Billiards

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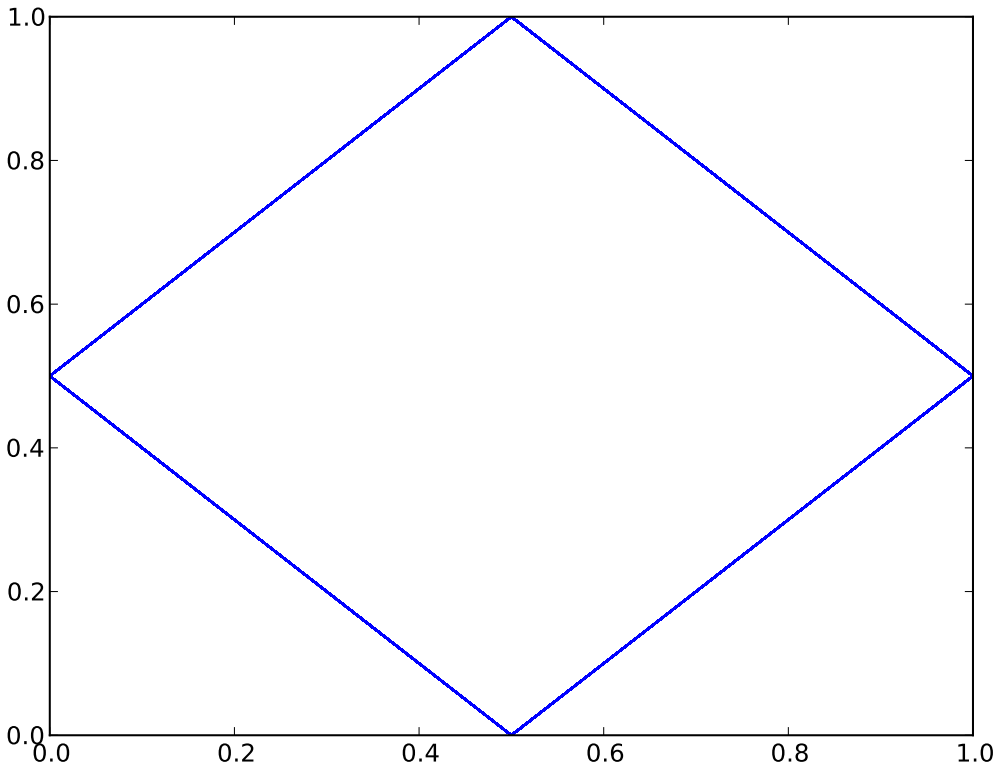
Introduction

- Billiard ball bouncing in a square
- Assume no gravity or friction
- Examine sequence of side collisions

