6.852: Distributed Algorithms
Fall, 2015

Lecture 12

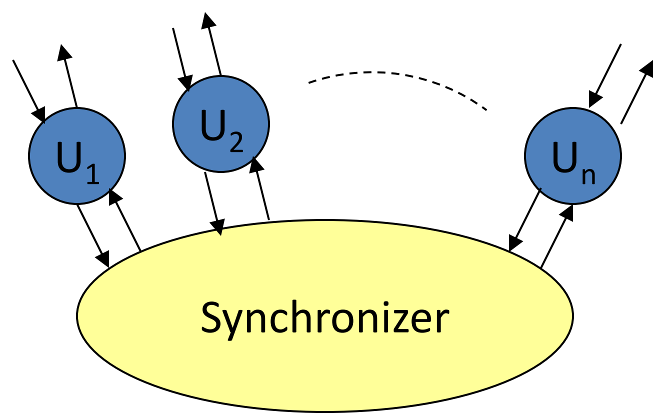
Last Time

- Asynchronous spanning tree algorithms:
  - Asynch Spanning Tree (not necessarily breadth-first)
  - Asynchronous BFS
  - Asynchronous Shortest Paths
- Important observation: In a distributed algorithm, fast execution of portions of the algorithm don't necessarily result in fastest execution overall.
- Different from sequential algorithms.
- Also GHS Minimum Spanning Tree algorithm.
- Questions?

Today's plan

- Simulating synchronous algorithms in asynchronous networks.
- Synchronizers
- Lower bound for global synchronization
- Reading: Chapter 16
- Next:
  - Logical time, state machine emulation, vector timestamps
  - Readings:
    - Chapter 18
    - [Lamport] Time, Clocks, and the Ordering of Events in a Distributed System
    - [Mattern] Vector timestamps

Simulating Synchronous Algorithms in Asynchronous Networks (Synchronizers)



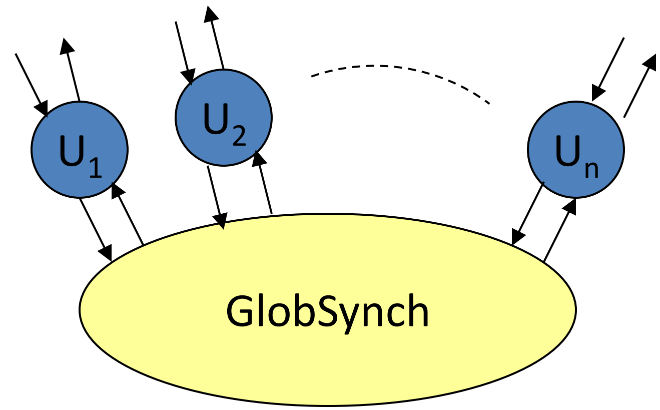
Minimum spanning tree, revisited

- In GHS, complications arise because different processes can be at very different levels at the same time.
- Alternative, simpler, more synchronized approach:
  - Keep levels of nearby nodes close, by restricting the asynchrony.
  - Each process uses a level variable to keep track of the level of its current component (according to its local knowledge).
  - Each process at level k delays all “interesting” processing until it hears that all its neighbors have reached $level \geq k$.
    - Looks (to each process) like global synchronization, but easier to achieve.
    - Each node inform its neighbors whenever it changes level.
- Resulting algorithm is simpler than GHS.
- Complexity:
  - Time: $O(n \log n)$, like GHS.
  - Messages: $O(|E| \log n)$, worse than GHS.

A strategy for designing asynchronous distributed algorithms

- Assume undirected graph $G = (V, E)$.
- Design a synchronous algorithm for G, then transform it into an asynchronous algorithm using local synchronization.
- Synchronize at every round (not every “level” as above).
- Method works only for non-fault-tolerant algorithms.
  - In fact, no general transformation can work for fault-tolerant algorithms.
  - E.g., simple fault-tolerant stopping agreement is solvable in synchronous networks, but unsolvable in asynchronous networks [FLP].
- Present a general strategy, and some special implementations.
  - Describe in terms of sub-algorithms, modeled as abstract services.
  - [Raynal book], [Awerbuch papers]
- Then a lower bound on the time for global synchronization.
  - Larger than upper bounds for local synchronization.

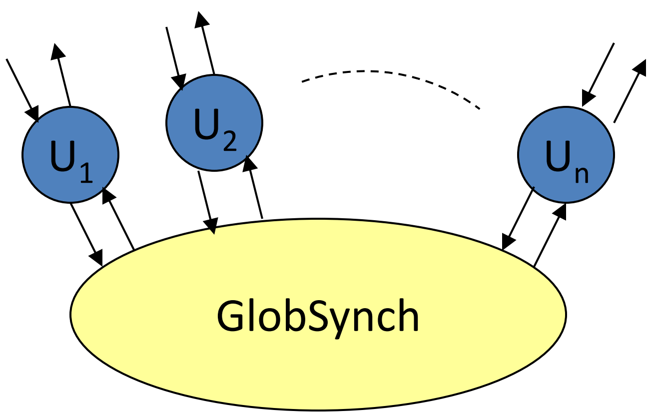
Synchronous model, reformulated in terms of I/O automata



- Global synchronizer automaton
- User process automata:
  - Processes of an algorithm that uses the synchronizer.
  - May have other inputs/outputs, for interacting with other programs.
- Interactions between user process i and synchronizer:
  - $usersend(T,r)_i$
    - T = set of (message, destination) pairs, destinations are neighbors of i.
    - T = empty set $\emptyset$, if no messages are sent by i at round r.
    - r = round number
  - $userrcv(T,r)_i$
    - T = set of (message, source) pairs, where source is a neighbor of i.
    - r = round number

