MAT267 Theorems

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Contents

Books referenced:

- *Ordinary Differential Equations* by Morris Tenenbaum and Harry Pollard: [TP]
- *Differential Equations, Dynamical Systems and An Introduction to Chaos* by Morris W. Hirsch, Stephen Smale and Robert L. Devaney: [HSD]

1 Existence and uniqueness for nonlinear systems

1.1 The (local) existence and uniqueness theorem:

Source: HSD Page 385

Consider the initial value problem

$$\begin{cases} X' = F(X) \\ X(t_0) = X_0 \end{cases}$$

where $X_0 \in \mathbb{R}^n$. Suppose $F : \mathbb{R}^n \to \mathbb{R}^n$ is C^1. Then, there exists a unique solution to this initial value problem for $a > 0$, $I = (t_0 - a, t_0 + a)$

$$X : I \to \mathbb{R}^n$$

satisfying the initial condition $X(t_0) = X_0$.

1.2 Corollaries to the local existence and uniqueness theorem

Source: Lecture 11

1. If $J \subseteq I$ with $t_0 \in J$ and we have a solution X_I unique in I and X_J unique in J, then the two solutions agree on J.
2. If $X(t)$ is a unique solution for $X' = F(X)$ with $F(0) = A$ defined on an interval $I \ni t_0$, then the maximal interval of existence for $X(t)$ is the union of all such intervals I that exist.
3. If $X(t)$ is the unique solution defined on the maximal interval $I_{max} = (a, b)$, then $\lim_{t \to b^-} x(t) = \pm\infty$

1.3 Definition (review): uniform continuity

Source: Lecture 12

A function $f : A \to \mathbb{R}$ is called uniformly continuous if it has the property

$$\forall \epsilon > 0. \exists \delta > 0. \forall x, y \in A : |x - y| < \delta \implies |f(x) - f(y)| < \epsilon$$

1.4 Definition (review): Lipschitz continuity

Source: Lecture 12

A function $f : A \to \mathbb{R}$ is called Lipschitz continuous if

$$\exists M. \forall x, y \in A : |f(x) - f(y)| \leq M|x - y|$$

1.5 Definition (review): uniform convergence

Source: Lecture 12

A sequence of functions $f_n : A \to \mathbb{R}$ converges uniformly on A to a function $f : A \to \mathbb{R}$ if

$$\forall \epsilon > 0. \exists N \in \mathbb{N}. \forall x \in A. \forall n \geq N : |f_n(x) - f(x)| < \epsilon$$

1.6 Definition: uniform boundedness

Source: Lecture 12

If $F = \{f_\alpha\}_{\alpha \in A}$ is a set of functions $f_\alpha : E \subseteq \mathbb{R}^n \to \mathbb{R}^m$, then we say that F is uniformly bounded if

$$\exists M. \forall x \in E. \forall \alpha \in A. |f_\alpha(x)| < M$$

1.7 Definition: equicontinuity

Source: Lecture 12

If $F = \{f_\alpha\}_{\alpha \in A}$ is a set of functions $f_\alpha : E \subseteq \mathbb{R}^n \to \mathbb{R}^m$, then we say that F is equicontinuous on E if

$$\forall \epsilon > 0. \exists \delta > 0. \forall y \in E. \forall \alpha \in A. |x - y| < \delta \implies |f_\alpha(y) - f_\alpha(x)| < \epsilon$$

1.8 Claims about equicontinuity:

Source: Lecture 12

1. Equicontinuity implies uniform continuity
2. Bounded gradients imply equicontinuity
3. A finite family of uniformly continuous functions is equicontinuous

