OPTIMAL PARTICLE PATHS AROUND POINTS IN $\mathbb{R}^2$ WITH CONSTRAINED ACCELERATION

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Abstract. Optimal particle paths in $\mathbb{R}^2$ between points and around points are investigated in this paper. A paramerization of a particle's motion around a point is defined. From this parameterization, we derive a general solution for an optimal path with constant speed between 2 points. The results of this paper can by applied towards calculating optimal object trajectories in physics when acceleration is constrained.

1. Introduction

Our model problem is a particle moving in $\mathbb{R}^2$ with constrained acceleration. The particle must navigate around cones (points in $\mathbb{R}^2$) to reach a final position. The goal is to minimize the total time spend navigating to the final position.

In this paper, we first present a polar coordinate representation of a particle's position around a point and derive a set of differential equations governing the motion of the particle in this coordinate system. This allows us to devise some simple lemmas about the motion of the particle.

After developing an intuition for particles and particle paths, we then move on to the more complicated problem of deriving governing rules for optimal paths between points.

Finally, we are able to tackle the problem of finding an optimal path around cones.

2. Notation

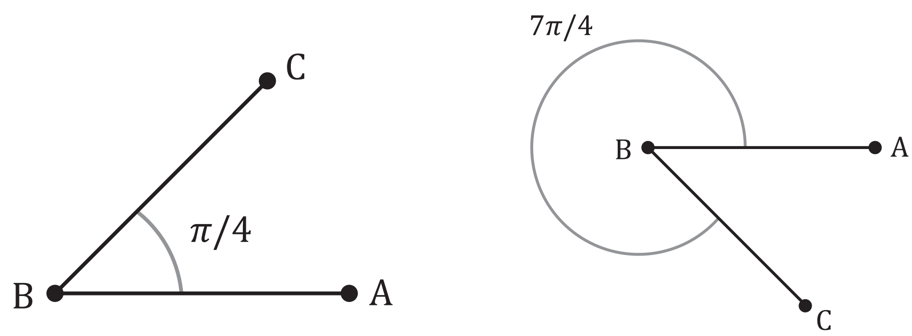
2.1. Vectors.

a**:** Scalar quantity.
a: Vector in n-dimensional space. $\|\mathbf{a}\| = a$.
$\hat{\mathbf{a}}$**:** Unit vector in n-dimensional space. $\hat{\mathbf{a}} = \mathbf{a}/a$ and $\|\hat{\mathbf{a}}\| = 1$.

Date: September 28, 2013.

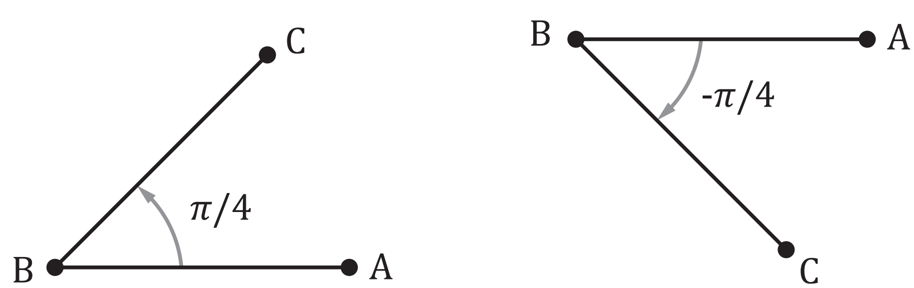
2.2. **Angles.** We define two different types of angle measurements: a standard measurement, and a directional measurement (all angles are measured in radians).

In the standard measurement, angles are in $[0, 2\pi]$ and are measured counterclockwise. In the directional measurement, angles are in $[-\pi, \pi]$ and the measurement direction is indicated by an arrow on the measurement.



(A) (B)

Figure 1. Standard angle notation (no arrow)



(A) (B)

Figure 2. Directional angle notation (arrow)

3. Particles and Paths

In this section, we will provide some basic definitions about particles and paths. These will lay the groundwork for thinking about optimal paths.

