MAT354: Complex Analysis - Notes

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3.1 Definition: polynomials, zeroes, orders

A function $f(z)$ is a polynomial if

$$\begin{aligned} f(z) &= a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0 \\ &= a_n (z - c_1)^{k_1} \cdots (z - c_n)^{k_n} \end{aligned}$$

The zeroes of the polynomial are the c_i's and each zero is said to have an order equal to the corresponding k_i.

3.2 Definition: rational functions

A function $R(z)$ is rational if

$$R(z) = \frac{P(z)}{Q(z)}$$

where P, Q are polynomials with no common factors.

3.3 Definition: poles and their orders

The poles of a rational function $R(z) = \frac{P(z)}{Q(z)}$ are precisely the zeroes of $Q(z)$. The order of the pole is the order of the corresponding zero of $Q(z)$.

4 Lecture 4

4.1 Definition: holomorphic

A function $f : \Omega \in \mathbb{C} \to \mathbb{C}$ where Ω open is said to be holomorphic at z if

$$\lim_{h \to 0} \frac{f(z+h) - f(z)}{h} = c$$

for some $c \in \mathbb{C}$.

4.2 Proposition: holomorphic $\iff$ Cauchy-Riemann equations

$f(z)$ is holomorphic at z if and only if the partial derivatives with respect to x, y ($z = x+iy$) satisfy the Cauchy-Riemann equations

$$\frac{\partial f}{\partial x} + i\frac{\partial f}{\partial y} = 0$$

4.3 Example: upper half plane to unit disk

The fractional linear transformation

$$\frac{z-i}{z+i}$$

maps the upper half plane to the unit disk

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10 Analytic functions

10.1 Definition: analytic

$f(z)$ analytic in open set Ω if it has a convergent power series representation at every point $z_0 \in \Omega$. In other words, for all $z_0 \in \Omega$, there exists a convergent power series

$$\sum_{n=0}^{\infty} a_n (z - z_0)^n$$

such that for all $|z - z_0| < r$ for some $r \leq$ radius of convergence

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

10.2 Proposition: primitive of an analytic function

If $f(z)$ has a convergent power series representation at z_0

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

then there is a convergent power series $g(z)$ at z_0 such that $g'(z) = f(z)$ in some disk $|z - z_0| < r$. Moreover, $g(z)$ is given by

$$g(z) = \sum_{n=0}^{\infty} \frac{a_n}{n+1} (z - z_0)^{n+1}$$

has the same radius of convergence as $f(z)$ and is uniquely determined up to a constant.

10.3 Proposition: convergent power series defines analytic function

If $f(z) = \sum a_n(z - z_0)^n$ is a convergent power series with radius of convergence R, then $f(z)$ is analytic in the disk $|z - z_0| < R$.

10.4 Proposition: Every analytic function is holomorphic

10.5 Proposition: zeroes of not identically zero analytic function are isolated

If f analytic and not identically zero, then its zeroes are isolated. We say that $x \in X$ is an isolated point of X if x has a neighbbourhood whose intersection with X reduces to the point x.

10.6 Definition: meromorphic

A function f is meromorphic in an open set Ω if it is well-defined and analytic in the complement of a discrete set. Moreover, in a neighbourhood of any point in Ω, it can be expressed as the quotient of two analytic functions $\frac{f}{g}$ with g not identically zero.

11 Lecture 11

11.1 Definition: integral of one-form over a curve

$$\int_\gamma \omega = \int_a^b F(t)dt$$

where $\omega = Pdx + Qdy$ (P, Q continuous functions on Ω) and $F(t) = P(x(t), y(t))x'(t) + Q(x(t), y(t))y'(t)$.

11.2 Lemma: points in domain can be connected by piecewise C^1 curve

Any two points in a connected and open $\Omega \in \mathbb{R}^2$ can be connected by a piecewise C^1 curve.

11.3 Proposition: primitive of one-form

A one-form ω has a primitive if and only if $\int_\gamma w = 0$ for every piecewise C^1 closed curve γ.

12 Lecture 12

12.1 Cauchy's Theorem

If $f(z)$ is a holomorphic in open $\Omega \in \mathbb{C}$, then $f(z)dz$ is closed.

12.2 Corollary 1 (of Cauchy's theorem):

A holomorphic function in open $\Omega \in \mathbb{C}$ locally has a primitive which is holomorphic.

12.3 Corollary 2 (of Cauchy's theorem): Generalisation of Cauchy's theorem

In Cauchy's theorem, it is enough to assume that f is continuous in Ω and holomorphic outside a line parallel to the x-axis.

