OPTIMAL PARTICLE PATHS AROUND POINTS IN $\mathbb{R}^2$ WITH CONSTRAINED ACCELERATION

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Abstract. This paper investigates optimal paths in $\mathbb{R}^2$ for particles to travel between points. A paramerization of a particle's motion around a point is defined. From this parameterization, we derive a general solution for an optimal path with constant speed between 2 points. The results of this paper can by applied towards calculating optimal object trajectories in physics when acceleration is constrained.

1. Introduction

In this paper, we shall examine optimal paths for a particle.

Our model problem is a particle moving in $\mathbb{R}^2$ with constrained acceleration. The particle must navigate around cones (points in $\mathbb{R}^2$) to reach a final position. The goal is to minimize the total time spend navigating to the final position.

We first present a polar coordinate representation of a particle's position around a point and derive a set of differential equations governing the motion of the particle in this coordinate system. This allows us to devise some simple lemmas about the motion of the particle.

After developing an intuition for particles and particle paths, we then move on to the more complicated problem of deriving governing rules for optimal paths between points.

Finally, we tackle the problem of finding an optimal path around cones.

1.1. Motivation.

The problem of finding an optimal trajectory is interesting because of its applications in physics. There are many instances where finding optimal paths is important. For instance, a race car driver wants to know the fastest way to get from one point to another so that he can win his race. For another example, imagine you are sending a spacecraft to a particular destination in space and would like the fastest means of getting there.

The space example directly motivates our model problem. On a spacecraft, there is a limited amount of acceleration that is possible

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(provided by thrusters). One would potentially like to navigate to a planet while moving around obstacles.

Understanding how to find optimal trajectories will provide greater insight into solving problems like these.

2. Notation

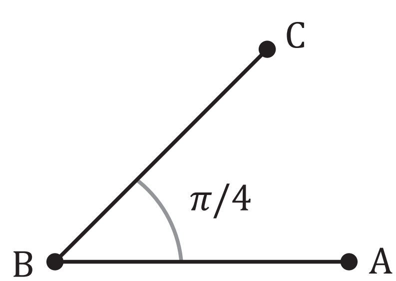
In this section, we will give some basic defintions which we will use throughout the paper. We will define our notational conventions here.

2.1. **Vectors.**

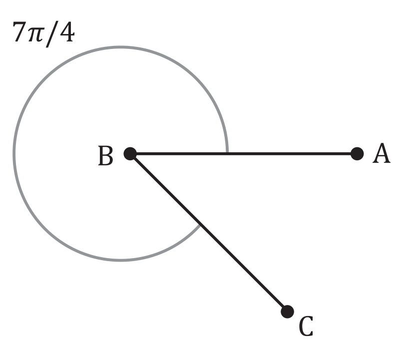
a**:** Scalar quantity.
$\mathbf{a}$**:** Vector in n-dimensional space. $\|\mathbf{a}\| = a$.
$\hat{\mathbf{a}}$**:** Unit vector in n-dimensional space. $\hat{\mathbf{a}} = \mathbf{a}/a$ and $\|\hat{\mathbf{a}}\| = 1$.

2.2. **Angles.** We define two different types of angle measurements: a standard measurement, and a directional measurement (all angles are measured in radians).

In the standard measurement, angles are in $[0, 2\pi]$ and are measured counterclockwise. In the directional measurement, angles are in $[-\pi, \pi]$ and the measurement direction is indicated by an arrow on the measurement.

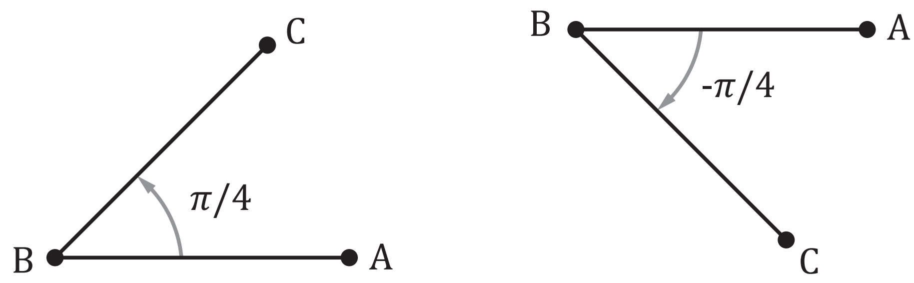


(A)



(B)

Figure 1. Standard angle notation (no arrow)



(A) (B)

FIGURE 2. Directional angle notation (arrow)

3. Particles and Paths

In this section, we will provide some basic definitions about particles and paths. These will lay the groundwork for thinking about optimal paths.

