OPTIMAL PARTICLE PATHS AROUND POINTS IN $\mathbb{R}^2$ WITH CONSTRAINED ACCELERATION

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Abstract. Optimal particle paths in $\mathbb{R}^2$ between points and around points are investigated in this paper. A paramerization of a particle's motion around a point is defined. From this parameterization, we derive a general solution for an optimal path with constant speed between 2 points. The results of this paper can by applied towards calculating optimal object trajectories in physics when acceleration is constrained.

1. INTRODUCTION

In this paper, we shall examine optimal paths for a particle.

Our model problem is a particle moving in $\mathbb{R}^2$ with constrained acceleration. The particle must navigate around cones (points in $\mathbb{R}^2$) to reach a final position. The goal is to minimize the total time spend navigating to the final position.

We first present a polar coordinate representation of a particle's position around a point and derive a set of differential equations governing the motion of the particle in this coordinate system. This allows us to devise some simple lemmas about the motion of the particle.

After developing an intuition for particles and particle paths, we then move on to the more complicated problem of deriving governing rules for optimal paths between points.

Finally, we tackle the problem of finding an optimal path around cones.

1.1. **Motivation.** The problem of finding an optimal trajectory is interesting because of its applications in physics. There are many instances where finding optimal paths is important. For instance, a race car driver wants to know the fastest way to get from one point to another so that he can win his race. For another example, imagine you are sending a spacecraft to a particular destination in space and would like the fastest means of getting there.

The space example directly motivates our model problem. On a spacecraft, there is a limited amount of acceleration that is possible

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(provided by thrusters). One would potentially like to navigate to a planet while moving around obstacles.

Understanding how to find optimal trajectories will provide greater insight into solving problems like these.

2. Notation

In this section, we will give some basic defintions which we will use throughout the paper. We will define our notational conventions here.

2.1. **Vectors.**

a: Scalar quantity.
a: Vector in n-dimensional space. $\|\mathbf{a}\| = a$.
â: Unit vector in n-dimensional space. $\hat{\mathbf{a}} = \mathbf{a}/a$ and $\|\hat{\mathbf{a}}\| = 1$.

2.2. **Angles.**
We define two different types of angle measurements: a standard measurement, and a directional measurement (all angles are measured in radians).

In the standard measurement, angles are in $[0, 2\pi]$ and are measured counterclockwise. In the directional measurement, angles are in $[-\pi, \pi]$ and the measurement direction is indicated by an arrow on the measurement.
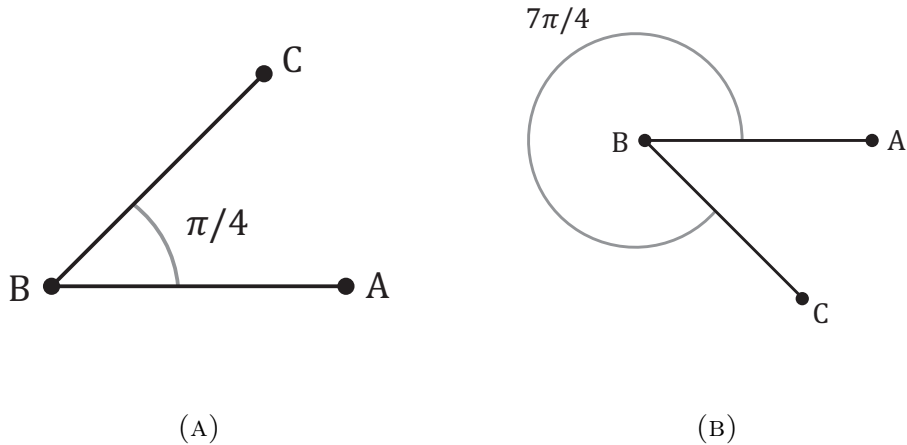

(A)   (B)

