OPTIMAL PARTICLE PATHS AROUND POINTS IN $\mathbb{R}^2$ WITH CONSTRAINED ACCELERATION

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Abstract. Optimal particle paths in $\mathbb{R}^2$ between points and around points are investigated in this paper. A paramerization of a particle's motion around a point is defined. From this parameterization, we derive a general solution for an optimal path with constant speed between 2 points. The results of this paper can by applied towards calculating optimal object trajectories in physics when acceleration is constrained.

1. Introduction

In this paper, we shall examine optimal paths for a particle.

Our model problem is a particle moving in $\mathbb{R}^2$ with constrained acceleration. The particle must navigate around cones (points in $\mathbb{R}^2$) to reach a final position. The goal is to minimize the total time spend navigating to the final position.

We first present a polar coordinate representation of a particle's position around a point and derive a set of differential equations governing the motion of the particle in this coordinate system. This allows us to devise some simple lemmas about the motion of the particle.

After developing an intuition for particles and particle paths, we then move on to the more complicated problem of deriving governing rules for optimal paths between points.

Finally, we tackle the problem of finding an optimal path around cones.

1.1. Motivation.

The problem of finding an optimal trajectory is interesting because of its applications in physics. There are many instances where finding optimal paths is important. For instance, a race car driver wants to know the fastest way to get from one point to another so that he can win his race. For another example, imagine you are sending a spacecraft to a particular destination in space and would like the fastest means of getting there.

The space example directly motivates our model problem. On a spacecraft, there is a limited amount of acceleration that is possible

Date: October 7, 2013.

(provided by thrusters). One would potentially like to navigate to a planet while moving around obstacles.

Understanding how to find optimal trajectories will provide greater insight into solving problems like these.

2. Notation

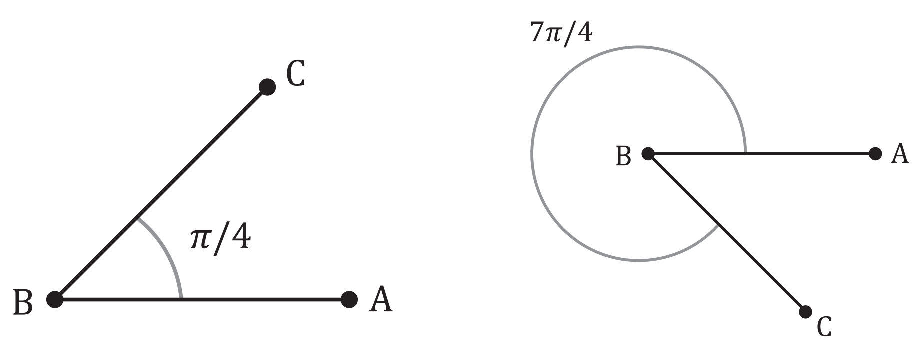
In this section, we will give some basic defintions which we will use throughout the paper. We will define our notational conventions here.

2.1. **Vectors.**

a**:** Scalar quantity.
a: Vector in n-dimensional space. $\|\mathbf{a}\| = a$.
â: Unit vector in n-dimensional space. $\hat{\mathbf{a}} = \mathbf{a}/a$ and $\|\hat{\mathbf{a}}\| = 1$.

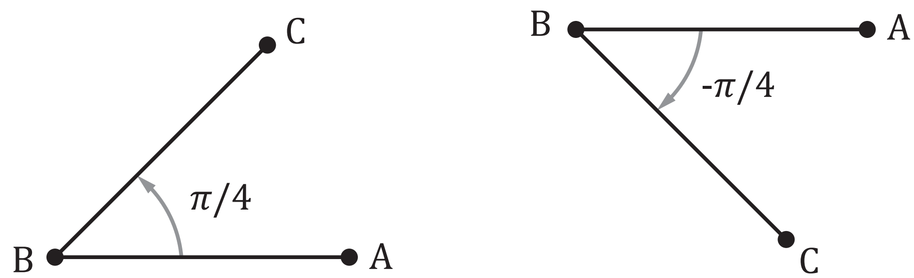
2.2. **Angles.** We define two different types of angle measurements: a standard measurement, and a directional measurement (all angles are measured in radians).

In the standard measurement, angles are in $[0, 2\pi]$ and are measured counterclockwise. In the directional measurement, angles are in $[-\pi, \pi]$ and the measurement direction is indicated by an arrow on the measurement.