Example

Example

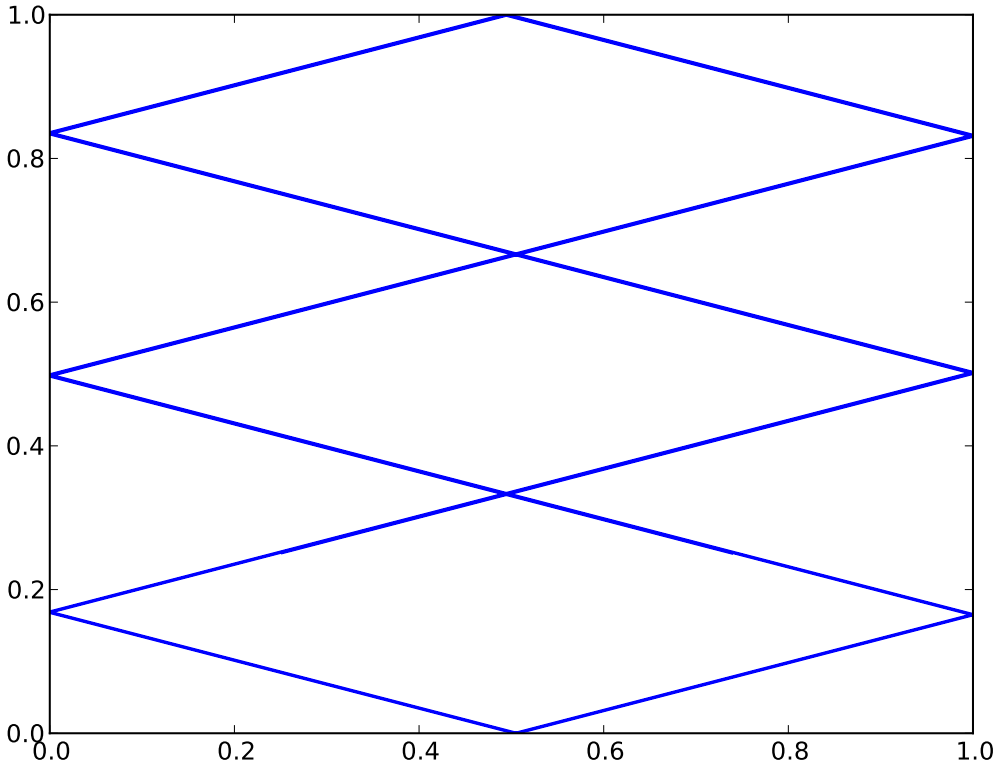
Examine the sequence: ‘abab‘



Another Example

Example

Examine the sequence: ‘aaabaaab‘



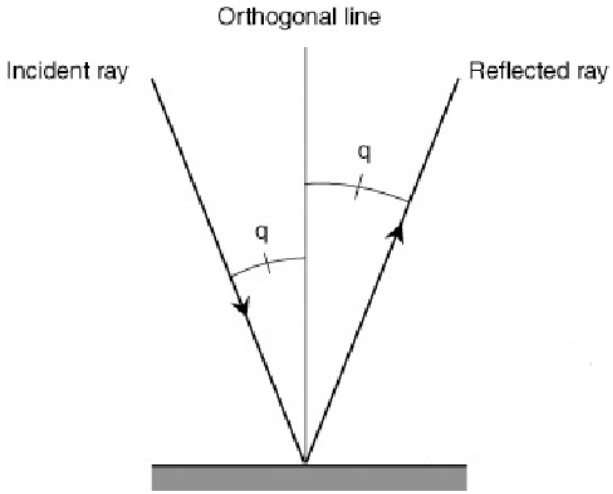
Presentation Outline

1. Introduction
   - Examples
   - Outline
   - Notation and Problem Statement
2. Lemmas
   - Tiling
   - 1-dimensional Problem
3. Future Research
   - Tileable Polygons
   - Non-Tileable Polygons
   - Circles

Notation

Definition

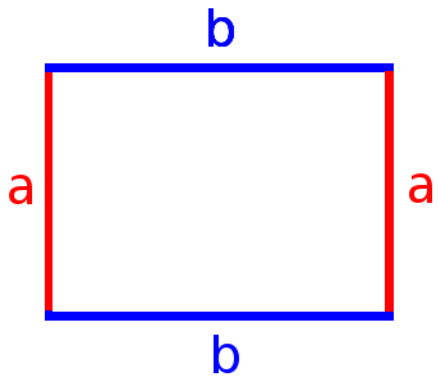
A table $T \in \mathbb{R}^2$ is the unit square. A ball $p \in T$ begins at position $\vec{r}_0 \in T$ with initial velocity $\vec{v}_0 \neq 0$. When the ball reaches an edge of the table, it reflects such that the component of its velocity normal to the edge is negated after the collision, and its component tangent to the edge is unchanged.



Notation

Definition

Opposite sides of the table are labeled a and b.



Notation

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Collision string: list of the sides of the table that a ball collides with, ordered by collision time. e.g. 'abaaabaaaab'.
Primary side: side appearing most often in a collision string.
secondary side the other side.

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Example

Collision string: 'aabaaabaabaaab', **Primary substrings**: 'aa', 'aaa'

Problem Statement

Problem: Characterize the properties of collision sequences.

