#1 (Jan 9)

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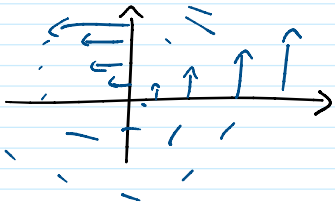
→ 3 asst, 1 final proj.

## Vector fields as dynamical systems

→ in a coord. system, $(x^1, \dots, x^n)$, vector field is

$V = \sum V_i(x^1, \dots, x^n) \frac{\partial}{\partial x^i}$ w/ $V_i$ smooth func.

eg. $V = x\frac{\partial}{\partial y} - y\frac{\partial}{\partial x}$



Corr. dynamical system is called the <u>flow</u> of $V$

→ at $t=0$ IC $p = (x^1(0), \dots, x^n(0))$

→ evolve as func. of $t \in \mathbb{R}$ into path $x(t) = (x^1(t), \dots, x^n(t))$

$$x: \begin{cases} \mathbb{R} \longrightarrow \mathbb{R}^n \\ t \qquad (x^1, \dots, x^n) \\ \text{Interval } (\varepsilon_-, \varepsilon_+) \text{ around } 0 \end{cases}$$

such that $\boxed{\frac{d}{dt} x(t) = V(x(t))}$ ☆

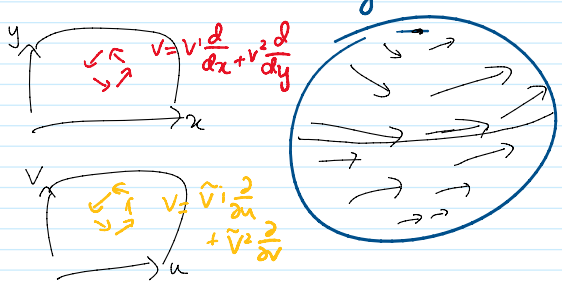
$\iff \frac{d}{dt} x^i(t) = V^i(x^1(t), \dots, x^n(t))$ "system of $n$ coupled $1^{st}$ order non-linear ODEs"

Why formulate in terms of v fields and not ODEs?

$V = V^i \frac{\partial}{\partial x^i} = \tilde{V}^i \frac{\partial}{\partial \tilde{x}^i}$ w/ $\tilde{x}^i = \tilde{x}^i(x^1, \dots, x^n)$

$x^i = x^i(\tilde{x}^1, \dots, \tilde{x}^n)$

Not only coord indep. but also allows def$^n$ of system of ODEs on topologically non-trivial space.




In this way, systems of $n$ coupled $1^{st}$ order nonlinear ODE generalize to $n$-dim manifolds

or $\underbrace{\text{n-dim manifolds}}_{\text{open sets of } \mathbb{R}^n \text{ glued by coord. transformations.}}$

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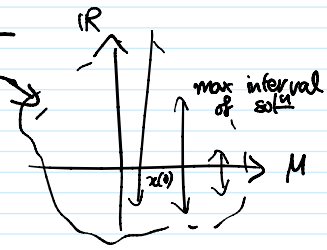
"Integrating" a vector field means to solve the ODE i.e. to find for any init. cond. $x(0)$ the maximal interval $I \subset \mathbb{R}$ and the integral curve satisfying $t=0 \Rightarrow x(0)$, $\frac{d}{dt} x(t) = V(x(t))$

<u>Theorem (Picard)</u>: $\exists!$ solution to this problem $C^\infty$ in init. conds.

i.e. $\exists!$ open set $U \subset M \times \mathbb{R}$ maximal such that

w/ $V \in \mathfrak{X}(M)$ set of v fields and $M$ mfld

and $\Phi : U \to M$ "flow" $C^\infty$ s.t.




$$\begin{cases} \Phi(x,0) = x \\ \frac{\partial}{\partial t} \Phi(x,t) = V(\Phi(x,t)) \end{cases}$$

If $U = M \times \mathbb{R}$, $V$ complete. If $M$ compact, any $V$ complete.

eg. incomplete, on $M = \mathbb{R}$

consider $V = x^2 \frac{\partial}{\partial x}$ flow exists only for finite time

<u>Major enterprise</u>: classification of v. fields (locally)

⓪ "non singular" ie nonvanishing

open nbhd $\subset \mathbb{R}^n$ if $V(0) \neq 0$ then $V$ nonvanishing on some nbhd.

<u>Thm (Frobenius)</u>: $\exists$ coords $(x^1, \dots, x^n)$ in nbhd s.t. $V = \frac{\partial}{\partial x^1}$

① near a "singular" point ie $V(0) = 0$

→ classical results focus on linearization of $V$ at $0$.

$$V^i(x^1, \dots, x^n) = \sum_{j=1}^n c^i_j x^j + h.o.t.$$

const

linearization of $V$: $V^{lin} = \sum_{i,j=1}^n c^i_j x^j \frac{\partial}{\partial x^i}$

main Q: is $V$ linearizable? ie. $\exists$? coord change making $V \cong V^{lin}$

eg. <u>Thm</u>: If $c^i_j = \delta^i_j$ then $V$ linearizable

Vector fields have 2 dramatically different behaviour types:

① Ergodic (deterministic chaos)

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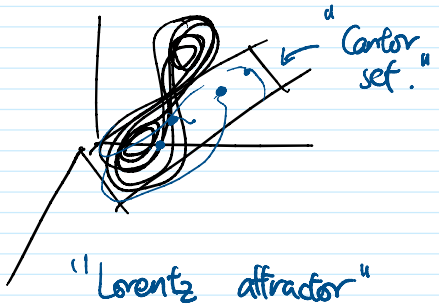
② Integrable ( integrable systems)

①: almost every integral curve is "dense" in some sense