Definition 3.1. *A n-dimensional path $\gamma(t) : \mathrm{R} \to \mathrm{R}^n$ is a function which maps a time $t \in \mathbb{R}$, $t \in [T_0(\gamma), T_f(\gamma)]$, to a position $\mathbf{X} \in \mathrm{R}^n$.*

Definition 3.2. *A planar path is a path in $\mathbb{R}^2$.*

Definition 3.3. *A path between two points, $\mathbf{X_1}$ and $\mathbf{X_2}$ is a path, $\gamma(t)$, where $\gamma(T_0(\gamma)) = \mathbf{X_1}$ and $\gamma(T_f(\gamma)) = \mathbf{X_2}$.*

Definition 3.4. *A particle, p, is an object in space with zero volume. A particle travels along a path, $\gamma(t)$ if the particle is at position $\gamma(t)$ at time t, for all $t \in [T_0(\gamma), T_f(\gamma)]$. If a particle is traveling along a path, $\gamma(t)$, then we define a position, $\mathbf{X}(t) \coloneqq \gamma(t)$, a velocity, $\mathbf{v}(t) := \frac{d\gamma(t)}{dt}$, and an acceleration, $\mathbf{a}(t) \coloneqq \frac{d^2\gamma(t)}{dt^2}$, for the particle, for all $t \in [T_0(\gamma), T_f(\gamma)]$.*

A particle is restricted if there are conditions on its position and the time derivatives of its position. Given a particle, a valid path is a path that the particle can travel along.

Definition 3.5. *Given a particle, a fastest path, $\hat{\gamma}(t)$, between two points, $\mathbf{X_1}$ and $\mathbf{X_2}$, is a valid path such that $T_f(\hat{\gamma}) \leq T_f(\gamma)$ for all valid paths, $\gamma(t)$, between $\mathbf{X_1}$ and $\mathbf{X_2}$.*

Definition 3.6. *A particle's speed is $v(t)$, and its direction of motion is $\hat{\mathbf{v}}$.*

Definition 3.7. *The centripetal acceleration,* $\mathbf{a_c}$*, of a particle,* p*, is the component of its acceleration in the direction perpendicular to its direction of motion.*

In rectangular coordinates, the sign of a_c *is defined to be the sign of the projection of* $\hat{\mathbf{a_c}}$ *onto* $\hat{\mathbf{x}}$*.*

In polar coordinates, the sign of a_c *is defined to be the sign of the projection of* $\hat{\mathbf{a_c}}$ *onto* $\hat{\mathbf{r}}$*.*

Definition 3.8. *The tangential acceleration,* a_t*, of a particle, is the component of the acceleration of the particle in its direction of motion.*

$$a_t := \frac{dv}{dt}$$

3.1. **Particle Motion in Polar Coordinates.** Unless otherwise specified, the motion of particle moving along planar paths will be described in polar coordinates, along with an extra parameter θ, as is shown in Figure 3.

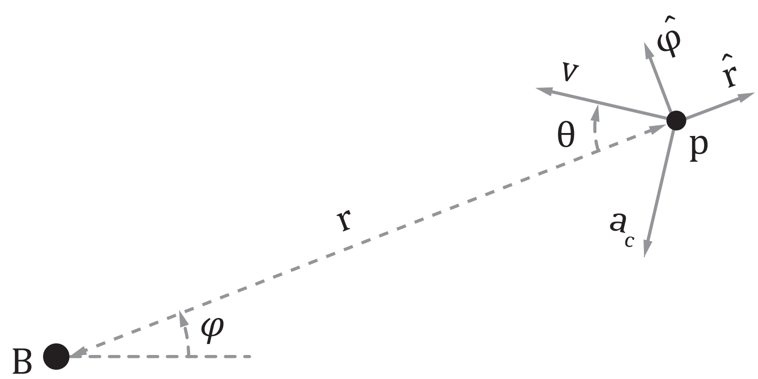


FIGURE 3. A particle moving in 2-dimensional polar coordinate system centered at B.