- Given a sequence of a's and b's, determine if it is a valid collision sequence.
- Given a valid collision sequence, determine a possible starting position and velocity.

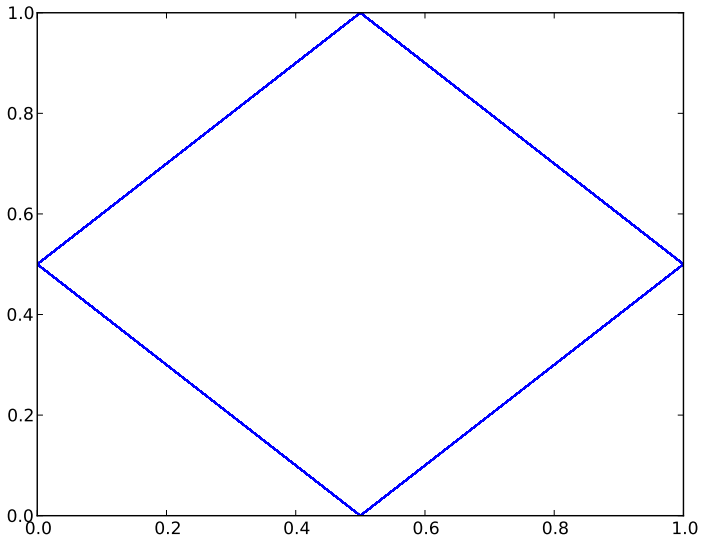
Representing Collision Strings

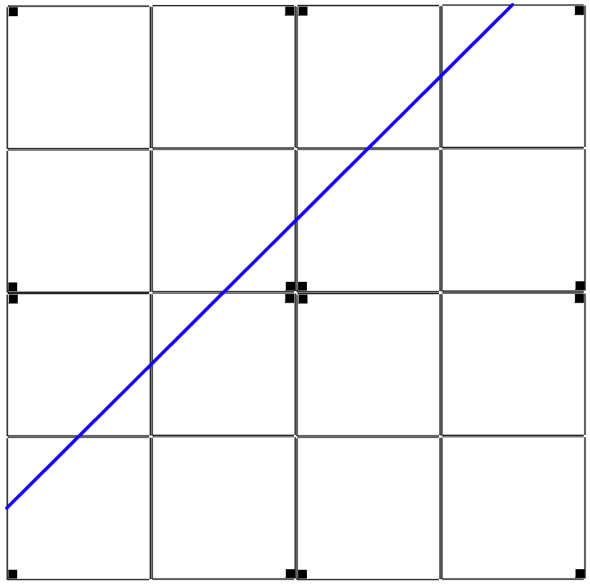
- Reflect squares about each side to create a tiling
- Solutions become lines in the plane
- Intersections become places where collisions occur

Representing Collision Strings

Example

Tiling of $\vec{x}_0 = (0, 0.5)$ and $\vec{v} = (0.25, 0.25)$.





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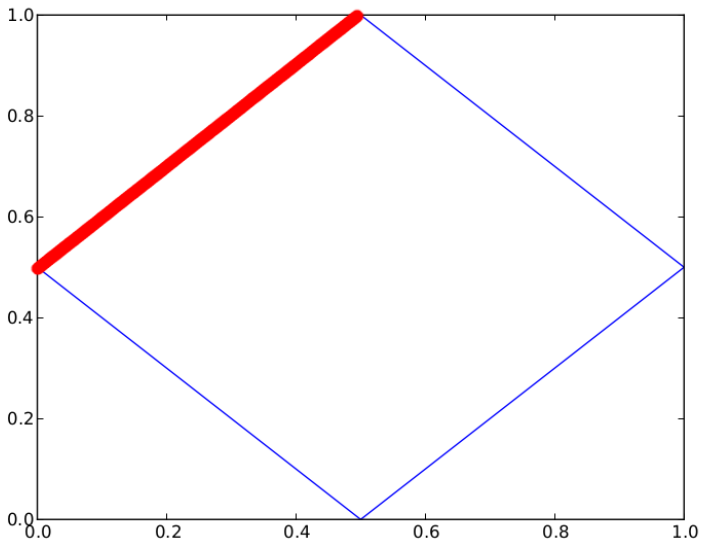

