# PHY472: Introduction to String Theory - Homework

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# Contents

## 1 Zwiebach's 6.7: Time evolution of a closed circular string

At $t = 0$, a closed string forms a circle of radius $R$ on the $(x, y)$ plane and has zero velocity. The time development of this string can be studied using the action

$$S = -T_0 \int dt \int_0^{\sigma_1} ds \left(\frac{ds}{d\sigma}\right) \sqrt{1 - \frac{v_\perp^2}{c^2}} \qquad \text{(Zwiebach 6.88)}$$

The string will remain circular, but its radius will be a time-dependent function $R(t)$. Give the Lagrangian $L$ as a function of $R(t)$ and its time derivative. Calculate the radius and velocity as functions of time. Sketch the spacetime surface traced by the string in a three-dimensional plot with $x, y$, and $ct$ axes. [Hint: calculate the Hamiltonian associated with $L$ and use energy conservation.]

### Solution

Since the action is the time integral of the Lagrangian, we have

$$L(t) = -T_0 \int_0^{\sigma_1} ds \sqrt{1 - \frac{v_\perp^2}{c^2}} \qquad \text{(Zwiebach 6.89)}$$

Since the string remains circular, we have that its transverse velocity will be given by the change in its radius

$$|v_\perp| = \left|\frac{dR(t)}{dt}\right| = |\dot{R}(t)|$$

and that the integral of the parameter $s$, which measures length along the string will evaluate to the circumference $2\pi R(t)$. Hence,

$$L(R, \dot{R}, t) = -2\pi T_0 R \sqrt{1 - \frac{\dot{R}^2}{c^2}}$$

Recall that the canonical momentum conjugate to $\dot{R}$ is given by

$$p = \frac{\partial L}{\partial \dot{R}} = \left[\frac{-2\pi T_0 R}{2}\left(1 - \frac{\dot{R}^2}{c^2}\right)^{-1/2}\left(-\frac{2\dot{R}}{c^2}\right)\right] = \frac{2\pi T_0 R \dot{R}}{c^2\sqrt{1 - \frac{\dot{R}^2}{c^2}}}$$

which allows us to write the Hamiltonian

$$H = p\dot{R} - L = \frac{2\pi T_0 R \dot{R}}{c^2\sqrt{1 - \frac{\dot{R}^2}{c^2}}}\dot{R} + 2\pi T_0 R \sqrt{1 - \frac{\dot{R}^2}{c^2}}$$

$$= \frac{2\pi T_0 R}{c^2}\left(\frac{\dot{R}^2}{\sqrt{1-\frac{\dot{R}^2}{c^2}}} + \frac{c^2-\dot{R}^2}{\sqrt{1-\frac{\dot{R}^2}{c^2}}}\right)$$
$$= \frac{2\pi T_0 R}{\sqrt{1-\frac{\dot{R}^2}{c^2}}}$$

At $t = 0$, we are given that $\dot{R}(0) = 0$. Assume that $R(0) = R_0$. Then,

$$H(0) = 2\pi T_0 R_0 = C$$

for some constant $C$. By conservation of energy, $H(t) = C = 2\pi T_0 R_0$ for all time.

$$H(t) - 2\pi T_0 R_0 = 0 \implies R^2 = R_0^2\left(1 - \frac{\dot{R}^2}{c^2}\right)$$
$$\frac{c^2}{R_0^2}R^2 + \dot{R}^2 = c^2$$

Note that

$$R(t) = R_0 \cos\left(\frac{c}{R_0}t\right) \implies \dot{R}(t) = -c\sin\left(\frac{c}{R_0}t\right)$$

solves the differential equation above.

The sketch of the spacetime traced by the string is provided in Figure 1 below.

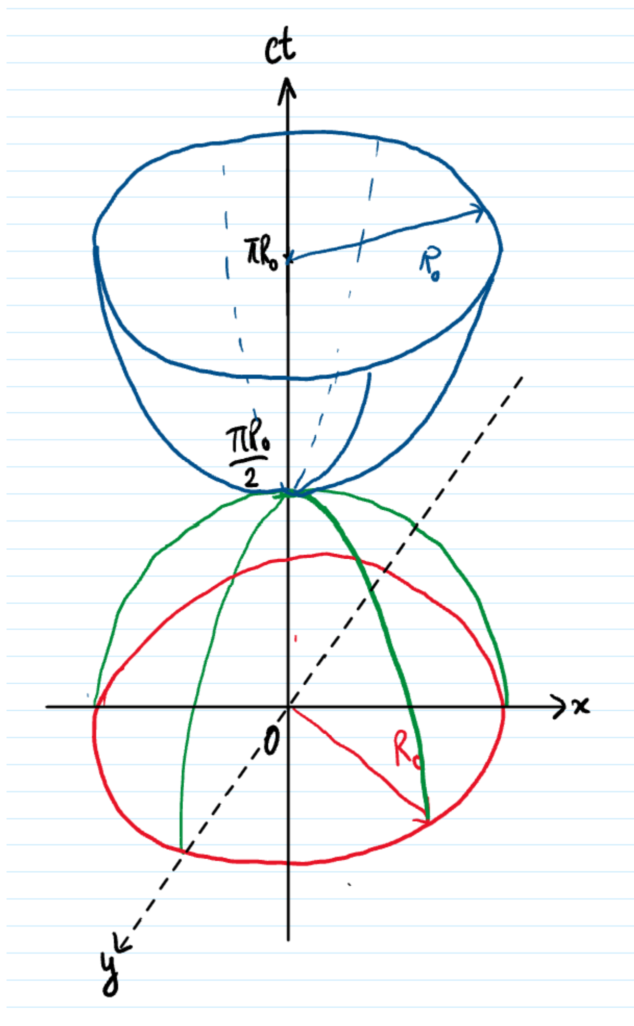



**Figure 1:** Sketch of the spacetime traced by the string in Zwiebach's 6.7

## 2 Zwiebach's 7.4: Kasey's relativistic jumping rope

Consider a relativistic open string with fixed endpoints:

$$\vec{X}(t,0) = \vec{x_1}, \qquad \vec{X}(t,\sigma_1) = \vec{x_2} \tag{1}$$

The boundary condition at $\sigma = 0$ is satisfied by the following solution to the wave equation:

$$\vec{X}(t,\sigma) = \vec{x_1} + \frac{1}{2}\left(\vec{F}(ct+\sigma) - \vec{F}(ct-\sigma)\right) \tag{2}$$

Here $\vec{F}$ is a vector function of a single variable.

(a) Use (2) and the boundary condition at $\sigma = \sigma_1$ to find a condition on $\vec{F}(u)$.

(b) Write down the constraint on $\vec{F}(u)$ that arises from the parameterization conditions (7.42)

$$\left(\frac{\partial \vec{X}}{\partial \sigma} \pm \frac{1}{c}\frac{\partial \vec{X}}{\partial t}\right)^2 = 1 \tag{Zwiebach 7.42}$$

As an application, consider Kasey's attempts to use a relativistic open string as a jumping rope. For this purpose, she holds the open string (in three spatial dimensions) with her right hand at the origin $\vec{x_1} = (0,0,0)$ and with her left hand at the point $z = L_0$ on the $z$ axis, or $\vec{x_2} = (0,0,L_0)$. As she starts jumping we observe that the tangent vector $\vec{X}'$ to the string at the origin rotates with constant angular velocity around the $z$ axis forming an angle $\gamma$ with it.

(c) Use the above information to write an expression for $\vec{F}'(u)$.

(d) Find $\sigma_1$ in terms of the length $L_0$ and the angle $\gamma$.

(e) Calculate $\vec{X}(t,\sigma)$ for the motion of Kasey's relativistic jumping rope.

(f) How is the energy distributed in the string as a function of $z$?

### Solution

Note that we have omitted the vector signs over all variables.

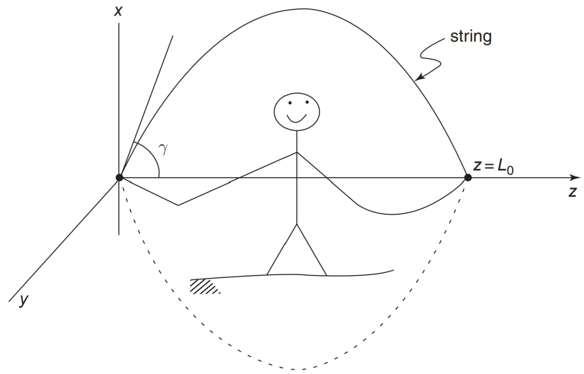



**Figure 2:** Zwiebach's 7.4: Kasey's relativistic jumping rope

(a) We have that

$$x_2 = x_1 + \frac{1}{2}\left(F(ct+\sigma_1) - F(ct-\sigma_1)\right)$$

If we set $u = ct - \sigma_1$, we can rewrite the above equation as a recursive equation

$$F(u+2\sigma_1) - F(u) = 2(x_2 - x_1) \tag{3}$$

The most general form for $F(u)$ is then given by

$$F(u) = G(u) + H(u)(x_2 - x_1)$$

where $G(u)$ and $H(u)$ are arbitrary vector and scalar functions of $u$ respectively. The recursive condition imposes the following conditions on $G$ and $H$

$$G(u) = G(u+2\sigma_1), \qquad H(u) = \frac{u}{\sigma_1}$$

(b) Note the derivatives of $X(t,\sigma)$

$$\frac{\partial X}{\partial t} = \frac{c}{2}\left(F'(ct+\sigma) - F'(ct-\sigma)\right)$$
$$\frac{\partial X}{\partial \sigma} = \frac{1}{2}\left(F'(ct+\sigma) + F'(ct-\sigma)\right)$$

Imposing the parameterization conditions, we get

$$[F'(ct+\sigma)]^2 = 1, \qquad [F'(ct-\sigma)]^2 = 1$$

Since $\sigma$ takes on continuous values, we conclude that

$$|F'(u)| = 1$$

(c) We are given that the string makes an angle $\gamma$ with the $z$ axis (as illustrated in Figure 2) and has angular velocity $\omega$.

