OPTIMAL PARTICLE PATHS AROUND POINTS IN $\mathbb{R}^2$ WITH CONSTRAINED ACCELERATION

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Abstract. Optimal particle paths in $\mathbb{R}^2$ between points and around points are investigated in this paper. A paramerization of a particle's motion around a point is defined. From this parameterization, we derive a general solution for an optimal path with constant speed between 2 points. The results of this paper can by applied towards calculating optimal object trajectories in physics when acceleration is constrained.

1. Introduction

In this paper, we shall examine optimal paths for a particle.

Our model problem is a particle moving in $\mathbb{R}^2$ with constrained acceleration. The particle must navigate around cones (points in $\mathbb{R}^2$) to reach a final position. The goal is to minimize the total time spend navigating to the final position.

We first present a polar coordinate representation of a particle's position around a point and derive a set of differential equations governing the motion of the particle in this coordinate system. This allows us to devise some simple lemmas about the motion of the particle.

After developing an intuition for particles and particle paths, we then move on to the more complicated problem of deriving governing rules for optimal paths between points.

Finally, we tackle the problem of finding an optimal path around cones.

1.1. **Motivation.**

The problem of finding an optimal trajectory is interesting because of its applications in physics. There are many instances where finding optimal paths is important. For instance, a race car driver wants to know the fastest way to get from one point to another so that he can win his race. For another example, imagine you are sending a spacecraft to a particular destination in space and would like the fastest means of getting there.

The space example directly motivates our model problem. On a spacecraft, there is a limited amount of acceleration that is possible

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(provided by thrusters). One would potentially like to navigate to a planet while moving around obstacles.

Understanding how to find optimal trajectories will provide greater insight into solving problems like these.

2. Notation

In this section, we will give some basic defintions which we will use throughout the paper. We will define our notational conventions here.

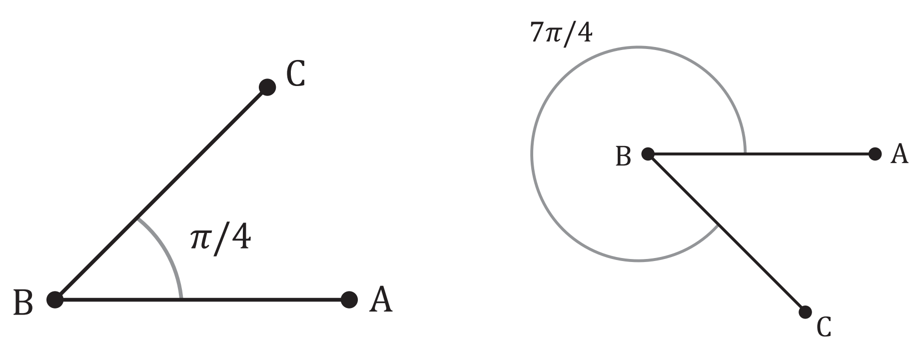
2.1. **Vectors.**

a**:** Scalar quantity.
a: Vector in n-dimensional space. $\|\mathbf{a}\| = a$.
â: Unit vector in n-dimensional space. $\hat{\mathbf{a}} = \mathbf{a}/a$ and $\|\hat{\mathbf{a}}\| = 1$.

2.2. **Angles.**

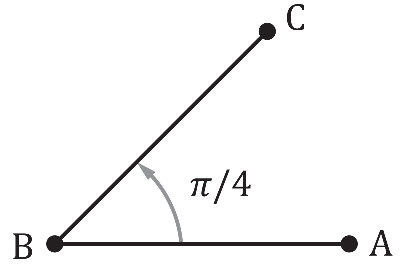
We define two different types of angle measurements: a standard measurement, and a directional measurement (all angles are measured in radians).

In the standard measurement, angles are in $[0, 2\pi]$ and are measured counterclockwise. In the directional measurement, angles are in $[-\pi, \pi]$ and the measurement direction is indicated by an arrow on the measurement.

