mathrm{Q}$, then the billiard ball will eventually return to it's original position $\mathbf{x}_0$ with its original velocity $\mathbf{v}_0$. Seeing why this is true is just a matter of using the tiling representation. We note that every second square in either the x or y direction is the same (because of the transitivity of reflection). Therefore, every second square will have a trajectory that exactly corresponds to the trajectory in the original square. Thus, when analyzing trajectories that are periodic, we must work modulo 2, since it takes two reflections in order to "straighten out" the table's reflection and get the original table's orientation back.

Thus, if $y_0 \equiv mk + y_0 \pmod 2$, then the y-intercept from one of the secondary squares is the same as the y-intercept from the original square. Since we know that the secondary squares have the same trajectories as in the original square, we see that the trajectory will have returned to its original position (since the velocity is the same). Thus, if $y_0 \equiv mk + y_0 \pmod 2$, then we know that the billiard ball will return to its original position with its original velocity. Theorem 3.1 also shows that if m is irrational, then the billiard ball will never return to its original position and velocity.

The formalized theorem is:

Theorem 3.1. *There exists a* $k \in \mathrm{N}$ *such that* $y_0 \equiv mk + y_0 \pmod 2$ *if and only if* $m \in \mathrm{Q}$.

Proof. Let us first show that if $m \in \mathrm{Q}$, then there exists a $k \in \mathrm{N}$ such that $y_0 \equiv mk + y_0 \pmod 2$. We simply need to show that $0 \equiv mk \pmod 2$ if $m \in \mathrm{Q}$. However, since we know $m \in \mathrm{Q}$, we can decompose it into $m = p/q$ where $p, q \in \mathrm{Z}$. Thus, we have $mk \pmod 2 \equiv \frac{pk}{q} \pmod 2$. Now we can choose:

$$k = \begin{cases} q & \text{if } p \pmod 2 \equiv 0 \\ 2q & \text{if } p \pmod 2 \equiv 1 \end{cases} \tag{4}$$

In this way, we see that $0 \equiv \frac{pk}{q} \pmod 2$, which proves the first half of the theorem.

Now let us show that if there exists a in $k \in \mathrm{N}$ such that $y_0 \equiv mk+y_0 \pmod 2$, then $m \in \mathrm{Q}$. If such a k exists, then we must have $0 \equiv mk \pmod 2$, which means that $mk = 2q$ for some $q \in \mathrm{Z}$. This means $m = \frac{2q}{k}$. Now it is clear that $m \in \mathrm{Q}$ because both its numerator and denominator are integers. □

We can also examine consecutive occurences of v and h. For example, can we have consecutive occurences of both v and h like in the sequence $hhvvhh$? In fact, we cannot as theorem 3.2 shows.

Theorem 3.2. *A valid collision sequence cannot have both consecutive occurrences of v and consecutive occurences of h.*

Proof. We have already shown in theorem 2.3 that the combined trajectory of a billiard ball must be a ray. This ray must lie on some line with some slope $m \in \mathrm{R}$ or with undefined slope (when the line is vertical). When the line is vertical, it is clear that the theorem holds, because only v collisions occur.

There are two cases left: $|m| < 1$ or $|m| \geq 1$. If $|m| \geq 1$, then the number of v-collisions between the xth and the $x + 1$st h-collision will be $\lfloor (m(x+1)+y_0) - (mx+y_0) \rfloor = \lfloor m \rfloor \geq 1$. Thus, we see that for any $x \in \mathrm{N}$, we must have at least 1 v-collision between the xth and $x + 1$st h-collisions, which proves the theorem for $|m| \geq 1$.

If $|m| < 1$, then the number of h-collisions between the yth and $y+1$st v-collisions will be $\lfloor ((y+1)-y_0)/m - (y-y_0)/m \rfloor = \lfloor 1/m \rfloor > 1$. This shows that there will be at least 1 h-collision between the yth and $y + 1$st v-collisions for all $y \in \mathrm{N}$. This completes the theorem. □

We therefore see that if v occurs consecutively in a collision sequence, then h cannot occur consecutively and vice versa. For example, the sequence $hvvhh$ has two consecutive occurrences of v and two consecutive occurrences of h, so it cannot be a valid collision sequence. However, the sequence $hvvhvvvh$ does not have consecutive occurrences of h, so theorem 3.2 does not rule it out as a valid collision sequence.