$\theta(t), \phi(t) \in [-\pi, \pi]$. From now on, $|\theta(t)|$ and $|\phi(t)|$ will be used, since there is a symmetry in the system about $\hat{\mathbf{r}}$.

Lemma 3.9. *The time derivative of* θ *is given by*

$$\frac{d\,|\theta(t)|}{dt} = \frac{a_c}{v}$$

Proof. If we look at a point, p, subject to only centripetal acceleration, a_c, the change in $\mathbf{v}$ over an infinitesimal time, dt, is shown in Figure 4 (the two vectors, $\mathbf{v}$ and $\mathbf{v} + \mathbf{dv}$, are superimposed).

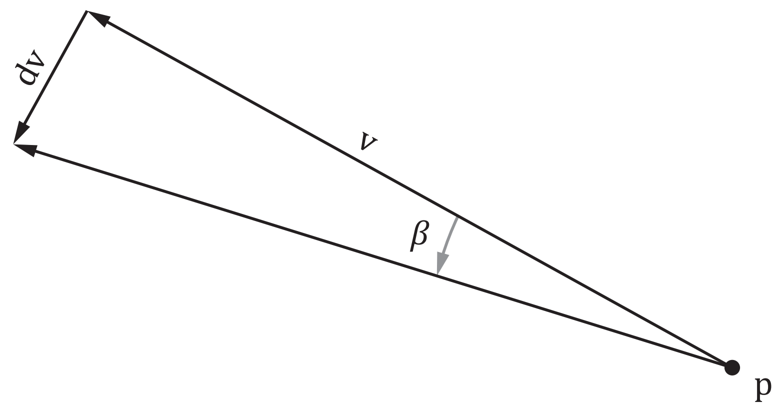


FIGURE 4. Tangential acceleration.

From the definition of a_c

$$\frac{dv}{dt} = a_c$$

Since $dv = d\,|\theta|\,v$, then

$$\frac{d\,|\theta|}{dt} = \frac{a_c}{v}$$

□

The proof of the lemma in the case where a_t is also nonzero is very similar, since the component of $\mathbf{dv}$ in the direction $\hat{\mathbf{v}}$ is negligible compared to v.

Returning again to Figure 3, the following equations can be derived

$$\frac{dr(t)}{dt} = -v(t)\,\cos\left(|\theta(t)|\right) \tag{1}$$

$$\frac{d\,|\phi(t)|}{dt} = \frac{v(t)}{r(t)}\,\sin\left(|\theta(t)|\right) \tag{2}$$

$$\frac{d\,|\theta(t)|}{dt} = \frac{d\,|\phi(t)|}{dt} - \frac{a_c(t)}{v(t)} \tag{3}$$

$$= \frac{v(t)}{r(t)}\,\sin\left(|\theta(t)|\right) - \frac{a_c(t)}{v(t)} \tag{4}$$

Applying the chain rule to (1)

$$\frac{d}{dt}\frac{dr(t)}{dt} = -\frac{dv(t)}{dt}\cos(|\theta(t)|) + v(t)\sin(|\theta(t)|)\frac{d|\theta(t)|}{dt} \tag{5}$$

$$= -a_t\cos(|\theta(t)|) + \frac{v(t)^2}{r(t)}\sin^2(|\theta(t)|) - \sin(|\theta(t)|)\,a_c(t) \tag{6}$$

Lemma 3.10. *Given a particle with the following restrictions:* $a = 0$, $v(t) = \bar{v}$, *for all* $t \geq T_0$*then the following equations hold*

$$\begin{cases} |\theta(t)| \to \pi \quad \textit{as} \quad t \to \infty & \textit{if } \theta(T_0) > 0 \\ |\theta(t)| = 0 \quad \textit{for all} \quad t \geq T_0 & \textit{if } \theta(T_0) = 0 \end{cases}$$

