6.852: Distributed Algorithms
Fall, 2015

Lecture 7

Today's plan

- Exponential Information Gathering (EIG) algorithm for Byzantine agreement.
- Lower bounds:
  - Number-of-processors lower bound for Byzantine agreement.
  - Connectivity bounds.
  - Weak Byzantine agreement.
  - Time lower bounds for stopping and Byzantine agreement.
- Reading:
  - Sections 6.3-6.7
  - [Aguilera, Toueg]
  - [Keidar, Rajsbaum]
- Next: Some other distributed agreement problems

Byzantine agreement

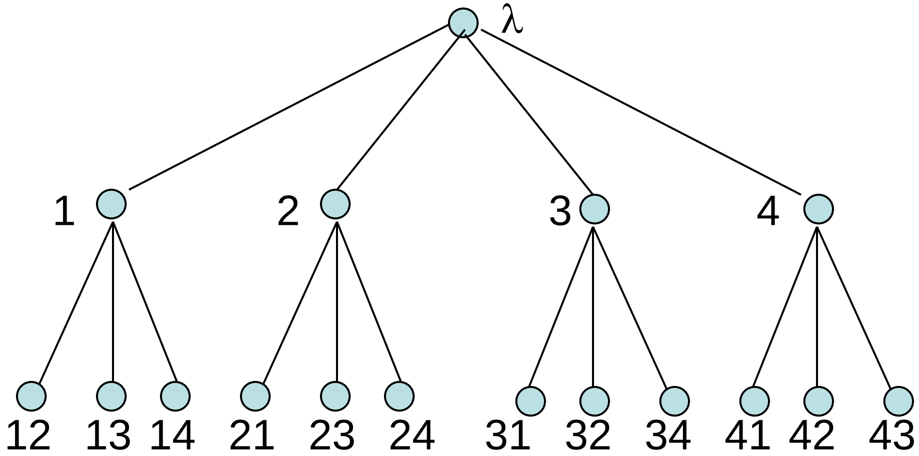
- Recall correctness conditions:
  - Agreement: No two nonfaulty processes decide on different values.
  - Validity: If all nonfaulty processes start with the same v, then v is the only allowable decision for nonfaulty processes.
  - Termination: All nonfaulty processes eventually decide.
- EIG algorithm for Byzantine agreement, using:
  - Exponential communication (in f)
  - f+1 rounds
  - n > 3f

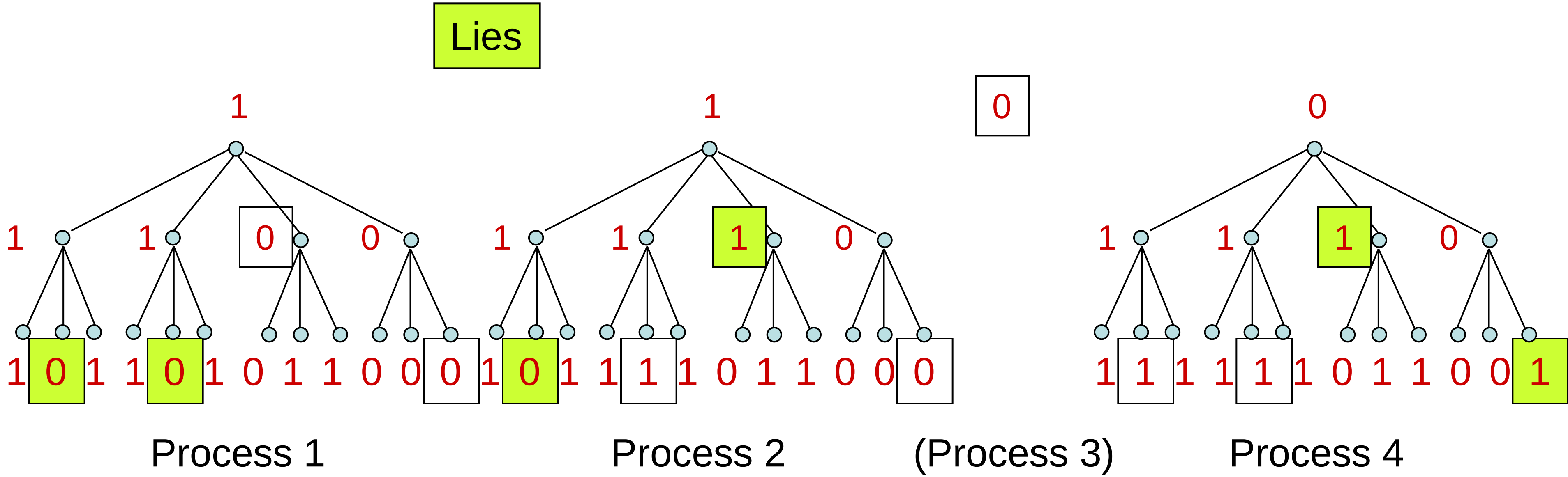
EIG algorithm for Byzantine agreement

- Assume $n > 3f$.
- Same EIG tree as before.
- Relay messages for $f+1$ rounds, as before.
- Decorate the tree with values from V, replacing any garbage messages with default value v_0.
- Call the decorations val(x), where x is any node label.
- New decision rule:
  - Redecorate the tree bottom-up, defining newval(x).
    - Leaf: newval(x) = val(x)
    - Non-leaf: newval(x) =
      - newval of strict majority of children in the tree, if majority exists,
      - v_0 otherwise.
  - Final decision: newval(λ) (newval at root)

Example: n = 4, f = 1

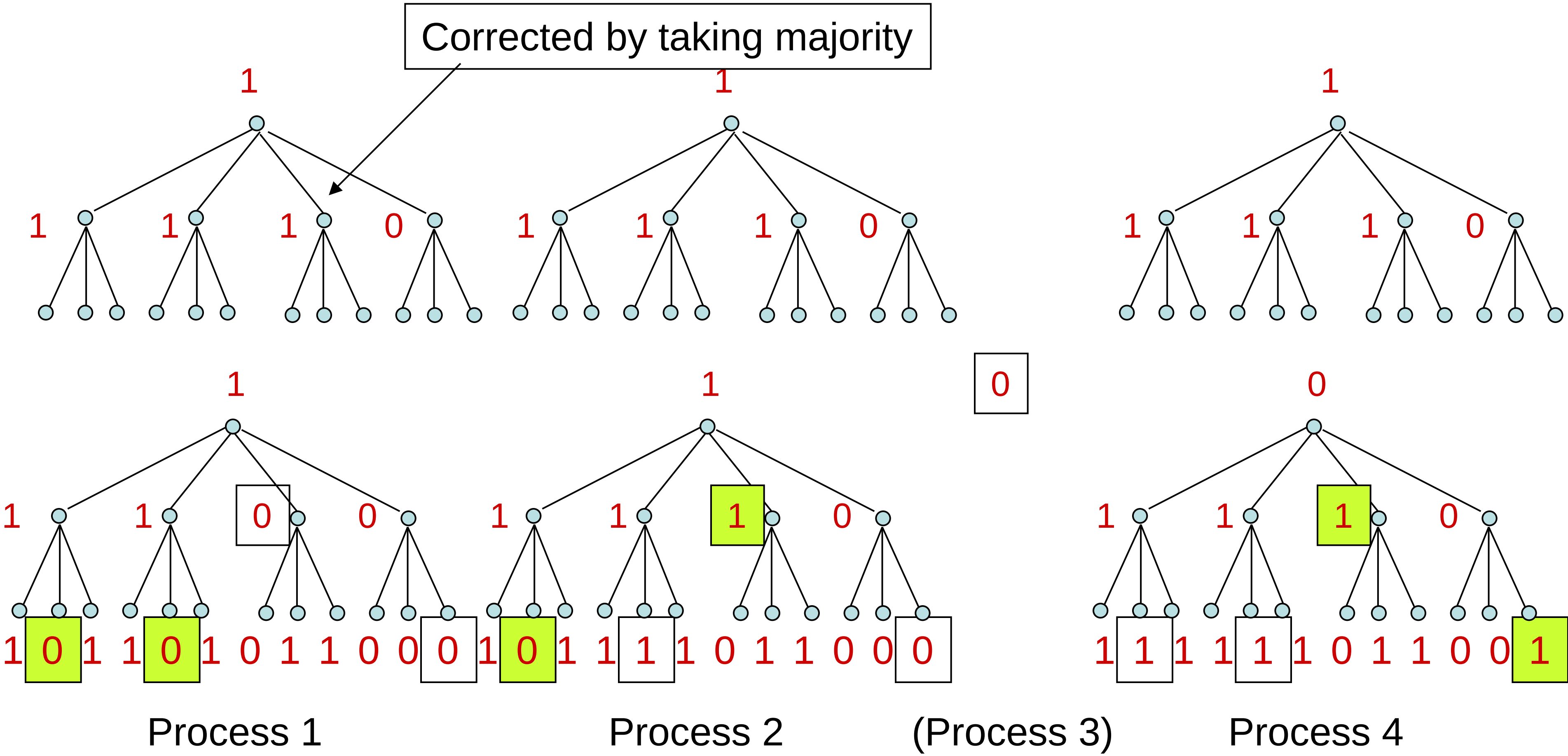
- $T_{4,1}$:
- Consider a possible execution in which p3 is faulty.
- Initial values 1 1 0 0
- Round 1
- Round 2