Behavior of $GlobSynch$

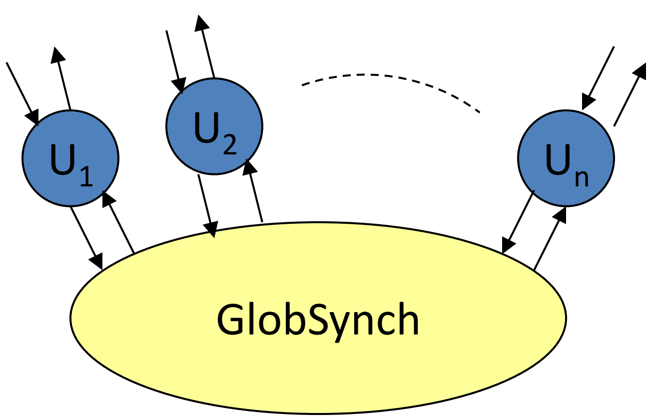
- Manages global synchronization of rounds:
  - Users send packages of all their round 1 messages, using $usersend(T, 1)$ actions.
  - GlobSynch waits for all round 1 messages, sorts them, then delivers to users, using $userrcv(T, 1)$ actions.
  - Users send round 2 messages, etc.
- Not exactly the same as the synchronous model:
  - $GlobSynch$ can receive round 2 messages from a user before it finishes delivering all the round 1 messages.
  - But it doesn't do anything with these until it's finished round 1 deliveries.
  - So, essentially the same.



- $GlobSynch$ synchronizes globally between each pair of rounds.

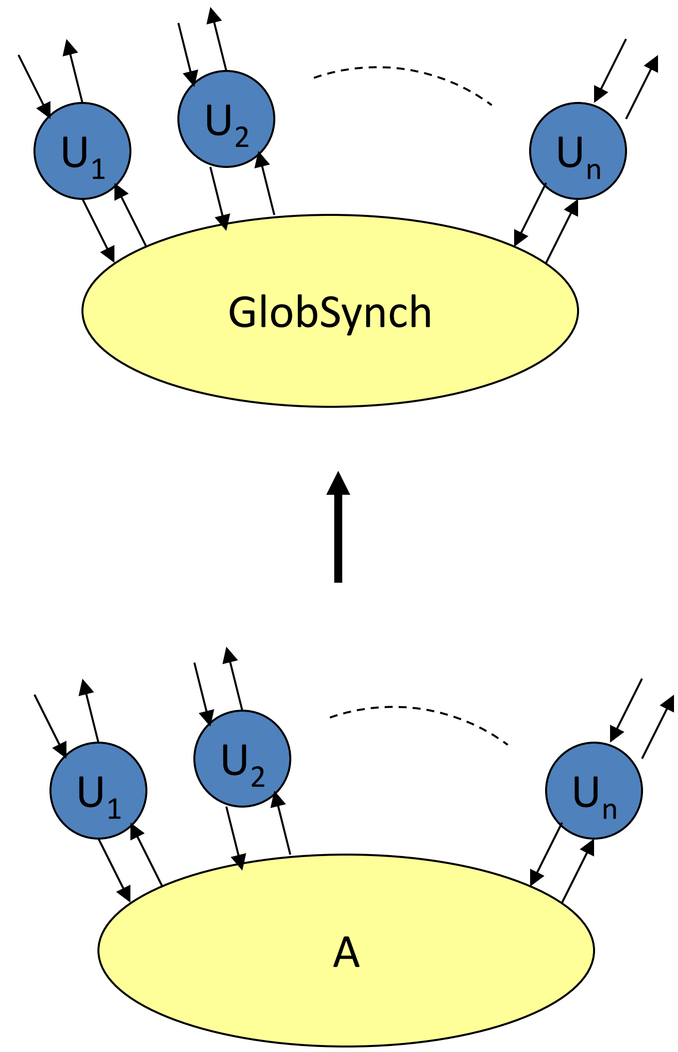
Requirements on each U_i

- Well-formed:
  - U_i sends the right kinds of messages, in the right order, at the right times.
- Liveness:
  - After receiving the messages for any round r, U_i eventually submits messages for round $r + 1$.
- Code for $GlobSynch$ in [book, p. 534].
  - State consists of:
    - A tray of messages for each (destination, round).
    - Some Boolean flags to keep track of which sends and rcvs have happened.
  - Transitions obvious.
  - Liveness expressed by tasks, one for each (destination, round).

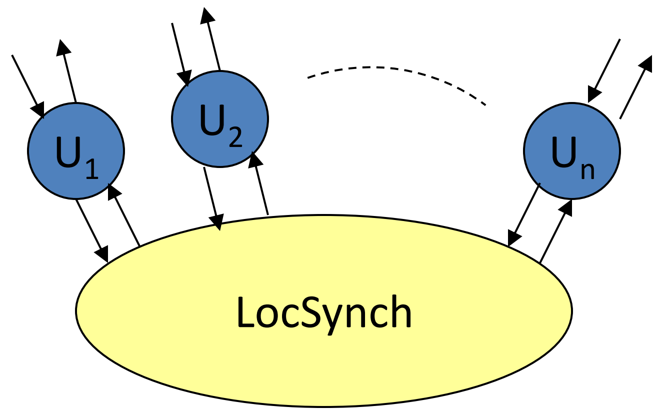


The Synchronizer Problem

- Design an automaton A that "implements" $GlobSynch$ in the sense that it "looks the same" to each U_i:
  - Has the right interface.
  - Exhibits the right behavior:
    - For every fair execution α of the U_is and A,
    - There exists a fair execution α' of the U_is and $GlobSynch$, such that
    - For every i, α is indistinguishable by U_i from α', written as $\alpha \sim_{U_i} \alpha'$.
- A "behaves like" $GlobSynch$, as far as any individual U_i can tell.
- Allows global reordering of events at different U_i.



Local Synchronizer, $LocSynch$



- Enforces local (not global) synchronization, still looks the same locally.
- Only difference from $GlobSynch$: the precondition for $usrrcv(T,r)_i$.
  - To deliver round r messages to user i, $LocSynch$ checks only that i's neighbors have sent round r messages.
  - Doesn't wait for all nodes.

