FASTEST PATHS BETWEEN POINTS IN $\mathbb{R}^2$ FOR A PARTICLE MOVING WITH CONSTANT SPEED AND BOUNDED CENTRIPETAL ACCELERATION

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Abstract. This paper investigates fastest paths for a particle moving between two points in $\mathbb{R}^2$. A modified polar coordinate system is defined, which we use to derive differential equations to model the particle's motion. These equations allow us to describe theorems about fastest paths between points. The results of this paper can by applied to modeling a particle's motion and to calculating shortest paths.

1. Introduction

The problem of finding fastest paths between two points (let's call them A and B) is interesting because many attributes of a fastest path are intuitively obvious, but rigorously proving them is much more difficult. One's first assumption would be that the fastest path between A and B, would be to travel in a straight line from A to B. This would be true in the simple case where the turning rate is not constrained, but it is not true in the general case when a particle has a constrained turning rate and starts with some nonzero velocity not directed towards B. In the general case, it is easier to turn towards B, the further one is away from it, so there is the possibility that it would be fastest for the particle to turn away from B for a short period of time, so that it can then turn towards it more quickly. In fact, if the particle is too close to B, it won't have time to turn towards it, and will pass by B.

1.1. The Problem (more precise terms will be defined later).

Given a particle in $\mathbb{R}^2$ starts at a position A in with an initial velocity, what is the path in for the particle to take to reach a final position B, such that the time spent traveling from A to B is minimized. The particle can have restrictions on its position, velocity, and acceleration. Our model problem has fixed speed and a bounded turning rate.

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1.2. **Motivation.** The problem of finding a fastest path for a particle with constant speed is useful because it is identical to the problem of finding a shortest path given constraints on the radius of curvature of the path, which appears in fields such as physics and computational geometry.

Intuitively, one could imagine a race car driver navigating a racetrack with constant speed, as a physical example of the problem. In fact, this example directly motivates our problem, because a race car driver has a limited turning rate which is created by friction between the wheels and the ground.

Understanding how to find fastest paths for a particle with constant speed will also provide insight into solving even more complicated problems which may have varying speed and more dimensions such as spacecraft trajectories.

1.3. **Outline.** We first present a modified polar coordinate system to represent the particle's motion around a point and then derive a set of differential equations governing the motion of the particle in this coordinate system. This allows us to devise some simple lemmas about the motion of the particle, and draw some conclusions about fastest paths.

2. Particles and Paths

We will first provide some basic definitions about particles and paths, which will lay the groundwork for thinking about fastest paths. First we will define what we mean by a particle and a path.

Definition 2.1. *A path* $\gamma(t) : [T_0(\gamma), T_f(\gamma)] \to \mathrm{R}^2$ *is a function which maps a time* $t \in [T_0(\gamma), T_f(\gamma)]$ *to a position* $\mathbf{X} \in \mathrm{R}^2$*.*

Definition 2.2. *A particle* p *is an object in space with zero volume. If a particle is traveling along a path,* $\gamma(t)$*, then we define a position,* $\mathbf{X}(t) := \gamma(t)$*, a velocity,* $\mathbf{v}(t) := \frac{d\gamma(t)}{dt}$*, and an acceleration,* $\mathbf{a}(t) := \frac{d^2\gamma(t)}{dt^2}$*, for the particle, for all* $t \in [T_0(\gamma), T_f(\gamma)]$*.*

A particle is restricted if there are conditions on its position and the time derivatives of its position. Given a particle, a valid path is a path that the particle can travel along without violating its restrictions.

Now we will make our problem statement more rigorous, by defining what a fastest path is.

Definition 2.3. *Let* p *be a particle with initial position* X_0*, a fastest path* $\hat{\gamma}(t)$ *to* $\mathbf{X_f}$*, is a valid path such that* $T_f(\hat{\gamma}) \leq T_f(\gamma)$ *for all valid paths* $\gamma(t)$ *to* $\mathbf{X_f}$*.*

Finally, we need to define some terms for a particle's motion which will be needed later on to describe restrictions on a particle's motion.