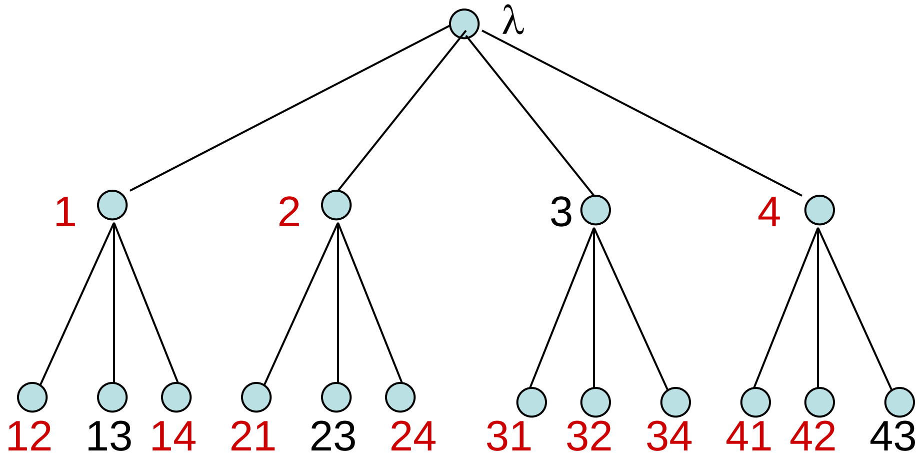
Example: n = 4, f = 1

- Now calculate newvals, bottom-up, choosing majority values, $v_0 = 0$ if no majority.



Correctness proof

- Lemma 1: If i, j, k are nonfaulty, then $val(x)_i = val(x)_j$ for every node label x ending with k.
- In example, such nodes are (in red):



- Proof: k sends same message to i and j and they decorate accordingly.

Proof, cont'd

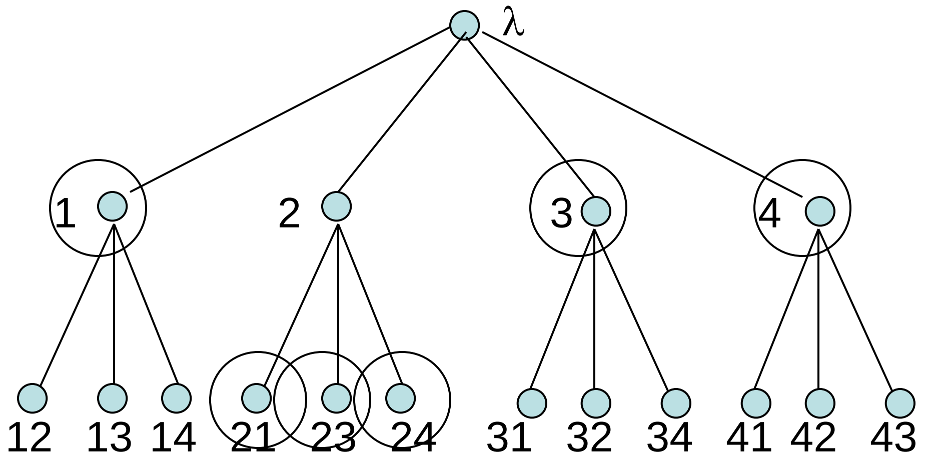
- Lemma 2: If x ends with a nonfaulty process index then $\exists v \in V$ such that $val(x)_i = newval(x)_i = v$ for every nonfaulty i.
- Proof: Induction on lengths of labels, bottom up.
  - Basis: Leaf.
    - Lemma 1 implies that all nonfaulty processes have same val(x).
    - newval = val for each leaf.
  - Inductive step: $|x| = r \leq f$ ($|x| = f+1$ at leaves)
    - Lemma 1 implies that all nonfaulty processes have same val(x), say v.
    - We need newval(x) = v everywhere also.
    - Every nonfaulty process j broadcasts same v for x at round r+1, so $val(xj)_i = v$ for every nonfaulty j and i.
    - By inductive hypothesis, also $newval(xj)_I = v$ for every nonfaulty j and i.
    - A majority of labels of x's children end with nonfaulty process indices:
      - Number of children of node x is $\geq n - f > 3f - f = 2f$.
      - At most f are faulty.
    - So, majority rule applied by i leads to $newval(x)_i = v$, for all nonfaulty i.

Main correctness conditions

- Validity:
  - If all nonfaulty processes begin with v, then all nonfaulty processes broadcast v at round 1, so $val(j)_i = v$ for all nonfaulty i, j.
  - By Lemma 2, also $newval(j)_i = v$ for all nonfaulty i,j.
  - Majority rule implies $newval(\lambda)_i = v$ for all nonfaulty i.
  - So all nonfaulty i decide v.
- Termination:
  - Obvious.
- Agreement:
  - Requires a bit more work:

Agreement

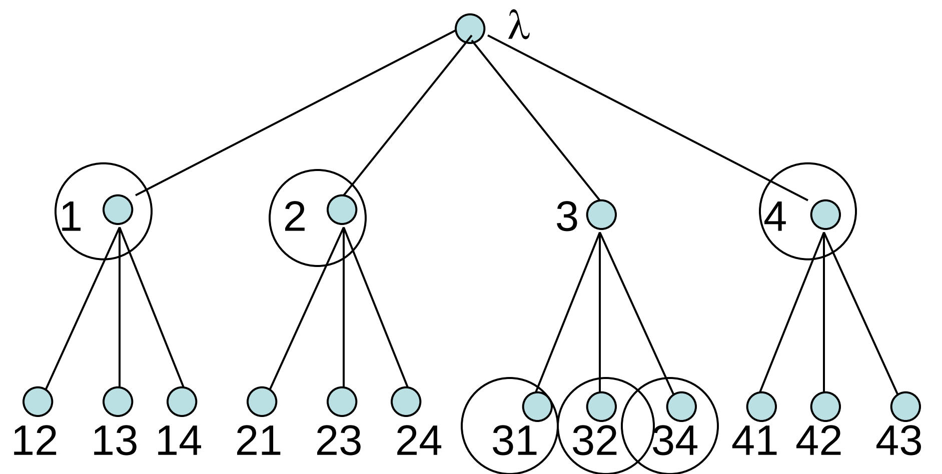
- Path covering: Subset of nodes containing at least one node on each path from root to leaf:



- Common node: One for which all nonfaulty processes have the same newval.
- If a node's label ends in a nonfaulty process index, Lemma 2 implies it's common.
- Others might be common too.

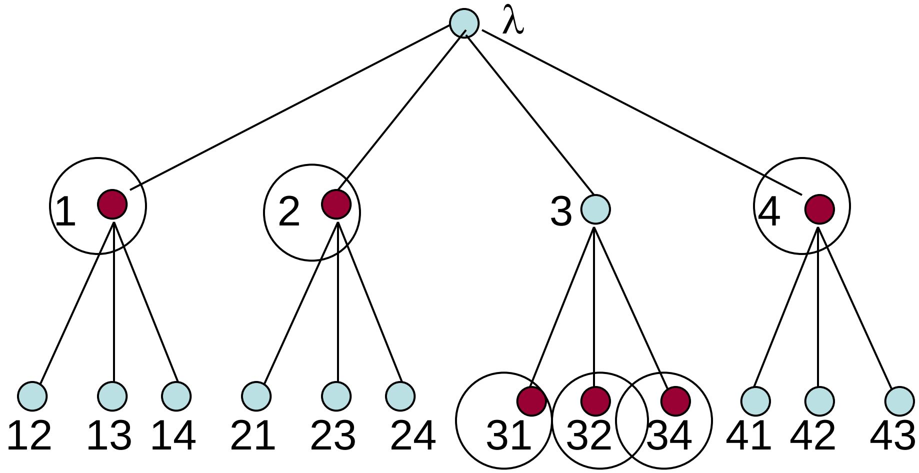
Agreement

- Lemma 3: There exists a path covering all of whose nodes are common.
- Proof:
  - Let C = nodes whose labels end with a nonfaulty process index.
  - By Lemma 2, every node in C is common.
  - Claim C is a path covering:
    - There are at most f faulty processes.
    - Each path contains f+1 labels ending with f+1 distinct indices.
    - So at least one of these labels ends with a nonfaulty process index.



