THE 18.821 MATHEMATICS PROJECT LAB REPORT [PROOFS]

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1. Theorems

1.1. Notation.

t: Time
l: Path length
s: Speed
a_t: Tangential acceleration
a_c: Centripetal acceleration
$\vec{x}$: Position
$\vec{v}$: Velocity
$\vec{a}$: Acceleration

2. Coordinates

2.1. Scalar Calculus.

$$s = \frac{dl}{dt} \tag{1}$$

$$a_t = \frac{ds}{dt} \tag{2}$$

(3)

2.2. Vector Calculus.

$$\boldsymbol{x} = r\hat{\boldsymbol{r}} \tag{4}$$

$$\vec{\boldsymbol{v}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}} \tag{5}$$

$$\vec{\boldsymbol{a}} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\boldsymbol{r}} + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right)\hat{\boldsymbol{\phi}} \tag{6}$$

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2.3. Relations.

$$l = \int_{t=0}^{T} \|\vec{\boldsymbol{v}}\| \ dt \tag{7}$$

$$s = \vec{\boldsymbol{v}} \tag{8}$$

$$a_t = \|\vec{\boldsymbol{a}}\| \frac{\vec{\boldsymbol{v}}}{\|\vec{\boldsymbol{v}}\|} \tag{9}$$

3. Traveling Between Points

Definition 3.1. *A description of a particle p is a set $D(t)$ which contains a particle's position $\vec{x} \in \mathrm{R}^2$ and its derivatives at some time t. In other words, we define $D(t) = \{\vec{x}(t), \frac{d\vec{x}}{dt}(t), \frac{d^2\vec{x}}{dt^2}(t), \ldots\}$.*

Definition 3.2. *A condition c on a particle p is a boolean function $c : \mathrm{D} \to \{0, 1\}$ which takes as input a description $D(t)$ of the particle p at some time t and outputs that either the condition is true or false.*

Definition 3.3. *A path $\gamma : \mathrm{R} \to \mathrm{R}^2$ is a function which maps some time t to a position $\gamma(t) \in \mathrm{R}^2$. The path is defined from time $t = 0$ until the end time of the path, denoted as $T_{f,\gamma}$.*

Definition 3.4. *A valid path $\gamma : \mathrm{R} \to \mathrm{R}^2$ for some particle p and conditions C is some path which at all times t such that $0 \leq t \leq T_{f,\gamma}$, all conditions in C on the particle are satisfied.*

Definition 3.5. *A valid targetted path $\gamma : \mathrm{R} \to \mathrm{R}^2$ for some point mass p, conditions C, starting point $\vec{x_1}$, and ending point $\vec{x_2}$ is a valid path where $\gamma(t) = \vec{x_1}$ and $\gamma(T_f(\gamma)) = \vec{x_2}$. In other words, it is a valid path which starts at $\vec{x_1}$ and ends at $\vec{x_2}$.*

Definition 3.6. *A fastest path $\hat{\gamma} : \mathrm{R} \to \mathrm{R}^2$ for a particular point mass p, a starting point $\vec{x_1}$, a destination point $\vec{x_2}$, and some set of conditions C is a valid targetted path $\hat{\gamma}$ such that $T_f(\hat{\gamma}) \leq T_f(\gamma)$ for all valid targetted paths γ with the same p, $\vec{x_1}$, $\vec{x_2}$, and C.*

Theorem 3.7. *Given points $(x_1, y_1), (x_2, y_2) \in \mathrm{R}^2$ and a point mass p whose initial position is (x_1, y_1) which moves with acceleration bounded by $\bar{a}$, the fastest path $\hat{\gamma}(t)$ which p can trace from (x_1, y_1) to (x_2, y_2) follows the straight line where all coordinates (x, y) on the straight line are given by:*

$$y = \frac{y_2 - y_1}{x_2 - x_1} x + y_1 \tag{10}$$

Proof. Let's transform the problem. We can reset our coordinate axes so that (x_1, y_1) is set to the origin and (x_2, y_2) is on the x-axis. In this new coordinate system, we have transformed the following:

$$(x_1, y_1) \rightarrow (0, 0) \tag{11}$$

$$(x_2, y_2) \rightarrow (x_2', 0) \tag{12}$$

For convenience of notation, we will now refer to x_2' as x_2.