- Lemma 1: For every fair execution α of the U_is and $LocSynch$, there is a fair execution α' of the U_is and $GlobSynch$, such that for each i, $\alpha \sim U_i \, \alpha'$.
- Proof:
  - Can't use a simulation relation, since the global order of external events need not be the same, and simulation relations preserve external order.
  - So consider a partial order of events and dependencies:

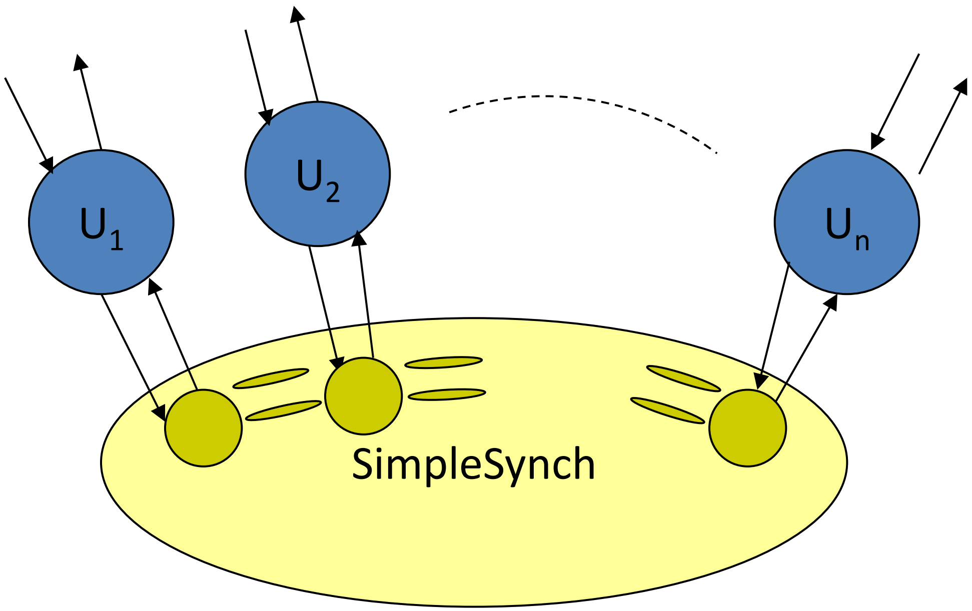
Proof sketch for Lemma 1

- Consider partial order of events and dependencies:
  - Each U_i event depends on previous U_i events.
  - $userrcv(*,r)_i$ depends on $usersend(*,r)_j$ for every neighbor j of i.
  - Take transitive closure.
- Claim: If we start with a (fair) execution of the $LocSynch$ system and reorder events while preserving these dependencies, the result is still a (fair) execution of the $LocSynch$ system.
- So, obtain α' by reordering the events of α so that:
  - These dependencies are preserved, and
  - The rounds are globally aligned: events associated with any round r precede those of round r+1.
- OK because round r events don't depend on round $r+1$ events.
- This reordering preserves the view of each U_i.
- Also satisfies the extra $userrcv$ precondition needed by $GlobSynch$.

Trivial distributed algorithm to implement $LocSynch$

- Processes, point-to-point channels.
- $SimpleSynch$ algorithm, process i:
  - After $usersend(T,r)_i$, send a message to each neighbor j containing round number r and any algorithm messages i has for j.
  - Send $(\emptyset, r)$ message if i has no basic algorithm messages for j.
  - Wait to receive round r messages from all neighbors.
  - Output $userrcv(T', r)$.

- **Lemma 2:** For every fair execution α of U_is and $SimpleSynch$, there is a fair execution α' of U_is and $LocSynch$, such that for each i, $\alpha \sim_{U_i} \alpha'$.
- In fact, indistinguishable by all the U_is together---preserves external order.

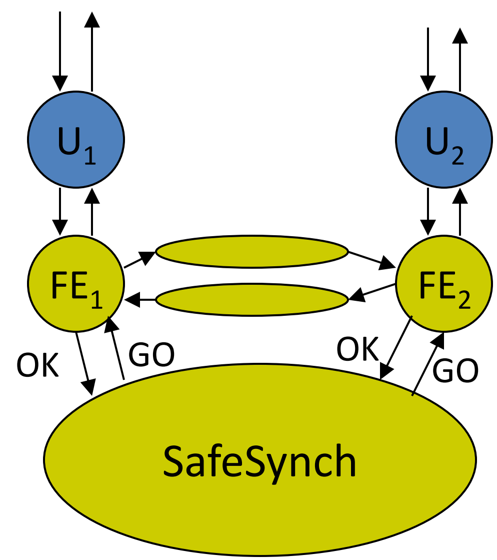


SimpleSynch, cont'd

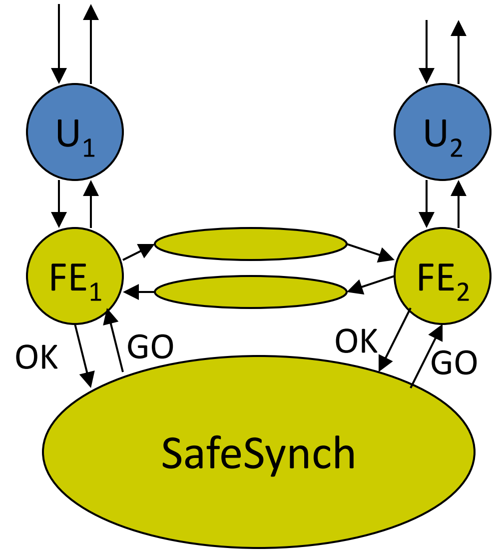
- Proof of Lemma 2:
  - No reordering needed, preserves order of external events.
  - Can use a simulation relation (for the safety part).
- Corollary: For every fair execution α of U_is and $SimpleSynch$, there is a fair execution α' of U_is and $GlobSynch$, such that for each i, $\alpha \sim_{U_i} \alpha'$.
- Proof: Combine Lemmas 1 and 2.
- Complexity:
  - Messages: $\leq 2\,|E|$ per simulated round.
  - Time:
    - Assume user always sends ASAP.
    - l, upper bound on time for each task of each process.
    - d, upper bound on time for first message in channel to be delivered
    - Then r rounds completed within time $r\ (d\ +\ O(l)\ )$.