Definition 3.1. *A n-dimensional path* $\gamma(t) : \mathrm{R} \to \mathrm{R}^n$ *is a function which maps a time* $t \in \mathbb{R}$, $t \in [0, T_{f,\gamma}]$, *to a position* $\mathbf{X} \in \mathrm{R}^n$.

Definition 3.2. *A n-dimensional particle, p, is an object with zero volume that travels along a n-dimensional path. Traveling along a path means that the particle is at position* $\lambda(t)$ *at time* t, *for all* $t \in [0, T_{f,\lambda}]$. *The particle may have conditions on its position, velocity, and acceleration in* $\mathbb{R}^n$.

Definition 3.3. *A valid path* $\gamma(t)$ *for a particle* p *is a path along which* p *can travel, such that all its conditions are satisfied at every point along the path.*

Definition 3.4. *A path between two points,* $\mathbf{X_1}$ *and* $\mathbf{X_2}$ *is a path,* $\gamma(t)$, *where* $\gamma(0) = \mathbf{X_1}$ *and* $\gamma(T_{f,\gamma}) = \mathbf{X_2}$.

Definition 3.5. *For a given particle, p, a fastest path,* $\hat{\gamma}(t)$, *between two points,* $\mathbf{X_1}$ *and* $\mathbf{X_2}$, *is a valid path such that* $T_{f,\hat{\gamma}} \leq T_{f,\gamma}$ *for all valid paths,* $\gamma(t)$, *between* $\mathbf{X_1}$ *and* $\mathbf{X_2}$.

Definition 3.6. *A particle's velocity is defined as*

$$\mathbf{v} := \frac{d\mathbf{X}}{dt}$$

Furthermore, a particle's speed is v, *and its direction of motion is* $\hat{\mathbf{v}}$

Definition 3.7. *A particle's acceleration is defined as*

$$\mathbf{a} := \frac{d\mathbf{v}}{dt}$$

Definition 3.8. *The centripetal acceleration,* $\mathbf{a_c}$, *of a particle, p, is the component of its acceleration in the direction perpendicular to its direction of motion.*

In rectangular coordinates, the sign of a_c *is defined to be the sign of the projection of* $\hat{\mathbf{a_c}}$ *onto* $\hat{\mathbf{x}}$.

In polar coordinates, the sign of a_c *is defined to be the sign of the projection of* $\hat{\mathbf{a_c}}$ *onto* $\hat{\mathbf{r}}$.

Definition 3.9. *The tangential acceleration,* $\mathbf{a_t}$, *of a particle, p, is the component of the acceleration of* p *in its direction of motion,* $\hat{\mathbf{v}}$.

$$a_t = \frac{dv}{dt}$$

3.1. **Particle Motion in Polar Coordinates.** Unless otherwise specified, the motion of 2-dimensional particles in this paper will be described in polar coordinates, along with an extra parameter θ as is shown in Figure 3.

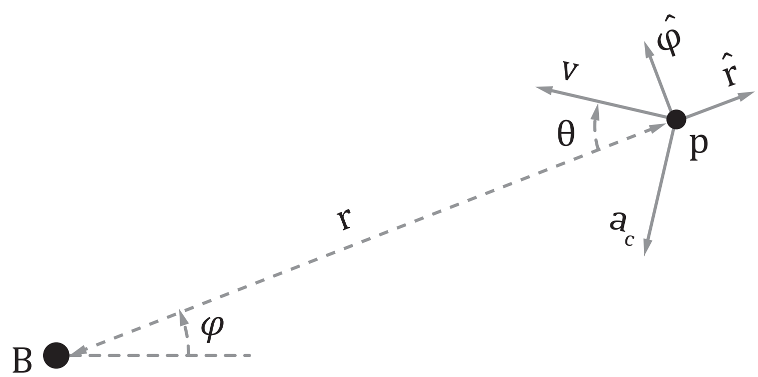


FIGURE 3. A particle moving in 2-dimensional polar coordinate system centered at B.