Now let us examine the particle's motion in the x direction. Let $a_t(t)$ be the tangential acceleration at time t in the x direction. Then we can obtain the speed of the particle $s(t)$ at time t in the x direction like so:

$$s(t) = \int_0^t a_t(t_1)dt_1 \tag{13}$$

To find the distance $d(t)$ travelled up to time t in the x direction, we can use the relation:

$$d(t) = \int_0^t s(t_2)dt_2 \tag{14}$$

$$= \int_0^t \int_0^t a_t(t_1)dt_1 dt_2 \tag{15}$$

Recall that the acceleration of the point mass p is bounded by $\bar{a}$. This means that $a_t(t) \leq \bar{a}$ for all t. Therefore, we see:

$$d(t) \leq \int_0^t \int_0^t \bar{a} dt_1 dt_2 \tag{16}$$

$$= \frac{\bar{a}t^2}{2} \tag{17}$$

Thus, in order to travel a distance of $d(T_f) = x_2$, it needs to be the case that $T_f \geq \sqrt{\frac{2x_2}{\bar{a}}}$. Moreover, equality holds if and only if $a_t(t) = \bar{a}$ for all $t \in [0, T_f(\gamma)]$.

If the point mass travels for time $t < \sqrt{\frac{2x_2}{\bar{a}}}$, then it is impossible for the point mass to reach $(x_2, 0)$ when starting at $(0, 0)$. This is because p cannot reach $(x_2, 0)$ in the x direction when $t < \sqrt{\frac{2x_2}{\bar{a}}}$ and any acceleration in the y direction would not enable this either.

This means that the fastest path is completed in time $T_f(\hat{\gamma}) = \sqrt{\frac{2x_2}{\bar{a}}}$. Let us examine the path taken by the point mass p on this fastest path. Recall that $a_t(t) = \bar{a}$ for all t along the fastest path. This means that there was no centripetal acceleration $|a_c| = 0$. In other words, the point mass never turned on its way to reaching the destination point. The only way this could have happened is if it travelled along the x axis in a straight line.

Now, we have seen that the fastest path in the transformed coordinates travels exactly on the x axis so that $y = 0$ anywhere along the fastest path. Notice, however, that the x axis in the transformed coordinates is given exactly by the following line:

$$y = \frac{y_2 - y_1}{x_2 - x_1} x + y_1 \tag{18}$$

Thus, we see that the fastest path in the original coordinates follows the above equation, which is what we wanted to show. ☐

Corollary 3.8. *The fastest path between two points* $(x_1, y_1), (x_2, y_2) \in \mathrm{R}^2$ *is unique.*

4. Turning Around a Cone - Constant Velocity

4.1. **Quadrilateral.** Before we attempt to tackle the problem of finding the optimal path around a cone at constant velocity, we may want to find some intuition for the problem with a simple, but related problem. In particular, if a particle is traveling around a cone, we would like to know what circle it traces as it goes around the cone.

To make a simple model of this problem, we will use a quadrilateral to model a path around a cone. We will construct a quadrilateral with particular constaints that models the constraints of a path around a cone, and we will try to find the parameters that minimize the perimeter around the quadrilateral. Let us call the sides of the quadrilateral s_1, s_2, s_3 and s_4. Imagine that s_1 and s_2 are on opposite sides of the quadrilateral with fixed lengths l_1 and l_2 respectively. Now we will constrain the problem to be related to the problem of a particle traveling around a cone: imagine s_1 and s_2 are connected by a bar s_b of length d. The bar will be perpendicular to s_1. The angle that s_b forms with s_2 will be called θ. We will try to find the optimal θ that minimizes the perimeter around the quadrilateral.

4.2. **Symmetric Quadrilateral.** Let us begin by solving the simplest version of this problem. Let us imagine that s_b is connected to the midpoints of s_1 and s_2. We know that the perimeter will be given by $l_1 + l_2 + l_3 + l_4$ where l_3 and l_4 are the lengths of the sides of s_3 and s_4, respectively. We are given l_1 and l_2, but we will need to compute l_3 and l_4 as functions of l_1, l_2, d, and θ.