Hence,

$$\begin{aligned}\left.\frac{\partial X}{\partial \sigma}\right|_{\sigma=0} &= F'(ct) = (\sin\gamma\cos\omega t, \sin\gamma\sin\omega t, \cos\gamma) \\ \implies F'(u) &= (\sin\gamma\cos\frac{\omega u}{c}, \sin\gamma\sin\frac{\omega u}{c}, \cos\gamma) \end{aligned} \tag{4}$$

(d) By Eq. (3), we must have that $F'(u)$ is $2\sigma_1$ periodic. In other words, we have the constraint

$$\begin{aligned}\cos\left(\frac{\omega u}{c}\right) &= \cos\left(\frac{\omega(u+2\sigma_1)}{c}\right) \\ &= \cos\left(\frac{\omega u}{c}\right)\cos\left(\frac{2\omega\sigma_1}{c}\right) - \sin\left(\frac{\omega u}{c}\right)\sin\left(\frac{2\omega\sigma_1}{c}\right) \\ &= \cos\left(\frac{\omega u}{c}\right) \qquad\qquad (\text{with } \sigma_1 = \frac{\pi c}{\omega})\end{aligned}$$

which is satisfied when $\sigma_1 = \frac{\pi c}{\omega}$. We are given that

$$F(2\sigma_1) - F(0) = (0, 0, 2L_0)$$

Consider the $z$-component of $F'(u)$. Note that

$$\int_0^{2\sigma_1} F_z'(u)du = 2\sigma_1\cos\gamma$$

but we also have that

$$\int_0^{2\sigma_1} F_z'(u)du = F_z(2\sigma_1) - F_z(0) = 2L_0$$

which finally gives

$$\sigma_1 = \frac{L_0}{\cos\gamma} \tag{5}$$

(e) Integrating Eq. (4) with respect to $u$ gives

$$F(u) = \left(\frac{c}{\omega}\sin\gamma\sin\frac{\omega u}{c}, -\frac{c}{\omega}\sin\gamma\cos\frac{\omega u}{c}, u\cos\gamma\right)$$

Recall the trigonometric identities

$$\sin(a+b) - \sin(a-b) = 2\cos a\sin b$$

$$\cos(a+b) - \cos(a-b) = -2\sin a \sin b$$

Given that $x_1 = (0,0,0)$, we can use Eq. (2) and the above identities to write

$$\begin{aligned}
X(t,\sigma) &= \frac{1}{2}[F(ct+\sigma) - F(ct-\sigma)] \\
&= \frac{1}{2}\left(\frac{2c}{\omega}\sin\gamma\left(\cos\omega t \sin\frac{\omega\sigma}{c}\right), \frac{2c}{\omega}\sin\gamma\left(\sin\omega t\sin\frac{\omega\sigma}{c}\right), 2\sigma\cos\gamma\right)
\end{aligned}$$

(f) Consider

$$\begin{aligned}
dz &= \cos\gamma d\sigma \\
&= \cos\gamma\frac{dE}{T_0} && \text{(Zwiebach 7.19)} \\
&= \frac{L_0}{\sigma_1}\frac{dE}{T_0} && \text{(Eq. (5))} \\
&= \frac{L_0}{E}dE && \text{(Zwiebach 7.21)} \\
\implies \ln E &= \frac{z}{L_0} \implies E(z) = e^{z/L_0}
\end{aligned}$$

## 3 Zwiebach's 9.3: Rotating open string in the light cone gauge

(a) From Zwiebach's Eq. (9.83), we know that

$$M^2 = \frac{1}{\alpha'} \sum_{n=1}^{\infty} n \alpha_n^{I*} \alpha_n^I$$

We are given $\alpha_1^I = (a, ia, 0, \cdots)$, $\alpha_{-1}^I = (a, -ia, 0, \cdots)$ and $\alpha_n^I = 0$ for all other $n \in \mathbb{Z}$. Hence,

$$M^2 = \frac{1}{\alpha'} 2a^2 \implies M = \sqrt{\frac{2}{\alpha'}} a$$

(b) From Zwiebach's Eq. (9.69), we have

$$X^I(\tau, \sigma) = x_0^I + \sqrt{2\alpha'} \alpha_0^I \tau + i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^I e^{-in\tau} \cos n\sigma$$

We are given that $x_0^I = 0$ and that $\alpha_n^I = 0$ for all $n \neq 1, -1$. This leaves us with

$$\begin{aligned} X^I(\tau, \sigma) &= i\sqrt{2\alpha'} \sum_{n \neq 0} \frac{1}{n} \alpha_n^I e^{-in\tau} \cos n\sigma \\ &= i\sqrt{2\alpha'} \left( \alpha_1^I e^{-i\tau} - \alpha_{-1}^I e^{i\tau} \right) \cos \sigma \end{aligned}$$

This directly gives

$$\begin{aligned} X^2(\tau, \sigma) &= i\sqrt{2\alpha'} a \left( e^{-i\tau} - e^{i\tau} \right) \cos \sigma = 2\sqrt{2\alpha'} a \sin \tau \cos \sigma \\ X^3(\tau, \sigma) &= -\sqrt{2\alpha'} a \left( e^{-i\tau} + e^{i\tau} \right) \cos \sigma = -2\sqrt{2\alpha'} a \cos \tau \cos \sigma \end{aligned}$$

