Problem Set 2, Part a

Due: Thursday, October 8, 2015
Problem sets will be collected in class. Please hand in each problem on a separate page.
Students who agree to let us hand out their writeups can help us by writing elegant and concise solutions and formatting them using LaTeX.

Readings:

Peleg book, Chapters 7 and 8; Section 24.2.
Stephan Holzer's class notes

For next week:
Section 5.1; Chapter 6; Aguilera, Toueg paper, listed in Handout 3; Keidar, Rajsbaum paper (skim).

Problems:

1. Design a deterministic distributed graph algorithm that colors any graph G in which all nodes' degrees are all at most 4. The algorithm should use a constant number of colors and work in time $O(log^*n)$. **Hint:** It can be similar to the 16-coloring $O(log^*n)$ algorithm for trees that was described in class.

2. Here we consider *deterministic* synchronous algorithms to color undirected *path graphs* G, which consist of some number n of nodes and $n-1$ undirected edges connecting the nodes in a line. Assume that processes at the nodes can distinguish their left and right neighbors, and can tell if they are endpoints of the line. Assume that the processes have UIDs from a large totally-ordered id space.

   (a) Prove that there is no deterministic algorithm that 2-colors all path graphs of any size n, and requires at most $n/4$ rounds.

   (b) Prove that there is no deterministic algorithm that 4-colors all path graphs of any size n, and requires only a constant number of rounds.

3. Consider a synchronous network based on an undirected graph $G = (V, E)$, with an upper bound Δ on the degree (number of neighbors) of any vertex in G. In this problem, we use a distributed *Maximal Independent Set (MIS)* algorithm, such as Luby's, as a subroutine to produce a coloring of the vertices of G in which no two neighboring vertices are colored with the same color. The total number of colors will be at most $\Delta + 1$.

   The algorithm works as follows: Each node associated with a vertex of G simulates $\Delta + 1$ nodes in a larger graph $G' = (V', E')$, which is a "Cartesian product" of G with a clique graph of size $\Delta + 1$. That is, G' has $(\Delta + 1)|V|$ vertices: $V' = \{(u, i) \mid u \in V \text{ and } 0 \leq i \leq \Delta\}$. Edges in E' connect the following vertex pairs:

   - (u, i) and (v, i), for any $(u, v) \in E$ and any particular i;
   - (u, i) and (u, j), for any $u \in V$ and any $i \neq j$.

   The nodes in G simulate an MIS algorithm on G', and produce an MIS S for G', where each node of G learns which of its simulated nodes correspond to vertices in S.

   (a) Prove that, for each vertex $u \in V$, S must contain exactly one of the $\Delta + 1$ vertices in V' of the form (u, i).

(b) Using the insight from Part (a), construct a coloring of the vertices of G, using colors $0, 1, \ldots, \Delta$. Prove that this is a valid coloring, that is, no two neighboring vertices of G are colored with the same color.

(c) Analyze the expected time and communication costs for solving the coloring problem in this way, including the cost of solving MIS using Luby's algorithm.

4. As we have seen in problem set 1b, in *SynchGHS* algorithm, $O(diam)$ rounds are not always sufficient to complete each level of computation. Consider a fully connected weighted graph $G = (V, E, \omega)$ (a CLIQUE) with nodes $V = \{1, \ldots, n\}$ and weights $\omega(i, j) = \min(i, j) + (|i - j| - 1) \cdot n$ with one exception: $\omega(n/2, n/2 + 1) = n$.

   (a) What does the MST of this graph look like?

   (b) What is the runtime of the *SynchGHS* algorithm on this graph?

   (c) What is the runtime of the fast MST algorithm on this graph?

   (d) Show that in a CLIQUE (independent of the weight function ω) an MST can be computed in the CONGEST model in time $O(\log n)$.