THE 18.821 MATHEMATICS PROJECT LAB REPORT [PROOFS]

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1. Introduction

sdafkjsadflkj

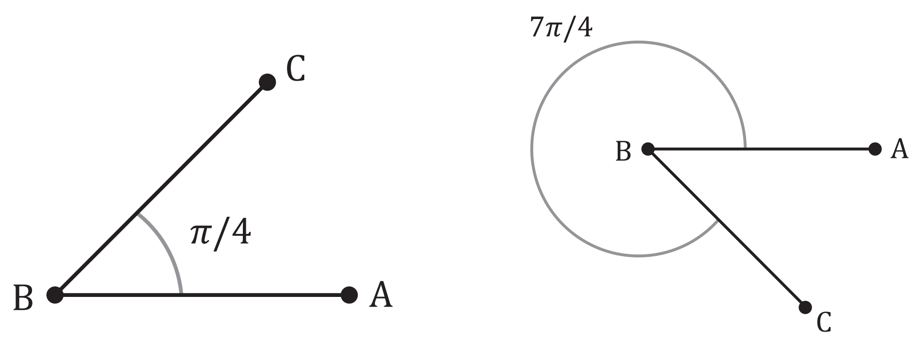
2. Notation

2.1. Vectors.

a**:** Scalar quantity.
a: Vector in n-dimensional space. $\|\mathbf{a}\| = a$.
â: Unit vector in n-dimensional space. $\hat{\mathbf{a}} = \mathbf{a}/a$ and $\|\hat{\mathbf{a}}\| = 1$.

2.2. **Angles.** Since many of the theorems in this paper involve polar coordinates, we define two different types of angle measurements: the standard measurement, and a directional measurement. All angles are measured in radians.

In the standard measurement, angles are in $[0, 2\pi]$ and are measured counterclockwise. In the directional measurement, angles are in $[-\pi, \pi]$ and the measurement direction is indicated by an arrow on the figure.



(a) (b)

Figure 1. Standard angle notation (no arrow)

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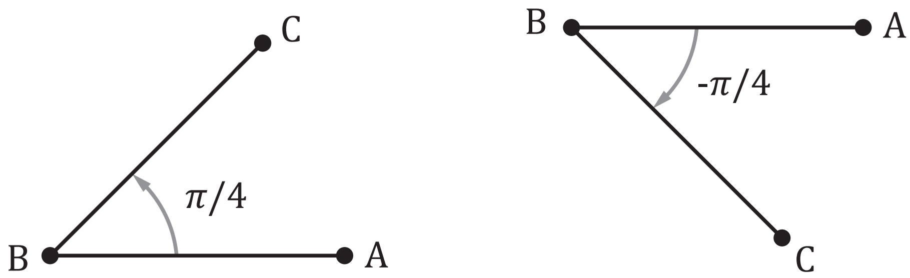


(A) (B)

FIGURE 2. Directional angle notation (arrow)

3. PARTICLES AND PATHS

Definition 3.1. *A n-dimensional path $\gamma(t) : \mathrm{R} \to \mathrm{R}^n$ is a function which maps a time $t \in \mathbb{R}$, $t \in [0, T_{f,\gamma}]$, to a position $\mathbf{X} \in \mathrm{R}^n$.*

Definition 3.2. *A n-dimensional particle, p, is an object with zero volume that travels along a n-dimensional path. The particle may have conditions on its position, velocity, and acceleration in $\mathbb{R}^n$.*

Definition 3.3. *A valid path $\gamma(t)$ for a particle p is a path such that all conditions on the particle are satisfied at every point along the path.*

Definition 3.4. *A path between two points, $\mathbf{X_1}$ and $\mathbf{X_2}$ is a path, $\gamma(t)$ where $\gamma(0) = \mathbf{X_1}$ and $\gamma(T_{f,\gamma}) = \mathbf{X_2}$.*

Definition 3.5. *For a given particle, p, a fastest path, $\hat{\gamma}(t)$, between two points, $\mathbf{X_1}$ and $\mathbf{X_2}$, is a valid path such that $T_f(\hat{\gamma}) \leq T_f(\gamma)$ for all valid paths, $\gamma(t)$, between $\mathbf{X_1}$ and $\mathbf{X_2}$.*

