#7 Functions and continuity

Friday, 24 September 2021 9:11 am

Hour 7: continuity

Read along: Spivak 11-14

$F: \mathbb{R}^n \to \mathbb{R}^m$

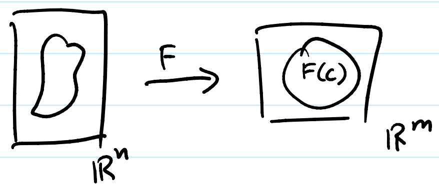
- a machine / procedure
- if $A \subset \mathbb{R}^n$ and $B \subset \mathbb{R}^m$

$\Rightarrow F: \underset{\text{domain}}{A} \to B$ is defined $\searrow$ codomain, range

$F: \mathbb{R}^n \to \mathbb{R}^m$, $F|_A : A \subset \mathbb{R}^n$

if $C \subset \mathbb{R}^n$

$\to F(C) = \{F(\gamma) : \gamma \in C\}$



The notion of an inverse (F^{-1}) is better behaved! (?)

$\to F^{-1}(D_1 \cup D_2) = F^{-1}(D_1) \cup F^{-1}(D_2)$

$\to F^{-1}(D_1 \cap D_2) = F^{-1}(D_1) \cap F^{-1}(D_2)$

$\to F^{-1}(D^c) = F^{-1}(D)^c$

on the other hand,

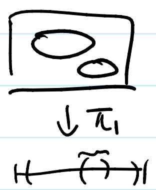
$F(C_1 \cup C_2) \overset{?}{=} F(C_1) \cup F(C_2)$ $\Rightarrow$ This is true

$F(C_1 \cap C_2) \overset{?}{=} F(C_1) \cap F(C_2)$

$F(C^c) \overset{?}{=} F(C)^c$

Counter for $\cap$: take $f: x \mapsto x^2$, C_1 = neg. reals, C_2 = pos. reals.

OR



Def: $\pi_i : \mathbb{R}^n \to \mathbb{R}$ s.t. $\pi_i(x_1, \ldots, x_n) = x_i$

For all $F: \mathbb{R}^n \to \mathbb{R}^m$, there are m functions automatically defined $F_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1, \ldots, m$ (aka coordinate functions)

$$F(x) = y = \begin{pmatrix} F_1(x) \\ \vdots \\ F_m(x) \end{pmatrix}$$

$\to F_i = \pi_i \circ F$

What is a graph?

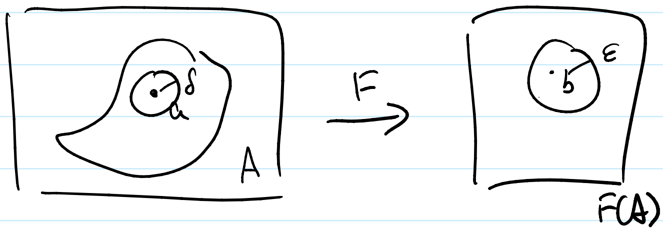
$\Gamma_f = \{(x, f(x)) : x \in \mathbb{R}\}$
$\subset \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$

$F : \mathbb{R} \to \mathbb{R}$

If $F : \mathbb{R}^n \to \mathbb{R}^m$ $\quad \Gamma_f = \{(x, f(x)) : x \in \mathbb{R}^n\}$
$\subset \mathbb{R}^n \times \mathbb{R}^m = \mathbb{R}^{n+m}$

$F : A \subset \mathbb{R}^n \to \mathbb{R}^m$; $a \in \overline{A}$

$\lim_{x \to a} f(x) = b$



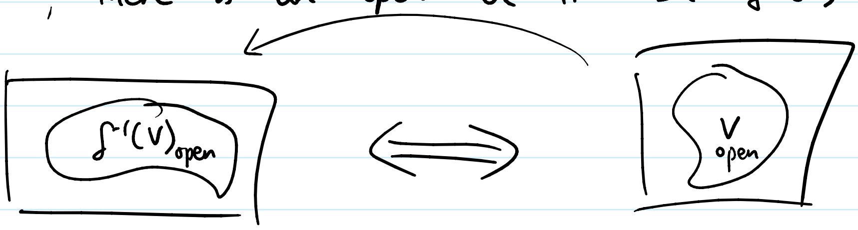
As a gets closer to the boundary, b gets closer to the boundary (?)

$\forall \varepsilon > 0 . \exists \delta > 0$ s.t $a \neq x \in B_\delta(a) \cap A \Rightarrow F(x) \in B_\varepsilon(b)$

If limit exists, it must be unique.
If a outside ' & set,

Def: $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is "continuous at $a \in A$" if $\lim_{x \to a} f(x) = f(a)$
"F is cont. on A" $\Leftrightarrow$ cont. at every $a \in A$
$\Leftrightarrow$ $\forall a \in A$ $\forall \varepsilon > 0$ $\exists \delta > 0$ $\forall x \in A$: $|x - a| < \delta \Rightarrow |f(x) - f(a)| < \varepsilon$.

Thm: $f : \mathbb{R}^n \to \mathbb{R}^m$ is continuos $\Leftrightarrow$ for every open $V \subset \mathbb{R}^m$, $f^{-1}(V)$ is also open. More generally, $f : A \subset \mathbb{R}^n \to \mathbb{R}^m$ is cont. $\Leftrightarrow$ for every open $V \subset \mathbb{R}^m$, there is an open $U \subset \mathbb{R}^n$ s.t $f^{-1}(V) = U \cap A$



Aside: Def $B \subset A$ is open in A if $\exists U$ open in $\mathbb{R}^n$ s.t $B = U \cap A$