# THE 18.821 MATHEMATICS PROJECT LAB REPORT [PROOFS]

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## 1. Introduction

sdafkjsadflkj

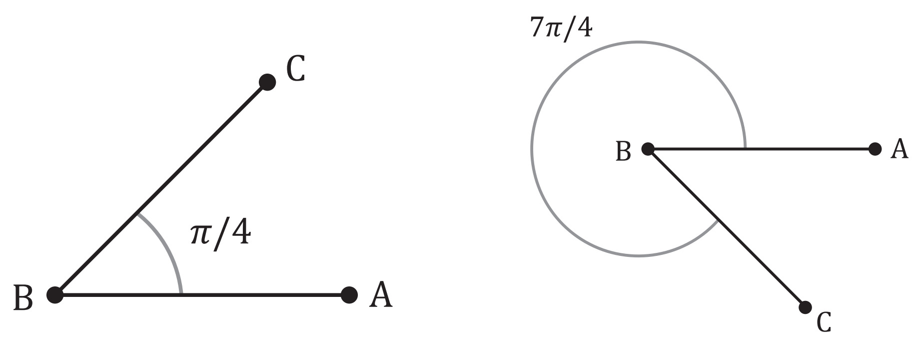
## 2. Notation

### 2.1. **Vectors.**

- $a$**:** Scalar quantity.
- $\mathbf{a}$**:** Vector in n-dimensional space. $\|\mathbf{a}\| = a$.
- $\hat{\mathbf{a}}$**:** Unit vector in n-dimensional space. $\hat{\mathbf{a}} = \mathbf{a}/a$ and $\|\hat{\mathbf{a}}\| = 1$.

2.2. **Angles.** Since many of the theorems in this paper involve polar coordinates, we define two different types of angle measurements: the standard measurement, and a directional measurement. All angles are measured in radians.

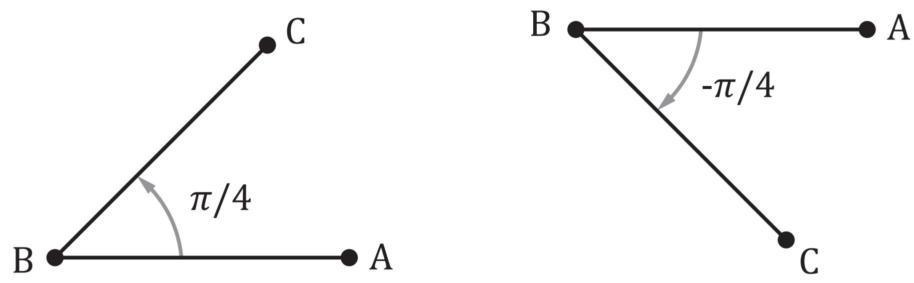
In the standard measurement, angles are in $[0, 2\pi]$ and are measured counterclockwise. In the directional measurement, angles are in $[-\pi, \pi]$ and the measurement direction is indicated by an arrow on the figure.




(a) (b)

Figure 1. Standard angle notation (no arrow)

*Date*: September 28, 2013.




(A) (B)

FIGURE 2. Directional angle notation (arrow)

## 3. Particles and Paths

In this section, we will make some basic definitions about particles and paths. These will lay the groundwork for thinking about optimal paths. We will begin by defining the description of a particle and then define various types of paths that a particle can take.

**Definition 3.1.** *A n-dimensional path $\gamma(t) : \mathrm{R} \to \mathrm{R}^n$ is a function which maps a time $t \in \mathbb{R}$, $t \in [0, T_{f,\gamma}]$, to a position $\mathbf{X} \in \mathrm{R}^n$.*

**Definition 3.2.** *A n-dimensional particle, $p$, is an object with zero volume that travels along a n-dimensional path. The particle may have conditions on its position, velocity, and acceleration in $\mathbb{R}^n$.*

**Definition 3.3.** *A valid path $\gamma(t)$ for a particle $p$ is a path such that all conditions on the particle are satisfied at every point along the path.*

