#1 Course introduction, inner products and norms

Friday, 10 September 2021 8:47 am

Hour 1: Course intro, inner products, norms

$\mathbb{R}^1 \to \mathbb{R}^n$, linear algebra, continuity, differentiability, integration

$\Rightarrow$ all leads to "the Stokes' theorem"

$$\int_c dW = \int_{\partial c} W \leftarrow \text{differential form}$$

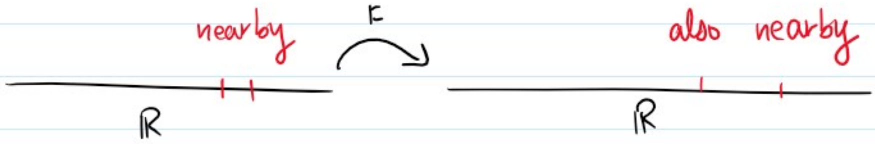
This is a generalization of the fundamental theorem of calculus.

$$\int_{[a,b]} F'(t)\,dt = F(b) - F(a) = \int_{\partial[a,b]} F$$

∂ = boundary

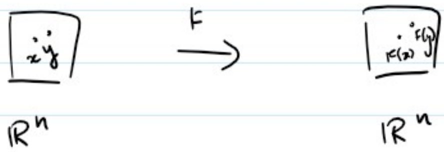
$$\partial[a,b] = \{b_+, a_-\}$$

Continuity in $\mathbb{R}$:



$\rightarrow$ generalize to $\mathbb{R}^n$

- How to measure distance



Def^n: For $x, y \in \mathbb{R}^n$, the "standard/Euclidean" inner product of x & y $\langle x, y\rangle$ as

$$\langle x, y\rangle = \sum_{i=1}^{n} x_i y_i$$

What is $\mathbb{R}^n$? dual vectors

$\rightarrow \left\{\begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}\right\}$ or $\rightarrow \{(x_1 \cdots x_n)\}$

$x_i \in \mathbb{R}$ $\qquad$ $x_i \in \mathbb{R}$

we define the norm squared of x $|x|^2$ as

$$|x|^2 = \langle x, x\rangle$$

and the norm of x $|x|$ as

$$|x| = \sqrt{|x|^2} = \sqrt{\sum x_i^2}$$

Proposition: If $x, y, z \in \mathbb{R}^n$ & $a, b \in \mathbb{R}$

bilinear $\rightarrow$ linear in each component

0. $\langle , \rangle$ is bilinear and $|\cdot|$ is semi-linear

?

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e.g. $\langle ax+by, z\rangle = a\langle x,z\rangle + b\langle y,z\rangle$

$\langle z, ax+by\rangle = a\langle z,x\rangle + b\langle z,y\rangle$ semi-linear

1. $|x| \geq 0$ & $|x| = 0$ iff $x = 0$ (positivity) $\rightarrow$ $|ax| = |a|\cdot|x|$

1.5. $\langle x,y\rangle = \langle y,x\rangle$ (symmetric)

2. $|\langle x,y\rangle| \leq |x||y|$, w/ equality iff x,y are lin. dep.

"Cauchy-Schwarz inequality"

3. "Triangle inequality": $|x+y| \leq |x| + |y|$

4. "Polarization identity": $\langle x,y\rangle = \dfrac{|x+y|^2 - |x-y|^2}{4}$

Proofs:

1. $|x| = \sqrt{\sum x_i^2} \geq 0$

$|x| = 0 \Rightarrow \sum x_i^2 = 0 \Rightarrow \forall i \; x_i^2 = 0 \Rightarrow x = 0$

2. $0 \leq \big||y|^2 x - \langle x,y\rangle y\big|^2 = |y|^4|x|^2 + \langle x,y\rangle^2|y|^2 - 2|y|^2\langle x,y\rangle^2 = |y|^2\left(|y|^2|x|^2 - \langle x,y\rangle^2\right)$

Aside: equality $\overset{?}{\Rightarrow}$ x,y dependent

$|s+t|^2 = \langle s+t, s+t\rangle$

$= \langle s,s\rangle + \langle s,t\rangle + \langle t,s\rangle + \langle t,t\rangle$

$= |s|^2 + 2\langle s,t\rangle + |t|^2$