(A) (B)

FIGURE 1. Standard angle notation (no arrow)



(A) (B)

FIGURE 2. Directional angle notation (arrow)

3. Particles and Paths

In this section, we will provide some basic definitions about particles and paths. These will lay the groundwork for thinking about optimal paths.

Definition 3.1. *A n-dimensional path* $\gamma(t) : \mathrm{R} \to \mathrm{R}^n$ *is a function which maps a time* $t \in \mathbb{R}$, $t \in [T_0(\gamma), T_f(\gamma)]$, *to a position* $\mathbf{X} \in \mathrm{R}^n$.

Definition 3.2. *A planar path is a path in* $\mathbb{R}^2$.

Definition 3.3. *A path between two points,* $\mathbf{X_1}$ *and* $\mathbf{X_2}$ *is a path,* $\gamma(t)$, *where* $\gamma(T_0(\gamma)) = \mathbf{X_1}$ *and* $\gamma(T_f(\gamma)) = \mathbf{X_2}$.

Definition 3.4. *A particle, p, is an object in space with zero volume. A particle travels along a path,* $\gamma(t)$ *if the particle is at position* $\gamma(t)$ *at time t, for all* $t \in [T_0(\gamma), T_f(\gamma)]$. *If a particle is traveling along a path,* $\gamma(t)$, *then we define a position,* $\mathbf{X}(t) := \gamma(t)$, *a velocity,* $\mathbf{v}(t) := \frac{d\gamma(t)}{dt}$, *and an acceleration,* $\mathbf{a}(t) := \frac{d^2\gamma(t)}{dt^2}$, *for the particle, for all* $t \in [T_0(\gamma), T_f(\gamma)]$.

A particle is restricted if there are conditions on its position and the time derivatives of its position. Given a particle, a valid path is a path that the particle can travel along.

Definition 3.5. *Given a particle, a fastest path,* $\hat{\gamma}(t)$, *between two points,* $\mathbf{X_1}$ *and* $\mathbf{X_2}$, *is a valid path such that* $T_f(\hat{\gamma}) \leq T_f(\gamma)$ *for all valid paths,* $\gamma(t)$, *between* $\mathbf{X_1}$ *and* $\mathbf{X_2}$.

Definition 3.6. *A particle's speed is* $v(t)$, *and its direction of motion is* $\hat{\mathbf{v}}$.

Definition 3.7. *The centripetal acceleration,* $\mathbf{a_c}$*, of a particle,* p*, is the component of its acceleration in the direction perpendicular to its direction of motion.*

In rectangular coordinates, the sign of a_c *is defined to be the sign of the projection of* $\hat{\mathbf{a_c}}$ *onto* $\hat{\mathbf{x}}$*.*

In polar coordinates, the sign of a_c *is defined to be the sign of the projection of* $\hat{\mathbf{a_c}}$ *onto* $\hat{\mathbf{r}}$*.*

Definition 3.8. *The tangential acceleration,* a_t*, of a particle, is the component of the acceleration of the particle in its direction of motion.*

$$a_t := \frac{dv}{dt}$$

3.1. **Particle Motion in Polar Coordinates.** Unless otherwise specified, the motion of particle moving along planar paths will be described in polar coordinates, along with an extra parameter θ, as is shown in Figure 3.

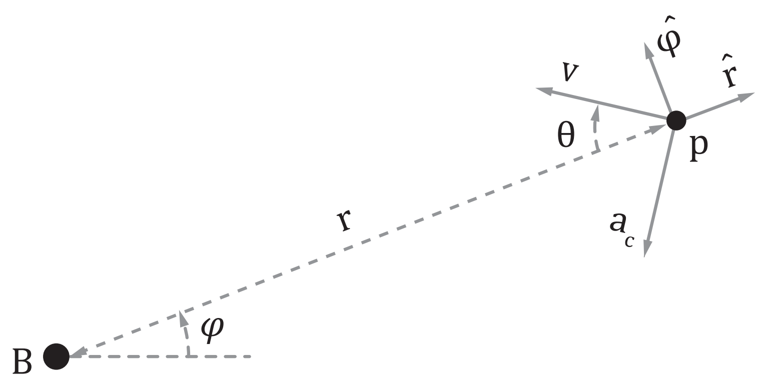


FIGURE 3. A particle moving in 2-dimensional polar coordinate system centered at B.

$\theta(t), \phi(t) \in [-\pi, \pi]$. From now on, $|\theta(t)|$ and $|\phi(t)|$ will be used, since there is a symmetry in the system about $\hat{\mathbf{r}}$.

