MAT157: Analysis I - Theorems

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Integration

Definition 12.1: A *partition* of $[a, b]$ is a finite subset $\mathcal{P} \subseteq [a, b]$ with $a, b \in \mathcal{P}$. A partition $\mathcal{Q}$ of $[a, b]$ is called a *refinement* of $\mathcal{P}$ if $\mathcal{P} \subseteq \mathcal{Q}$

Proposition 12.2: If $\mathcal{P}, \mathcal{Q}$ are partitions of $[a, b]$, with $\mathcal{P} \subseteq \mathcal{Q}$, then

$$L(f, \mathcal{P}) \leq L(f, \mathcal{Q}), \quad U(f, \mathcal{P}) \geq U(f, \mathcal{Q})$$

Proposition 12.3: For any two partitions $\mathcal{P}_1, \mathcal{P}_2$, one has that $L(f, \mathcal{P}_1) \leq U(f, \mathcal{P}_2)$.

Lemma 12.5: For any bounded function $f : [a, b] \to \mathbb{R}$, $L(f) \leq U(f)$.

Definition 12.6: The bounded function $f : [a, b] \to \mathbb{R}$ is called *integrable* if $L(f) = U(f)$. In this case, the common number $L(f) = U(f)$ is denoted by

$$\int_a^b f$$

or (in Leibniz notation)

$$\int_a^b f(x)dx$$

Therorem 12.10: (Riemann criterion). A bounded function $f : a, b \to \mathbb{R}$ is integrable if and only if for all $\epsilon > 0$, there exists a partition $\mathcal{P}$ such that

$$U(f, \mathcal{P}) - L(f, \mathcal{P}) < \epsilon$$

Lemma 12.11: Let $A \subseteq \mathbb{R}$ be a subset, and let $f : A \to \mathbb{R}$ be a bounded function. If

$$M = \sup\{f(x) : x \in A\}, \quad m = \inf\{f(x) : x \in A\}$$

Then,

$$M - m = \sup\{|f(x) - f(y)| : x, y \in A\}$$

Theorem 12.12: If $f : [a, b] \to \mathbb{R}$ is continuous, then f is integrable.

Theorem 12.13: If $f : [a, b] \to \mathbb{R}$ is weakly increasing or weakly decreasing, then f is integrable.

Lemma 12.131: Given a set A and functions $f, g : A \to \mathbb{R}$, we have that

$$\inf(f(A)) + \inf(g(A)) \leq \inf((f + g)(A))$$

and

$$\sup(f(A)) + \sup(g(A)) \geq \sup((f+g)(A))$$

Theorem 12.14: (Linearity of the integral). Suppose $f, g : [a, b] \to \mathbb{R}$ are bounded and integrable. Then $f + g$ is integrable, and

$$\int_a^b (f+g) = \int_a^b f + \int_a^b g$$

Furthermore, given $\lambda \in \mathbb{R}$, the function λf is integrable, and

$$\int_a^b (\lambda f) = \lambda \int_a^b f$$

Remark 12.15: Put differently, the theorem says that the set $\mathcal{R}([a, b])$ of bounded, integrabe functions on $[a, b]$ is a subspace of the vector space of all functions, with integration a linear functional on that vector space. (An element of the dual space.)

Remark 12.16: One consequence of this result is that if f_1, f_2 differ only at finitely many points, then f_1 is integrable if and only if f_2 is integrable, and in this case

$$\int_a^b f_1 = \int_a^b f_2$$

Theorem 12.17: (Properties of the integral). Suppose $f, g : [a, b] \to \mathbb{R}$ are bounded and integrable. Then:

(a) $\max(f, g), \min(f, g), |f|$ are integrable.

(b) The product fg is integrable.

(c) If $[c, d] \subseteq [a, b]$, the the restriction $f|_{[c,d]}$ is integrable.

