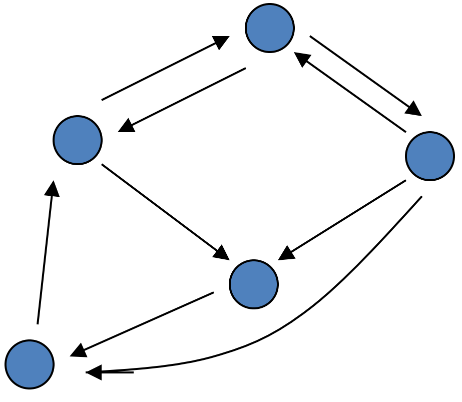
6.852: Distributed Algorithms
Fall, 2015

Lecture 11

Today's plan

- Basic asynchronous network algorithms, general networks:
  - Leader election
  - (Arbitrary) spanning trees
  - Breadth-first spanning trees
  - Shortest-paths spanning trees
  - Minimum Spanning Trees (MSTs)
- Readings:
  - Chapter 15
  - [Gallager, Humblet, Spira]
- Next time:
  - Synchronizers
  - Reading: Chapter 16.

Leader Election in General Networks



Leader election in general networks

- Consider undirected graphs.
- We can get an asynchronous version of the synchronous $FloodMax$ algorithm:
  - Simulate rounds with local counters.
  - Need to know the diameter for termination.
- We'll see several better asynchronous algorithms later:
  - Don't need to know diameter.
  - In some cases, better message complexity.
- Depend on techniques such as:
  - Breadth-first search
  - Convergecast using a spanning tree
  - Synchronizers to simulate synchronous algorithms
  - Consistent global snapshots

Spanning Trees and Searching

Spanning trees and searching

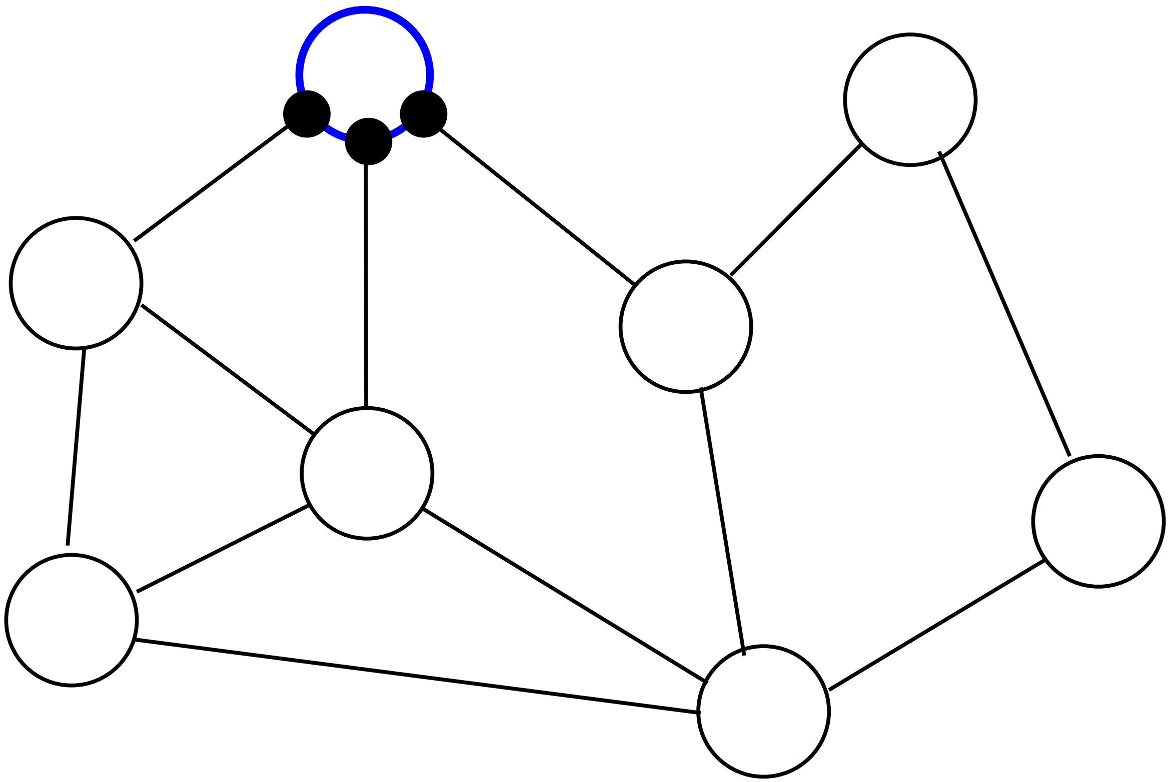
- Start with the simple task of setting up some (arbitrary) spanning tree with a (given) root i_0.
- Assume:
  - Undirected, connected graph (i.e., bidirectional communication).
  - Root i_0
  - Size and diameter unknown.
  - UIDs, with comparisons for equality.
  - Can recognize when in-edges and out-edges connect to the same neighbor.
- Require: Each process should output its parent in tree, with a $parent$ output action.
- Starting point: $SynchBFS$ algorithm:
  - i_0 floods a $search$ message; parent of a node is the first neighbor from which it receives a $search$ message.
- If we try to run the same algorithm in an asynchronous network, then we still get a spanning tree, but not necessarily a breadth-first tree.

$AsynchSpanningTree$, Process i

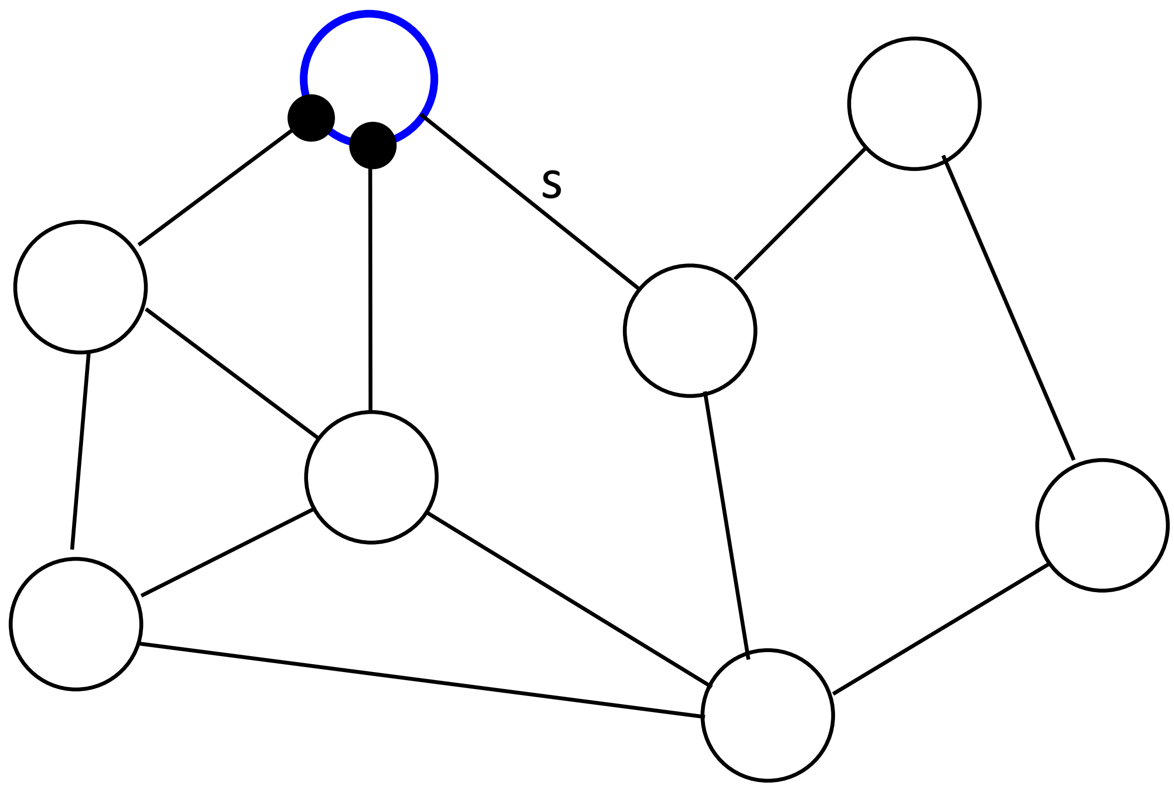
- Signature
  - ***in*** receive(search)$_{j,i}$, j ∈ nbrs
  - ***out*** send(search)$_{i,j}$, j ∈ nbrs
  - ***out*** parent(j)$_i$, j ∈ nbrs
- State
  - parent: nbrs U { ⊥ }, init ⊥
  - reported: Boolean, init false
  - for each j ∈ nbrs:
    - send(j) ∈ {search, ⊥}, init search if i = i_0, else ⊥

- send(search)$_{i,j}$
  pre: send(j) = search
  eff: send(j) := ⊥
- receive(search)$_{j,i}$
  eff: if i ≠ i_0 and parent = ⊥ then
  parent := j
  for k ∈ nbrs - { j } do
  send(k) := search
- parent(j)$_i$
  pre: parent = j
  reported = false
  eff: reported := true

AsynchSpanningTree



AsynchSpanningTree



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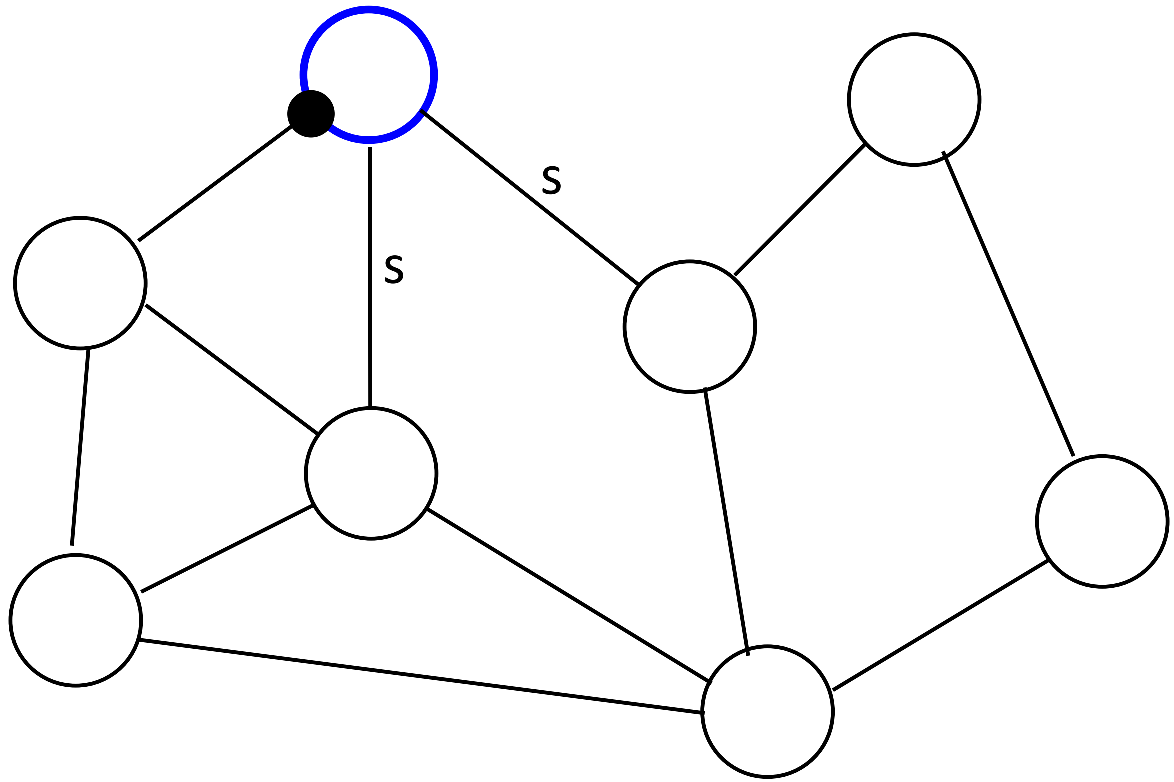