Lemma 3.9. *The time derivative of* θ *is given by*

$$\frac{d\,|\theta(t)|}{dt} = \frac{a_c}{v}$$

Proof. If we look at a point, p, subject to only centripetal acceleration, a_c, the change in $\mathbf{v}$ over an infinitesimal time, dt, is shown in Figure 4 (the two vectors, $\mathbf{v}$ and $\mathbf{v} + \mathbf{dv}$, are superimposed).

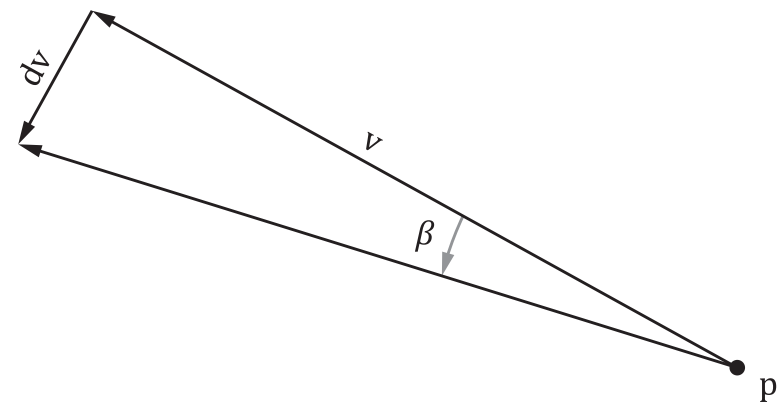


FIGURE 4. Tangential acceleration.

From the definition of a_c

$$\frac{dv}{dt} = a_c$$

Since $dv = d\,|\theta|\,v$, then

$$\frac{d\,|\theta|}{dt} = \frac{a_c}{v}$$

□

The proof of the lemma in the case where a_t is also nonzero is very similar, since the component of $\mathbf{dv}$ in the direction $\hat{\mathbf{v}}$ is negligible compared to v.

Returning again to Figure 3, the following equations can be derived

$$\frac{dr(t)}{dt} = -v(t)\,\cos\left(|\theta(t)|\right) \tag{1}$$

$$\frac{d\,|\phi(t)|}{dt} = \frac{v(t)}{r(t)}\,\sin\left(|\theta(t)|\right) \tag{2}$$

$$\frac{d\,|\theta(t)|}{dt} = \frac{d\,|\phi(t)|}{dt} - \frac{a_c(t)}{v(t)} \tag{3}$$

$$= \frac{v(t)}{r(t)}\,\sin\left(|\theta(t)|\right) - \frac{a_c(t)}{v(t)} \tag{4}$$

Applying the chain rule to (1)

$$\frac{d}{dt}\frac{dr(t)}{dt} = -\frac{dv(t)}{dt}\cos\left(|\theta(t)|\right) + v(t)\sin\left(|\theta(t)|\right)\frac{d\,|\theta(t)|}{dt} \tag{5}$$

$$\begin{aligned} &= -a_t\cos\left(|\theta(t)|\right) + \frac{v(t)^2}{r(t)}\sin^2\left(|\theta(t)|\right) \\ &\quad - v(t)\sin\left(|\theta(t)|\right)\frac{a_c(t)}{v(t)} \end{aligned} \tag{6}$$

Lemma 3.10. *Given a particle with the following restrictions: $a = 0$, $v(t) = \bar{v}$, for all $t \geq T_0$then the following equations hold*

$$\begin{cases} |\theta(t)| \to \pi \quad as \quad t \to \infty & if\ \theta(T_0) > 0 \\ |\theta(t)| = 0 \quad for\ all \quad t \geq T_0 & if\ \theta(T_0) = 0 \end{cases}$$

Furthermore,

$$\begin{cases} r(t) \to \infty \quad as \quad t \to \infty & if\ \theta(T_0) > 0 \\ r(t) \to 0 \quad as \quad t \geq T_0 & if\ \theta(T_0) = 0 \end{cases}$$

Proof. The first part of the lemma is proven by referring to (4), and noting that $\frac{d|\theta(t)|}{dt} = 0$ when $|\theta(t)| = 0, \pi$, and $\frac{d|\theta(t)|}{dt} > 0$ otherwise. If $\theta(T_0) \neq 0$, then $\theta(t)$ is bounded above by π and strictly increasing, because its derivative is strictly positive, so $|\theta(t)| \to \pi \quad \text{as} \quad t \to \infty$. If $\theta(T_0) = 0$, then $|\theta(t)| = 0 \quad \text{for all} \quad t \geq T_0$.