Now, the length of the string is given by

$$\ell = 4a\sqrt{2\alpha'}$$

(c) From Zwiebach's 9.77, we know that the transverse Virasoro modes $L_n^\perp$ are given by

$$L_n^\perp = \frac{1}{2} \sum_{p \in \mathbb{Z}} \alpha_{n-p}^I \alpha_p^I$$

Since only $\alpha_1^I$ and $\alpha_{-1}^I$ are nonzero, the only potentially nonzero transverse Virasoro modes are $L_0^\perp$ and $L_{\pm 2}^\perp$. They are

$$L_0^\perp = \frac{1}{2}(\alpha_{-1}^I \alpha_1^I + \alpha_1^I \alpha_{-1}^I) = \frac{1}{2}(a^2 + a^2 + a^2 + a^2) = 2a^2$$

$$L_2^\perp = \frac{1}{2}(\alpha_1^I)^2 = \frac{1}{2}(a^2 - a^2) = 0$$
$$L_{-2}^\perp = \frac{1}{2}(\alpha_{-1}^I)^2 = \frac{1}{2}(a^2 - a^2) = 0$$

Finally, from Zwiebach's 9.80,

$$\begin{aligned} X^-(\tau,\sigma) &= x_0^- + \frac{1}{p^+}L_0^\perp\tau + \frac{i}{p^+}\sum_{n\neq 0}\frac{1}{n}L_n^\perp e^{-in\tau}\cos n\sigma \\ &= \frac{2a^2\tau}{p^+} \qquad\qquad (x_0^- = L_{n\neq 0}^\perp = 0) \end{aligned}$$

(d) From Zwiebach's 9.62, we know that for open strings,

$$X^+(\tau,\sigma) = 2\alpha' p^+\tau$$

To have $X^1 = 0$, we must have that

$$X^+ = X^- \implies \alpha' p^+ = \frac{a^2}{p^+} \implies p^+ = \frac{a}{\sqrt{\alpha'}}$$

With $X^1 = 0$, we have that $X^\pm = X^0/\sqrt{2} = t/\sqrt{2}$. This gives us a relation between $t$ and $\tau$:

$$t = 2\sqrt{2}\alpha' p^+\tau = 2a\sqrt{2\alpha'}\tau$$

(e) Zwiebach's 7.59 states that

$$\ell = \frac{2c}{\omega} = \frac{2E}{\pi T_0}$$

We know that

$$\omega = \frac{\tau}{t} = \frac{1}{2a\sqrt{2\alpha'}}$$

which allows us to verify

$$\ell = 4a\sqrt{2\alpha'} = \frac{2}{\omega}$$

in natural units. From Zwiebach's 8.76, we also know that

$$T_0 = \frac{1}{2\pi\alpha'} \implies \frac{\ell}{E} = 4\alpha'$$

which is true if we check with $E = M$ as above.

## 4 Zwiebach's 10.6: Field equations and particle states for Kalb-Ramond field

(a) Let's swap the first two indices

$$\begin{aligned} H_{\nu\mu\rho} &= \partial_\nu B_{\mu\rho} + \partial_\mu B_{\rho\nu} + \partial_\rho B_{\nu\mu} \\ &= -\partial_\mu B_{\nu\rho} - \partial_\nu B_{\rho\mu} - \partial_\rho B_{\mu\nu} \\ &= -H_{\mu\nu\rho} \end{aligned}$$

Similarly, we swap the last two indices

$$\begin{aligned} H_{\mu\rho\nu} &= \partial_\mu B_{\rho\nu} + \partial_\rho B_{\nu\mu} + \partial_\nu B_{\mu\rho} \\ &= -\partial_\mu B_{\nu\rho} - \partial_\nu B_{\rho\mu} - \partial_\rho B_{\mu\nu} \\ &= -H_{\mu\nu\rho} \end{aligned}$$

Finally, the first and the third

$$\begin{aligned} H_{\rho\nu\mu} &= \partial_\rho B_{\nu\mu} + \partial_\nu B_{\mu\rho} + \partial_\mu B_{\rho\nu} \\ &= -\partial_\mu B_{\nu\rho} - \partial_\nu B_{\rho\mu} - \partial_\rho B_{\mu\nu} \\ &= -H_{\mu\nu\rho} \end{aligned}$$

showing that $H_{\mu\nu\rho}$ is totally antisymmetric. To prove that $H_{\mu\nu\rho}$ is invariant under the gauge transformation, we check

$$\begin{aligned} \delta H_{\mu\nu\rho} &= \partial_\mu \delta B_{\nu\rho} + \partial_\nu \delta B_{\rho\mu} + \partial_\rho \delta B_{\mu\nu} \\ &= \partial_\mu(\partial_\nu \epsilon_\rho - \partial_\rho \epsilon_\nu) + \partial_\nu(\partial_\rho \epsilon_\mu - \partial_\mu \epsilon_\rho) + \partial_\rho(\partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu) \end{aligned}$$

Since partials commute, the above expression goes to 0, as desired.

(b) Using $\epsilon'_\mu = \epsilon_\mu + \partial_\mu \lambda$, we check

$$\begin{aligned} \delta' B_{\mu\nu} &= \partial_\mu \epsilon'_\nu - \partial_\nu \epsilon'_\mu \\ &= \partial_\mu(\epsilon_\nu + \partial_\nu \lambda) - \partial_\nu(\epsilon_\mu + \partial_\mu \lambda) \\ &= \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu = \delta B_{\mu\nu} \end{aligned}$$

Hence, $\epsilon'_\mu$ and $\epsilon_\mu$ generate the same gauge transformation.