Figure 1. Standard angle notation (no arrow)
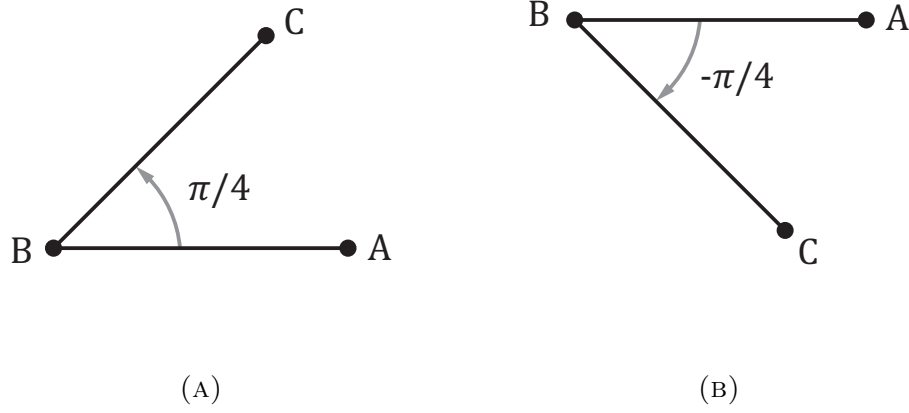
(A)   (B)

FIGURE 2. Directional angle notation (arrow)

3. PARTICLES AND PATHS

In this section, we will provide some basic definitions about particles and paths. These will lay the groundwork for thinking about optimal paths.

Definition 3.1. *A n-dimensional particle, p, is an object with zero volume that travels along a n-dimensional path. Traveling along a path means that the particle is at position $\lambda(t)$ at time t, for all $t \in [0, T_{f,\lambda}]$.*

Definition 3.2. *A description of a particle p is a set $D(t)$ which contains a particle's position $\vec{x} \in \mathrm{R}^n$ and its derivatives at some time t.*

Definition 3.3. *A condition c on a particle p is a boolean function $c : \mathrm{D} \to \{0, 1\}$ which takes as input a description $D(t)$ of the particle p at some time t. We call a condition satisfied if the condition's output is 1 at time t.*

Definition 3.4. *A n-dimensional path $\gamma(t) : \mathrm{R} \to \mathrm{R}^n$ is a function which maps a time $t \in \mathbb{R}$, $t \in [0, T_{f,\gamma}]$, to a position $\mathbf{X} \in \mathrm{R}^n$.*

Definition 3.5. *A valid path $\gamma(t)$ for a particle p is a path along which p can travel, such that all its conditions are satisfied at every point along the path.*

Definition 3.6. *A path between two points, $\mathbf{X_1}$ and $\mathbf{X_2}$ is a path, $\gamma(t)$, where $\gamma(0) = \mathbf{X_1}$ and $\gamma(T_{f,\gamma}) = \mathbf{X_2}$.*

Definition 3.7. *For a given particle, p, a fastest path, $\hat{\gamma}(t)$, between two points, $\mathbf{X_1}$ and $\mathbf{X_2}$, is a valid path such that $T_{f,\hat{\gamma}} \leq T_{f,\gamma}$ for all valid paths, $\gamma(t)$, between $\mathbf{X_1}$ and $\mathbf{X_2}$.*

Definition 3.8. *A particle's velocity is defined as*

$$\mathbf{v} := \frac{d\mathbf{X}}{dt}$$

Furthermore, a particle's speed is v, and its direction of motion is $\hat{\mathbf{v}}$

Definition 3.9. *A particle's acceleration is defined as*

$$\mathbf{a} := \frac{d\mathbf{v}}{dt}$$

Definition 3.10. *The centripetal acceleration, $\mathbf{a_c}$, of a particle, p, is the component of its acceleration in the direction perpendicular to its direction of motion.*

In rectangular coordinates, the sign of a_c is defined to be the sign of the projection of $\hat{\mathbf{a}}_\mathbf{c}$ onto $\hat{\mathbf{x}}$.

In polar coordinates, the sign of a_c is defined to be the sign of the projection of $\hat{\mathbf{a}}_\mathbf{c}$ onto $\hat{\mathbf{r}}$.

Definition 3.11. *The tangential acceleration, $\mathbf{a_t}$, of a particle, p, is the component of the acceleration of p in its direction of motion, $\hat{\mathbf{v}}$.*