Agreement

- Lemma 4: If there's a common path covering of the subtree rooted at any node x, then x is common.
- Proof:
  - By induction, from the leaves up.
  - "Common-ness" propagates upward.
- Example:



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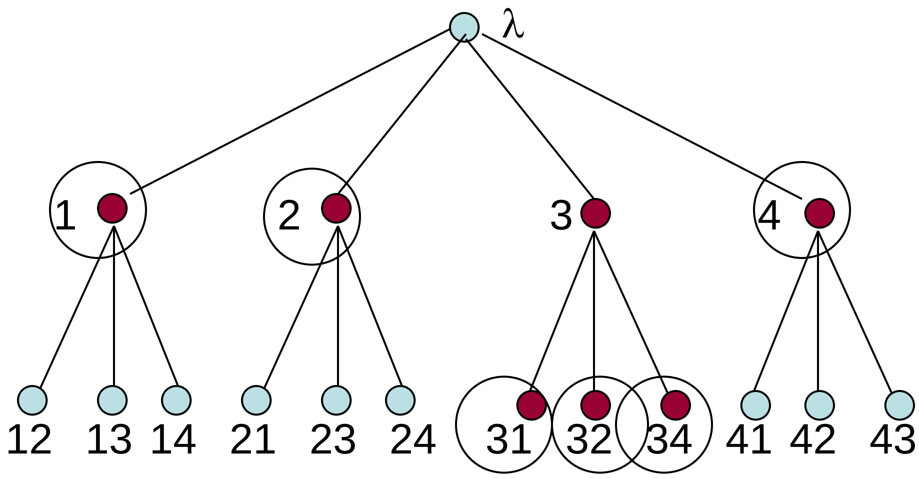

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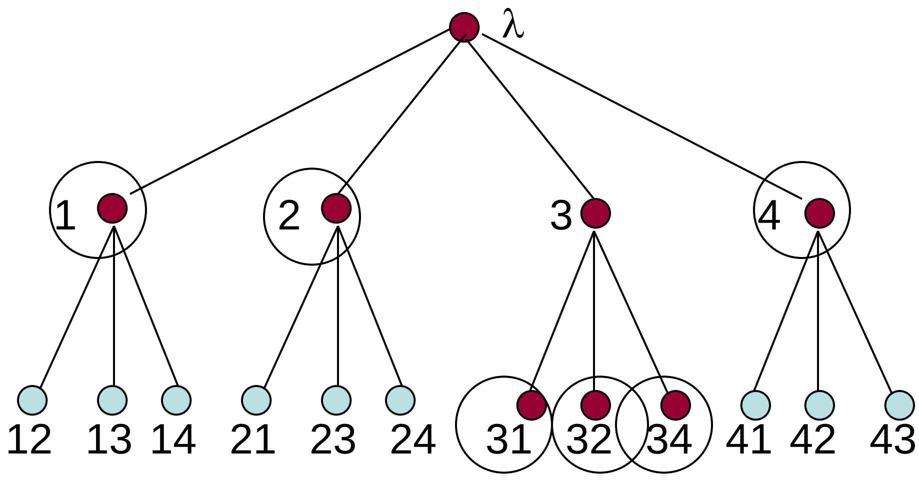


Agreement

- Lemma 3: There exists a path covering all of whose nodes are common.
- Lemma 4: If there's a common path covering of the subtree rooted at any node x, then x is common.
- Lemma 5: The root is common.
- Proof: By Lemmas 3 and 4.

- Thus, all nonfaulty processes get the same newval(λ).
- Yields Agreement.

Complexity bounds

- As for EIG for stopping agreement:
  - Time: f+1
  - Communication: $O(n^{f+1})$
- Number of processes: $n > 3f$
- Q: Is $n > 3f$ necessary?

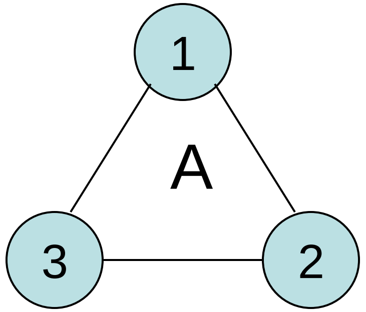
Lower bound on the number of processes for Byzantine Agreement

Number of processes for Byzantine agreement

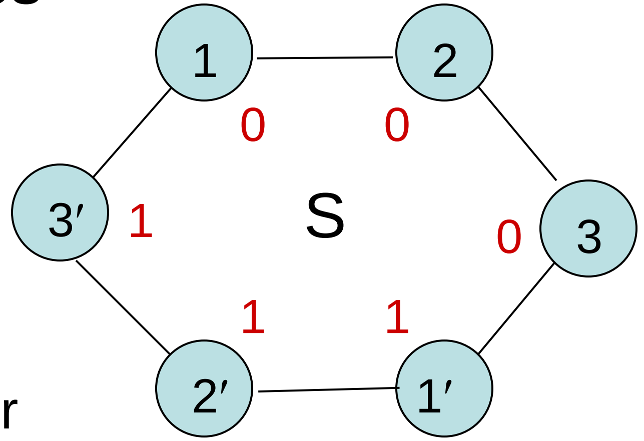
- $n > 3f$ is necessary!
  - Holds for any n-node (undirected) graph.
  - For graphs with low connectivity, may need even more processes.
  - Number of failures that can be tolerated for Byzantine agreement in an undirected graph G has been completely characterized, in terms of number of nodes and graph connectivity.

- Theorem 1: 3 processes cannot solve Byzantine Agreement with 1 possible failure.

Proof (3 vs. 1 BA)



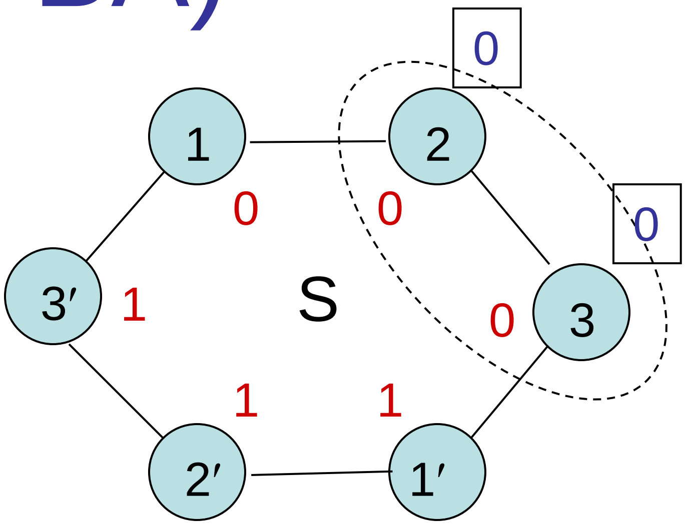
- By contradiction. Suppose algorithm A, consisting of processes 1, 2, 3, solves BA with 1 possible failure.
- Construct new system S from 2 copies of A, with initial values as follows:

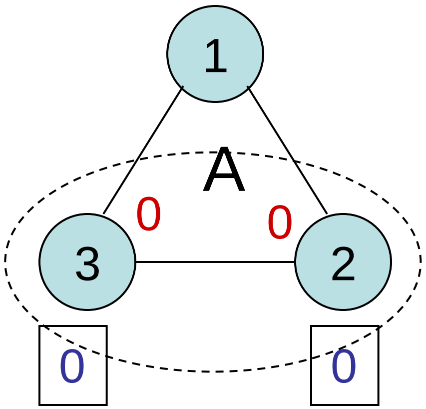


- What is S?
  - A synchronous system of some kind.
  - Not required to satisfy any particular correctness conditions.
  - Not necessarily a correct BA algorithm for the 6-node ring.
  - Just some synchronous system, which runs and does something.
  - We'll use it to get our contradiction.

Proof (3 vs 1 BA)

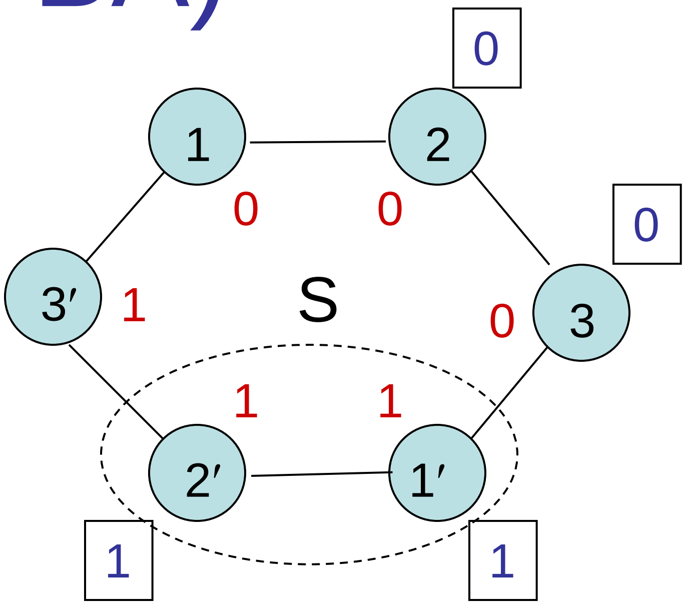
- Consider 2 and 3 in S:
- Looks to them like:
  - They’re in A, with a faulty process 1.
  - 1 emulates 1′-2′-3′-1 from S.
- In A, 2 and 3 must decide 0
- So by indistinguishability, they decide 0 in S also.