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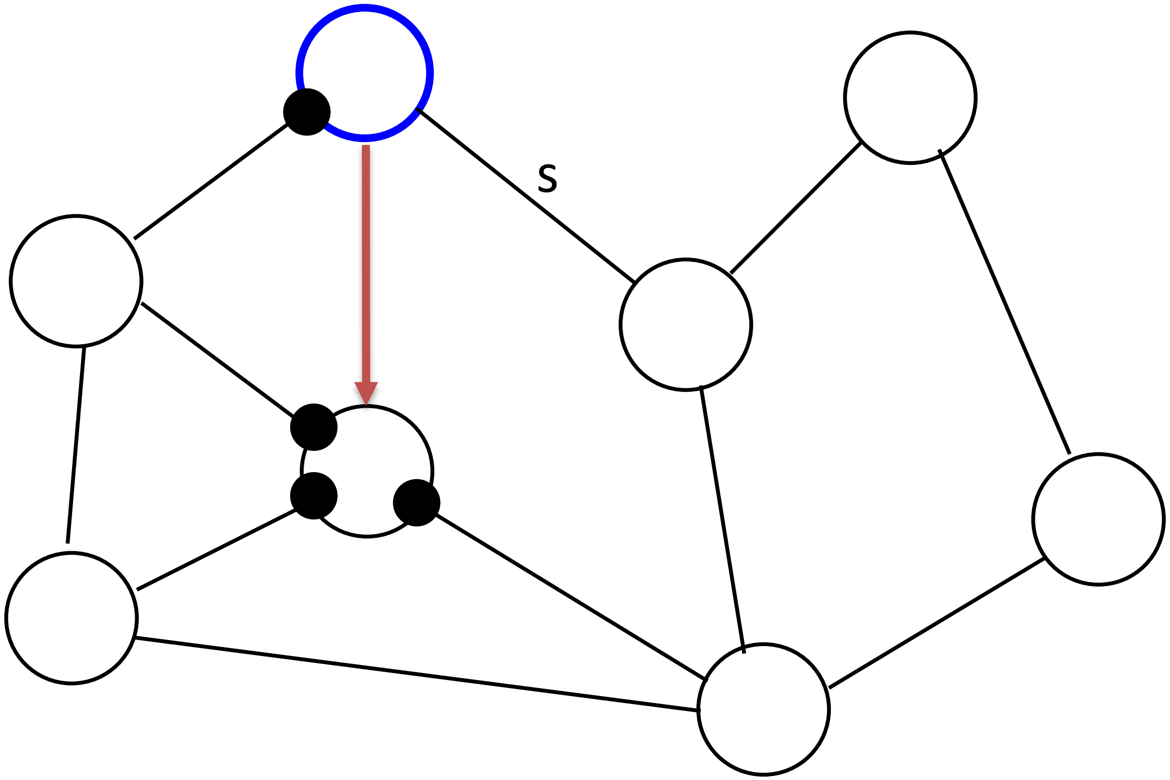

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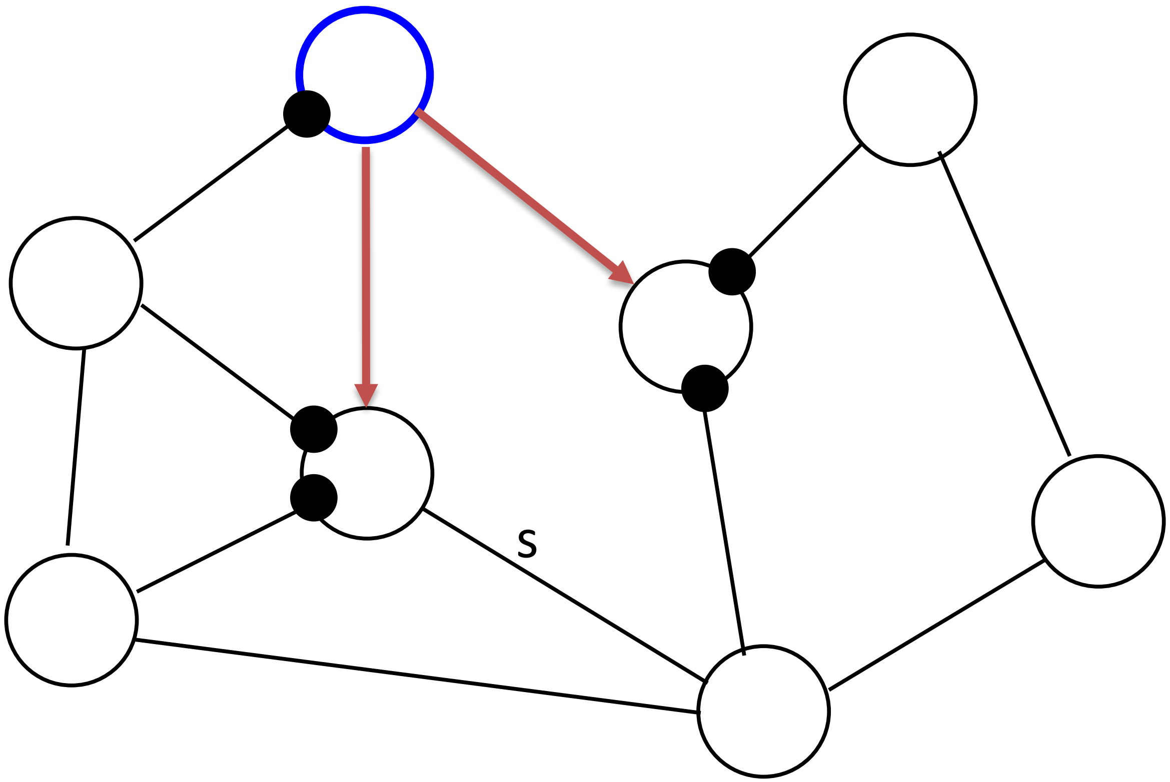

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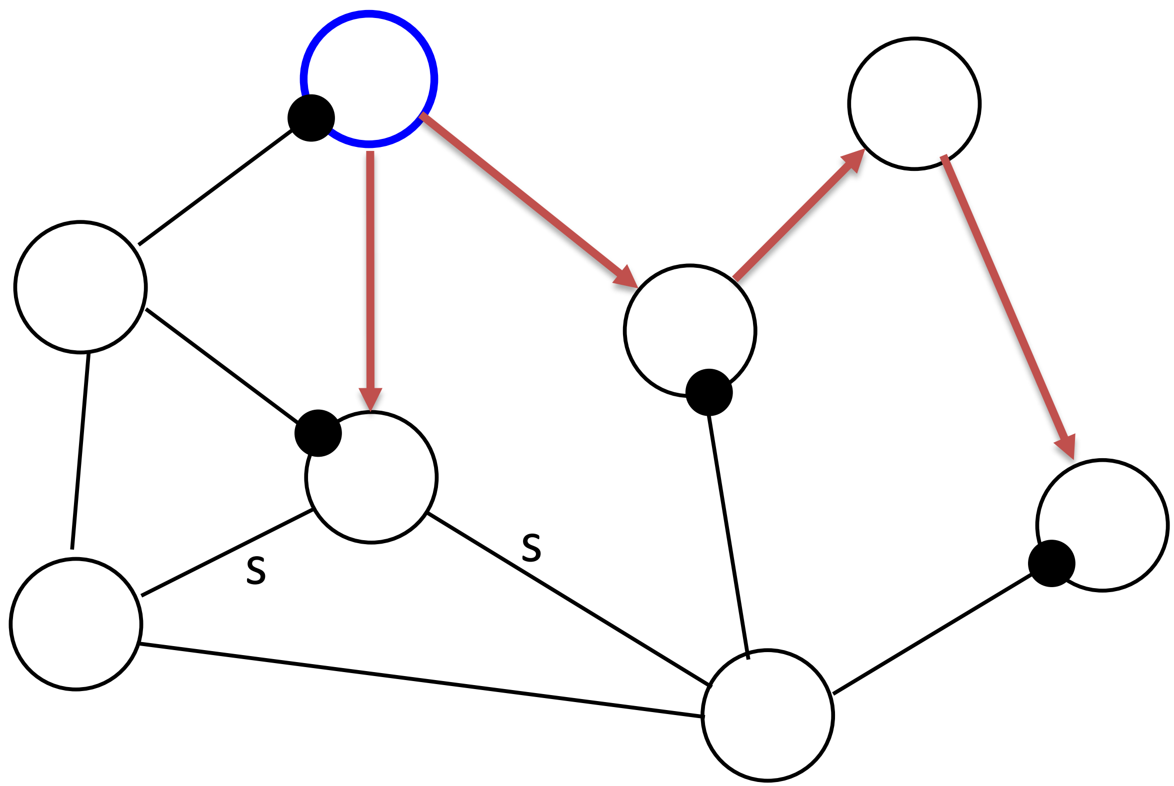

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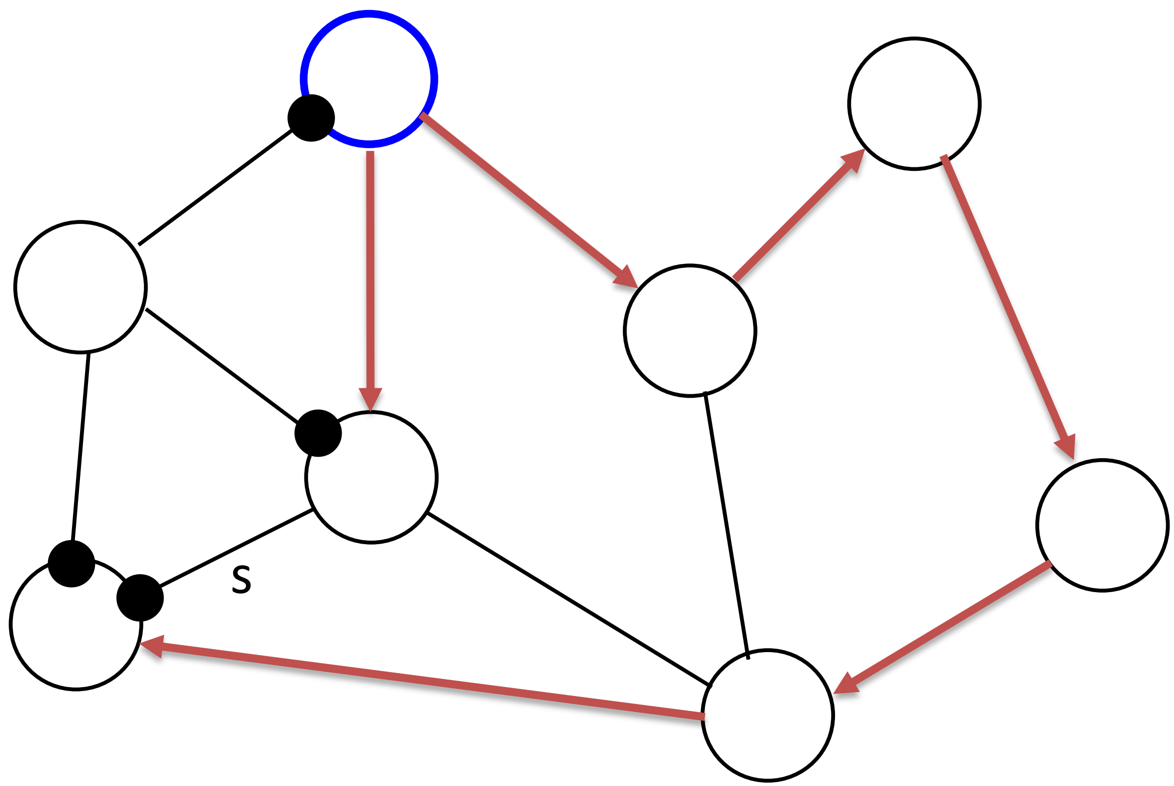