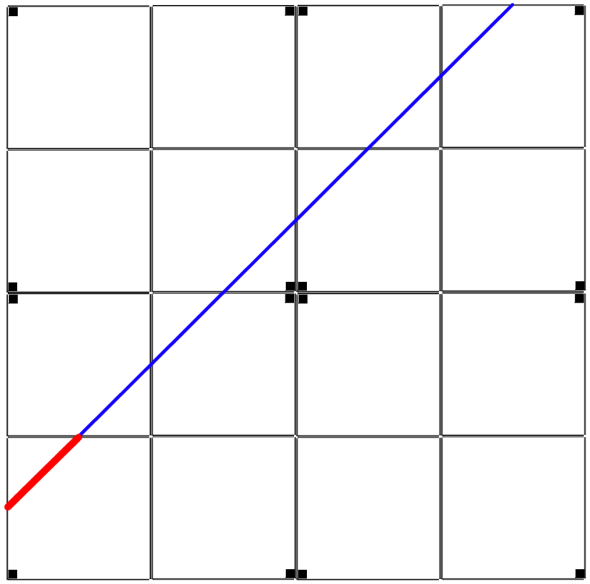

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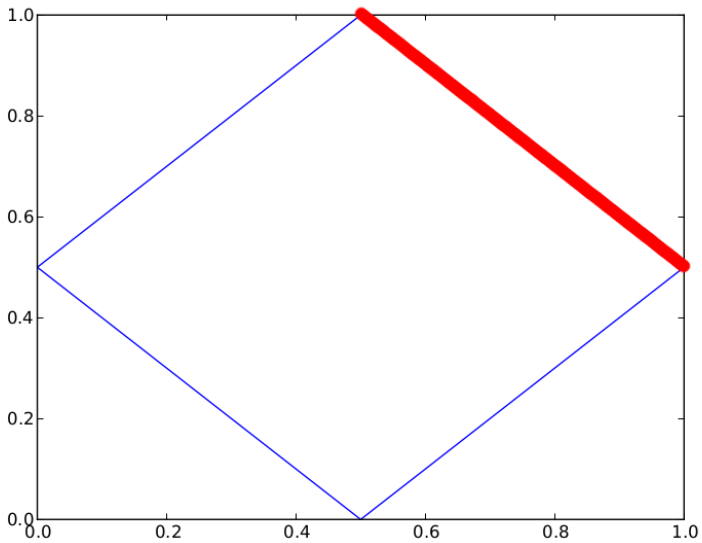