Definition 2.4. *The centripetal acceleration* $\mathbf{a_c}(t)$ *of a particle, p, is the component of its acceleration in the direction perpendicular to its direction of motion.*

Definition 2.5. *The tangential acceleration* $a_t(t)$ *of a particle is the component of the acceleration of the particle in its direction of motion.*

$$a_t(t) := \frac{dv(t)}{dt} \tag{1}$$

2.1. **Particle Motion in Polar Coordinates.** Before we introduce the equations governing a particle's motion in $\mathbb{R}^2$, we must first define a coordinate system to describe its motion. Unless otherwise specified, the motion of particles in this paper will be described in a modified polar coordinate system that contain an extra parameter θ, that is shown in Figure 1. We will refer to this system as the *B centered particle coordinate system*.

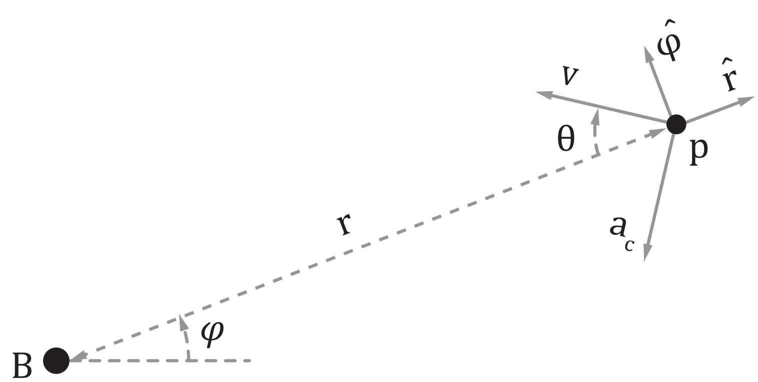

FIGURE 1. A particle moving in a B centered particle coordinate system.

The sign of $a_c(t)$ is defined to be the sign of the projection of $\hat{\mathbf{a_c}}(t)$ onto $-\hat{\mathbf{r}}(t)$; in other words, it is positive when pointing towards B. $\theta(t), \phi(t) \in [-\pi, \pi]$, and from now on, we will only use $|\theta(t)|$ and $|\phi(t)|$, since there is a symmetry in the system about $\hat{\mathbf{r}}$.

Lemma 2.6. *The time derivative of* $|\theta(t)|$ *is given by*

$$\frac{d|\theta(t)|}{dt} = \frac{d|\phi(t)|}{dt} + \frac{a_c(t)}{v(t)} \tag{2}$$

Proof. $\theta(t)$ is composed of $\phi(t)$ plus the angle between $\hat{\mathbf{v}}(t)$ and the horizontal, which we can temporarily call $\beta(t)$. $\phi(t)$ is the component of $\theta(t)$ in the polar coordinate system, and changes as the point moves with respect to the point B. $\beta(t)$ can be thought of as the component of $\theta(t)$ in the Cartesian coordinate system, and it only changes when $a_c(t)$ is nonzero.

Differentiating, we get

$$\frac{d|\theta(t)|}{dt} = \frac{d|\phi(t)|}{dt} + \frac{d\beta(t)}{dt}$$

To find $\frac{d\beta(t)}{dt}$, we can look at a point, p, subject to nonzero centripetal acceleration, and zero tangential acceleration. The change in $\mathbf{v}(\mathbf{t})$ over an infinitesimal time, dt, is shown below (the two vectors, $\mathbf{v}(\mathbf{t})$ and $(\mathbf{v}(\mathbf{t}) + \mathbf{dv}(\mathbf{t}))$, are superimposed).

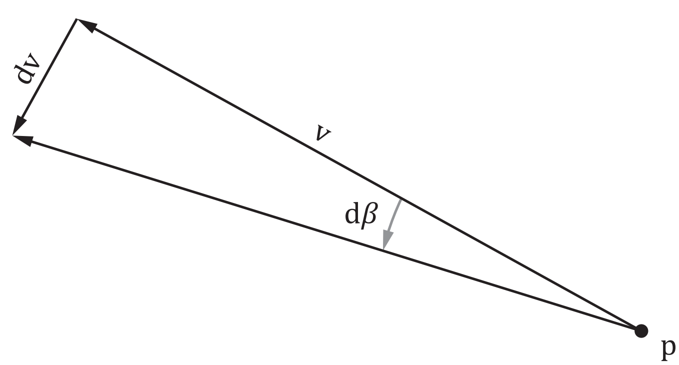

Figure 2

$a_c(t)$ is defined to be the component of acceleration perpendicular to the direction of motion, so