Furthermore,

$$\begin{cases} r(t) \to \infty \quad \textit{as} \quad t \to \infty & \textit{if } \theta(T_0) > 0 \\ r(t) \to 0 \quad \textit{as} \quad t \geq T_0 & \textit{if } \theta(T_0) = 0 \end{cases}$$

Proof. The first part of the lemma is proven by referring to (4), and noting that $\frac{d|\theta(t)|}{dt} = 0$ when $|\theta(t)| = 0, \pi$, and $\frac{d|\theta(t)|}{dt} > 0$ otherwise. If $\theta(T_0) \neq 0$, then $\theta(t)$ is bounded above by π and strictly increasing, because its derivative is strictly positive, so $|\theta(t)| \to \pi \quad \text{as} \quad t \to \infty$. If $\theta(T_0) = 0$, then $|\theta(t)| = 0 \quad \text{for all} \quad t \geq T_0$.

For the first case of the second part of the lemma,

$$\frac{d|\theta(t)|}{dt} = 0 \qquad \text{for } t \geq t_0$$

So

$$|\theta(t)| = 0 \qquad \text{for } t \geq t_0$$

[TODO]

□

4. Traveling Between Points

Now that we have defined our notation, we are ready to think about the problem of optimal trajectories.

The simplest problem to tackle first, is the optimal trajectory from a starting position, $\mathbf{X_0}$ to a final position, $\mathbf{X_f}$. Working in a polar coordinate system centered on $\mathbf{X_f}$, it is clear, that our goal is achieved when $r = 0$. Furthermore, $\theta(t)$ must be 0 before reaching $\mathbf{X_f}$, otherwise, it will grow exponentially fast as $r(t) \to 0$, which can be seen in equation (1).

Lemma 4.1. *If $a_t = 0$ and $a_c \leq \bar{a}_c$, then the minimum radius of curvature of a particle's trajectory is given by*

$$R_{min} = \frac{v^2}{\bar{a}_c} \tag{7}$$

Proof. This comes directly from the definition of radius of curvature for a particle without any tangential acceleration:

$$R = \frac{v^2}{a_c}$$

and the fact that

$$a_c \leq \bar{a}_c \to \frac{1}{\bar{a}_c} \geq \frac{1}{\bar{a}_c}$$

□

We can also investigate how the particle moves along a fastest path. In particular, we can make some statements about the change in θ, the angle between the particle's velocity vector and the straight line between the particle and it's destination.

Theorem 4.2. *A 2-dimensional particle, p, traveling along a fastest path, $\hat{\gamma(t)}$, from X_1 to X_2 has $\frac{d|\theta(t)|}{dt} \leq 0$ for all $t \in [0, T_{f,\gamma}]$, unless $\|X_1 - X_2\| < 2R_{min}$.*

Proof. Since $\theta(T_{f,\gamma}) = 0$, then if $\frac{d|\theta(t)|}{dt} > 0$ for $t \in [t_1, t_2]$, since $\theta T_{f,\gamma} < \theta t_1$, then by Lemma **??** $\theta(t_1) = \theta(t_2)$ for some $t_2 \in [t_1, T_{f,\gamma}$. Since all of the time derivatives in (1) - (4) are all monotonic for $r \in (0, \infty)$, there is no local maximum for them, so the optimal strategy should be the same, regardless of r. In other words, there can not be multiple cases for optimal strategies based on r. □

Now we will show that the fastest path between two points goes in a straight line if there is no initial speed. This will formalize the natural intuition that straight lines are the fastest path when there are no obstacles between the start and finish position.

Theorem 4.3. *Given a particle $p \in \mathrm{R}^2$ with initial position of $\mathbf{X_1}$, no initial speed, and moves with bounded tangential acceleration, $a_t \leq \bar{a}_t$, and infinite centripetal acceleration, the fastest path $\hat{\gamma}(t)$ which p can trace from $\mathbf{X_1}$ to $\mathbf{X_2}$ lies on the line segment from $\mathbf{X_1}$ to $\mathbf{X_2}$:*

Proof. Without loss of generality, we can define a cartesian coordinate system, where the origin is at $\mathbf{X_1}$ and the positive x-axis passes through $\mathbf{X_2}$. Let us say that the two points in our cartesian coordinate system become $(0,0)$ and $(x_2,0)$ respectively.