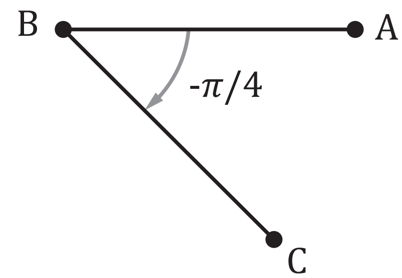


(A) (B)

Figure 1. Standard angle notation (no arrow)



(A)



(B)

FIGURE 2. Directional angle notation (arrow)

3. PARTICLES AND PATHS

In this section, we will provide some basic definitions about particles and paths. These will lay the groundwork for thinking about optimal paths.

Definition 3.1. *A n-dimensional path $\gamma(t) : \mathrm{R} \to \mathrm{R}^n$ is a function which maps a time $t \in \mathbb{R}$, $t \in [T_0(\gamma), T_f(\gamma)]$, to a position $\mathbf{X} \in \mathrm{R}^n$.*

Definition 3.2. *A planar path is a path in $\mathbb{R}^2$.*

Definition 3.3. *A path between two points, $\mathbf{X_1}$ and $\mathbf{X_2}$ is a path, $\gamma(t)$, where $\gamma(T_0(\gamma)) = \mathbf{X_1}$ and $\gamma(T_f(\gamma)) = \mathbf{X_2}$.*

Definition 3.4. *A particle, p, is an object in space with zero volume. A particle travels along a path, $\gamma(t)$ if the particle is at position $\gamma(t)$ at time t, for all $t \in [T_0(\gamma), T_f(\gamma)]$. If a particle is traveling along a path, $\gamma(t)$, then we define a position, $\mathbf{X} \coloneqq \gamma(t)$, a velocity, $\mathbf{v} \coloneqq \frac{d\gamma(t)}{dt}$, and an acceleration, $\mathbf{a} \coloneqq \frac{d^2\gamma(t)}{dt^2}$, for the particle, for all $t \in [T_0(\gamma), T_f(\gamma)]$.*

A particle is restricted, if there are conditions on its position, $\mathbf{X}$ and the time derivatives of its position. Given a particle, a valid path is a path that the particle can travel along.

Definition 3.5. *Given a particle, a fastest path, $\hat{\gamma}(t)$, between two points, $\mathbf{X_1}$ and $\mathbf{X_2}$, is a valid path such that $T_f(\hat{\gamma}) \leq T_f(\gamma)$ for all valid paths, $\gamma(t)$, between $\mathbf{X_1}$ and $\mathbf{X_2}$.*

Definition 3.6. *A particle's speed is v, and its direction of motion is $\hat{\mathbf{v}}$.*

Definition 3.7. *The centripetal acceleration, $\mathbf{a_c}$, of a particle, p, is the component of its acceleration in the direction perpendicular to its direction of motion.*

In rectangular coordinates, the sign of a_c is defined to be the sign of the projection of $\hat{\mathbf{a_c}}$ onto $\hat{\mathbf{x}}$.

In polar coordinates, the sign of a_c is defined to be the sign of the projection of $\hat{\mathbf{a_c}}$ onto $\hat{\mathbf{r}}$.

Definition 3.8. *The tangential acceleration, a_t, of a particle, is the component of the acceleration of the particle in its direction of motion.*

$$a_t \coloneqq \frac{dv}{dt}$$

Definition 3.9. *A particle with constant speed restrictions and bounded centripetal acceleration is a particle with the following conditions: $\mathbf{X} \in \mathbb{R}^2$, $a_t = 0$, and $\|a_c\| \leq \bar{a_c}$, for some $\bar{a_c} \geq 0$.*

3.1. **Particle Motion in Polar Coordinates.** Unless otherwise specified, the motion of 2-dimensional particles in this paper will be described in polar coordinates, along with an extra parameter θ as is shown in Figure 3.

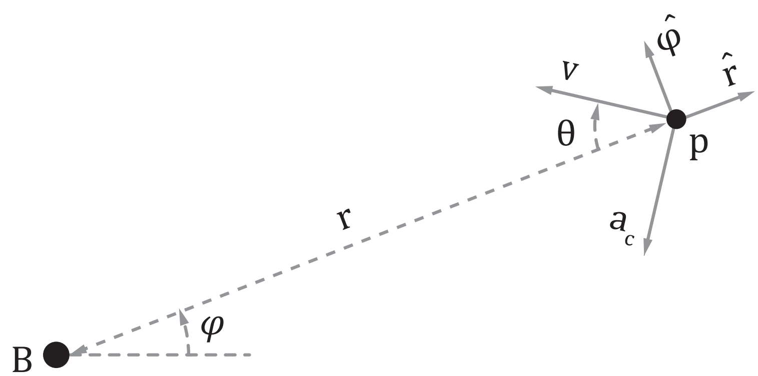


FIGURE 3. A particle moving in 2-dimensional polar coordinate system centered at B.