(d) (Monotonicity) If $f \leq g$, then

$$\int_a^b f \leq \int_a^b g$$

In particular,

$$|\int_a^b f| \leq \int_a^b |f|$$

For any subset $A \in \mathbb{R}$, let $\mathcal{X}_A$ be the characteristic function (or indicator function)

$$\mathcal{X}_A(x) = \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases}$$

Lemma 12.19: Let $f : [a,b] \to \mathbb{R}$ be bounded and $[c,d] \subseteq [a,b]$ a sub interval. Then f is integrable over $[c,d]$ if and only if $\mathcal{X}_{[c,d]}$ is integrable over $[a,b]$. In this case,

$$\int_c^d f = \int_a^b \mathcal{X}_{[c,d]} f$$

Theorem 12.20 Suppose $a < b < c$, and let $f : [a,c] \to \mathbb{R}$ be bounded. Suppose f is integrable over both $[a,b]$ and $[b,c]$. Then f is integrable over $[a,c]$ and

$$\int_a^c f = \int_a^b f + \int_b^c f$$

Remark 12.21 Until now, we defined $\int_a^b f$ only when $a < b$. It is convenient to include the possibility $a \geq b$, by putting

$$\int_a^a f = 0, \quad \int_a^b f = -\int_b^a f \qquad \text{(if } a > b\text{)}$$

With these conventions, the formula in theorem 12.20 holds without any restrictions on a, b, c.

The fundamental theorem of calculus

Theorem 12.24 Suppose $f : [a, b] \to \mathbb{R}$ is bounded and integrable. Then the function $F : [a, b] \to \mathbb{R}$ given by

$$F(x) = \int_a^x f(t)dt$$

is continuous on $[a, b]$.

Theorem (Lipschitz continuity) A function $F : A \to \mathbb{R}$ is called Lipschitz continuous if there exists $M > 0$ with property

$$\forall x, y \in A : |F(x) - F(y)| \leq M|x - y|$$

Lipschitz continuity implies uniform continuity. Assume the function is Lipschitz continuous and take $\delta = \frac{\epsilon}{M+1}$.

Theorem 12.26 (First fundamental theorem of calculus) Suppose $f : [a, b] \to \mathbb{R}$ is bounded and integrable, and let

$$F(x) = \int_a^x f(t)dt$$

for $x \in [a, b]$. If f is continuous at $x_0 \in (a, b)$, then F is differentiable at x_0, with

$$F'(x_0) = f(x_0)$$

Lemma 12.27 Let $f : J \to \mathbb{R}$ be continuous at $x_0 \in J$ (where J is an interval), and define m_h, M_h as

$$m_h = \inf\{f(x)|x \in [a, b], |x - x_0| \leq |h|\}$$
$$M_h = \sup\{f(x)|x \in [a, b], |x - x_0| \leq |h|\}$$

Then, $\lim_{h\to 0} m_h = f(x_0)$ and $\lim_{h\to 0} M_h = f(x_0)$.

Remark 12.28 Theorem 12.26 also gives, for $x \in (a, b)$,

$$\frac{d}{dx}\Big|_{x=x_0} \int_x^b f = -f(x_0)$$

by writing the integral on the left as $\int_a^b f - \int_a^x f$.

Corollary 12.29 Let $J \subseteq \mathbb{R}$ be an open interval. Then any continuous function $f : J \to \mathbb{R}$ is the derivative of a function $F : J \to \mathbb{R}$. Furthermore, F is unique up to an additive constant.

Definition 12.30 Suppose $J \subseteq \mathbb{R}$ is an open interval, and $f : J \to \mathbb{R}$ is a function. A function $F : J \to \mathbb{R}$ is called a primitive or anti-derivative of f if

$$F' = f$$

Definition 12.33 The function $\log : (0, \infty) \to \mathbb{R}$ given by

$$\log(x) = \int_1^x \frac{1}{t} dt$$

is called the (natural) logarithm.

Example 12.34 What is $\int_1^x \log(t)dt$? Note that

$$\frac{d}{dx}(x \log(x)) = \log(x) + 1$$

Since $1 = \frac{dx}{dx}$, this shows that $F(x) = x\log(x) - x$ is a primitive of $\log(x)$. Now, the integration follows from FTC:

$$\int_1^x \log(t)dt = F(x) - F(1) = x\log(x) - x + 1$$

where we used $\log(1) = 0$.