(c) In momentum space, we have

$$\begin{aligned} \epsilon^{\mu\prime}(p) &= \epsilon^\mu(p) + ip^\mu \lambda(p) \\ \implies \epsilon^{+\prime}(p) &= \epsilon^+(p) + ip^+ \lambda(p) \end{aligned}$$

and making the choice

$$\lambda(p) = i\frac{\epsilon^+(p)}{p^+} \implies {\epsilon^+}'(p) = 0$$

as desired.

(d) We vary the action

$$\begin{aligned}
\delta S &= -\frac{1}{3}\int d^D x \,(H^{\mu\nu\rho}\delta H_{\mu\nu\rho}) \\
&= -\frac{1}{3}\int d^D x H^{\mu\nu\rho}(\partial_\mu \delta B_{\nu\rho} + \partial_\nu \delta B_{\rho\mu} + \partial_\rho \delta B_{\mu\nu}) \\
&= -\int d^D x H^{\mu\nu\rho}\partial_\mu \delta B_{\nu\rho} \qquad \text{(reordering and relabelling)}
\end{aligned}$$

Since we want this quantity to be 0 for all $\delta B_{\nu\rho}$, we need $\partial_\mu H^{\mu\nu\rho}$ to be vanish

$$\begin{aligned}
\partial_\mu H^{\mu\nu\rho} &= 0 \\
\partial_\mu(\partial^\mu B^{\nu\rho} + \partial^\nu B^{\rho\mu} + \partial^\rho B^{\mu\nu}) &= 0
\end{aligned}$$

In momentum space, this is

$$p^2 B^{\nu\rho} + p_\mu p^\nu B^{\rho\mu} + p_\mu p^\rho B^{\mu\nu} = 0$$

(e) By antisymmetry of $B^{\mu\nu}$, we have that $B^{++} = 0$. We want to choose $\epsilon^\mu(p)$ such that we also have $B^{+-} = B^{+I} = 0$.

Since we have $\delta B^{\mu\nu} = \partial_\mu \epsilon_\nu - \partial_\nu \epsilon_\mu$, we know that

$$\begin{aligned}
\delta B^{++} &= 0 \\
\delta B^{+-} &= \partial_+ \epsilon_- - \partial_- \epsilon_+ \\
\delta B^{+I} &= \partial_+ \epsilon_I - \partial_I \epsilon_+
\end{aligned}$$

Hence, we can take $\epsilon^+ = 0$ and then the equations above determine $\epsilon_-$ and $\epsilon_I$ completely. Now, the only non-zero field components are $B^{--}, B^{IJ}, B^{-I}$.

In this gauge, define $A^\mu = p_\nu B^{\mu\nu}$, which gives

$$A^+ = 0, \qquad A^- = p_I B^{-I} + p_- B^{--}, \qquad A^I = p_J B^{IJ} + p_- B^{I-}$$

From the equation of motion derived in (d), we can now write

$$p^2 B^{\nu\rho} = -p^\nu A^\rho - p^\rho A^\nu$$

When $\nu\rho = +-$, we get $p^+ A^- = 0 \implies A^- = 0$

When $\nu\rho = +I$, we get $p^+ A^I = 0 \implies A^I = 0$.

Hence, $A^\mu = 0 \implies p^2 B^{\nu\rho} = 0$.

Finally, $A^- = 0 \implies B^{-I} \propto B^{--}$ and $A^I \implies B^{IJ} \propto B^{-J}$, implying that there is only one degree of freedom in the components of $B^{\mu\nu}$, namely $B^{-I}$.

(f) Since each of the independent fields $B^{\mu\nu}$ can be expanded in terms of the oscillatory modes, we can write the one-particle states of the Kalb-Ramond field as

$$\sum_{I,J=2}^{D-1} \xi_{IJ} a_p^{IJ\dagger} |\Omega\rangle$$

Since $a^{IJ\dagger} = -a^{JI\dagger}$, we conclude that $\xi_{IJ}$ is an antisymmetric matrix.

## 5 Zwiebach's 11.7: Transformations generated by light-cone gauge Lorentz generators $M^{+-}$ and $M^{-I}$

It is useful to know the following commutation relations

$$\begin{aligned}
[p^-, p^+] = [p^+, p^I] = [p^-, p^I] = 0 \\
[x_0^-, p^I] = [x_0^-, x_0^I] = [p^+, x_0^I] = 0 \\
[x_0^-, p^-] = i\frac{p^-}{p^+} \\
[x_0^-, p^+] = -i \\
[x_0^I, p^-] = i\frac{p^I}{p^+} \\
[x_0^-, \frac{1}{p^+}] = \frac{i}{p^{+2}}
\end{aligned}$$

(a) $M^{+-}$ is given by

$$M^{+-} = -\frac{1}{2}(x_0^- p^+ + p^+ x_0^-) = -x_0^- p^+ - \frac{1}{2}i$$

where we used $[x_0^-, p^+] = -i$ in the final step. Note that

$$x^+(\tau) = \frac{p^+\tau}{m^2}, \quad x^-(\tau) = x_0^- + \frac{p^-\tau}{m^2} \qquad \text{(Zwiebach's 11.63)}$$

$$x^I(\tau) = x_0^I + \frac{p^I}{m^2}\tau \qquad \text{(Zwiebach's 11.15)}$$

Hence, the commutators of interest are

$$\begin{aligned}
[M^{+-}, x^+(\tau)] &= [-x_0^- p^+, \frac{p^+\tau}{m^2}] = \frac{\tau}{m^2}(-x_0^- p^{+2} + p^+ x_0^- p^+) = ix^+(\tau) \\
[M^{+-}, x^-(\tau)] &= [-x_0^- p^+, x_0^- + \frac{p^-\tau}{m^2}] \\
&= x_0^-(-p^+ x_0^- + x_0^- p^+) + \frac{\tau}{m^2}[-x_0^- p^+, p^-] \\
&= -ix_0^- - \frac{\tau}{m^2} i\frac{p^-}{p^+}p^+ \\
&= -ix^-(\tau) \\
[M^{+-}, x^I(\tau)] &= [-x_0^- p^+, x_0^I + \frac{p^I}{m^2}\tau] = 0
\end{aligned}$$