For the first case of the second part of the lemma,

$$\frac{d\,|\theta(t)|}{dt} = 0 \qquad \text{for } t \geq t_0$$

So

$$|\theta(t)| = 0 \qquad \text{for } t \geq t_0$$

[TODO]

□

4. Traveling Between Points

Now that we have defined our notation, we are ready to think about the problem of optimal trajectories.

Definition 4.1. *A fixed speed particle is a particle with the following conditions:* $\mathbf{X} \in \mathbb{R}^2$, $a_t = 0$, $v(t) = \bar{v}$.

Definition 4.2. *A particle with bounded centripetal acceleration is a particle with the following condition:* $\|a_c\| \leq \bar{a}_c$*, for some* $\bar{a}_c \geq 0$.

The simplest problem to tackle first, is the optimal trajectory from a starting position, $\mathbf{X_0}$ to a final position, $\mathbf{X_f}$. Working in a polar coordinate system centered on X_f, it is clear, that our goal is achieved when $r = 0$. Furthermore, $\theta(t)$ must be 0 before reaching X_f, otherwise, it will grow exponentially fast as $r(t) \to 0$, which can be seen in equation (1).

Lemma 4.3. *If* $a_t = 0$ *and* $a_c \leq a_{c,max}$*, then the minimum radius of curvature of a particle's trajectory is given by*

$$R_{min} = \frac{v^2}{a_{c,max}} \tag{7}$$

Proof. This comes directly from the definition of radius of curvature

$$R = \frac{v^2}{a_c}$$

and the fact that

$$a_c \leq a_{c,max} \to \frac{1}{a_c} \geq \frac{1}{a_{c,max}}$$

□

Theorem 4.4. *A 2-dimensional particle, p, traveling along a fastest path,* $\hat{\gamma(t)}$*, from* X_1 *to* X_2 *has* $\frac{d|\theta(t)|}{dt} \leq 0$ *for all* $t \in [0, T_{f,\gamma}]$*, unless* $\|X_1 - X_2\| < 2R_{min}$.

Proof. Since $\theta(T_{f,\gamma}) = 0$, then if $\frac{d|\theta(t)|}{dt} > 0$ for $t \in [t_1, t_2]$, since $\theta T_{f,\gamma} < \theta t_1$, then by Lemma **??** $\theta(t_1) = \theta(t_2)$ for some $t_2 \in [t_1, T_{f,\gamma}$. Since all of the time derivatives in (1) - (4) are all monotonic for $r \in (0, \infty)$, there is no local maximum for them, so the optimal strategy should be the same, regardless of r. In other words, there can not be multiple cases for optimal strategies based on r. □

Now we will show that the fastest path between two points goes in a straight line if there is no initial speed. This will formalize the natural intuition that straight lines are the fastest path when there are no obstacles between the start and finish position.

Theorem 4.5. *Given a particle* $p \in \mathrm{R}^2$ *with initial position of* $\mathbf{X_1}$*, no initial speed, and moves with bounded tangential acceleration,* $a_t \leq \bar{a}_t$*, and infinite centripetal acceleration, the fastest path* $\hat{\gamma}(t)$ *which* p *can trace from* $\mathbf{X_1}$ *to* $\mathbf{X_2}$ *lies on the line segment from* $\mathbf{X_1}$ *to* $\mathbf{X_2}$*:*

Proof. Without loss of generality, we can define a cartesian coordinate system, where the origin is at $\mathbf{X_1}$ and the positive x-axis passes through $\mathbf{X_2}$. Let us say that the two points in our cartesian coordinate system become $(0, 0)$ and $(x_2, 0)$ respectively.

Now let us examine the particle's motion in the $\hat{\mathbf{x}}$ direction. The speed of the particle is the following

$$v_x(t) = \int_0^t a_x(t_1)dt_1 \tag{8}$$

To find the distance $l_x(t)$ travelled in the $\hat{\mathbf{x}}$ direction, we can use the relation:

$$l_x(t) = \int_0^t v_x(t_2)dt_2 \tag{9}$$

$$= \int_0^t \int_0^t a_x(t_1)dt_1dt_2 \tag{10}$$

Recall that the tangential acceleration of the point mass p is bounded by $\bar{a}_t$. This means that $a_t(t) \leq \bar{a}_t$ for all t. Therefore, we see:

$$d(t) \leq \int_0^t \int_0^t \bar{a}_t dt_1dt_2 \tag{11}$$

$$= \frac{\bar{a}_t t^2}{2} \tag{12}$$

Thus, in order to travel a distance of $l_x(T_f(\hat{\gamma})) = d' = d(\mathbf{X_2}, \mathbf{X_1})$, it needs to be the case that $T_f(\hat{\gamma}) \geq \sqrt{\frac{2d'}{\bar{a}_t}}$. It is possible to travel from $\vec{X_2}$ to $\vec{X_1}$ in time $T_{min} = \sqrt{\frac{2d'}{\bar{a}_t}}$ if and only if $a_t(t) = \bar{a}_t$ and $\theta(t) = 0$ for all $t \in [T_0(\gamma), T_{min}]$. In other words, the particle must be accelerating at the maximum possible tangential acceleration of $\bar{a}_t$ and it must be accelerating in the straight line direction to $\vec{X_2}$ at all times.

If the particle travels for time $t < T_{min}$, then it is impossible for the point mass to reach $\vec{X_2}$ when starting at $\vec{X_1}$. This is because p cannot reach $\vec{X_2}$ in the x direction when $t < \sqrt{\frac{2d'}{\bar{a}}}$ because it is impossible for p to reach any point whose x-coordinate is x_2, so it is obviously impossible to reach $(x_2, 0) = \vec{X_2}$.

Moreover, if there exists some time $t < T_{min}$ when $\theta(t) \neq 0$, then it is also impossible for the point mass to reach $\vec{X_2}$. Suppose that there are two particles p_1 and p_2 both accelerating at $a_t(t) = \bar{a}_t$. Imagine p_1 satisfies $\theta_{p_1}(t) = 0$ for all t and that p_2 satisfies $\theta_{p_2}(t) = 0$ except for times in some interval $[t_0, t_1]$, the particle p_2 sets a constant non-zero angle $\zeta_{p2}(t) \neq 0$ with the x-axis for $t \in [t_0, t_1]$. Denote d_p as the distance that is left to be travelled by particle p between times t_1 and T_{min}. We can find d_{p_2} in terms of d_{p_1} by using trigonometry:

$$d_{p_2}^2 = d_{p_1}^2 + \left(\frac{\bar{a}_t (t_1 - t_0)^2}{2} \sin \zeta_{p_2} \right)^2 \tag{13}$$

Since $d_{p_2} \geq 0$ by being a distance, $\sin \zeta_{p_2} \neq 0$ by setting $\zeta_{p_2} > 0$, and $t_1 - t_0 > 0$, we know that $d_{p_2} > d_{p_1}$. This means that at time T_{min} when p_1 has reached its destination at $\vec{X}_2$, particle p_2 still has $d_{p_2} - d_{p_1}$ distance left to travel. This means that it is impossible for $\theta(t) > 0$ on the fastest path.

This means that the fastest path is completed in time $T_f(\hat{\gamma}) = \sqrt{\frac{2x_2}{\bar{a}}}$. Let us examine the path taken by the point mass p on this fastest path. Recall that $a_t(t) = \bar{a}$ for all t along the fastest path. This means that there was no centripetal acceleration $|a_c| = 0$. In other words, the point mass never turned on its way to reaching the destination point. The only way this could have happened is if it travelled along the x axis in a straight line.

Thus, we see that the fastest path is the straight line between $\vec{X}_1$ and $\vec{X}_2$. $\square$

Corollary 4.6. *Given points $\mathbf{X_1}, \mathbf{X_f} \in \mathrm{R}^2$ and a particle p whose initial position is $\mathbf{X_1}$ which moves under conditions set forth in 4.5, the fastest path $\hat{\gamma}(t)$ which p can trace to $\mathbf{X_f}$ is unique.*

Proof. We have already shown that any fastest path between $\mathbf{X_1}$ and $\mathbf{X_f}$ follows the straight line between them. Moreover, we showed that when travelling along the fastest path, the particle must have tangential acceleration of $\bar{a}_t$. Since we have starting position $\vec{X}_1$ and initial speed of 0, the acceleration of the particle uniquely defines a path for the particle.