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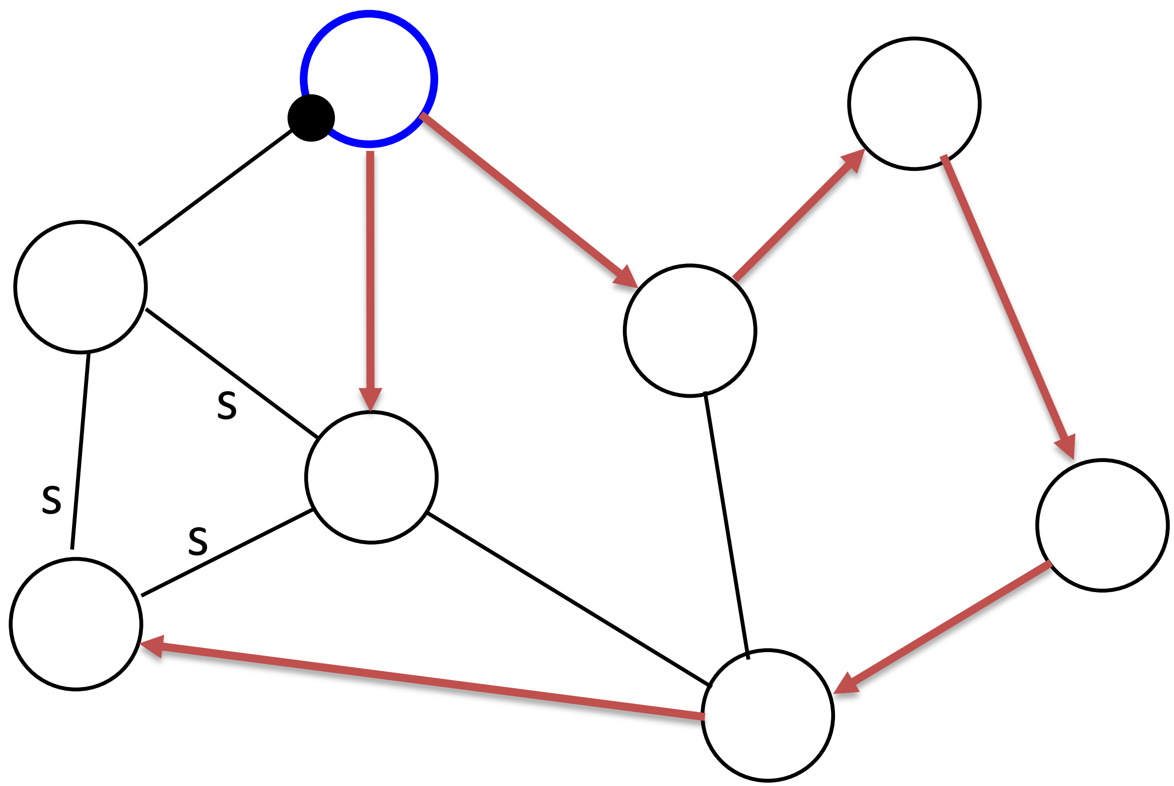
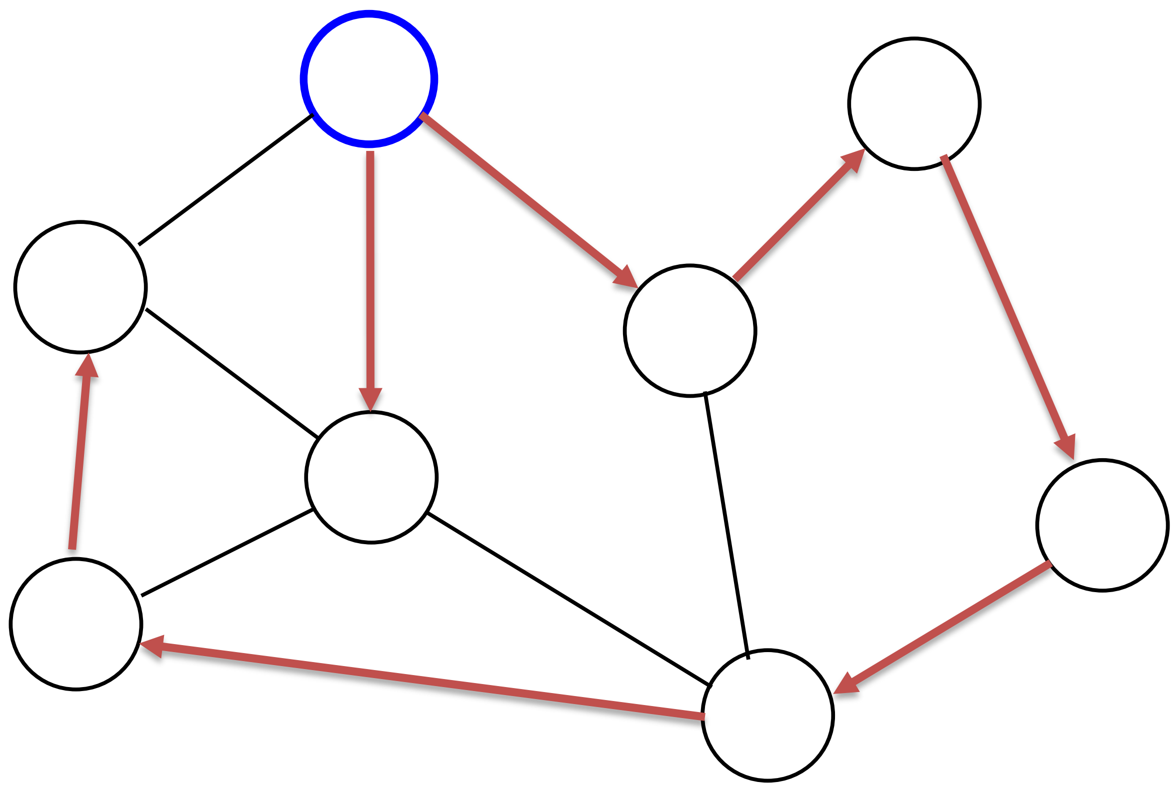
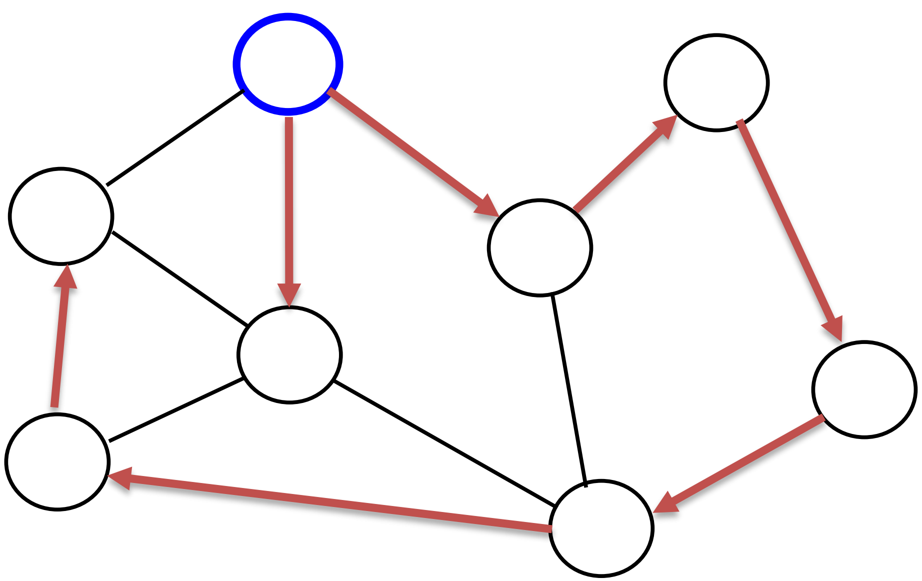


AsynchSpanningTree



AsynchSpanningTree

- Complexity
  - Messages: $O(\,|E|\,)$
  - Time: $diam\,(l + d)\ +\ l$
- Anomaly: Paths may be longer than the diameter!
  - Messages may travel faster along longer paths, in asynchronous networks.



Applications of AsynchSpanningTree

- Similar to those for synchronous BFS
- Message broadcast: Piggyback on $search$ message.
- Child pointers: Add responses to $search$ messages, easy because of bidirectional communication.
- Use precomputed tree for broadcast/convergecast
  - Convergecast works as in the synchronous setting.
  - Now the timing anomaly becomes significant.
  - $O(\ h(l+d)\ )$ time complexity.
  - $O(n)$ message complexity.
  - See book for details.

h = height of tree; may be as large as n

More applications

- Asynchronous broadcast/convergecast:
  - Can also construct spanning tree while using it to broadcast a message and also to collect responses.
  - E.g., to tell the root when the bcast is done, or to collect aggregated data.
  - See book, p. 499-500, $AsynchBcastAck$.
  - Complexity:
    - $O(|E|)$ message complexity.
    - $O(\ n\ (l+d)\ )$ time complexity, timing anomaly.
    - See book for details.
- Elect leader when nodes have no info about the network (no knowledge of $n, diam$, etc.; no root, no spanning tree):
  - All independently initiate $AsynchBcastAck$, use it to determine max, max elects itself.

Breadth-First Spanning Trees

Breadth-first spanning trees

- Assume (same as above):
  - Undirected, connected graph (i.e., bidirectional communication).
  - Root i_0.
  - Size and diameter unknown.
  - UIDs, with comparisons.
- Require: Each process should output its parent in a breadth-first spanning tree.
- $AsynchSpanningTree$ does not guarantee that the spanning tree constructed is breadth-first.
  - Long paths may be traversed faster than short ones.
- Now modify each process to keep track of distance, change parent when it hears of a shorter path.
  - Relaxation algorithm (like Bellman-Ford).
  - Must inform neighbors of changes.
  - Eventually, tree stabilizes to a breadth-first spanning tree.

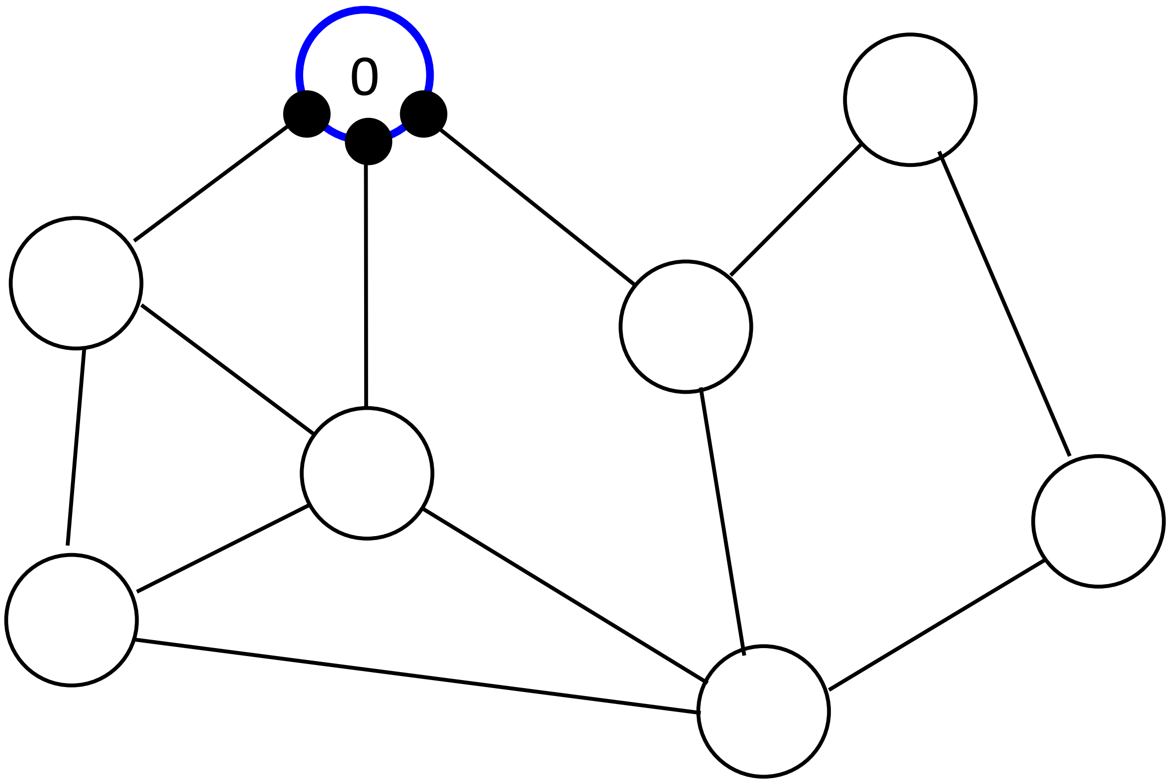
AsynchBFS

- Signature
  - ***in*** receive(m)$_{j,i}$, m $\in$ **N,** j $\in$ nbrs
  - ***out*** send(m)$_{i,j}$, m $\in$ **N**, j $\in$ nbrs
- State
  - dist: N U { ∞ }, initially 0 if i = i_0, else ∞
  - parent: nbrs U { $\perp$ }, init $\perp$
  - for each j $\in$ nbrs:
    - send(j): FIFO queue of N, initially (0) if i = i_0, else empty

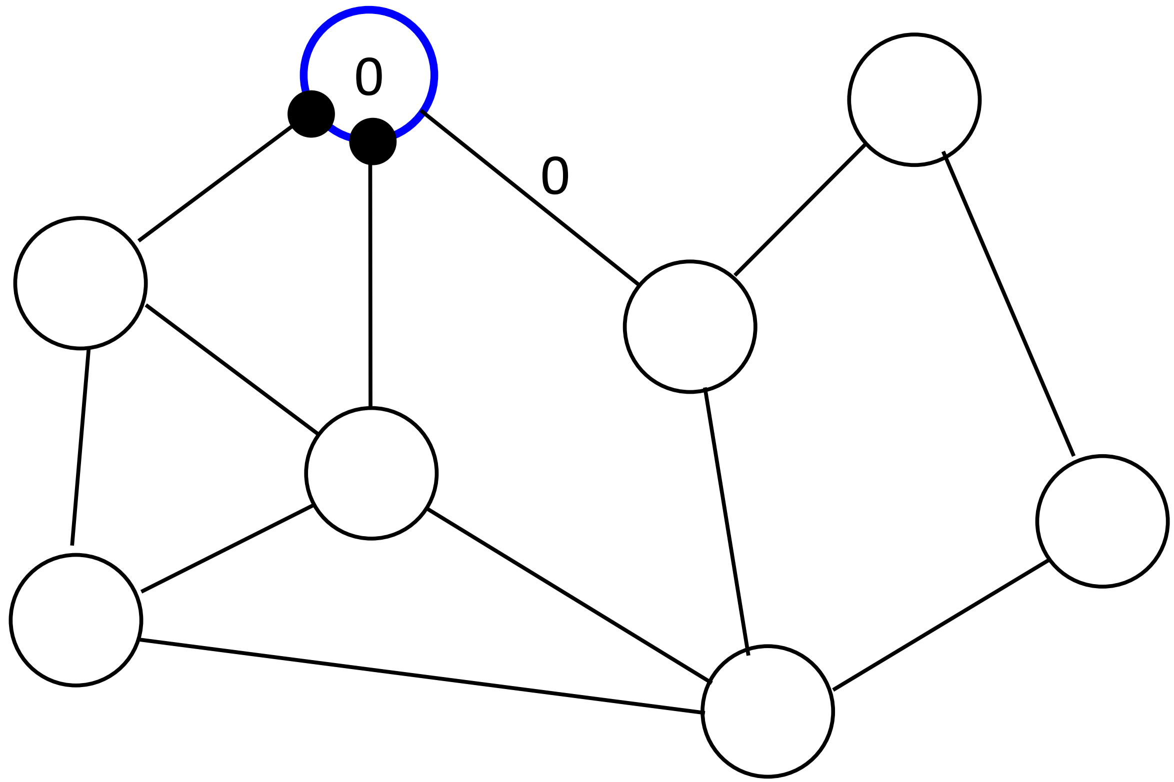
- send(m)$_{i,j}$
  pre: m = head(send(j))
  eff: remove head of send(j)
- receive(m)$_{j,i}$
  eff: if m+1 < dist then
  dist := m +1
  parent := j
  for k $\in$ nbrs - { j } do
  add dist to send(k)