$\theta(t), \phi(t) \in [-\pi, \pi]$. From now on, $|\theta|$ and $|\phi|$ will be used, since there is a symmetry in the system about $\hat{\mathbf{r}}$.

Lemma 3.10. *The time derivative of θ is given by*

$$\frac{d|\theta(t)|}{dt} = \frac{a_c}{v}$$

Proof. If we look at a point, p, subject to only centripetal acceleration, a_c, the change in $\mathbf{v}$ over an infinitesimal time, dt, is shown in Figure 4 (the two vectors, $\mathbf{v}$ and $\mathbf{v} + \mathbf{dv}$, are superimposed).

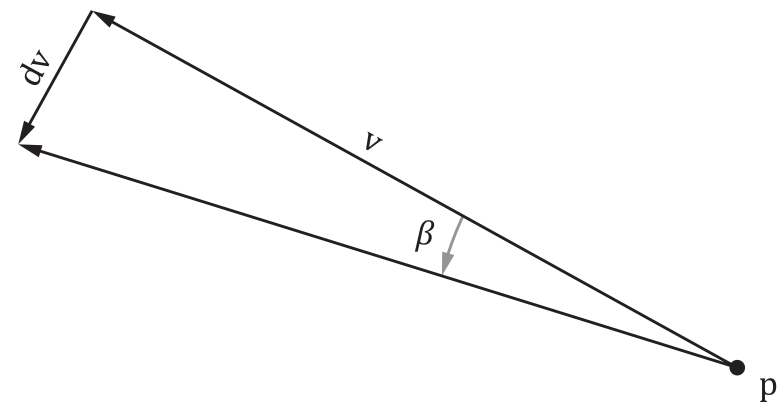


FIGURE 4. Tangential acceleration.

From the definition of a_c

$$\frac{dv}{dt} = a_c$$

Since $dv = d|\theta|v$, then

$$\frac{d|\theta|}{dt} = \frac{a_c}{v}$$

□

The proof of the lemma in the case where a_t is also nonzero is very similar, since the component of $\mathbf{dv}$ in the direction $\hat{\mathbf{v}}$ is negligible compared to v.

Returning again to Figure 3, the following equations can be derived

$$\frac{dr(t)}{dt} = -v(t)\cos(|\theta(t)|) \tag{1}$$

$$\frac{d|\phi(t)|}{dt} = \frac{v(t)}{r(t)}\sin(|\theta(t)|) \tag{2}$$

$$\frac{d|\theta(t)|}{dt} = \frac{d|\phi(t)|}{dt} - \frac{a_c(t)}{v(t)} \tag{3}$$

$$= \frac{v(t)}{r(t)}\sin(|\theta(t)|) - \frac{a_c(t)}{v(t)} \tag{4}$$

Applying the chain rule to (1)

$$\frac{d}{dt}\frac{dr(t)}{dt} = -\frac{dv(t)}{dt}\cos(|\theta(t)|) + v(t)\sin(|\theta(t)|)\frac{d|\theta(t)|}{dt} \tag{5}$$

$$= -\frac{dv(t)}{dt}\cos(|\theta(t)|) + \frac{v(t)^2}{r(t)}\sin^2(|\theta(t)|) - v(t)\sin(|\theta(t)|)\frac{a_c(t)}{v(t)} \tag{6}$$

Lemma 3.11. *For a particle, p, with bounded centripetal and tangential acceleration, then:*

1. The function $|\theta(t)|$ is continuous.

2. Given $t_1, t_2 \in \mathbb{R}^2$, s.t. $t_1 < t_2$, $\theta(t_1) > a$ and $\theta(t_2) < a$, then $\theta(t_c) = a$ for some $t_c \in [t_1, t_2]$.

