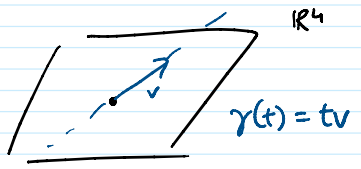
#3 (Jan 16)

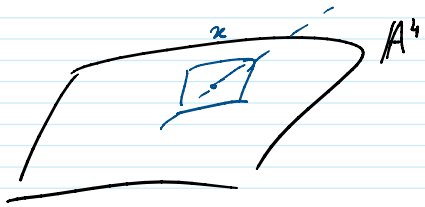
Tuesday, 16 January 2024 10:11 am

Today: Galilean Spacetime & phase portraits for 1D systems

free action → no group element can have fixed points except Id.

Spacetime = $\mathbb{A}^4$, an affine space for $\mathbb{R}^4$





a choice of "basepoint" $x_0 \in \mathbb{A}^4$ immediately identifies $\mathbb{A}^4 \xrightarrow{\cong} \mathbb{R}^4$

$x_0 + v \longleftarrow\!\shortmid\ v$

$\mathbb{R}^4$ equipped w/ linear function $t : \mathbb{R}^4 \to \mathbb{R}$

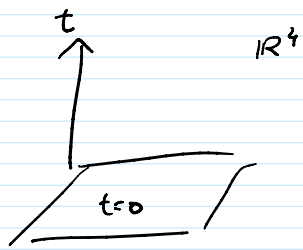
→ time coord. ⇒ well-defined notion of time interval between events

If $x, y \in \mathbb{A}^4$, $y - x \in \mathbb{R}^4$ and $t(y-x) \in \mathbb{R}$

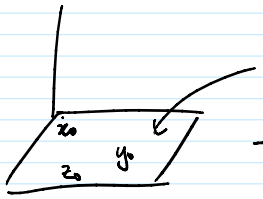
Finally, on kernel of t, have pos. def inner product $(\cdot,\cdot)$ "Euclidean" inner product

→ ($\mathbb{A}^4$, $\mathbb{R}^4$ affine space, t time coord., $(\cdot,\cdot)$)

affine space time



<u>Consequences</u>: if we fix an event $x_0 \in \mathbb{A}^4$, then $\mathbb{R}^4 \xrightarrow{x_0} \mathbb{A}^4$



3d space of events simult with x_0 $(t=0)$

→ have a notion of distance btwn simult. events.

$$d(y_0, z_0) = \sqrt{(y_0 - z_0, y_0 - z_0)}$$

<u>Galilean symmetries</u>:

- $(\vec{x}, t) \mapsto (\vec{x} + \vec{s}, t+s)$ where $(\vec{s}, s) \in \mathbb{R}^4$
- $(\vec{x}, t) \mapsto (A\vec{x}, t)$ for $A \in O(3)$ group of orthog. matrices, $AA^T = I$

  → rot. (det 1), refl. (det −1)

  ↳ $SO(3)$

t

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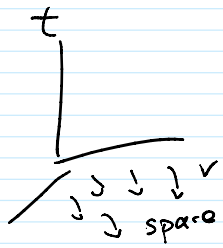


- uniform rectilinear motion $(\vec{x}, t) \mapsto (\vec{x} + t\vec{v}, t)$

→ composition of these 3 actions form Galilean group of symmetries of spacetime.

Q: why not $(x,t) \mapsto (\overset{\in O(3)}{A(t)}\vec{x}, t)$?

<u>Convenient notation</u>: $\begin{pmatrix} t \\ \vec{x} \end{pmatrix} \longmapsto \left(\begin{array}{c|c} 1 & 0 \\ \hline \vec{v} & A \end{array}\right)\begin{pmatrix} t \\ \vec{x} \end{pmatrix} + \begin{pmatrix} s \\ \vec{s} \end{pmatrix}$

$$= \begin{pmatrix} ct + s \\ A\vec{x} + t\vec{v} + \vec{s} \end{pmatrix}$$

→ must show it has this form

MAT461 Hamiltonian 16 Jan

1D systems

→ states = that information which determines motion

Newton, Galileo: x position & v velocity at a moment in time.

Newton's model: 2nd order ODE

$$\ddot{x}(t) = F(x, \dot{x}, t)$$

This ODE can be written in 1st order form $x = x$, $y = \dot{x}$

vector field on phase space

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ F \end{pmatrix}_{(x,y,t)}$$

Phase space: $\{(x,y)\}$ "tangent space TX" & $X = \mathbb{R}$

$\to (x,y)$ has tangent vector $y\frac{\partial}{\partial x}$ to

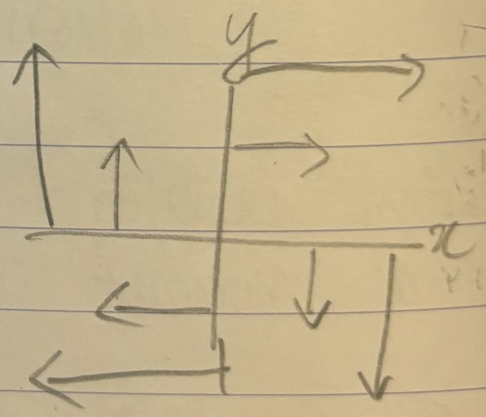
$X = \mathbb{R}$ $\quad$ TX

$\rightarrow$ (x,y) denotes tangent vector $y\frac{\partial}{\partial x}$ to the point x

ie. Newton's law is a v. field on phase space. Integral curves of v. field are phase trajectories.

eg. $\ddot{x} = -x$

$$\Leftrightarrow \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} y \\ -x \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$



$\rightarrow$ integral curve through an initial state $\begin{pmatrix} x(0) \\ y(0) \end{pmatrix}$ is

$$\dot{v} = Av \Rightarrow v = e^{tA} v(0)$$

$$= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \cos t + \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \sin t$$

$$= \begin{pmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{pmatrix}$$

$$\Rightarrow \quad x(t) = x(0)\cos t + \dot{x}(0)\sin t$$

$$\dot{x}(t) = -x(0)\sin t + \dot{x}(0)\cos t$$

<u>Sage math</u> <u>CoCale</u>