1.9 Lemma: Mean-value theorem for vector-valued functions

Source: Wikipedia

Let $U \subseteq \mathbb{R}^n$ be open, $f : U \to \mathbb{R}^m$ continuously differentiable, and $x \in U, h \in \mathbb{R}^n$ vectors such that the line segment $x + th$ (where $0 \leq t \leq 1$) remains in U. Then, we have

$$f(x+h) - f(x) = \Big(\int_0^1 Df(x+th)dt \Big) \cdot h$$

1.10 Lemma: mean-value inequality

Source: Wikipedia

If the norm of $Df(x + th)$ is bounded by some constant M for $t \in [0, 1]$, then

$$||f(x+h) - f(x)|| \leq M||h||$$

1.11 Definition: complete metric space

Source: Munkes' Topology Page 264

Let (X, d) be a metric space. A sequence (x_n) of points of X is said to be a **Cauchy sequence** in (X, d) if it has the property that given $\epsilon > 0$, there is an integer N such that $\forall n, m \geq N$,

$$d(x_n, x_m) < \epsilon$$

The metric space (X, d) is said to be **complete** if every Cauchy sequence in X converges [to a point in X].

1.12 Claim: $(C[0, 1], || \cdot ||_\infty)$ is complete

Source: Lecture 12

Continuous functions on $[0, 1]$, denoted $C[0, 1]$, with the infinity norm $|| \cdot ||_\infty$ is a complete metric space.

1.13 Theorem: uniformly convergent continuous functions on compact sets

Source: Lecture 12

Let $f_k : E \subseteq \mathbb{R}^n \to \mathbb{R}^m$ be continuous functions. If E is compact, and each f_k converges uniformly to f, then $\{f_k\}$ is uniformly bounded and equicontinuous.

1.14 Theorem: Ascoli-Arzelà

Source: Lecture 12

If $F = \{f_\alpha\}_{\alpha \in A}$, where each $f_\alpha : E \subseteq \mathbb{R}^n \to \mathbb{R}^m$ for E bounded, is an infinite family of functions that is uniformly bounded and equicontinuous, then there exists a subsequence of functions $\{f_n\}$ in F that converge uniformly in E.

1.15 Definition: relatively compact

Source: Lecture 12

A set B in a metric space is said to be relatively compact if $\overline{B}$ is compact.

1.16 Corollary to Ascoli-Arzelà

Source: Lecture 12

A set $A \subseteq C[K, \mathbb{R}^m)$ (with $K \subseteq \mathbb{R}^n$ compact) is relatively compact in $|| \cdot ||_\infty \iff A$ is uniformly bounded and equicontinuous.

1.17 Lemma: function approximation

Source: Lecture 12

Let $f : B(0, r) \in \mathbb{R}^n \to \mathbb{R}^n$ be a continuous function. Let $F : \mathbb{R}^n \to \mathbb{R}^n$ such that

$$F(x) = \begin{cases} f(x) & |x| \leq r \\ f(r \cdot \frac{x}{|x|}) & |x| > r \end{cases}$$

Then, F is continuous on $\mathbb{R}^n$ and $F\Big|_{\overline{B(0,r)}} = f$

1.18 Banach contraction mapping theorem

Source: Tutorial 6

Let $d(x,y)$ be a metric on a set X. Assume that the metric space X is complete. Let $F: X \to X$ be a function and $0 \leq q < 1$ a constant. If

$$d(F(x), F(y)) \leq q d(x,y) \qquad (\forall x, y \in X)$$

Then, F has a unique fixed point (i.e. $F(x) = x$ has exactly one solution in X). Such a function F is called a contraction on X

1.19 Brouwer's fixed point theorem

Source: Lecture 12

Let $T: \overline{B(0,1)} \to \overline{B(0,1)}$ such that $B(0,1) \subseteq \mathbb{R}^n$ be a continuous function. Then, T has at least one fixed point (i.e. $\exists z$ such that $T(z) = z$)

1.20 Schauder-Tychonoff fixed-point theorem

Source: Lecture 12

Let $E = (C([0,1], \mathbb{R}^n), ||\cdot||_\infty)$. Let $B = \{f \in E : ||f||_\infty \leq 1\}$ be the unit ball in E. Let $T: B \to B$ be a continuous map (with respect to $||\cdot||_\infty$) (i.e. convergent sequences go to convergent sequences) such that $T(B)$ is relatively compact. Then, T has a fixed point.