$$a_t = \frac{dv}{dt}$$

3.1. **Particle Motion in Polar Coordinates.** Unless otherwise specified, the motion of 2-dimensional particles in this paper will be described in polar coordinates, along with an extra parameter θ as is shown in Figure 3.
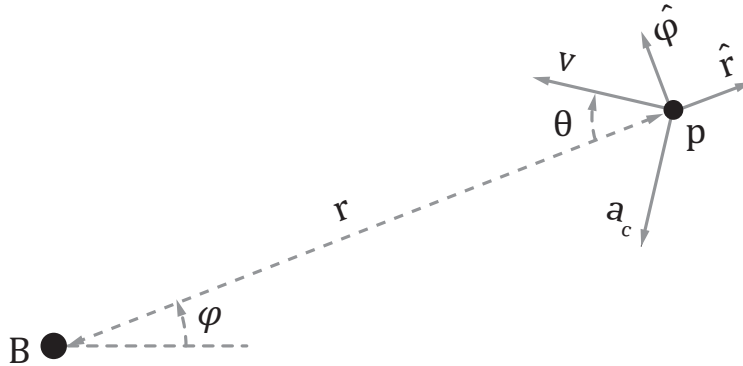

FIGURE 3. A particle moving in 2-dimensional polar coordinate system centered at B.

$\theta(t), \phi(t) \in [-\pi, \pi]$. From now on, $|\theta|$ and $|\phi|$ will be used, since there is a symmetry in the system about $\hat{\mathbf{r}}$.

Lemma 3.12. *The time derivative of θ is given by*

$$\frac{d\,|\theta(t)|}{dt} = \frac{a_c}{v}$$

Proof. If we look at a point, p, subject to only centripetal acceleration, a_c, the change in $\mathbf{v}$ over an infinitesimal time, dt, is shown in Figure 4 (the two vectors, $\mathbf{v}$ and $\mathbf{v} + \mathbf{dv}$, are superimposed).
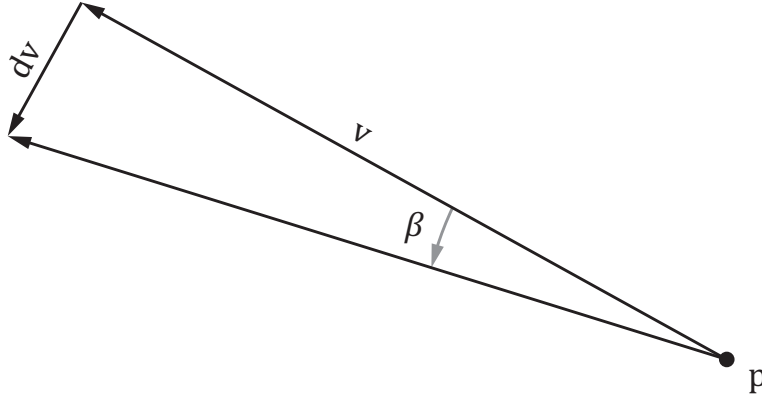

FIGURE 4. Tangential acceleration.

From the definition of a_c

$$\frac{dv}{dt} = a_c$$

Since $dv = d\,|\theta|\,v$, then

$$\frac{d\,|\theta|}{dt} = \frac{a_c}{v}$$

$\square$

The proof of the lemma in the case where a_t is also nonzero is very similar, since the component of $\mathbf{dv}$ in the direction $\hat{\mathbf{v}}$ is negligible compared to v.

Returning again to Figure 3, the following equations can be derived

$$(1) \qquad \frac{dr(t)}{dt} = -v(t) \, \cos\left(|\theta(t)|\right)$$

$$(2) \qquad \frac{d\,|\phi(t)|}{dt} = \frac{v(t)}{r(t)} \, \sin\left(|\theta(t)|\right)$$

$$(3) \qquad \frac{d\,|\theta(t)|}{dt} = \frac{d\,|\phi(t)|}{dt} - \frac{a_c(t)}{v(t)}$$

$$(4) \qquad \qquad = \frac{v(t)}{r(t)} \, \sin\left(|\theta(t)|\right) - \frac{a_c(t)}{v(t)}$$

Applying the chain rule to (1)

$$(5) \qquad \frac{d}{dt}\frac{dr(t)}{dt} = -\frac{dv(t)}{dt} \cos\left(|\theta(t)|\right) + v(t)\sin\left(|\theta(t)|\right)\frac{d\,|\theta(t)|}{dt}$$

$$= -\frac{dv(t)}{dt} \cos\left(|\theta(t)|\right) + \frac{v(t)^2}{r(t)} \sin^2\left(|\theta(t)|\right)$$

$$(6)$$

$$- v(t)\sin\left(|\theta(t)|\right)\frac{a_c(t)}{v(t)}$$

Lemma 3.13. *For a particle, p, with bounded centripetal and tangential acceleration, then:*

  1. The function $|\theta(t)|$ is continuous.

  2. Given $t_1, t_2 \in \mathbb{R}^2$, s.t. $t_1 < t_2$, $\theta(t_1) > a$ and $\theta(t_2) < a$, then $\theta(t_c) = a$ for some $t_c \in [t_1, t_2]$.

