#8 Continuity and differentiability

Monday, 27 September 2021 9:11 am

Hour 8: Continuity, differentiability
Read along: Spivak 11-19

<u>Def</u>: $F: A \subset \mathbb{R}^n \to \mathbb{R}^n$ is cont. : $\forall a \in A \; \forall \varepsilon > 0 \; \exists \delta > 0 \; \forall x \in A : |x-a| < \delta$
$\Rightarrow |f(x) - f(a)| < \varepsilon$

<u>Thm</u>: $F: A \subset \mathbb{R}^n \to \mathbb{R}^m$ is cont. iff whenever $U \subset \mathbb{R}^m$ is open, $\exists V \subset \mathbb{R}^n$ open s.t. $F^{-1}(U) = V \cap A$ "$F^{-1}(U)$ is open in A".

<u>Proof</u>: (in the case where $A = \mathbb{R}^n$)
$\Rightarrow$ Assume $U \subset \mathbb{R}^m$ is open, WTS $F^{-1}(U)$ is open
Pick $a \in F^{-1}(U)$, then $F(a) \in U$ so pick $\varepsilon > 0$ s.t. $B_\varepsilon(F(a)) \subset U$.
By continuity, find $\delta > 0$ s.t. the condition of continuity is satisfied (s.t. $F(B_\delta(a)) \subset B_\varepsilon(F(a)) \subset U$.) So $a \in B_\delta(a) \subset F^{-1}(U)$. So $F^{-1}(U)$ open.

☆ Balls are always open

$\Leftarrow$ Given $a \in \mathbb{R}^n$ and $\varepsilon > 0$, consider $B_\varepsilon(F(a))$, it is open. So $a \in F^{-1}(B_\varepsilon(F(a))$ is open so there exists a $\delta > 0$ s.t. $B_\delta(a) \subset F^{-1}(B_\varepsilon(F(a)))$

QED

<u>Example 1/Thm</u>:

$$\begin{tikzcd} \mathbb{R}^n \arrow[r, "f"] \arrow[rr, bend right, "g \circ f"'] & \mathbb{R}^m \arrow[r, "g"] & \mathbb{R}^p \end{tikzcd}$$

f, g cont. $\Rightarrow g \circ f$ continuous

Given $U \subset \mathbb{R}^p$ open, WTS $(g \circ f)^{-1}(U)$ open.

$$(g \circ f)^{-1}(U) = f^{-1}(g^{-1}(U)) = f^{-1}(\text{open}) = \text{open}$$

QED

<u>Example 2/Thm</u>: If $F: \mathbb{R}^n \to \mathbb{R}^m$ is cont. and $C \subset \mathbb{R}^n$ is compact, then $F(C)$ is also compact.
"A cont image of a compact is compact".

We know Compact in $\mathbb{R}^n \Leftrightarrow$ closed and bounded.

<u>Corollary</u>: A cont. function on a compact set C is bounded
<u>Proof</u>: $F(C)$ by ex 2 is compact, so it is bounded. So F is bounded.

<u>Proof of ex. 2</u>: (sketch)
Given an open cover $\{U_\alpha\}$ of $F(C)$, $\{F^{-1}(U_\alpha)\}$ is an open cover of C, hence it has a finite subcover, which in itself corresponds to a finite subcover for $F(C)$

Compactness in terms of closed sets, check

F cont $\Leftrightarrow$ (U open $\Rightarrow F^{-1}(U)$ open)
$\Leftrightarrow$ (D is closed $\Rightarrow F^{-1}(D)$ is closed)
$\Downarrow$

F cont $\Leftrightarrow$ (U open $\Rightarrow$ $F^{-1}(U)$ open)

$\Leftrightarrow$ (D is closed $\Rightarrow$ $F^{-1}(D)$ is closed)

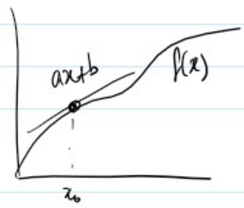
$\Downarrow$

D^c is open $\Rightarrow$ $F^{-1}(D^c)$ is open

$\Rightarrow$ $F^{-1}(D)^c$ is open

$\Rightarrow$ $F^{-1}(D)$ closed

<u>Differentiability</u>:



$$\lim_{x \to x_0} \frac{f(x) - ax - b}{|x - x_0|} = 0$$

<u>Defⁿ</u> in $\mathbb{R}^n$: same