Note: instead of assuming $T(B)$ to be relatively compact, we could have just as well assumed it equicontinous and uniformly bounded for we have the corollary to the Ascoli-Arzela theorem.

1.21 Claim: IEq $\iff$ IVP

Source: Lecture 13

The initial value problem (IVP): Recall that the initial value problem is as follows. Given a continuous function $f: [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}^n$ with $v = \xi_0 \in \mathbb{R}^n$, find, if possible, a continuously differentiable function $x: [t_0, t_1] \to \mathbb{R}^n$ such that

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = \xi_0 \end{cases}$$

The integral equation (IEq): Consider the following problem. Given a continuous function $f: [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}^n$ with $v = \xi_0 \in \mathbb{R}^n$, find, if possible, a continuously differentiable function $x : [t_0, t_1] \to \mathbb{R}^n$ such that $\forall t \in [t_0, t_1]$,

$$x(t) = \xi_0 + \int_{t_0}^{t} f(s, x(s))ds$$

Note: It is often reformulated as follows: If

$$(Tx)(t) = \xi_0 + \int_{t_0}^{t} f(s, x(s))ds$$

then find a fixed-point $Tx = x$.

1.22 Theorem: Cauchy-Peano

Source: Lecture 13, Wikipedia

Let $f : [t_0, t_1] \times \mathbb{R}^n \to \mathbb{R}^n$ be a continuous and bounded function with $\xi_0 \in \mathbb{R}^n$. Then, the initial value problem

$$\begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = \xi_0 \end{cases}$$

has at least one solution $x(t)$ on $[t_0, t_1]$.

1.23 Euler's polygonal/method:

Source: Lecture 13

For $x'(t) = f(t, x(t))$, we can find a numerical solution at x_{j+1} by

$$\begin{aligned} \frac{x_{j+1} - x_j}{t_{j+1} - t_j} &= f(t_j, x_j) \qquad \text{(take } t_j = j\tau \text{ for all } j\text{)} \\ x_{j+1} &= x_j + \tau f(j\tau, x_j) \end{aligned}$$

This is a recursion that we can solve, given initial conditions. The piecewise affine (linear) function obtained is a good approximation of the solution as $\tau \to 0$.

1.24 Zorn's lemma

Source: Lecture 14

There exists a maximal element (x^*, I^*) which is a continuation of (x, I). Then, for all possible continuations $(\overline{x}, \overline{I})$ of (x, I), we can write

$$I^* = \bigcup \overline{I} \quad \text{and} \quad x^*|_{\overline{I}_i} = \overline{x}_i(t)$$

This holds analogously for extension to the left.

1.25 Theorem: existence and continuation of solutions

Source: Lecture 14

Let $h > 0$, $a > 0$ and define $A = [t_0 - h, t_0 + h] \times \overline{B_a(\xi_0)}$. Let $f : A \to \mathbb{R}^n$ be a continuous function with $m := \sup\{|f(t,\xi)| : (t,\xi) \in A\} < \infty$. Let $(t_0, \xi_0) \in \mathbb{R} \times \mathbb{R}^n$. Then,

$$(IVP) : \begin{cases} x'(t) = f(t, x(t)) \\ x(t_0) = \xi_0 \end{cases}$$

has at least one solution x defined on $I = [t_0 - \min(h, \frac{a}{m}), t_0 + \min(h, \frac{a}{m})]$ and any solution to the IVP defined on $J \subseteq I$, where J is a neighbourhood of t_0, can be continued to I.

1.26 Theorem: Picard–Lindelöf

Source: Lecture 14, TP Theorem 58.5[1], Wikipedia[2]

Let $D \in \mathbb{R} \times \mathbb{R}^n$ be a closed rectangle with $(x_0, y_0) \in D$. Let $f : D \to \mathbb{R}^n$ be a function that is continuous in x and Lipschitz in y. Consider the initial value problem

$$\begin{cases} y'(x) = f(x, y(x)) \\ y(x_0) = y_0 \end{cases}$$

Then, there exists $a > 0$ such that the initial value problem has a unique solution $y(x)$ on $[x_0 - a, x_0 + a]$.