Definition 3.6. *The centripetal acceleration, $\mathbf{a_c}$, of a particle, p, is the component of the acceleration of p perpendicular to its direction of motion, $\hat{\mathbf{v}}$. The sign of a_c is difined as the sign of the projection of $\hat{\mathbf{a_c}}$ onto $\hat{\mathbf{r}}$.*

Definition 3.7. *The tangential acceleration, $\mathbf{a_t}$, of a particle, p, is the component of the acceleration of p in its direction of motion, $\hat{\mathbf{v}}$.*

3.1. **Particle Motion in Polar Coordinates.** The motion of the 2-dimensional particles in this paper will typically be described in the polar coordinate system, shown in figure 3.

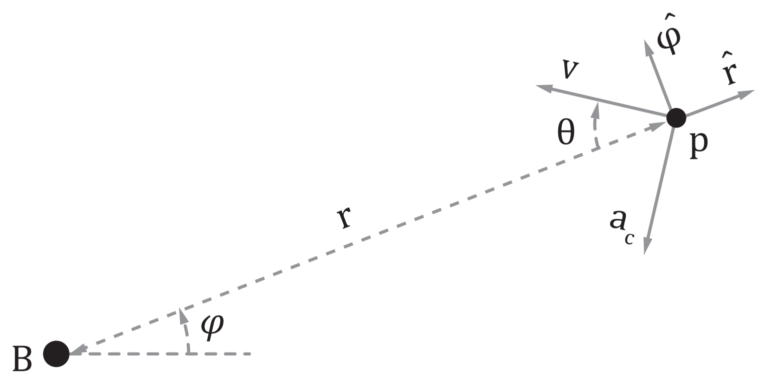
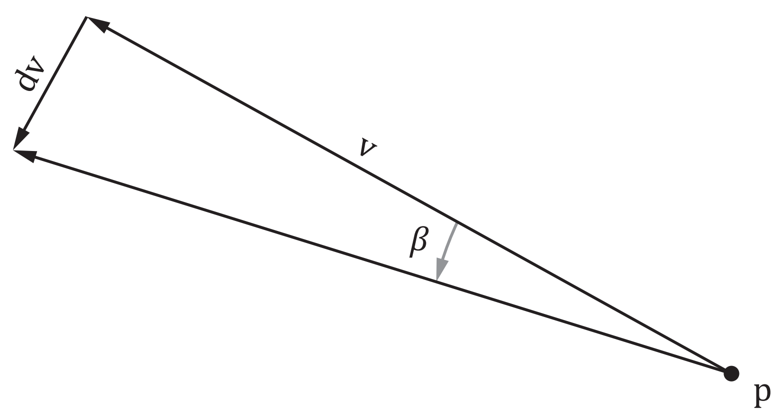


FIGURE 3. A particle moving in 2-dimensional polar coordinate system centered at B.

Lemma 3.8. *The time derivative of θ is given by*

$$\frac{d\beta}{dt} = \frac{a_c}{s}$$



Proof.

FIGURE 4. Tangential acceleration.

If we look at a point, p, subject to only a centripetal acceleration, a_c, the change in $\mathbf{v}$ over an infinitesimal time, dt, is shown in the figure 4 (the two vectors, $\mathbf{v}$ and $\mathbf{v}+\mathbf{dv}$, are superimposed). From the definition of a_c

$$\frac{dv}{dt} = a_c$$

So...

$$\frac{d\beta}{dt} = \frac{a_c}{s}$$

□

The proof of the lemma in the case where a_t is also nonzero is very similar, since the component of $\mathbf{dv}$ in the direction $\hat{\mathbf{v}}$ is negligable compared to v.