Proof. First off, it should be noted that a solution to (4) exists because the right hand side is Lipschitz continuous. The proof of 1. follows directly from the fact that the derivative of $|\theta(t)|$ exists and is bounded. 2. is just a restatement of the intermediate value theorem. □

Lemma 3.12. *For a particle, p, with nonzero speed, and zero centripetal acceleration for $t \geq t_0$, then*

$$\begin{cases} |\theta(t)| \to \pi \quad \textit{as} \quad t \to \infty & \textit{if } \theta(t_0) > 0 \\ |\theta(t)| = 0 \quad \textit{for all} \quad t \geq t_0 & \textit{if } \theta(t_0) = 0 \end{cases}$$

Furthermore,

$$\begin{cases} r(t) \to \infty \quad \textit{as} \quad t \to \infty & \textit{if } \theta(t_0) > 0 \\ r(t) \to 0 \quad \textit{as} \quad t \geq t_0 & \textit{if } \theta(t_0) = 0 \end{cases}$$

Proof. For the first case, $\frac{d|\theta(t)|}{dt} > 0$, except when $|\theta(t)| = 0, \pi$, which means that $\theta(t)$ is monotonically increasing. Since $\phi(t)$ is bounded above py π, $\phi(t) \to \pi$ as ∞.

The second case is the easiest to check. If we plug $\theta(t_0) = 0$ into 4, we get

$$\frac{d|\theta(t)|}{dt} = 0 \qquad \text{for } t \geq t_0$$

So

$$|\theta(t)| = 0 \qquad \text{for } t \geq t_0$$

[TODO]

□

4. Traveling Between Points

The simplest problem to tackle first, is the optimal trajectory from a current position, $\mathbf{X_0}$ to a final position, $\mathbf{X_f}$. Working in a polar coordinate system centered on X_f, it is clear, that our goal is achieved when $r = 0$. Furthermore, $\theta(t)$ must be 0 before reaching X_f, otherwise, it will grow exponentially fast as $r(t) \to 0$, which can be seen in equation (1).

Lemma 4.1. *If $a_t = 0$ and $a_c \leq a_{c,max}$, then the minimum radius of curvature of a particle's trajectory is given by*

$$R_{min} = \frac{v^2}{a_{c,max}} \tag{7}$$

Proof. This comes directly from the definition of radius of curvature

$$R = \frac{v^2}{a_c}$$

and the fact that

$$a_c \leq a_{c,max} \to \frac{1}{a_c} \geq \frac{1}{a_{c,max}}$$

□

Theorem 4.2. *A 2-dimensional particle, p, traveling along a valid path, $\hat{\gamma(t)}$, from X_1 to X_2 has $\frac{d|\theta(t)|}{dt} < 0$ for all $t \in [0, T_{f,\gamma}]$, unless $\|X_1 - X_2\| < 2R_{min}$.*

Proof. Since $\theta(T_{f,\gamma}) = 0$, then if $\frac{d|\theta(t)|}{dt} > 0$ at some point along the path, then $\theta T_{f,\gamma} < \theta t_0 < \theta t_c$, so [TODO] □

Theorem 4.3. *Given a 2-dimensional particle, p, it is always optimal to minimize $\frac{d|\theta(t)|}{dt}$ at all $t \in [0, T_\gamma]$.*

TODO. □

Theorem 4.4. *Given a 2-dimensional particle, p, with bounded centripetal acceleration, $a_c \leq a_{c,max}$ and zero tangential acceleration, every valid optimal path, $\hat{\gamma}(t)$ between two points X_1 and X_2, has a radius of curvature of 0 or $a_{c,max}$ at every point along it.*

TODO. □

Corollary 4.5. *Given a 2-dimensional particle, p, with bounded centripetal acceleration, $a_c \leq a_{c,max}$ and zero tangential acceleration, every valid optimal trajectory is constructed from line segments and circular sections of radius R_{min}, that are tangent at their intersections.*

TODO. □

Theorem 4.6. *Given a 2-dimensional particle p whose initial position is $\mathbf{X_1}$ and moves with bounded tangential acceleration, $a_t \leq a_{t,max}$, and infinite centripetal acceleration, the fastest path $\hat{\gamma}(t)$ which p can trace from $\mathbf{X_1}$ to $\mathbf{X_2}$ is the line segment from $\mathbf{X_1}$ to $\mathbf{X_2}$:*

Proof. Without loss of generality, we can define a cartesian coordinate system, where the origin is at $\mathbf{X_1}$ and the positive x-axis passes through $\mathbf{X_2}$.