Proof. First off, it should be noted that a solution to (4) exists because the right hand side is Lipschitz continuous. The proof of 1. follows directly from the fact that the derivative of $|\theta(t)|$ exists and is bounded. 2. is just a restatement of the intermediate value theorem. $\qquad \square$

Lemma 3.14. *For a particle, p, with nonzero speed, and zero centripetal acceleration for $t \geq t_0$, then*

$$\begin{cases} |\theta(t)| \to \pi & as \quad t \to \infty & if \ \theta(t_0) > 0 \\ |\theta(t)| = 0 & for \ all \quad t \geq t_0 & if \ \theta(t_0) = 0 \end{cases}$$

  Furthermore,

$$\begin{cases} r(t) \to \infty & as \quad t \to \infty & if \ \theta(t_0) > 0 \\ r(t) \to 0 & as \quad t \geq t_0 & if \ \theta(t_0) = 0 \end{cases}$$

Proof. For the first case, $\frac{d|\theta(t)|}{dt} > 0$, except when $|\theta(t)| = 0, \pi$, which means that $\theta(t)$ is monotonically increasing. Since $\phi(t)$ is bounded above py π, $\phi(t) \to \pi$ as ∞.

The second case is the easiest to check. If we plug $\theta(t_0) = 0$ into 4, we get

$$\frac{d\,|\theta(t)|}{dt} = 0 \qquad \text{for } t \geq t_0$$

So

$$|\theta(t)| = 0 \qquad \text{for } t \geq t_0$$

[TODO]

$\square$

4. Traveling Between Points

Now that we have defined our notation, we are ready to think about the problem of optimal trajectories.

The simplest problem to tackle first, is the optimal trajectory from a starting position, $\mathbf{X_0}$ to a final position, $\mathbf{X_f}$. Working in a polar coordinate system centered on X_f, it is clear, that our goal is achieved when $r = 0$. Furthermore, $\theta(t)$ must be 0 before reaching X_f, otherwise, it will grow exponentially fast as $r(t) \to 0$, which can be seen in equation (1).

Lemma 4.1. *If $a_t = 0$ and $a_c \leq a_{c,max}$, then the minimum radius of curvature of a particle's trajectory is given by*

$$(7) \qquad\qquad R_{min} = \frac{v^2}{a_{c,max}}$$

Proof. This comes directly from the definition of radius of curvature

$$R = \frac{v^2}{a_c}$$

and the fact that

$$a_c \leq a_{c,max} \rightarrow \frac{1}{a_c} \geq \frac{1}{a_{c,max}}$$

$\square$

Theorem 4.2. *A 2-dimensional particle, p, traveling along a valid path, $\hat{\gamma}(t)$, from X_1 to X_2 has $\frac{d|\theta(t)|}{dt} < 0$ for all $t \in [0, T_{f,\gamma}]$, unless $\|X_1 - X_2\| < 2R_{min}$.*

Proof. Since $\theta(T_{f,\gamma}) = 0$, then if $\frac{d|\theta(t)|}{dt} > 0$ for $t \in [t_1, t_2]$, since $\theta T_{f,\gamma} < \theta t_1$, then by Lemma 3.13 $\theta(t_1) = \theta(t_2)$ for some $t_2 \in [t_1, T_{f,\gamma}]$. Since all of the time derivatives in (1) - (4) are all monotonic for $r \in (0, \infty)$, there is no local maximum for them, so the optimal strategy should be the same, regardless of r. In other words, there can not be multiple cases for optimal strategies based on r.   □

Theorem 4.3. *Given a 2-dimensional particle, p, it is always optimal to minimize $\frac{d|\theta(t)|}{dt}$ at all $t \in [0, T_\gamma]$.*

TODO.   □

Theorem 4.4. *Given a 2-dimensional particle, p, with bounded centripetal acceleration, $a_c \leq a_{c,max}$ and zero tangential acceleration, every valid optimal path, $\hat{\gamma}(t)$ between two points X_1 and X_2, has a radius of curvature of 0 or $a_{c,max}$ at every point along it.*

TODO.   □

Corollary 4.5. *Given a 2-dimensional particle, p, with bounded centripetal acceleration, $a_c \leq a_{c,max}$ and zero tangential acceleration, every valid optimal trajectory is constructed from line segments and circular sections of radius R_{min}, that are tangent at their intersections.*

TODO.   □

Now we will show that the fastest path between two points goes in a straight line. This will formalize the natural intuition that straight lines are the fastest path when there are no obstacles between the start and finish position.

