Group Theory Notes

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Contents

1 List of groups with definitions and representations

1.1 $SO(3) : \{R \in Aut(\mathbb{V}), R^T R = 1, det(R) = 1\}$

- "Special Orthogonal Group" ("3D rotation group")
- $SO(3) \subset Aut(\mathbb{R}^3)$
- Fundamental/defining/vector representation is the identity map
- Lie algebra $\mathfrak{so}(3) = \{r \in End(\mathbb{V}); r^T = -r\}$
  - an (imaginary) basis: $J_k := i\vec{e}_k^{\times} \in i\mathfrak{so}(3) \subset \mathfrak{so}(3, \mathbb{C})$
  - defining representation $\rho_{\mathfrak{so}(3)}(J_k) = i\vec{e}_k^{\times} \in i\mathfrak{so}(3) \subset iEnd(\mathbb{R}^3)$
  - Lie bracket expansion: $[\![J_i, J_j]\!] = i\epsilon_{ijk}J_k$
  - N-dimensional irrep labelled by non-negative 'spin' $j = \frac{1}{2}(N-1)$
    - representation space spanned by

$$|m\rangle\,, \qquad m \in \{-j, -j+1, \cdots, j-1, j\}$$

    - action of generators $J_z, J_\pm = J_x \pm iJ_y$

$$\rho(J_z)\,|m\rangle = m\,|m\rangle$$
$$\rho(J_\pm)\,|m\rangle = c_m^\pm\,|m \pm 1\rangle$$

      where

$$c_m^\pm = \sqrt{(j \mp m)(j \pm m + 1)}$$

    - unitarity

$$\rho(J_z)^\dagger = \rho(J_z), \qquad \rho(J_+)^\dagger = \rho(J_-)$$

    - application: spherical harmonics (TODO)
    - character polynomial

$$P_j(q) = \sum_{k=0}^{2j} q^{2k-2j} = \frac{q^{2j+1} - q^{-2q-1}}{q - q^{-1}}$$

    - the tensor product of two representations with spin j and j' yields

$$P_{j\otimes j'}(q) = P_j(q)P_{j'}(q) = \frac{q^{2j+1} - q^{-2j-1}}{q - q^{-1}}\frac{q^{2j'+1} - q^{-2j'-1}}{q - q^{-1}}$$

$$= \sum_{k=0}^{2j'} P_{j+j'-k}(q)$$

w in the last step, assume $j' \leq j$. If not, $P_{-k} = -P_{k-1}$

– Casimir invariant $C = J_k \otimes J_k = J_z \otimes J_z + \frac{1}{2} J_+ \otimes J_- + \frac{1}{2} J_- \otimes J_+$ with

$$\rho(C) |m\rangle = j(j+1) |m\rangle$$

1.2 $SU(2)$

- double cover of $SO(3)$
- Lie algebra $\mathfrak{su}(2) = \{m \in End(\mathbb{C}^2); m = -m^\dagger, tr(m) = 0\}$
  - an (imaginary) basis: $J_k = \frac{1}{2}\sigma_k \in i\mathfrak{su}(2)$ where σ_k Pauli matrices
  - defining representation $\rho_{\mathfrak{su}(2)}(J_k) = \frac{1}{2}\sigma_k \in i\mathfrak{su}(2) \subset End(\mathbb{C}^2)$
  - Lie bracket expansion: $[\![J_i, J_j]\!] = i\epsilon_{ijk} J_k$

1.3 $Sp(1)$

- Lie algebra $\mathfrak{sp}(1)$
  - defining representation $\rho_{\mathfrak{sp}(1)}(-iJ_x) = \frac{1}{2}\hat{\mathbf{i}}$ and similarly $J_y \to \hat{\mathbf{j}}$ and $J_z \to \hat{\mathbf{k}}$
    - symplectic

$$\rho_{\mathfrak{sp}(1)}(a)^\dagger = -\rho_{\mathfrak{sp}(1)}(a)$$

1.4 S_3

- $|S_3| = 3! = 6$ elements
- all elements can be written in terms of two elementary permutations σ_1, σ_2 which satisfy $(\sigma_1)^2 = (\sigma_2)^2 = (\sigma_1\sigma_2)^2 = 1$
- can be viewed as symmetry group of an equilateral triangle
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1.5 Other groups