$$\frac{dv(t)}{dt} = a_c(t)$$

Since $dv(t) = -d\beta(t)v(t)$, then

$$\frac{d\beta(t)}{dt} = -\frac{a_c(t)}{v(t)}$$

The proof of the lemma in the case where $a_t(t)$ is nonzero is nearly identical, since the component of $\mathbf{dv}(t)$ in the direction $\hat{\mathbf{v}}(t)$ is negligible compared to $v(t)$.

□

Returning again to Figure 1, the following equations can be derived

$$\frac{dr(t)}{dt} = -v(t)\,\cos(|\theta(t)|) \tag{3}$$

$$\frac{d|\phi(t)|}{dt} = \frac{v(t)}{r(t)}\,\sin(|\theta(t)|) \tag{4}$$

$$\frac{d|\theta(t)|}{dt} = \frac{d|\phi(t)|}{dt} - \frac{a_c(t)}{v(t)} \tag{5}$$

$$= \frac{v(t)}{r(t)}\,\sin(|\theta(t)|) - \frac{a_c(t)}{v(t)} \tag{6}$$

Applying the chain rule to (3)

$$\frac{d}{dt}\frac{dr(t)}{dt} = -\frac{dv(t)}{dt}\cos(|\theta(t)|) + v(t)\sin(|\theta(t)|)\frac{d|\theta(t)|}{dt} \tag{7}$$

$$= -a_t(t)\cos(|\theta(t)|) + \frac{v(t)^2}{r(t)}\sin^2(|\theta(t)|) - \sin(|\theta(t)|)a_c(t) \tag{8}$$

3. Traveling Between Points

Now that we have defined a coordinate system, we are ready to think about particle paths. All particles in this section have bounded tangential acceleration, $|a_t(t)| \leq \bar{a}_t$, for some $\bar{a}_t \geq 0$.

The simplest problem to tackle first is the fastest path from $\mathbf{X_0}$ to $\mathbf{X_f}$ when centripetal acceleration is unbounded, which means that a particle can change its direction of motion instantaneously. The following theorem makes this notion more precise.

Theorem 3.1. *Let p be a particle with bounded tangential acceleration and $\theta(T_0) = \theta_0$. For all $\theta_f \in [0, \pi]$ and $\epsilon > 0$, there exists a valid path $\gamma(t)$, such that $|\theta(T_0(\gamma) + \epsilon)| \to \theta_f$ as $\epsilon \to 0$.*

Proof. Define a a particle's centripetal acceleration to be the following

$$a_c(t) = v(t)\frac{\theta(T_0(\gamma)) - \theta(T_f(\gamma))}{\epsilon}$$

Now, using Equation 6

$$|\theta(T_0(\gamma)+\epsilon)| = \theta(T_0(\gamma)) + \int_{t'=T_0(\gamma)}^{T_0(\gamma)+\epsilon} \left(\frac{d|\phi(t')|}{dt} - \frac{\theta(T_0(\gamma)) - \theta_f}{\epsilon} \right) dt'$$

Taking the limit as $\epsilon \to 0$

$$|\theta(T_0(\gamma)+\epsilon)| \to \theta_f \quad \text{as} \quad \epsilon \to 0$$

□

Theorem 3.2. *Let p be a particle with bounded tangential acceleration and initial position X_0. A path γ_1 to $\mathbf{X_f}$ is a fastest path if, for every other valid path γ_2 to $\mathbf{X_f}$, $|\theta_1(t)| \leq |\theta_2(t)|$ and $v_1(t) \leq v_2(t)$. The fastest path is unique if all the inequalities are strict.*

Furthermore, if, at some time $t_s \geq T_0(\hat{\gamma})$ along a fastest path $\hat{\gamma}$, $|\theta(t_s)| = 0$, then $|\theta(t)| = 0$ for all $t \geq t_s$.