Theorem 3.2 allows us to make a simplification for our collision sequences. Since one of either v or h must occur non-consecutively, we can abitrarily assign v to be the side that occurs non-consecutively by rotating the billiard table and creating the opposite collision sequence. For example, the sequence $vhhhvhhhv$ is the same as the sequence $hvvvhvvvh$ when one rotates the billiard table by $\pi/2$. Therefore, from now on, we can confine all our sequences to have only non-consecutive occurrences of v (i.e. sequences like $hvvvhvvvh$ become $vhhhvhhhv$) without loss of generality.

4. 1-Dimensional Representation

Rather than looking at the explicit representation equation of lines in the plane, we can gain much more insight from looking at the parametric representation. To simplify our analysis, we will choose our time parameter such that v collisions occur every $\Delta t = 1$ and h collisions occur every $\Delta t = m$. The equation for a line $y(x) = m\,x + y_0$ is equivalent to the following parametric system

$$x(t) = \frac{1}{m}\,t \tag{5}$$

$$y(t) = t + y_0 \tag{6}$$

Now v and h collisions in the 2-dimensional plane can be projected onto the 1-dimensional t axis as is shown in Figure 5.

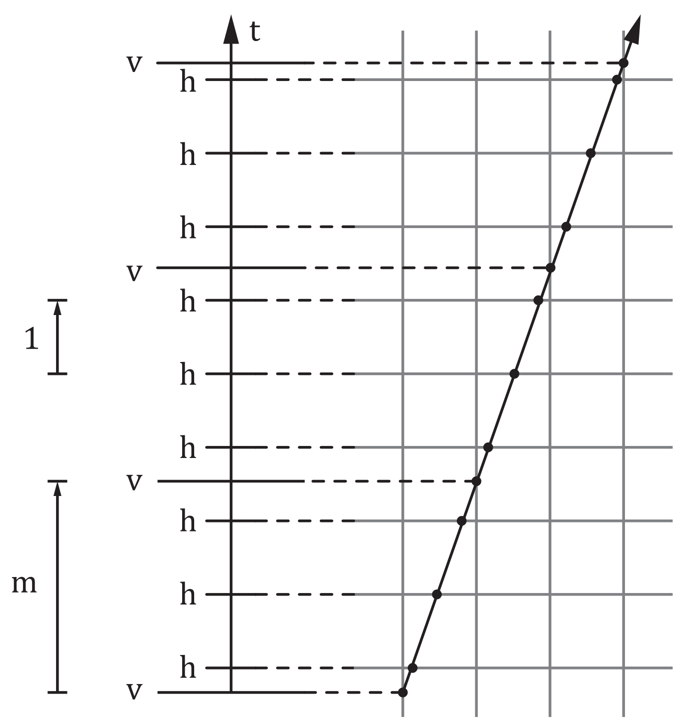

FIGURE 5. Projecting onto the t axis.

For the sake of space, we will rotate the t axis so that it is horizontal, as is shown in Figure 6. The problem of mapping collision sequences to lines in the plane becomes a problem of fitting regularly spaced tick marks into intervals on the real number line.

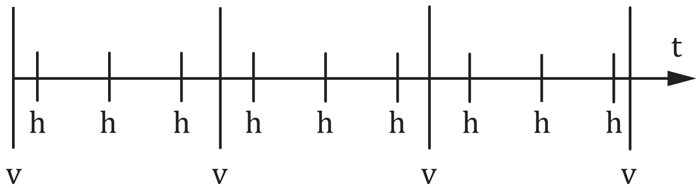

FIGURE 6. The 1-dimensional problem.

To start off analyzing this 1-dimensional problem, we need a basic lemma about counting ticks marks inside an interval.

Lemma 4.1. *Let $l_2 < l_1$. The number of real numbers at regular spacing l_2 inside an open interval of length l_1 is in* $\left\{ \left\lfloor \frac{l_1}{l_2} \right\rfloor, \left\lceil \frac{l_1}{l_2} \right\rceil \right\}$.

Proof. Let $A \subset \mathbb{Z}$ represent the set of all possible numbers of real numbers at regular spacing l_2 inside an open interval of length l_1. Given some $a \in A$, let $b \in \mathbb{R}$ be defined such that the following holds

$$l_1 = (a-1)l_2 + b \tag{7}$$

We know that $(a-1)l_2 < l_1$, so $b \geq 0$. Also, $b < 2\,l_2$, because otherwise, there must be more than a real numbers inside the interval, contradicting our original assumption.