$M^{+-}$ generates a Lorentz transformation of these coordinates as follows

$$\begin{aligned}
\delta x^{\mu} &= [-i\epsilon M^{+-}, x^{\mu}] \\
\implies \delta x^{+} = \epsilon x^{+}, \delta x^{-} &= -\epsilon x^{-}, \delta x^{I} = 0
\end{aligned}$$

as expected.

(b) $M^{-I}$ is defined as

$$\begin{aligned}
M^{-I} &= x_0^- p^I - \frac{1}{2}(x_0^I p^- + p^- x_0^I) \\
&= x_0^- p^I - x_0^I p^- + \frac{1}{2} i \frac{p^I}{p^+}
\end{aligned}$$

Hence, the commutators of interest are

$$\begin{aligned}
[M^{-I}, x^{+}(\tau)] &= \left[ x_0^- p^I - x_0^I p^- + \frac{1}{2} i \frac{p^I}{p^+}, \frac{p^+ \tau}{m^2} \right] \\
&= \frac{\tau}{m^2} [x_0^- p^I, p^+] \\
&= -i \frac{\tau p^I}{m^2} \\
&= i(x_0^I - x^I) \\
[M^{-I}, x^{-}(\tau)] &= \left[ x_0^- p^I - x_0^I p^- + \frac{1}{2} i \frac{p^I}{p^+}, x_0^- + \frac{p^- \tau}{m^2} \right] \\
&= -x_0^I [p^-, x_0^-] + \frac{1}{2} i [\frac{p^I}{p^+}, x_0^-] + \frac{\tau}{m^2} \left( [x_0^-, p^-] p^I + [-x_0^I, p^-] p^- \right) \\
&= i x_0^I \frac{p^-}{p^+} + \frac{1}{2} \frac{1}{p^{+2}} p^I + \frac{\tau}{m^2} \left( i \frac{p^-}{p^+} p^I - i \frac{p^I}{p^+} p^- \right) \\
&= \frac{1}{p^+} \left( i x_0^I p^- + \frac{p^I}{2p^+} \right) \\
[M^{-I}, x^{J}(\tau)] &= \left[ x_0^- p^I - x_0^I p^- + \frac{1}{2} i \frac{p^I}{p^+}, x_0^J + \frac{p^J}{m^2} \tau \right] \\
&= x_0^- [p^I, x_0^J] - x_0^I [p^-, x_0^J] + \frac{i}{2} [p^I, x_0^J] \frac{1}{p^+} - \frac{\tau}{m^2} [x_0^I, p^J] p^- \\
&= -i x_0^- \delta_{IJ} + i x_0^I \frac{p^J}{p^+} + \frac{1}{2p^+} \delta_{IJ} - i \frac{\tau}{m^2} \delta_{IJ} p^- \\
&= -i \delta_{IJ} x^-(\tau) + \frac{1}{p^+} \left( i x_0^I p^J + \frac{\delta_{IJ}}{2} \right)
\end{aligned}$$

We know that the expected Lorentz transformations are

$$\begin{aligned}
[M^{-I}, x^{+}(\tau)] &= -ix^{I}(\tau) \\
[M^{-I}, x^{-}(\tau)] &= 0 \\
[M^{-I}, x^{I}(\tau)] &= -i\delta_{IJ}x^{-}(\tau)
\end{aligned} \tag{11.80}$$

Note that these differ from our results above by exactly the following term in each case

$$\frac{im^2}{2}\left(\frac{x_0^I}{p^+}\frac{dx^\mu}{d\tau} + \frac{dx^\mu}{d\tau}\frac{x_0^I}{p^+}\right)$$

which corresponds to a reparameterization by

$$\lambda = \frac{im^2 x_0^I}{p^+}$$

## 6 Zwiebach's 12.10: Reparameterization and constraints

(a) Check

$$\dot{\xi^\tau} = me^{im\tau}\cos m\sigma = \xi^{\sigma\prime}$$
$$\dot{\xi^\sigma} = ime^{im\tau}\sin m\sigma = \xi^{\tau\prime}$$

(b) First note that

$$\frac{d\tau}{d\tau'} = 1 - \epsilon\dot{\xi}^\tau, \qquad \frac{d\tau}{d\sigma'} = -\epsilon\xi^{\tau\prime}, \qquad \frac{d\sigma}{d\sigma'} = 1 - \epsilon\xi^{\sigma\prime}, \qquad \frac{d\sigma}{d\tau'} = -\epsilon\dot{\xi}^\sigma$$