We can find l_3 by using the fact that it forms a right triangle. We know that $l_3^2 = m^2 + n^2$. Finding l_4 is similar. Therefore, we just need to find m and n. This can be done by using the fact that $m = \frac{l_1}{2} - \frac{l_2}{2}\sin\theta$. This is just the upper half of s_1 minus the projection of s_2 onto s_1. We can find n similarly: $n = d - \frac{l_2}{2}\cos\theta$. This is just the bar s_b minus the projection of s_2 onto the bar. Thus by substituting, we have:

$$l_3 = \sqrt{\left(\frac{l_1}{2} - \frac{l_2}{2}\sin\theta\right)^2 + \left(d - \frac{l_2}{2}\cos\theta\right)^2} \tag{19}$$

We can do a similar analysis on l_4, only remembering that n for l_4 gets extended by the projection onto s_4 instead of shrunken. We therefore have:

$$l_4 = \sqrt{\left(\frac{l_1}{2} - \frac{l_2}{2}\sin\theta\right)^2 + \left(d + \frac{l_2}{2}\cos\theta\right)^2} \tag{20}$$

Now, to minimize the perimeter with respect to θ, we want to minimize $l_1 + l_2 + l_3 + l_4$. Since we know that l_1 and l_2 are fixed, we really want to minimize $l_3 + l_4$ with respect to θ. The other thing to note is that we're minimizing positive distances. We will invoke the following lemma so that we can simplify our expression for $\min l_3 + l_4$:

Lemma 4.1. *If $f(t), g(t) > 0$ and $k(t) > 0$ is a strictly monotonically increasing function for all t, then* $\arg\min_t k(f(t)) + k(g(t)) = \arg\min_t f(t) + g(t)$.

Proof. Let t_1, t_2 be such that $f(t_1)+g(t_1) < f(t_2)+g(t_2)$. In this proof, we will show that $k(f(t_1)) + k(g(t_1)) < \sqrt{f(t_2)} + \sqrt{g(t_2)}$. Since $k(t)$ is a strictly monotonically increasing function when $t > 0$, we know that $k(x) < k(y)$ if and only if $x < y$ (assuming we can confine x, y to be non-negative).

Because this is the case, we see that $k(f(x)) < k(f(y))$ if and only if $f(x) < f(y)$ (the same goes for g). Thus, we see that if we have found the minimum t_m to $\arg\min_t k(f(t)) + k(g(t))$, then it is the case that $k(f(t_m)) + k(g(t_m)) < k(f(t)) + k(g(t))$ for all $t \neq t_m$ (again where $t > 0$). Following our train of logic, we see that $f(t_m) + g(t_m) < f(t) + g(t)$ for all $t \neq t_m$, which means that t_m is a minimum of $f(t) + g(t)$. Thus by finding a minimum t_m to $k(f(t)) + k(g(t))$, we also found a minimum to $f(t) + g(t)$. □

Since we've proven this lemma, we can invoke it upon $\min l_3 + l_4$. Since $l_3 = \sqrt{z_3}$ and $l_4 = \sqrt{z_4}$, we can use $k(t) = \sqrt{t}$ and we can write $\min l_3 + l_4 = \min z_3 + z_4$ by using our lemma. Thus, we now want to solve the problem:

$$\arg\min_\theta 2\left(\frac{l_1}{2} - \frac{l_2}{2}\sin\theta\right)^2 + \left(d - \frac{l_2}{2}\cos\theta\right)^2 + \left(d + \frac{l_2}{2}\cos\theta\right)^2 \tag{21}$$

Now, we can expand out our expression and use the fact that $\sin^2\theta + \cos^2\theta = 1$ to obtain a much simpler (but equivalent) minimization problem:

$$\arg\min_{\theta} \frac{l_1^2}{2} + \frac{l_2^2}{2} + 2d^2 - l_1 l_2 \sin\theta \tag{22}$$

We note that l_1, l_2, and d are all constants which are given to us in the problem. Therefore, the minimization problem really boils down to

$$\arg\min_{\theta} -l_1 l_2 \sin\theta = \arg\max_{\theta} \sin\theta \tag{23}$$

Thus, we see that $\theta = \frac{\pi}{2}$ minimizes the perimeter of the quadrilateral in the symmetric case.

References

[1] http://en.wikipedia.org/wiki/Polar_coordinate_system