Now let us examine the particle's motion in the $\hat{\mathbf{x}}$ direction. The speed of the particle is the following

$$v_x(t) = \int_0^t a_x(t_1)dt_1 \tag{8}$$

To find the distance $l_x(t)$ travelled in the $\hat{\mathbf{x}}$ direction, we can use the relation:

$$l_x(t) = \int_0^t v_x(t_2)dt_2 \tag{9}$$

$$= \int_0^t \int_0^t a_x(t_1)dt_1dt_2 \tag{10}$$

Recall that the tangential acceleration of the point mass p is bounded by $\bar{a}_t$. This means that $a_t(t) \leq \bar{a}_t$ for all t. Therefore, we see:

$$d(t) \leq \int_0^t \int_0^t \bar{a}_t dt_1dt_2 \tag{11}$$

$$= \frac{\bar{a}_t t^2}{2} \tag{12}$$

Thus, in order to travel a distance of $l_x(T_f(\hat{\gamma})) = d' = d(\mathbf{X_2}, \mathbf{X_1})$, it needs to be the case that $T_f(\hat{\gamma}) \geq \sqrt{\frac{2d'}{\bar{a}_t}}$. It is possible to travel from $\vec{X_2}$ to $\vec{X_1}$ in time $T_{min} = \sqrt{\frac{2d'}{\bar{a}_t}}$ if and only if $a_t(t) = \bar{a}_t$ and $\theta(t) = 0$ for all $t \in [T_0(\gamma), T_{min}]$. In other words, the particle must be accelerating at the maximum possible tangential acceleration of $\bar{a}_t$ and it must be accelerating in the straight line direction to $\vec{X_2}$ at all times.

If the particle travels for time $t < T_{min}$, then it is impossible for the point mass to reach $\vec{X_2}$ when starting at $\vec{X_1}$. This is because p cannot reach $\vec{X_2}$ in the x direction when $t < \sqrt{\frac{2d'}{\bar{a}}}$ because it is impossible for p to reach any point whose x-coordinate is x_2, so it is obviously impossible to reach $(x_2, 0) = \vec{X_2}$.

Moreover, if there exists some time $t < T_{min}$ when $\theta(t) \neq 0$, then it is also impossible for the point mass to reach $\vec{X_2}$. Suppose that there are two particles p_1 and p_2 both accelerating at $a_t(t) = \bar{a}_t$. Imagine p_1 satisfies $\theta_{p_1}(t) = 0$ for all t and that p_2 satisfies $\theta_{p_2}(t) = 0$ except for times in some interval $[t_0, t_1]$, the particle p_2 sets a constant non-zero angle $\zeta_{p2}(t) \neq 0$ with the x-axis for $t \in [t_0, t_1]$. Denote d_p as the distance that is left to be travelled by particle p between times t_1 and T_{min}. We can find d_{p_2} in terms of d_{p_1} by using trigonometry:

$$d_{p_2}^2 = d_{p_1}^2 + \left(\frac{\bar{a}_t(t_1 - t_0)^2}{2}\sin\zeta_{p_2}\right)^2 \tag{13}$$

Since $d_{p_2} \geq 0$ by being a distance, $\sin\zeta_{p_2} \neq 0$ by setting $\zeta_{p_2} > 0$, and $t_1 - t_0 > 0$, we know that $d_{p_2} > d_{p_1}$. This means that at time T_{min} when p_1 has reached its destination at $\vec{X}_2$, particle p_2 still has $d_{p_2} - d_{p_1}$ distance left to travel. This means that it is impossible for $\theta(t) > 0$ on the fastest path.