Check that

$$\frac{\partial X}{\partial \tau'} = \frac{\partial X}{\partial \tau}\frac{\partial \tau}{\partial \tau'} + \frac{\partial X}{\partial \sigma}\frac{\partial \sigma}{\partial \tau'} = \frac{\partial X}{\partial \tau} - \epsilon\left(\frac{\partial X}{\partial \tau}\dot{\xi}^\tau + \frac{\partial X}{\partial \sigma}\dot{\xi}^\sigma\right)$$
$$\frac{\partial X}{\partial \sigma'} = \frac{\partial X}{\partial \tau}\frac{\partial \tau}{\partial \sigma'} + \frac{\partial X}{\partial \sigma}\frac{\partial \sigma}{\partial \sigma'} = \frac{\partial X}{\partial \sigma} - \epsilon\left(\frac{\partial X}{\partial \tau}\xi^{\tau\prime} + \frac{\partial X}{\partial \sigma}\xi^{\sigma\prime}\right)$$

Using results from part (a), we can write

$$\begin{aligned}\frac{\partial X}{\partial \tau'} \pm \frac{\partial X}{\partial \sigma'} &= \frac{\partial X}{\partial \tau} \pm \frac{\partial X}{\partial \sigma} - \epsilon\left[\left(\frac{\partial X}{\partial \tau} \pm \frac{\partial X}{\partial \sigma}\right)\dot{\xi}^\tau + \left(\frac{\partial X}{\partial \sigma} \pm \frac{\partial X}{\partial \tau}\right)\dot{\xi}^\sigma\right] \\ &= \left(\frac{\partial X}{\partial \sigma} \pm \frac{\partial X}{\partial \tau}\right)\left(1 - \epsilon\dot{\xi}^\tau - \epsilon\dot{\xi}^\sigma\right)\end{aligned}$$

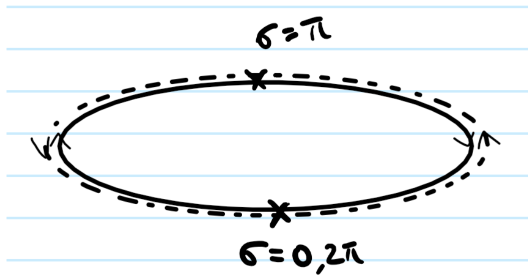
Now, see that the classical constraints yield

$$\left(\frac{\partial X}{\partial \tau'} \pm \frac{\partial X}{\partial \sigma'}\right)^2 = \left(\frac{\partial X}{\partial \sigma} \pm \frac{\partial X}{\partial \tau}\right)^2\left(1 - \epsilon\dot{\xi}^\tau - \epsilon\dot{\xi}^\sigma\right)^2 = 0$$

from which the constraints in the $(\tau', \sigma')$ coordinates follow (by adding and subtracting the $\pm$ terms from each other).

## 7 Zwiebach's 13.5: Unoriented closed strings

(a) The following image shows how the first and second strings are related to each other.




(b) Zwiebach's 13.24 gives

$$X^\mu(\tau, \sigma) = x_0^\mu + \sqrt{2\alpha'}\alpha_0^\mu \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{e^{-in\tau}}{n}(\alpha_n^\mu e^{in\sigma} + \overline{\alpha}_n^\mu e^{-in\sigma})$$

Since $\sigma$ only occurs in the exponential terms at the end, we can immediately conclude

$$\Omega x_0^I \Omega^{-1} = x_0^I, \qquad \Omega \alpha_0^I \Omega^{-1} = \alpha_0^I$$

Using the identity $e^{\pm in(2\pi - \sigma)} = e^{\mp in\sigma}$ then gives

$$\Omega \alpha_n^I \Omega^{-1} = \overline{\alpha}_n^I, \qquad \Omega \overline{\alpha}_n^I \Omega^{-1} = \alpha_n^I$$

(c) From Zwiebach's 13.24, we have

$$X^-(\tau, \sigma) = x_0^- + \sqrt{2\alpha'}\alpha_0^- \tau + i\sqrt{\frac{\alpha'}{2}} \sum_{n \neq 0} \frac{e^{-in\tau}}{n}(\alpha_n^- e^{in\sigma} + \overline{\alpha}_n^- e^{-in\sigma})$$

Using Zwiebach's 13.37 and 13.40, we can write

$$\overline{\alpha}_n^- = \frac{1}{p^+\sqrt{2\alpha'}} \sum_{p \in \mathbb{Z}} \overline{\alpha}_p^I \overline{\alpha}_{n-p}^I, \qquad \alpha_n^- = \frac{1}{p^+\sqrt{2\alpha'}} \sum_{p \in \mathbb{Z}} \alpha_p^I \alpha_{n-p}^I$$

which, when we use (b), immediately gives us

$$\Omega \overline{\alpha}_n^- \Omega^{-1} = \alpha_n^-, \qquad \Omega \alpha_n^- \Omega^{-1} = \overline{\alpha}_n^-$$

Moreover, Zwiebach's 13.41 tells us that

$$\alpha_0^- = \overline{\alpha}_0^- \implies \Omega \alpha_0^- \Omega^{-1} = \alpha_0^-$$

With our assumption that $\Omega x_0^- \Omega^{-1}$, we have our desired $\Omega X^-(\tau, \sigma)\Omega^{-1} = X^-(\tau, 2\pi - \sigma)$.