1.27 Claim: the Picard map

Source: Lecture 14

[1] TP's Theorem 58.5 assumes that f is bounded, but this is not necessary because we can take a closed subset of D, on which f would be bounded.

[2] Also known as Picard's existence theorem, the Cauchy-Lipschitz theorem or the existence and uniqueness theorem.

Let $f : D \to \mathbb{R}^n$ be a function that is continuous in x and Lipschitz in y with Lipschitz constant K. There exists a map $\Gamma : C[x_0 - a, x_0 + a] \to C[x_0 - a, x_0 + a]$ such that

$$\Gamma y(x) = y_0 + \int_{x_0}^{x} f(s, y(s))ds$$

Moreover, if we find a $y \in C[x_0 - a, x_0 + a]$ such that $\Gamma y = y$, then y is a solution of the IEq ($\iff$ y is a solution of the IVP).

1.28 Claim: Picard map satisfies conditions of BCMT

Source: Lecture 14

$$\exists 0 \leq q < 1. \forall x, y \in C[x_0 - a, x_0 + a]. \ ||\Gamma(y) - \Gamma(z)|| \leq q||y - z||$$

1.29 Theorem: Gronwall's inequality

Source: Lecture 15

If $U'(x) \leq K(x)U(x)$ where $K(x)$ is continuous and $U(x)$ is differentiable for $x \geq x_0$, then,

$$U(x) \leq U(x_0)e^{\int_{x_0}^{x} K(t)dt}$$

That is, $U \leq$ the solution to the case with equality.

1.30 Osgood's uniqueness theorem

Source: Mary Pugh Notes: Osgood's uniqueness theorem, Lecture 16

Consider the initial value problem

$$\begin{cases} y' = f(x, y) \\ y(x_0) = y_0 \end{cases}$$

Let $D \subseteq \mathbb{R} \times \mathbb{R}^n$ be an open set containing (x_0, y_0). Assume that for all $(x, y_1), (x, y_2) \in D$,

$$|f(x, y_1) - f(x, y_2)| \leq \phi(|y_1 - y_2|)$$

for some continuous function $\phi : [0, \infty) \to [0, \infty)$ such that $\phi(u) > 0$ and $\phi(0) = 0$ and

$$\int_0^1 \frac{du}{\phi(u)} = +\infty$$

Then, no more than one solution passes through (x_0, y_0).

1.31 Theorem: behaviour at finite maximal time of existence

Source: HSD Page 146

Let $U \subseteq \mathbb{R}^n$ be an open set, and let $F : U \to \mathbb{R}^n$ be C^1. Let $X(t)$ be a solution of $X' = F(X)$ defined on a maximal open interval $J = (\alpha, \beta) \subseteq \mathbb{R}$ with $\beta < \infty$. Then, given any closed and bounded set $K \subseteq U$, there is some $t \in (\alpha, \beta)$ with $X(t) \notin K$.

1.32 Lemma: locally Lipschitz

Source: Lecture 16

A continuously differentiable function $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^m$ is locally Lipschitz on all compact $K \subseteq D$ with the Lipschitz constant L

$$L = \sup_K ||Df(x)||$$

1.33 Theorem: global existence and uniqueness

Source: Lecture 16

For $D \subseteq \mathbb{R}^n$ open and connected, $f : D \to \mathbb{R}^n$ a locally Lipschitz vector field and $v \in D$, the IVP

$$(IVP) : \begin{cases} x' = f(x) \\ x(0) = v \end{cases}$$

has a unique solution on the maximal interval of existence. If the maximal interval of existence does not cover all time and $\overline{T}$ is the maximum time for which it is defined, then

$$\lim_{t \to \overline{T}^-} |x(t)| + \frac{1}{d(x(t), \partial D)} = +\infty$$

In other words, either $|x(t)| \to \infty$ or $x(t) \to \partial D$.