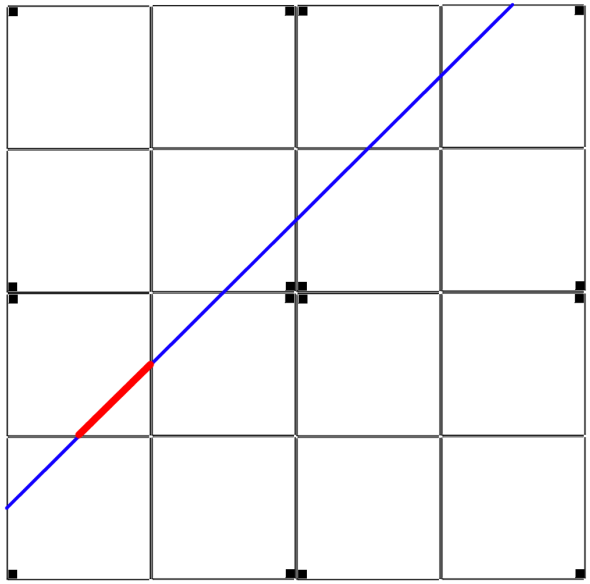

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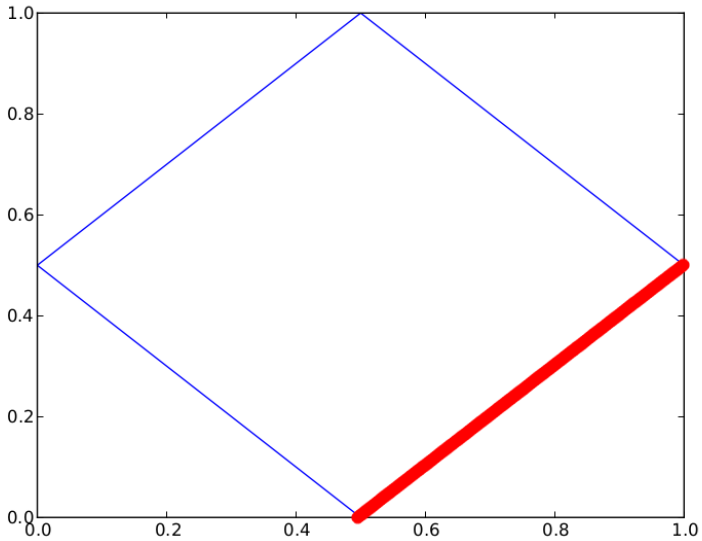

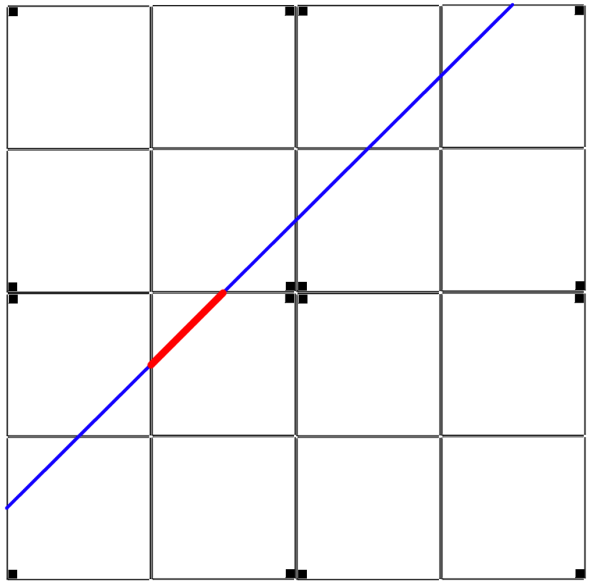

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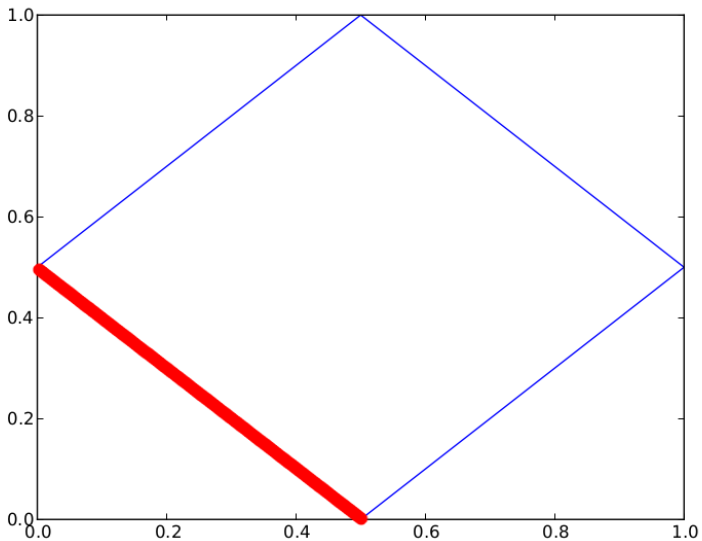