Theorem 12.36 (Second fundamental theorem of calculus). Suppose $f : [a, b] \to \mathbb{R}$ is bounded and integrable, and suppose $F : [a, b] \to \mathbb{R}$ is a continuous function that is differentiable on (a, b) with $F' = f$ on (a, b). Then

$$\int_a^b f = F(b) - F(a)$$

Definition 12.38 The length of the curve sugment of $f : [a, b] \to \mathbb{R}$ from $(a, f(a))$ to $(b, f(b))$ is defined as

$$l(f) = \sup\{l(f, \mathcal{P}) | \mathcal{P} \text{ is a partition}\}$$

provided that the set $\{l(f, \mathcal{P}) | \mathcal{P} \text{ is a partition}\}$ is bounded above. We put $(f) = \infty$ otherwise.

Theorem 12.39 Suppose $f : J \to \mathbb{R}$ is a differentiable function, with f' continuous. For $a, b \in J$ with $a < b$, the length of the corresponding curve segment is finite, and is given by

$$l(f) = \int_a^b \sqrt{1 + (f')^2}$$

Remark 12.40 More generally, suppose $f : [a, b] \to \mathbb{R}$ is a continuous function such that f' exists and is continuous over (a, b). Then it may be that $\sqrt{1 + (f')^2}$ is unbounded. We still have

$$l(f) = \int_a^b \sqrt{1 + (f')^2}$$

where the integral is defined as a limit if needed.

Area and circumference of a circle

Definition 12.41 The number π is defined as

$$\pi = 4\int_0^1 \sqrt{1-x^2}dx$$

Definition (circumference of unit circle) We'll calculate it as 4 times the length of the arc in the upper right orthant, $f : [0,1] \to \mathbb{R}$, $f(x) = \sqrt{1-x^2}$. We have that $f'(x) = -\frac{x}{\sqrt{1-x^2}}$, and so

$$1 + f'(x)^2 = 1 + \frac{x^2}{1-x^2} = \frac{1}{1-x^2}$$

Hence, the length of the arc is

$$l(f) = \int_0^1 \frac{1}{\sqrt{1-x^2}}dx$$

Since the function is unbounded near $x = 1$, we define the integral as

$$\lim_{\epsilon \to 0^+} \int_0^{1-\epsilon} \frac{1}{\sqrt{1-x^2}}dx$$

Using the following trick,

$$\frac{1}{\sqrt{1-x^2}} = 2\sqrt{1-x^2} - \frac{d}{dx}(x\sqrt{1-x^2})$$

we get that the circumference of a unit circle is 2π. Doing the calculation more generally for a circle of radius $r > 0$, we get an area of πr^2 and a circumference $2\pi r$.

Improper integrals

Definition 12.44 Suppose the restriction of $f : (a, b) \to \mathbb{R}$ to any closed subinterval is bounded and integrable. Then, we define

$$\int_a^b f = \lim_{\epsilon \to 0^+} \int_{a+\epsilon}^c f + \lim_{\epsilon \to 0^+} \int_c^{b-\epsilon} f$$

for any $c \in (a, b)$ provided that both of these limits exist.

Remarks 12.45

- This definition does not depend on the choice of c: for any choice c', the integrals change by the opposite amount:

$$\int_{a+\epsilon}^{c'} f = \int_{a+\epsilon}^{c} f + \int_c^{c'} f, \qquad \int_{c'}^{b-\epsilon} f = \int_c^{b-\epsilon} f - \int_c^{c'} f$$

- Note that we did not define the integral as $\lim_{\epsilon \to 0^+} \int_{a+\epsilon}^{b-\epsilon} f$; we really insist that the limits of both of the integrals exists separately.

- If $f = F'$ for a differentiable function $F : (a, b) \to \mathbb{R}$, the fundamental theorem of calculus gives us

$$\int_a^b F' = F(b) - F(a)$$

  where $F(a) = \lim_{t \to a^+} F(t)$ and $F(b) = \lim_{t \to b^-} F(t)$ (the integral converges provided both of these limits exist).

