THE 18.821 MATHEMATICS PROJECT LAB REPORT [PROOFS]

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1. Theorems

1.1. Notation.

t: Time
l: Path length
s: Speed
a_t: Tangential acceleration
a_c: Centripetal acceleration
$\vec{x}$: Position
$\vec{v}$: Velocity
$\vec{a}$: Acceleration

2. Coordinates

2.1. Scalar Calculus.

$$s = \frac{dl}{dt} \tag{1}$$

$$a_t = \frac{ds}{dt} \tag{2}$$

(3)

2.2. Vector Calculus.

$$\boldsymbol{x} = r\hat{\boldsymbol{r}} \tag{4}$$

$$\vec{\boldsymbol{v}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}} \tag{5}$$

$$\vec{\boldsymbol{a}} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\boldsymbol{r}} + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right)\hat{\boldsymbol{\phi}} \tag{6}$$

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2.3. **Relations.**

$$l = \int_{t=0}^{T} \|\vec{\boldsymbol{v}}\| \, dt \tag{7}$$

$$s = \vec{\boldsymbol{v}} \tag{8}$$

$$a_t = \|\vec{\boldsymbol{a}}\| \frac{\vec{\boldsymbol{v}}}{\|\vec{\boldsymbol{v}}\|} \tag{9}$$

3. Traveling Between Points

Definition 3.1. *A path $\gamma : \mathrm{R} \to \mathrm{R}^2$ is a function which maps some time t to a position $\gamma(t) \in \mathrm{R}^2$. The path is defined from time $t = 0$ until the end time of the path, denoted as $T_f(\gamma)$.*

Definition 3.2. *A valid path $\gamma : \mathrm{R} \to \mathrm{R}^2$ for some point mass p and conditions C is some path which at all times t such that $0 \leq t \leq T_f(\gamma)$, all conditions in C are satisfied.*

Definition 3.3. *A valid targetted path $\gamma : \mathrm{R} \to \mathrm{R}^2$ for some point mass p, conditions C, starting point $\vec{x_1}$, and ending point $\vec{x_2}$ is a valid path where $\gamma(t) = \vec{x_1}$ and $\gamma(T_f(\gamma)) = \vec{x_2}$. In other words, it is a valid path which starts at $\vec{x_1}$ and ends at $\vec{x_2}$.*

Definition 3.4. *A fastest path $\hat{\gamma} : \mathrm{R} \to \mathrm{R}^2$ for a particular point mass p, a starting point $\vec{x_1}$, a destination point $\vec{x_2}$, and some set of conditions C is a valid targetted path $\hat{\gamma}$ such that $T_f(\hat{\gamma}) \leq T_f(\gamma)$ for all valid targetted paths γ with the same p, $\vec{x_1}$, $\vec{x_2}$, and C.*

Theorem 3.5. *Given points $(x_1, y_1), (x_2, y_2) \in \mathrm{R}^2$ and a point mass p whose initial position is (x_1, y_1) which moves with acceleration bounded by $\bar{a}$, the fastest path $\hat{\gamma}(t)$ which p can trace from (x_1, y_1) to (x_2, y_2) follows the straight line where all coordinates (x, y) on the straight line are given by:*

$$y = \frac{y_2 - y_1}{x_2 - x_1} x + y_1 \tag{10}$$

Proof. Let's transform the problem. We can reset our coordinate axes so that (x_1, y_1) is set to the origin and (x_2, y_2) is on the x-axis. In this new coordinate system, we have transformed the following:

$$(x_1, y_1) \;\to\; (0, 0) \tag{11}$$

$$(x_2, y_2) \;\to\; (x_2', 0) \tag{12}$$

For convenience of notation, we will now refer to x_2' as x_2.

Now let us examine the particle's motion in the x direction. Let $a_t(t)$ be the tangential acceleration at time t in the x direction. Then we can obtain the speed of the particle $s(t)$ at time t in the x direction like so:

$$s(t) = \int_0^t a_t(t_1)dt_1 \tag{13}$$

To find the distance $d(t)$ travelled up to time t in the x direction, we can use the relation:

$$\begin{aligned} d(t) &= \int_0^t s(t_2)dt_2 && (14) \\ &= \int_0^t \int_0^t a_t(t_1)dt_1dt_2 && (15) \end{aligned}$$

Recall that the acceleration of the point mass p is bounded by $\bar{a}$. This means that $a_t(t) \leq \bar{a}$ for all t. Therefore, we see:

$$\begin{aligned} d(t) &\leq \int_0^t \int_0^t \bar{a} dt_1dt_2 && (16) \\ &= \frac{\bar{a}t^2}{2} && (17) \end{aligned}$$

Thus, in order to travel a distance of $d(T_f) = x_2$, it needs to be the case that $T_f \geq \sqrt{\frac{2x_2}{\bar{a}}}$. Moreover, equality holds if and only if $a_t(t) = \bar{a}$ for all $t \in [0, T_f(\gamma)]$.

If the point mass travels for time $t < \sqrt{\frac{2x_2}{\bar{a}}}$, then it is impossible for the point mass to reach $(x_2, 0)$ when starting at $(0, 0)$. This is because p cannot reach $(x_2, 0)$ in the x direction when $t < \sqrt{\frac{2x_2}{\bar{a}}}$ and any acceleration in the y direction would not enable this either.

This means that the fastest path is completed in time $T_f(\hat{\gamma}) = \sqrt{\frac{2x_2}{\bar{a}}}$. Let us examine the path taken by the point mass p on this fastest path. Recall that $a_t(t) = \bar{a}$ for all t along the fastest path. This means that there was no centripetal acceleration $|a_c| = 0$. In other words, the point mass never turned on its way to reaching the destination point. The only way this could have happened is if it travelled along the x axis in a straight line.

Now, we have seen that the fastest path in the transformed coordinates travels exactly on the x axis so that $y = 0$ anywhere along the fastest path. Notice, however, that the x axis in the transformed coordinates is given exactly by the following line:

$$y = \frac{y_2 - y_1}{x_2 - x_1}x + y_1 \tag{18}$$

Thus, we see that the fastest path in the original coordinates follows the above equation, which is what we wanted to show. $\square$

Corollary 3.6. *The fastest path between two points* $(x_1, y_1), (x_2, y_2) \in \mathrm{R}^2$ *is unique.*

References

[1] http://en.wikipedia.org/wiki/Polar_coordinate_system