This means that the fastest path is completed in time $T_f(\hat{\gamma}) = \sqrt{\frac{2x_2}{\bar{a}}}$. Let us examine the path taken by the point mass p on this fastest path. Recall that $a_t(t) = \bar{a}$ for all t along the fastest path. This means that there was no centripetal acceleration $|a_c| = 0$. In other words, the point mass never turned on its way to reaching the destination point. The only way this could have happened is if it travelled along the x axis in a straight line.

Thus, we see that the fastest path is the straight line between $\vec{X}_1$ and $\vec{X}_2$. $\square$

We can also prove that this fastest path is unique without very much extra work.

Corollary 4.4. *Given points $\mathbf{X_1}, \mathbf{X_f} \in \mathrm{R}^2$ and a particle p whose initial position is $\mathbf{X_1}$ which moves under conditions set forth in Theorem 4.3, the fastest path $\hat{\gamma}(t)$ which p can trace to $\mathbf{X_f}$ is unique.*

Proof. We have already shown in Theorem 4.3 that any fastest path between $\mathbf{X_1}$ and $\mathbf{X_f}$ follows the straight line between them. Moreover, we showed that when travelling along the fastest path, the particle must have tangential acceleration of $\bar{a}_t$. Since we have starting position $\vec{X}_1$ and initial speed of 0, the acceleration of the particle uniquely defines a path for the particle.

There is only a single function of the acceleration $a(t) = \bar{a}_t$ which can be satisfied when the particle is moving along a fastest path, therefore, there is only a single possible fastest path. $\square$

5. Constant Speed Conditions

In the previous section, we made statements about fastest paths between two points when the initial velocity was set to zero. We saw that our physical intuition was confirmed and that a straight line was the fastest way to travel between two points.

This is a simple case of the more general problem. We shall now examine optimal trajectories when the initial velocity is non-zero but when the centripetal acceleration is bounded. These conditions are much closer to physical reality, because it is often the case that you want to find a fastest path after you have already started moving. Moreover, Theorem 4.3 assumed that centripetal acceleration was unbounded so that the particle could turn on a dime in any direction. This is not realistic because most physical objects have momentum when they are travelling at high speeds, making it much more difficult to turn.

Assuming bounded acceleration makes the problem more interesting, and also much more difficult if we allow for tangential acceleration. In this section, we will eliminate tangential acceleration so that the particle will travel at constant speed with bounded acceleration.

To begin proving things about the fastest path, we first formalize the conditions we will be using:

Definition 5.1. *A fixed speed particle is a particle such that:* $\mathbf{X} \in \mathbb{R}^2$*,* $a_t = 0$*,* $v(t) = \bar{v}$*.*

Definition 5.2. *A particle with bounded centripetal acceleration is a particle with the following condition:* $\|a_c\| \leq \bar{a_c}$*, for some* $\bar{a_c} \geq 0$*.*

Now we shall use the following shorthand throughout this section to talk about the conditions on a particle:

Definition 5.3. *A particle with constant speed conditions is a fixed speed particle with bounded centripetal acceleration.*

To prove things about the fastest path of a particle under constant speed conditions, we shall make Conjecture 5.4 about how fastest paths behave.

Conjecture 5.4. *Let particles* p_1, p_2 *have initial velocities* $\mathbf{v}_1, \mathbf{v}_2$ *and starting positions* $\mathbf{X}_1, \mathbf{X}_2$ *respectively. Additionally, let* p_1, p_2 *have constant speed conditions and let the end position be* $\mathbf{X}_f$ *such that* $d(\mathbf{X}_f, \mathbf{X}_1) = d(\mathbf{X}_f, \mathbf{X}_2) > R_{min}$*. If* $d(\mathbf{X}_1, \mathbf{X}_f) \leq d(\mathbf{X}_2, \mathbf{X}_f)$ *and* $\theta_{p_1}(T_0) < \theta_{p_2}(T_0)$*, then* $T_f(\hat{\gamma_1}) < T_f(\hat{\gamma_2})$*.*

This conjecture comes from physical intuition. If two particles p_1 and p_2 are attempting to arrive at a particular ending point $\mathbf{X}_f$, and p_2 is farther away than p_1 and also has an initial angle θ to $\mathbf{X}_f$ that is larger than p_1's initial angle, then p_2 must take a longer time to turn towards $\mathbf{X}_f$ and must travel a longer distance than p_1. Because of this, it seems to be the case that p_1's fastest path is faster than p_2's fastest path.