Now let us examine the particle's motion in the $\hat{\mathbf{x}}$ direction. The speed of the particle is the following

$$v_x(t) = \int_0^t a_x(t_1)dt_1 \tag{8}$$

To find the distance $l_x(t)$ travelled in the $\hat{\mathbf{x}}$ direction, we can use the relation:

$$d(t) = \int_0^t v_x(t_2)dt_2 \tag{9}$$

$$= \int_0^t \int_0^t a_x(t_1)dt_1dt_2 \tag{10}$$

Recall that the tangential acceleration of the point mass p is bounded by $a_{t,max}$. This means that $a_t(t) \leq \bar{a}$ for all t. Therefore, we see:

$$d(t) \leq \int_0^t \int_0^t \bar{a}dt_1dt_2 \tag{11}$$

$$= \frac{\bar{a}t^2}{2} \tag{12}$$

Thus, in order to travel a distance of $d(T_f) = x_2$, it needs to be the case that $T_f \geq \sqrt{\frac{2x_2}{\bar{a}}}$. Moreover, equality holds if and only if $a_t(t) = \bar{a}$ for all $t \in [0, T_f(\gamma)]$.

If the point mass travels for time $t < \sqrt{\frac{2x_2}{\bar{a}}}$, then it is impossible for the point mass to reach $(x_2, 0)$ when starting at $(0, 0)$. This is because p cannot reach $(x_2, 0)$ in the x direction when $t < \sqrt{\frac{2x_2}{\bar{a}}}$ and any acceleration in the y direction would not enable this either.

This means that the fastest path is completed in time $T_f(\hat{\gamma}) = \sqrt{\frac{2x_2}{\bar{a}}}$. Let us examine the path taken by the point mass p on this fastest path. Recall that $a_t(t) = \bar{a}$ for all t along the fastest path. This means that there was no centripetal acceleration $|a_c| = 0$. In other words, the point mass never turned on its way to reaching the destination point. The only way this could have happened is if it travelled along the x axis in a straight line.

Now, we have seen that the fastest path in the transformed coordinates travels exactly on the x axis so that $y = 0$ anywhere along the fastest path. Notice, however, that the x axis in the transformed coordinates is given exactly by the following line:

$$y = \frac{y_2 - y_1}{x_2 - x_1} x + y_1 \tag{13}$$

Thus, we see that the fastest path in the original coordinates follows the above equation, which is what we wanted to show. □

Corollary 4.7. *Given points $(x_1, y_1), (x_2, y_2) \in \mathrm{R}^2$ and a particle p whose initial position is (x_1, y_1) which moves with acceleration bounded by $\bar{a}$, the fastest path $\hat{\gamma}(t)$ which p can trace from (x_1, y_1) to (x_2, y_2) is unique.*

Proof. We have already shown that any fastest path between (x_1, y_1) and (x_2, y_2) follows the straight line given by $y = \frac{y_2-y_1}{x_2-x_1}x+y_1$. Moreover, we showed that when travelling along the fastest path, the particle must have acceleration along the straight line of $\bar{a}$. Since we have starting position (x_1, y_1) and initial speed of 0, the acceleration of the particle $a(t)$ uniquely defines a path for the particle.

