#6 Compactness of products, compactness in R^n

Wednesday, 22 September 2021 9:06 AM

Hour 6 : Compactness

Read along : Spivak 1-10

<u>Compact</u>: Every open cover has a finite subcover

<u>Thm</u>: $[a,b]$ is compact

<u>Pf</u>: $G := \{ q \in [a,b] : [a,q]$ has a finite subcover $\}$

$\gamma = \sup G$ exists ! $\gamma = b$ shown

<u>Claim</u>: $b = \gamma \in G$

<u>Proof</u>: b is covered by $\mathcal{U} = \{u_\alpha\}$. Hence, some interval $(b^-, b^+) \ni b$ is covered by one set $U_\alpha \in \mathcal{U}$.

As $\sup G = b > b^-$ $\exists g' \in G$ s.t. $b^- < g' \leq b$

$\Rightarrow \underbrace{[a, g']}_{\text{covered by finite subcover}} \cup \underbrace{(b^-, b]}_{\text{covered by one } U_\alpha} = [a,b] \}$ covered with finitely subcover.

<u>Theorem</u>: If $A \subset \mathbb{R}^n$ is compact & $B \subset \mathbb{R}^m$ is compact. Then $A \times B \subset \mathbb{R}^{n+m}$ is compact.

<u>Corollary</u>: Closed rectangles $R = \prod_{i=1}^{n} [a_i, b_i]$ are compact.

<u>Proof</u>: Suppose $\mathcal{U} = \{U_\alpha\}$ is an open cover of $A \times B$

WLOG each U_α is itself an open rectangle.

* Every open set is a union of open rectangles

<u>Lemma</u>: For every $x \in A$, we can find an open set $N_x \ni x$ s.t. $N_x \times B$ can be covered w/ finitely many of the U_α's.

Write $U_\alpha = \underbrace{V_\alpha}_{\text{open rect. in } \mathbb{R}^n} \times \underbrace{W_\alpha}_{\text{open rect. in } \mathbb{R}^m}$.

Consider $\{ W_\alpha : x \in V_\alpha \}$, it covers B which is compact.

So, find $\alpha_1, \dots, \alpha_p$ s.t. $W_{\alpha_1}, \dots, W_{\alpha_p}$ cover B, so $U_{\alpha_1}, \dots, U_{\alpha_p}$ cover $\{x\} \times B$

$V_{\alpha_1} \times W_{\alpha_1} \dots$

Let $N_x = \bigcap_{i=1}^{p} V_{\alpha_i} \subset V_{\alpha_i}$

$x \in N_x \Leftarrow x \in V_{\alpha_i}$

$$\begin{aligned} \text{Now}, \ N_x \times B &\subset \bigcup_{i=1}^{p} N_x \times W_{\alpha_i} \\ &\subset \bigcup_{i=1}^{p} V_{\alpha_i} \times W_{\alpha_i} \\ &= \bigcup_{i=1}^{p} U_{\alpha_i} \end{aligned}$$

Now, $\{N_x\}_{x \in A}$ is an open cover of A. By compactness, we can find finitely many N_x's such that their union covers all of A. $(x_1, \dots, x_q)$ s.t. $\bigcup_{j=1}^{q} N_{x_j} \supset A$ ie. $\bigcup_{j=1}^{q} N_{x_j} \times B \supset A \times B$

For each $j \in \{1, \dots, q\}$, find U_{ji} where $i \in \{1, \dots, p_j\}$ such that.

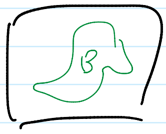
$\bigcup_{i=1}^{p_j} U_{ji} \supset N_{x_j} \times B$. Now $\bigcup_{j=1}^{q} \bigcup_{i=1}^{p_j} U_{ji} \supset A \times B$. ∎

For each $j \in \{1, \dots, k\}$, find U_{ji} where $i \in \{1, \dots, p_j\}$ such that:
$\bigcup_{i=1}^{p_j} U_{ji} \supset N_{x_j} \times B$. Now $\bigcup_{j=1}^{p} \bigcup_{i=1}^{p_j} U_{ji} \supset A \times B$. ■

<u>Proposition</u>: A closed subset of a compact set is compact.

(<u>Corollary</u>: Every closed and bounded subset of $\mathbb{R}^n$ is compact)
If B bounded, $\exists M \in \mathbb{R}$ $\forall b \in B$ $|b| < M$
$\Leftrightarrow$ B is contained in some closed rect.

<u>Proof</u>: Suppose C is compact, $B \subset C$ is closed



(Choose random open cover of C. There is a finite subcover. For all sets in finite subcover, if $x \in B$, choose it. The chosen sets are the finite subcover of B) .

Suppose $\{U_\alpha\}$ is an open cover of B, then $\{U_\alpha\} \cup \{B^c\}$ is an open cover of C.

It has a finite subcover $U_{\alpha_1}, \dots, U_{\alpha_p}$ & maybe $\cancel{B^c}$
consider $U_{\alpha_1}, \dots, U_{\alpha_p}$
this is a finite cover of $B \subset C$.