**Definition 3.4.** *A path between two points, $\mathbf{X_1}$ and $\mathbf{X_2}$ is a path, $\gamma(t)$ where $\gamma(0) = \mathbf{X_1}$ and $\gamma(T_{f,\gamma}) = \mathbf{X_2}$.*

**Definition 3.5.** *For a given particle, $p$, a fastest path, $\hat{\gamma}(t)$, between two points, $\mathbf{X_1}$ and $\mathbf{X_2}$, is a valid path such that $T_f(\hat{\gamma}) \leq T_f(\gamma)$ for all valid paths, $\gamma(t)$, between $\mathbf{X_1}$ and $\mathbf{X_2}$.*

**Definition 3.6.** *The centripetal acceleration, $\mathbf{a_c}$, of a particle, $p$, is the component of the acceleration of $p$ perpendicular to its direction of motion, $\hat{\mathbf{v}}$. The sign of $a_c$ is difined as the sign of the projection of $\hat{\mathbf{a_c}}$ onto $\hat{\mathbf{r}}$.*

**Definition 3.7.** *The tangential acceleration, $\mathbf{a_t}$, of a particle, $p$, is the component of the acceleration of $p$ in its direction of motion, $\hat{\mathbf{v}}$.*

3.1. **Particle Motion in Polar Coordinates.** The motion of the 2-dimensional particles in this paper will typically be described in the polar coordinate system, shown in figure 3.

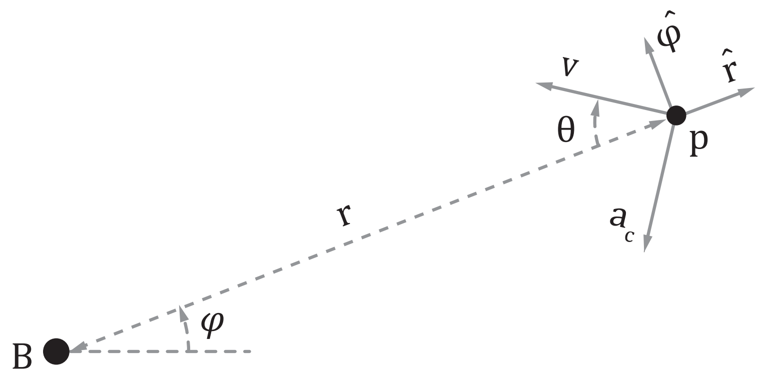
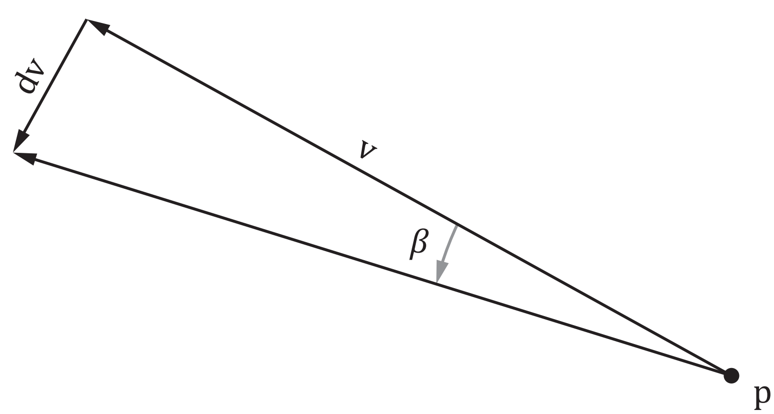



FIGURE 3. A particle moving in 2-dimensional polar coordinate system centered at B.

**Lemma 3.8.** *The time derivative of* $\theta$ *is given by*

$$\frac{d\beta}{dt} = \frac{a_c}{s}$$




*Proof.*

FIGURE 4. Tangential acceleration.