Applying some trig, we get the following equations

$$\begin{aligned}
\frac{dr(t)}{dt} &= -s(t)\,\cos(\theta(t)) \\
\frac{d\phi(t)}{dt} &= s(t)\,\sin(\theta(t)) \\
\frac{d\theta(t)}{dt} &= \frac{d\phi(t)}{dt} - \frac{a_c(t)}{s(t)} \\
\frac{d}{dt}\frac{dr(t)}{dt} &= -\frac{ds(t)}{dt}\cos(\theta(t)) + s(t)\sin(\theta(t))\frac{d\theta(t)}{dt}
\end{aligned}$$

Lemma 3.9. *For a particle, p, with bounded centripetal and tangential acceleration, then:*

(1) The functions $\phi(t)$ and $\theta(t)$ are continuous.

(2) For two times t_1 and t_2, s.t. $t_1 < t_2$, $\theta(t_1) > a$ and $\theta(t_2) < a$, then $\theta(t_c) = a$ for some $t_c \in [t_1, t_2]$.

Proof. The proof of (1) follows directly from the fact that the derivative of $\theta(t)$ exist. (2) is just a restatement of the intermediate value theorem.

Lemma 3.10. *For a particle, p, with nonzero speed, and no centripetal acceleration for $t \geq t_0$, then*

$$\begin{cases} \theta(t) \to \pi \quad as \quad t \to \infty & if\ \theta(t_0) > 0 \\ \theta(t) = 0 \quad for \quad t \geq t_0 & if\ \theta(t_0) = 0 \end{cases}$$

4. Traveling Between Points

Theorem 4.1. *Given points, $X_1, X_2 \in \mathbb{R}^2$ and a particle p initially at X_1 with infinite centripetal acceleration bounded tangential acceleration, $|a_t| \leq \bar{a}$, then the fastest path $\hat{\gamma}(t)$ which p can trace from X_1 to X_2 is the line segment from X_1 to X_2.*

Proof. Let's transform the problem. We can reset our coordinate axes so that (x_1, y_1) is set to the origin and (x_2, y_2) is on the x-axis. In this new coordinate system, we have transformed the following:

$$(x_1, y_1) \rightarrow (0,0) \tag{1}$$

$$(x_2, y_2) \rightarrow (x_2', 0) \tag{2}$$

For convenience of notation, we will now refer to x_2' as x_2.

Now let us examine the particle's motion in the x direction. Let $a_t(t)$ be the tangential acceleration at time t in the x direction. Then we can obtain the speed of the particle $s(t)$ at time t in the x direction like so:

$$s(t) = \int_0^t a_t(t_1)dt_1 \tag{3}$$

To find the distance $d(t)$ travelled up to time t in the x direction, we can use the relation:

$$\begin{aligned} d(t) &= \int_0^t s(t_2)dt_2 && (4)\\ &= \int_0^t \int_0^t a_t(t_1)dt_1dt_2 && (5) \end{aligned}$$

Recall that the acceleration of the particle p is bounded by $\bar{a}$. This means that $a_t(t) \leq \bar{a}$ for all t. Therefore, we see:

$$\begin{aligned} d(t) &\leq \int_0^t \int_0^t \bar{a}dt_1dt_2 && (6)\\ &= \frac{\bar{a}t^2}{2} && (7) \end{aligned}$$

Thus, in order to travel a distance of $d(T_f) = x_2$, it needs to be the case that $T_f \geq \sqrt{\frac{2x_2}{\bar{a}}}$. Moreover, equality holds if and only if $a_t(t) = \bar{a}$ for all $t \in [0, T_f(\gamma)]$.

If the particle travels for time $t < \sqrt{\frac{2x_2}{\bar{a}}}$, then it is impossible for the particle to reach $(x_2, 0)$ when starting at $(0,0)$. This is because p cannot reach $(x_2, 0)$ in the x direction when $t < \sqrt{\frac{2x_2}{\bar{a}}}$ and any acceleration in the y direction would not enable this either.

This means that the fastest path is completed in time $T_f(\hat{\gamma}) = \sqrt{\frac{2x_2}{\bar{a}}}$. Let us examine the path taken by the particle p on this fastest path. Recall that $a_t(t) = \bar{a}$ for all t along the fastest path. This means that there was no centripetal acceleration $|a_c| = 0$. In other words, the particle never turned on its way to reaching the destination point. The only way this could have happened is if it travelled along the x axis in a straight line.