Reducing the communication

- General Safe Synchronizer strategy [Awerbuch].
- If there's no message from U_i to U_j at round r of the underlying synchronous algorithm, try to avoid sending such messages in the simulating asynchronous algorithm.
- Can't just omit them, since each process must determine, for each round r, when it has received all of its round r messages.
- Key idea: Separate the functions of:
  - Sending the actual messages, and
  - Determining when the round is over.
- Algorithm decomposes into:
  - $\underbrace{\text{Front Ends + channels}}_{\text{For the actual messages}} + \underbrace{SafeSynch}_{\text{For deciding when round is finished}}$



Safe Synchronizers

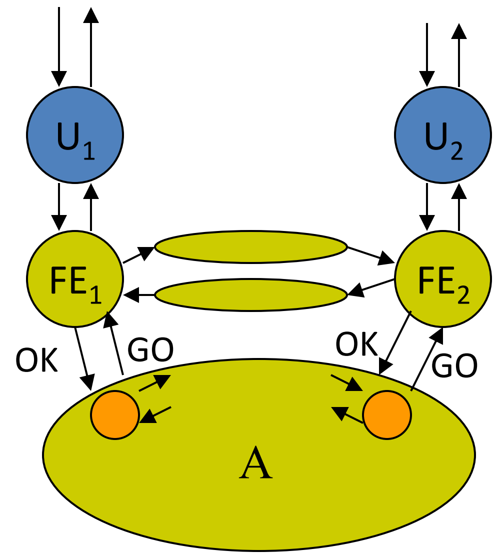


- Front End:
  - Sends, receives algorithm messages for each round r.
  - Sends acks for received messages.
  - Waits to receive acks for its own messages.
- Notes:
  - Sends only actual algorithm messages, no dummies.
  - acks double the messages, but can still be a win.
- Front End, cont'd:
  - When FE receives acks for all its round r messages, it's safe: it knows that all its messages have been received by its neighbors.
  - Then sends OK for round r to $SafeSynch$.
  - Before responding to user, FE must know that it has received all its neighbors' messages for round r.
  - Suffices to know that all its neighbors are safe, that is, that they know that their messages have been received.
- $SafeSynch$:
  - Tells each FE when its neighbors are safe.
  - After it has received OK from i and all its neighbors, sends GO to i.

Correctness of SafeSynch

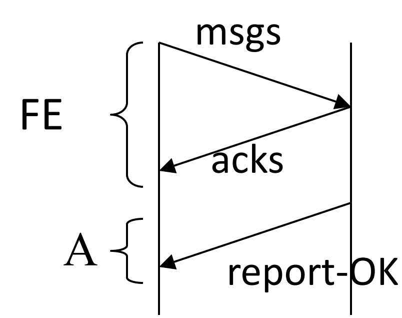
- Lemma 3: For every fair execution α of the $SafeSynch$ system, there is a fair execution α' of the $LocSynch$ system, such that for each i, $\alpha \sim U_i \, \alpha'$.
- (Actually, indistinguishable to all the U_is together.)
- Corollary: For every fair execution α of the $SafeSynch$ system, there is a fair execution α' of the $GlobSynch$ system, such that for each i, $\alpha \sim U_i \, \alpha'$.
- We must still implement $SafeSynch$ with a distributed algorithm…
- Three $SafeSynch$ implementations: Synchronizers A, B, and Γ [Awerbuch].
- All implement $SafeSynch$, in the sense that the resulting systems are indistinguishable to each U_i (in fact, to all the U_is together).

SafeSynch Implementations



- $SafeSynch$'s job: After receiving OK for round r at location i and all its neighbors, send GO for round r at location i.

- Synchronizer A:
  - When process i receives OK_i, sends to neighbors.
  - When process i hears that it and all its neighbors have received OKs, outputs GO_i.
- Obviously implements $SafeSynch$.
- Complexity: To emulate r rounds:
  - Messages: $\leq \underbrace{2m}_{\text{Messages and acks by FEs}} + \underbrace{2\, r\, |E|}_{\text{Messages within } \mathrm{A}}$, if synchronous algorithm sends m messages in r rounds.



  - Time: $\leq r\,(3d + O(l))$

Comparisons

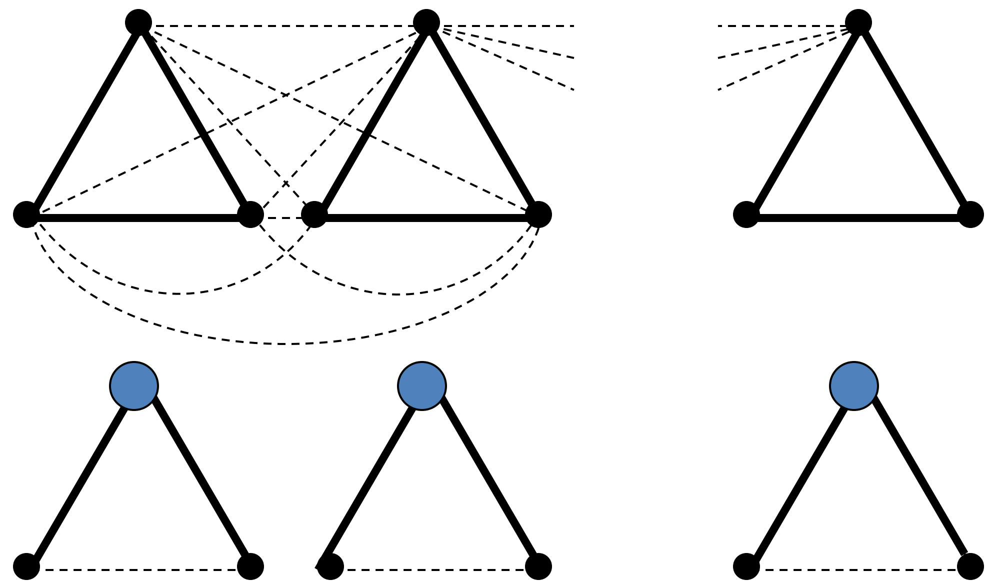
- To emulate r rounds:
  - $SafeSynch$ system with Synchronizer A
    - Messages: $2m + 2r|E|$
    - Time: $r(3d + O(l))$
  - $SimpleSynch$
    - Messages: $2r|E|$
    - Time: $r(d + O(l))$
- So Synchronizer A hasn't improved anything.
- Next, Synchronizer B, with lower message complexity, but higher time complexity.
- Then Synchronizer Γ, does well in terms of both messages and time, in an important subclass of networks (those with a "cluster" structure).

