STAT 515 Spring 2025
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Set Theory

1. Course Overview and Some Notation

Statistics is often used for:

1. **Estimation** of unknown **parameters** from a population of interest (e.g. the mean height of professional basketball players, the proportion of likely voters who favor one political candidate or the other)
2. **Inference** of an unknown parameter (e.g. uncertainty quantification, hypothesis testing)
3. **Measures of association**. If we have two variables of interest (e.g. years of experience and annual salary, e.g. highest level of education completed and level of personal happiness) can we say that these two things are significantly **associated** with each other?

This course aims to develop the tools needed to perform the three statistical procedures above. However, before getting there, we first need to develop the basic building block of statistics: **probability**.

We first define some notation. For two numbers a and b, where $a < b$:

- (a, b) means the interval of all numbers x where $a < x < b$
- $[a, b]$ means the interval of all numbers x where $a \leq x \leq b$
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Note that we can set a to be $-\infty$ and b to be ∞.

- $(-\infty, \infty)$ is the set of **all real** numbers x where $-\infty < x < \infty$
- If a is real, then $[a, \infty)$ is the set of all real numbers x where $a \leq x < \infty$
- If b is real, then $(-\infty, b]$ is the set of all real numbers x where $-\infty < x \leq b$

2. Basics of Sets

A **set** is a collection of elements. A set can either be **finite *or* infinite.**

- If a set A consists of any real number in an interval, then we use "(", "[", ")", "]" to denote the set.
- If a set A can only take specific values, then we use "{" and "}" to denote it.
- Elements can be numbers but they do not need to be. e.g. the set of rainbow colors is {Red, Orange, Yellow, Green, Blue, Indigo, Violet}.

Example:

- $A = [0,1]$ means the set of all real numbers between 0 and 1 (**infinite** number of elements)
- $A = \{0, 1\}$ means the set containing <u>ONLY</u> "0" or "1" (only **two** elements)

The **empty set** is a set containing **no** elements. We denote it by either "{ }" or "∅"

A set B is said to be a **subset** of a set A if **all** the elements in B are also contained in the set A.

- The empty set ∅ is always considered to be a subset of any nonzero set.
- A set A is always a subset of itself.

Example:

- Let $A = [0,1]$. Then $B = (0.5, 0.7]$, $C = [0.2, 0.5]$, $D = \{0.5\}$, and $E = [0,1]$ are all subsets of A.

- If $A = \{0, 1\}$, then there are four total subsets. What are they?

In this class, we will mainly discuss sets in the context of statistics, where they are commonly referred to as **events.**

A **statistical experiment** is a process which generates a single outcome where:

1. There is more than one *possible* outcome;
2. It is known in advance what the possible outcomes are;
3. The outcome to be generated cannot be predicted with 100% certainty.

The **sample space** S of an experiment is the set of **all** possible outcomes of the experiment.

- A sample space S can either contain a finite *or* an infinite number of outcomes.
- The outcomes in a sample space S are called **sample points**

Example. Rolling two dice and recording the rolls is a statistical experiment. We can write each possible outcome in the form (roll 1, roll 2). The sample space S consists of 36 outcomes:

$$S = \begin{Bmatrix} (1,1) & (1,2) & (1,3) & (1,4) & (1,5) & (1,6) \\ (2,1) & (2,2) & (2,3) & (2,4) & (2,5) & (2,6) \\ (3,1) & (3,2) & (3,3) & (3,4) & (3,5) & (3,6) \\ (4,1) & (4,2) & (4,3) & (4,4) & (4,5) & (4,6) \\ (5,1) & (5,2) & (5,3) & (5,4) & (5,5) & (5,6) \\ (6,1) & (6,2) & (6,3) & (6,4) & (6,5) & (6,6) \end{Bmatrix}$$

Example. In a statistical experiment, we roll two dice. The outcome is the sum of the two rolls. What is the sample space S in this case?

Example. In a statistical experiment, you buy a new light bulb. The outcome is the amount of time in days until the light bulb dies. The sample space consists of all values greater than or equal to 0, i.e.

$$S = [0, \infty)$$

An **event** A is any subset of the sample space S of a statistical experiment. Recall that S is the set of all possible outcomes of the experiment.

- If the sample space is **finite**, then we can describe A by listing the set of outcomes in A, separated by commas and inside left and right brackets.

Example. We roll two dice and record the rolls. An event of interest is rolling snake eyes (i.e. rolling two ones). In this case,

$$A = \{(1,1)\}$$

Example. We roll two dice and record the sum of the two rolls. An event of interest is the set of all pairs whose sum is equal to seven. In this case,

$$A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

Example. We have a list of all track and field events. An event of interest is the set of events that involve throwing. In this case,

$$A = \{\text{javelin}, \text{discus}, \text{shot put}, \text{hammer throw}\}$$

Example. You buy a new light bulb. The event is "the light bulb lasts at least 300 days." In this case,

$$A = [300, \infty)$$

Going forward, we shall use "set" and "event" interchangeably.

3. Containment and Equality of Events

For two events A and B, we say that A is **contained** in B if every outcome in A is also in B.

- This is the same as saying that A is a subset of B.
- We write "$A \subset B$".

We say that two events A and B are **equal** if they contain exactly the same elements. In other words, "$A \subset B$" and "$B \subset A$".

Example. Let A be the set of all aquatic sports (e.g. swimming, diving, water polo, etc.). featured at the Summer Olympics and B be the set of **all** sports featured at the summer Olympics. Clearly, $A \subset B$.

Example. Let $A =$ the set of all odd numbers, and let B = the set of all numbers of the form $2k + 1$, where k is an integer. Then $A = B$.