Rearranging Equation 7, we get

$$a = \frac{l_1}{l_2} + 1 - \frac{b}{l_2} \tag{8}$$

$$\frac{l_1}{l_2} - 1 < a < \frac{l_1}{l_2} + 1 \tag{9}$$

$$a \in \left\{ \left\lfloor \frac{l_1}{l_2} \right\rfloor, \left\lceil \frac{l_1}{l_2} \right\rceil \right\} \tag{10}$$

□

5. Some Useful Sequences

To analyze the 1-dimensional problem, we will derive two groups of sequences from the original collision sequence, which we will call the β and δ groups. These group will allow us to validate any collision sequence. The first sequence in the β group is defined below.

Definition 5.1. *Given a collision sequence α, define a sequence $\beta^{(0)}$ such that each element $\beta_i^{(0)}$ is the number of h's between the i^{th} and $(i+1)^{th}$ v in α. From Lemma 4.1, each element in $\beta^{(0)}$ can be one of two different values, which we will refer to as $\beta_{min}^{(0)}, \beta_{max}^{(0)}$.*

Graphically, $\beta_i^{(0)}$ represents the number of tick marks in each interval. An example of this is shown in Figure 7.

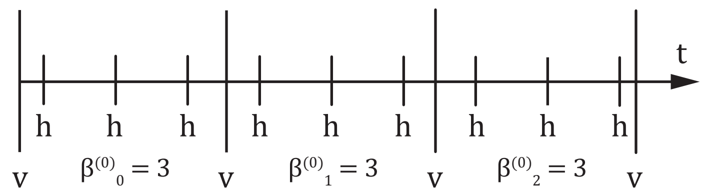

FIGURE 7. An example $\beta^{(0)}$ sequence.

The first sequence in the δ group, $\delta^{(0)}$, is defined such that each element is spaced $\left\{\frac{m}{1}\right\}$ apart. $\delta^{(0)}$ will also include an offset of $-(1-y_0)$ for reasons that will become clear shortly. In terms of our β group, $\left\{\frac{m}{1}\right\}$ can be written as

$$\left\{\frac{m}{1}\right\} = \frac{m}{1} - \beta_{min}^{(0)} \tag{11}$$

Definition 5.2. *Given a $\beta^{(0)}$ sequence, $\delta^{(0)}$ is defined as follows*

$$\delta_i^{(0)} := \begin{cases} -(1-y_0) & \text{for} \quad i = 0 \\ i(m - \beta_{min}^{(0)}) - (1-y_0) & \text{for} \quad i \geq 1 \end{cases} \tag{12}$$

Visually, each element $\delta_i^{(0)}$ is the distance between the beginning of the i^{th} interval and the $(i\,\beta_{min}^{(0)})^{th}$ tick mark (with positive distance measured right to left). Figure 8 shows the $\delta_i^{(0)}$ sequence on top of the original parametric representation (top of the Figure) as well as by itself (bottom of the Figure).