Synchronizer B

- Assumes rooted spanning tree of the graph, height h.
- Algorithm:
  - All processes convergecast OK to the root, using spanning tree edges.
  - Root then broadcasts permission to GO, again using the spanning tree.
- Obviously implements $SafeSynch$ (overkill).
- Complexity: To emulate r rounds, in which synchronous algorithm sends m messages:
  - Messages: $\underbrace{2\,m}_{\text{Messages and acks by } FEs} \;+\; \underbrace{2\,r\,n}_{\text{Messages within B}}$
  - Beats A: $2m + 2\,r\,|E|$
  - Time: $\leq r\,(\underbrace{2d + O(l)}_{\text{FEs}}) + 2h\,\underbrace{(d + O(l))}_{\text{B, convergecast and broadcast}})$

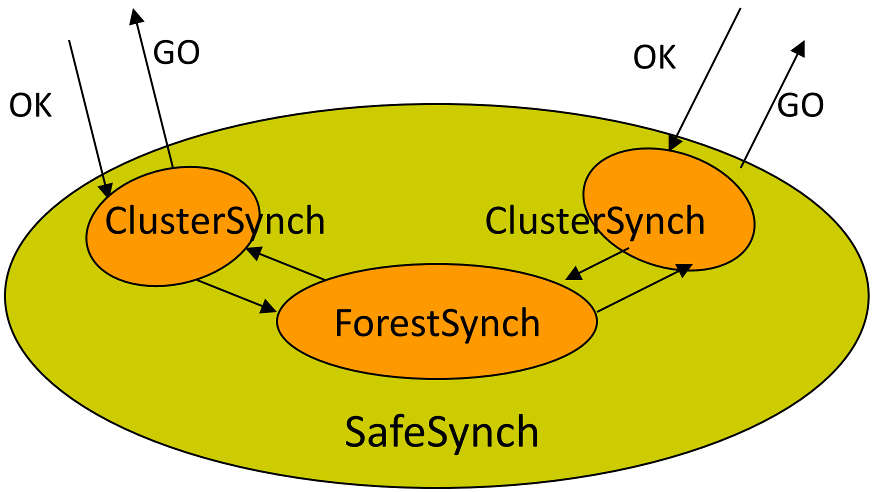
Synchronizer Γ

- Hybrid of A and B.
- In "clustered" (almost partitionable) graphs, can get performance advantages of both:
  - Time like A, communication like B.
- Assume spanning forest of rooted trees, each tree spanning a "cluster" of nodes.
- Example:
  - Clusters = triangles
  - All edges between adjacent triangles in the line.
  - Spanning forest:



- Use B within each cluster, A between clusters.

Decomposition of Γ



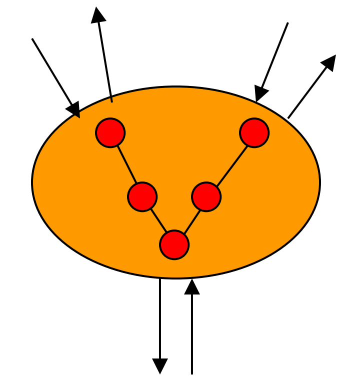
- $ClusterSynch$:
  - After receiving OKs from everyone in the cluster, sends $clusterOK$ to $ForestSynch$.
  - After receiving $clusterGO$ from $ForestSynch$, sends GO to everyone in the cluster.
  - Similar to B.
- $ForestSynch$:
  - Essentially, a safe synchronizer for the "Cluster Graph" G':
    - Nodes of G' are the clusters.
    - Edge between two clusters if and only if they contain nodes that are adjacent in G.
  - Send $clusterGO$ to a cluster after receiving $clusterOK$ from that cluster and all its neighboring clusters.
- Lemma: Automaton Γ Implements $SafeSynch$
- Proof idea:
  - Must show: If $GO(r)_i$ occurs, then there must be a previous $OK(r)_i$, and also a previous $OK(r)_j$ for every neighbor j of i.

Γ Implements SafeSynch

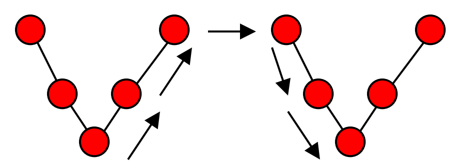
- Show: If $GO(r)_i$ occurs, then there must be a previous $OK(r)_{i,}$ and also previous $OK(r)_j$ for every neighbor j of i.
- Must be a previous $OK(r)_i$:
  - $GO(r)_i$ preceded by $clusterGO(r)$ for i's cluster ($ClusterSynch$),
  - Which is preceded by $clusterOK(r)$ for i's cluster ($ForestSynch$),
  - Which is preceded by $OK(r)_i$ ($ClusterSynch$).
- Must be a previous $OK(r)_j$ for any neighbor j in the same cluster as i.
  - $GO(r)_i$ preceded by $clusterGO(r)$ for i's cluster ($ClusterSynch$),
  - Which is preceded by $clusterOK(r)$ for i's cluster ($ForestSynch$),
  - Which is preceded by $OK(r)_j$ ($ClusterSynch$).
- Must be a previous $OK(r)_j$ for any neighbor j in a different cluster.
  - Then the two clusters are neighboring clusters in the cluster graph G', because i and j are neighbors in G.
  - $GO(r)_i$ preceded by $clusterGO(r)$ for i's cluster ($ClusterSynch$),
  - Which is preceded by $clusterOK(r)$ for j's cluster ($ForestSynch$),
  - Which is preceded by $OK(r)_j$ ($ClusterSynch$).