Proof. Referring back to Equation 3

$$r(t) = r(T_0) + \int_{t'=T_0}^{t} -v(t')\cos(|\theta(t')|)dt'$$

Thus, if $|\theta_1(t)| \leq |\theta_2(t)|$ and $v_1(t) \leq v_2(t)$, then $r_1(t) \leq r_2(t)$, and $T_f(\gamma_1) \leq T_f(\gamma_2)$, so γ_1 is a fastest path. If all the inequalities are strict, then $T_f(\gamma_1) < T_f(\gamma_2)$, so every other path is longer, and thus γ_1 is the unique fastest path.

The second part of the theorem follows directly from the first part.

□

Now we will formalize the natural intuition that straight lines are the fastest paths between points when centripetal acceleration is unbounded.

Theorem 3.3. *Let p be a particle with bounded tangential acceleration, initial position $\mathbf{X_0}$, and initial velocity $\mathbf{v}(t)$. The line segment from $\mathbf{X_0}$ to $\mathbf{X_f}$ with $a_c(t) = \bar{a}_c$ for all $t \in [T_0(\gamma_1), T_f(\gamma_1)]$ is a fastest path $\hat{\gamma}(t)$ to $\mathbf{X_f}$. Furthermore, the fastest path for this system is unique.*

Proof. Because of Theorem 3.1, there is a valid path $\gamma_1(t)$, which has $\theta_1(t) = 0$ for $t \in (T_0(\gamma_1), T_f(\gamma_1)]$.

If we let $a_{t,1}(t) = \bar{a}_t$ for all $t \in [T_0(\gamma_1), T_f(\gamma_1)]$, then $v_1(t) = \bar{a}_t t$, so for every other valid path γ_2

$$
\begin{aligned}
v_2(t) &= \int_{t'=T_0(\gamma_2}^{t} a_{t,2}(t')dt' \\
&\leq \bar{a}_t t \\
&\leq v_1(t)
\end{aligned}
$$

Thus, all the conditions are satisfied for Theorem 3.2, so $\gamma_1(t)$ is a fastest path. The θ inequality is strict, so the fastest path is unique. Geometrically, since $\theta(t) = 0$, the path is the line segment from $\mathbf{X_0}$ to $\mathbf{X_f}$. □

4. Constant Speed Problems

We will now examine the more general case of particles with bounded centripetal acceleration, $|a_c(t)| \leq \bar{a_c}$, for some $\bar{a_c} \geq 0$. This case is much more relevant to physical problems, because objects typically have momentum which limits their ability to turn quickly. To keep the problem manageable, all particle in this section have zero tangential acceleration, $a_t(t) = 0$, and $v(t) = \bar{v}$.

Lemma 4.1. *Given a particle with constant speed and bounded centripetal acceleration, the minimum radius of curvature of the particle's trajectory is given by*

$$
R_{min} = \frac{\bar{v}^2}{\bar{a}_c} \tag{9}
$$

Proof. This comes directly from the definition of radius of curvature for a particle with zero tangential acceleration:

$$
R(t) = \frac{\bar{v}^2}{a_c(t)}
$$

and

$$
a_c(t) \leq \bar{a}_c \to \frac{1}{a_c} \geq \frac{1}{\bar{a}_c}
$$

□

At the beginning of our paper we mentioned that there is a case when a particle is too close to the ending position and can't turn fast enought, so it ends up passing by it. We formalize this notion in the following lemma.

Lemma 4.2. *Let particle p have constant speed, bounded centripetal acceleration, an initial angle* θ_0*, starting position* $\mathbf{X_0}$*, and ending position* $\mathbf{X_f}$*. Every valid path where* $\theta(t)$ *is minimized for all* $t \in [T_0(\hat{\gamma_\mathbf{p}}), T_f(\hat{\gamma_\mathbf{p}})]$ *has* $d(\mathbf{X_0}, \mathbf{X_f}) \geq 2R_{min}\cos(\theta_0)$.

Proof. The following figure will help illustrate the theorem.

