

#6 Compactness of products, compactness in \mathbb{R}^n

Wednesday, 22 September 2021 9:06 AM

Hour 6 : Compactness

Read along: Spivak 1-10

Compact: Every open cover has a finite subcover

Claim: $[a, b]$ is compact

Pf: $G := \{q \in [a, b] : [a, q] \text{ has a finite subcover}\}$

$$\gamma = \sup G_1 \text{ exists! } \gamma = b \text{ shown}$$

Claim: $b = \gamma \in G_1$

Proof: b is covered by $U = \{U_\alpha\}$. Hence, some interval

$(b^-, b^+) \supseteq b$ is covered by one set $U_\alpha \in U$.

As $\sup G_1 = b > b^- \exists q' \in G_1$ s.t. $b^- < q' \leq b$

$\Rightarrow \underbrace{[a, q']}_{\substack{\text{covered by} \\ \text{finite subcover}}} \cup \underbrace{(b^-, b]}_{\substack{\text{covered} \\ \text{by one} \\ \text{one } U_\alpha}} = [a, b]$ covered with finitely

subcover.

Theorem: If $A \subset \mathbb{R}^n$ is compact & $B \subset \mathbb{R}^m$ is compact.
Then $A \times B \subset \mathbb{R}^{n+m}$ is compact.

Corollary: Closed rectangles $R = \prod_{i=1}^n [a_i, b_i]$ are compact.

Proof: Suppose $U = \{U_\alpha\}$ is an open cover of $A \times B$
wLOG each U_α is itself an open rectangle.

* Every open set is a union of open rectangles

Lemma: For every $x \in A$, we can find an open set $N_x \ni x$ s.t. $N_x \times B$ can be covered w/ finitely many of the U_α 's.

Write $U_\alpha = \underbrace{V_\alpha}_{\substack{\text{open rect.} \\ \text{in } \mathbb{R}^n}} \times \underbrace{W_\alpha}_{\substack{\text{open rect. in } \mathbb{R}^m}}$.

Consider $\{W_\alpha : x \in V_\alpha\}$, it covers B which is compact.

So, find a_1, \dots, a_p s.t. W_{a_1}, \dots, W_{a_p} cover B , so V_{a_1}, \dots, V_{a_p} cover $\{x\} \times B$

$V_{a_1} \times W_{a_1} \dots$

$$\text{Let } N_x = \bigcap_{i=1}^p V_{a_i} \subset V_{a_i}$$

$$\begin{aligned} \text{Now, } N_x \times B &\subset \bigcup_{i=1}^p N_x \times W_{a_i} \\ &\subset \bigcup_{i=1}^p V_{a_i} \times W_{a_i} \\ &= \bigcup_{i=1}^p U_{a_i} \end{aligned}$$

Now, $\{N_x\}_{x \in A}$ is an open cover of A . By compactness, we can find finitely many N_x 's such that their union covers all of A . (x_1, \dots, x_q) s.t. $\bigcup_{j=1}^q N_{x_j} \supset A$ ie. $\bigcup_{j=1}^q N_{x_j} \times B \supset A \times B$

For each $j \in \{1, \dots, q\}$, find $U_{j,i}$ where $i \in \{1, \dots, p\}$ such that.

$$\bigcup_{i=1}^p U_{j,i} \supset N_{x_j} \times B. \text{ Now } \bigcup_{i=1}^p \bigcup_{j=1}^q U_{j,i} \supset A \times B$$

or when $j = 1, \dots, k_j$, since v_{ji} where $i = 1, \dots, l_j$ same value.
P: $\bigcup_{i=1}^l v_{ji} \supset N_{x_j} \times B$. Now $\bigcup_{j=1}^p \bigcup_{i=1}^{l_j} v_{ji} \supset A \times B$.

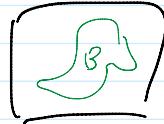
Proposition: A closed subset of a compact set is compact.

(Corollary: Every closed and bounded subset of \mathbb{R}^n is compact)

If B bounded, $\exists M \in \mathbb{R} \quad \forall b \in B \quad |b| < M$

$\Leftrightarrow B$ is contained in some closed rect.

Proof: Suppose C is compact, $B \subset C$ is closed



Choose random open cover of C . There is a finite subcover. For all sets in finite subcover, if $x \in B$, choose it. The chosen sets are the finite subcover of B .

Suppose $\{U_\alpha\}$ is an open cover of B , then $\{U_\alpha\} \cup \{B^c\}$ is an open cover of C .

It has a finite subcover U_{i_1}, \dots, U_{i_p} & B^c
consider U_{i_1}, \dots, U_{i_p}
this is a finite cover of $B \subset C$.