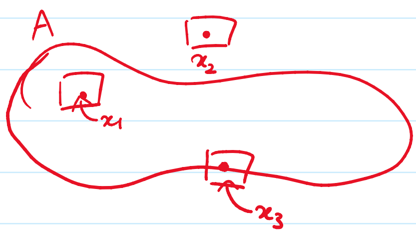
Hour 5: Interior, exterior, closure, boundary, compactness, Heine-Borel

Given $A \subset \mathbb{R}^n$ & $x \in \mathbb{R}^n$, trichotomy: (exactly one of the following is true).

1) $\exists$ open rectangle R s.t. $x \in R \subset A$ } $x \in$ interior of $A = \operatorname{int} A = \mathring{A}$
2) $\exists$ open rectangle R s.t. $x \in R \subset A^c$ } $x \in$ exterior of $A = \operatorname{ext} A$
3) Every open rectangle s.t. $x \in R$, has $R \cap A^c = \emptyset$ } $x \in$ boundary of $A = \operatorname{Bd} A$
   & $R \cap A \neq \emptyset$

$\ast$ complement of exterior = union of ① & ③ aka closure of A
$\operatorname{cl} A = (\operatorname{ext} A)^c$

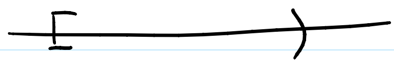


<u>Claim</u>: $\operatorname{cl}(A) = \bar{A}$ has the following properties:
$x \in \bar{A}$ iff:
→ every open rect. R containing $x \in R$ satisfies $R \cap A \neq \emptyset$

<u>Claim</u>: ① $\operatorname{int} A \cup \operatorname{ext} A \cup \operatorname{Bd} A = \mathbb{R}^n$ (in fact, it is a disjoint union)
→ represented by $\dot{\cup}$

② $\operatorname{cl} A = A \cup \operatorname{Bd} A$
③ $\operatorname{int} A = A \setminus \operatorname{Bd} A$

<u>Example</u>: $[0,1) \subset \mathbb{R}$



$\operatorname{int} A = (0,1)$, $\operatorname{ext} A = (-\infty, 0) \cup (1, \infty)$, $\operatorname{Bd} A = \{0,1\}$
$\operatorname{cl} A = [0,1]$

<u>Compactness</u>:
<u>Def</u>^{n}: An open cover of a set $A \subset \mathbb{R}^n$ is a collection $\{U_\alpha\}$ of open sets s.t. $\bigcup_\alpha U_\alpha \supset A$. A subcover of $\{U_\alpha\}_{\alpha \in I}$ is a collection $\{U_\alpha\}_{\alpha \in I'}$ where $I' \subset I$
$\bigcup_{\alpha \in I'} U_\alpha \supset A$

<u>Def</u>^{n}: A is called compact if every open cover of A has a finite sub cover.

<u>Examples</u>:

Examples:

① If $F \subset \mathbb{R}^n$ is finite, then it is compact.

② ~~$\mathbb{R}$ is compact~~

~~$\Rightarrow \mathbb{R} = (-\infty, 1) \cup (-1, \infty)$~~ wrong! EVERY open cover...

③ ~~$\mathbb{R}$ is compact~~

~~$\{(n,\infty)\}_{n\in\mathbb{Z}} \cup \{(-\infty,-n)\}_{n\in\mathbb{Z}}$~~ wrong again! EVERY

④ $\mathbb{R}$ is not compact

ex. $\bigcup_{n\in\mathbb{Z}} (n-1, n+1)$

ex. $\bigcup_{n\in\mathbb{Z}} (-n, n)$

Generally, , ,  are compact

Thm: Heine-Borel

$[0,1]$ is compact

How do we know such a g exists at all? $\Rightarrow 0 \in G$

Proof: Let $\{U_\alpha\}_{\alpha\in J}$ be an open cover of $I = [0,1]$

Define $\underset{\text{Good}}{G} = \{ g \in [0,1] : \exists\, J' \subset J$ finite subcover $\bigcup_{\alpha\in J'} U_\alpha \supset [0,g]\}$

Set $\gamma = \sup G$. (to have a sup a set has to be bounded and nonempty, G bounded $\subset [0,1]$ and $0 \in G \Rightarrow$ non empty).

We claim that $\gamma = 1$. Suppose not and $\gamma < 1$.

As $\gamma \in I$, $\exists \beta \in J$ such that $\gamma \in U_\beta$ and as U_β is open, we can find g' & g'' s.t. $\gamma \in (g', g'') \subset [g', g''] \subset U_\beta$

$$[0, g''] = \underbrace{[0, g']}_{\text{has a finite subcover}} \cup \underbrace{[g', g'']}_{\text{covered by } U_\beta} \quad \text{has a finite subcover}$$

$\Rightarrow g'' \in G \Rightarrow \sup G = g'' > \gamma$ CONTRADICTION

$\Rightarrow$ SUP IS ONE!