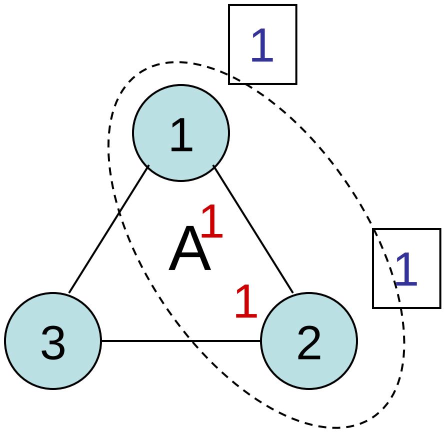




Proof (3 vs 1 BA)

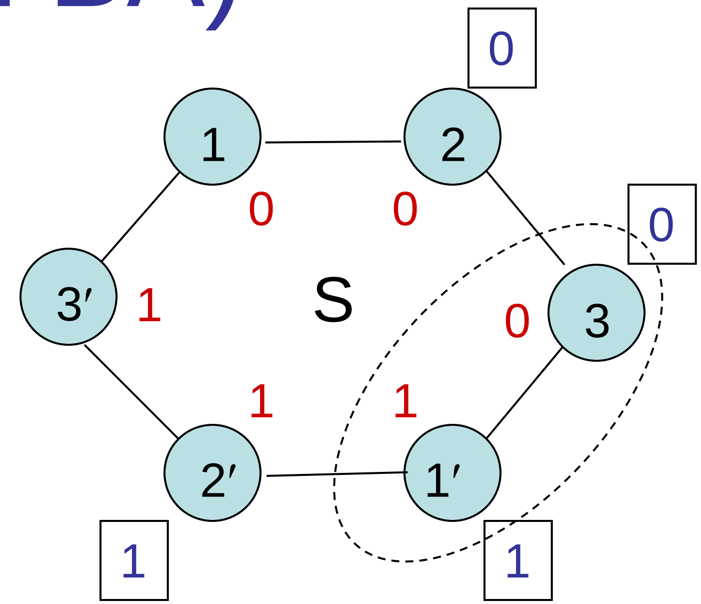
- Now consider 1′ and 2′ in S.
- Looks to them like:
  - They're in A with a faulty process 3.
  - 3 emulates 3′-1-2-3 from S.
- They must decide 1 in A, so they decide 1 in S also.



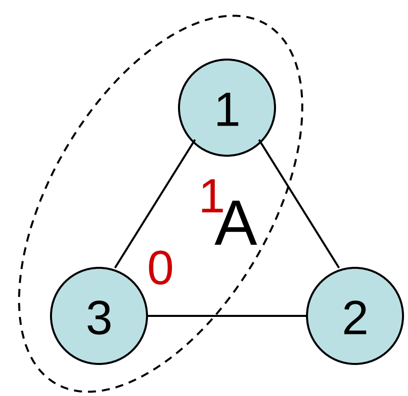


Proof (3 vs 1 BA)

- Finally, consider 3 and 1′ in S:
- Looks to them like:
  - They're in A, with a faulty process 2.
  - 2 emulates 2′-3′-1-2 from S.
- In A, 3 and 1 must agree.
- So by indistinguishability, 3 and 1′ agree in S also.



- But we already know that process 1′ decides 1 and process 3 decides 0, in S.
- Contradiction!

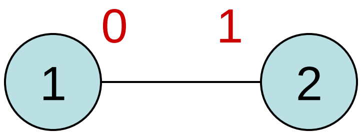


Discussion

- We get this contradiction even if the original algorithm A is assumed to “know n”.
- That simply means that:
  - The processes in A have the number 3 hard-wired into their state.
  - Their correctness properties are required to hold only when they are actually configured into a triangle.
- We are allowed to use these processes in a different configuration S---as long as we don’t claim any particular correctness properties for S.

Impossibility for n = 3f

- Theorem 2: n processes can't solve BA, if $n \leq 3f$.
- Proof:
  - Similar construction, with f processes treated as a group.
  - Or, can use a reduction:
    - Show how to transform a solution for $n \leq 3f$ to a solution for 3 vs. 1.
    - Since 3 vs. 1 is impossible, this yields a contradiction.

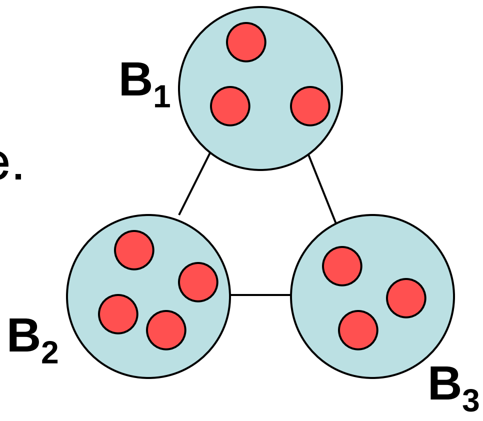


- Treat n = 2 as a special case:
  - n = 2, f = 1
  - Each could be faulty, requiring the other to decide on its own value.
  - Or both nonfaulty, which requires agreement, contradiction.

- So from now on, assume $3 \leq n \leq 3f$.
- Assume a Byzantine Agreement algorithm A for (n,f).
- Transform it into a BA algorithm B for (3,1).

Transforming A to B

- Algorithm:
  - Partition A-processes into groups I_1, I_2, I_3, where $1 \leq |I_1|, |I_2|, |I_3| \leq f$.
  - Each B_i process simulates the entire I_i group.
  - B_i initializes all processes in I_i with B_i's initial value.
  - At each round, B_i simulates sending messages:
    - Local: Just simulate locally.
    - Remote: Package and send.
  - If any simulated process decides, B_i decides the same (use any).

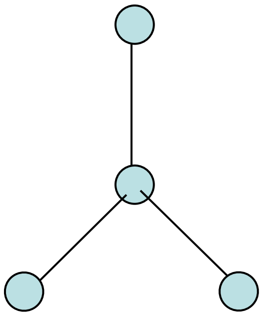


- Show B satisfies correctness conditions:
  - Consider any execution of B with at most 1 faulty process.
  - Simulates an execution of A with at most f faulty processes.
  - Correctness conditions must hold in the simulated execution of A.
  - Show these all carry over to B's execution.

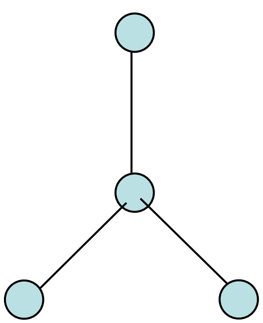
B's correctness

- Termination:
  - If B_i is nonfaulty in B, then it simulates only nonfaulty processes of A (at least one).
  - Those terminate, so B_i does also.
- Agreement:
  - If B_i, B_j are nonfaulty processes of B, they simulate only nonfaulty processes of A.
  - Agreement in A implies all these agree.
  - So B_i, B_j agree.
- Validity:
  - If all nonfaulty processes of B start with v, then so do all nonfaulty processes of A.
  - Then validity of A implies that all nonfaulty A processes decide v, so the same holds for B.

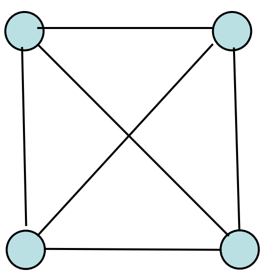
General graphs and connectivity bounds



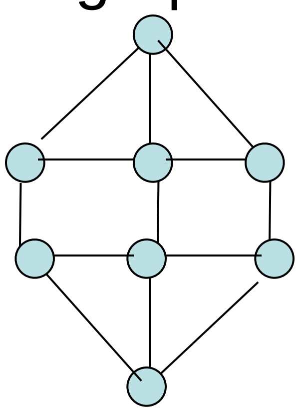
- n > 3f isn't the whole story:
  - 4 processes, can't tolerate 1 fault:
- Theorem 3: BA is solvable in an n-node graph G, tolerating f faults, if and only if both of the following hold:
  - n > 3f, and
  - conn(G) > 2f.
- conn(G) = minimum number of nodes of G whose removal results in either a disconnected graph or a 1-node graph.
- Examples:



conn = 1



conn = 3



conn = 3

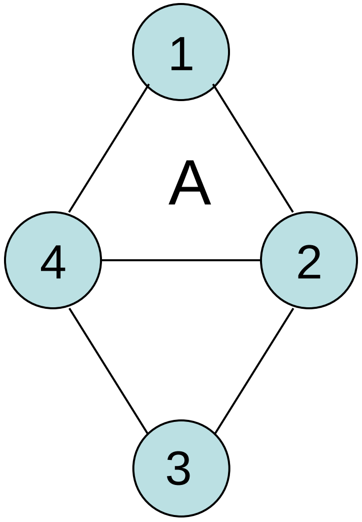
Proof: “If” direction

- Theorem 3: BA is solvable in an n-node graph G, tolerating f faults, if and only if $n > 3f$ and $conn(G) > 2f$.
- Proof (“if”):
  - Suppose both hold.
  - Then we can simulate a total-connectivity algorithm.
  - Key is to emulate reliable communication from any node i to any other node j.
  - Rely on Menger’s Theorem, which says that a graph is c-connected (that is, has $conn \geq c$) if and only if each pair of nodes is connected by $\geq c$ node-disjoint paths.
  - Since $conn(G) \geq 2f + 1$, we have $\geq 2f + 1$ node-disjoint paths between i and j.
  - To send a message, send it on all these paths (assumes graph is known).
  - Majority must be correct, so take majority message.

Proof: "Only if" direction

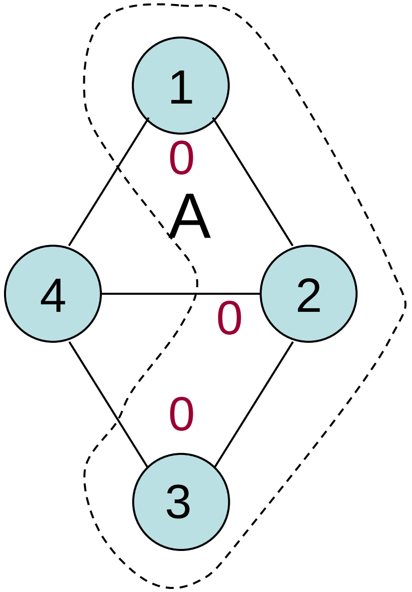
- Theorem 3: BA is solvable in an n-node graph G, tolerating f faults, if and only if $n > 3f$ and $conn(G) > 2f$.