Note: No parent output actions---no one knows when the algorithm is done

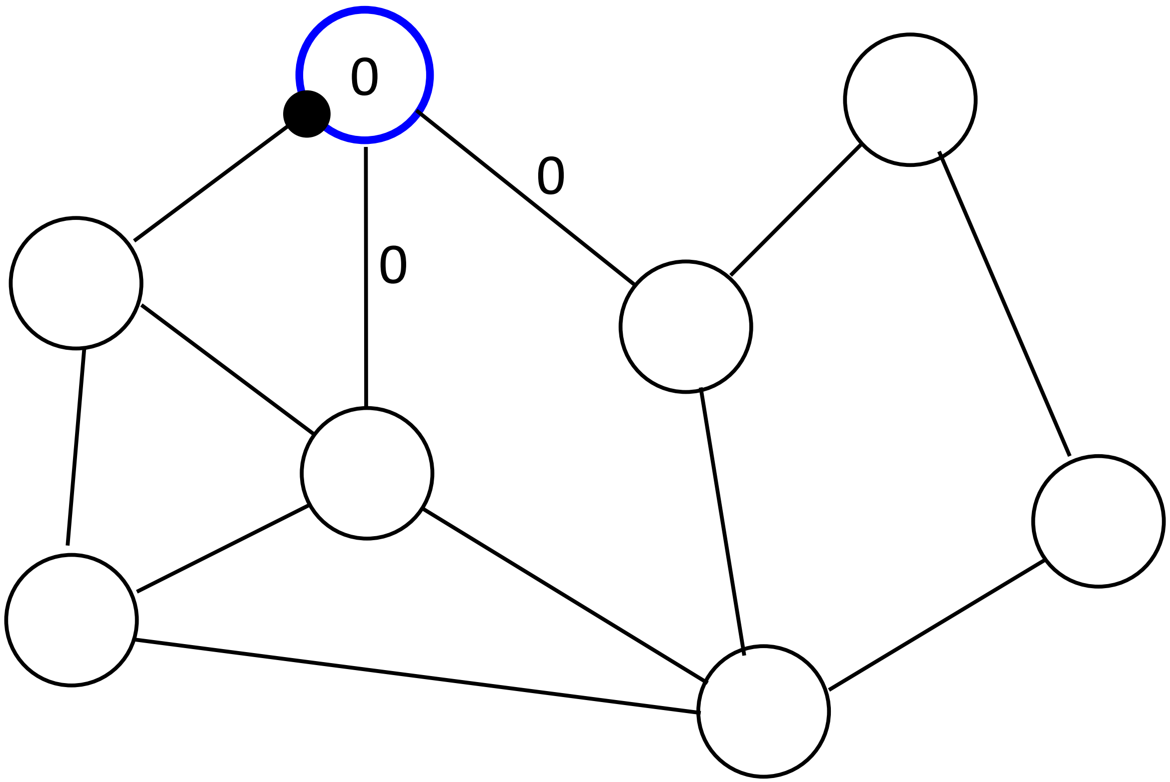
AsynchBFS



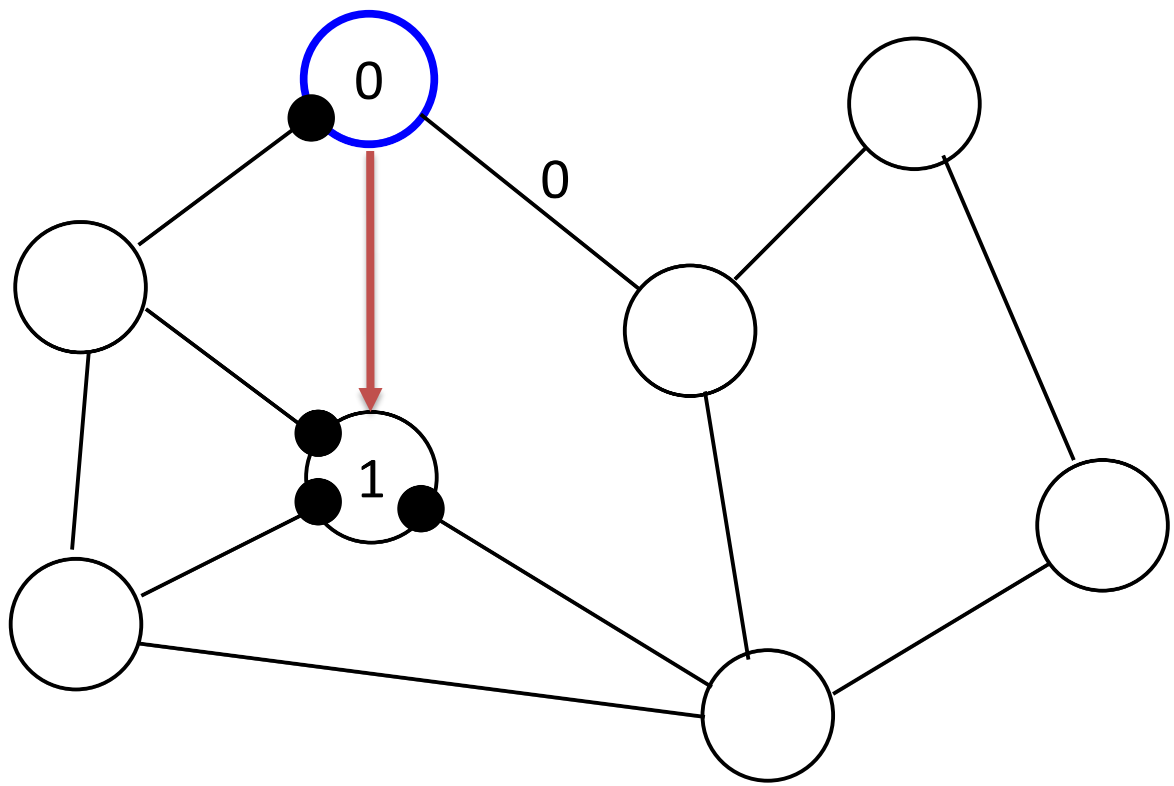
AsynchBFS



AsynchBFS



AsynchBFS



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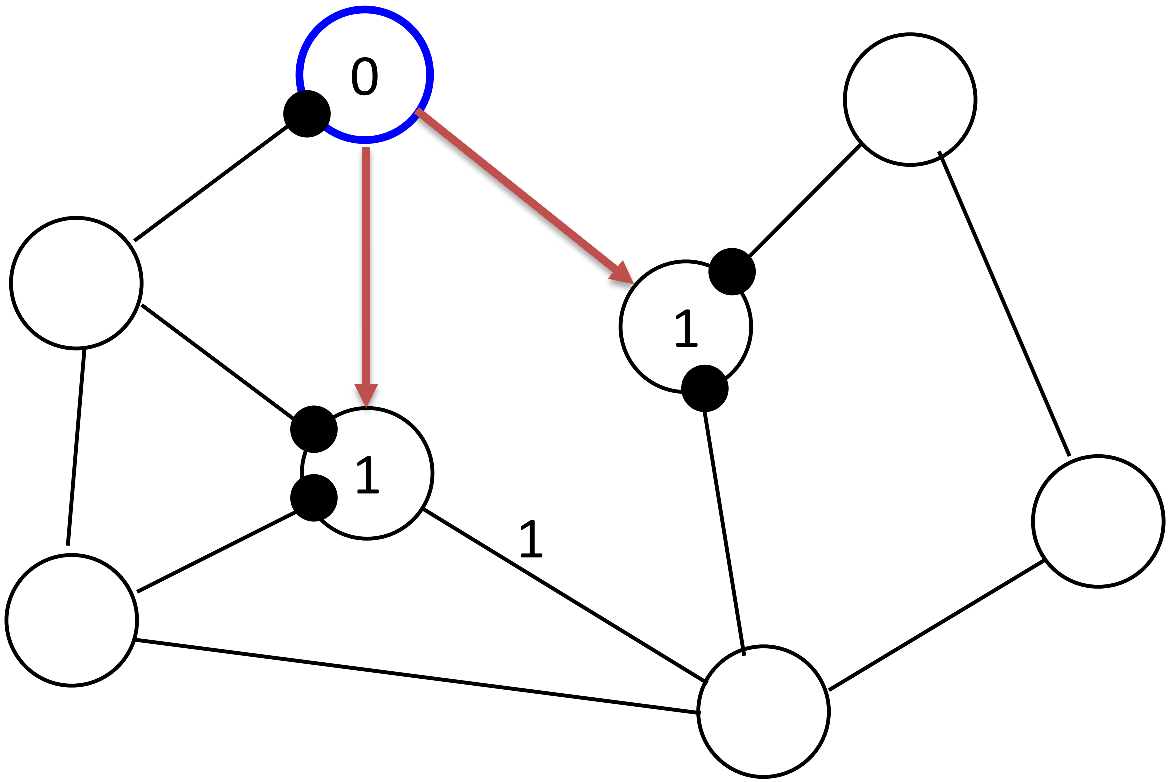
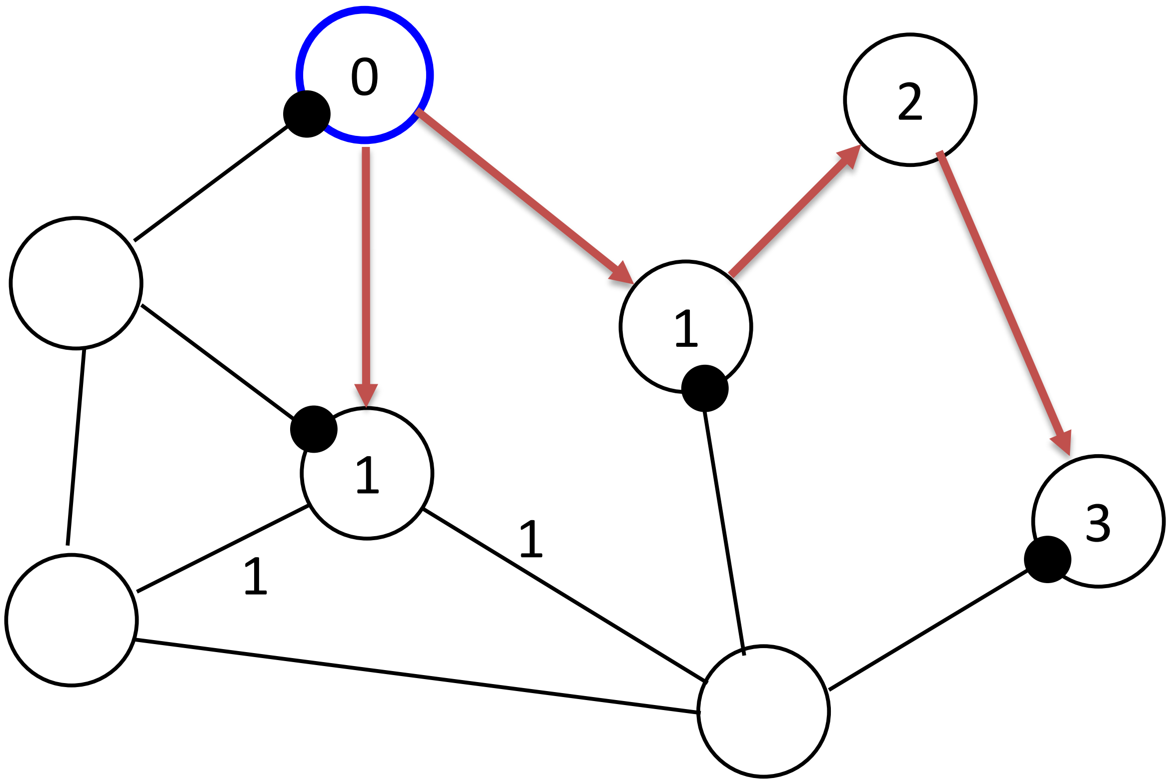
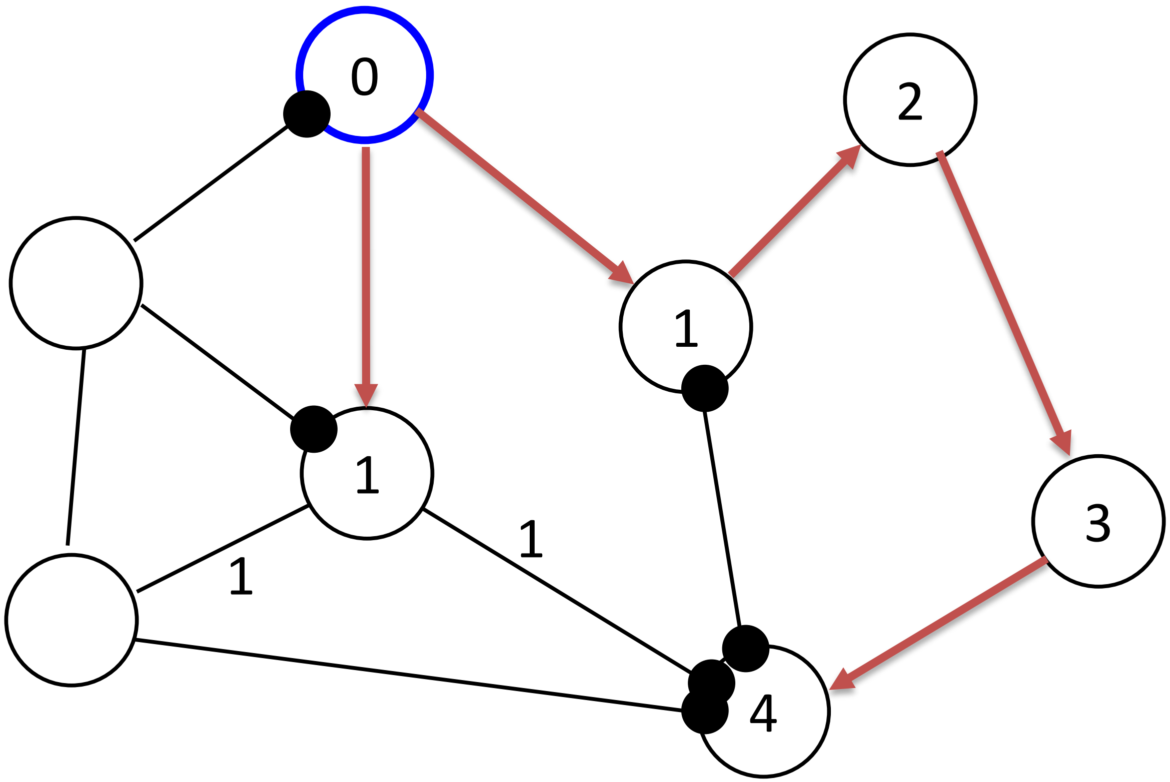


