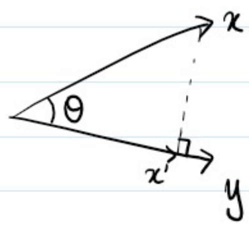
#2 More on inner products and norms

Monday, 13 September 2021 9:30 am

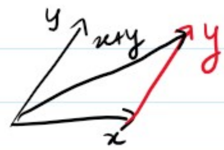
Hour 2: More on inner products and norms



$$\langle x,y\rangle = |x||y|\cos\theta$$
$$= |x'||y|$$

<u>Proofs (from last lecture):</u>

3. Triangle inequality: $|x+y| \le |x| + |y|$



As both sides ≥ 0, we can instead prove

$$|x+y|^2 \le (|x|+|y|)^2$$
$$\langle x+y, x+y\rangle \le |x|^2 + 2|x||y| + |y|^2$$
$$2\langle x,y\rangle \le 2|x||y|$$
$$\langle x,y\rangle \le |x||y|$$

We know $|\langle x,y\rangle| \le |x||y|$, Cauchy-Schwarz, hence triangle inequality follows.

4. Polarization identity: $\langle x,y\rangle = \dfrac{|x+y|^2 - |x-y|^2}{4}$.

Trivial.

<u>Def^n</u>: If $x, y \in \mathbb{R}^n$, $d(x,y)$ = "distance between x and y" $= |x-y|$

→ becomes a definition later on

<u>Thm</u>:

① d is "symmetric" (ie $d(x,y) = d(y,x)$)

② d is "positive definite" (ie $d(x,y) \ge 0$, w/ equality iff $x=y$)

③ Triangle inequality: $d(x,z) \le d(x,y) + d(y,z)$

<u>Proof</u>:

① $d(x,y) = |x-y| = |-(y-x)| = |-1| \cdot |y-x| = |y-x| = d(y,x)$

② Norm always ≥ 0.

$$d(x,y) = 0 \iff |x-y| = 0 \iff x-y=0$$
$$\iff x=y$$

③ $|x-z| \le |x-y| + |y-z|$

$|x-y| + |y-z| \ge |x-y+y-z|$ (we know $|p| + |q| \ge |p+q|$)

$= |x-z|$

$|x-y| + |y-z| \geq |x-y+y-z|$ (we know $|p| + |q| \geq |p+q|$)
$= |x-z|$

$|x| = \sqrt{\sum x_i^2}$ commonly known as $|x|_{L^2}$ or Euclidean norm

Similarly,

$|x|_{L^1} = \sum |x_i|$ $\qquad d_1(x,y) = |x-y|_1$

$|x|_\infty = \max |x_i|$ $\qquad d_\infty(x,y) = |x-y|_\infty$

<u>Exercise</u>: d_1 & d_∞ have the exact same properties.