Implementing $ClusterSynch, ForestSynch$

- $ClusterSynch$:
  - Use variant of Synchronizer B on cluster tree:
    - Convergecast OKs to root on the cluster tree,
    - root outputs $clusterOK$, receives $clusterGO$,
    - root broadcasts GO on the cluster tree.



- $ForestSynch$:
  - Clusters run Synchronizer A.
    - But clusters can't actually run anything…
    - So cluster roots run A.
    - Simulate communication channels between neighboring clusters by indirect communication paths between the roots.
    - These paths must exist: Run through the trees and across edges that join the clusters.



- $clusterOK$ and $clusterGO$ are internal actions of the cluster root processes.

Putting the pieces together

- In Γ, process i emulates $FrontEnd_i$, process i in $ClusterSynch$ algorithm, and process i in $ForestSynch$ algorithm.
  - Formally, it's the composition of three I/O automata.
- Real channel $C_{i\ j}$ emulates channel from $FrontEnd_i$ to $FrontEnd_j$, channel from i to j in the $ClusterSynch$ algorithm, and channel from i to j in the $ForestSynch$ algorithm.
- Orthogonal decompositions of Γ:
  - Physical: Nodes and channels.
  - Logical: FEs, $ClusterSynch$, and $ForestSynch$
  - Same system, two views.
  - Works because composition of I/O automata is associative, commutative.
- Such decompositions are common for complex distributed algorithms:
  - Each node runs pieces of algorithms at several layers.

- Theorem 1: For every fair execution α of the Γ (or A, or B) system, there is a fair execution α' of the $GlobSynch$ system, such that for each i, $\alpha \sim U_i\ \alpha'$.

Complexity of Γ

- Consider r rounds, in which the synchronous algorithm sends m messages.
- Let:
  - h = max height of a cluster tree
  - e' = total number of edges on shortest paths between roots of neighboring clusters.
- Messages: $2m + O(r\,(n + e'))$

Messages between roots,
In $ForestSynch$ algorithm

Messages and acks by FEs

Messages in cluster trees,
In $ClusterSynch$ algorithm

- Time: $O\,(r\,h\,(d + l))$
- If $n + e' << |E|$, then Γ's message complexity is much better than A's.
- If h << height of spanning tree of entire network, then Γ's time complexity is much better than B's.
- Both of these are true for "nicely clustered" networks.

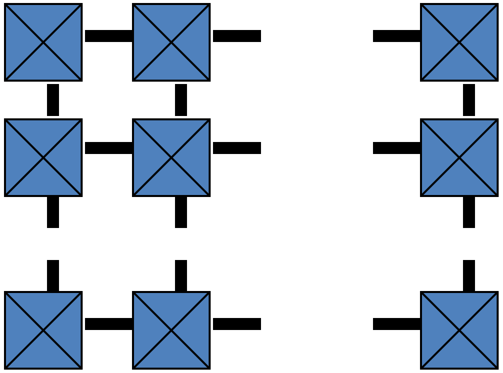
Comparison of Costs

- r rounds
- m messages sent by synchronous algorithm
- d, message delay
- Ignore local processing time l.
- e' = total length of paths between roots of neighboring clusters
- h = height of global spanning tree
- h' = max height of cluster tree

	Messages	Time
Α	2 m + 2 r |E|	O(r d)
Β	2 m + 2 r n	O(r h d)
Γ	2 m + O(r (n + e′))	O(r h′ d)

Example

- $p \times p$ grid of complete k-node graphs, with all nodes of neighboring k-node graphs connected.
- Clusters = k-node graphs
- $h = O(p)$
- $h' = O(1)$
- $e' = O(p^2)$



	Messages	Time
A	$2m + O(r\,p^2 k^2)$	$O(r\,d)$
B	$2m + O(r\,p^2 k)$	$O(r\,p\,d)$
Γ	$2m + O(r\,p^2 k)$	$O(r\,d)$

Synchronizer Applications

Application 1: Breadth-First Search

- Recall:
  - $SynchBFS$:
    - Constructs BFS tree.
    - $O(\ |E|\ )$ messages, $O(\ diam\ )$ rounds
  - When run in asynchronous network:
    - Constructs a spanning tree, but not necessarily a BFS tree.
  - Modified version, with corrections:
    - Constructs BFS tree.
    - $O(\ n\ |E|\ )$ messages, $O(\ diam\ n\ d\ )$ time (counting pileups)
- BFS using Synchronizers:
  - Runs more like $SynchBFS$, avoids corrections, pileups
  - With Synchronizer A:
    - $O(\ diam\ |E|\ )$ messages, $O(\ diam\ d\ )$ time
  - With Synchronizer B :
    - Better communication, but worse time.
  - With Synchronizer Γ :
    - Better overall, in clustered graphs.

Application 2: Broadcast/Ack

- Assume known leader, but no spanning tree.
- Recall:
  - Synchronous Bcast/Ack:
    - Constructs spanning tree while broadcasting
    - $O(\ |E|\ )$ messages, $O(\ diam\ )$ rounds
  - Asynchronous Bcast/Ack:
    - Timing anomaly: Construct non-minimum-hop paths, on which acks travel.
    - $O(\ |E|\ )$ messages, $O(\ n\ d\ )$ time
- Bcast/Ack using Synchronizers:
  - Using (e.g.) Synchronizer A:
    - Avoids timing anomaly.
    - Bcast travels on min-hop paths, so Acks follow min-hop paths.
    - $O(\ diam\ |E|\ )$ messages, $O(\ diam\ d\ )$ time