Now, we have seen that the fastest path in the transformed coordinates travels exactly on the x axis so that $y = 0$ anywhere along the fastest path. Notice, however, that the x axis in the transformed coordinates is given exactly by the following line:

$$y = \frac{y_2 - y_1}{x_2 - x_1} x + y_1 \tag{8}$$

Thus, we see that the fastest path in the original coordinates follows the above equation, which is what we wanted to show. □

Corollary 4.2. *The fastest path between two points* $(x_1, y_1), (x_2, y_2) \in \mathbb{R}^2$ *is unique.*

5. Turning Around Cones

5.1. **Quadrilateral.** Before we attempt to tackle the problem of finding the optimal path around a cone at constant velocity, we may want to find some intuition for the problem with a simple, but related problem. In particular, if a particle is traveling around a cone, we would like to know what circle it traces as it goes around the cone.

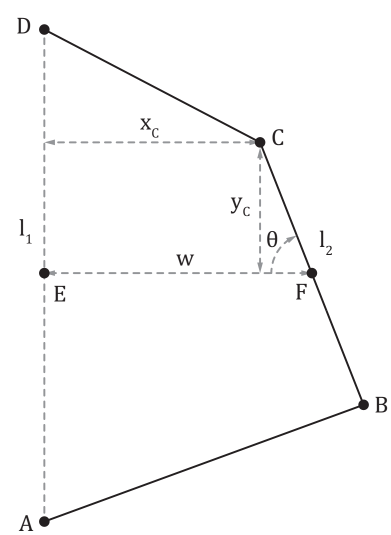


Figure 5. Quadrilateral path. A → B → C → D.

To make a simple model of this problem, we will use a quadrilateral to model a path around a cone. We will construct a quadrilateral with particular constaints that models the constraints of a path around a cone, and we will try to find the parameters that minimize the perimeter around the quadrilateral. Let us call the sides of the quadrilateral

s_1, s_2, s_3 and s_4. Imagine that s_1 and s_2 are on opposite sides of the quadrilateral with fixed lengths l_1 and l_2 respectively. Now we will constrain the problem to be related to the problem of a particle traveling around a cone: imagine s_1 and s_2 are connected by a bar s_b of length d. The bar will be perpendicular to s_1. The angle that s_b forms with s_2 will be called θ. We will try to find the optimal θ that minimizes the perimeter around the quadrilateral.

5.2. **Symmetric Quadrilateral.** Let us begin by solving the simplest version of this problem. Let us imagine that s_b is connected to the midpoints of s_1 and s_2. We know that the perimeter will be given by $l_1 + l_2 + l_3 + l_4$ where l_3 and l_4 are the lengths of the sides of s_3 and s_4, respectively. We are given l_1 and l_2, but we will need to compute l_3 and l_4 as functions of l_1, l_2, d, and θ.

We can find l_3 by using the fact that it forms a right triangle. We know that $l_3^2 = m^2 + n^2$. Finding l_4 is similar. Therefore, we just need to find m and n. This can be done by using the fact that $m = \frac{l_1}{2} - \frac{l_2}{2}\sin\theta$. This is just the upper half of s_1 minus the projection of s_2 onto s_1. We can find n similarly: $n = d - \frac{l_2}{2}\cos\theta$. This is just the bar s_b minus the projection of s_2 onto the bar. Thus by substituting, we have:

$$l_3 = \sqrt{\left(\frac{l_1}{2} - \frac{l_2}{2}\sin\theta\right)^2 + \left(d - \frac{l_2}{2}\cos\theta\right)^2} \tag{9}$$

We can do a similar analysis on l_4, only remembering that n for l_4 gets extended by the projection onto s_4 instead of shrunken. We therefore have:

$$l_4 = \sqrt{\left(\frac{l_1}{2} - \frac{l_2}{2}\sin\theta\right)^2 + \left(d + \frac{l_2}{2}\cos\theta\right)^2} \tag{10}$$

Now, to minimize the perimeter with respect to θ, we want to minimize $l_1 + l_2 + l_3 + l_4$. Since we know that l_1 and l_2 are fixed, we really want to minimize $l_3 + l_4$ with respect to θ. The other thing to note is that we're minimizing positive distances. We will invoke the following lemma so that we can simplify our expression for $\min l_3 + l_4$:

Lemma 5.1. *If $f(t), g(t) > 0$ and $k(t) > 0$ is a strictly monotonically increasing function for all t, then* $\arg\min_t k(f(t)) + k(g(t)) = \arg\min_t f(t) + g(t)$.