Note: The integral $\int_0^1 t^\lambda dt$ converges if and only if $\lambda > -1$.

Definition 12.44 (Continued) A similar discussion applies to integrals over unbounded intervals. Given a function $f : [a, \infty) \to \mathbb{R}$, we define

$$\int_a^\infty f = \lim_{x \to \infty} \int_a^x f$$

provided that f is bounded and integrable over all intervals $[a, x]$ with $a < x$, and provided that the limit exists. In this case, the integrals are said to converge. Of course, it might happen that $\lim_{b\to\infty}\int_a^b f = \pm\infty$ as an improper limit, in which case we would write $\lim_a^\infty f = \pm\infty$.

Given $f : (a,\infty) \to \mathbb{R}$, bounded and integrable on each closed interval inside (a,∞), we define

$$\int_a^\infty f = \lim_{x\to a^+}\int_x^c f + \lim_{x\to\infty}\int_c^x f$$

for any $c \in (a,\infty)$, provided that both limits on the right hand side exist. In a similar way, we define $\int_{-\infty}^b f, \int_{-\infty}^\infty f$ for functions defined on the appropriate domain.

Remark 12.46

- The last definition does not depend on the choice of c, since for a different choice c',

$$\int_{-\infty}^{c'} f = \int_{-\infty}^{c} f + \int_c^{c'} f$$
$$\int_{c'}^{\infty} f = \int_{c}^{\infty} f - \int_c^{c'} f$$

- Note that we did not define $\int_{-\infty}^\infty f$ as $\lim_{T\to\infty}\int_{-T}^T f$. This would seem a rather 'arbitrary' definition; even if the limit exists, we may get different answers if we approach $\pm\infty$ differently. For example, you may check that for any $C \in \mathbb{R}$,

$$\lim_{T\to\infty}\int_{-T}^{T+\frac{C}{T}} x dx = C$$

- The fundamental theorem, FTC2, applies as before, provided we interpret $F(a), F(b)$ as limits. For example,

$$\int_a^\infty F'(t)dt = F(t)\Big|_a^\infty$$

where, by definition, $F(\infty) = \lim_{t\to\infty} F(t)$. The integral converges if and only if this limit exists. Similarly,

$$\int_{-\infty}^\infty F'(t)dt = F(t)\Big|_{-\infty}^\infty = F(\infty) - F(-\infty)$$

The right hand side is computed as $\lim_{T\to\infty} F(T) - \lim_{T\to\infty} F(-T)$; note that we require that both of these limits exist, rather than just $\lim_{T\to\infty}(F(T) - F(-T))$.

Example 12.47 The integral $\int_1^\infty t^\lambda dt$ converges if and only if $\lambda < -1$.

Theorem 12.49 (Limit comparison test for integrals). Let $a < b$ (where possibly $b = \infty$), and suppose that $f, g : [a, b) \to \mathbb{R}$ are both $\neq 0$ everywhere, and are integrable on each $[a, x]$ for $a \leq x < b$. Suppose that

$$\lim_{x \to b^-} \frac{f(x)}{g(x)} = C$$

exists, with $C \neq 0$. Then,

$$\int_a^b f(t)dt \text{ converges } \Leftrightarrow \int_a^b g(t)dt \text{ converges}$$

Similar for functions defined on $(a, b]$ (where possibly $a = -\infty$).

Theorem (Symmetry considerations)

- Suppose $f : [a, b] \to \mathbb{R}$ is integrable, and let $h(x) = f(x - c)$ for some $c \in \mathbb{R}$. Then $h : [a + c, b + c] \to \mathbb{R}$ is integrable, and

$$\int_a^b f = \int_{a+c}^{b+C} h$$

- Suppose $f : [a, b] \to \mathbb{R}$ is integrable, and let $g(x) = f(-x)$. Then g is integrable over $[-b, -a]$, and

$$\int_a^b f = \int_{-b}^{-a} g$$

Logarithms and exponentials

Definition 13.1 The logarithm function

$$\log : (0, \infty) \to \mathbb{R}$$

is defined by the integral

$$\log(x) = \int_1^x \frac{1}{t} dt$$

Equivalently, log is the unique differentiable function with the property $\log'(x) = \frac{1}{x}$ and $\log(1) = 0$.