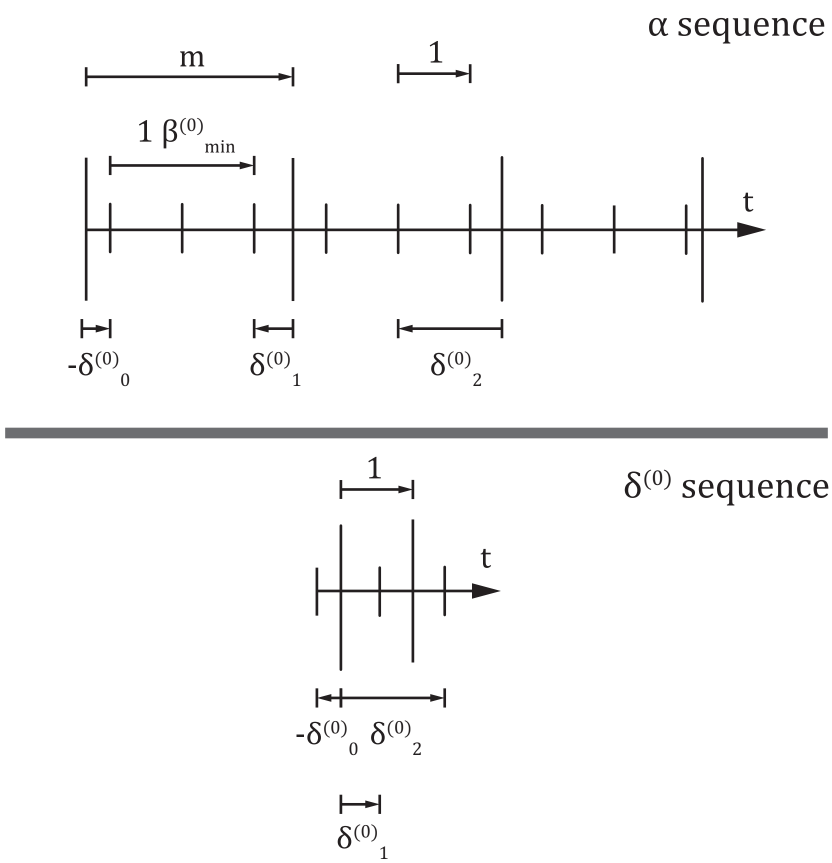

FIGURE 8. Generating the $\delta^{(0)}$ sequence.

Our next step is to define $\beta^{(j)}, \delta^{(j)}$ for $j \geq 1$, which will be done in an inductive manner. Before continuing, let's create a new sequence that will be helpful for defining the rest of the β and δ groups.

Definition 5.3. *Assume that the β group is fully defined, then a is defined as follows*

$$a_j := \begin{cases} m & \textit{for} \quad i = -2 \\ 1 & \textit{for} \quad i = -1 \\ a_{j-2} - \beta^{(j)}_{min} a_{j-1} & \textit{for} \quad i \geq 0 \end{cases} \tag{13}$$

a_{-2}, a_{-1} are the interval size and tick spacing in our original problem. a_0 is the spacing of elements of $\delta^{(0)}$, which is illustrated in Figure 9.

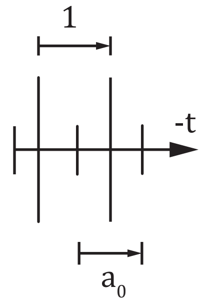

FIGURE 9. a_0.

$\delta^{(j)}$ will be defined such that the spacing of elements of the sequences will be a_j for $j \geq 1$. For the moment, the reader should assume that, given some j, then $\delta^{(j-1)}$ exists and its elements are strictly increasing and evenly spaced. An example $\delta^{(j-1)}$ sequence is shown in Figure 10.

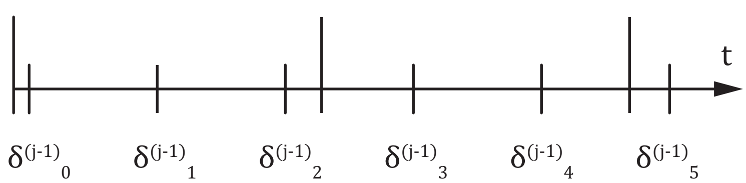

FIGURE 10. An example $\delta^{(j-1)}$ sequence.

Definition 5.4. *Given a collision sequence α, for some $j > 0$ assume that $\beta^{(j-1)}$ is defined and each element in the sequence is either $\beta^{(j-1)}_{min}$ or $\beta^{(j-1)}_{max}$. The sequence $\beta^{(j)}$ is defined such that each element $\beta^{(j)}_i$ is 1 more than the number of occurrences of $\beta^{(j-1)}_{min}$ between the i^{th} and $(i+1)^{th}$ occurrence of $\beta^{(j-1)}_{max}$ in the $\beta^{(j-1)}$ sequence. From Lemma 4.1, each element in $\beta^{(j)}$ can be one of two different values, which we will refer to as $\beta^{(j)}_{min}, \beta^{(j)}_{max}$.*

If, for some j_f, the length of $\beta^{(j_f-1)}$ is 1, then $\beta^{(j_f-1)}$ is called the terminating β sequence, and all subsequent $\beta^{(j)}$ for $j \geq j_f$ are undefined.

$\beta^{(j)}$ is much simpler to understand visually. Let's return to the example $\delta^{(j-1)}$ sequence, shown in Figure 10. We need to understand how the $\delta^{(j-1)}$ sequence is related to the $\beta^{(j-1)}$ sequence.