1.34 Corollary to global existence and uniqueness

Source: HSD Page 397

Let C be a compact subset of the open set $\mathcal{O} \subseteq \mathbb{R}^n$ and let $F : \mathcal{O} \to \mathbb{R}^n$ be C^1. Let $Y_0 \in C$ and suppose that every solution curve of the form $Y : [0, \beta] \to \mathcal{O}$ with $Y(0) = Y_0$ lies entirely in C. Then, there is a solution $Y : [0, \infty) \to \mathcal{O}$ satisfying $Y(0) = Y_0$, and $Y(t) \in C$ for all $t \geq 0$, so this solution is defined for all (forward) time.

2 Solving nonlinear systems

2.1 Theorem: Gronwall's inequality (generalised)

Source: MAT267 Lecture 17

If $f : [a, b] \to \mathbb{R}$, $g : [a, b] \to \mathbb{R}^+$ and $y : [a, b] \to \mathbb{R}$ are continuous functions such that $\forall t \in [a, b]$,

$$y(t) \leq f(t) + \int_a^t g(s)y(s)ds$$

Then, $\forall t \in [a, b]$,

$$y(t) \leq f(t) + \int_a^t f(s)g(s) \exp\Big(\int_s^t g(u)du\Big)ds$$

In particular, if $f(t) \equiv k$ for some constant k, then,

$$y(t) \leq k \exp\Big(\int_a^t g(s)ds\Big)$$

2.2 Theorem: bound on difference between functions

Source: MAT267 Lecture 18

Let $x_1 : [a, b] \to \mathbb{R}^n$ and $x_2 : [a, b] \to \mathbb{R}^n$ be differentiable functions such that

$$|x_1(a) - x_2(a)| \leq \delta$$

Let $f : [a, b] \times \mathbb{R}^n \to \mathbb{R}^n$ be a function that is Lipschitz in the second variable such that for $t \in [a, b]$ and $\xi_1, \xi_2 \in \mathbb{R}^n$,

$$|f(t, \xi_1) - f(t, \xi_2)| \leq L|\xi_1 - \xi_2|$$

Assume that for $t \in [a, b]$,

$$|x_1'(t) - f(t, x_1(t))| \leq \epsilon_1$$
$$|x_2'(t) - f(t, x_2(t))| \leq \epsilon_2$$

Then,

$$|x_1(t) - x_2(t)| \leq \delta e^{L(t-a)} + (\epsilon_1 + \epsilon_2)\frac{e^{L(t-a)} - 1}{L}$$

2.3 Corollary (to 3.2, when $\delta = 0$)

Source: MAT267 Lecture 18, HSD Chapter 17.5

Let $A(t)$ be a continuous family of $n \times n$ matrices. Let $(t_0, X_0) \in I \times \mathbb{R}^n$, Then, the initial value problem

$$X' = A(t)X, \qquad\qquad X(t_0) = X_0$$

has a unique solution on all of I. Moreover, the solution is given by

$$X(t) = X_0 \exp\Big(\int_{t_0}^{t} A(s)ds\Big)$$

2.4 Proposition: approximation of nearby solutions

Source: MAT267 Lecture 18, HSD 17.6

Let J be the closed interval containing 0 on which $X(t)$ solves the initial value problem

$$\begin{aligned} x' &= f(x) \\ x(0) &= x_0 \end{aligned}$$

Then, $u(t,\xi)$ solves the variational equation

$$\begin{aligned} u' &= A(t)u \\ u(0,\xi) &= \xi \end{aligned}$$

where $A(t) = Df(x(t))$. Assume $\xi, x_0 + \xi \in D$ where $f : D \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is defined. Assume $y(t)$ solves

$$\begin{aligned} y'(t) &= f(y) \\ y(0) &= x_0 + \xi \end{aligned}$$

Then,

$$\lim_{\xi \to 0} \frac{|y(t,\xi) - x(t) - u(t,\xi)|}{|\xi|}$$

converges to 0 uniformly for $t \in J$. Alternatively,

$$\forall \epsilon > 0 \exists \delta > 0 : |\xi| \le \delta \implies |y(t,\xi) - x(t) - u(t,\xi)| \le \epsilon|\xi| \qquad (\forall t \in J)$$

2.5 Theorem: the flow of $x' = f(x)$ for $f \in C^1$ is C^1

If $\phi(t, X)$ is the flow of $x' = f(x)$ where $f : D \to \mathbb{R}^n$ is C^1, then the flow is a C^1 function, that is $\frac{\partial \phi}{\partial t}$ and $\frac{\partial \phi}{\partial x}$ exist and are continuou in t and X.