AsynchBFS



AsynchBFS



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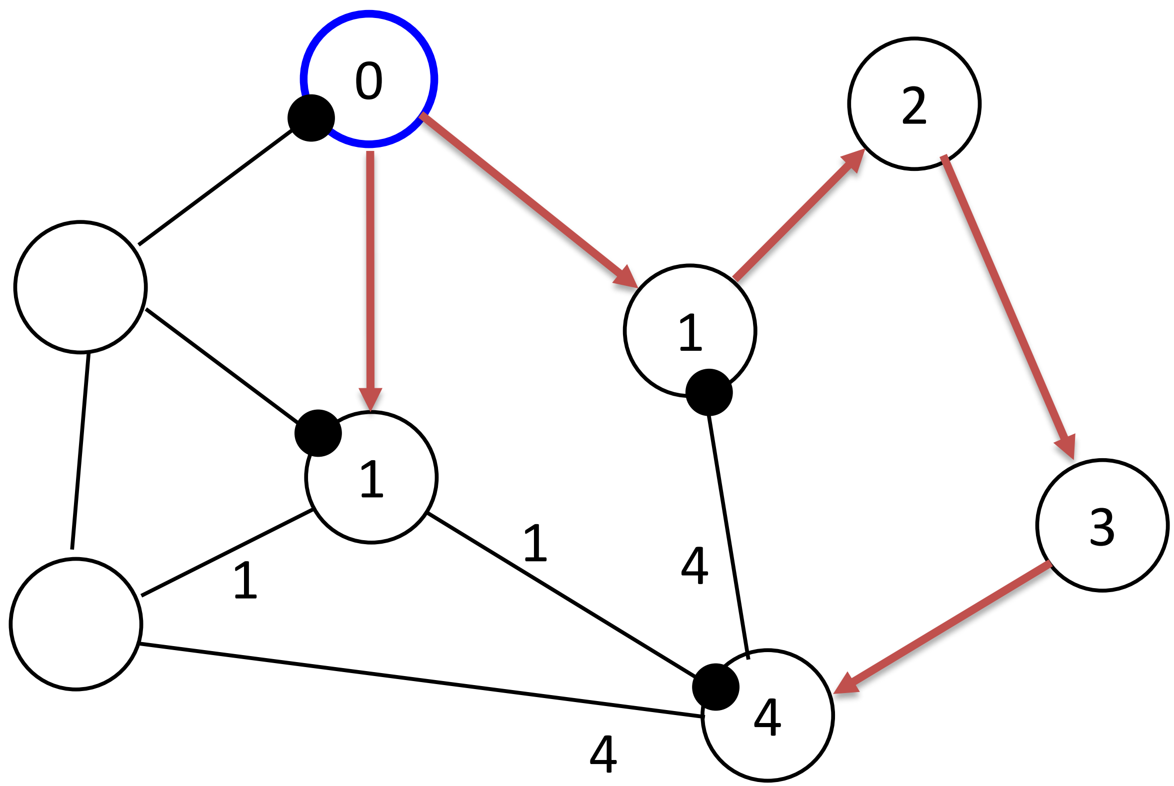

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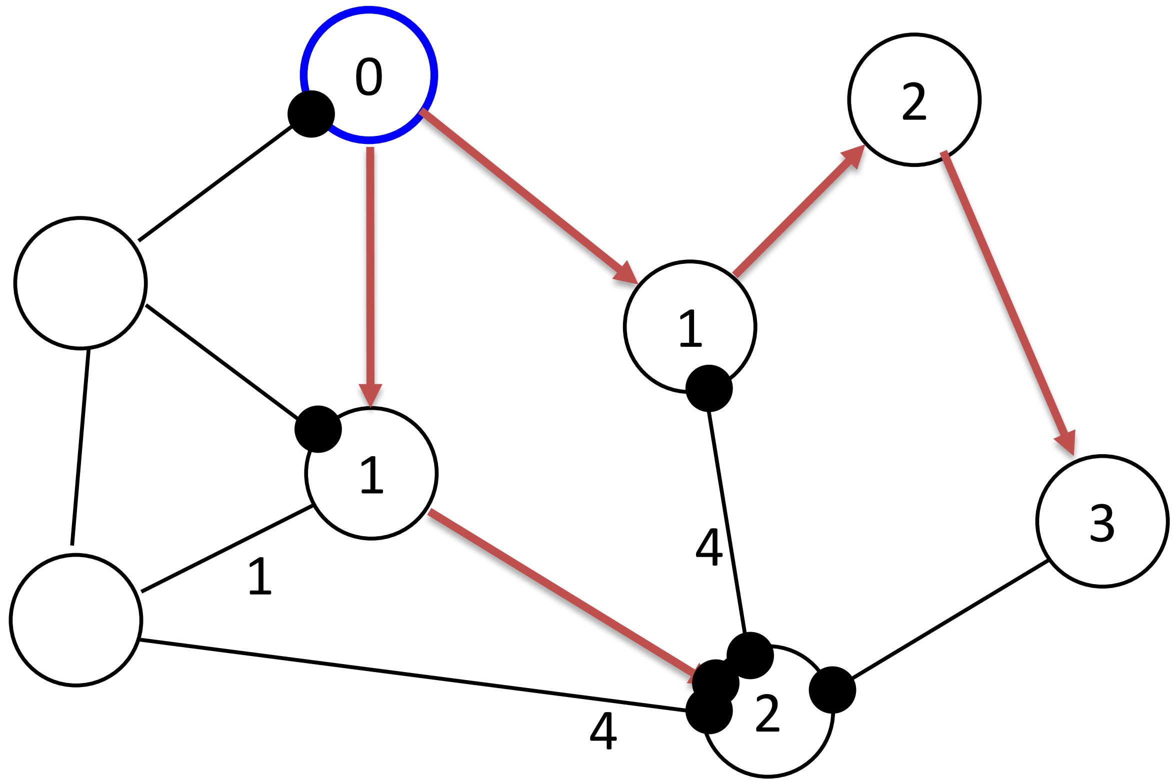

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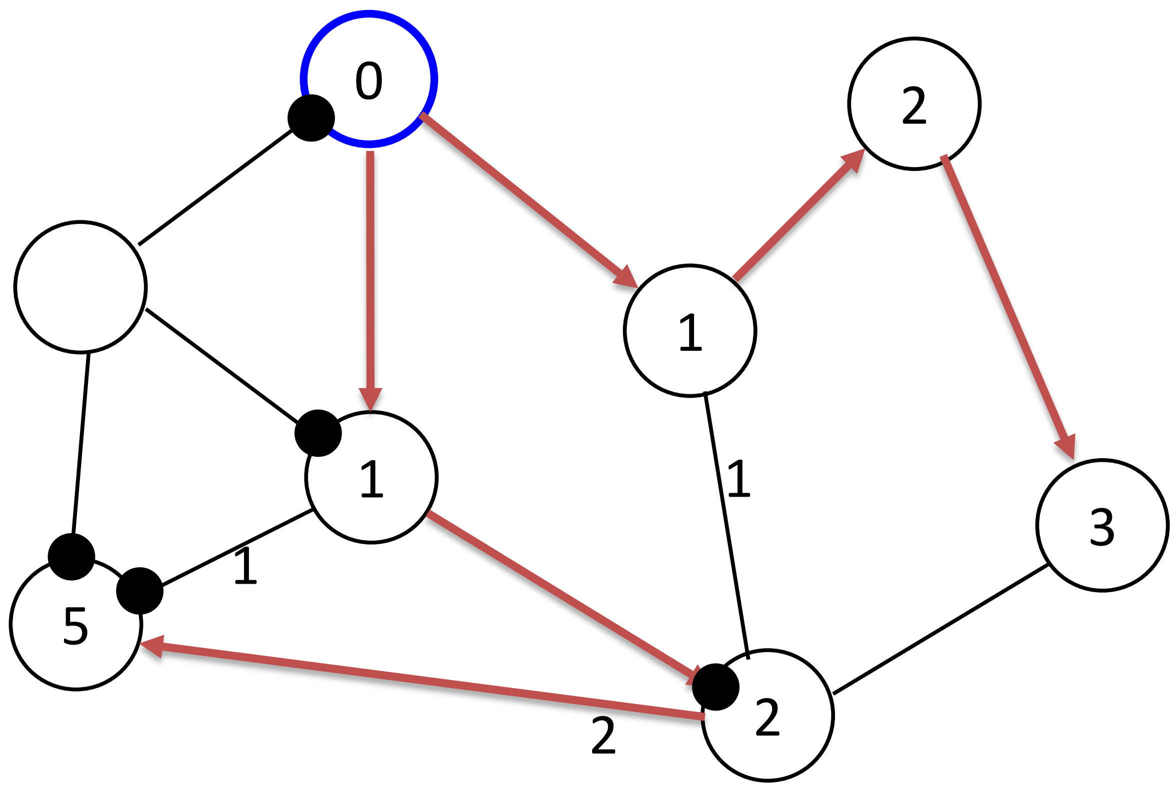

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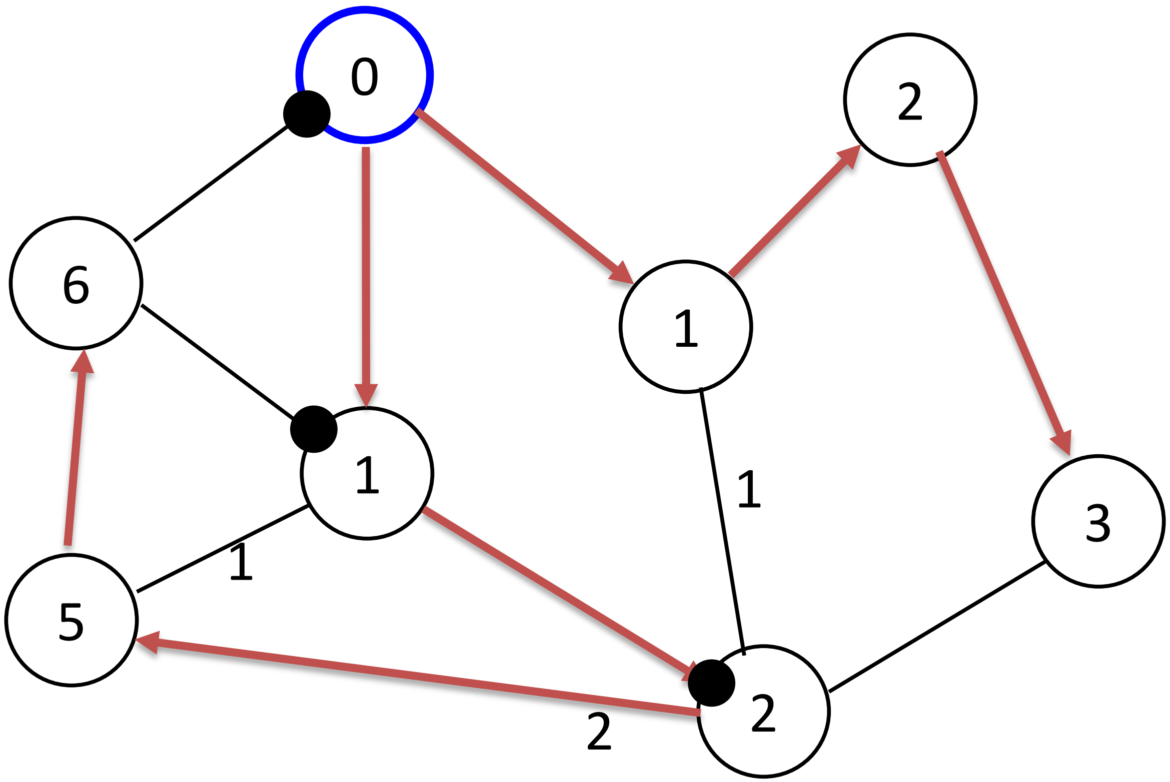

