#4 Open and closed sets in R^n

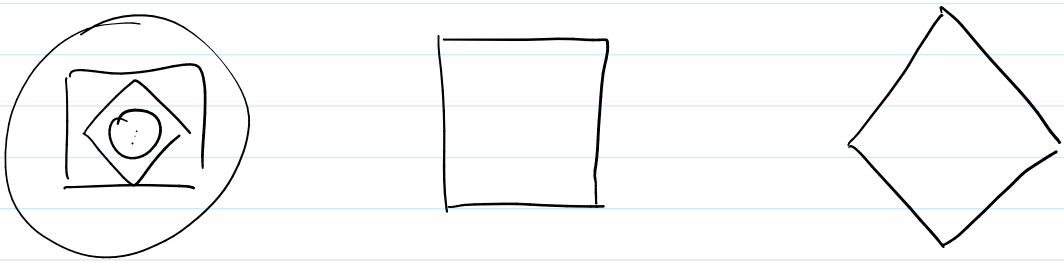
Thursday, 16 September 2021 4:11 pm

Hour 4: open and closed sets in $\mathbb{R}^n$
Read along: Spivak 1-10
HW1 on Crowdmark tonight, piazza

Theorem: Defining "open" using rectangles is equivalent to defining open using balls.

Proof:

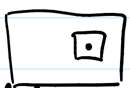
"Every open rectangle is open using the open ball def^n"
"Every open ball is open using the open rect. def^n"



ball using $|\cdot|_2$ ball using $|\cdot|_\infty$ ball using $|\cdot|_1$

Def^n: A set B is "closed" if $\mathbb{R}^n \setminus B = B^c$ is open.

Theorem:
① $\emptyset$, $\mathbb{R}^n$ are clopen
② Any union of open sets is open
Any intersection of closed sets is closed
③ A finite intersection of open sets is open.
A finite union of closed sets is closed.

Proof:
① $\mathbb{R}^n$ is open  $(x \in \prod (x_i - 1, x_i + 1) \subset \mathbb{R}^n)$
$\Rightarrow \emptyset$ is closed (by def).
$\emptyset$ is open (no points, vacuously true)
"Every horse in an empty set of horses has a horn"
$\Rightarrow \mathbb{R}^n$ is closed

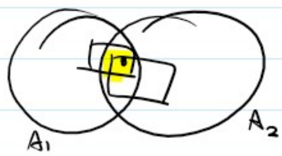
"Every horse in an empty set of horses has a horn"
$\Rightarrow \mathbb{R}^n$ is closed

② Suppose $\{A_\alpha\}_{\alpha \in I}$, where I is an indexing set, is a collection of open sets. $A = \bigcup_{\alpha \in I} A_\alpha = \{x : \exists\, \alpha \in I,\ x \in A_\alpha\}$ is open.
Let $x \in A$. Find α such that $x \in A_\alpha$. Find an open rect. R s.t. $x \in R \subset A_\alpha \subset A$.

Reminder: If Y_α is any collection of subsets of some universe U. Then, $(\bigcup Y_\alpha)^c = \bigcap Y_\alpha^c$ and $(\bigcap Y_\alpha)^c = \bigcup Y_\alpha^c$ "De Morgan's laws"

Suppose $\{B_\alpha\}_{\alpha \in I}$ is a collection of closed sets, we need to show $\bigcap B_\alpha$ is closed. $(\bigcap B_\alpha)^c = \bigcup B_\alpha^c$ is open. $\Rightarrow \bigcap B_\alpha$ is closed.

③ Suppose A_1 & A_2 are open.



Lemma: The intersection of two open rectangles, if non-empty, is an open rectangle

Prove as an exercise.

Suppose $x \in A_1 \cap A_2$ by openness of A_i. $\exists$ open rects R_i s.t. $x \in R_i \subset A_i$ for $i = 1,2$. Then, $x \in \underbrace{R_1 \cap R_2}_{\text{an open rect.}} \subset A_1 \cap A_2$

Suppose A_i $i \in 1, \dots, n$ are open.

$$\bigcap_{i=1}^{n} A_i = \underbrace{\left(\bigcap_{i=1}^{n-1} A_i\right)}_{\substack{\text{By induction,} \\ \text{this is open}}} \cap A_n$$ which is open by the previous part.

Suppose B_i $i \in 1, \dots, n$ are closed

$$\left(\bigcup_{i=1}^{n} B_i\right)^c = \bigcap_{i=1}^{n} B_i^c$$ which is open, which proves it.

$$\bigcap_{n>0} \left(-\frac{1}{n}, 1 + \frac{1}{n}\right) = [0,1]$$

$\rightarrow$ inf intersection of open sets that is closed.