If this conjecture is true, then we can provide the strategy for obtaining the fastest path between two points $\mathbf{X}_0$ and $\mathbf{X}_f$.

Theorem 5.5. *Let particle p have initial velocity $\mathbf{v}$ and starting position $\mathbf{X}_0$. Let p have constant speed conditions and let the end position be $\mathbf{X}_f$ such that $d(\mathbf{X}_f, \mathbf{X}_0) > R_{min}$. The fastest path $\hat{\gamma_\mathbf{p}}$ for particle p to reach $\mathbf{X}_f$ minimizes $\theta(t)$ for all $t \in [T_0(\hat{\gamma_\mathbf{p}}), T_f(\hat{\gamma_\mathbf{p}})]$.*

Proof. Let p_1 be a particle whose path γ_{p_1} is determined by the minimizing $|\theta_{p_1}(t)|$ for all times $t \in [T_0(\gamma_{p_1}), T_f(\gamma_{p_1})]$. We shall show that γ_{p_1} is in fact the fastest path to $\mathbf{X}_f$.

To do this, we will invoke a lemma which is closely related to Theorem 4.3:

Lemma 5.6. *Let particle p have initial velocity $\mathbf{v}$ and starting position $\mathbf{X}_0$ with constant speed conditions. If the end position is $\mathbf{X}_f$ and $\theta(T_0(\gamma)) = 0$, then the fastest path $\hat{\gamma_\mathbf{p}}$ for particle p follows the straight line between $\mathbf{X}_0$ and $\mathbf{X}_f$.*

The lemma says that if a particle starts with initial velocity pointing directly towards $\mathbf{X}_f$, then it is optimal to continue moving towards $\mathbf{X}_f$ in a straight line. The proof of this lemma is very similar to the proof for Theorem 4.3, so we will omit it.

This lemma does allow us to determine something crucial about the fastest path of $\hat{\gamma_\mathbf{p}}$. In particular, if the particle p ever obtains $\theta_p(t_s) = 0$ at some time t_s, then it will be the case that $\theta_p(t') = 0$ for all t' such that $t_s < t' < T_f(\hat{\gamma_\mathbf{p}})$. In words, if $\theta_p(t_s) = 0$ for any $t_s < T_f(\hat{\gamma_\mathbf{p}})$, then the particle will follow a straight line to $\mathbf{X}_f$ after time t_s. Moreover that it must be the case that the particle reaches $\theta_p = 0$ at some time (or else the particle would never reach $\mathbf{X}_f$). Thus, there must exist time t_s where $\theta_p(t') = 0$ for all $t' > t_s$ in the fastest path $\hat{\gamma_\mathbf{p}}$. Note that γ_{p_1} will contain this point. This is a necessary, but not sufficient condition for being a fastest path.

To show that γ_{p_1} truly is a fastest path, we must examine what happens before t_s, i.e. for all times t' such that $T_0(\gamma_{p_1}) < t' < t_s$. This becomes the important period because we know that after t_s (which necessarily occurs in a fastest path), the motion of the particle is in a straight line.

For particle p_1, we know that $|a_c(t')| = \bar{a}_c$ for all $t' < t_s$ because p_1 is minimizing $|\theta_{p_1}(t)|$. Thus, we see that p_1 traces out an arc of a circle with radius $R_{min} = \frac{\bar{v}}{\bar{a}_c}$ because we know that $|a_c(t')| = \bar{a}_c$ for all $t' < t_s$.

Now let us introduce a particle p_2 with the same conditions as p_1. However, at some time $t_b < t_s$, p_2 will choose $a_c^{p_2}$ such that $|a_c^{p_2}| \neq \bar{a}_c$ so that p_2's centripetal acceleration diverges from p_1's centripetal acceleration. We shall show that p_2's fastest path after this divergence is worse than p_1's fastest path, which will allow us to conclude that deviations from p_1 are suboptimal.