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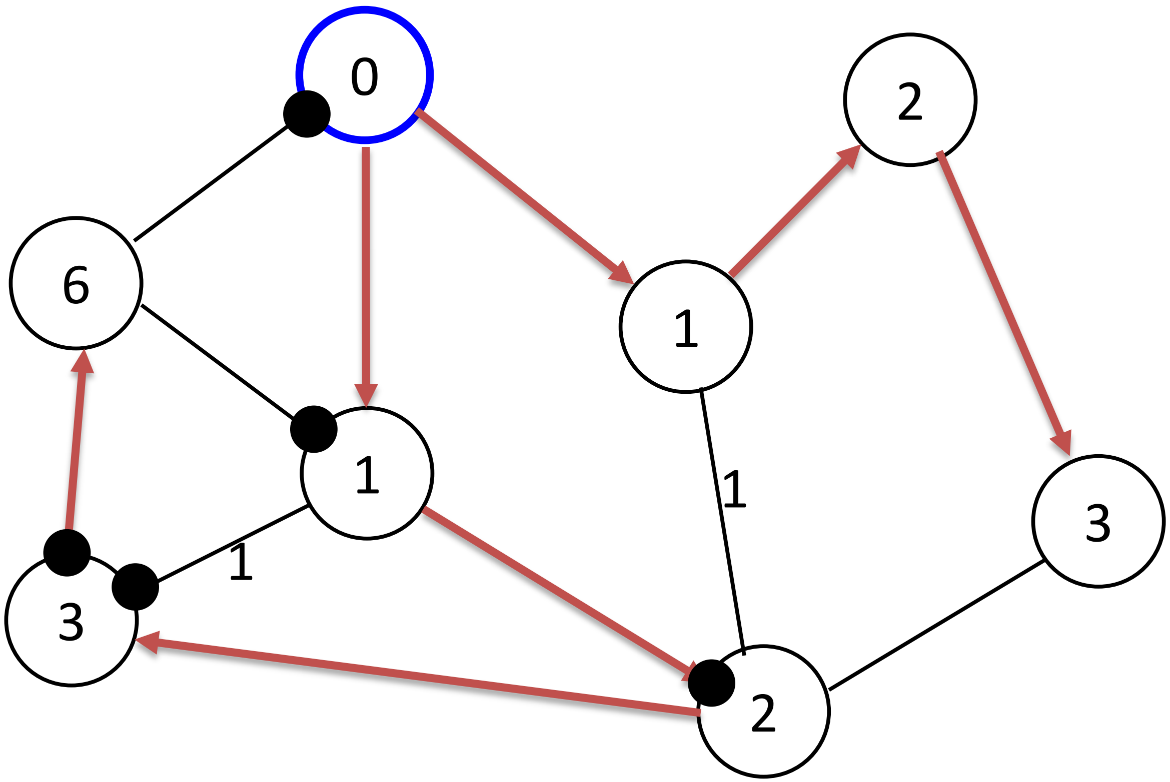
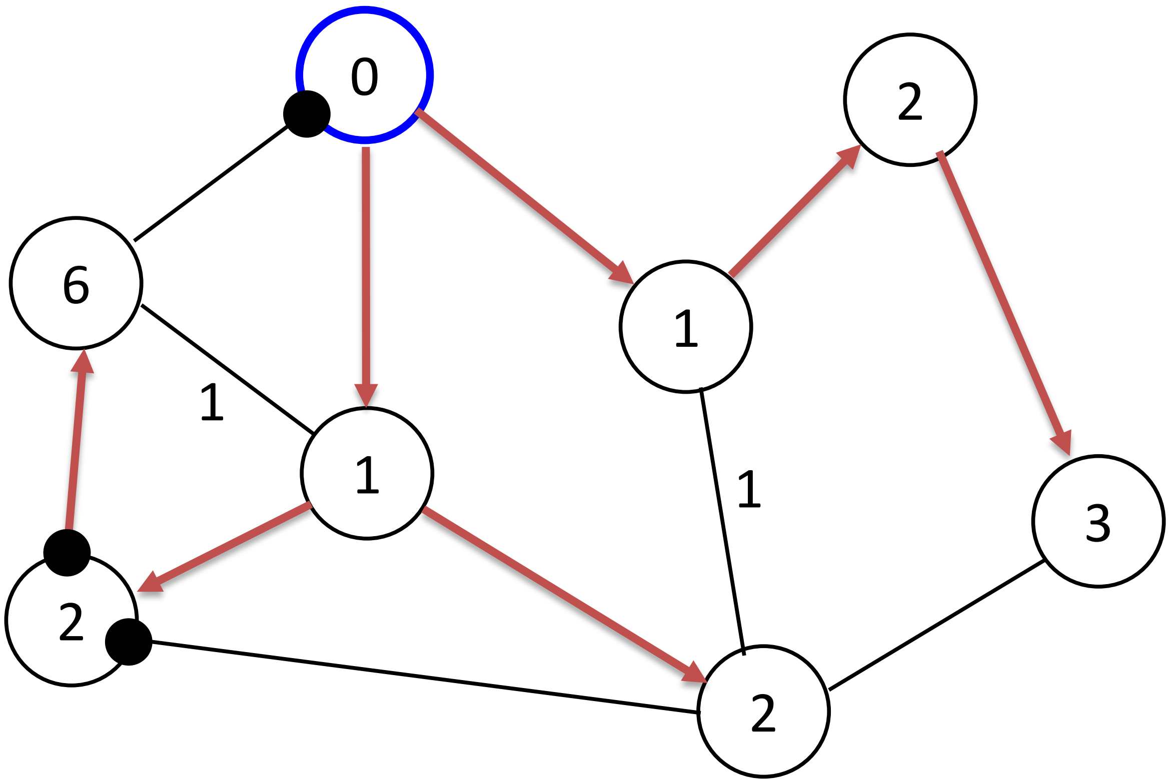
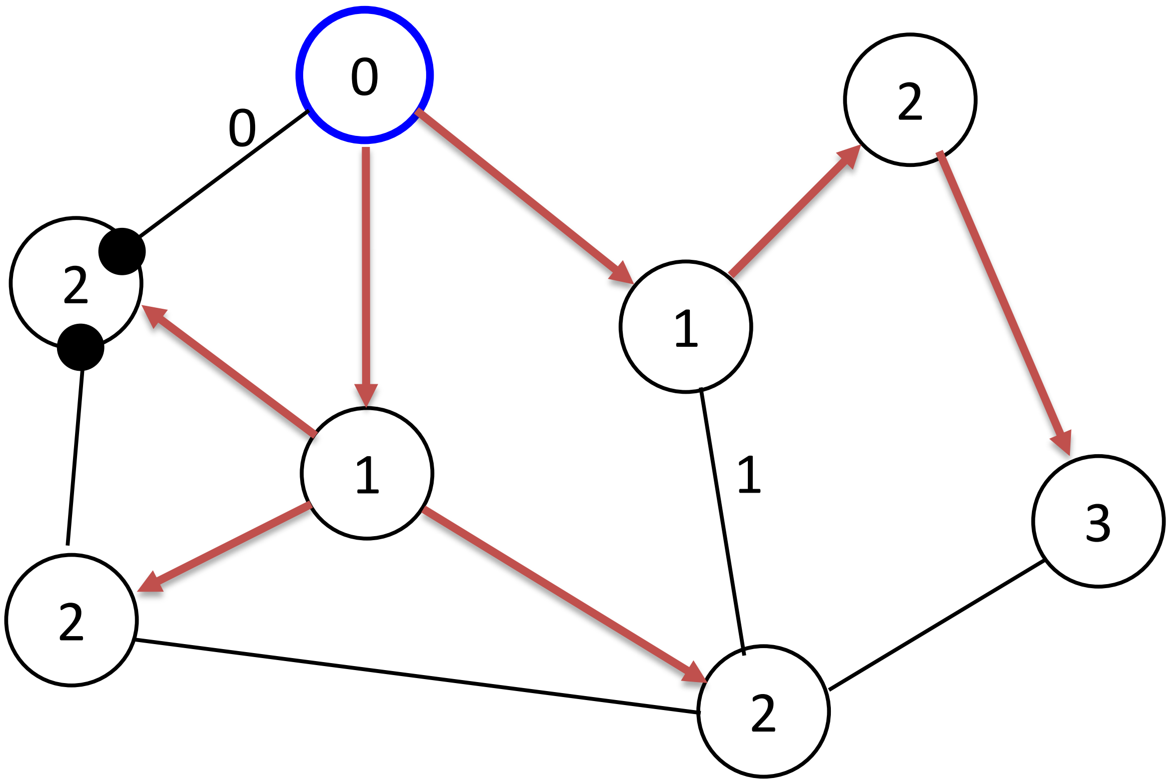


AsynchBFS



AsynchBFS



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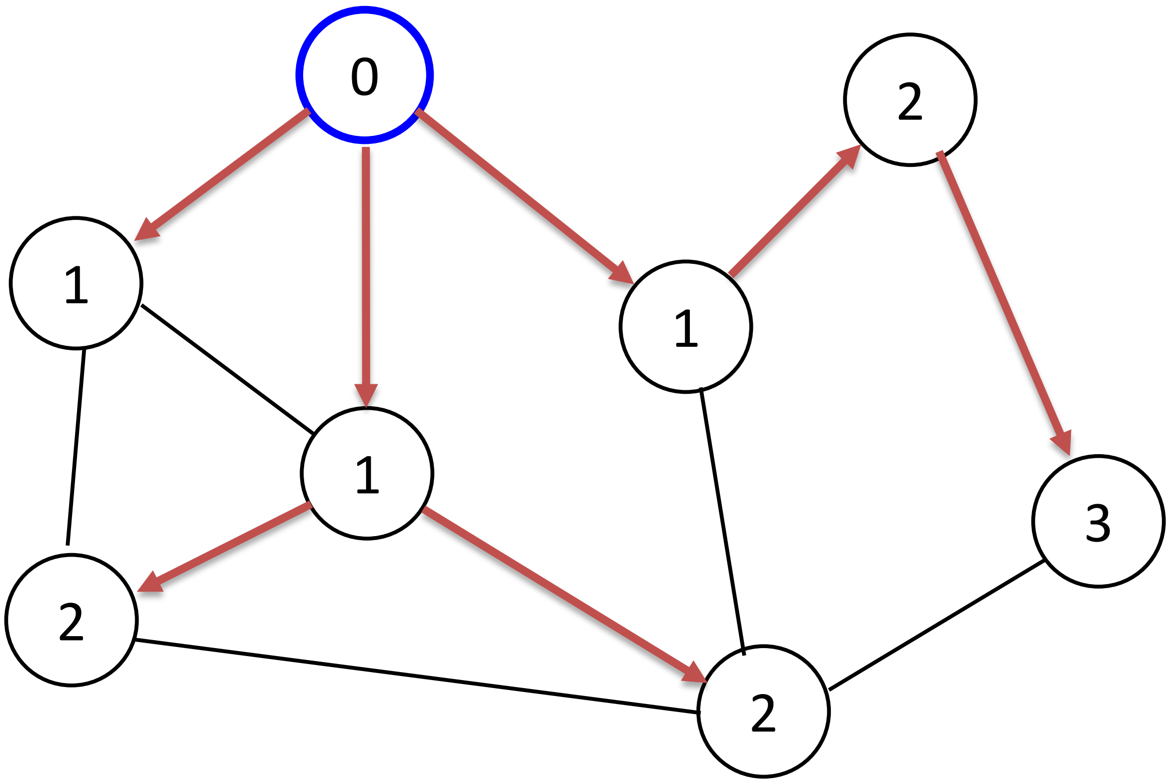


AsynchBFS

- Complexity:
  - Messages: $O(n\ |E|)$
    - May send $O(n)$ messages on each link (one for each distance estimate).
  - Time: $O(diam\ n\ (l+d))$ (taking pileups into account).
  - We can reduce complexity if we know an upper bound D on diameter:
    - Allow only distance estimates $\leq D$.
    - Messages: $O(D\ |E|)$; Time: $O(diam\ D\ (l+d))$
- Termination:
  - No one knows when this is done, so they can't produce $parent$ outputs.
  - Can augment with acks for search messages, convergecast back to i_0.
  - i_0 learns when the tree has stabilized, tells everyone else.
  - A bit tricky:
    - Tree grows and shrinks.
    - Some processes may participate many times, as they learn improvements.
    - Bookkeeping needed.
    - Complexity?