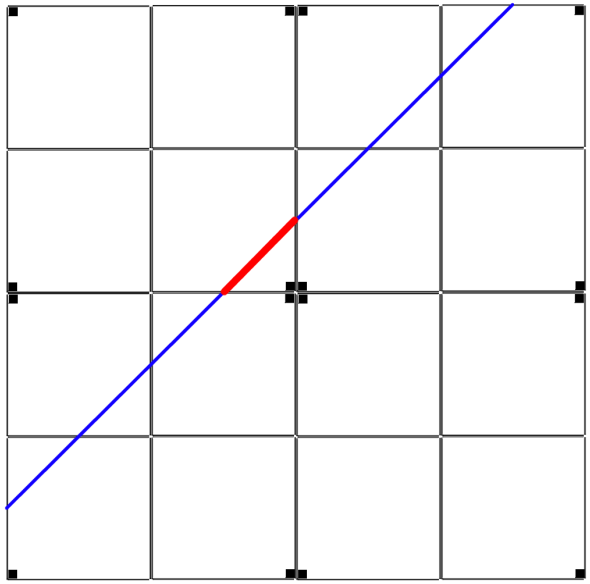

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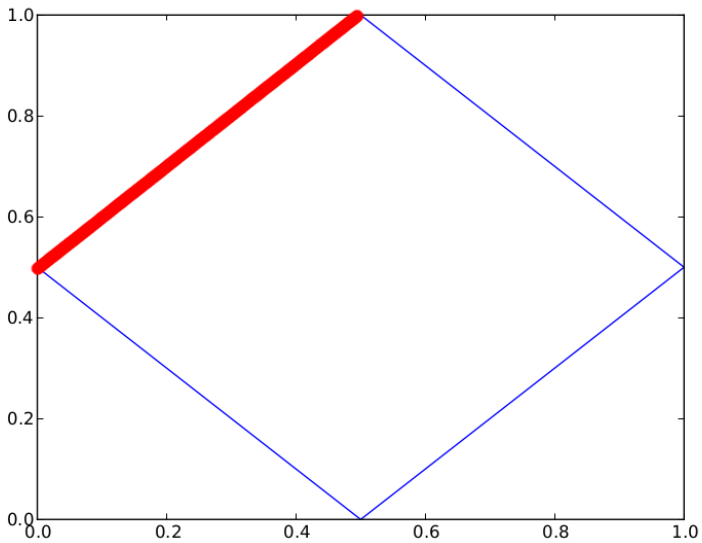