Theorem 4.6. *Given a 2-dimensional particle p whose initial position is X_1 and moves with bounded tangential acceleration, $a_t \leq a_{t,max}$, and infinite centripetal acceleration, the fastest path $\hat{\gamma}(t)$ which p can trace from X_1 to X_2 lies on the line segment from X_1 to X_2:*

Proof. Without loss of generality, we can define a cartesian coordinate system, where the origin is at X_1 and the positive x-axis passes through X_2.

Now let us examine the particle's motion in the $\hat{x}$ direction. The speed of the particle is the following

$$(8) \qquad\qquad v_x(t) = \int_0^t a_x(t_1)dt_1$$

To find the distance $l_x(t)$ travelled in the $\hat{\mathbf{x}}$ direction, we can use the relation:

$$(9) \qquad d(t) = \int_0^t v_x(t_2)dt_2$$

$$(10) \qquad = \int_0^t \int_0^t a_x(t_1)dt_1 dt_2$$

Recall that the tangential acceleration of the point mass p is bounded by $a_{t,max}$. This means that $a_t(t) \leq \bar{a}$ for all t. Therefore, we see:

$$(11) \qquad d(t) \leq \int_0^t \int_0^t \bar{a}dt_1 dt_2$$

$$(12) \qquad = \frac{\bar{a}t^2}{2}$$

Thus, in order to travel a distance of $d(T_f) = x_2$, it needs to be the case that $T_f \geq \sqrt{\frac{2x_2}{\bar{a}}}$. Moreover, equality holds if and only if $a_t(t) = \bar{a}$ for all $t \in [0, T_f(\gamma)]$.

If the point mass travels for time $t < \sqrt{\frac{2x_2}{\bar{a}}}$, then it is impossible for the point mass to reach $(x_2, 0)$ when starting at $(0,0)$. This is because p cannot reach $(x_2, 0)$ in the x direction when $t < \sqrt{\frac{2x_2}{\bar{a}}}$ and any acceleration in the y direction would not enable this either.

This means that the fastest path is completed in time $T_f(\hat{\gamma}) = \sqrt{\frac{2x_2}{\bar{a}}}$. Let us examine the path taken by the point mass p on this fastest path. Recall that $a_t(t) = \bar{a}$ for all t along the fastest path. This means that there was no centripetal acceleration $|a_c| = 0$. In other words, the point mass never turned on its way to reaching the destination point. The only way this could have happened is if it travelled along the x axis in a straight line.

Now, we have seen that the fastest path in the transformed coordinates travels exactly on the x axis so that $y = 0$ anywhere along the fastest path. Notice, however, that the x axis in the transformed coordinates is given exactly by the following line:

$$(13) \qquad y = \frac{y_2 - y_1}{x_2 - x_1}x + y_1$$

Thus, we see that the fastest path in the original coordinates follows the above equation, which is what we wanted to show. $\qquad \square$

Corollary 4.7. *Given points $(x_1, y_1), (x_2, y_2) \in \mathrm{R}^2$ and a particle p whose initial position is (x_1, y_1) which moves with acceleration bounded*

by $\bar{a}$, the fastest path $\hat{\gamma}(t)$ which p can trace from (x_1, y_1) to (x_2, y_2) is unique.

Proof. We have already shown that any fastest path between (x_1, y_1) and (x_2, y_2) follows the straight line given by $y = \frac{y_2 - y_1}{x_2 - x_1} x + y_1$. Moreover, we showed that when travelling along the fastest path, the particle must have acceleration along the straight line of $\bar{a}$. Since we have starting position (x_1, y_1) and initial speed of 0, the acceleration of the particle $a(t)$ uniquely defines a path for the particle.