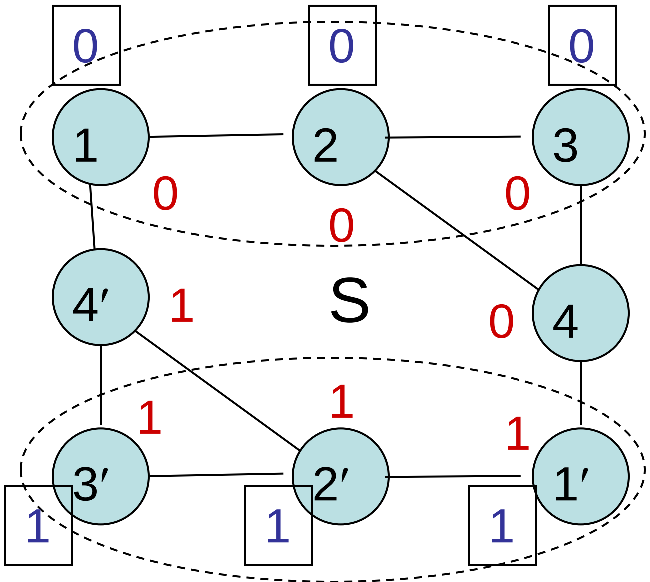
- Proof ("only if"):
  - We already showed $n > 3f$; remains to show $conn(G) > 2f$.
  - Show key idea with simple case, conn = 2, f = 1.
  - Canonical example:
    - Disconnect 1 and 3 by removing 2 and 4:
  - Proof by contradiction.
  - Assume some algorithm A that solves BA in this canonical graph, tolerating 1 failure.



Proof (conn = 2, 1 failure)

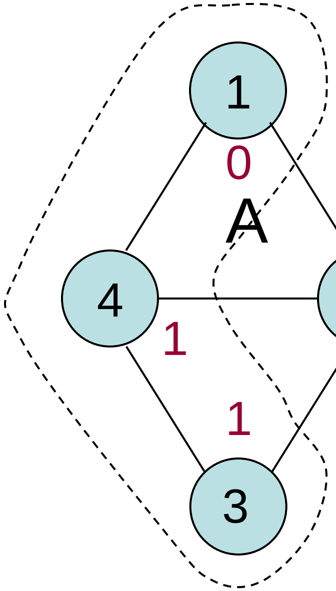
- Now construct S from two copies of A.
- Consider 1, 2, and 3 in S:
  - Looks to them like they're in A, with a faulty process 4.
  - In A, 1, 2, and 3 must decide 0
  - So they decide 0 in S also.
- Similarly, 1′, 2′, and 3′ decide 1 in S.

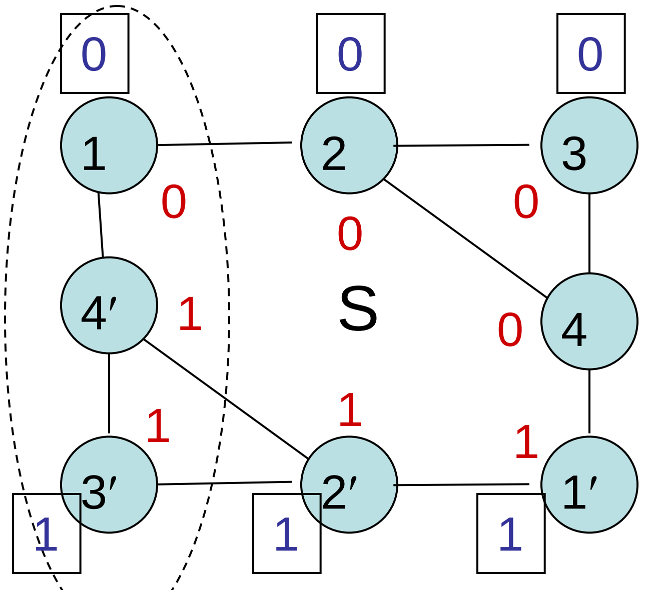




Proof (conn = 2, 1 failure)

- Finally, consider 3′, 4′, and 1 in S:
  - Looks to them like they're in A, with a faulty process 2.
  - In A, they must agree, so they also agree in S.
  - But 3′ decides 0 and 1 decides 1 in S, contradiction.
- Therefore, we can't solve BA in this canonical graph, with 1 failure.
- As before, we can generalize to conn(G) $\leq$ 2f, or use a reduction.





Other Byzantine processor bounds

- The bounds n > 3f and conn > 2f are fundamental for consensus-style problems with Byzantine failures.
- Same bounds hold, in synchronous settings with f Byzantine faulty processes, for:
  - Byzantine Firing Squad synchronization problem,
  - Weak Byzantine Agreement, and
  - Approximate agreement.
- Also, in timed (partially synchronous settings), for maintaining clock synchronization.
- Proofs all use similar “pasting” methods.

Weak Byzantine Agreement [Lamport]

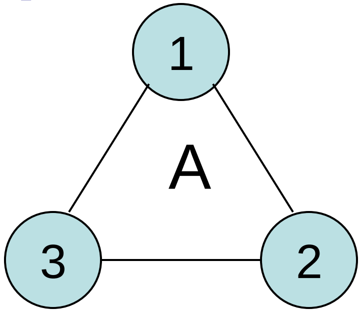
- Correctness conditions for BA:
  - Agreement: No two nonfaulty processes decide on different values.
  - Validity: If all nonfaulty processes start with the same v, then v is the only allowable decision for nonfaulty processes.
  - Termination: All nonfaulty processes eventually decide.
- Correctness conditions for Weak BA:
  - Agreement: Same as for BA.
  - Validity: If all processes are nonfaulty and start with the same v, then v is the only allowed decision value.
  - Termination: Same as for BA.
- Limits the situations where the decision is forced to go a certain way.
- Similar style to one direction of the validity condition for the 2-Generals problem.

WBA Processor Bounds

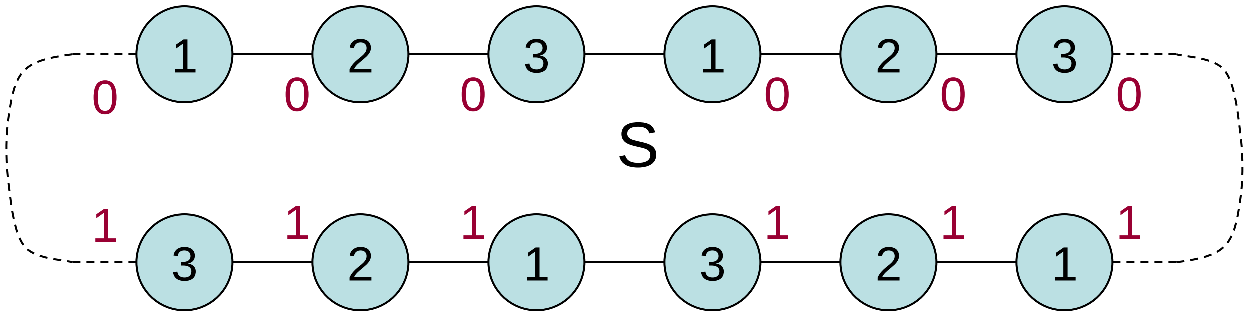
- Theorem 4: Weak BA is solvable in an n-node graph G, tolerating f faults, if and only if $n > 3f$ and $conn(G) > 2f$.
- Same bounds as for BA.

- Proof:
  - "If": Follows from results for ordinary BA.
  - "Only if":
    - By constructions like those for ordinary BA, but slightly more complicated.
    - Show 3 vs. 1 here, rest LTTR.

Proof (3 vs. 1 Weak BA)

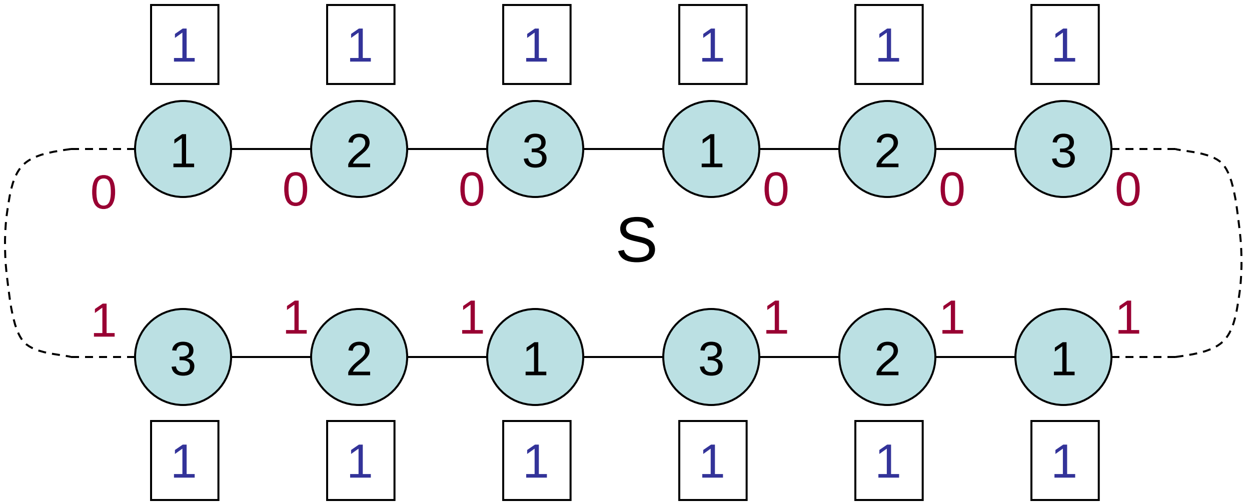


- By contradiction. Suppose algorithm A, consisting of procs 1, 2, 3, solves WBA with 1 fault.
- Let α_0 = execution in which everyone starts with 0 and there are no failures; results in decision 0.
- Let α_1 = execution in which everyone starts with 1 and there are no failures; results in decision 1.
- Let b = an upper bound on number of rounds for all processes to decide, in both α_0 and α_1.
- Construct new system S from 2b copies of A:



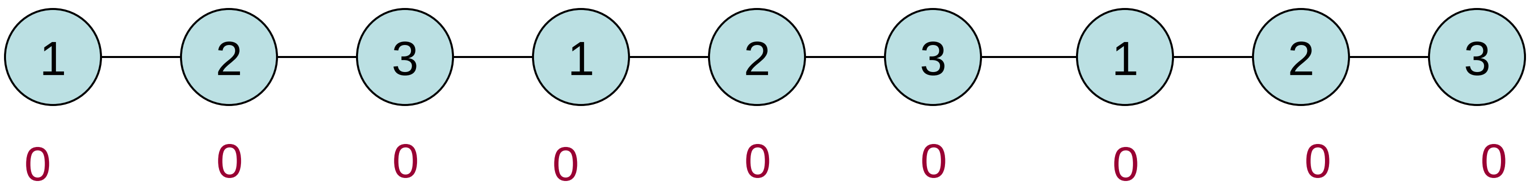
Proof (3 vs. 1 Weak BA)

- Claim: Any two adjacent processes in S must decide the same thing..
  - Because it looks to them like they are in A, and they must agree in A.
- So everyone decides the same in S.
- WLOG, all decide 1.



Proof (3 vs. 1 Weak BA)

- Now consider a block of 2b + 1 consecutive processes that begin with 0:



- Claims:
  - To all but the endpoints, the execution of S is indistinguishable from α_0, the failure-free execution of A in which everyone starts with 0, for one round.
  - To all but two at each end, indistinguishable from α_0 for two rounds.
  - To all but three at each end, indist. from α_0 for three rounds.
  - …
  - To midpoint, indistinguishable for b rounds.
- But b rounds are enough for the midpoint to decide 0, contradicting the fact that everyone decides 1 in S.

Lower bound on the number of rounds for Byzantine agreement

Lower bound on number of rounds

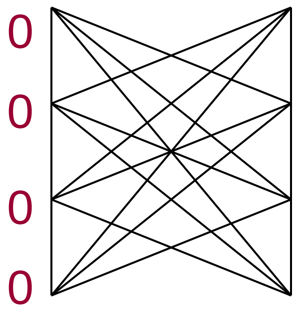
- Notice that f+1 rounds are used in all the agreement algorithms we've seen so far---both stopping and Byzantine.
- That's inherent: f+1 rounds are needed in the worst-case, even for simple stopping failures.
- Assume an f-round stopping agreement algorithm A tolerating f faults, and get a contradiction.
- Restrictions on A (WLOG):
  - n-node complete graph.
  - Decisions at end of round f.
  - $V = \{0,1\}$
  - All-to-all communication at every round $\leq$ f.

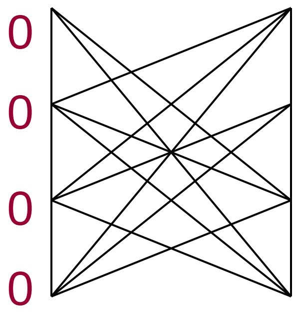
Special case: f = 1

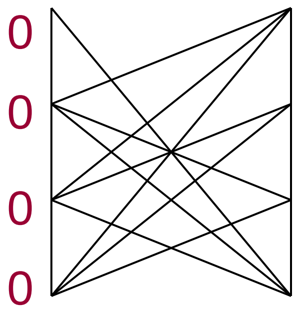
- Theorem 5: Suppose $n \geq 3$. There is no n-process 1-fault stopping agreement algorithm in which nonfaulty processes always decide at the end of round 1.
- Proof: Suppose A exists.
  - Construct a chain of executions, each with at most one failure, such that:
    - First has (unique) decision value 0.
    - Last has decision value 1.
    - Any two consecutive executions in the chain are indistinguishable to some process i that is nonfaulty in both. So i must decide the same in both executions, and the two must have the same decision values.
  - So decision values in first and last executions must be the same.
  - Contradiction.

Round lower bound, f = 1

- α_0: All processes have input 0, no failures.
- …
- α_k (last one): All inputs 1, no failures.
- Start the chain from α_0.
- Next execution, α_1, removes message $1 \rightarrow 2$.
  - α_0 and α_1 indistinguishable to everyone except 1 and 2; since $n \geq 3$, there is some other process.
  - These processes are nonfaulty in both executions.
- Next execution, α_2, removes message $1 \rightarrow 3$.
  - α_1 and α_2 indistinguishable to everyone except 1 and 3, hence to some nonfaulty process.
- Next, remove message $1 \rightarrow 4$.
  - Indistinguishable to some nonfaulty process.

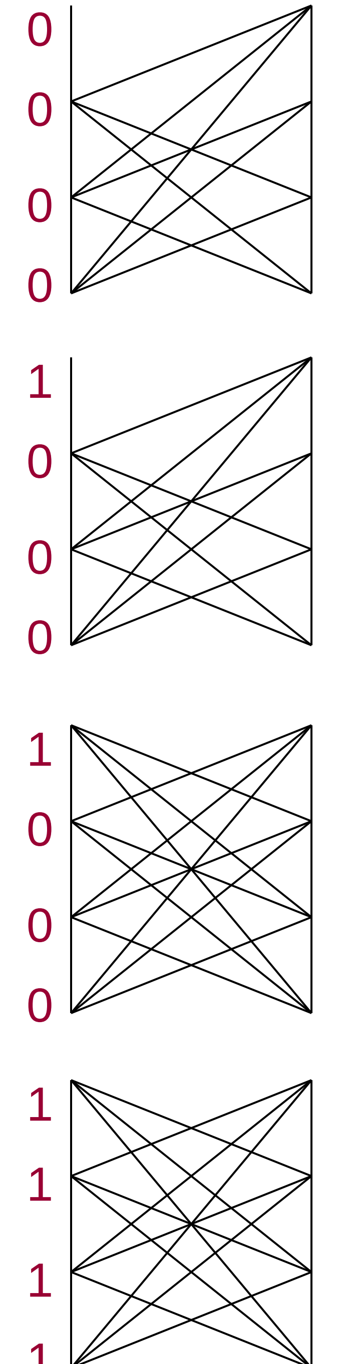






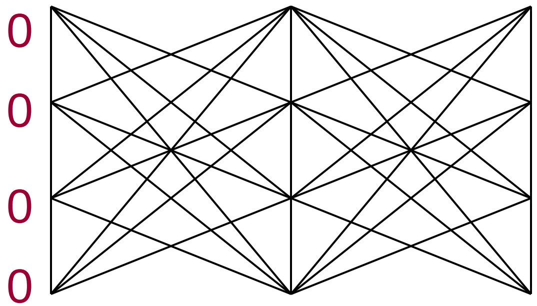
Continuing…

- Having removed all of process 1's messages, change 1's input from 0 to 1.
  - Looks the same to everyone else.
- We can't just keep removing messages, since we are allowed at most one failure in each execution.
- So, we continue by replacing missing messages, one at a time.
- Repeat with process 2, 3, and 4, eventually reach the last execution: all inputs 1, no failures.



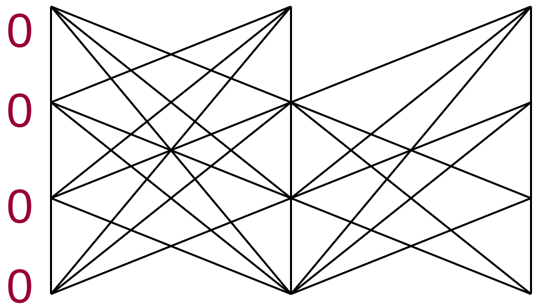
Special case: f = 2

- Theorem 6: Suppose n ≥ 4. There is no n-process 2-fault stopping agreement algorithm in which nonfaulty processes always decide at the end of round 2.
- Proof: Suppose A exists.
  - Construct another chain of executions, each with at most 2 failures.
    - This chain is longer and more complicated.
  - Start with α_0: All processes have input 0, no failures, 2 rounds:
  - Work toward α_n, all 1's, no failures.
  - Each consecutive pair is indistinguishable to some nonfaulty process.
  - Use intermediate execs $\alpha_{i,}$ in which:
    - Processes 1,…,i have initial value 1.
    - Processes i+1,…,n have initial value 0.
    - No failures.