1. $SO(2)$: orthogonal matrices with unit determinant
2. (Reals, +)
3. $O(2)$
   - dihedral group is a subgroup
4. (Reals/0, x)
5. S_n, $n > 1$
   - "symmetric group"
   - all $n!$ permutations of a set of n elements
6. U(1): set of unitary 1x1 matrices (unitary: conjugate transpose = inverse)
7. $G^(1) = R_\phi^(1) : \phi \in R/2piZ$
8. $Aut(\mathbb{R}^3)$
9. Spin(3)
10. $O(3) = \{R \in Aut(\mathbb{R}^3); R^T R = 1\}$
    - "group of reflections"
    - Lie algebra $\mathfrak{so}(3) = \{r \in End(\mathbb{V}); r^T = -r\}$
11. $C_n = \mathbb{Z}_N = \mathbb{Z}/n\mathbb{Z}$, $n > 1$
    - "cyclic group"
    - integers modulo n under addition
    - abelian
12. $SU(2)_L \times SU(2)_R$
    - double cover of $SO(4)$
13. $\{e\}$
    - "trivial group"
14. A_n, $n > 1$
    - "alternating group"
    - all $n!/2$ even permutations of a set of n elements

2 Definitions

- Lie group: a group whose set G is a differentiable manifold and whose composition rule and inversion are smooth maps on this manifold
- universal cover of a group: a bigger group which contains the original group and is simply connected
- abelian: ab = ba
- automorphism: invertible linear transformation from $\mathbb{V}$ to itself
- endomorphism: linear map from $\mathbb{V}$ to itself
- composition rule: group operation (eg addition for group of reals equipped with +)
- For any Lie group there is a corresponding Lie algebra with a Lie bracket
  - tangent space of a group at the identity
  - not associative
  - when the Lie algebra is given in terms of matrices, i.e. $\mathfrak{g} \subset End(\mathbb{V})$, the matrix commutator and the Lie bracket coincide.
- quaternions $\mathbb{H}$: number field spanned by $1, \hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}$ with
  - products given by $\hat{\mathbf{i}}\hat{\mathbf{j}} = -\hat{\mathbf{j}}\hat{\mathbf{i}} = \hat{\mathbf{k}}$ and cyclic permutations ($\implies$ non commutativity)
  - closely related to Pauli matrices

$$(1, \hat{\mathbf{i}}, \hat{\mathbf{j}}, \hat{\mathbf{k}}) \equiv (1, -i\sigma_x, -i\sigma_y, -i\sigma_z)$$

- Casimir operator $C \in \mathfrak{g} \otimes \mathfrak{g}$ with $[\![a, C]\!] = 0, \forall a \in \mathfrak{g}$ where

$$[\![a, b \otimes c]\!] := [\![a, b]\!] \otimes c + b \otimes [\![a, c]\!]$$

  - any representation $\rho : \mathfrak{g} \to End(\mathbb{V})$ can be lifted to $\rho : \mathfrak{g} \otimes \mathfrak{g} \to End(\mathbb{V})$ with

$$\rho(a \otimes b) := \rho(a)\rho(b)$$

- tensor product of Lie algebra representations $\rho_\otimes : \mathfrak{g} \to End(\mathbb{V}_\otimes)$ defined as

$$\rho_\otimes := \sum_{k=1}^{N} 1 \otimes \cdots \otimes 1 \otimes \rho_k \otimes 1 \otimes \cdots \otimes 1$$

  where $\rho_k : \mathfrak{g} \to End(\mathbb{V}_k), k = 1, \cdots, N$

- direct sum of Lie algebra representations $\rho_\oplus : \mathfrak{g} \to End(\mathbb{V}_\oplus)$ defined as