If we look at a point, $p$, subject to only a centripetal acceleration, $a_c$, the change in $\mathbf{v}$ over an infinitesimal time, $dt$, is shown in the figure 4 (the two vectors, $\mathbf{v}$ and $\mathbf{v}+\mathbf{dv}$, are superimposed). From the definition of $a_c$

$$\frac{dv}{dt} = a_c$$

So...

$$\frac{d\beta}{dt} = \frac{a_c}{s}$$

□

The proof of the lemma in the case where $a_t$ is also nonzero is very similar, since the component of $\mathbf{dv}$ in the direction $\hat{\mathbf{v}}$ is negligeable compared to $v$.

Returning again to figure 3, the following equations can be derived

$$\frac{dr(t)}{dt} = -s(t)\cos(\theta(t)) \tag{1}$$

$$\frac{d\phi(t)}{dt} = s(t)\sin(\theta(t)) \tag{2}$$

$$\frac{d\theta(t)}{dt} = \frac{d\phi(t)}{dt} - \frac{a_c(t)}{s(t)} \tag{3}$$

$$\frac{d}{dt}\frac{dr(t)}{dt} = -\frac{ds(t)}{dt}\cos(\theta(t)) + s(t)\sin(\theta(t))\frac{d\theta(t)}{dt} \tag{4}$$

**Lemma 3.9.** *For a particle, p, with bounded centripetal and tangential acceleration, then:*

*(1) The functions $\phi(t)$ and $\theta(t)$ are continuous.*

*(2) For two times $t_1$ and $t_2$, s.t. $t_1 < t_2$, $\theta(t_1) > a$ and $\theta(t_2) < a$, then $\theta(t_c) = a$ for some $t_c \in [t_1, t_2]$.*

*Proof.* First off, it should be noted that a solution to 3 exists because the rhs is Lipschitz continuous. The proof of (1) follows directly from the fact that the derivative of $\theta(t)$ exist and is bounded. (2) is just a restatement of the intermediate value theorem. □

**Lemma 3.10.** *For a particle, p, with nonzero speed, and no centripetal acceleration for $t \geq t_0$, then*

$$\begin{cases} \theta(t) \to \pi \quad \textit{as} \quad t \to \infty & \textit{if} \ \theta(t_0) > 0 \\ \theta(t) = 0 \quad \textit{for} \quad t \geq t_0 & \textit{if} \ \theta(t_0) = 0 \end{cases}$$

*Proof.* The second case is the easiest to check,

□

## 4. Traveling Between Points

**Lemma 4.1.** *If $a_t = 0$ and $\|a_c\| \leq a_{c,max}$, then the minimum radius of curvature of a point trajectory is given by*

$$R_{min} = \frac{v^2}{a_{c,max}} \tag{5}$$

**Theorem 4.2.** *For a particle, p, traveling along an optimal path, $\hat{\gamma}(t)$ between two points $X_1$ and $X_2$,*

**Lemma 4.3.** *Every optimal trajectory is constructed from line segments and circular sections of radius $R_{min}$. Every point along an optimal trajectory where $a_t = 0$ has $R = \infty$ or $R = R_{min}$.*

*Proof.* If we look at a position, p, on a trajectory. We can align a rectangular coordinate system with this point, such that $\hat{y} = \hat{v}$.

$$a_c = \ddot{r} - r\dot{\phi}^2 \tag{6}$$

$$a_t = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right) \tag{7}$$

$$0 = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right) \tag{8}$$

$$r^2\dot{\phi} = constant \tag{9}$$

□

**Theorem 4.4.** *Given points $(x_1, y_1), (x_2, y_2) \in \mathrm{R}^2$ and a particle p whose initial position is $(x_1, y_1)$ which moves with acceleration bounded by $\bar{a}$, the fastest path $\hat{\gamma}(t)$ which p can trace from $(x_1, y_1)$ to $(x_2, y_2)$ follows the straight line where all coordinates $(x, y)$ on the straight line are given by:*

$$y = \frac{y_2 - y_1}{x_2 - x_1}x + y_1 \tag{10}$$

*Proof.* Let's transform the problem. We can reset our coordinate axes so that $(x_1, y_1)$ is set to the origin and $(x_2, y_2)$ is on the x-axis. In this new coordinate system, we have transformed the following:

$$(x_1, y_1) \rightarrow (0, 0) \tag{11}$$

$$(x_2, y_2) \rightarrow (x_2', 0) \tag{12}$$

For convenience of notation, we will now refer to $x_2'$ as $x_2$.

