Introduction to JuMP

Modeling Optimization Problems

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Getting Started

- All code and slides are in GIT repo `https://github.com/jalving/JuMPTutorial_2018.git`
- We will use `Julia` notebooks that will be executed on `JuliaBox`
- Access `http://juliabox.com`
- Click on Sync tab and clone GIT repo `https://github.com/jalving/JuMPTutorial_2018.git`
- Click on Jupyter tab and access notebooks under JuMPTutorial_2018
- Open `install_packages.ipynb`, click on "Cell" and then on "Run All"

Optimization Overview

Standard Nonlinear Programming (NLP) Problem

$$\min_{x} f(x) \qquad \text{Objective function (e.g., economics)}$$

$$s.t.\ \ c(x) = 0 \qquad \text{Equality constraints (e.g. physical equations)}$$

$$g(x) \geq 0 \qquad \text{Inequality constraints (e.g. physical limits)}$$

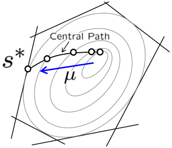
This tutorial considers PDE-constrained optimization problems, which we will cast as standard NLPs.

Example NLP

$$\begin{aligned} \min_{x_1, x_2} \ & x_1^2 + x_2^2 \\ s.t. \ & x_1 + x_2 = 5 \\ & e^{x_1} \leq 4 \end{aligned}$$

Ipopt - Interior Point OPTimizer

- To solve NLPs we will use the interior-point NLP solver `Ipopt`.
- `Ipopt` uses a logarithmic barrier to transform the NLP into an equality-constrained subproblem:



$$\min_x \phi^{\mu}(x) := f(x) - \mu \sum_{j=1}^{m} ln(g_j(x))$$

$$s.t.\ c(x) = 0$$

- The barrier subproblem is solved for decreasing values of μ to reach a solution of the original NLP. This is done by solving:

$$\begin{aligned} \nabla_x \mathcal{L}(x,\lambda) = \nabla_x \phi^{\mu}(x) + \nabla_x c(x)\lambda &= 0 \\ c(x) &= 0 \end{aligned} \implies \begin{bmatrix} H(x,\lambda) & \nabla_x c(x) \\ \nabla_x c(x)^T & \end{bmatrix} \begin{bmatrix} \Delta x \\ \Delta\lambda \end{bmatrix} = -\begin{bmatrix} \nabla_x \mathcal{L}(x,\lambda) \\ c(x) \end{bmatrix}$$

- Highly efficient sparse linear algebra and globalization strategies enable solution of complex NLPs.
- `Ipopt` can solve NLPs with *millions* of variables and constraints.

Julia and JuMP

`Julia`

- Language for high-performance scientific computing
- Extensive library of tools for optimization, statistics, plotting, graphs,...
- Performance comparable to C
- Designed to enable parallelism

`JuMP`

- Algebraic modeling language for optimization written in `Julia`
- Syntax close to natural mathematical expressions
- Processes problems at speeds comparable to `GAMS` and `AMPL`
- Support for open source & commercial solvers (e.g. `Ipopt`, `Gurobi`)
- Automatic computation of derivatives

Solving Optimization Problems with JuMP

Consider our previous NLP example:

$$\min_{x_1,x_2} \; x_1^2 + x_2^2$$
$$s.t. \; x_1 + x_2 = 5$$
$$e^{x_1} \leq 4$$

```julia
using JuMP
using Ipopt
m = Model()
@variable(m, x1)
@variable(m, x2)
@objective(m, Min, x1^2 + x2^2)
@constraint(m, x1+x2 == 5)
@NLconstraint(m, exp(x1) <= 4)
solve(m)
```

- `Ipopt` internally solves a sequence of barrier problems of the form:

$$\min_{x_1,x_2} \; x_1^2 + x_2^2 - \mu \log(4 - e^{x_1})$$
$$s.t. \; x_1 + x_2 = 5$$

- Example implemented in Julia notebook `simple_model.ipynb`

Solving Optimization Problems with JuMP

`JuMP` also enables compact syntax expressions:

$$\min_{x_1, x_2} \sum_{j \in \{1,2\}} x_j^2$$
$$s.t. \sum_{j \in \{1,2\}} x_j = 5$$
$$e^{x_1} \leq 4$$

```julia
using JuMP
using Ipopt
m = Model()
S = collect(1:2)
@objective(m,Min,sum(x[j]^2 for j in S))
@variable(m,x[S])
@constraint(m,sum(x[j] for j in S) == 5)
@NLconstraint(m,exp(x[1]) <= 4)
solve(m)
```

Example implemented in Julia notebook `simple_model_set.ipynb`

MILP Problems

Standard Mixed Integer Linear Programming Problem