Theorem 13.2 (Properties of log).

(a) For all $a, b > 0$,

$$\log(ab) = \log(a) + \log(b)$$
$$\log(a/b) = \log(a) - \log(b)$$

(b) For $\lambda \in \mathbb{Q}$ and all $x > 0$,

$$\log(x^\lambda) = \lambda \log(x)$$

(c) The logarithm function is increasing, and concave.

(d) The limits for $x \to \infty$, $x \to 0^+$ are

$$\lim_{x\to\infty} \log(x) = \infty, \qquad \lim_{x\to 0^+} \log(x) = -\infty$$

Note: The logarithm function log is slow:

- For $x \to \infty$, the logarithm function approaches ∞ more slowly than any positive power of x.
- For $x \to 0^+$, the logarithm function approaches $-\infty$ more slowly than any negative power of x approaches infinity.

Remark 13.3 (Logarithm at base a). For $a > 0$ define

$$\log_a(x) = \frac{\log(x)}{\log(a)}$$

The function ahs the property $\log_a(a) = 1$ and $\log_a(a^n) = n$.

Definition 13.4 The Euler number e is the unique solution of $\log(e) = 1$.

Definition 13.5 The exponential function $\exp : \mathbb{R} \to \mathbb{R}$ is the inverse function to the log function.

Theorem 13.6 (Properties of the exponential function).

(a) The exponential function is increasing and convex.

(b) For $a, b \in \mathbb{R}$,

$$\exp(a + b) = \exp(a) \exp(b)$$

(c) The limits at $\pm\infty$ are

$$\lim_{x\to\infty} \exp(x) = \infty, \qquad \lim_{x\to-\infty} \exp(x) = 0$$

(d) The exponential function satisfied the differential equation

$$\exp'(x) = \exp(x)$$

and is the unique such solution satisfying $\exp(0) = 1$.

Definition 13.7 For all $a > 0$ and $x \in \mathbb{R}$, we define

$$a^x = \exp(x \log(a))$$

Theorem 13.8 Basic properties of irrational powers: For $a, a_1, a_2 > 0$ and $x, x_1, x_2 \in \mathbb{R}$,

- $a^{x_1+x_2} = a^{x_1} a^{x_2}$
- $(a^{x_1})^{x_2} = a^{x_1 x_2}$
- $a^0 = 1$, $a^1 = a$

- $(a_1 a_2)^x = (a_1)^x (a_2)^x$
- $1^x = 1$

Definition

$$\frac{d}{dx}a^x = \frac{d}{dx}\exp\left(x\log a\right) = a^x \cdot \log a$$

When we take $a = e$, we get $\frac{d}{dx}e^x = e^x$. Hence, $\exp(x) = e^x$ and we use e^x as shorthand for $\exp(x)$.

Proposition 13.10

More about exponential functions

Trigonometic functions

Irrationality of π and e

Tangent and cotangent function

Definition 13.40 (Tangent and cotangent function)

The tangent function $x \to \tan(x)$ is defined for $x \neq \frac{\pi}{2} + k\pi$, $k \in \mathbb{Z}$, by

$$\tan(x) = \frac{\sin(x)}{\cos(x)}$$

The cotangent function $x \to \cot(x)$ is defined for $x \neq k\pi$, $k \in \mathbb{Z}$, by

$$\cot(x) = \frac{\cos(x)}{\sin(x)}$$

Properties of tangent and cotangent

- Both tan, cos are periodic with period π (not just 2π):
$$\tan(x+\pi) = \tan(x), \qquad \cot(x+\pi) = \cot(x)$$
- $\tan(x + \dfrac{\pi}{2}) = -\cot(x)$
- $\tan'(x) = 1 + \tan^2(x)$ and $\cot'(x) = -(1 + \cot^2(x))$.
- tan is increasing for $\pi/2 < x < \pi/2$, is an odd function, $\tan'(0) = 1$, $\lim_{x\to\pi/2^-} \tan(x) = \infty$
- cot is decreasing for each of the intervals, is an odd function and $\lim_{x\to\pi^-} \cot(x) = -\infty$