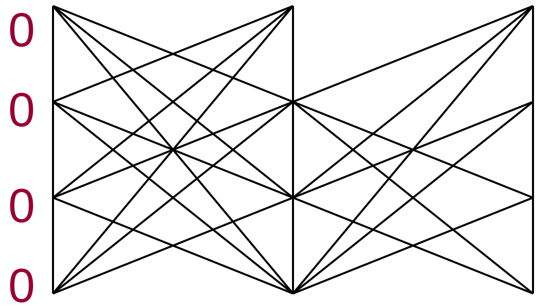
Special case: f = 2

- Show how to connect α_0 and α_1.
  - That is, change process 1's initial value from 0 to 1.
  - Other intermediate steps are essentially the same.
- Start with α_0, work toward killing p1 at the beginning, to change its initial value, by removing messages.
- Then replace the messages, working back up to α_1.
- Start by removing p1's round 2 messages, one by one.
- Q: Continue by removing p1's round 1 messages?
- No, because consecutive executions would not look the same to anyone:
  - E.g., removing $1 \rightarrow 2$ at round 1 allows p2 to tell everyone about the failure, at round 2.

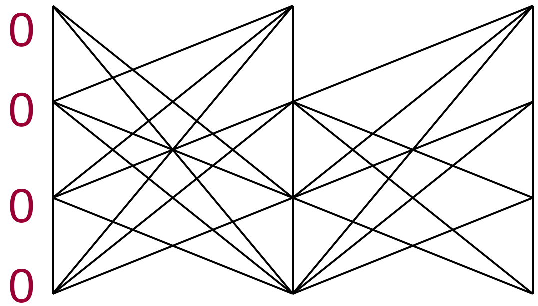


Special case: f = 2

- Removing 1 $\rightarrow$ 2 at round 1 allows p2 to tell all other processes about the failure:



vs.



- Distinguishable to everyone.
- So we must do something more elaborate.
- Recall that now we can allow 2 processes to fail in some executions.
- Use many steps to remove a single round 1 message 1 $\rightarrow$ i; in these steps, both 1 and i will be faulty.

Removing p1’s round 1 messages

- Start with execution where p1 sends to everyone at round 1, and only p1 is faulty.
- Remove round 1 message $1 \rightarrow 2$:
  - p2 starts out nonfaulty, so sends all its round 2 messages.
  - Now make p2 faulty.
  - Remove p2’s round 2 messages, one by one, until we reach an execution where $1 \rightarrow 2$ at round 1, but p2 sends no round 2 messages.
  - Now remove the round 1 message $1 \rightarrow 2$.
    - Executions look the same to everyone but 1 and 2 (and they’re all nonfaulty).
  - Replace the round 2 messages from p2, one by one, until p2 is no longer faulty.
- Repeat to remove p1’s round 1 messages to p3, p4,…
- After removing all of p1’s round 1 messages, change p1’s initial value from 0 to 1, as needed.

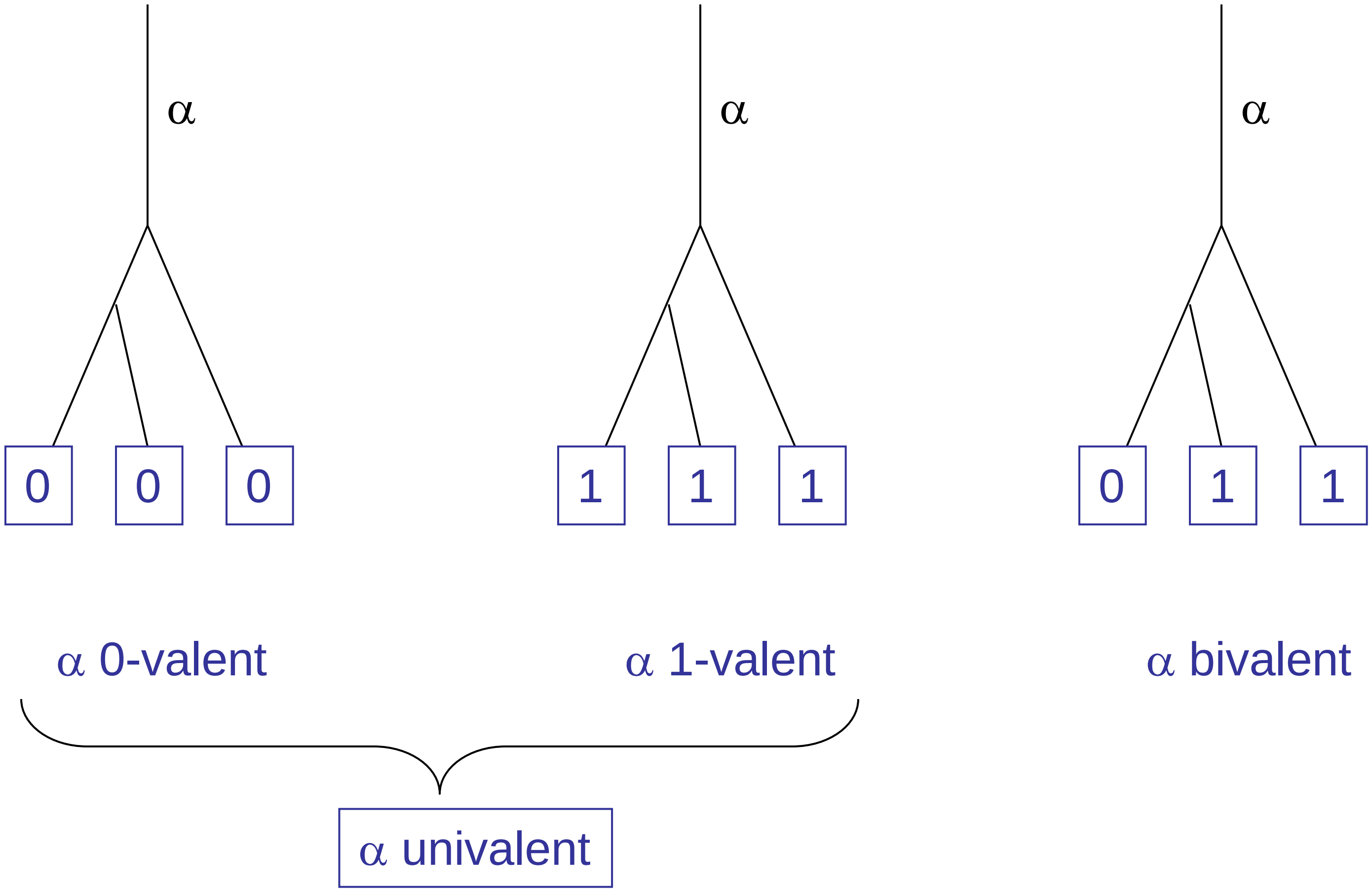
General case: Any f

- Theorem 7: Suppose $n \geq f + 2$. There is no n-process f-fault stopping agreement algorithm in which nonfaulty processes always decide at the end of round f.
- Proof: Suppose A exists.
  - Same ideas, longer chain.
  - Must fail f processes in some executions in the chain, in order to remove all the required messages, at all rounds.
  - Construction in book, LTTR.
- Newer proof [Aguilera, Toueg]:
  - Uses ideas from [Fischer, Lynch, Paterson] impossibility of consensus (which you will see later).
  - They assume strong validity, but their proof works for our weaker validity condition also.

[Aguilera,Toueg] proof

- By contradiction. Assume A solves stopping agreement for f failures and everyone decides after exactly f rounds.
- Consider only executions in which at most one process fails during each round.
- Recall: Failure at a round allows process to send any subset of the messages, or to send all but halt before changing state.
- Regard vector of initial values as a 0-round execution.
- Definitions (adapted from [FLP]): α, an execution that completes some finite number (possibly 0) of rounds, is:
  - 0-valent, if 0 is the only decision that can occur in any execution (of the kind we consider) that extends α.
  - 1-valent, if 1 is the only decision that can occur in…
  - Univalent, if α is either 0-valent or 1-valent (essentially decided).
  - Bivalent, if both decisions occur in some extensions (undecided).

Univalence and Bivalence



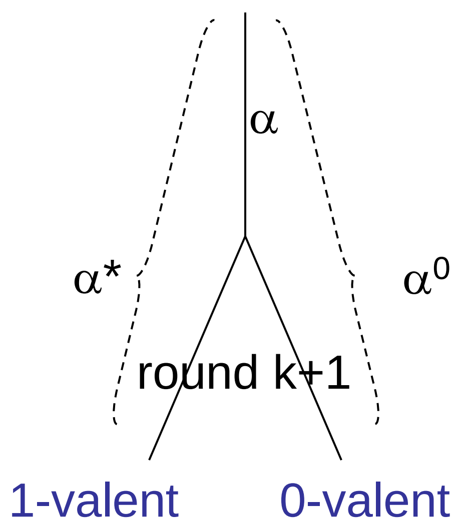
Initial bivalence

- Lemma 1: There is some 0-round execution (vector of initial values) that is bivalent.

- Proof (derived from [FLP]):
  - Assume for contradiction that all 0-round executions are univalent.
  - 000…0 is 0-valent.
  - 111…1 is 1-valent.
  - So there must be two 0-round executions that differ in the value of just one process, i, such that one is 0-valent and the other is 1-valent.
  - But this is impossible, because if i fails at the start, no one else can distinguish the two 0-round executions.