There is only a single function $a(t) = \bar{a}$ which the acceleration can satisfy when the particle is moving along a fastest path, therefore, there is only a single possible fastest path. □

5. Turning Around Cones

Now we are finally able to tackle the model problem of optimal paths around cones. First we will solve a simpler but related problem of minimizing the path length around a quadrilateral.

5.1. Quadrilateral.

We would like to minimize the path length around the quadrilateral shown in Figure 5.

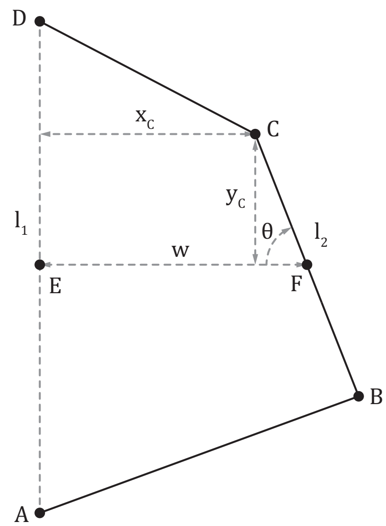


FIGURE 5. Quadrilateral path. A → B → C → D.

We will constrain the problem such that it is similar to the problem of a particle traveling around a cone. Sides $\overline{BC}$, $\overline{DA}$, and $\overline{EF}$ have fixed lengths l_1, l_2, and w respectively. Points E and F bisect $\overline{DA}$ and $\overline{BC}$ respectively. The goal is

$$\arg\min_{\theta}[l_{AB} + l_{BC} + l_{CD}]$$

$$x_c = w - \frac{l_2}{2\cos\theta} \tag{14}$$

$$y_c = \frac{l_2}{2} - \frac{l_2}{2\sin\theta} \tag{15}$$

$$l_{AB} = \sqrt{x_c^2 + y_c^2} \tag{16}$$

$$= \sqrt{\left(\frac{l_1}{2} - \frac{l_2}{2}\cos\theta\right)^2 + \left(w - \frac{l_2}{2}\sin\theta\right)^2} \tag{17}$$

$$l_{BC} = l_2 \tag{18}$$

$$l_{CD} = \sqrt{\left(\frac{l_1}{2} + \frac{l_2}{2}\cos\theta\right)^2 + \left(w - \frac{l_2}{2}\sin\theta\right)^2} \tag{19}$$

Since we know that l_{BC} is fixed the problem reduces to

$$\arg\min_{\theta}[l_{AB} + l_{BC} + l_{CD}]$$

Because all distances are positive, we can use Lemma 6.1, defined in the Appendix, to simplify our problem:

If we define $l_{AB} = \sqrt{z_{AB}}$ and $l_{CD} = \sqrt{z_{CD}}$, we can use 6.1 with $k(t) = \sqrt{t}$ to simplify our problem to

$$\arg\min_{\theta} \left[2\left(\frac{l_1}{2} - \frac{l_2}{2}\sin\theta\right)^2 + \left(w - \frac{l_2}{2\cos\theta}\right)^2 + \left(w + \frac{l_2}{2}\cos\theta\right)^2 \right] \tag{20}$$

Now, we can expand out our expression and use Lemma the fact that $\sin^2\theta + \cos^2\theta = 1$ to obtain a much simpler (but equivalent) minimization problem:

$$\arg\min_{\theta} \frac{l_1^2}{2} + \frac{l_2^2}{2} + 2d^2 - l_1 l_2 \sin\theta \tag{21}$$

We note that l_1, l_2, and d are all constants which are given to us in the problem. Therefore, the minimization problem really boils down to

$$\arg\min_{\theta} -l_1 l_2 \sin\theta = \arg\max_{\theta} \sin\theta \tag{22}$$

Theorem 5.1. *Given a particle with zero tangential acceleration and bounded centripetal acceleration, $a_c \leq a_{c,max}$, and a cone setup consisting of 3 cones, A, B, and C, at locations $\mathbf{X_A}$, $\mathbf{X_B}$, and $\mathbf{X_C}$ as is shown in Figure 6*