2.6 Definition: stable equilibrium

Source: MAT267 Lecture 20, HSD 8.4

Let $(\Phi_t)_{t \geq 0}$ be a dynamical system on $D \subseteq \mathbb{R}^n$. In other words,

$$\Phi_0 = I \text{ and } \Phi_{s \circ t} = \Phi_s \circ \Phi_t$$

Let a be an equilibrium (also known as a steady state or a fixed point). Alternatively, let a be such that $\forall t. \Phi_t(a) = a$. We say that a is stable if

$$\forall \epsilon > 0. \exists \delta > 0 : |x - a| < \delta \implies \sup_{t \geq 0} |\Phi_t(x) - a| < \epsilon$$

2.7 Definition: unstable equilibrium

Source: MAT267 Lecture 20, HSD 8.4

The equilibrium at $x(t) \equiv a$ is unstable if

$$\exists \epsilon > 0 \forall \delta > 0 : |x - a| < \delta \text{ AND } \sup_{t \geq 0} |\Phi_t(x) - a| \geq \epsilon$$

In other words, we need to find a sequence of points $\{x_j\}$ that converges to a such that $\sup_{t \geq 0} |\Phi_t(x_j) - a| \geq \epsilon$ for all $j = 1, 2, 3, \cdots$

2.8 Definition: asymptotically stable

Source: MAT267 Lecture 20, HSD 8.4

The equilibrium at $x(t) \equiv a$ is asymptotically stable (also known as attractive) if

1. a is stable
2. $\exists \delta > 0 \forall x : |x - a| < \delta \implies \lim_{t \to \infty} |\Phi_t(x) - a| = 0$.

2.9 Definition: hyperbolic

Source: HSD 4.2, MAT267 Lecture 20

A matrix A is hyperbolic if none of its eigenvalues have nonzero real part. The system $X' = AX$ is then also called hyperbolic.

2.10 Theorem: Hartman-Grobman / linearization

Source: MAT267 Lecture 20, HSD Page 168, Wikipedia

Consider $x' = f(x)$ with $f \in C^1$ and an equilibrium point at a. Let $A = Df(a)$. Assume that A is hyperbolic. Let Φ_t be the dynamical system for $x' = f(x)$ and $\phi_t = e^{tA}$ the dynamical system for its linearization. Then, there exists a neighbourhood $U \ni a$ and $V \ni 0$ such that there exists a homeomorphism $h : U \to V$ with

$$\Phi_t = h^{-1} \circ \phi_t \circ h$$

on U if t is small enough.

2.11 Lemma: change of coordinates at equilibrium

Source: HSD Q8.11, MAT267 Lecture 20

Consider the system $X' = F(X)$ where $X \in \mathbb{R}^n$. Suppose that F has an equilibrium point at X_0. Then, there is a change of coordinates that moves X_0 to the origin and converts the system to

$$X' = AX + G(X)$$

where A is an $n \times n$ matrix, which is the canonical form of $DF(X_0)$ and where $G(X)$ satisfies

$$\lim_{|X| \to 0} \frac{|G(X)|}{|X|} = 0$$

2.12 Lemma: negative distinct eigenvalues $\iff$ asymptotically stable

Source: MAT267 Lecture 20

Consider a system $x = f(x)$ with $f \in C^1$ and an equilibrium at a. If $A = Df(a)$ has negative and distinct real eigenvalues, then a is asymptotically stable (also known as a sink).