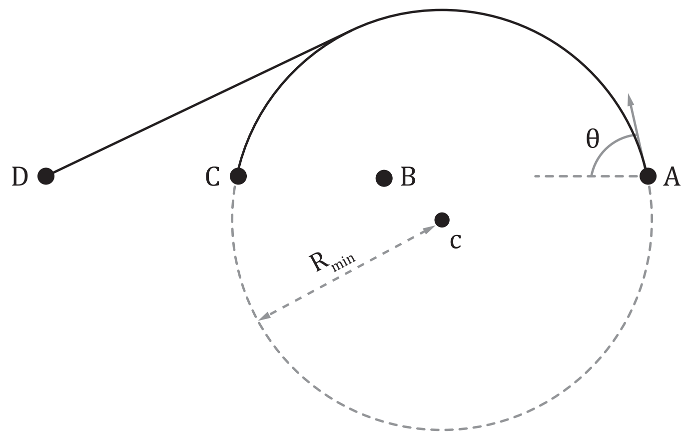

FIGURE 3

One should notice that if X_f is any position inside the circle of radius R_{min}, such as point B, then the path is invalid. It is not possible to turn inside the circle, because the centripetal acceleration is bounded. Thus, C is the closest valid ending position. C is a distance $2R_{min}\cos(\theta_0)$ away from A.

□

In order to continue, we must first make a conjecture about how fastest paths behave. Unfortunately, our research has not yet yielded a rigorous proof of this conjecture, so we will provide some intuition behind it.

Conjecture 4.3. *Let* p_1, p_2 *be particles with constant speed, bounded centripetal acceleration, starting positions* $\mathbf{X_1}, \mathbf{X_2}$ *respectively, and a final position* $\mathbf{X}_f$*, such that* $d(\mathbf{X_1}, \mathbf{X}_f), d(\mathbf{X_2}, \mathbf{X}_f) \geq 2R_{min}\cos(\theta_0)$*. If* $r_1(T_0) \leq r_2(T_0)$ *and* $|\theta_1(T_0)| < |\theta_1(T_0)|$*, then* $T_f(\hat{\gamma_1}) < T_f(\hat{\gamma_2})$.

In words, this conjecture states that a particle p_2 that is further away from X_f cannot overtake another particle p_1 that is closer to X_f, if $\theta_1(t) \leq \theta_2(t)$. The intuition behind this is that p_2 must travel further than p_1 to reach X_f, because it is further away. Furthermore, one could also imagine, that if this conjecture were not true, then it would be fastest to move away from X_f for some period of time, which seems intuitively wrong.

The special case when $d(\mathbf{X_1}, \mathbf{X}_f), d(\mathbf{X_2}, \mathbf{X}_f) < (1/2)R_{min}\cos(\theta_0)$ is omitted, because some special strategy will be needed in this situation. We leave this case as an open problem in our paper.

Assuming that Conjecture 4.3 is true, we will move on to prove one final theorem about fastest paths.

Theorem 4.4. *Let p be a particle with constant speed, bounded centripetal acceleration, initial angle θ_0, starting position $\mathbf{X}_0$, and ending position $\mathbf{X}_f$, such that $d(\mathbf{X}_0, \mathbf{X}_f) \geq 2R_{min}\cos(\theta_0)$. The unique fastest path $\hat{\gamma_\mathbf{p}}$ for particle p to reach $\mathbf{X}_f$ minimizes $\theta(t)$ for all $t \in [T_0(\hat{\gamma_\mathbf{p}}), T_f(\hat{\gamma_\mathbf{p}})]$.*

Proof. Let p_1 be a particle whose path γ_1 minimizes $|\theta_1(t)|$ for all times $t \in [T_0(\gamma_1), T_f(\gamma_1)]$. We shall show that γ_1 is the unique fastest path.

Let t_s be the smallest time such that $\theta_p(t) = 0$. From 3.2, we know that $\theta_p(t) = 0$ for $t \geq t_s$, so the particle moves in a straight line towards X_f for $t \geq t_s$. To show that $\gamma_1(t)$ truly is a fastest path, we must examine what happens for $t < t_s$.

We know that $|a_{c,1}(t)| = \bar{a}_c$ for all $t < t_s$ because p_1 follows a path that minimizes $|\theta_1(t)|$.

Now let us introduce a particle p_2 with the same conditions as p_1, but different centripetal acceleration. For some small enough $\epsilon > 0$, we can now find a time interval $t \in [t_b - \epsilon, t_b + \epsilon]$ where that $a_{c,2}(t) < a_{c,1}$. We shall show that the fastest path for p_2, $\hat{\gamma_2(t)}$, after t_b is strictly worse than the fastest path for p_1, $\hat{\gamma}_1$, which will allow us to conclude that $\gamma_1(t)$ is the unique fastest path.