Now let us examine the particle's motion in the $x$ direction. Let $a_t(t)$ be the tangential acceleration at time $t$ in the $x$ direction. Then we can obtain the speed of the particle $s(t)$ at time $t$ in the $x$ direction like so:

$$s(t) = \int_0^t a_t(t_1)dt_1 \tag{13}$$

To find the distance $d(t)$ travelled up to time $t$ in the $x$ direction, we can use the relation:

$$\begin{aligned} d(t) &= \int_0^t s(t_2)dt_2 & (14)\\ &= \int_0^t \int_0^t a_t(t_1)dt_1dt_2 & (15)\end{aligned}$$

Recall that the acceleration of the point mass $p$ is bounded by $\bar{a}$. This means that $a_t(t) \leq \bar{a}$ for all $t$. Therefore, we see:

$$\begin{aligned} d(t) &\leq \int_0^t \int_0^t \bar{a}dt_1dt_2 & (16)\\ &= \frac{\bar{a}t^2}{2} & (17)\end{aligned}$$

Thus, in order to travel a distance of $d(T_f) = x_2$, it needs to be the case that $T_f \geq \sqrt{\frac{2x_2}{\bar{a}}}$. Moreover, equality holds if and only if $a_t(t) = \bar{a}$ for all $t \in [0, T_f(\gamma)]$.

If the point mass travels for time $t < \sqrt{\frac{2x_2}{\bar{a}}}$, then it is impossible for the point mass to reach $(x_2, 0)$ when starting at $(0, 0)$. This is because $p$ cannot reach $(x_2, 0)$ in the $x$ direction when $t < \sqrt{\frac{2x_2}{\bar{a}}}$ and any acceleration in the $y$ direction would not enable this either.

This means that the fastest path is completed in time $T_f(\hat{\gamma}) = \sqrt{\frac{2x_2}{\bar{a}}}$. Let us examine the path taken by the point mass $p$ on this fastest path. Recall that $a_t(t) = \bar{a}$ for all $t$ along the fastest path. This means that there was no centripetal acceleration $|a_c| = 0$. In other words, the point mass never turned on its way to reaching the destination point. The only way this could have happened is if it travelled along the $x$ axis in a straight line.

Now, we have seen that the fastest path in the transformed coordinates travels exactly on the $x$ axis so that $y = 0$ anywhere along the fastest path. Notice, however, that the $x$ axis in the transformed coordinates is given exactly by the following line:

$$y = \frac{y_2 - y_1}{x_2 - x_1}x + y_1 \tag{18}$$

Thus, we see that the fastest path in the original coordinates follows the above equation, which is what we wanted to show. □

**Corollary 4.5.** *Given points $(x_1, y_1), (x_2, y_2) \in \mathrm{R}^2$ and a particle $p$ whose initial position is $(x_1, y_1)$ which moves with acceleration bounded by $\bar{a}$, the fastest path $\hat{\gamma}(t)$ which $p$ can trace from $(x_1, y_1)$ to $(x_2, y_2)$ is unique.*

*Proof.* We have already shown that any fastest path between $(x_1, y_1)$ and $(x_2, y_2)$ follows the straight line given by $y = \frac{y_2 - y_1}{x_2 - x_1} x + y_1$. Moreover, we showed that when travelling along the fastest path, the particle must have acceleration along the straight line of $\bar{a}$. Since we have starting position $(x_1, y_1)$ and initial speed of 0, the acceleration of the particle $a(t)$ uniquely defines a path for the particle.

There is only a single function $a(t) = \bar{a}$ which the acceleration can satisfy when the particle is moving along a fastest path, therefore, there is only a single possible fastest path. □

## 5. Turning Around Cones

5.1. **Quadrilateral.** Before we attempt to tackle the problem of finding the optimal path around a cone at constant velocity, we may want to find some intuition for the problem with a simple, but related problem. In particular, if a particle is traveling around a cone, we would like to know what circle it traces as it goes around the cone.

