THE 18.821 MATHEMATICS PROJECT LAB REPORT [PROOFS]

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1. Theorems

1.1. Notation.

t: Time
$l(t)$: Path length
$s(t)$: Speed
$a_t(t)$: Tangential acceleration
$a_c(t)$: Centripetal acceleration
$\vec{X}(t)$: Position (x, y) coordinate.
$\vec{v}(t)$: Velocity
$\vec{a}(t)$: Acceleration

Note: For the rest of this paper, it is assumed that all particle parameters $l, s, \ldots$ are parameterized by t. Also, define the following calculus notation

$$\dot{a} = \frac{da}{dt} \tag{1}$$

$$\ddot{a} = \frac{d^2a}{dt^2} \tag{2}$$

Definition 1.1. *Rectangular Coordinates*

$$R = \left| \frac{\left(\dot{x}^2 + \dot{y}^2\right)^{3/2}}{\dot{x}\ddot{y} - \dot{y}\ddot{x}} \right| \tag{3}$$

$$= \left| \frac{s^2}{a_t} \right| \tag*{(for $a_c =$)}$$

Date: September 27, 2013.

1.2. Scalar Calculus in Polar Coordinates.

$$s = \frac{dl}{dt} \tag{4}$$

$$a_t = \frac{ds}{dt} \tag{5}$$

$$\tag{6}$$

1.3. Vector Calculus in Polar Coordinates.

$$\boldsymbol{x} = r\hat{\boldsymbol{r}} \tag{7}$$

$$\vec{\boldsymbol{v}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}} \tag{8}$$

$$\vec{\boldsymbol{a}} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\boldsymbol{r}} + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right)\hat{\boldsymbol{\phi}} \tag{9}$$

1.4. Relations.

$$l = \int_{t=0}^{T} \|\vec{\boldsymbol{v}}\| \, dt \tag{10}$$

$$s = \|\vec{\boldsymbol{v}}\| \tag{11}$$

$$a_t = \|\vec{\boldsymbol{a}}\| \cdot \hat{\boldsymbol{v}} = \|\vec{\boldsymbol{a}}\| \cdot \frac{\vec{\boldsymbol{v}}}{\|\vec{\boldsymbol{v}}\|} \tag{12}$$

$$a_c = \|\vec{\boldsymbol{a}}\| \times \hat{\boldsymbol{v}} = \|\vec{\boldsymbol{a}}\| \times \frac{\vec{\boldsymbol{v}}}{\|\vec{\boldsymbol{v}}\|} \tag{13}$$

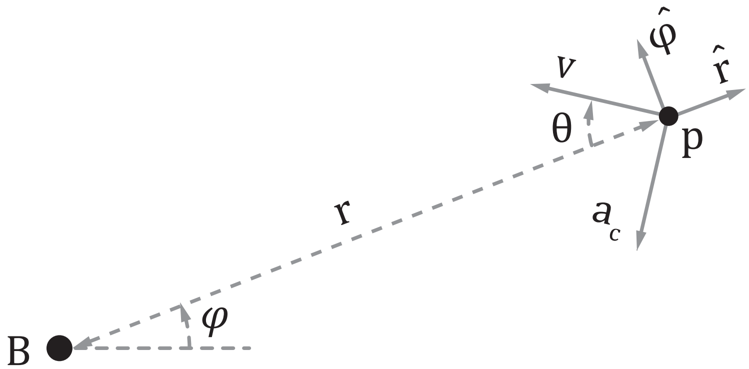
2. Lemmas and Definitions

Lemma 2.1. *If $a_t = 0$ and $\|a_c\| \leq a_{c,max}$, then the minimum radius of curvature of a point trajectory is given by*

$$R_{min} = \frac{v^2}{a_{c,max}} \tag{14}$$

Lemma 2.2. *Every optimal trajectory is constructed from line segments and circular sections of radius R_{min}. Every point along an optimal trajectory where $a_t = 0$ has $R = \infty$ or $R = R_{min}$.*

Proof. If we look at a position, p, on a trajectory. We can align a rectangular coordinate system with this point, such that $\hat{y} = \hat{v}$.



Proof.

FIGURE 1. Particle Parameterization.

$$a_c = \ddot{r} - r\dot{\phi}^2 \tag{15}$$

$$a_t = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right) \tag{16}$$

$$0 = \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right) \tag{17}$$

$$r^2\dot{\phi} = constant \tag{18}$$

□

3. Theorems

Theorem 3.1. *If $a_t = 0$ and $\|a_c\| \leq a_{c,max}$, it is always optimal to minimize the turning radius.*

Proof.

$$\vec{\boldsymbol{v}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}} \tag{19}$$

$$s = \sqrt{\dot{r}^2 + r^2\dot{\phi}^2} \tag{20}$$

$$\dot{r} = s^2 - r^2\dot{\phi}^2 \tag{21}$$

□

References

[1] http://en.wikipedia.org/wiki/Polar_coordinate_system