Layered BFS

- Asynchrony leads to many corrections, which lead to lots of communication.
- Idea: Slow down communication, grow the tree in synchronized phases.
  - In phase k, incorporate all nodes at distance k from i_0.
  - i_0 synchronizes between incorporating nodes at distance k and $k + 1$.
- Phase 1:
  - i_0 sends $search$ messages to neighbors.
  - Neighbors set $dist$:= 1, send acks to i_0.
- Phase $k + 1$:
  - Assume phases $1, \ldots, k$ are completed: each node at distance $\leq k$ knows its parent, and each node at distance $\leq k - 1$ also knows its children.
  - i_0 broadcasts $newphase$ message along tree edges, to distance-k processes.
  - Each of these sends $search$ message to all neighbors except its parent.
  - When any non-i_0 process receives its first $search$ message, it sets $parent$:= sender and sends ack; sends $nacks$ for subsequent $search$ messages.
  - When distance-k process receives $acks/nacks$ for all its $search$ messages, it designates nodes that sent $acks$ as its children.
  - Distance-k processes convergecast back to i_0 along the depth k tree to say that they're done; include a bit saying whether any new nodes were found.

Layered BFS

- Terminates: When i_0 learns, in some phase, that no new nodes were found.
- Obviously produces BFS tree, in $diam$ phases.
- Complexity:
  - Messages: $O(|E| + n\,diam)$

Each edge is explored at most once in each direction by search/ack.

Each tree edge is traversed at most once in each phase by newphase/convergecast.

  - Time:
    - Simplified analysis:
      - Neglect local computation time l
      - Assume every message in a channel is delivered in time d (ignore congestion delays).
    - $O(diam^2\, d)$

$LayeredBFS$ vs $AsynchBFS$

- Message complexity:
  - $AsynchBFS$: $O(diam\ |E|)$, assuming diameter is known, $O(n\ |E|)$ if not
  - $LayeredBFS$: $O(|E|\ +\ n\ diam)$
- Time complexity:
  - $AsynchBFS$: $O(diam\ d)$
  - $LayeredBFS$: $O(diam^2\ d)$

- Can also define "hybrid" algorithm (in book)
  - Add m layers in each phase instead of just one.
  - Within each phase, layers get constructed asynchronously.
  - Intermediate performance.

Shortest-Paths Spanning Trees

Shortest paths

- Assumptions:
  - Same as for BFS, plus edge weights.
  - $weight(i, j)$, nonnegative real, same in both directions.
- Require:
  - Output shortest distance and parent in shortest-paths tree.
- Use Bellman-Ford asynchronously
  - Used to establish routes in ARPANET 1969-1980.
  - Can augment with convergecast as for BFS, for termination.
  - But worst-case complexity is very, very bad…

AsynchBellmanFord

- Signature
  - ***in*** receive(w)$_{j,i}$, w $\in$ $\mathbf{R}^{\geq 0}$, j $\in$ nbrs
  - ***out*** send(w)$_{i,j}$, w $\in$ $\mathbf{R}^{\geq 0}$, j $\in$ nbrs
- State
  - dist: $R^{\geq 0}$ U { ∞ }, initially 0 if i = i_0, else ∞
  - parent: nbrs U { $\perp$ }, init $\perp$
  - for each j $\in$ nbrs:
    - send(j): FIFO queue of $R^{\geq 0}$; init (0) if i = i_0, else empty
- Transitions
  - send(w)$_{i,j}$

    pre: w = head(send(j))

    eff: remove head of send(j)
  - receive(w)$_{j,i}$

    eff: if w + weight(j,i) < dist then

    dist := w + weight(j,i)

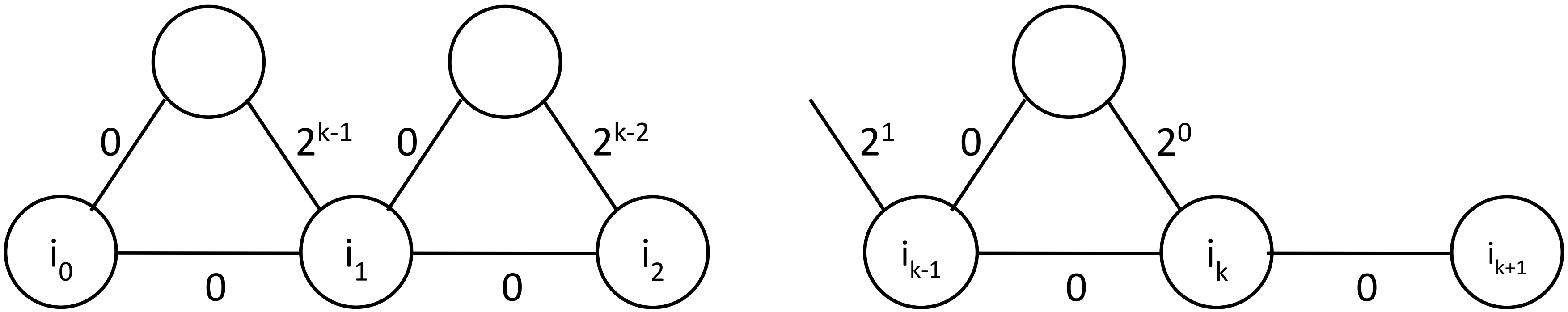
    parent := j

    for k $\in$ nbrs - { j } do

    add dist to send(k)

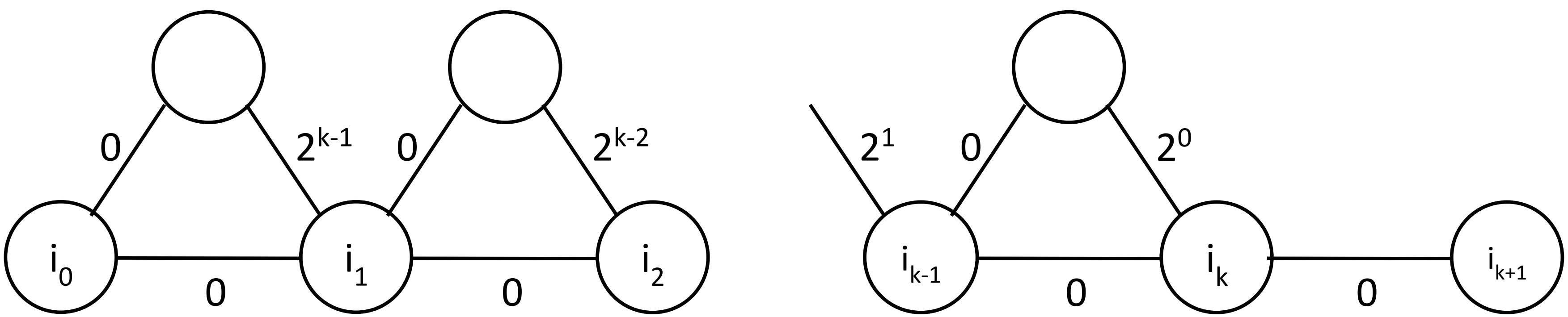
AsynchBellmanFord

- Termination:
  - Use convergecast (as for AsynchBFS).
- Complexity:
  - $O(n!)$ simple paths from i_0 to any other node, which is $O(n^n)$.
  - So the number of messages sent on any channel is $O(n^n)$.
  - So message complexity = $O(n^n\,|E|)$, time complexity = $O(n^n\, n\, (l + d))$.
  - Q: Are the message and time complexity really exponential in n?
  - A: Yes: In some execution of the network below, i_k sends 2^k messages to i_{k+1}, so message complexity is $\Omega(2^{n/2})$ and time complexity is $\Omega(2^{n/2}\, d)$.



Exponential time/message complexity

- In some execution, i_k sends 2^k messages to i_{k+1}, so message complexity is $\Omega(2^{n/2})$ and time complexity is $\Omega(2^{n/2}\, d)$.
- Possible distance estimates for i_k are $2^k - 1, 2^k - 2, \dots, 0$.
- Moreover, i_k can take on all these estimates in sequence:
  - First, messages traverse upper links, $2^k - 1$.
  - Then last lower message arrives at i_k, $2^k - 2$.
  - Then lower message $i_{k-2} \rightarrow i_{k-1}$ arrives, reduces i_{k-1}'s estimate by 2, message $i_{k-1} \rightarrow i_k$ arrives on upper links, $2^k - 3$.
  - Etc. Count down in binary.
  - If this happens quickly, get pileup of 2^k search messages in $C_{k,k+1}$.



Shortest Paths

- Moral: Unrestrained asynchrony can cause problems.
- Return to this problem after we have better synchronization methods.

- Now, another good illustration of the problems introduced by asynchrony:

Minimum Spanning Tree

Minimum spanning tree

- Assumptions:
  - $G = (V, E)$ connected, undirected.
  - Weighted edges, weights known to endpoint processes, weights distinct.
  - UIDs
  - Processes don't know $n, diam$.
  - Can identify in-edge and out-edge connecting to the same neighbor.
  - Input: $wakeup$ actions, occurring at any time at one or more nodes.
  - Process wakes up when it first receives either a $wakeup$ input or a protocol message.
- Requires:
  - Produce MST, where each process knows which of its incident edges belong to the tree.
  - Guaranteed to be unique, because of unique weights.

- [Gallager-Humblet-Spira]: Recommended reading!

Recall synchronous algorithm

- Proceeds in phases (levels).
- After each phase, we have a spanning forest, in which each component tree has a leader.
- In each phase, each component finds min weight outgoing edge ($MWOE$), then components merge using all $MWOE$s to get components for next phase.
- In more detail:
  - Each node is initially in component by itself (level 0 components).
  - Phase 1 (produces level 1 components):
    - Each node uses its min weight edge as the component $MWOE$.
    - Send $connect$ message across $MWOE$.
    - There is a unique edge that is the $MWOE$ of two components.
    - Leader of new component is higher-id endpoint of this unique edge.
  - Phase $k+1$ (produces level $k+1$ components):

Synchronous algorithm

- Phase $k + 1$ (produces level $k + 1$ components):
  - Leader of each component initiates search for $MWOE$ (broadcast $initiate$ on tree edges).
  - Each node finds its $mwoe$:
    - Send $test$ on potential edges, wait for $accept$ (different component) or $reject$ (same component).
    - Test edges one at a time in order of weight.
  - Report to leader (convergecast $report$); remember direction of best edge.
  - Leader picks $MWOE$ for component.
  - Send $changeroot$ message to $MWOE$'s endpoint, using remembered best edges.
  - Send $connect$ message across $MWOE$.
  - There is a unique edge that is the $MWOE$ of two components.
  - Leader of new component is higher-id endpoint of this unique edge.
  - Wait sufficient time for phase to end.

Synchronous algorithm

- Complexity is good:
  - Messages: $O(n \log n + |E|)$
  - Time (rounds): $O(n \log n)$

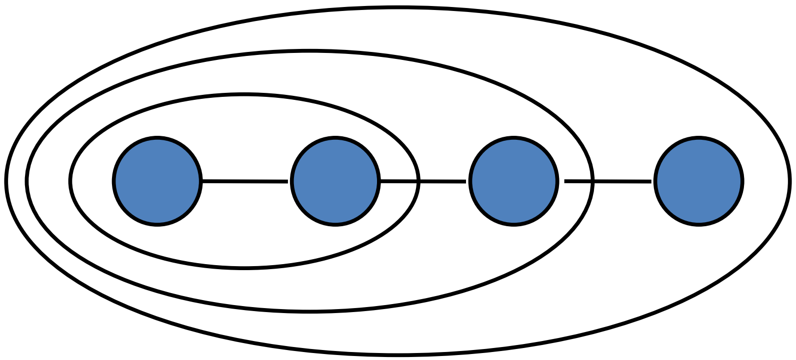
- Low message complexity depends on the way nodes test their incident edges, in order of weight, not retesting the same edge once it's rejected.

- Q: How to run this algorithm asynchronously?

Running the algorithm asynchronously

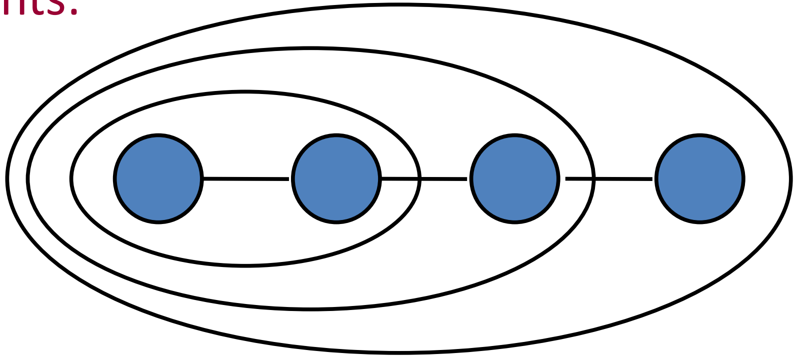
- Problems arise:
  - Inaccurate information about outgoing edges:
    - In the synchronous algorithm, when a node tests its edges, it knows that its neighbors are already up to the same level, and have up-to-date information about their component.
    - In asynchronous version, neighbors could lag behind; they might be in same component but not yet know this.
  - Less "balanced" combination of components:
    - In synchronous algorithm, level k components have $\geq 2^k$ nodes, and level $k + 1$ components are constructed from at least two level k components.
    - In asynchronous version, components at different levels could be combined.
    - Can lead to more messages overall.