13 Lecture 13

13.1 Definition: homotopic

If $\gamma_0, \gamma_1 : [0,1] \to \Omega$ are two continuous closed curves with the same endpoints

$$\gamma_0(0) = \gamma_1(0), \qquad \gamma_0(1) = \gamma_1(1)$$

then they are said to be homotopic in Ω with fixed endpoints if there is a continuous function $\gamma : I_s \times I_t \to \Omega$ such that

$$\begin{aligned} \gamma(0,t) &= \gamma_0(t), & \gamma(1,t) &= \gamma_1(t) \\ \gamma(s,0) &= \gamma_0(0) = \gamma_1(0), & \gamma(s,1) &= \gamma_0(1) = \gamma_1(1) \end{aligned}$$

If γ_1 is constant, we say that the curves are homotopic as closed curves or that they are homotopic to a point.

13.2 Theorem: invariance of integral of closed form under homotopy

If ω is a closed differential form in Ω and $\gamma_0, \gamma_1 : [0,1] \to \Omega$ are continuous homotopic curves with fixed endpoints or as closed curves, then

$$\int_{\gamma_0} \omega = \int_{\gamma_1} \omega$$

14 Lecture 14

14.1 Lemma (for homotopy integral invariance theorem)

Let ω be a closed form in Ω and $\gamma : [a,b] \times [c,d] \to \Omega$ a continuous curve. Then, there is a continuous function $f : [a,b] \times [c,d] \to \mathbb{C}$ such that for every $(s_0, t_0) \in [a,b] \times [c,d]$, there is a primitive F of ω defined in a neighbourhood of $\gamma(s_0, t_0)$ such that $f(s,t) = F(\gamma(s,t))$ in some neighbourhood of (s_0, t_0). Moreover, f is unique up to addition of a constant.

14.2 Definition: simply connected

Ω is simply-connected if Ω is connected and any closed curve in Ω is null-homotopic.

14.3 Corollary: simply-connected

In simply-connected open sets, any closed form has a primitive.

14.4 Definition: winding number

The winding number of a closed curve γ with regards to any point $a \notin \gamma$ is defined as

$$w(\gamma, a) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z-a} \in \mathbb{Z}$$

14.5 Theorem: Cauchy's Integral Formula

Let $\Omega \in \mathbb{C}$ open, $a \in C$, $f(z)$ holomorphic in Ω and γ a closed curve in Ω not containing a, homotopic to a point in Ω. Then,

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = w(\gamma, a)f(a)$$

14.6 Corollary (Cauchy's Integral Formula)

If $f(z)$ is holomorphic in some neighbourhood of a closed disk D and γ is the boundary of the disk (in positive sense), then

$$\frac{1}{2\pi i}\int_\gamma \frac{f(z)}{z-a}dz = \begin{cases} f(a) & a \text{ in circle} \\ 0 & a \text{ not in circle} \end{cases}$$

14.7 Cauchy's Theorem (General)

Consider $f(z)$ continuous in Ω. TFAE

1. $f(z)$ is holomorphic in Ω
2. $f(z)dz$ is closed
3. $f(z) = \frac{1}{2\pi i}\int_\gamma \frac{f(\xi)}{(\xi - z)}d\xi$ when $z \in$ interior of closed disk $D \in \Omega$ with oriented boundary γ.

14.8 Morera's theorem

If $f(z)dz$ is closed then $f(z)$ locally has a primitive $g(z)$ which is holomorphic.

14.9 Corollary: Morera's theorem

Continuous function which is holomorphic except on a line is holomorphic everywhere.

15 Lecture 15

15.1 Theorem: Holomorphic functions have convergent power series expansions

A holomorphic function $f(z)$ in a disk $|z| < R$ has a convergent power series expansion in the disk.

15.2 Corollary: Every holomorphic function is analytic

15.3 Definition: Cauchy's inequalities

TODO

15.4 Liouville's theorem

A bounded holomorphic function in $\mathbb{C}$ is constant.

15.5 Fundamental Theorem of Algebra

Every non-constant polynomial has at least one root.

15.6 Schwarz's Reflection Principle

TODO

16 Lecture 16

17 Lecture 17

17.1 Definition: isolated singularity

A holomorphic function $f(z)$ in a punctured disk $0 < |z| < R$, has an isolated singularity at 0 if f cannot be extended to be holomorphic in $|z| < R$.

The isolated singularity is a pole if there are only finitely many negative exponents in the Laurent series, and is an essential singularity if there are infinitely many such exponents.

17.2 Theorem (Weierstrass)

If 0 is an essential singularity of f then for any $\epsilon > 0$, $f(0 < |z| < \epsilon)$ is dense in $\mathbb{C}$.

17.3 Picard's Theorem

The holomorphic image of a punctured disk, $f(0 < |z| < \epsilon)$ omits at most one value from $\mathbb{C}$.

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19 Lecture 19