$\theta(t), \phi(t) \in [-\pi, \pi]$. From now on, $|\theta(t)|$ and $|\phi(t)|$ will be used, since there is a symmetry in the system about $\hat{\mathbf{r}}$.

Lemma 3.10. *The time derivative of θ is given by*

$$\frac{d|\theta(t)|}{dt} = \frac{a_c}{v}$$

Proof. If we look at a point, p, subject to only centripetal acceleration, a_c, the change in $\mathbf{v}$ over an infinitesimal time, dt, is shown in Figure 4 (the two vectors, $\mathbf{v}$ and $\mathbf{v} + \mathbf{dv}$, are superimposed).

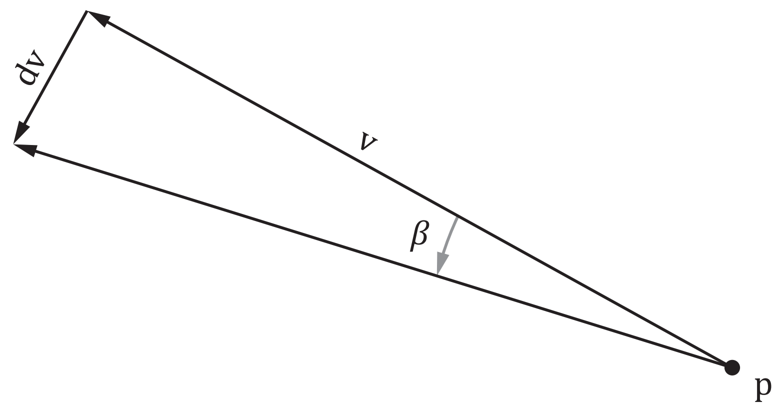


FIGURE 4. Tangential acceleration.

From the definition of a_c

$$\frac{dv}{dt} = a_c$$

Since $dv = d|\theta|v$, then

$$\frac{d|\theta|}{dt} = \frac{a_c}{v}$$

□

The proof of the lemma in the case where a_t is also nonzero is very similar, since the component of $\mathbf{dv}$ in the direction $\hat{\mathbf{v}}$ is negligible compared to v.

Returning again to Figure 3, the following equations can be derived

$$\frac{dr(t)}{dt} = -v(t)\,\cos(|\theta(t)|) \tag{1}$$

$$\frac{d|\phi(t)|}{dt} = \frac{v(t)}{r(t)}\,\sin(|\theta(t)|) \tag{2}$$

$$\frac{d|\theta(t)|}{dt} = \frac{d|\phi(t)|}{dt} - \frac{a_c(t)}{v(t)} \tag{3}$$

$$= \frac{v(t)}{r(t)}\,\sin(|\theta(t)|) - \frac{a_c(t)}{v(t)} \tag{4}$$

Applying the chain rule to (1)

$$\frac{d}{dt}\frac{dr(t)}{dt} = -\frac{dv(t)}{dt}\cos(|\theta(t)|) + v(t)\sin(|\theta(t)|)\frac{d|\theta(t)|}{dt} \tag{5}$$

$$\begin{aligned} &= -a_t\cos(|\theta(t)|) + \frac{v(t)^2}{r(t)}\sin^2(|\theta(t)|) \\ &\qquad - v(t)\sin(|\theta(t)|)\frac{a_c(t)}{v(t)} \end{aligned} \tag{6}$$

Lemma 3.11. *For a particle, p, with bounded centripetal and tangential acceleration, then:*
1. The function $|\theta(t)|$ is continuous.
2. Given $t_1, t_2 \in \mathbb{R}^2$, s.t. $t_1 < t_2$, $\theta(t_1) > a$ and $\theta(t_2) < a$, then $\theta(t_c) = a$ for some $t_c \in [t_1, t_2]$.