Referring back to Figure 8, it can be seen that $\beta_i^{(j-1)}$ is equal to $\beta_{min}^{(j-1)}$ plus the number of tick marks included in the $\delta_{i+1}^{(j-1)}$ interval minus the number of tick marks included in the $\delta_i^{(j-1)}$ interval. More precisely

$$\beta_i^{(j-1)} = \left\lfloor \delta_{i+1}^{(j-1)} \right\rfloor + \beta_{min}^{(j)} - \left\lfloor \delta_i^{(j-1)} \right\rfloor \tag{14}$$

Figure 11 plots the $\beta^{(j-1)}$ sequence in place of the $\delta^{(j)}$ sequence. Reffering back to the definition of $\beta^{(j)}$, it can be seen that $\beta^{(j)}$ counts the number of tick marks in each interval on the $\delta^{(j-1)}$ plot.

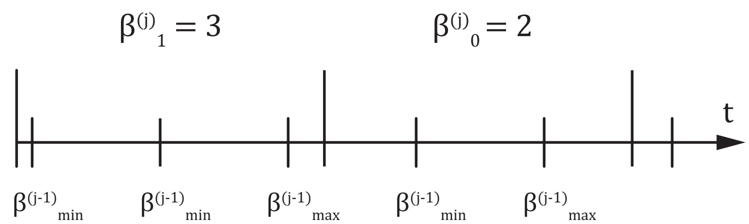

FIGURE 11. A general $\beta^{(j)}$ sequence.

As was promised earlier, we will finish defining the δ group. In words, each $\delta_i^{(j)}$ is the distance between the beginning of the i^{th} interval and the $(i\,\beta_{min}^{(j)})^{th}$ tick mark in the $\beta^{(j)}$ visual representation.

Definition 5.5. *$\delta^{(j)}$ is defined more precisely as*

$$\delta_i^{(j)} := \begin{cases} 1(1-y_0) & \textit{for} \quad i = 0 \\ i(a_{j-2} - \beta_{min}^{(j)} * a_{j-1}) - (1-y_0) & \textit{for} \quad i \geq 1 \end{cases} \tag{15}$$

6. Satisfiability Conditions

In this paper, we will only present a satisfiability condition for collision sequences, where α starts and ends with v, and all $\beta^{(j)}$ start and end with $\beta_{max}^{(j)}$. This is only a minor restriction, and applying our results to the general case would only require some special treatment for the ends of the collision sequence.

Theorem 6.1. *A collision sequence is valid if the following is true for all j*

$$\beta_i^{(j)} \in \left\{ \left\lfloor \frac{a_{j-2}}{a_{j-1}} \right\rfloor, \left\lceil \frac{a_{j-2}}{a_{j-1}} \right\rceil \right\} \tag{16}$$

Proof. This follows directly from Lemma 4.1 and the definition of $\beta^{(j)}$. □

Theorem 6.2. *For every valid collision sequence,* $\lim_{n\to\infty} a_n = 0$

Proof. From the definition of a_j

$$a_j = \beta^{(j-2)}_{max}\, a_{j-1} - a_{j-2} \tag{17}$$

$$= \left\lceil \frac{a_{j-2}}{a_{j-1}} \right\rceil a_{j-1} - a_{j-2} \tag{18}$$

Now from the definition of the ceiling function, we know that

$$0 \leq \left\lceil \frac{a_{j-2}}{a_{j-1}} \right\rceil - \frac{a_{j-2}}{a_{j-1}} < 1 \tag{19}$$

Rearranging, we get

$$0 \leq \left\lceil \frac{a_{j-2}}{a_{j-1}} \right\rceil a_{j-1} - a_{j-2} < a_{j-1} \tag{20}$$

Combining the above equation with Equation 17, we get

$$0 \leq a_j < a_{j-1} \tag{21}$$

Thus a is strictly decreasing and bounded below by 0, so $\lim_{n\to\infty} a_n = 0$. □

7. Continued Fractions

Finally, we will show how collision sequences relate to continued fractions. A simple continued fraction of a real number r (hereafter referred to as a continued fraction) is the expression given below in equation 22:

$$r = k_1 + \frac{1}{k_2 + \frac{1}{k_3 + \cdots}} \tag{22}$$

Where $k_1 = \lfloor r \rfloor, r_1 = \frac{1}{r-k_1}, k_2 = \lfloor r_1 \rfloor, r_2 = \frac{1}{r_1 - k_2}, \ldots$. This process terminates when r_i becomes an integer. The integers k_i are called "partial quotients" and a real number r can be expressed in its continued fraction form $r = [k_1, k_2, k_3, \ldots]$. As an example, we can find the continued fraction for $r = 3.245$.