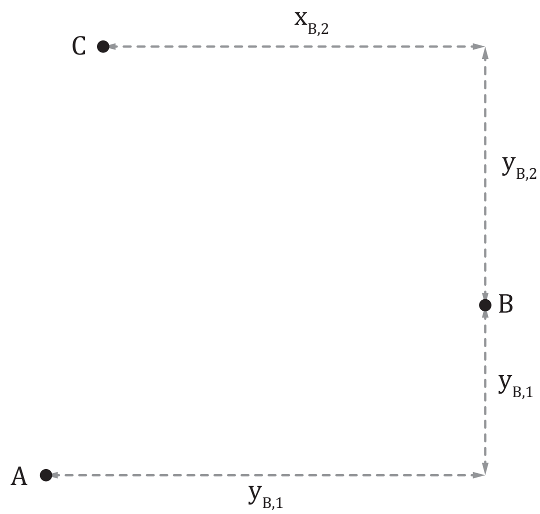


FIGURE 6. 3 cone setup.

the optimal path around the cones is shown in Figure 7

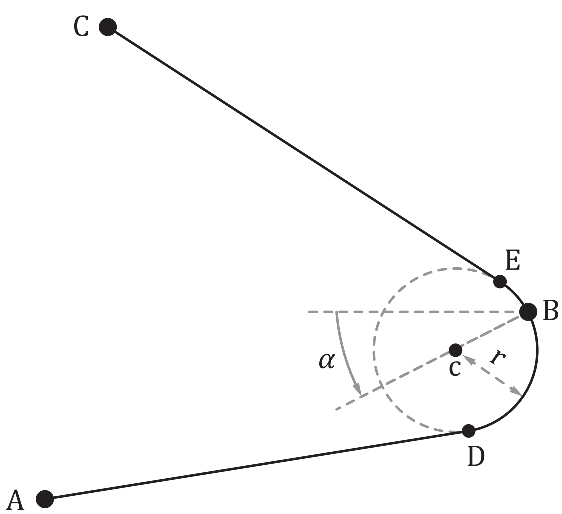


FIGURE 7. 3 cone optimal path.

TODO: Finish proof. Since speed is constant, the optimal path will be the shortest path. By Corollary 4.5, the optimal path can only include line segments and circular sections. Thus, the problem reduces to a trigonometric problem.

The turn must touch cone B, otherwise the trajectory will either not go around the cone, or be longer than necessary. Therefor, the center of the turn, c, must be a distance r away from cone B.

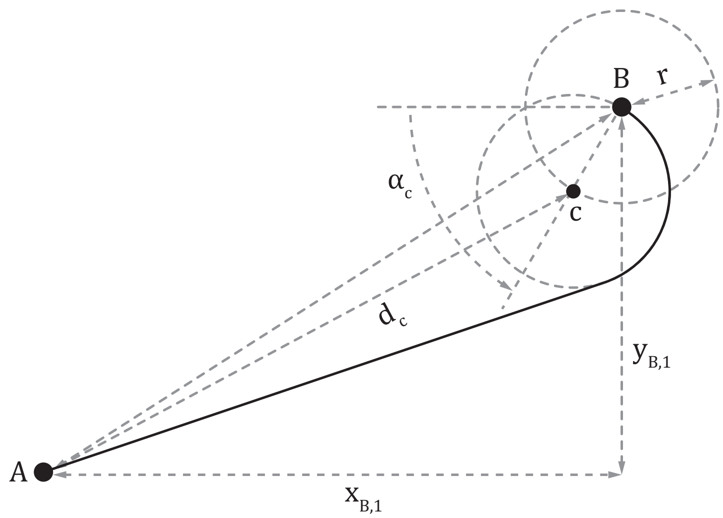


FIGURE 8. Position of center of turn.

Now that c is defined by α, let us determine the total path length as a function of α.