$$\rho_\oplus := \rho_1 \oplus \cdots \oplus \rho_N$$

- character polynomials: define a group element $g(q)$ depending on a formal variable q

$$g(q) = \exp(2\log(q) J_z) = q^{2J_z}$$

  define the character as

$$P_\rho(q) = \mathrm{tr}\,\rho(g(q)) = \sum_k n_k q^{2m_k}$$

  where m_k eigenvalues of J_z and n_k corresponding multiplicities

  - $q = 1$ gives dimension
  - have the property

$$P_{\rho\otimes\rho'}(q) = P_\rho(q) P_{\rho'}(q)$$
$$P_{\rho\oplus\rho'}(q) = P_\rho(q) + P_{\rho'}(q)$$

  -

3 Isomorphisms

- $\mathfrak{so}(3) \equiv \mathfrak{su}(2) \equiv \mathfrak{sp}(1)$
- $Spin(3) \equiv SU(2) \equiv Sp(1) \equiv S^3 \equiv D_3$
- $SO(3) \equiv SU(2)/\mathbb{Z}_2$
- icosahedral $\equiv A_5$
- $T \equiv A_4$
- $T_d \equiv O \equiv S_4$

4 Notation

- $\mathfrak{gothic}$ script represents Lie algebras
- $\vec{v}^\times$ denotes the 3×3 anti-symmetric matrix that defines the cross product of $\vec{v}$ with an arbitrary vector $\vec{w}$ via matrix multiplication: $\vec{v}^\times \vec{w} = \vec{v} \times \vec{w}$

$$\vec{v}^\times := \begin{bmatrix} 0 & -v_z & +v_y \\ +v_z & 0 & -v_x \\ -v_y & +v_x & 0 \end{bmatrix}$$

5 Maps

- adjoint action: $Ad(R) : \mathfrak{g} \to \mathfrak{g}$ such that $Ad(R)(a_1) := Ra_1R^{-1} =: a_2$ with $a_1, a_2 \in \mathfrak{g}$
- adjoint representation: $Ad : G \to Aut(\mathfrak{g})$
- adjoint action $ad : \mathfrak{g} \to End(\mathfrak{g})$ is a linear map such that
  - $ad(a_2)a_1 = a_2a_1 - a_1a_2$
- "Lie bracket" $[\![,]\!] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$
  - anti-symmetric: $[\![a, b]\!] = -[\![b, a]\!]$
  - Jacobi identity satisfied: $[\![a, [\![b, c]\!]]\!] + [\![b, [\![c, a]\!]]\!] + [\![c, [\![a, b]\!]]\!] = 0$
- exponential map $\exp : \mathfrak{g} \to G$
  - If matrix group $G \subset Aut(\mathbb{V})$ or if representation $\rho : G \to Aut(\mathbb{V})$, it coincides with the matrix exponential $\exp : End(\mathbb{V}) \to Aut(\mathbb{V})$

$$\exp a = \sum_{n=0}^{\infty} \frac{1}{n!} a^n$$

  - $\exp a \exp b = expC(a, b)$ where $C(a, b)$ Baker-Campbell-Hausdorff formula:

$$C(a, b) = a + b + \frac{1}{2}[\![a, b]\!] + \frac{1}{12}[\![a, [\![a, b]\!]]\!] + \frac{1}{12}[\![b, [\![b, a]\!]]\!] + \cdots$$

  - $(\exp a)^{-1} = \exp(-a)$
- Lie algebra representation $\rho : \mathfrak{g} \to End(\mathbb{V})$ such that

$$\rho([\![a, b]\!]) = [\rho(a), \rho(b)]$$

  - trivial representation $\rho_0 : \mathfrak{g} \to End(\mathbb{V})$ with $\rho_0(J_k) = 0$
  - adjoint representation $ad : \mathfrak{g} \to End(\mathfrak{g})$ with $[\![a, b]\!] = ad(a)b$