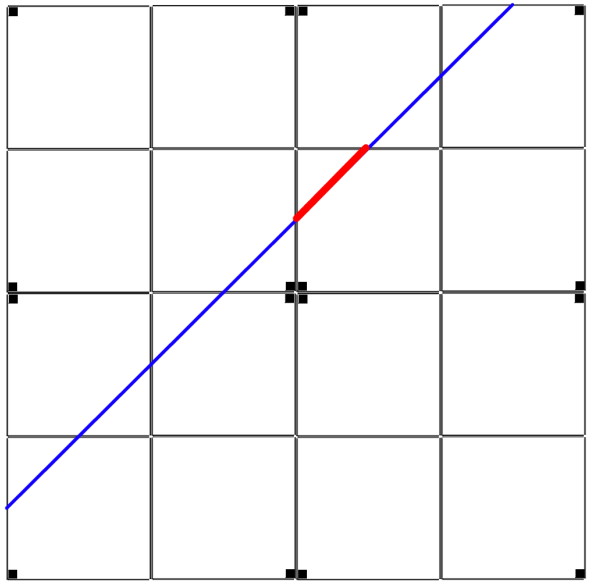

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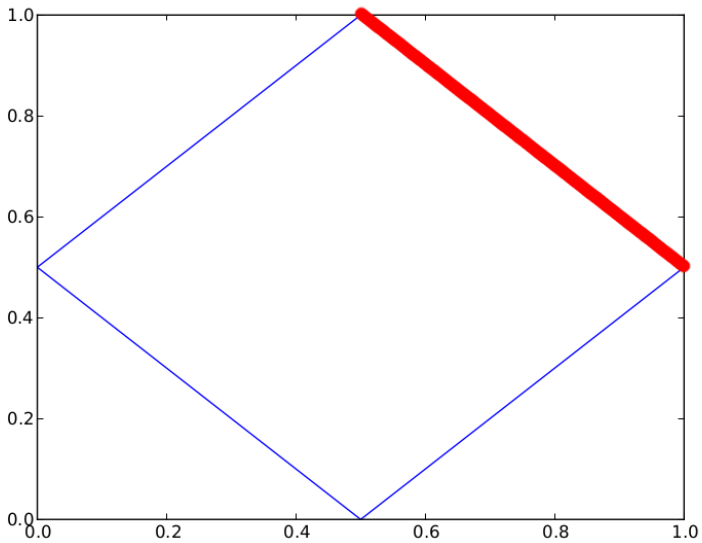

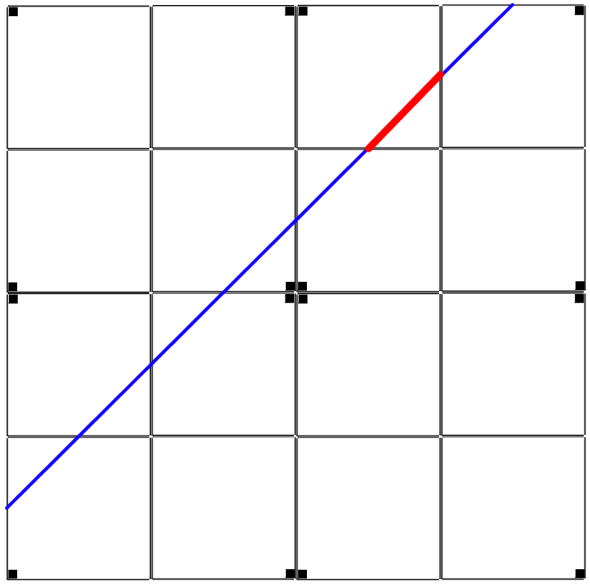

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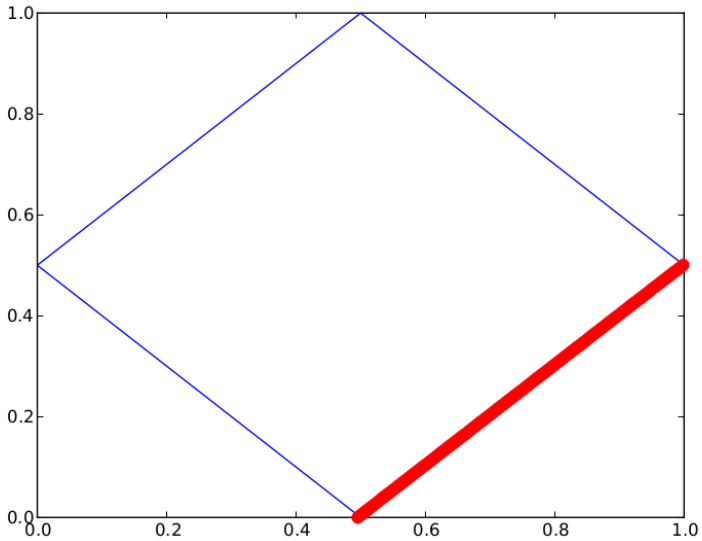