Proof. Let t_1, t_2 be such that $f(t_1) + g(t_1) < f(t_2) + g(t_2)$. In this proof, we will show that $k(f(t_1)) + k(g(t_1)) < \sqrt{f(t_2)} + \sqrt{g(t_2)}$. Since $k(t)$ is a strictly monotonically increasing function when $t > 0$, we know that

$k(x) < k(y)$ if and only if $x < y$ (assuming we can confine x, y to be non-negative).

Because this is the case, we see that $k(f(x)) < k(f(y))$ if and only if $f(x) < f(y)$ (the same goes for g). Thus, we see that if we have found the minimum t_m to $\arg\min_t k(f(t)) + k(g(t))$, then it is the case that $k(f(t_m)) + k(g(t_m)) < k(f(t)) + k(g(t))$ for all $t \neq t_m$ (again where $t > 0$). Following our train of logic, we see that $f(t_m) + g(t_m) < f(t) + g(t)$ for all $t \neq t_m$, which means that t_m is a minimum of $f(t) + g(t)$. Thus by finding a minimum t_m to $k(f(t)) + k(g(t))$, we also found a minimum to $f(t) + g(t)$. $\square$

Since we've proven this lemma, we can invoke it upon $\min l_3 + l_4$. Since $l_3 = \sqrt{z_3}$ and $l_4 = \sqrt{z_4}$, we can use $k(t) = \sqrt{t}$ and we can write $\min l_3 + l_4 = \min z_3 + z_4$ by using our lemma. Thus, we now want to solve the problem:

$$\arg\min_\theta \quad 2 \quad \left(\frac{l_1}{2} - \frac{l_2}{2}\sin\theta\right)^2 \tag{11}$$

$$+ \quad \left(d - \frac{l_2}{2}\cos\theta\right)^2 + \left(d + \frac{l_2}{2}\cos\theta\right)^2 \tag{12}$$

Now, we can expand out our expression and use the fact that $\sin^2\theta + \cos^2\theta = 1$ to obtain a much simpler (but equivalent) minimization problem:

$$\arg\min_\theta \frac{l_1^2}{2} + \frac{l_2^2}{2} + 2d^2 - l_1 l_2 \sin\theta \tag{13}$$

We note that l_1, l_2, and d are all constants which are given to us in the problem. Therefore, the minimization problem really boils down to

$$\arg\min_\theta -l_1 l_2 \sin\theta = \arg\max_\theta \sin\theta \tag{14}$$

Thus, we see that $\theta = \frac{\pi}{2}$ minimizes the perimeter of the quadrilateral in the symmetric case.

Theorem 5.2. *Given a cone setup consisting of 3 cones at locations X_A, X_B, and X_C.*

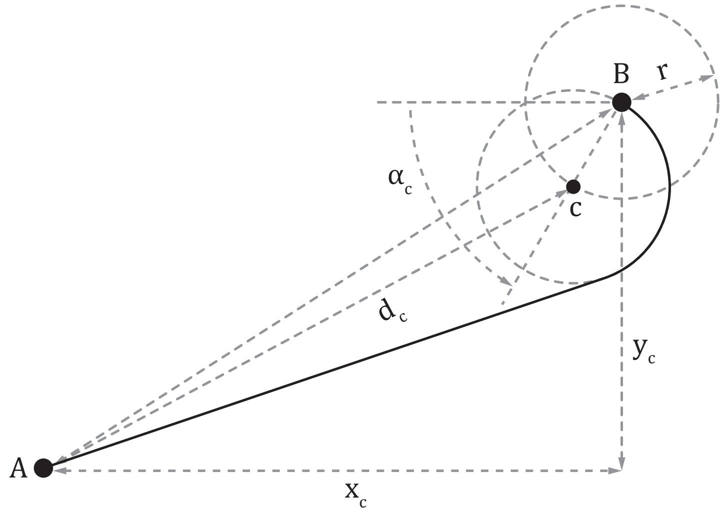


FIGURE 6. A.