Proof. First off, it should be noted that a solution to (4) exists because the right hand side is Lipschitz continuous. The proof of 1. follows directly from the fact that the derivative of $|\theta(t)|$ exists and is bounded. 2. is just a restatement of the intermediate value theorem. □

Lemma 3.12. *For a particle, p, with nonzero speed, and zero centripetal acceleration for $t \geq t_0$, then*

$$\begin{cases} |\theta(t)| \to \pi \quad \text{as} \quad t \to \infty & \text{if } \theta(t_0) > 0 \\ |\theta(t)| = 0 \quad \text{for all} \quad t \geq t_0 & \text{if } \theta(t_0) = 0 \end{cases}$$

Furthermore,

$$\begin{cases} r(t) \to \infty \quad \text{as} \quad t \to \infty & \text{if } \theta(t_0) > 0 \\ r(t) \to 0 \quad \text{as} \quad t \geq t_0 & \text{if } \theta(t_0) = 0 \end{cases}$$

Proof. For the first case, $\frac{d|\theta(t)|}{dt} > 0$, except when $|\theta(t)| = 0, \pi$, which means that $\theta(t)$ is monotonically increasing. Since $\phi(t)$ is bounded above py π, $\phi(t) \to \pi$ as ∞.

The second case is the easiest to check. If we plug $\theta(t_0) = 0$ into 4, we get

$$\frac{d|\theta(t)|}{dt} = 0 \qquad \text{for } t \geq t_0$$

So

$$|\theta(t)| = 0 \qquad \text{for } t \geq t_0$$

[TODO]

□

4. Conclusion

In this paper, we have formalized many of the things that intuition would tell us. Namely, we have shown that a straight line is the fastest way to get between two points. This fact is unsurprising because of the fact that acceleration in a single direction (toward the finish point) is the least wasteful means of getting to the finish.

We developed a number of simple lemmas governing the motion of a volume-less particle with bounded acceleration. We also showed that symmetric paths tend to be better than non-symmetric paths (through our simple quadrilateral path minimization problem).

Finally, we found an optimal path for a particle to trace around three cones when the particle has constant velocity.

Appendix

4.1. **Analysis.**

Lemma 4.1. *If* $f(t), g(t) > 0$ *and* $k(t) > 0$ *is a strictly monotonically increasing function for all* t*, then* $\arg\min_t k(f(t)) + k(g(t)) = \arg\min_t f(t) + g(t)$.

Proof. Let t_1, t_2 be such that $f(t_1)+g(t_1) < f(t_2)+g(t_2)$. In this proof, we will show that $k(f(t_1)) + k(g(t_1)) < \sqrt{f(t_2)} + \sqrt{g(t_2)}$. Since $k(t)$ is a strictly monotonically increasing function when $t > 0$, we know that $k(x) < k(y)$ if and only if $x < y$ (assuming we can confine x, y to be non-negative).

Because this is the case, we see that $k(f(x)) < k(f(y))$ if and only if $f(x) < f(y)$ (the same goes for g). Thus, we see that if we have found the minimum t_m to $\arg\min_t k(f(t)) + k(g(t))$, then it is the case that $k(f(t_m)) + k(g(t_m)) < k(f(t)) + k(g(t))$ for all $t \neq t_m$ (again where $t > 0$). Following our train of logic, we see that $f(t_m) + g(t_m) < f(t) + g(t)$ for all $t \neq t_m$, which means that t_m is a minimum of $f(t) + g(t)$. Thus by finding a minimum t_m to $k(f(t)) + k(g(t))$, we also found a minimum to $f(t) + g(t)$. □

4.2. **Coordinate Systems.**

4.2.1. *Radius of Curvature.* The radius of curvature, R, of a curve at a point is a measure of the radius of the circular arc which best approximates the curve at that point.

For $a_t = 0$

$$R = \left|\frac{v^2}{a_c}\right|$$

4.3. Vector Calculus in Polar Coordinates.

$$\boldsymbol{x} = r\hat{\boldsymbol{r}}$$

$$\vec{\boldsymbol{v}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}$$

$$\vec{\boldsymbol{a}} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\boldsymbol{r}} + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right)\hat{\boldsymbol{\phi}}$$

References

[1] http://en.wikipedia.org/wiki/Polar_coordinate_system