There is only a single function $a(t) = \bar{a}_t$ which the acceleration can satisfy when the particle is moving along a fastest path, therefore, there is only a single possible fastest path. $\square$

5. Constant Speed Conditions

Conjecture 5.1. *Let particles p_1, p_2 have initial velocities $\mathbf{v}_1, \mathbf{v}_2$ and starting positions $\mathbf{x}_1, \mathbf{x}_2$ respectively. Additionally, let p_1, p_2 have constant speed conditions and let the end position be $\mathbf{x}_f$ such that $d(\mathbf{x}_f, \mathbf{x}_1) = d(\mathbf{x}_f, \mathbf{x}_2) > R_{min}$. If $d(\mathbf{x}_1, \mathbf{x}_f) \leq d(\mathbf{x}_2, \mathbf{x}_f)$ and $\theta_{p_1}(T_0) < \theta_{p_2}(T_0)$, then $T_f(\hat{\gamma_\mathbf{1}}) < T_f(\hat{\gamma_\mathbf{2}})$.*

Theorem 5.2. *Let particle p have initial velocity $\mathbf{v}$ and starting position $\mathbf{x}_0$. Let p have constant speed conditions and let the end position be $\mathbf{x}_f$ such that $d(\mathbf{x}_f, \mathbf{x}_0) > R_{min}$. The fastest path $\hat{\gamma_{\mathbf{p}}}$ for particle p to reach $\mathbf{x}_f$ minimizes $\theta(t)$ for all $t \in [T_0, T_f(\hat{\gamma_{\mathbf{p}}})]$.*

Corollary 5.3. *The fastest path $\hat{\gamma_{\mathbf{p}}}$, given the conditions set forth in theorem 5.2, will trace out a path which contains an arc of a circle with radius $\bar{v}^2/\bar{a}^2$ connected to a straight line tangent to the arc.*

Corollary 5.4. *Given conditions set forth in theorem 5.2, we have:*

$$T_f(\hat{\gamma_{\mathbf{p}}}) = v_0 \left(\left(\pi - \arccos \frac{R_{min}}{d - R_{min}} \right) 2\pi R_{min} + \sqrt{d^2 - 2R_{min} d} \right) \tag{14}$$

6. Conclusion

In this paper, we have formalized many of the things that intuition would tell us. Namely, we have shown that a straight line is the fastest way to get between two points. This fact is unsurprising because of the fact that acceleration in a single direction (toward the finish point) is the least wasteful means of getting to the finish.

We developed a number of simple lemmas governing the motion of a volume-less particle with bounded acceleration. We also showed that symmetric paths tend to be better than non-symmetric paths (through our simple quadrilateral path minimization problem).

Finally, we found an optimal path for a particle to trace around three cones when the particle has constant velocity.

Appendix

6.1. Analysis.

Lemma 6.1. *If $f(t), g(t) > 0$ and $k(t) > 0$ is a strictly monotonically increasing function for all t, then $\arg\min_t k(f(t)) + k(g(t)) = \arg\min_t f(t) + g(t)$.*

Proof. Let t_1, t_2 be such that $f(t_1) + g(t_1) < f(t_2) + g(t_2)$. In this proof, we will show that $k(f(t_1)) + k(g(t_1)) < \sqrt{f(t_2)} + \sqrt{g(t_2)}$. Since $k(t)$ is a strictly monotonically increasing function when $t > 0$, we know that $k(x) < k(y)$ if and only if $x < y$ (assuming we can confine x, y to be non-negative).

Because this is the case, we see that $k(f(x)) < k(f(y))$ if and only if $f(x) < f(y)$ (the same goes for g). Thus, we see that if we have found the minimum t_m to $\arg\min_t k(f(t)) + k(g(t))$, then it is the case that $k(f(t_m)) + k(g(t_m)) < k(f(t)) + k(g(t))$ for all $t \neq t_m$ (again where $t > 0$). Following our train of logic, we see that $f(t_m) + g(t_m) < f(t) + g(t)$

for all $t \neq t_m$, which means that t_m is a minimum of $f(t) + g(t)$. Thus by finding a minimum t_m to $k(f(t))+k(g(t))$, we also found a minimum to $f(t) + g(t)$. □

6.2. Coordinate Systems.

6.2.1. *Radius of Curvature.* The radius of curvature, R, of a curve at a point is a measure of the radius of the circular arc which best approximates the curve at that point.

For $a_t = 0$

$$R = \left| \frac{v^2}{a_c} \right|$$

6.3. Vector Calculus in Polar Coordinates.

$$\boldsymbol{x} = r\hat{\boldsymbol{r}}$$

$$\vec{\boldsymbol{v}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}$$

$$\vec{\boldsymbol{a}} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\boldsymbol{r}} + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right)\hat{\boldsymbol{\phi}}$$

References

[1] http://en.wikipedia.org/wiki/Polar_coordinate_system