2.13 Definition: Lyapunov functions

Source: MAT267 Lecture 20

A function $L : U \subseteq \mathbb{R}^n \to \mathbb{R}^n$ is called a Lyapnunov function for a system $x' = f(x)$ on a neighbourhood $U \ni 0$ if L is nonincreasing along solutions. In other words,

$$\frac{d}{dt}L(x(t)) \leq 0$$

We say that the Lyapunov function is strict if

$$\frac{d}{dt}L(x(t)) < 0$$

2.14 Theorem: implicit function theorem

Source: MAT257

Given $f : \mathbb{R}^n_x \times \mathbb{R}^k_y \to \mathbb{R}^k$ continuouly differentiable near $(a, b) \in \mathbb{R}^n \times \mathbb{R}^k$ with $f(a, b) = 0$ and $\frac{\partial f}{\partial y}$ invertible. Then, there is an open neighbourhood $A \ni a$ and a neighbourhood of $B \ni b$ and a unique function (continuously differentiable) $g : A \to B$ such that $f(x, g(x)) = 0$ for all $x \in A$. Finally,

$$g' = -\left(\frac{\partial f}{\partial y}\right)^{-1} \cdot \frac{\partial f}{\partial x}$$

2.15 Definition: nullclines

For a system in the form

$$\begin{aligned} x_1' &= f_1(x_1, \cdots, x_n) \\ &\vdots \\ x_n' &= f_n(x_1, \cdots, x_n) \end{aligned}$$

the x_j nullcline is the set of points where $x_j' = f_j(x_1, \cdots, x_n) = 0$.

2.16 Definition: ω- and α- limit sets

The set of all points that are limit points of a given solution is called the set of ω-limit points, or the ω-limit set, of the solution $X(t)$. Similarly, we define the set of α-limit points, or the α-limit set, of a solution $X(t)$ to be the set of all points Z such that $\lim_{n\to\infty} X(t_n) = Z$ for some sequence $t_n \to -\infty$.

2.17 Theorem: saddle-node bifurcation

Suppose f_a is a function that depends on a in a C^∞ fashion and suppose $x' = f_a(x)$ is a first-order differential equation. If

1. $f_{a_0}(x_0) = 0$
2. $f'_{a_0}(x_0) = 0$
3. $f''_{a_0}(x_0) \neq 0$
4. $\dfrac{\partial f_{a_0}}{\partial a} \neq 0$

then this differential equation undergoes a saddle-node bifurcation at $a = a_0$.

2.18 Proposition: properties of limit sets

Source: HSD P217

1. If X and Z lie on the same solution curve, then $\omega(X) = \omega(Z)$ and $\alpha(X) = \alpha(Z)$;
2. If D is a closed, positively invariant set and $Z \in D$, then $\omega(Z) \subseteq D$, and similarly for negatively invariant sets and α-limits;
3. A closed invariant set, in particular, a limit set, contains the α-limit and ω-limit sets of every point in it.

2.19 Definition: Hamiltonian systems

Source: MAT267 Lecture 24, HSD 9.4

A Hamiltonian system on $\mathbb{R}^2$ is a system of the form

$$\begin{aligned} x' &= \frac{\partial H}{\partial y}(x, y) \\ y' &= -\frac{\partial H}{\partial x}(x, y) \end{aligned}$$

where $H : \mathbb{R}^2 \to \mathbb{R}$ is a C^∞ function called the Hamiltonian function.

2.20 Proposition: equivalent definition of Hamiltonian systems

Source: MAT267 Lecture 24

An equivalent definition of Hamiltonian systems is a differential equation of the form

$$x'' = -\nabla V(x)$$

where $V : D \to \mathbb{R}$ is a C^2 function.

2.21 Proposition: Lyapunov function for Hamiltonian systems

Source: MAT267 Lecture 24

The Lyapunov function for a Hamiltonian system is given by

$$L(x, y) = \frac{1}{2}|y|^2 + V(x)$$

This has the property that

$$\frac{dL}{dt}(x, y) = 0$$