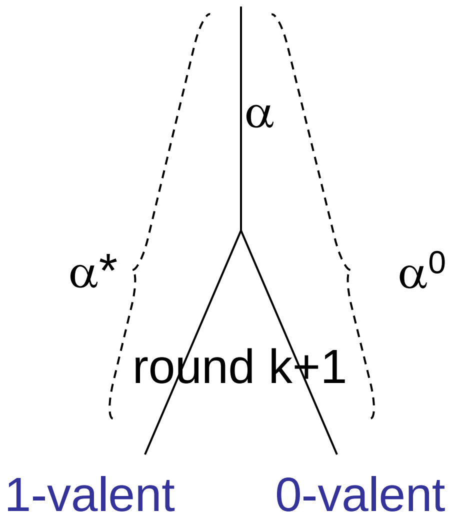
Bivalence through f-1 rounds

- Lemma 2: For every k, $0 \leq k \leq f-1$, there is a bivalent k-round execution.
- Proof: By induction on k.
  - Base (k=0): Lemma 1.
  - Inductive step: Assume for k, show for k+1, where k < f -1.
    - Assume a bivalent k-round execution α.
    - Assume for contradiction that every 1-round extension of α (with at most one new failure) is univalent.
    - Let α^* be the 1-round extension of α in which no new failures occur in round k+1.
    - By assumption, this is univalent, say WLOG that it's 1-valent.
    - Since α is bivalent, there must be another 1-round extension of α, α^0, that is 0-valent.



Bivalence through f-1 rounds

- In α^0, some single process, say i, fails in round k+1, by not sending to some set of processes, say J = $\{j_1, j_2, \ldots j_m\}$.
- Define a chain of (k+1)-round executions, $\alpha^0, \alpha^1, \alpha^2, \ldots, \alpha^m$.
- Each α^l in this sequence is the same as α^0 except that i also sends messages to $j_1, j_2, \ldots j_l$.
  - Adding in messages from i, one at a time.



- Each α^l is univalent, by assumption.
- Since α^0 is 0-valent, either:
  - At least one of these is 1-valent, or
  - All are 0-valent.

Case 1: At least one α^l is 1-valent

- Then there must be some l such that α^{l-1} is 0-valent and α^l is 1-valent.
- But α^{l-1} and α^l differ after round k+1 only in the state of one process, j_l.
- We can extend both α^{l-1} and α^l by simply failing j_l at beginning of round k+2.
  - There is actually a round k+2 because we've assumed $k < f-1$, so $k+2 \leq f$.
- And no one left alive can tell the difference!
- Contradiction for Case 1.

Case 2: Every α^l is 0-valent

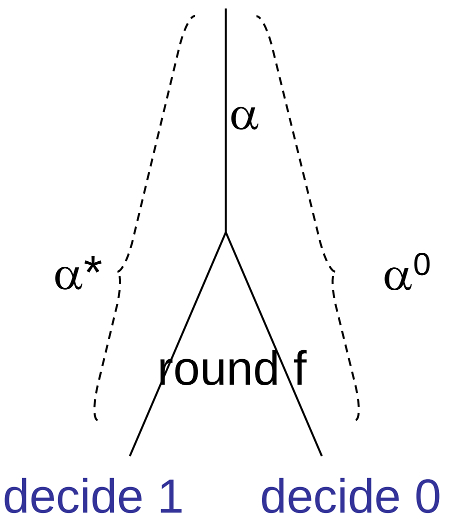
- Then compare:
  - α^m, in which i sends all its round k+1 messages and then fails, with
  - α^* , in which i sends all its round k+1 messages and does not fail.
- No other differences, since only i fails at round k+1 in α^m.
- α^m is 0-valent and α^* is 1-valent.
- Extend to full f-round executions:
  - α^m, by allowing no further failures,
  - α^*, by failing i right after round k+1 and then allowing no further failures.
- No one can tell the difference.
- Contradiction for Case 2.

Bivalence through f-1 rounds

- So we've proved, so far:
- Lemma 2: For every k, $0 \leq k \leq f-1$, there is a bivalent k-round execution.

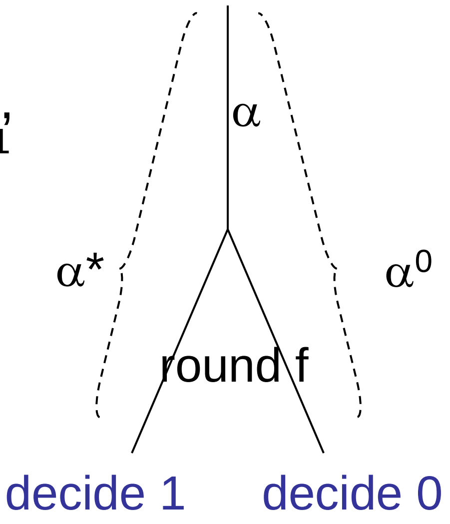
Disagreement after f rounds

- Lemma 3: There is an f-round execution in which two nonfaulty processes decide differently.
- Proof:
  - Use Lemma 2 to get a bivalent (f-1)-round execution α with $\leq$ f-1 failures.
  - In every 1-round extension of α, everyone who hasn't failed must decide (and agree).
  - Let α^* be the 1-round extension of α in which no new failures occur in round f.
  - Everyone who is still alive decides after α^*, and they must decide the same thing. WLOG, say they decide 1.
  - Since α is bivalent, there must be another 1-round extension of α, say α^0, in which some nonfaulty process (and so, all nonfaulty processes) decide 0.



Disagreement after f rounds

- In α^0, some single process i fails in round f.
- Let j, k be two nonfaulty processes.
- Define a chain of three f-round executions, $\alpha^0, \alpha^1, \alpha^*$, where α^1 is identical to α^0 except that i sends to j in α^1 (it might not in α^0).
- Then $\alpha^1 \sim^k \alpha^0$.
- Since k decides 0 in α^0, k also decides 0 in α^1.
- Also, $\alpha^1 \sim^j \alpha^*$.
- Since j decides 1 in α^*, j also decides 1 in α^1.
- Yields disagreement in α^1, contradiction!



- So we've proved:
- Lemma 3: There is an f-round execution in which two nonfaulty processes decide differently.
- Which immediately yields the lower bound result.

Early-stopping agreement algorithms

- Tolerate f failures, but in executions with $f' < f$ failures, terminate correspondingly faster.
- [Dolev, Reischuk, Strong 90] Gave a stopping agreement algorithm in which all nonfaulty processes terminate in at most $\min(f' + 2, f+1)$ rounds.
  - If $f' + 2 \leq f$, decide "early", within $f' + 2$ rounds; in any case decide within $f+1$ rounds.
- [Keidar, Rajsbaum 02] Lower bound of $f' + 2$ for early-stopping agreement.
  - Not just $f' + 1$. Early stopping requires an extra round.

Early-stopping agreement algorithms

- Tolerate f failures, but in executions with $f' < f$ failures, terminate correspondingly faster.
- [Keidar, Rajsbaum 02] Lower bound of $f' + 2$ for early-stopping agreement.
  - Not just $f' + 1$. Early stopping requires an additional round.
- Theorem 1: Assume $0 \le f' \le f - 2$ and $f < n$. Every early-stopping agreement algorithm tolerating f failures has an execution with f' failures in which some nonfaulty process doesn't decide by the end of round $f' + 1$.

Special case: $f' = 0$

- Special Case Theorem 2: Assume $2 \leq f < n$. Every early-stopping agreement algorithm tolerating f failures has a failure-free execution in which some nonfaulty process does not decide by the end of round 1.

- Definition: Let α be an execution that completes some finite number (possibly 0) of rounds. Then val(α) is the unique decision value in the extension of α with no new failures.
  - Different from bivalence defs---now consider value in just one particular extension.

Special case: $f' = 0$

- Theorem 2: Assume $2 \leq f < n$. Every early-stopping agreement algorithm tolerating f failures has a failure-free execution in which some nonfaulty process does not decide by the end of round 1.
- Definition: $val(\alpha)$ is the decision value in the extension of α with no new failures.
- Proof of Theorem 2:
  - Assume executions in which at most one process fails per round.
  - Identify 0-round executions with vectors of initial values.
  - Assume, for contradiction, that everyone decides by the end of round 1, in all failure-free executions.
  - val(000…0) = 0, val(111…1) = 1.
  - So there must be two 0-round executions α^0 and α^1, that differ in the value of just one process i, such that $val(\alpha^0) = 0$ and $val(\alpha^1) = 1$.

Special case: $f' = 0$

- 0-round executions α^0 and α^1, differing only in the initial value of process i, such that $val(\alpha^0) = 0$ and $val(\alpha^1) = 1$.
- In the ff extensions of α^0 and α^1, all nonfaulty processes decide by the end of round 1.
- Define:
  - β^0, 1-round extension of α^0, in which process i fails, sends only to j.
  - β^1, 1-round extension of α^1, in which process i fails, sends only to j.
- Then:
  - β^0 looks to j like ff extension of α^0, so j decides 0 in β^0 by round 1.
  - β^1 looks to j like ff extension of α^1, so j decides 1 in β^1 by round 1.
- β^0 and β^1 are indistinguishable to all processes except i, j.
- Define:
  - γ^0, infinite extension of β^0, in which process j fails right after round 1.
  - γ^1, infinite extension of β^1, in which process j fails right after round 1.
- By agreement, all nonfaulty processes must decide 0 in γ^0, 1 in γ^1.
- But γ^0 and γ^1 are indistinguishable to all nonfaulty processes, so they can't decide differently, contradiction.

Next time…

- Other kinds of consensus problems:
  – k-agreement
  – Approximate agreement (skim)
  – Distributed commit
- Reading:
  – Chapter 7 (just skim 7.2)