Application 3: Shortest paths

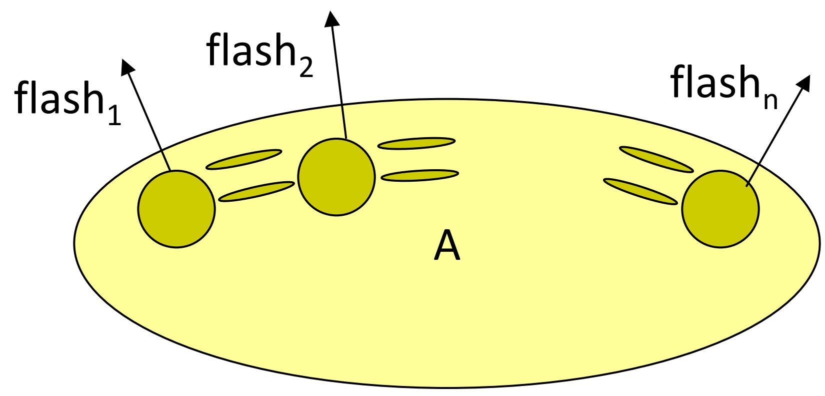
- Assume weights on edges.
- Without termination detection.
- Recall:
  - Synchronous Bellman-Ford:
    - Makes some corrections, due to low-cost high-hop-count paths.
    - $O(n\,|E|)$ messages, $O(n)$ rounds
  - Asynchronous Bellman-Ford
    - Many more corrections possible (exponential), due to message delays.
    - (Worst-case) message complexity is exponential in n.
    - Time complexity also exponential in n, counting message pileups.
- Using (e.g.) Synchronizer A:
    - Behaves like Synchronous Bellman-Ford.
    - Avoids corrections due to message delays.
    - Still has corrections due to low-cost high-hop-count paths.
    - $O(n\,|E|)$ messages, $O(n\,d)$ time
    - Big improvement.

Further reading

- To read more:
  - See Awerbuch's extensive work on
    - Applications of synchronizers.
    - Distributed algorithms for clustered networks.
  - Also work by Peleg.
  - [Awerbuch, Peleg] Dijkstra Prize paper on clustered networks.

- Q: This work used a strategy of purposely slowing down portions of a system in order to improve overall performance. In which situations is this strategy a win?

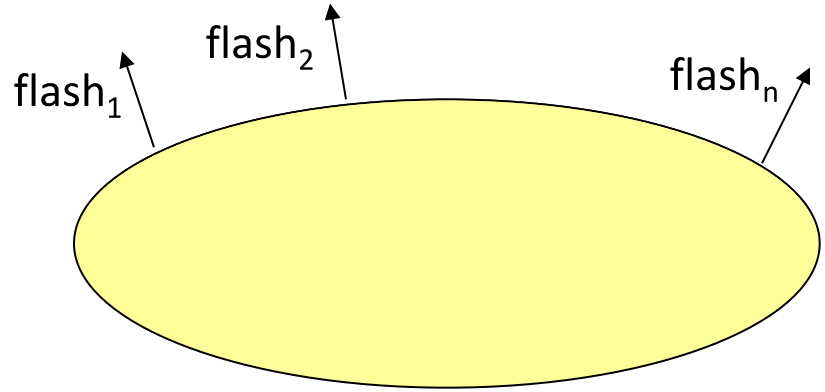
Lower Bound on Time for Synchronization



Lower bound on time for synchronization

- A, B, Γ emulate synchronous algorithms only in a local sense:
  - Looks the same to individual users,
  - Not to the combination of all users---can reorder events at different users.
- Good enough for many applications (e.g., data management).
- Not for others (e.g., embedded systems).
- Now a theoretical result showing that global synchronization is inherently more costly than local synchronization, in terms of time complexity.
- Approach:
  - Define a toy global synchronization problem, the k-Session Problem.
  - Show that this problem has a fast synchronous algorithm, and thus, a fast algorithm using $GlobSynch$.
    - Time $O(\ k\ d\ )$, assuming $GlobSynch$ takes steps ASAP.
  - Prove that all asynchronous distributed algorithms for this problem are slow.
    - Time $\Omega(\ k\ diam\ d\ )$.
  - Implies $GlobSynch$ has no fast distributed implementation.
- Contrast:
  - Synchronizer A, $SimpleSynch$ are fast distributed impls of $LocSynch$.

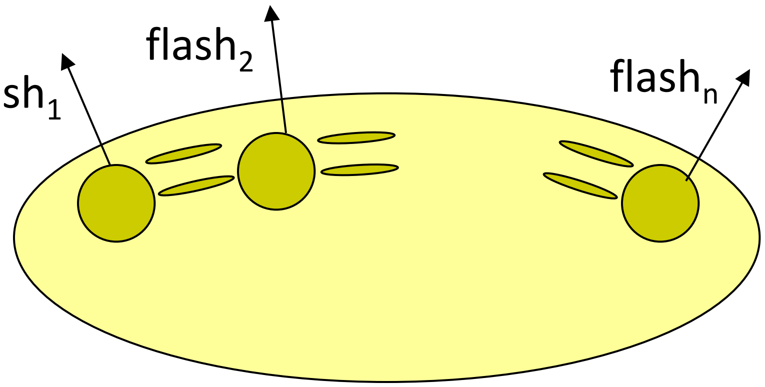
k-Session Problem



- Session:
  - Any sequence of $flash$ events containing at least one $flash_i$ event for each location i.
- k-Session problem:
  - In every fair execution, perform at least k separate sessions, and eventually halt.
- Original motivation:
  - Synchronization of this kind is useful for performing parallel matrix computations that require enough interleaving of process steps, but tolerate extra steps.

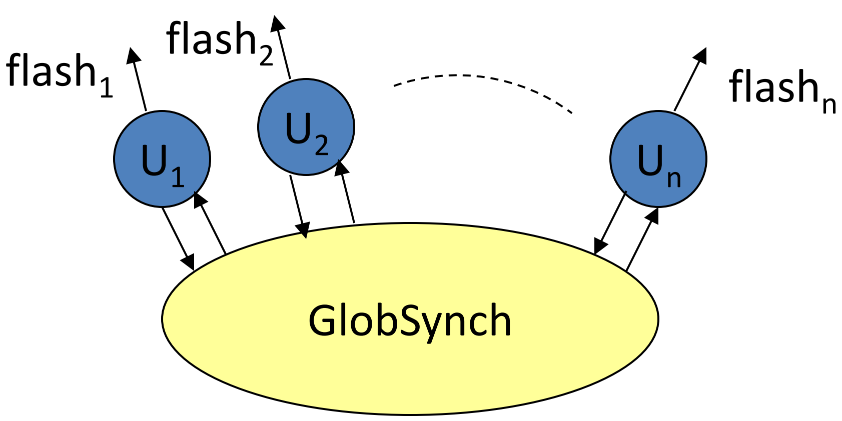
Application: Boolean matrix computation

- $n = m^3$ processes cooperate to compute the transitive closure of an $m \times m$ Boolean matrix M.
- $p_{i,j,k}$ repeatedly does:
  - read $M(i,k)$, read $M(k,j)$
  - If both are 1 then write 1 in $M(i,j)$
- Each $flash_{i,j,k}$ in the abstract session problem represents a chance for $p_{i,j,k}$ to read or write a matrix entry.
- With enough interleaving ($O(\log n)$ sessions), this is guaranteed to compute the transitive closure.