To make a simple model of this problem, we will use a quadrilateral to model a path around a cone. We will construct a quadrilateral with particular constaints that models the constraints of a path around a cone, and we will try to find the parameters that minimize the perimeter around the quadrilateral. Let us call the sides of the quadrilateral $s_1, s_2, s_3$ and $s_4$. Imagine that $s_1$ and $s_2$ are on opposite sides of the quadrilateral with fixed lengths $l_1$ and $l_2$ respectively. Now we will constrain the problem to be related to the problem of a particle traveling around a cone: imagine $s_1$ and $s_2$ are connected by a bar $s_b$ of length $d$. The bar will be perpendicular to $s_1$. The angle that $s_b$ forms with $s_2$ will be called $\theta$. We will try to find the optimal $\theta$ that minimizes the perimeter around the quadrilateral.

We will use figure as reference.

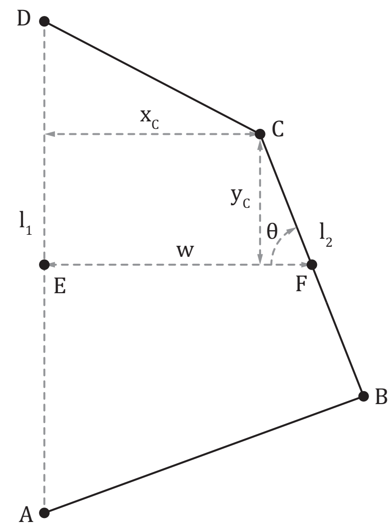



FIGURE 5. Quadrilateral path. A → B → C → D.

5.2. **Symmetric Quadrilateral.** Let us begin by solving the simplest version of this problem. Let us imagine that $s_b$ is connected to the midpoints of $s_1$ and $s_2$. We know that the perimeter will be given by $l_1 + l_2 + l_3 + l_4$ where $l_3$ and $l_4$ are the lengths of the sides of $s_3$ and $s_4$, respectively. We are given $l_1$ and $l_2$, but we will need to compute $l_3$ and $l_4$ as functions of $l_1, l_2, d$, and $\theta$.

We can find $l_3$ by using the fact that it forms a right triangle. We know that $l_3^2 = m^2+n^2$. Finding $l_4$ is similar. Therefore, we just need to find $m$ and $n$. This can be done by using the fact that $m = \frac{l_1}{2} - \frac{l_2}{2}\sin\theta$. This is just the upper half of $s_1$ minus the projection of $s_2$ onto $s_1$. We can find $n$ similarly: $n = d - \frac{l_2}{2}\cos\theta$. This is just the bar $s_b$ minus the projection of $s_2$ onto the bar. Thus by substituting, we have:

$$l_3 = \sqrt{\left(\frac{l_1}{2} - \frac{l_2}{2}\sin\theta\right)^2 + \left(d - \frac{l_2}{2}\cos\theta\right)^2} \tag{19}$$

We can do a similar analysis on $l_4$, only remembering that $n$ for $l_4$ gets extended by the projection onto $s_4$ instead of shrunken. We therefore have:

$$l_4 = \sqrt{\left(\frac{l_1}{2} - \frac{l_2}{2}\sin\theta\right)^2 + \left(d + \frac{l_2}{2}\cos\theta\right)^2} \tag{20}$$

Now, to minimize the perimeter with respect to $\theta$, we want to minimize $l_1 + l_2 + l_3 + l_4$. Since we know that $l_1$ and $l_2$ are fixed, we really want to minimize $l_3 + l_4$ with respect to $\theta$. The other thing to note is that we're minimizing positive distances. We will invoke the following lemma so that we can simplify our expression for $\min l_3 + l_4$:

**Lemma 5.1.** *If $f(t), g(t) > 0$ and $k(t) > 0$ is a strictly monotonically increasing function for all $t$, then* $\arg\min_t k(f(t)) + k(g(t)) = \arg\min_t f(t) + g(t)$.