Since the Hamiltonian for the closed string is given as

$$H = L_0^\perp + \overline{L}_0^\perp - 2$$

it does not change under action of the twist operator and we say that orientation reversal is a symmetry of closed string theory.

(d) When $N^\perp = \overline{N}^\perp = 0$, the only state in the state space is $|p^+, \vec{p}_T\rangle$. Since there is only one state, the twist eigenvalue is $+1$.

When $N^\perp = \overline{N}^\perp = 1$, Zwiebach's 13.64 gives the general form of the state as

$$\sum_{I,J} R_{IJ} a_1^{I\dagger} \overline{a}^{J\dagger} |p^+, \vec{p}_T\rangle$$

Under action of the twist operator, the barred and unbarred raising operators get swapped and $R_{IJ} \to R_{JI}$. If we write $R_{IJ}$ as in Zwiebach's 13.65,

$$R_{IJ} = S_{IJ} + A_{IJ}$$

then the symmetric states have eigenvalue $+1$ while the antisymmetric states have eigenvalue $-1$.

When $N^\perp = \overline{N}^\perp = 2$, there are three distinct contributions to the state space:

$$\sum_{I,J} R_{I,J} a_2^{I,\dagger} \overline{a}_2^{J,\dagger} |p^+, \vec{p}_T\rangle$$
$$\sum_{IJK} \left[ R_{I,JK} a_2^{I\dagger} \overline{a}_1^{J\dagger} \overline{a}_1^{K\dagger} + \overline{R}_{I,JK} \overline{a}_2^{I\dagger} a_1^{J\dagger} a_1^{K\dagger} \right] |p^+, \vec{p}_T\rangle$$
$$\sum_{IJKL} R_{IJ,KL} a_1^{I\dagger} a_1^{J\dagger} \overline{a}_1^{K\dagger} \overline{a}_1^{L\dagger} |p^+, \vec{p}_T\rangle$$

In each case, we have a square matrix which can be decomposed into a sum of its symmetric and antisymmetric parts. Hence, as in the $N^\perp = 1$ case, we have eigenvalue $+1$ for the symmetric states and $-1$ for the antisymmetric states.

To obtain the unoriented theory, we would have to discard the eigenvalue $-1$ states. Hence, the massless fields are the graviton and the dilaton only.

# 8 Zwiebach's 14.3: Massive level in the open superstring

(a) We need to count all possible choices of $b_i$'s such that there are no repetitions. We know that ordering does not matter because one list of $b_i$'s only differs from another reordered list of the same $b_i$'s by a constant factor.

Hence, the number of inequivalent $b^{i_1}b^{i_2}$'s is $8C2 = 28$.

The number of inequivalent $b^{i_1}b^{i_2}b^{i_3}$'s is $8C3 = 56$.

The number of inequivalent $b^{i_1}b^{i_2}b^{i_3}b^{i_4}$'s is $8C4 = 70$.

(b) Consider $\alpha' M^2 = 1$. For the NS sector, we have

$$\alpha' M^2 = N^\perp - \frac{1}{2}$$

Hence, we get $N^\perp = 3/2$. From Zwiebach's 14.38, we know that there are three states that contribute to the NS sector; they are

$$\{\alpha^I_{-1}b^J_{-1/2}, b^I_{-3/2}, b^I_{-1/2}, b^J_{-1/2}, b^K_{-1/2}\}|NS\rangle$$

They contribute (respectively) $8 \times 8 + 8 + 56 = 128$ states in total.

For the R sector, we have

$$\alpha' M^2 = N^\perp$$

Hence, we get $N^\perp = 1$. From Zwiebach's 14.54, there are two ways to construct such (fermionic) states

$$\{\alpha^I_{-1}, d^I_{-1}\}|R_a\rangle$$

They contribute (respectively) $8 \times 8 + 8 \times 8 = 128$ states (there are 8 $|R_a\rangle$ states) in total, which agrees with the NS sector.

Now consider $\alpha' M = 2$. For the NS sector, we get $N^\perp = 5/2$. There are 7 ways to construct the contributing states

$$\{\alpha^I_{-2}b^J_{-1/2}, \alpha^I_{-1}\alpha^J_{-1}b^K_{-1/2}, \alpha^I_{-1}b^J_{-1/2}b^K_{-1/2}b^L_{-1/2}, \alpha^I_{-1}b^J_{-3/2},$$
$$b^I_{-1/2}b^J_{-1/2}b^K_{-1/2}b^L_{-1/2}b^M_{-1/2}, b^I_{-3/2}b^J_{-1/2}b^K_{-1/2}, b^I_{-5/2}\}|NS\rangle$$

They contribute $(8 \times 8) + (28 \times 8) + (8 \times 56) + (8 \times 8) + 8C5 + (8 \times 28) + 8 = 1088$ states.

For the $R$ sector, we get $N^\perp = 2$. From Zweibach's 14.54, there are 5 ways to construct the contributing states

$$\{\alpha^I_{-2}, \alpha^I_{-1}\alpha^J_{-1}, d^I_{-1}, d^J_{-1}\}|R_a\rangle$$
$$\{\alpha^I_{-1}d^J_{-1}, d^I_{-2}\}|R_{\bar{a}}\rangle$$

They contribute $8 \times (8 + 28 + 28 + 8 \times 8 + 8) = 1088$ states in total, as desired.