There is only a single function $a(t) = \bar{a}$ which the acceleration can satisfy when the particle is moving along a fastest path, therefore, there is only a single possible fastest path. $\qquad\square$

5. Turning Around Cones

Now we are finally able to tackle the model problem of optimal paths around cones. First we will solve a simpler but related problem of minimizing the path length around a quadrilateral.

5.1. **Quadrilateral.** We would like to minimize the path length around the quadrilateral shown in Figure 5. Solving this problem will hopefully give us intuition for solving the problem of actual paths. We allow the quadrilateral's perimeter to vary based on an angle θ which has a physical representation of measuring how symmetric the system is.
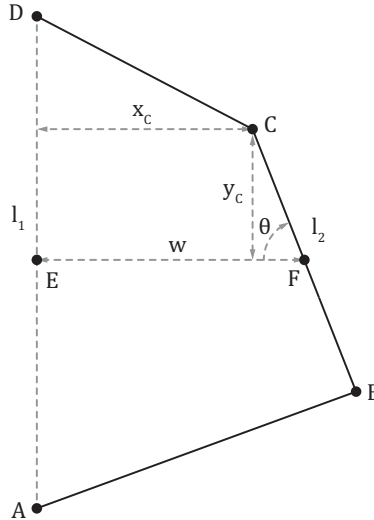

FIGURE 5. Quadrilateral path. A $\rightarrow$ B $\rightarrow$ C $\rightarrow$ D.

We will constrain the problem such that it is similar to the problem of a particle traveling around a cone. Sides $\overline{BC}$, $\overline{DA}$, and $\overline{EF}$ have fixed lengths l_1, l_2, and w respectively. Points E and F bisect $\overline{DA}$ and $\overline{BC}$ respectively. The goal is

$$\arg\min_{\theta} \left[l_{AB} + l_{BC} + l_{CD} \right]$$

Thus, we want to choose the angle which will minimize the path traveled by a particle along the perimeter of the quadrilateral. Intuitively, a symmetric quadrilateral should provide the fastest route, but we will formalize this intuition. We do this by deriving the following equations using basic trigonometry:

$$(14) \qquad x_c = w - \frac{l_2}{2}\cos\theta$$

$$(15) \qquad y_c = \frac{l_2}{2} - \frac{l_2}{2}\sin\theta$$

$$(16) \qquad l_{AB} = \sqrt{x_c^2 + y_c^2}$$

$$(17) \qquad = \sqrt{\left(\frac{l_1}{2} - \frac{l_2}{2}\cos\theta\right)^2 + \left(w - \frac{l_2}{2}\sin\theta\right)^2}$$

$$(18) \qquad l_{BC} = l_2$$

$$(19) \qquad l_{CD} = \sqrt{\left(\frac{l_1}{2} + \frac{l_2}{2}\cos\theta\right)^2 + \left(w - \frac{l_2}{2}\sin\theta\right)^2}$$

Since we know that l_{BC} is fixed, our minimization problem reduces to

$$\arg\min_{\theta} \left[l_{AB} + l_{CD} \right]$$

Because all distances are positive, we can use Lemma 6.1, defined in the Appendix, to simplify our problem:

If we define $l_{AB} = \sqrt{z_{AB}}$ and $l_{CD} = \sqrt{z_{CD}}$, we can use 6.1 with $k(t) = \sqrt{t}$ to simplify our problem to

$$(20) \quad \arg\min_{\theta} \left[2\left(\frac{l_1}{2} - \frac{l_2}{2}\sin\theta\right)^2 \right.$$

$$\left. + \left(w - \frac{l_2}{2}\cos\theta\right)^2 + \left(w + \frac{l_2}{2}\cos\theta\right)^2 \right]$$

Now, we can expand out our expression and use Lemma the fact that $\sin^2 \theta + \cos^2 \theta = 1$ to obtain a much simpler (but equivalent) minimization problem:

$$(21) \qquad \arg\min_{\theta} \frac{l_1^2}{2} + \frac{l_2^2}{2} + 2d^2 - l_1 l_2 \sin\theta$$

We note that l_1, l_2, and d are all constants which are given to us in the problem. Therefore, the minimization problem really boils down to

$$(22) \qquad \arg\min_{\theta} -l_1 l_2 \sin\theta = \arg\max_{\theta} \sin\theta$$

Thus we see that the optimal angle for minimizing the path length is $\theta = \frac{\pi}{2}$. In other words, it is better to be have a symmetric path. This works well with our intuition, and so now we are ready to solve problems related to particles traveling around cones.