$$dB = \sqrt{x_B + y_B} \tag{15}$$

Applying the triangle inequality...

$$d_c = \alpha_c - \tan^{-1}\left(\frac{y_B}{x_B}\right) \tag{16}$$

(17)

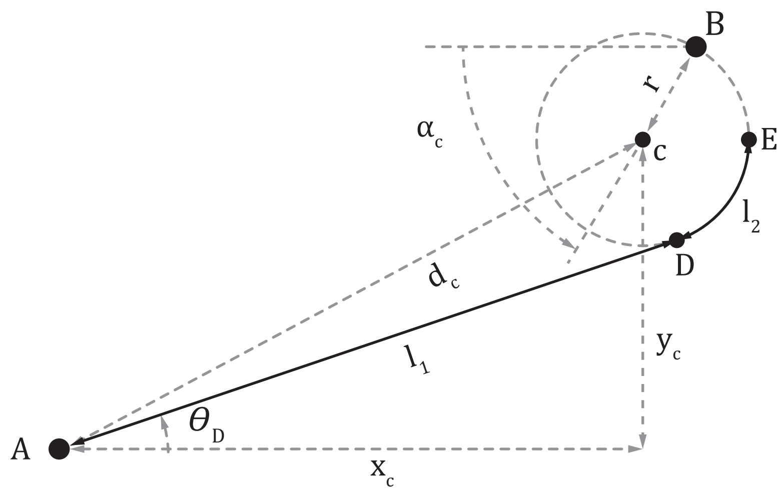


FIGURE 7. A.

$$l_1 = \sqrt{d_c - r} \tag{18}$$
$$x_c = x_B - r\cos\alpha_c \tag{19}$$
$$y_c = y_B - r\sin\alpha_c \tag{20}$$
$$d_c = \alpha_c - \tan^{-1}\left(\frac{y_c}{x_c}\right) \tag{21}$$
(22)

Proof. APPENDIX

5.3. **Coordinate Systems.**

5.3.1. *Time Derivatives.*

$$\dot{a} = \frac{da}{dt}$$
$$\ddot{a} = \frac{d^2a}{dt^2}$$

5.3.2. *Radius of Curvature.* The radius of curvature, R, of a curve at a point is a measure of the radius of the circular arc which best approximates the curve at that point.

$$R = \left| \frac{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} \right| \tag{23}$$

(for $a_c =$)
$$= \left| \frac{s^2}{a_t} \right|$$

5.4. Scalar Calculus in Polar Coordinates.

$$s = \frac{dl}{dt} \tag{24}$$

$$a_t = \frac{ds}{dt} \tag{25}$$

(26)

5.5. Vector Calculus in Polar Coordinates.

$$\boldsymbol{x} = r\hat{\boldsymbol{r}} \tag{27}$$

$$\vec{\boldsymbol{v}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}} \tag{28}$$

$$\vec{\boldsymbol{a}} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\boldsymbol{r}} + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right)\hat{\boldsymbol{\phi}} \tag{29}$$

5.6. Relations.

$$l = \int_{t=0}^{T} \|\vec{\boldsymbol{v}}\|\, dt \tag{30}$$

$$s = \|\vec{\boldsymbol{v}}\| \tag{31}$$

$$a_t = \|\vec{\boldsymbol{a}}\| \cdot \hat{\boldsymbol{v}} = \|\vec{\boldsymbol{a}}\| \cdot \frac{\vec{\boldsymbol{v}}}{\|\vec{\boldsymbol{v}}\|} \tag{32}$$

$$a_c = \|\vec{\boldsymbol{a}}\| \times \hat{\boldsymbol{v}} = \|\vec{\boldsymbol{a}}\| \times \frac{\vec{\boldsymbol{v}}}{\|\vec{\boldsymbol{v}}\|} \tag{33}$$

References

[1] http://en.wikipedia.org/wiki/Polar_coordinate_system