In a statistical experiment, let S be the sample space, i.e. the set of all possible outcomes. Let A be an event in this experiment, so that $A \subset S$.

- The **complement** A^c of the event A is the set of all outcomes in S which are NOT in A.
- We may also write

  $$A^c = S \backslash A,$$

  where "$S \backslash A$" means every element in S that is NOT in A.
- It is clear that $S^c = S \backslash S = \emptyset$

Example. Roll two dice. Let A be the event that the sum of the two rolls is less than or equal to seven. Then

$$A = \left\{\begin{matrix} (1,1),(1,2),(1,3),(1,4),(1,5),(1,6), \\ (2,1),(2,2),(2,3),(2,4),(2,5), \\ (3,1),(3,2),(3,3),(3,4), \\ (4,1),(4,2),(4,3), \\ (5,1),(5,2), \\ (6,1) \end{matrix}\right\},$$

and

$$A^c = \left\{\begin{matrix} \\ (2,6) \\ (3,5),(3,6) \\ (4,4),(4,5),(4,6) \\ (5,3),(5,4),(5,5),(5,6) \\ (6,2),(6,3),(6,4),(6,5),(6,6) \end{matrix}\right\}.$$

Example. There are 7 continents in the world, and penguins can be naturally found in the wild on four of them. These continents are

$$A = \{\text{Antarctica}, \text{South America}, \text{Africa}, \text{Australia}\}.$$

Then the set of continents where penguins cannot be naturally found is

$$A^c = \{\text{North America}, \text{Europe}, \text{Asia}\}.$$

Example. You buy a new light bulb. The event A is “the light bulb lasts at least 300 days.” In this case,

$$A = [300, \infty)$$

Since the sample space is $S = [0, \infty)$, the complement of A is “the light bulb lasts less than 300 days.” Then

$$A^c = [0,300).$$

4. Unions and Intersections

Two set operations that we use very often to make new sets are the **union** and the **intersection.**

The **union** of two events A and B is the set of all outcomes in either A **OR** in B **OR** in both A and B. We denote the union of A and B as

$$A \cup B$$

The **intersection** of two events A and B is the set of all outcomes in both A **AND** B. We denote the intersection of A and B as

$$A \cap B$$

Based on this definition, we can observe a few things:

- If S is the sample space of an experiment, then $S = A \cup A^c$.
- For any set A:
  - $A \cup \emptyset = A$
  - $A \cap \emptyset = \emptyset$

Example. We roll two dice. Let

$$A = \{\text{the sum of the rolls is 7}\}$$

and

$$B = \{\text{at least one of the rolls is a 2}\}$$

Then

$$A = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$$

and

$$B = \left\{\begin{matrix}(1,2), (2,2), (3,2), (4,2), (5,2), (6,2), \\ (2,1), (2,3), (2,4), (2,5), (2,6)\end{matrix}\right\}$$

Then it is clear that

$$A \cup B = \left\{\begin{matrix}(1,6), (2,5), (3,4), (4,3), (5,2), \\ (6,1), (1,2), (2,2), (3,2), (4,2), \\ (6,2), (2,1), (2,3), (2,4), (2,6)\end{matrix}\right\},$$

and

$$A \cap B = \{(2,5), (5,2)\}.$$

Example. You buy a new lightbulb. Let A be the event that the lightbulb lasts at least 300 days, and let B be the event that the lightbulb lasts strictly less than 400 days. Then

$$A = [300, \infty) \qquad \text{and} \qquad \mathrm{B} = [0,400)$$

Then

$$A \cup B = [0, \infty) = S$$

and

$$A \cap B = [300, 400)$$

We say that two events A and B are **mutually exclusive** if they cannot both occur simultaneously. One event automatically excludes the other. A formal definition is given below.

A and B are said to be **mutually exclusive** if

$$A \cap B = \emptyset$$

In other words, the intersection of A and B can only be the empty set.

Example. We roll two dice and take the sum of the two dice as the outcome. We define the event A as the set of all even outcomes and the event B as the set of all odd outcomes. Then

$$A = \{2,4,6,8,10,12\} \quad \text{and} \quad B = \{3,5,7,9,11\}$$

Clearly, A and B are mutually exclusive since

$$A \cap B = \emptyset.$$

Example. You buy a lightbulb. Let A be the event that the lightbulb lasts more than 300 days and let the event B be the event that the lightbulb lasts less than 150 days. Then

$$A = (300, \infty) \quad \text{and} \quad B = [0,150)$$

Then A and B are mutually exclusive. Intuitively, this is because a lightbulb cannot simultaneously last more than 300 days **AND** less than 150 days.

5. DeMorgan's Laws

For two events A and B, we have the following two results, also known as **DeMorgan's Laws.**

$$(A \cup B)^c = A^c \cap B^c \quad \text{and} \quad (A \cap B)^c = A^c \cup B^c$$

These laws can be established pictorially.

Example. We roll two dice and take the sum of the two dice as the outcome. We define the event A as the set of all outcomes that are multiples of two and the event B as the set of all outcomes that are multiples of three.

Then

$$A = \{2,4,6,8,10,12\} \quad \text{and} \quad B = \{3,6,9,12\}$$

The set $(A \cup B)^c = \{5,7,11\}$ can be determined by either taking the complement of $A \cup B$ **OR** by using the first of DeMorgan's Laws.

The set $(A \cap B)^c = \{2, 3, 4, 5, 7, 8, 9, 10, 11\}$ can be determined by either taking the complement of $A \cup B$ **OR** by using the first of DeMorgan's Laws.