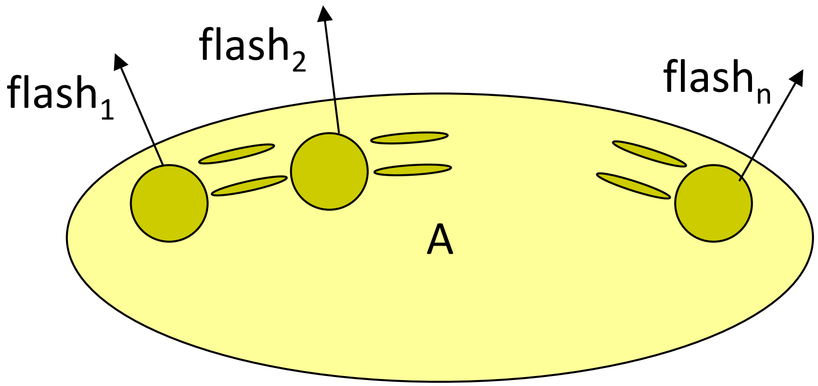
Synchronous solution

- Fast algorithm using $GlobSynch$:
  - Just flash once at every round.
  - k sessions done in time $O(\ k\ d\ )$, assuming $GlobSynch$ takes steps ASAP.



Asynchronous lower bound

- Consider a distributed algorithm A that solves the k-session problem.
- Consists of process automata and FIFO send/receive channel automata.



- Assume:
  - d = upper bound on time to deliver any message (don't count pileups)
  - l = local processing time, $l \ll d$
- Define time measure $T(A)$:
  - Timed execution α: Fair execution with times labeling events, subject to upper bound of d on message delay, l for local processing.
  - $T(\alpha)$ = time of last flash in α.
  - $T(A)$ = supremum, over all timed executions α, of $T(\alpha)$.

Lower bound

- Theorem 2: If A solves the k-session problem then
  $$T(A) \geq (k-1)\, diam\, d.$$
- Factor of $diam$ worse than the synchronous algorithm.
- Definition: Slow timed execution: All message deliveries take exactly the upper bound time d.
- Proof: By contradiction.
  - Suppose $T(A) < (k-1)\, diam\, d$.
  - Consider α, any slow timed execution of A.
  - α contains at least k sessions.
  - α contains no flash event at a time $\geq (k-1)\, diam\, d$.
  - So we can decompose $\alpha = \underbrace{\alpha_1\, \alpha_2\, \ldots \alpha_{k\text{-}1}}_{\alpha'}\, \alpha''$, where:
    - Time of last event in α' is $< (k-1)\, diam\, d$.
    - No flash events occur in α''.
    - The difference between the times of the first and last events in each α_r is strictly less than $diam\, d$.

Lower bound, cont'd

- Now reorder events in α, while preserving dependencies:
  - Events of same process.
  - Send and corresponding receive.
- Reordered execution will have strictly fewer than k sessions, which will yield a contradiction.
- Fix processes, j_0 and j_1, with $dist(j_0, j_1) = diam$ (maximum distance apart).
- Reorder within each α_r separately:
  - For α_1: Reorder to $\beta_1 = \gamma_1 \delta_1$, where:
    - γ_1 contains no event of j_0, and
    - δ_1 contains no event of j_1.
  - For α_2: Reorder to $\beta_2 = \gamma_2 \delta_2$, where:
    - γ_1 contains no event of j_1, and
    - δ_1 contains no event of j_0.
  - And alternate thereafter.

Lower bound, cont'd

- If the reordering yields a fair execution of A (can ignore timing), then we get a contradiction, because it contains at most $k-1$ sessions:
  - No session entirely within γ_1, (no event of j_0).
  - No session entirely within $\delta_1 \gamma_2$ (no event of j_1).
  - No session entirely within $\delta_2 \gamma_3$ (no event of j_0).
  - …
  - Thus, every session must span some γ_r - δ_r boundary.
  - But, there are only $k-1$ such boundaries.

- It remains only to construct the reordering…

Constructing the reordering

- For example, consider α_r for r odd (analogous construction for r even).
- Need $\beta_r = \gamma_r\ \delta_r$, where γ_r contains no event of j_0, δ_r no event of j_1.

- If α_r contains no event of j_0 then don't reorder, define $\gamma_r = \alpha_r$, $\delta_r = \lambda$.
- If α_r contains no event of j_1 then don't reorder, define $\gamma_r = \lambda$, $\delta_r = \alpha_r$.
- Now assume α_r contains at least one event of each.
- Let π be the first event of j_0, φ the last event of j_1 in α_r.

- Claim: φ does not depend on π.
- Why: There is insufficient time for messages to travel from j_0 to j_1:
  - Execution α is slow (message deliveries take time d).
  - Time between π and φ is $< diam\ d$.
  - j_0 and j_1 are $diam$ hops apart.

- Then, we can reorder α_r to β_r, in which π comes after φ.
- Consequently, in β_r, all events of j_1 precede all events of j_0.
- Define γ_r to be the part ending with φ, δ_r the rest.

Next time…

- Time, clocks, and the ordering of events in a distributed system.
- State-machine emulation.
- Vector timestamps.
- Reading:
  - Chapter 18
  - [Lamport] Time, Clocks…paper
  - [Mattern] paper