    - Example: One component might keep merging with level 0 single-node components. After each merge, the number of messages sent in the tree is proportional to the component's size. Leads to $\Omega(n^2)$ messages overall.



Running the algorithm asynchronously

- Problems arise:
  - Inaccurate information about outgoing edges.
  - Less “balanced” combination of components:



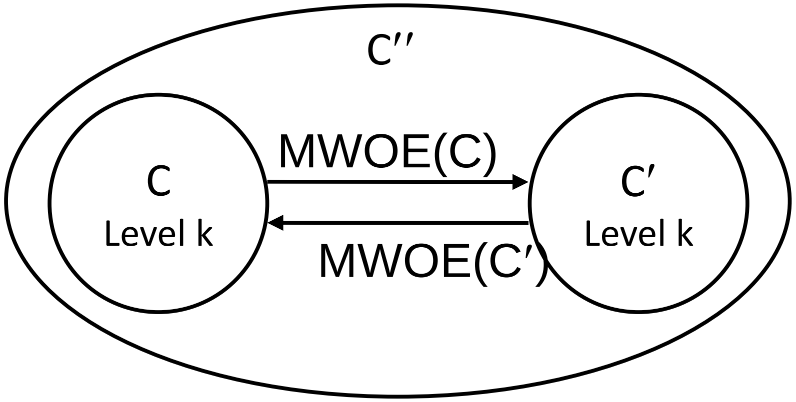
  - Concurrent overlapping searches/convergecasts:
    - When nodes are out of synch, concurrent searches for MWOEs could interfere with each other (we’ll see this).
- These problems result from nodes being out-of-synch, at different levels.
- We could try to synchronize levels, but carefully, so as not to hurt the time and message complexity too much.

GHS algorithm (asynchronous)

- Same basic ideas as before:
  - Form components, combine along $MWOE$s.
  - Within any component, processes cooperate to find component $MWOE$.
  - Broadcast from leader, convergecast, etc.
- Introduce synchronization to prevent nodes from getting too far ahead of their neighbors.
  - Associate a level with each component, as before.
  - Number of nodes in a level k component $\geq 2^k$, as before.
  - Now, each level $k+1$ component will be (initially) formed from exactly two level k components.
  - Level numbers are used for synchronization, and for determining who is in the same component.
- Complexity:
  - Messages: $O(|E| + n \log n)$
  - Time: $O(n \log n \, (d + l))$

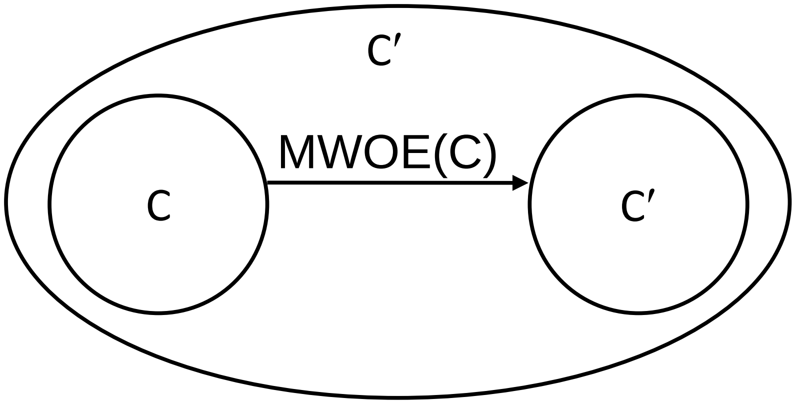
GHS algorithm

- Combine pairs of components in two ways, merging and absorbing.
- Merging:



  - C and C' have same level k, and have a common $MWOE$.
  - Result is a new merged component C'', with level $k+1$.

GHS algorithm



- Absorbing:
  - $level(C) < level(C')$, and C's $MWOE$ leads to C'.
  - Result is to absorb C into C'.
  - Not creating a new component---just adding C to existing C'.
  - C "catches up" with the more advanced C'.
  - Absorbing is cheap, local.

- Merging and absorbing ensure that the number of nodes in any level k component $\geq 2^k$.
- Merging and absorbing are both allowable operations in computing the MST, because they are allowed by the general theory for MSTs.

Liveness

- Q: Why are merging and absorbing sufficient to ensure that the construction is eventually completed?
- Lemma: After any allowable finite sequence of merges and absorbs, either the forest consists of one tree (so we're done), or some merge or absorb is enabled.
- Proof:
  - Consider the current "component digraph":
    - Nodes = components
    - Directed edges correspond to $MWOE$s
  - Then there must be some C, C' whose $MWOE$s point to each other. (Why?)
  - These $MWOE$s must be the same edge. (Why?)
  - Can combine, using either merge or absorb: If same level, merge, else absorb.
- So, merging and absorbing are enough.
- Now, how to implement them with a distributed algorithm?

Component names and leaders

- For every component with $level > 1$, define the core edge of the component's tree.
- Defined in terms of the merge and absorb operations used to construct the component:
  - After merge: Use the common $MWOE$.
  - After absorb: Keep the old core edge of the higher-level component.
- "The edge along which the most recent merge occurred."

- Component name: $(core, level)$
- Leader: Endpoint of core edge with higher id.

Determining whether an edge is outgoing

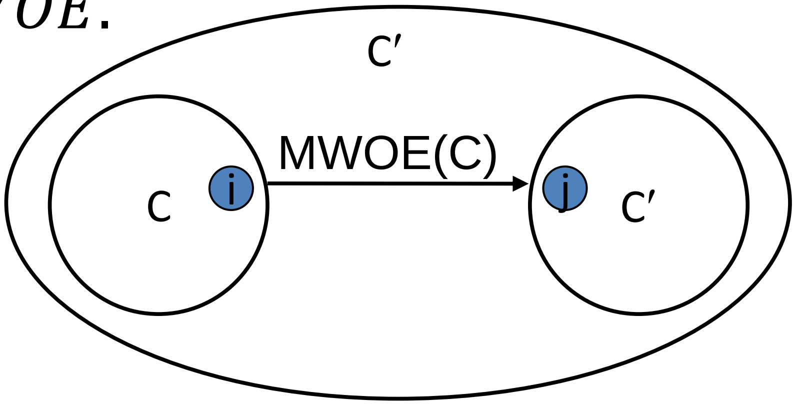
- Suppose i wants to know whether the edge (i, j) is outgoing from i's current component.
- At that point, i's component name info is up-to-date:
  - Component is in "search mode".
  - i has received an $initiate$ message from the leader, which included the component name.
- So i sends j a $test$ message.
- Three cases:
  - If j's current $(core, level)$ is the same as i's, then j knows that it is in the same component as i.
  - If j's $(core, level)$ is different from i's and j's level is $\geq$ i's, then j knows that j is in a different component from i.
    - Each component has only one core per level.
    - No one in the same component currently has a higher level than i does, since the component is still searching for its $MWOE$.
  - If j's level is < i's, then j doesn't know if it is in the same or a different component. So it doesn't yet respond---it waits to catch up to i's level.

Liveness, again

- Q: Can the extra delays imposed here affect the progress argument?
- No:
  - We can redo the progress argument, this time considering only those components with the lowest current level k.
  - All processes in these components must succeed in determining their $mwoe$s, so these components succeed in determining the component $MWOE$.
  - If any of these level k components' $MWOE$s leads to a higher level, then we can absorb.
  - If not then all lead to other level k components, so as before, we must have two components that point to each other; so we can merge.

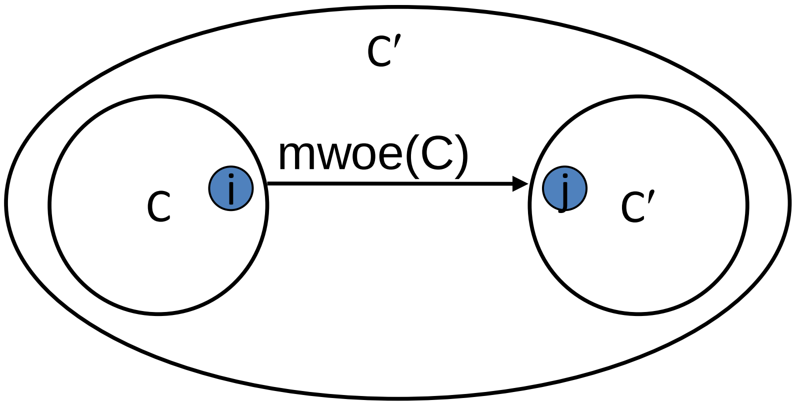
Interference between $MWOE$ searches

- Suppose C gets absorbed into C' via an edge from i to j, while C' is working on determining its $MWOE$.



- Two cases:
  - When the absorb occurs, j has not yet reported its local $mwoe$.
    - Then it's not too late for C' to include C in its $MWOE$ search. So j passes the *initiate* message into C.
  - j has already reported its local $mwoe$.
    - Then it's too late to include C in the search.
    - But it doesn't matter: the $MWOE$ for the combined component can't be outgoing from a node in C anyhow!
    - Why not?

Interference between $MWOE$ searches



- If j has already reported its local $mwoe$, then the $MWOE$ for the combined component is not outgoing from a node in C.

- Claim 1: j 's reported $mwoe$ is not the edge (i, j).
- Proof:
- j 's $mwoe$ must lead to a node with $level \geq level(C')$.
- But i 's level $< level(C')$ when the $absorb$ occurs.
- So j 's $mwoe$ must be a different edge, with weight $< \ weight(i, j)$.

- Claim 2: $MWOE$ for combined component is not outgoing from a node in C.
- Proof:
- The weight of the $MWOE$ of the combined component is $\leq$ the weight of j 's $mwoe$, so is $< \ weight(i, j)$.
- Since (i, j) is the $MWOE$ of C, there are no edges outgoing from C with $weight \ < \ weight(i, j)$.
- So $MWOE$ of combined component isn't outgoing from C.

A few details

- Specific messages:
  - $initiate$: Broadcast from leader to find $MWOE$; piggyback the component name on the message.
  - $report$: Convergecast the responses back to the leader.
  - $test$: Asks whether an edge is outgoing from the component.
  - $accept/reject$: Answers.
  - $changeroot$: Sent from leader to endpoint of $MWOE$.
  - $connect$: Sent across the $MWOE$, to connect components.
    - We say $merge$ occurs when a $connect$ message has been sent both ways on the edge (2 nodes must have same level).
    - We say $absorb$ occurs when a $connect$ message has been sent on the edge from a lower-level to a higher-level node.

Test-Accept-Reject Protocol

- Bookkeeping: Each process i keeps a list of incident edges in order of weight, classified as:
  - $branch$ (in the MST),
  - $rejected$ (leads to same component, not in the MST), or
  - $unknown$ (not yet classified).
- Process i tests only $unknown$ edges, in order of weight:
  - Sends $test$ message, with $(core, level)$; recipient j compares.
  - If same $(core, level)$, j sends $reject$ (same component), and i reclassifies edge as $rejected$.
  - If $(core, level)$ pairs are unequal and $level(j) \geq level(i)$ then j sends $accept$ (different component). i does not reclassify the edge.
  - If $level(j) \ < \ level(i)$ then j delays responding, until $level(j) \geq level(i)$.
- Retesting is possible, for accepted edges.
- Reclassify edge as $branch$ as a result of $changeroot$ message.

Complexity

- As for synchronous version.
- Messages: $O(|E| + n \log n)$
  - $4|E|$ for $test + reject$ messages (one pair for each direction of every edge)
  - n $initiate$ messages per level (broadcast: only sent on tree edges)
  - n $report$ messages per level (convergecast)
  - $2n$ $test + accept$ message pairs per level (one pair per node)
  - n $changeroot + connect$ messages per level (leader to $MWOE$ path)
  - $\log n$ levels
  - Total: $4|E| + 5n \log n$
- Time: O(n log n (l + d))

Proving Correctness

- GHS MST is hard to prove, because it's complicated.
- GHS paper includes informal arguments.
  - Pretty convincing, but not formal.
  - Also simulated the algorithm extensively.
- Many successful attempts to formalize, all complicated
  - Many invariants because many variables and actions.
  - Some use simulation relations.
  - Recent proof by Moses and Shimony.

Minimum spanning tree

- Application to leader election:
  - Convergecast from leaves until messages meet at node or edge.
  - Works with any spanning tree, not just MST.
  - E.g., in asynchronous ring, this yields $O(n \log n)$ messages for leader election.
- Lower bounds on message complexity for MST:
  - $\Omega(n \log n)$, from leader election lower bound and the reduction above.

Next time

- Synchronizers
- Reading: Chapter 16