We can examine the difference in radius r and angle $theta$ between p_1 and p_2 after t_b, and use Conjecture 5.4 to finish our proof. Recall equations 6 and 4 which we derived in a previous section. We can use 6 to obtain the following:

$$\frac{d^2 r_{p_1}(t)}{dt^2} - \frac{d^2 r_{p_2}(t)}{dt^2} = \tag{14}$$

$$-sin(|\theta(t_b)|)\,(\bar{a}_c - a_c^{p2}(t_b)) < 0 \tag{15}$$

The inequality was obtained because $\bar{a}_c - a_c^{p_2}(t_b) > 0$. This directly implies that $r_1(t_b + \epsilon) < r_2(t_b + \epsilon)$ for some small $\epsilon > 0$. Therefore the distance left to travel for p_1 is less than the distance left for p_2 at $t_b + \epsilon$ time, i.e. $d(\gamma_{p_1}(t_b + \epsilon), \mathbf{X}_f) < d(\gamma_{p_2}(t_b + \epsilon), \mathbf{X}_f)$.

We can also use 4 to obtain the following for the difference in angles:

$$\frac{d|\theta_{p_1}(t_b)|}{dt} - \frac{|\theta_{p_1}(t_b)|}{dt} = \tag{16}$$

$$-\frac{1}{v(t_b)}\,(\bar{a}_c - a_c^{p_2}(t_b)) < 0 \tag{17}$$

Again, we used the fact that $\bar{a}_c - a_c^{p_2}(t_b) > 0$ to show the inequality. This implies that $\theta_{p_1}(t_b + \epsilon) < \theta_{p_2}(t_b + \epsilon)$.

Now, we have satisfied the conditions of Conjecture 5.4 since $d(\gamma_{p_1}(t_b + \epsilon), \mathbf{X}_f) < d(\gamma_{p_2}(t_b + \epsilon), \mathbf{X}_f)$ and $\theta_{p_1}(t_b + \epsilon) < \theta_{p_2}(t_b + \epsilon)$. This implies that $T_f(\gamma_{p_1}) < T_f(\gamma_{p_2})$. Thus, the fastest path before t_s must be to travel along γ_{p_1}. However, we know that the fastest path after t_s is to travel along the straight line (which γ_{p_1} does). Therefore, we see that $\gamma_{p_1} = \hat{\gamma}_p$ is the fastest path. □

Corollary 5.7. *The fastest path $\hat{\gamma_{\mathbf{p}}}$, given the conditions set forth in theorem 5.5, will trace out a path which contains an arc of a circle with radius $\bar{v}^2/\bar{a}^2$ connected to a straight line tangent to the arc.*

Corollary 5.8. *Given conditions set forth in theorem 5.5, we have:*

$$T_f(\hat{\gamma_{\mathbf{p}}}) = v_0 \left(\left(\pi - \arccos \frac{R_{min}}{d - R_{min}} \right) 2\pi R_{min} + \sqrt{d^2 - 2R_{min} d} \right) \tag{18}$$

6. Conclusion

In this paper, we have formalized many of the things that intuition would tell us. Namely, we have shown that a straight line is the fastest way to get between two points. This fact is unsurprising because of the fact that acceleration in a single direction (toward the finish point) is the least wasteful means of getting to the finish.

We developed a number of simple lemmas governing the motion of a volume-less particle with bounded acceleration. We also showed that symmetric paths tend to be better than non-symmetric paths (through our simple quadrilateral path minimization problem).

Finally, we found an optimal path for a particle to trace around three cones when the particle has constant velocity.

Appendix

6.1. Coordinate Systems.

6.2. Vector Calculus in Polar Coordinates.

$$\begin{aligned} \boldsymbol{x} &= r\hat{\boldsymbol{r}} \\ \vec{\boldsymbol{v}} &= \dot{r}\hat{\boldsymbol{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}} \\ \vec{\boldsymbol{a}} &= \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\boldsymbol{r}} + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right)\hat{\boldsymbol{\phi}} \end{aligned}$$

References

[1] http://en.wikipedia.org/wiki/Polar_coordinate_system