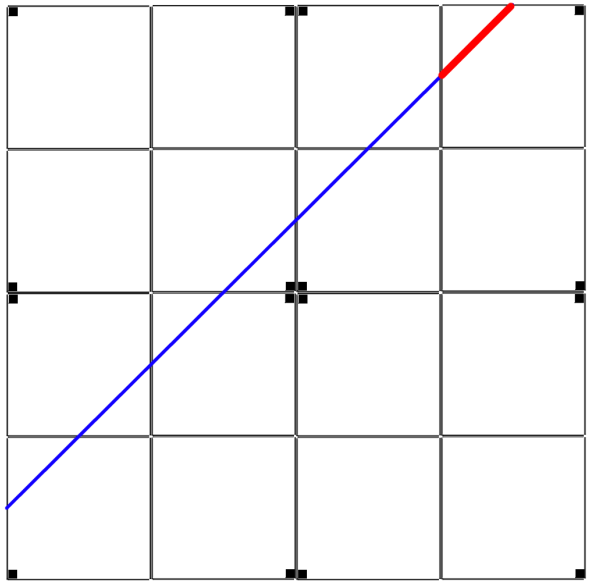

Sequence Characterization

$$\underbrace{aaa}_{3}\ b\ \underbrace{aa}_{2}\ b\ \underbrace{aa}_{2}\ b\ \underbrace{aaa}_{3}\ b\ \underbrace{aa}_{2}\ b$$

$$\underbrace{aa}_{2}\ b\ \underbrace{aaa}_{3}\ b\ \underbrace{aa}_{2}\ b\ \underbrace{aaa}_{3}\ \ldots$$

Sequence Characterization

$$3 \underbrace{22}_{2} 3 \underbrace{22}_{2} 3 \underbrace{2}_{1} 3 \ldots$$
$$\ldots$$

Extensions to Tileable Polygons

Other Tileable Polygons:

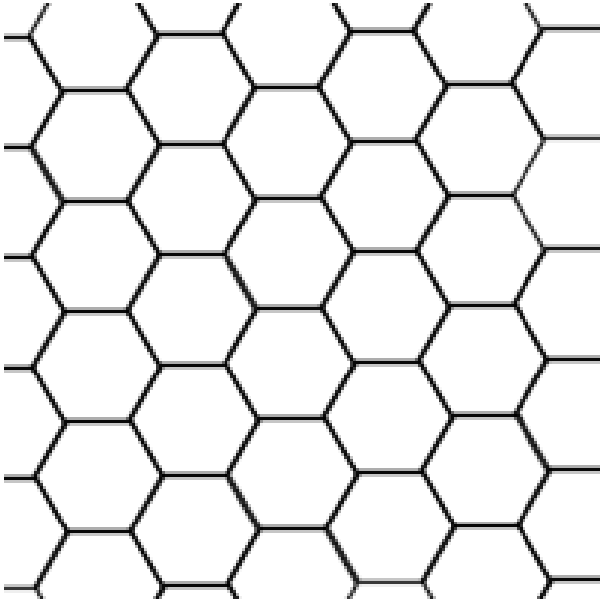

Figure : Regular Hexagons

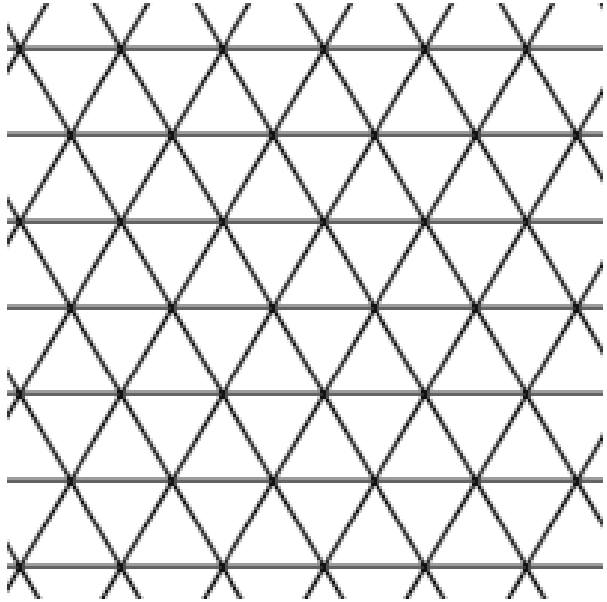

Figure : Equilateral Triangles

Extensions to Non-Tileable Polygons

- Irregular triangles
- Pentagons
- Octagons

Extensions to Circles

- Characterize how particle bounces around circle
- Analog to a, b might be sequence of collision points as you move around circle.