Because γ_1 minimizes $|\theta(t)|$, $|\theta_1(t_b)| \leq |\theta_2(t_b)|$. If $t_b > t_s$, then from 3.2 we know that $\gamma_2(t)$ is worse than γ_1. If $t_b \leq t_s$, we will prove that $r_1(t_b) < r_2(t_b)$.

Looking at Equation 6

$$|\theta(t_b)| = |\theta(t_b - \epsilon)| + \int_{t'=t_b-\epsilon}^{t_b} \left(\frac{\bar{v}}{r(t)} \sin(|\theta(t)|) - \frac{a_c(t)}{\bar{v}} \right) dt'$$

Now we can find the difference of the two θ's and take the limit as $\epsilon \to 0$

$$\begin{aligned} |\theta_2(t_b)| - |\theta_1(t_b)| &\to \int_{t'=t_b-\epsilon}^{t_b} -\frac{a_{c,2}(t) - a_{c,1}(t)}{\bar{v}} dt' \quad \text{as} \quad \epsilon \to 0 \\ &\geq 0 \quad \text{as} \quad \epsilon \to 0 \end{aligned}$$

Furthermore,

(10)
$$r(t_b) = r(t_b - \epsilon) + \int_{t'=t_b-\epsilon}^{t_b} -v(t')\cos(|\theta(t')|)dt'$$
Taking the limit of the difference again
(11)
$$r_2(t_b) - r_1(t_b) \to \int_{t'=t_b-\epsilon}^{t_b} -v(t')(\cos(|\theta_2(t')|) - \cos(|\theta_1(t')|))dt' \quad \text{as} \quad \epsilon \to 0$$
We know that $\theta_2(t_b) \geq \theta_1(t_b)$ as $\epsilon \to 0$, so
(12)
$$r_2(t_b) - r_1(t_b) \geq 0 \quad \text{as} \quad \epsilon \to 0$$

Now we can invoke Conjecture 4.3 to show that γ_1 is a faster path than γ_2, which finishes the proof. □

5. Conclusion

In this paper, we developed a number of simple lemmas governing the motion of a particle with bounded acceleration. We also formalized many of the things that intuition tells us are true. Namely, we have shown that a straight line is the fastest path for a particle to take to get to a point, if centripetal acceleration is unbounded. This fact is unsurprising, since the particle should be accelerating towards the point as much as possible, which occurs when it is moving directly towards the point.

We posed the conjecture that it is also optimal for a particle to orient itself towards the point as quickly as possible, in the case when centripetal acceleration is bounded. Assuming this conjecture to be true, we were able to draw more conclusions about optimal paths under the new constraint of bounded centripetal acceleration.

5.1. Future Work.

Much of this paper was spend defining the terminology, coordinate system, and basic equation governing a particle. The intention was to build a strong framework for future research. Future work should first involve a rigorous proof of 4.3. Furthermore, the special case of optimal paths for when the particle is ẗoo closeẗo the final point, should be researched. This special case is very interesting, because it involves moving away from the final point for a period of time, and the shape of the optimal path does not appear to be intuitive.

To make the problem more general, one could extend it to include particles moving with varying speed. Equations 3 - 8 still apply in this case, but the analysis is more complex.

Appendix

5.2. **Vectors.**

a: Vector in n-dimensional space.
â: Unit vector in n-dimensional space. $\hat{\mathbf{a}} = \mathbf{a}/a$ and $\|\hat{\mathbf{a}}\| = 1$.
a**:** Scalar quantity. Usually $a = \|\mathbf{a}\|$, but occasionally we will also define a sign that is dependent on the direction of $\hat{\mathbf{a}}$.

5.3. **Vector Calculus in Polar Coordinates.**

$$\boldsymbol{x} = r\hat{\boldsymbol{r}}$$

$$\vec{\boldsymbol{v}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}$$

$$\vec{\boldsymbol{a}} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\boldsymbol{r}} + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right)\hat{\boldsymbol{\phi}}$$

References

[1] http://en.wikipedia.org/wiki/Polar_coordinate_system