Theorem 5.1. *Given a particle with zero tangential acceleration and bounded centripetal acceleration, $a_c \leq a_{c,max}$, and a cone setup consisting of 3 cones, A, B, and C, at locations $\mathbf{X_A}$, $\mathbf{X_B}$, and $\mathbf{X_C}$ as is shown in Figure 6*
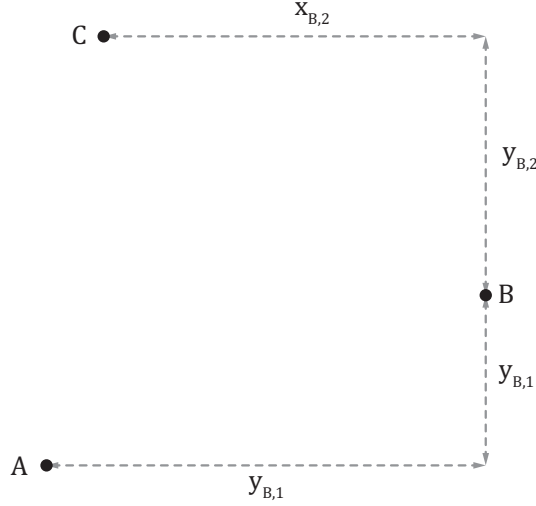

FIGURE 6. 3 cone setup.

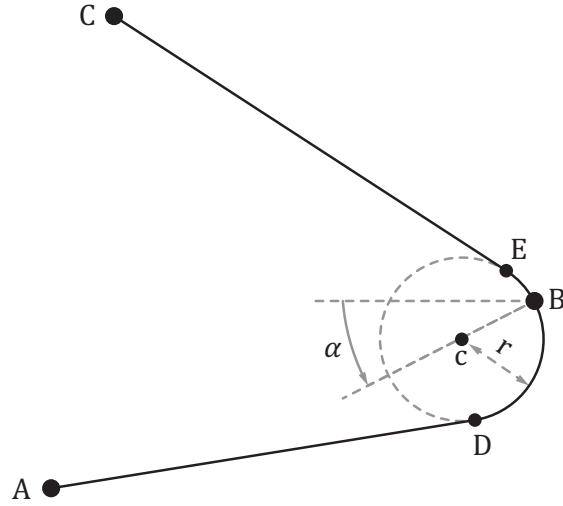*the optimal path around the cones is shown in Figure 7*

FIGURE 7. 3 cone optimal path.

TODO: Finish proof. Since speed is constant, the optimal path will be the shortest path. By Corollary 4.5, the optimal path can only include line segments and circular sections. Thus, the problem reduces to a trigonometric problem.

   The turn must touch cone B, otherwise the trajectory will either not go around the cone, or be longer than necessary. Therefor, the center of the turn, c, must be a distance r away from cone B.
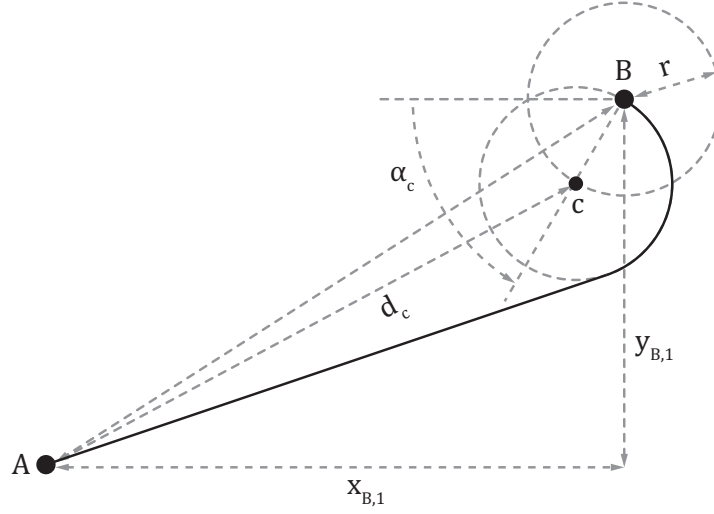
FIGURE 8. Position of center of turn.