Example 7.1.

$$3.245 = 3 + \cfrac{1}{4 + \cfrac{1}{12 + \frac{1}{4}}} \tag{23}$$

In this example, we see that $k_1 = \lfloor r \rfloor = \lfloor 3.245 \rfloor = 3$ *by the definition. Thus, we can define* $r_1 = \frac{1}{3.245-3} = \frac{1}{0.245} = \frac{200}{49}$ *and find* $k_2 = \lfloor r_1 \rfloor = \lfloor \frac{200}{49} \rfloor = 4$. *Now we can recursively define* $r_2 = \frac{1}{\frac{200}{49}-4} = \frac{49}{4}$. *As we continue onwards in this process, we find* $k_3 = 12$, $r_3 = 4$, *and* $k_4 = 4$. *Thus, the continued fraction representation is* $3.245 = [3, 4, 12, 4]$.

If we use the continued fraction representation for the slope m of a combined trajectory for a billiard ball, then we have:

$$m = \lfloor m \rfloor + \cfrac{1}{\lfloor 1/\{m\} \rfloor + \cfrac{1}{\lfloor 1/\{1/\{m\}\} \rfloor + \cdots}} \tag{24}$$

Where the partial quotients are given by:

$$\begin{aligned}
k_1 &= \lfloor m \rfloor \\
k_2 &= \left\lfloor \frac{1}{\{m\}} \right\rfloor \\
k_3 &= \left\lfloor \frac{1}{\left\{ \frac{1}{\{m\}} \right\}} \right\rfloor \\
k_4 &= \left\lfloor \frac{1}{\left\{ \frac{1}{\left\{ \frac{1}{\{m\}} \right\}} \right\}} \right\rfloor \\
&\vdots
\end{aligned}$$

We see that the partial quotients are given by the recursive formula: $k_j = \left\lfloor \frac{1}{m - [k_1, k_2, \ldots, k_{j-1}]} \right\rfloor$. In fact, a more interesting observation is that the sequence of partial quotients k_j form exactly the sequence of the minimum number of tick marks in each $\beta^{(j)}$ subproblem:

Theorem 7.2. *Given the continued fraction representation* $m = [k_1, k_2, k_3, \ldots]$ *for the slope* $m \in \mathrm{R}$ *of the combined trajectory* T *of a billiard ball, we must have* $k_j = \beta^{(j)}_{min}$.

Proof. We will proceed by induction. This is clearly true for $k_1 = \lfloor m \rfloor$ by lemma 4.1, since for $\beta^{(j)}_{min}$ the spacing of the larger tick marks are m and the spacing of the smaller tick marks is 1.

Now suppose we have shown our hypothesis to be true for all $k_1, k_2, \ldots, k_{j-1}$. This means that $\beta^{(i)}_{min} = k_i$ for all $i \leq j-1$. By definition 5.4 (of the sequence $\beta^{(j)}$), we see that $\beta^{(j)}_i$ is one more than the number of occurrences of $\beta^{(j-1)}_{max}$ between the ith and $(i+1)$st occurrence of $\beta^{(j-1)}_{min}$ in the $\beta^{(j-1)}$ sequence.

Now, we can find an expression for k_j based on our definition of continued fractions:

$$k_j = \left\lfloor \frac{1}{m - [k_1, k_2, \ldots, k_{j-1}]} \right\rfloor \tag{25}$$

$$= \left\lfloor \frac{1}{m - [\beta^{(1)}_{min}, \beta^{(2)}_{min}, \ldots, \beta^{(j-1)}_{min}]} \right\rfloor \tag{26}$$

However, notice that $d = m - [\beta^{(1)}_{min}, \ldots, \beta^{(j-1)}_{min}]$ is equal to the spacing between tick marks in the $\beta^{(j)}$ sequence. Thus, we see that $\beta^{(j)}_{min} = \lfloor 1/d \rfloor$. However, this is exactly what we wanted to show, since we know that $k_j = \lfloor 1 \rfloor / d \rfloor$, so we see that $\beta^{(j)}_{min} = k_j$. □

In fact, one can see that the process of finding $\beta^{(j)}_i$ for the $\beta^{(j)}$ subproblems exactly mirrors the process of finding the partial quotients k_j in a continued fraction of m.

8. Conclusion

TODO