#4 (Jan 18)

Thursday, 18 January 2024 11:12 am

Phase space $= \underbrace{TA^1_{\text{space}}}_{\text{tangent}} = \{ (x,\dot{x}) : x \in A^1_{\text{space}} , \dot{x} \in T_x A^1_{\text{space}} \}$

<u>1D systems</u>

$\ddot{x} = f(x)$ when force indep. of $\dot{x}$, it turns out to be <u>conservative</u>

i.e. we can define an energy function on phase space

$$E = \underbrace{\tfrac{1}{2}\dot{x}^2}_{KE} + \underbrace{U(x)}_{PE} , \qquad U(x) = -\int_{x_0}^{x} f(\tilde{x})\, d\tilde{x}$$

<u>Theorem</u>: E is preserved by evolution (i.e. phase flow)

<u>Proof</u>: Consider $E(x(t))$ where $x(t)$ satisfies EOM

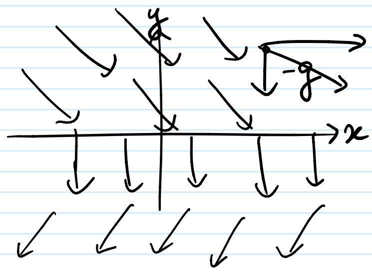
$$\dot{E} = \dot{x}\ddot{x} + \frac{\partial U}{\partial x}\dot{x} = \dot{x}\big(\ddot{x} - f(x)\big) = 0$$

<u>Corollary</u>: a trajectory (i.e. a sol$^{\underline{n}}$) must remain on a level set of $E(x,y)$.

eg. $f(x) = -kx \qquad U(x) = +\int_{x_0}^{x} kx = \frac{1}{2}kx^2$

$f(x) = -m\underset{\text{constant } 9.8\ m/s^2}{g}$

$V = y\frac{\partial}{\partial x} - g\frac{\partial}{\partial y}$



$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} + \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

$$\dot{v} = Av + b$$

linear solution: $e^{tA}v(0)$

full solution: $v(t) = e^{tA}\left(\int_0^t e^{-sA} b\, ds\right) v(0)$

$$e^{t\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}} = 1 + t\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{t^2}{2!}\cancel{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}^2}^{0} + \cancel{\cdots}^{0}$$

$$= \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$$

$$\int_0^t \begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 0 \\ -g \end{pmatrix} ds = \begin{pmatrix} g\int_0^t s\, ds \\ -g\int_0^t ds \end{pmatrix} = g\begin{pmatrix} \frac{1}{2}t^2 \\ -t \end{pmatrix}$$

$$v(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}\left(g\begin{pmatrix} \frac{1}{2}t^2 \\ -t \end{pmatrix} + \begin{pmatrix} x(0) \\ y(0) \end{pmatrix}\right)$$

$$= \Big(-\tfrac{1}{2}t^2 g + x(0) \Big)$$

$$= \begin{pmatrix} -\frac{1}{2}t^2 g + x(0) \\ -tg + \dot{x}(0) \end{pmatrix}$$

$$\Rightarrow \quad x(t) = x(0) - \frac{1}{2}gt^2 \quad , \quad \dot{x}(t) = \dot{x}(0) - gt$$