*Proof.* Let $t_1, t_2$ be such that $f(t_1)+g(t_1) < f(t_2)+g(t_2)$. In this proof, we will show that $k(f(t_1)) + k(g(t_1)) < \sqrt{f(t_2)} + \sqrt{g(t_2)}$. Since $k(t)$ is a strictly monotonically increasing function when $t > 0$, we know that $k(x) < k(y)$ if and only if $x < y$ (assuming we can confine $x, y$ to be non-negative).

Because this is the case, we see that $k(f(x)) < k(f(y))$ if and only if $f(x) < f(y)$ (the same goes for $g$). Thus, we see that if we have found the minimum $t_m$ to $\arg\min_t k(f(t)) + k(g(t))$, then it is the case that $k(f(t_m)) + k(g(t_m)) < k(f(t)) + k(g(t))$ for all $t \neq t_m$ (again where $t > 0$). Following our train of logic, we see that $f(t_m) + g(t_m) < f(t) + g(t)$ for all $t \neq t_m$, which means that $t_m$ is a minimum of $f(t) + g(t)$. Thus by finding a minimum $t_m$ to $k(f(t))+k(g(t))$, we also found a minimum to $f(t) + g(t)$. $\square$

Since we've proven this lemma, we can invoke it upon $\min l_3 + l_4$. Since $l_3 = \sqrt{z_3}$ and $l_4 = \sqrt{z_4}$, we can use $k(t) = \sqrt{t}$ and we can write $\min l_3 + l_4 = \min z_3 + z_4$ by using our lemma. Thus, we now want to solve the problem:

$$\arg\min_\theta \quad 2 \quad \left(\frac{l_1}{2} - \frac{l_2}{2}\sin\theta\right)^2 \tag{21}$$

$$+ \quad \left(d - \frac{l_2}{2}\cos\theta\right)^2 + \left(d + \frac{l_2}{2}\cos\theta\right)^2 \tag{22}$$

Now, we can expand out our expression and use the fact that $\sin^2\theta + \cos^2\theta = 1$ to obtain a much simpler (but equivalent) minimization problem:

$$\arg\min_\theta \frac{l_1^2}{2} + \frac{l_2^2}{2} + 2d^2 - l_1 l_2 \sin\theta \tag{23}$$

We note that $l_1, l_2$, and $d$ are all constants which are given to us in the problem. Therefore, the minimization problem really boils down to

$$\arg\min_\theta -l_1 l_2 \sin\theta = \arg\max_\theta \sin\theta \tag{24}$$

**Theorem 5.2.** *Given a cone setup consisting of 3 cones at locations $X_A$, $X_B$, and $X_C$.*

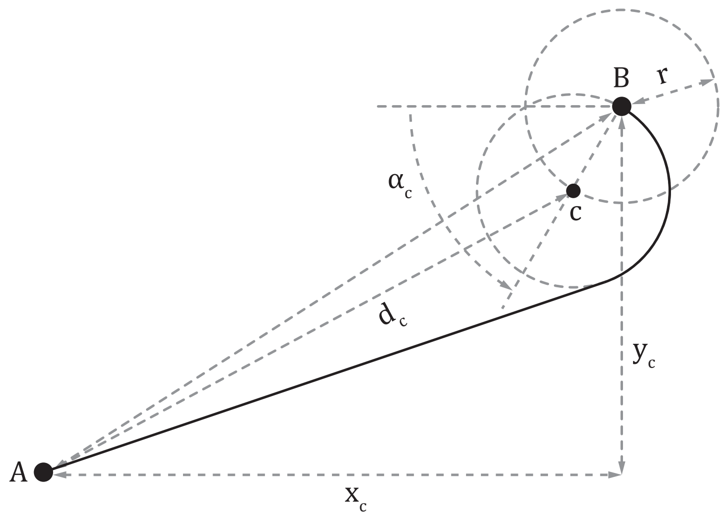



FIGURE 6. A.