Definition of inverses

Since tan is increasing on $(-\dfrac{\pi}{2}, \dfrac{\pi}{2})$, the inverse function is defined. It is denoted arctan : $\mathbb{R} \to \mathbb{R}$. By definition, it is increasing, odd, bounded and $\lim_{x\to\pm\infty} = \pm\dfrac{\pi}{2}$.

$$\arctan'(x) = \frac{1}{\tan'(y)} = \frac{1}{1+\tan^2(y)} = \frac{1}{1+x^2}$$

Hyperbolic trigonometric functions

Definition:

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \qquad \cosh(x) = \frac{e^x + e^{-x}}{2}$$
$$\tanh(x) = \frac{\sinh(x)}{\cosh(x)} = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \qquad \coth(x) = \frac{\cosh(x)}{\sinh(x)} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

The first three functions are defined for all $x \in \mathbb{R}$, the last function for $x \neq 0$.

Properties:

- $\sinh'(x) = \cosh(x), \qquad \cosh'(x) = +\sinh(x)$
- $f''(x) - f(x) = 0$

Infinite series

Definition: Given a sequence of numbers $a_1, a_2, a_3, \cdots$ one can consider the corresponding series given by sums

$$s_1 = a_1, \quad s_2 = a_1 + a_2, \quad s_3, a_1 + a_2 + a_3, \cdots$$
$$s_n = \sum_{k=1}^{n} a_k$$

We refer to a sequence s_n obtained in this way as a series, and we say that the series converges (with limit l) is the sequence s_n converges (with limit l). If the limit exists (possibly as an improper limit), we write

$$\sum_{k=1}^{\infty} a_k = l$$

for this limit. One sometimes refers to the formal expresssion $\sum_{k=1}^{\infty} a_k$ itself as the series (even before one has established convergence), and to

$$s_n = \sum_{k=1}^{n} a_k$$

as its partial sums. Thus, a series converges if and only if the sequence of partial sums converges.

Remark 18.1: Actually, every sequence s_n can be regarded as a series: Letting

$$a_1 = s_1, \ a_2 = s_2 - s_1, \ a_3 = s_3 - s_2, \cdots$$

we have that $s_n = s_{n-1} + a_n$ for all n, hence $s_n = \sum_{k=1}^{n} a_k$. Nevertheless, certain s_n's arise 'more naturally' as sums.

Remark 18.2: In practice, it is often useful to let the series (or sequence) start at $n = 0$ or at some other value $n_0 \in \mathbb{Z}$, so we'll consider any such expression

$$\sum_{n=n_0}^{\infty} a_n$$

as a series. We typically make this allowance without any special comment.

18.2 Criteria for convergence of series

Proposition 18.4:

A necessary condition for convergence of a series $\sum_{k=0}^{\infty} a_k$ is that

$$\lim_{n\to\infty} a_n = 0$$

In particular, the sequence $\{a_n\}$ must be bounded.

Proposition 18.6 (Boundedness criterion):

Suppose $\{a_n\}$ is a sequence of numbers with $a_n \geq 0$. Then the sequence $\sum_{k=0}^{\infty} a_k$ converges if and only if the sequence of partial sums $s_n = \sum_{k=0}^{n} a_k$ is bounded above.

Proposition 18.8 (Comparison test): Suppose $\{a_n\}, \{b_n\}$ are sequences of numbers with

$$0 \leq a_n \leq b_n$$

Then

$$\text{convergence of } \sum_{k=0}^{\infty} b_k \Rightarrow \text{ convergence of } \sum_{k=0}^{\infty} a_k$$

More generally, this conclusion holds true if there exists N with $n \geq N \Rightarrow 0 \leq a_n \leq b_n$.