Now that c is defined by α, let us determine the total path length as a function of α.
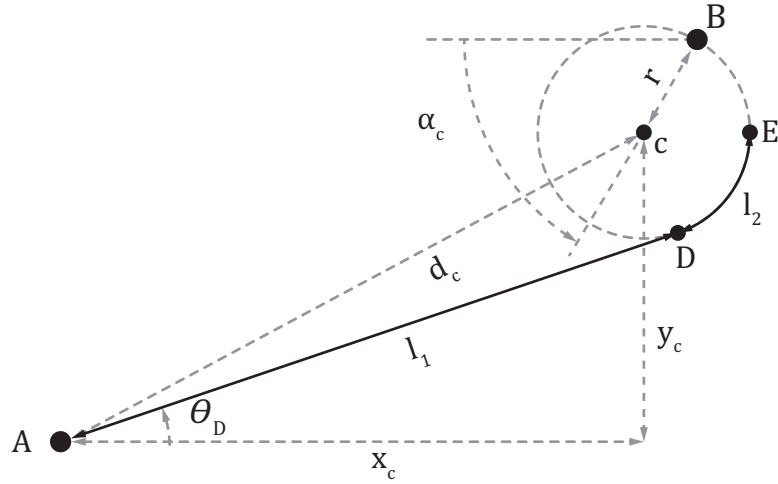

FIGURE 9. Path length.

$$x_c = x_B - r \cos \alpha_c$$
$$y_c = y_B - r \sin \alpha_c$$
$$l_1 = \sqrt{d_c - r}$$
$$l_2 = r \left(\alpha_c - \tan^{-1} \left(\frac{y_c}{x_c} \right) \right)$$

$\square$

6. Conclusion

In this paper, we have formalized many of the things that intuition would tell us. Namely, we have shown that a straight line is the fastest way to get between two points. This fact is unsurprising because of the fact that acceleration in a single direction (toward the finish point) is the least wasteful means of getting to the finish.

We developed a number of simple lemmas governing the motion of a volume-less particle with bounded acceleration. We also showed that symmetric paths tend to be better than non-symmetric paths (through our simple quadrilateral path minimization problem).

Finally, we found an optimal path for a particle to trace around three cones when the particle has constant velocity.

Appendix

6.1. Analysis.

Lemma 6.1. *If $f(t), g(t) > 0$ and $k(t) > 0$ is a strictly monotonically increasing function for all t, then $\arg\min_t k(f(t)) + k(g(t)) = \arg\min_t f(t) + g(t)$.*

Proof. Let t_1, t_2 be such that $f(t_1) + g(t_1) < f(t_2) + g(t_2)$. In this proof, we will show that $k(f(t_1)) + k(g(t_1)) < \sqrt{f(t_2)} + \sqrt{g(t_2)}$. Since $k(t)$ is a strictly monotonically increasing function when $t > 0$, we know that $k(x) < k(y)$ if and only if $x < y$ (assuming we can confine x, y to be non-negative).

Because this is the case, we see that $k(f(x)) < k(f(y))$ if and only if $f(x) < f(y)$ (the same goes for g). Thus, we see that if we have found the minimum t_m to $\arg\min_t k(f(t)) + k(g(t))$, then it is the case that $k(f(t_m)) + k(g(t_m)) < k(f(t)) + k(g(t))$ for all $t \neq t_m$ (again where $t > 0$). Following our train of logic, we see that $f(t_m) + g(t_m) < f(t) + g(t)$ for all $t \neq t_m$, which means that t_m is a minimum of $f(t) + g(t)$. Thus

by finding a minimum t_m to $k(f(t)) + k(g(t))$, we also found a minimum to $f(t) + g(t)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

6.2. **Coordinate Systems.**

6.2.1. *Radius of Curvature.* The radius of curvature, R, of a curve at a point is a measure of the radius of the circular arc which best approximates the curve at that point.

For $a_t = 0$

$$R = \left| \frac{v^2}{a_c} \right|$$

6.3. **Vector Calculus in Polar Coordinates.**

$$\boldsymbol{x} = r\hat{\boldsymbol{r}}$$

$$\vec{v} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}$$

$$\vec{a} = \left(\ddot{r} - r\dot{\phi}^2 \right)\hat{\boldsymbol{r}} + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi} \right)\hat{\boldsymbol{\phi}}$$

REFERENCES

[1] http://en.wikipedia.org/wiki/Polar_coordinate_system