$$dB = \sqrt{x_B + y_B} \tag{25}$$

*Applying the triangle inequality...*

$$d_c = \alpha_c - \tan^{-1}\left(\frac{y_B}{x_B}\right) \tag{26}$$

(27)

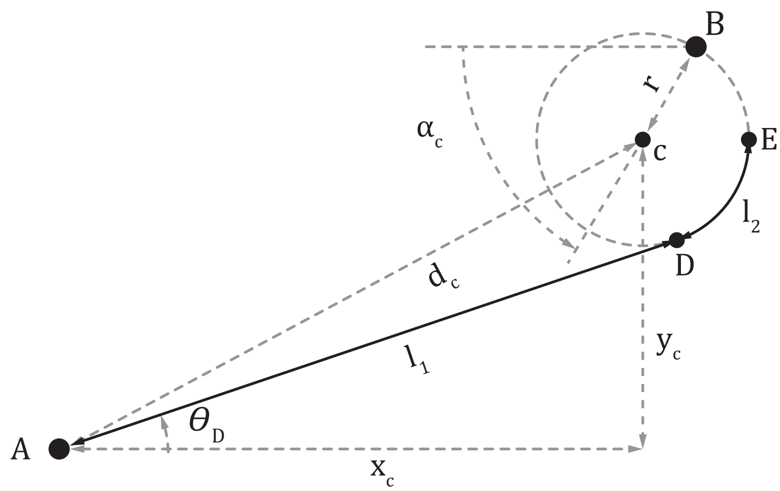



FIGURE 7. A.

$$l_1 = \sqrt{d_c - r} \tag{28}$$

$$x_c = x_B - r \cos \alpha_c \tag{29}$$

$$y_c = y_B - r \sin \alpha_c \tag{30}$$

$$d_c = \alpha_c - \tan^{-1}\left(\frac{y_c}{x_c}\right) \tag{31}$$

(32)

*Proof.* APPENDIX

## 5.3. **Coordinate Systems.**

### 5.3.1. *Time Derivatives.*

$$\dot{a} = \frac{da}{dt}$$

$$\ddot{a} = \frac{d^2a}{dt^2}$$

5.3.2. *Radius of Curvature.* The radius of curvature, $R$, of a curve at a point is a measure of the radius of the circular arc which best approximates the curve at that point.

$$R = \left| \frac{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} \right| \tag{33}$$

$$= \left| \frac{s^2}{a_t} \right| \tag{for $a_c =$}$$

### 5.4. Scalar Calculus in Polar Coordinates.

$$s = \frac{dl}{dt} \tag{34}$$

$$a_t = \frac{ds}{dt} \tag{35}$$

(36)

### 5.5. Vector Calculus in Polar Coordinates.

$$\boldsymbol{x} = r\hat{\boldsymbol{r}} \tag{37}$$

$$\vec{\boldsymbol{v}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}} \tag{38}$$

$$\vec{\boldsymbol{a}} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\boldsymbol{r}} + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right)\hat{\boldsymbol{\phi}} \tag{39}$$

### 5.6. Relations.

$$l = \int_{t=0}^{T} \|\vec{\boldsymbol{v}}\| \, dt \tag{40}$$

$$s = \|\vec{\boldsymbol{v}}\| \tag{41}$$

$$a_t = \|\vec{\boldsymbol{a}}\| \cdot \hat{\boldsymbol{v}} = \|\vec{\boldsymbol{a}}\| \cdot \frac{\vec{\boldsymbol{v}}}{\|\vec{\boldsymbol{v}}\|} \tag{42}$$

$$a_c = \|\vec{\boldsymbol{a}}\| \times \hat{\boldsymbol{v}} = \|\vec{\boldsymbol{a}}\| \times \frac{\vec{\boldsymbol{v}}}{\|\vec{\boldsymbol{v}}\|} \tag{43}$$

## References

[1] http://en.wikipedia.org/wiki/Polar_coordinate_system