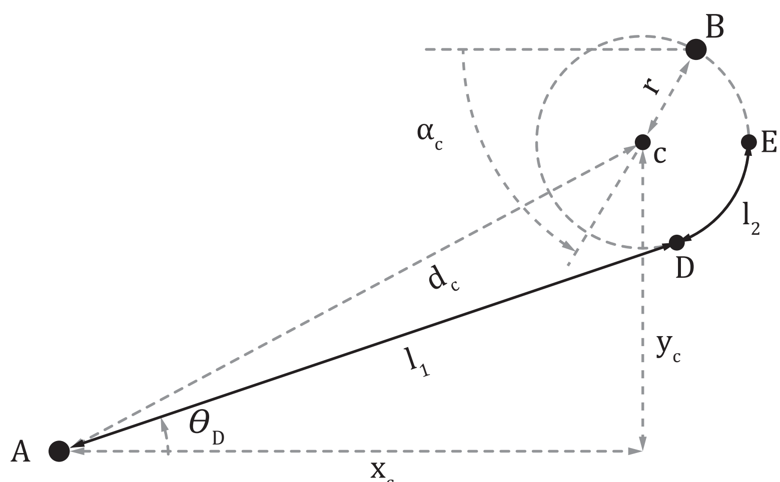


FIGURE 9. Path length.

$$\begin{aligned}
x_c &= x_B - r\cos\alpha_c \\
y_c &= y_B - r\sin\alpha_c \\
l_1 &= \sqrt{d_c - r} \\
l_2 &= r\left(\alpha_c - \tan^{-1}\left(\frac{y_c}{x_c}\right)\right)
\end{aligned}$$

□

6. Conclusion

Appendix

6.1. Analysis.

Lemma 6.1. *If $f(t), g(t) > 0$ and $k(t) > 0$ is a strictly monotonically increasing function for all t, then $\arg\min_t k(f(t)) + k(g(t)) = \arg\min_t f(t) + g(t)$.*

Proof. Let t_1, t_2 be such that $f(t_1)+g(t_1) < f(t_2)+g(t_2)$. In this proof, we will show that $k(f(t_1)) + k(g(t_1)) < \sqrt{f(t_2)} + \sqrt{g(t_2)}$. Since $k(t)$ is a strictly monotonically increasing function when $t > 0$, we know that $k(x) < k(y)$ if and only if $x < y$ (assuming we can confine x, y to be non-negative).

Because this is the case, we see that $k(f(x)) < k(f(y))$ if and only if $f(x) < f(y)$ (the same goes for g). Thus, we see that if we have found the minimum t_m to $\arg\min_t k(f(t)) + k(g(t))$, then it is the case that $k(f(t_m)) + k(g(t_m)) < k(f(t)) + k(g(t))$ for all $t \neq t_m$ (again where $t > 0$). Following our train of logic, we see that $f(t_m)+g(t_m) < f(t)+g(t)$ for all $t \neq t_m$, which means that t_m is a minimum of $f(t) + g(t)$. Thus by finding a minimum t_m to $k(f(t))+k(g(t))$, we also found a minimum to $f(t) + g(t)$. □

6.2. Coordinate Systems.

6.2.1. *Radius of Curvature.* The radius of curvature, R, of a curve at a point is a measure of the radius of the circular arc which best approximates the curve at that point.

For $a_t = 0$

$$R = \left|\frac{v^2}{a_c}\right|$$

6.3. Vector Calculus in Polar Coordinates.

$$\boldsymbol{x} = r\hat{\boldsymbol{r}}$$

$$\vec{\boldsymbol{v}} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\phi}\hat{\boldsymbol{\phi}}$$

$$\vec{\boldsymbol{a}} = \left(\ddot{r} - r\dot{\phi}^2\right)\hat{\boldsymbol{r}} + \frac{1}{r}\frac{d}{dt}\left(r^2\dot{\phi}\right)\hat{\boldsymbol{\phi}}$$

References

[1] http://en.wikipedia.org/wiki/Polar_coordinate_system