#3 Linear transformations and matrices, rectangles in Rn

Monday, 13 September 2021 9:30 am

Hour 3: More linear algebra
Read along: Spivak 1-5

MAT240 review:

$\left\{ \begin{array}{l} \text{Linear } \mathbb{R}^n \to \mathbb{R}^m \\ T: \mathbb{R}^n \to \mathbb{R}^m \text{ s.t.} \\ T(ax+by) = aT(x) + bT(y) \end{array} \right\}$ $\underset{L_A \leftarrow A}{\overset{T \mapsto M_T}{\rightleftarrows}}$ bijection $\quad M_{m\times n}(\mathbb{R}) = \left\{ A = \underbrace{\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \\ a_{m1} & \cdots & a_{mn} \end{pmatrix}}_{n} \Big\} m \right\}$

this is a vector space — this is also a vector space

→ depends on the basis chosen for $\mathbb{R}^n$ and $\mathbb{R}^m$
– we choose the standard basis $e_i = \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}$ where 1 in position i.

by
$A \in M_{m\times n} \mapsto L_A(x) = Ax \quad {}^m(\;)_n (\;)_1^n = {}^m(\;)_1$
where $x \in \mathbb{R}^n$

If T is a linear transformation,

$$M_T = \underbrace{\begin{pmatrix} Te_1 \,|\, Te_2 \,|\, \cdots \,|\, Te_n \end{pmatrix}}_{n} \Big\} m \in M_{m\times n}$$

where did I see the word homomorphism before? The topology book?

Q: How is bijection diff. from homomorphism?

Theorem:
0. $T = L_{(M_T)}$, $M_{(L_A)} = A$ (this is called a "homomorphism")
1. $A \mapsto L_A$ is linear, $T \mapsto M_T$ is linear
→ $L_{aA+bB} = aL_A + bL_B$, $M_{aT+bS} = aM_T + bM_S$
2. $T: \mathbb{R}^n \to \mathbb{R}^m$, $S: \mathbb{R}^m \to \mathbb{R}^p$ (or $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m \xrightarrow{S} \mathbb{R}^p$, $\underbrace{\hphantom{xxxx}}_{S\circ T}$)
→ $M_S \cdot M_T = M_{S\circ T}$

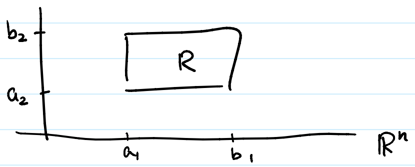
the exponential map is a homomorphism
$e^{x+y} = e^x e^y$
$\mathbb{R}, + \to \mathbb{R}_{>0}, \cdot$

End of review.

In 157, we looked at intervals on the real line.

$\overset{}{\underset{a}{[}} \text{———} \underset{b}{]} \text{———} \mathbb{R}$

In $\mathbb{R}^n$ we talk about rectangles.



Given $a_i \le b_i$ where $i = 1, \ldots, n$, $R = \prod_{i=1}^n [a_i, b_i] = \{ x \in \mathbb{R}^n : \forall i \; a_i \le x_i \le b_i \}$
R= The closed rectangle corresponding to a_i, b_i

Why the multiplication notation?
Set theory: If X & Y are sets, $X \times Y = \{(x,y) : x \in X, y \in Y\}$
If X, Y & Z are sets, $\underset{((x,y),z)}{(X\times Y)\times Z} \neq \underset{(x,(y,z))}{X \times (Y \times Z)}$ (strictly speaking)

BUT we lie and pretend $((x,y),z) = (x,y,z) = (x,(y,z))$
$\Rightarrow (X\times Y)\times Z = X \times (Y \times Z)$

This is by agreement, not by strict truth.
$\mathbb{R} = \underbrace{\mathbb{R} \times \mathbb{R} \times \cdots \times \mathbb{R}}_{n} = \{ (x_1, \ldots, x_n) : \forall i \; x_i \in \mathbb{R} \}$

So, $\prod_{i=1}^n [a_i, b_i] = \{ (x_1, \ldots, x_n) : \forall i \; x_i \in [a_i, b_i] \}$

Likewise, the "open" rectangle defined by a_i, b_i is $\prod_{i=1}^n (a_i, b_i) = \prod (a_i, b_i)$ (notation)
$= \{ x \in \mathbb{R}^n : \forall i \; a_i < x_i < b_i \}$

Likewise, the "open" rectangle defined by a_i, b_i is $\prod_{i=1}^{n} (a_i, b_i) \overset{\text{(notation)}}{=} \prod (a_i, b_i)$
$= \{ x \in \mathbb{R}^n : \forall i \;\; a_i < x_i < b_i \}$

<u>Def</u>^{n}: $A \subset \mathbb{R}^n$ is called "open"
if $\forall a \in A$, $\exists$ $\overset{\text{open}}{\text{rect.}}$ R s.t. $x \in R \subset A$